The Golden-Thompson inequality — historical aspects and random matrix applications

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Abstract

The Golden-Thompson inequality, $\operatorname{Tr}(e^{A+B}) \leq \operatorname{Tr}(e^A e^B)$ for A, B Hermitian matrices, appeared in independent works by Golden and Thompson published in 1965. Both of these were motivated by considerations in statistical mechanics. In recent years the Golden-Thompson inequality has found applications to random matrix theory. In this survey article we detail some historical aspects relating to Thompson's work, giving in particular an hitherto unpublished proof due to Dyson, and correspondence with Pólya. We show too how the 2×2 case relates to hyperbolic geometry, and how the original inequality holds true with the trace operation replaced by any unitarily invariant norm. In relation to the random matrix applications, we review its use in the derivation of concentration type lemmas for sums of random matrices due to Ahlswede-Winter, and Oliveira, generalizing various classical results.

1 Introduction

Let A and B be $N \times N$ Hermitian matrices. The Golden-Thompson inequality asserts that

$$\operatorname{Tr}\left(e^{A+B}\right) \le \operatorname{Tr}\left(e^{A}e^{B}\right). \tag{1.1}$$

This result has been known since 1964–1965, and has its origins in considerations from statistical mechanics. Thus Golden, in the paper 'Lower bounds for the Helmoltz function' [12] proved (1.1) for A and B Hermitian and non-negative definite. Independently Thompson, in 'Inequality with applications to statistical mechanics' [41], proved (1.1) without any requirement that the Hermitian matrices A and B be non-negative definite. In a further remarkable coincidence, also published in 1965 and motivated by statistical mechanics, Symanzik in the paper 'Proof of refinements of an inequality of Feynman' [37] derived (1.1) in the case of $A = -\frac{d^2}{dx^2}$, B = V(x) and thus particular Hermitian operators in Hilbert space.

This latter case is easy to illustrate. First, with $H = -\frac{d^2}{dx^2} + V(x)$, (1.1) reduces to

$$\operatorname{Tr} e^{-\beta H} \le \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \, e^{-\beta p^2} \int_{-\infty}^{\infty} dx \, e^{-\beta V(x)},$$
 (1.2)

which in words says that the classical partition function is an upper bound for the corresponding quantum partition function. If we take $V(x) = x^2$, the integral over position on the RHS is, like the integral over momentum, a Gaussian, while the energy levels of the quantum problem are $E_n = 2n + 1$, $n = 0, 1, \ldots$, similarly allowing the LHS to be made explicit to give

$$\frac{1}{\sinh \beta} \le \frac{1}{\beta}.\tag{1.3}$$

In recent years, renewed prominence has been given to the Golden-Thompson inequality, due to its applicability in random matrix theory. The latter is the main research area of coauthor Forrester of the present paper. In 1979 Forrester was a student in the second year linear algebra course of coauthor Thompson—the same Thompson as in Golden-Thompson—at the University of Melbourne. The inequality (1.1), with the additional information imparted to the class that Golden was a person rather than an adjective, was presented as enrichment knowledge in one of the lectures. Upon learning of the renewed interest in (1.1) due to its application in random matrix theory, the suggestion was then made from Forrester to Thompson of a review article, with the initial intention of familiarising potential users in random matrix theory with the surrounding mathematics.

Moreover, it is fair to say that the lineage in relation to Thompson's paper [41] is rather rich and of independent interest. Part of this, as recorded in the acknowledgements of [41], relates to an until now unpublished proof of (1.1) by Dyson. In the early 1960's Dyson wrote a series of seminal papers relating to random matrix theory and its application to nuclear physics (these were numbered by Roman numerals I up to V, except the foundational paper 'The three fold way', logically number zero in the series and so lacking labelling by a Roman numeral [10]). It is thus a happy coincidence that (1.1) should find application in this field, and pleasing too that the opportunity to write this article should have arisen during Freeman Dyson's 90th birthday year.

In Section 2 we detail the mathematics relating to the 2×2 case of (1.1) (in particular its relation to hyperbolic geometry), Dyson's proof of (1.1), Thompson's simplification and we comment too on Golden's proof. Thompson's proof makes use of results of Weyl and Pólya, and some correspondence with the latter is quoted. The topic of Section 3 is inequalities of a similar type to (1.1). One such result is that (1.1) holds with the trace operation replaced by any unitarily invariant norm, and a further relation to hyperbolic geometry of a generalization of this latter result in noted. The application of (1.1) to deriving operator norm bounds on certain sums of random matrices is given in Section 4.

2 Proof and generalizations

2.1 The 2 by 2 case

Upon being lead to (1.1) by a different consideration in statistical mechanics than the one mentioned in [41] — specifically from mathematical structures appearing in the variational Quasi-Chemical Equilibrium Theory of superconductivity as observed during his PhD thesis [40] — Thompson first took up the challenge of proving the 2×2 case. This case can be further simplified by the observation that 2×2 matrices X can be written as the sum of a multiple of the identity, say $c_X \mathbb{I}$, and a traceless matrix \tilde{X} . Since in general $e^{A+B} = e^A e^B$ when A and B commute, a simple corollary of the definition of e^X in terms of a power series, we see that $\operatorname{Tr} e^{A+B} = e^{c_A+c_B}\operatorname{Tr} e^{\tilde{A}+\tilde{B}}$ and similarly $\operatorname{Tr} (e^A e^B) = e^{c_A+c_B}\operatorname{Tr} (e^{\tilde{A}}e^{\tilde{B}})$. Thus in the 2×2 case it suffices to restrict attention to A and B traceless matrices in (1.1).

The latter can each be parametrised by three real numbers a_1, a_2, a_3 and b_1, b_2, b_3 say, by writing

$$A = a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3, \qquad B = b_1 \sigma_1 + b_2 \sigma_2 + b_3 \sigma_3, \tag{2.4}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the 2×2 Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

An elementary calculation shows $A^2 = ||\vec{a}||^2 \mathbb{I}$, $||\vec{a}||^2 := \sum_{i=1}^3 a_i^2$, and thus (see e.g. [42])

$$\exp A = \cosh ||\vec{a}|| \mathbb{I} + \frac{\sinh ||\vec{a}||}{||\vec{a}||} A.$$

From this, and the analogous equation for B, after substituting in (1.1) and making essential use of A and B being traceless to simplify the RHS, we see that our task in establishing (1.1) in the 2×2 case reduces to proving that

$$\cosh ||\vec{a} + \vec{b}|| \le \cosh ||\vec{a}|| \cosh ||\vec{b}|| - \cos \theta \sinh ||\vec{a}|| \sinh ||\vec{b}||, \quad \cos \theta := -\frac{\vec{a} \cdot \vec{b}}{||\vec{a}|| \, ||\vec{b}||}, \quad (2.5)$$

where as usual $\vec{a} \cdot \vec{b} := \sum_{i=1}^{3} a_i b_i$.

At this stage a connection with hyperbolic geometry is revealed. One recognises the RHS of (2.5) as $\cosh ||\vec{c}||$, in a hyperbolic geometry of curvature -1, as given by the law of cosines for a triangle of side lengths $||\vec{a}||, ||\vec{b}||, ||\vec{c}||$, and angle between $||\vec{a}||$ and $||\vec{b}||$ equal to θ (all measured in the hyperbolic geometry). So now, after first taking the inverse hyperbolic cosine of both sides and then squaring, we have reduced (2.5) to the statement that for a triangle in a hyperbolic geometry

$$||\vec{c}||^2 \ge ||\vec{a}||^2 + ||\vec{b}||^2 - 2||\vec{a}|| \, ||\vec{b}|| \cos \theta. \tag{2.6}$$

The latter is well known [15].

We remark that it is also possible to prove (2.5) directly. Thus first introduce the parameter ϵ by the scalings $\vec{a} \mapsto \epsilon \vec{a}$ and $\vec{b} \mapsto \epsilon \vec{b}$. Then check that the inequality holds term-by-term in like powers of ϵ (which are all even). The ϵ^0 and ϵ^2 terms are in fact equalities; from then on one simply needs Cauchy's inequality.

2.2 Dyson's proof

At the time of being led to (1.1), Thompson had just began a postdoc at UCSD and Dyson was visiting there on sabbatical. The result (1.1) was shown to Dyson in late 1964 as a conjecture for $N \geq 3$, and soon after he produced a proof. Working from the original notes, we record this proof in the following working.

Lemma 1. For any square matrices X, Y

$$|\operatorname{Tr}(XY)|^2 \le \operatorname{Tr}(X^{\dagger}X)\operatorname{Tr}(Y^{\dagger}Y).$$

Proof. This is Cauchy's inequality.

Lemma 2. Let P be any product of 2n factors which may be X or X^{\dagger} in any order. Then

$$|\operatorname{Tr} P| \le \operatorname{Tr} (XX^{\dagger})^n.$$

Proof. Among all the products P choose one for which |Tr P| is the greatest. If $P = (XX^{\dagger})^n$ or $(X^{\dagger}X)^n$ there is nothing to prove. Otherwise there appears somewhere in P a pair of consecutive factors X or X^{\dagger} . Permute the factors cyclically so that the two X or X^{\dagger} appear in positions n, n+1. Now write P = QR, where Q is the product of the first n factors, R the remainder.

By Lemma 1,

$$|\operatorname{Tr} P|^2 \le \operatorname{Tr} (Q^{\dagger} Q) \operatorname{Tr} (R^{\dagger} R).$$

But $Q^{\dagger}Q:=P',\,R^{\dagger}R:=P''$ are both products of the same type as P, which means that

$$|\operatorname{Tr} P'| \leq |\operatorname{Tr} P|, \qquad |\operatorname{Tr} P''| \leq |\operatorname{Tr} P|.$$

Therefore

$$|\operatorname{Tr} P| = |\operatorname{Tr} P'| = |\operatorname{Tr} P''|.$$

Now the number of neighbour-pairs XX^{\dagger} in P, P', P'' may be denoted by k, k', k'' respectively. We count the last and the first factors as neighbours in the trace. We have then

$$k' + k'' = 2k + 1, \ 2k + 2$$

according as the first and the last factors in P were different or the same. In any case at least one of P' and P'' has more neighbour-pairs XX^{\dagger} than P.

Now apply this argument to the product P with maximum $|\operatorname{Tr} P|$ and with the maximum number of neighour-pairs XX^{\dagger} . We have a contradiction unless $P = (XX^{\dagger})^n$ or $(X^{\dagger}X)^n$. This proves the lemma.

Lemma 3. For any two Hermitian matrices A and B

$$\operatorname{Tr}(A^{2^k}B^{2^k}) \ge \operatorname{Tr}(AB)^{2^k}.$$

Proof. Taking X = AB, $X^{\dagger} = BA$ in Lemma 2, we have

$$|\text{Tr}(AB)^{2n}| \le \text{Tr}(ABBA)^n = \text{Tr}(A^2B^2)^n.$$
 (2.7)

Next take $X = A^2B^2$, so that

$$|\operatorname{Tr}(AB)^{4n}| \le \operatorname{Tr}(A^2B^2B^2A^2)^n = \operatorname{Tr}(A^4B^4)^n.$$

Repeating this argument leads at once to Lemma 3.

Proof of (1.1). Write in Lemma 3 $A' = (1 + 2^{-k}A)$ for $A, B' = (1 + 2^{-k}B)$ for B and take the limit $k \to \infty$.

Before we proceed to give Thompson's [41] simplification of the above working, for the purpose of historical links, suppose that in Lemma 3 we write $A' = e^{A/2^k}$ for A and $B' = e^{B/2^k}$ for B. Then taking the limit $k \to \infty$ on the RHS asks us to compute $\lim_{k\to\infty} (e^{A/2^k}e^{B/2^k})^{2^k}$. For this one can use the Lie-Trotter formula [23, 45],

$$\lim_{n \to \infty} (e^{A/n} e^{B/n})^n = e^{A+B}, \tag{2.8}$$

which according to some sources [6] was found by Lie in 1875.

2.3 Thompson's proof

We have just seen that the last step in Dyson's proof makes use of a fundamental matrix identity. Thompson's [41] simplification similarly calls upon fundamental matrix relations, now between the eigenvalues $|\lambda_1| \geq \cdots \geq |\lambda_N|$ of a matrix X and its singular values $\mu_1 \geq \cdots \geq \mu_N$ (i.e. positive square roots of the eigenvalues of $X^{\dagger}X$), due to Weyl [47].

Lemma 4. Let w(x) be an increasing function of positive argument x, with the further property that $w(e^{\xi})$ is a convex function of ξ . One has

$$\sum_{i=1}^{k} w(\mu_i) \ge \sum_{i=1}^{k} w(|\lambda_i|), \qquad k = 1, \dots, N.$$
(2.9)

Taking $w(x) = x^{2s}$, (s = 1, 2, ...), k = N, (2.9) reads

$$\sum_{i=1}^{N} \mu_i^{2s} \ge \sum_{i=1}^{N} |\lambda_i|^{2s} \ge |\sum_{i=1}^{N} \lambda_i^{2s}|, \tag{2.10}$$

where the final relation is just the triangle inequality, or equivalently

$$\operatorname{Tr}(X^{\dagger}X)^{s} \ge |\operatorname{Tr}X^{2s}|, \qquad s = 1, 2, \dots$$
 (2.11)

The inequality (2.11) is given as Lemma 1 in [41]. Armed with (2.11), one now has an alternative to Dyson's Lemma 2 to deduce the result of Lemma 3 and thus (1.1). The argument is precisely the same [41].

2.4 Golden's proof

As in the above workings, Golden (a physical chemist at Brandeis during the years 1951–1981 [34]) in [12] first establishes the inequality of Lemma 3, then concludes the argument through use of (2.8). And as in Dyson's strategy, explicit use too is made of a Cauchy inequality, though not that in Lemma 1, but rather

$$0 \le \operatorname{Tr} X^2 \le \operatorname{Tr} X X^{\dagger}$$

(cf. (2.11) with s=1) which is restricted to the case that X is non-negative definite.

In fact in the case of non-negative definite matrices A, B the inequality (2.7) underlying the proof of Lemma 3 can be generalized to the Araki-Lieb-Thirring inequality [2, 25]

$$0 \le \text{Tr} (A^{1/2}BA^{1/2})^{rs} \le \text{Tr} (A^{r/2}B^rA^{r/2})^s, \qquad r \ge 1, \ s > 0$$

(take r = 2, s = n [3]).

2.5 The George Pólya link

Weyl's result Lemma 4 is deduced in two steps. The first is to establish the inequalities $\prod_{i=1}^k |\lambda_i| \leq \prod_{i=1}^k \mu_i$. The second is to apply the following lemma, with $a_i = \log \mu_i$, $b_i = \log |\lambda_i|$ and $w(x) = \omega(e^{\xi})$.

Lemma 5. Given two sequences of real numbers a_1, \ldots, a_m and b_1, \ldots, b_m such that $b_1 \ge b_2 \ge \cdots \ge b_m$ and

$$b_1 + \dots + b_q \le a_1 + \dots + a_q, \qquad q = 1, \dots, m$$
 (2.12)

the inequality

$$\omega(b_1) + \dots + \omega(b_m) \le \omega(a_1) + \dots + \omega(a_m)$$

holds for any convex increasing function $\omega(x)$.

Pólya [31] pointed out that this lemma can be deduced from an inequality of Hardy, Littlewood and Pólya [14], and rediscovered a few years later by Karamata [20]. The latter differs from Lemma 5 in that the last inequality in (2.12), namely the case q=m, is required to be an equality, and that $\omega(x)$ is assumed to be convex but need not be increasing.

After isolating (2.11) as a simplification to Dyson's Lemma 2 in the pathway to the proof of (1.1), and thus the role played by earlier results of Weyl and Pólya, Thompson wrote to George Pólya (who was actually a Great Uncle by marriage) with this proof, and an enquiry about the possibility of it having already appeared in the literature. We quote from the subsequent reply:

Zurich, Feb. 5 '65

Dear Colin,

I was much pleased to receive your letter. The theorem

$$\operatorname{tr}(e^A e^B) \ge \operatorname{tr}(e^{A+B)}$$

is new to me, and I find it very neat. It, is true, I did not look up the literature --- I had, or rather I am still having, a bad spell of sciatica which makes very difficult for me to sit in the library & I have very few books in my hotel room. I must even confess that I remember only vaguely the notes by Weyl and by myself you are quoting. ... We should arrive in Stanford March 13. In Stanford, I can consult Loewner who knows much more about matrices than myself. With my sciatica, the trip may present serious complications --- but we shall see. ...

Your Uncle George

Please give my best regards to Prof. Dyson.

A letter dated the Rockefeller Institute, Tuesday, March 2 (1965) from Pólya to Thompson, contains some follow up on previous knowledge of (1.1): "I asked several mathematicians about that nice result (1.1) e.g. Richard Brauer (Harvard) Radamacher (Philadelphia) etc. but nobody knew it." And then, again sent from the Rockefeller Institute, on March 5 (1965): "I was just shown Physical Review vol. 137 No. 4B 22 Feb. 1965 p. B1127." The latter is of course Golden's article.

Dyson suggested to Thompson that with the simplified proof and additional results that he should submit anyhow (by himself), with an appropriate cover letter to the Editor, and indeed the paper was accepted.

3 Related inequalities and generalizations

Let $\phi(X)$ be a continuous real valued function of the eigenvalues of X, and suppose

$$\phi((X^{\dagger}X)^s) \ge |\phi(X^{2s})|, \qquad s = 1, 2, \dots$$
 (3.13)

We know from Lemma 2.10 that

$$\phi(X) := \sum_{i=1}^{k} |\lambda_i| \tag{3.14}$$

is an explicit example. As noted by Thompson [42], the proof of (1.1) via Lemma 3 now yields

$$\phi(e^{A+B}) \le \phi(e^A e^B). \tag{3.15}$$

An equivalent viewpoint on (3.15) in the case that ϕ is given by (3.14) is that

$$e^{A+B} \prec_w e^A e^B, \tag{3.16}$$

where for X, Y Hermitian $X \prec_w Y$ (in words Y weakly majorizes X) means that $\sum_{i=1}^k \lambda_i(X) \leq \sum_{i=1}^k \lambda_i(Y)$ (k = 1, ..., N). Choosing $\phi(X) = \sum_{i=1}^N |\lambda_i|^p$, $p \geq 1$, one has in the case that X is Hermitian and non-negative definite $\phi(X) = (\operatorname{Tr}(XX^{\dagger})^{p/2}) = (||X||_p)^p$ (here $||\cdot||_p$ denotes the so-called Schatten p-norm) and we see that (3.15) gives

$$||e^{A+B}||_{p} \le ||e^{A}e^{B}||_{p}. \tag{3.17}$$

We remark that (3.17) can equivalently be written [16]

$$\operatorname{Tr} e^{A+B} \le \operatorname{Tr} \left(e^{pB/2} e^{pA} e^{pB/2} \right)^{1/p}.$$
 (3.18)

In the case $p \to \infty$ (3.17) was first established by Segal [35], while Leonard [22] and Thompson [42] showed that it holds for general unitary invariant norms.

A generalisation of (3.17) is [4]

$$||A - B|| \le ||\log(e^{-B/2}e^Ae^{-B/2})|| \tag{3.19}$$

valid for A, B Hermitian and the norm $||\cdot||$ unitarily invariant. In the case $||\cdot|| = ||\cdot||_2$ and with A and B positive definite (3.19) reads

$$||A - B||_2 \le \left(\sum_{i=1}^N \left(\log \lambda_i(e^A e^{-B})\right)^2\right)^{1/2} =: \delta_2(e^A, e^B),$$
 (3.20)

where $\{\lambda_i(X)\}$ denotes the eigenvalues of X. The significance of δ_2 is that it is the distance associated with the invariant Riemannian metric $ds^2 = \left(\operatorname{Tr}(A^{-1}dA)\right)^2$. Since $||A||_2 = \left(\sum_{i=1}^N (\log \lambda_i(e^A))^2\right)^{1/2}$, it follows from the definition in (3.20) that $||A||_2 = \delta_2(e^A, \mathbb{I})$. This in turn allows (3.20), after squaring both sides, to be interpreted as a law of cosines for triangles in a hyperbolic geometry [28, 21, 4, 5], as already seen in (2.6).

On another front, using essentially the same working as used by Thompson to obtain (3.15), Cohen et al. [7] generalized (3.15) to

$$|\phi(e^{A+B})| \le \phi\left(e^{(A+A^{\dagger})/2}e^{(B+B^{\dagger})/2}\right),$$
 (3.21)

where it is no longer required that A, B are Hermitian. Note that the special case B = 0, ϕ given by (3.14) with k = 1 yields [7, Corollary 5]

$$\lambda_1 \left(\frac{1}{2} (A + A^{\dagger}) \right) \ge \operatorname{Re} \lambda_1(A).$$
 (3.22)

We remark that in distinction to (3.17) with $p = \infty$ (3.21) for ϕ given by (3.14) with k = 1 is not a statement about matrix norms. Thus one has that $\lambda_1(A)| = ||A||_{\infty}$ for A Hermitian, but more generally $|\lambda_1(A)| \leq ||A||_{\infty}$. This latter property has been used in [8] to prove (3.17) with $p = \infty$.

A basic question related to (1.1) is the condition for equality. It has already been remarked that when A, B commute, $e^A e^B = e^{A+B}$, and so the former is a sufficient condition for equality. It can be proved that this is also a necessary condition [36]. One way to see this is to expand (1.1) to increasing higher orders of the matrix products [39, p. 253]. To third order both sides agree, while the fourth order terms differ by a term proportional to Tr[A,B]. Another basic question relates to extending (1.1) or the underlying inequality from Lemma 3, to involve three or more matrices. In the original paper [41] a counter-example involving 3×3 real symmetric matrices to the obvious generalization of the latter, $|\text{Tr}(ABC)^{2^k}| \leq \text{Tr}(A^{2^k}B^{2^k}C^{2^k})$, is given. And a counter-example to the obvious generalization of (1.1) to three matrices

$$\operatorname{Tr}\left(e^{A+B+C}\right) \le |\operatorname{Tr}\left(e^{A}e^{B}e^{C}\right)|,\tag{3.23}$$

involving the 2×2 Pauli matrices, is given in [42]. In relation to this latter setting, a result of Lieb [24] gives

$$\operatorname{Tr}\left(e^{A+B+C}\right) \le \int_0^\infty \operatorname{Tr}\left(e^A(t+e^{-C})^{-1}e^B(t+e^{-C})^{-1}\right)dt,$$
 (3.24)

which one sees reduces to (1.1) when C = 0. Moreover, (3.24) shows that if [B, C] = 0, then (3.23) (without the need for the absolute value on the RHS) holds true. However, since in this circumstance $e^{B+C} = e^B e^C$ as noted in the first paragraph of Section 2.1, then (3.23) is just (1.1) with B replaced by B + C.

4 Applications to random matrix theory

The first application of the Golden-Thompson inequality to random matrix theory was made by Ahlswede and Winter [1]. This contribution was later popularized by it featuring in a blog article of Tao [38], and this blog article in turn became a section in the book on random matrices of the latter [39].

To motivate this line of research, let us recall [11, Preface] that random matrix theory has its historical origin in the work of Hurwitz [17], who computed the volume form of a general unitary matrix parametrized in terms of Euler angles. This work was influential in the development of the concept of the Haar measure on classical groups, and more generally locally compact topological groups, in the work of Haar, von Neumann, Weyl, Weil, Seigel and others [26, pg. 143]. In the case of positive definite matrices the Haar measure is induced by the invariant Riemann metric noted below (3.20). One consequence of the work of Hurwitz is that the precise probability density function on the four non-zero, non-unit entries, of a set of N(N-1)/2 unimodular matrices can be read off, such that when multiplied in an appropriate order, and further multiplied by the scalar $e^{i\alpha_0}$, $-\pi \le \alpha_0 < \pi$ chosen uniformly at random, the corresponding unitary matrix corresponds to sampling U(N) with Haar measure [48].

There are alternative ways to sample U(N) with Haar measure [27]. One is to begin with an $N \times N$ complex standard Gaussian matrix (all entries independent with standard

complex Gaussian entries) X, and to form the matrix $V := (X^{\dagger}X)^{-1/2}X$. Another is to apply the Gram-Schmidt algorithm to the columns of X. Granted these facts, a natural question is to quantify properties of $U \in U(N)$ in common with X. A celebrated result of Dyson [9] is that the local eigenvalue statistics of V and $\frac{1}{2}(X + X^{\dagger})$ (the latter in the so-called bulk; see e.g. [11, Ch. 5]) agree in the $N \to \infty$ limit. At the level of the distribution of the entries, the top $k \times k$ block of U, U_k say, has distribution with probability density proportional to [49]

 $\left(\det(\mathbb{I}_k - U_k^{\dagger} U_k)\right)^{N-2k}.$

Thus for $N \to \infty$, and with k fixed, the probability density function of $G_k := \sqrt{N}U_k$ is proportional to $e^{-G_k^{\dagger}G_k}$, and hence the entries of G_k are independent complex standard Gaussians in distribution. The analogue of this result for Haar distributed real orthogonal matrices, telling us that in the limit $N \to \infty$ the top $k \times k$ block, when scaled by \sqrt{N} , is distributed as a real Gaussian matrix of independent standard Gaussian entries, is known to hold for $k = \mathrm{o}(\sqrt{N})$ in the variation norm [18].

Related to this interplay between Gaussian and unitary random matrices is the question of quantifying the orthogonality between columns of a standard complex Gaussian matrix. This has application, for example, to the problem of bringing a convex body to nearisotropic position [32], and that of analyzing low rank approximations of matrices [33, 13]. Denote by $X_{N,k}$ the first k columns of an $N \times N$ complex standard Gaussian matrix X, and form the empirical covariance matrix

$$\Sigma := \frac{1}{N} X_{N,k}^{\dagger} X_{N,k}. \tag{4.25}$$

Since all entries are independent, with zero mean and unit standard derivation, we see immediately that $\mathbb{E} \Sigma = \mathbb{I}_k$. To measure the distance from the mean, introduce the operator norm $||X||_{\text{op}} = \sup_{||\vec{x}||=1} ||X\vec{x}||$. This is related to the Schatten *p*-norm by $||X||_{\text{op}} = ||X||_{\infty}$ and so for X Hermitian we have

$$||X||_{\text{op}} = \max\left\{-\lambda_{\min}(X), \lambda_{\max}(X)\right\}. \tag{4.26}$$

For fixed k and large N it follows from the central limit theorem applied to the elements of Σ that $||\Sigma - I_k||_{\text{op}}$ goes to zero at a rate proportional to $1/\sqrt{N}$. Of interest is a large deviation bound relating to $||\Sigma - I_k||_{\text{op}}$, which is valid for general N and k. We first observe

$$\Sigma = \frac{1}{N} \left[\sum_{p=1}^{N} \bar{x}_{i,p} x_{p,j} \right]_{i,j=1,\dots,k} = \frac{1}{N} \left[\sum_{p=1}^{N} \bar{x}_{i,p} x_{p,j} \right]_{i,j=1,\dots,k} = \frac{1}{N} \sum_{p=1}^{N} (\vec{X}^{(p)})^{\dagger} \vec{X}^{(p)}, \qquad (4.27)$$

where $\vec{X}^{(p)}$ is the row vector formed by the p-th row of $X_{N,k}$. This exhibits Σ as the average of independent and identically distributed (rank one) random matrices. Thus the task at hand can be interpreted as a noncommutative analogue of classical questions relating to the distribution of the scalar average $S := \frac{1}{N} \sum_{p=1}^{N} s_p$ for $\{s_p\}$ identical and independently distributed. Specifically, if we take as our task that of bounding $\Pr(||\Sigma - \mathbb{I}_k||_{\text{op}} > \epsilon)$ then

we are seeking a non-commutative analogue of the Chernoff inequality (large deviation formula) from classical probability theory

$$\Pr\left(s_1 + \dots + s_N \ge \epsilon\right) \le \max\left(e^{-\epsilon^2/4}, e^{-\epsilon\sigma/2}\right),\tag{4.28}$$

where $\epsilon > 0$, and the s_i are independent, identically distributed random variables taking values in [-1, 1] with mean zero and variance σ^2/N .

For this task, we begin by making use of (4.26) to observe

$$\Pr(||\Sigma - \mathbb{I}_k||_{\text{op}} > \epsilon) \le \Pr(\lambda_{\max}(\Sigma - \mathbb{I}_k) > \epsilon) + \Pr(-\lambda_{\min}(\Sigma - \mathbb{I}_k) > \epsilon). \tag{4.29}$$

We now consider the first term on the RHS of (4.29). Bernstein's trick of effectively exponentiating the statement $||\Sigma - \mathbb{I}_k||_{\text{op}} > \epsilon$ without changing its probability tells us that for any c > 0

$$\Pr(\lambda_{\max}(\Sigma - \mathbb{I}_k) > \epsilon) = \Pr(e^{c\lambda_{\max}(\Sigma - \mathbb{I}_k)} > e^{c\epsilon}) = \Pr(\lambda_{\max}(e^{c(\Sigma - \mathbb{I}_k)}) > e^{c\epsilon}).$$

But

$$\lambda_{\max}(e^{c(\Sigma - \mathbb{I}_k)}) \le \operatorname{Tr} e^{c(\Sigma - \mathbb{I}_k)} \tag{4.30}$$

and so

$$\Pr(\lambda_{\max}(\Sigma - \mathbb{I}_k) > \epsilon) \le \Pr(\operatorname{Tr} e^{c\lambda_{\max}(\Sigma - \mathbb{I}_k)} > e^{c\epsilon}) \le e^{-c\epsilon} \mathbb{E}(\operatorname{Tr} e^{c(\Sigma - \mathbb{I}_k)}), \tag{4.31}$$

where the final bound follows from Chebyshev's inequality. The analogous argument applied to the second term on the RHS of (4.29) gives

$$\Pr(-\lambda_{\min}(\Sigma - \mathbb{I}_k) > \epsilon) \le e^{-c\epsilon} \mathbb{E}\left(\operatorname{Tr} e^{-c(\Sigma - \mathbb{I}_k)}\right). \tag{4.32}$$

So far the working has mimicked that of one of the standard proofs of (4.28) (see e.g. [39]). To proceed further requires a new idea, and this was forthcoming in the work of Ahlswede and Winter [1], by way of application of the Golden-Thompson inequality (1.1) to bound the average on the RHS of (4.31).

Lemma 6. For any $\mu \in \mathbb{R}$,

$$\mathbb{E}(\operatorname{Tr} e^{\mu(\Sigma - \mathbb{I}_k)}) \le k \prod_{p=1}^N ||\mathbb{E}(e^{\mu((\vec{X}^{(p)})^{\dagger} \vec{X}^{(p)} - \mathbb{I}_k)/N})||_{\text{op}}.$$
(4.33)

Proof. Write $\Sigma - \mathbb{I}_k =: \Sigma^{(N)}$ and $\Sigma^{(N)} = \sum_{p=1}^N S^{(p)}$ so that $S^{(p)} := ((\vec{X}^{(p)})^{\dagger} \vec{X}^{(p)} - \mathbb{I}_k) / N$. Then, since $\Sigma^{(N)} = \Sigma^{(N-1)} + S^{(N)}$ we have

$$\mathbb{E}(\operatorname{Tr} e^{\mu \Sigma^{(N)}}) = \mathbb{E}(\operatorname{Tr} e^{\mu \Sigma^{(N-1)} + \mu S^{(N)}}).$$

Applying the Golden-Thompson inequality (1.1) to the RHS gives

$$\mathbb{E}(\operatorname{Tr} e^{\mu \Sigma^{(N)}}) \le \mathbb{E}(\operatorname{Tr} e^{\mu \Sigma^{(N-1)}} e^{\mu S^{(N)}}). \tag{4.34}$$

Each of the $S^{(p)}$ are independent, so taking the expectation with respect to $S^{(N)}$, and furthermore making use of the fact that \mathbb{E} are Tr commute (to be able to take advantage of this fact explains the use of (4.30)), we see that (4.34) can be rewritten to read

$$\mathbb{E}(\operatorname{Tr} e^{\mu \Sigma^{(N)}}) \leq \mathbb{E}_{S^{(1)},\dots,S^{(N-1)}} \operatorname{Tr}(e^{\mu \Sigma^{(N-1)}} \mathbb{E}_{S^{(N)}} e^{\mu S^{(N)}})
\leq ||E_{S^{(N)}} e^{\mu S^{(N)}}||_{\operatorname{op}} E_{S^{(1)},\dots,S^{(N-1)}} \operatorname{Tr}(e^{\mu \Sigma^{(N-1)}}),$$
(4.35)

where the final inequality follows from the general fact that for A positive definite $\operatorname{Tr} AB \leq ||B||_{\operatorname{op}}\operatorname{Tr} A$. Applying (4.35) recursively, with base case N=1 for which $\operatorname{Tr} e^{\mu\Sigma^{(0)}}=\operatorname{Tr} \mathbb{I}_k=k$, gives (4.33).

Substituting (4.33) in (4.31) and (4.32) and recalling (4.29) gives

$$\Pr(||\Sigma - \mathbb{I}_k||_{\text{op}} > \epsilon) \le k e^{-\epsilon c} \Big(\prod_{l=1}^N ||\mathbb{E}(e^{cS^{(l)}})||_{\text{op}} + \prod_{l=1}^N ||\mathbb{E}(e^{-cS^{(l)}})||_{\text{op}} \Big).$$

The remaining task is to bound $||\mathbb{E}(e^{cS^{(l)}})||_{\text{op}}$, and then to minimize the RHS with respect to c > 0. This is straightforward; the details can be found in e.g. [46]. The final result, under the simplifying assumption that $||S^{(l)}|| \leq 1$ for each $l = 1, \ldots, N$ is [32]

$$\Pr(||\Sigma - \mathbb{I}_k||_{\text{op}} > \epsilon) \le k \max(e^{-\epsilon^2/(4\sigma^2)}, e^{-\epsilon/2}), \tag{4.36}$$

where $\sigma^2 := \sum_{l=1}^N ||\operatorname{Var} S^{(l)}||_{\operatorname{op}}$. This is a noncommutative matrix analogue of the Chernoff inequality (4.28).

There is a significant refinement of the Ahlswede-Winter argument due to Oliveira [29, 30]. One setting where it shows itself is in bounding the operator norm of the random matrix sum $Z^{(N)} := \sum_{p=1}^{N} \epsilon_p A^{(p)}$ where the $A^{(p)}$ are fixed $d \times d$ Hermitian matrices while $\{\epsilon_p\}$ are independent Rademacher (uniformly distributed on [-1,1]) or standard Gaussian random variables. In accordance with the argument leading to (4.31), the key task is to bound $\mathbb{E}(\operatorname{Tr} e^{\mu Z^{(N)}})$.

Lemma 7. In terms of the above notation, for all $\mu \in \mathbb{R}$,

$$\mathbb{E}(\operatorname{Tr} e^{\mu Z^{(N)}}) \le \operatorname{Tr} \left(e^{\mu^2 \sum_{p=1}^N \epsilon_p(A^{(p)})^2} \right). \tag{4.37}$$

Proof. For j = 1, ..., N define

$$D^{(j)} = D^{(0)} + \sum_{p=1}^{j} \left(\mu \epsilon_p A^{(p)} - \frac{1}{2} \mu^2 A^{(p)} \right), \tag{4.38}$$

with $D^{(0)} := \frac{1}{2}\mu^2 \sum_{p=1}^N (A^{(p)})^2$.

From these definitions

$$\mathbb{E}(\operatorname{Tr} e^{\mu D^{(N)}}) = \mathbb{E}\operatorname{Tr} e^{D^{(N-1)} + \mu \epsilon_N A^{(N)} - \mu^2 (A^{(N)})^2/2}.$$

Making use of the Golden-Thompson inequality (1.1) on the RHS, then proceeding as in the derivation of (4.35) shows

$$\mathbb{E}(\operatorname{Tr} e^{\mu D^{(N)}}) \leq ||\mathbb{E}_{\epsilon_{N}}(e^{\mu \epsilon_{N} A^{(N)} - \mu^{2}(A^{(N)})^{2}/2})||_{\operatorname{op}}\operatorname{Tr}(e^{\mu D^{(N-1)}})$$

$$= ||e^{-\mu^{2}(A^{(N)})^{2}/2}\mathbb{E}_{\epsilon_{N}}(e^{\mu \epsilon_{N} A^{(N)}})||_{\operatorname{op}}\operatorname{Tr}(e^{\mu D^{(N-1)}}), \tag{4.39}$$

where in obtaining the equality, use has been made of $-\mu^2(A^{(N)})^2/2$ and $\mu\epsilon_N A_N$ commuting.

For ϵ_N a Rademacher or standard Gaussian random variable, explicit computation reveals

$$||e^{-\mu^2(A^{(N)})^2/2}\mathbb{E}_{\epsilon_N}(e^{\mu\epsilon_N A^{(N)}})||_{\text{op}} = ||f(A^{(N)})||_{\text{op}}$$
 (4.40)

where $0 \le f(t) \le 1$ for all $t \in \mathbb{R}$ and thus (4.40) is bounded by 1. This fact allows us to deduce from (4.39) the recurrence relation

$$\mathbb{E}(\operatorname{Tr} e^{\mu D^{(N)}}) \le \mathbb{E}(\operatorname{Tr} e^{\mu D^{(N-1)}}). \tag{4.41}$$

Applying (4.41) recursively a total of N times and noting $D^{(N)} = Z^{(N)}$ gives (4.37).

Note that a direct adaptation of the proof of Lemma 6 would give

$$\mathbb{E}(e^{\mu Z^{(N)}}) \le de^{\mu^2 \sum_{\mu=1}^N ||(A^{(p)})^2||_{\text{op}}}.$$

This is demonstrated in [30] to always be weaker than (4.37)

The Chernoff inequality (4.28) is one of a large number of probabilistic bounds relating to sums of scalar random variables. We have seen that the Golden-Thompson inequality (1.1) allows for the derivation of a matrix generalization (4.36). It turns out that (4.37) is key to deriving a bound for $(\mathbb{E}||Z^{(N)}||^p)^{1/p}$, which is the matrix generalization of the classical Khintchine inequality [30]. Recently Tropp [43, 44] has shown that a result of Lieb [24] relating to convex trace functions offers a powerful alternative to the Golden-Thompson inequality in the derivation of matrix generalization of further probabilistic bounds. On the other hand, as commented in [44], the approach of Oliveira [29] using the Golden-Thompson inequality to establish a matrix generalization of Freedman's inequality (a martingale version of Bernstein's inequality) has the advantage of of extending to the fully noncommutative setting [19].

We conclude by quantifying the values of the sides of (1.1) when averaged over suitable matrix ensembles. We restrict attention to the case N=2 with A and B traceless as in (2.4), and the real numbers a_1, \ldots, b_3 all standard Gaussians. We read off from the two sides of (2.5) that

$$\mathbb{E}\operatorname{Tr}(e^{A+B}) = \left(\frac{1}{2\pi}\right)^3 \int_{\mathbb{R}^6} e^{-||\vec{a}||^2/2 - ||\vec{b}||^2/2} \cosh||\vec{a} + \vec{b}|| \, d\vec{a} \, d\vec{b}$$

$$\mathbb{E}\operatorname{Tr}(e^A e^B) = \left(\frac{1}{2\pi}\right)^3 \int_{\mathbb{R}^6} e^{-||\vec{a}||^2/2 - ||\vec{b}||^2/2} \cosh||\vec{a}|| \cosh||\vec{b}|| \, d\vec{a} \, d\vec{b}$$

(the term involving $\cos \theta$ on the RHS of (2.5) averages to zero). Making use of polar coordinates to evaluate the integrals shows

$$\frac{\mathbb{E}\operatorname{Tr}(e^A e^B)}{\mathbb{E}\operatorname{Tr}(e^{A+B})} = \frac{4}{3}.$$
(4.42)

An obvious question is the behaviour of the ratio (4.42) as a function of the matrix size N for A, B from the Gaussian unitary ensemble, for example. The analogous question for the inequality (3.22) can readily be answered in the case that A is a standard real or complex Gaussian matrix. Then well known results for the largest eigenvalues in the Gaussian orthogonal and unitary ensemble, and for the spectral radius of the real and complex Ginibre ensemble, (see e.g. [11]) tell us that

$$\lim_{N \to \infty} \frac{\mathbb{E} \lambda_1 \left(\frac{1}{2} (A + A^{\dagger}) \right)}{\mathbb{E} \operatorname{Re} \lambda_1 (A)} = \sqrt{2}.$$

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