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Physics 8.022   Fall 2002

Special Relativity

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Abstract

This write-up summarizes the derivation and use of the Lorentz transformation to relate two inertial frames. The main results are Lorentz length contraction and the time dilation. The derivation of the way of adding velocities is also given.

Inertial Frames and Einstein's Postulate

Inertial frames are frames related by a constant velocity along any axis. An object is at rest in an inertial frame when its position in space does not change with time. We consider two inertial frames: the 26-100 (S) frame in which we are at rest and the Space Shuttle frame (S') which moves relative to S with a velocity u along the x axis and in which the Shuttle is at rest. Newton postulated that the laws of physics are the same in all inertial frames; specific predictions made in one frame must also be true in all other inertial frames.

Based on the result of the Michelson-Morley experiment, Einstein postulated: *There is a universal constant, c, which is the same in all inertial frames.* This velocity corresponds to the velocity of light, $c = 3 \times 10^{10}$ cm/s. We will now investigate the implications of Einstein's postulate.

Galilean Transformation

Let the coordinates (x, t) denote the location of an event in the 26-100 frame and (x', t') denote the location of the same event in the Shuttle frame. The Galilean transformation which relates the two coordinate systems is:

$$
\begin{aligned}
x' &= x - ut \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
\tag{1}
$$

which tells us we just need to take into account the relative translation of the coordinate systems to transform the coordinates of a given event from one frame to another. Suppose a body moves with velocity v' inside the Shuttle. What velocity v do we in 26-100 see it move with? Using the transformation Eq. 1 from Shuttle coordinates to 26-100 coordinates gives:

$$
\begin{aligned}
x &= x' + ut' \\
\rightarrow dx &= dx' + udt' \\
\rightarrow \frac{dx}{dt} &= v = \frac{dx'}{dt'} + u\frac{dt'}{dt'} = v' + u
\end{aligned}
$$

This just says we in 26-100 see the body moving with the velocity v' it has in the shuttle frame plus the velocity of the Shuttle frame (u) with respect to 26-100. *If Einstein's postulate that the velocity of light is the same in all inertial frames (based on the result of the Michelson-Morley experiment) is correct, then the Galilean transformation is not the correct way to transform space-time coordinates between two inertial frames.*

Derivation of the Lorentz Transformation

We wish to find a transformation between the 26-100 frame (S) and the Shuttle frame (S') which such that the velocity of light c is the same in both coordinate systems. The most general linear transformation (i.e. depending on x and t, but not higher powers like x^2, etc.) [1] is:

$$x' = Ax + Bt \tag{2}$$
$$y' = y$$
$$z' = z$$
$$t' = Dx + Et \tag{3}$$

remember the velocity u is along the x-axis, so the other coordinates must remain the same: $y' = y$, $z' = z$. We have chosen at $x = x' = 0$ at $t = t' = 0$. The origin of the S' frame $(x' = 0)$ moves along the trajectory $x = -ut$ in the S frame. Putting this into the Eq. 2 above gives

$$0 = Aut + Bt \rightarrow B = -Au \rightarrow x' = A(x - ut).$$

In the frame S' the origin $(x = 0)$ of S describes a trajectory along $x' = ut'$ and putting this into equation Eq. 2 above gives $-ut' = A(x - ut)$.

Substituting this result into Eq. 3 gives $-u(Dx + Et) = A(x - ut)$ and $x = 0$ implies $-uEt = -Aut \rightarrow E = A$. Now we have

$$x' = A(x - ut) \tag{4}$$
$$t' = Dx + At. \tag{5}$$

Next, we apply Einstein's Postulate: *The velocity of light is the same in all inertial frames.* A pulse of light emitted at $t' = 0$ along the x' axis in the S' frame follows the trajectory $x' = ct'$. We use this to relate Eqs. 4 and 5:

$$A(x - ut) = c(Dx + At). \tag{6}$$

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[1]It can be shown that transformations involving terms like x^2 lead to transformations which are not invariant for translations of the coordinate systems.

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The same pulse of light would be seen as following the trajectory $x = ct$ in the S frame. Use this to eliminate x in Eq. 6: $A(c - u)t = c(Dc + A)t \rightarrow D = -Au/c^2$ and we have

$$x' = A(x - ut) \tag{7}$$
$$t' = A(-\frac{ux}{c^2} + t). \tag{8}$$

Finally, what happens to a light pulse emitted along the y axis at time $t = 0$ in frame S? It follows a trajectory $y = ct$ and $x = 0$ in frame S and a trajectory $x'^2 + y'^2 = ct'^2$ in frame S'. If we substitute for x', t' using Eq. 7 and 8 and $y' = y = ct$, we have:

$$A^2(x - ut)^2 + c^2 t^2 = c^2 A^2 (-\frac{ux}{c^2} + t)$$

and $x = 0$ implies

$$A = \frac{1}{\sqrt{1 - u^2/c^2}}.$$

Since this expression occurs so often, pros use $\gamma = 1/\sqrt{1 - \beta^2}$ where $\beta = u/c$. The positive value of the square root corresponds to the physical solution and we have the final form of the Lorentz transformation:

$$x' = \frac{1}{\sqrt{1 - u^2/c^2}}(x - ut) \tag{9}$$
$$y' = y \tag{10}$$
$$z' = z \tag{11}$$
$$t' = \frac{1}{\sqrt{1 - u^2/c^2}}(t - \frac{ux}{c^2}).$$

Going from frame S' to S, we will have

$$x = \frac{1}{\sqrt{1 - u^2/c^2}}(x' + ut') \tag{12}$$
$$t = \frac{1}{\sqrt{1 - u^2/c^2}}(t' + \frac{ux'}{c^2}).$$

Simple examples of the Lorentz Transformations

Particle at rest or moving at constant velocity

Fig. 1 shows the coordinate system in the 26-100 frame. A particle at rest at the point x_o follows a trajectory in a coordinate system (called a *world line*) as shown in the figure. An *event* represents an occurrence at a *specific coordinate and time*, for example at (x_o, t_o) in the figure. Suppose a particle moves with velocity u along the x axis in 26-100 (frame S). Fig. 2 shows the world line for a particle moving in 26-100 frame with velocity u passing through (x_o, t_o). Now suppose a particle is at rest in the Shuttle (frame S') at coordinate $x' = 0$. To the people in the shuttle, the coordinate system appears as in Fig. 3.

We put primes to indicate this is the rest frame in the Shuttle. A particle at rest at x'_o in the Shuttle describes the world line shown.

Coordinate systems

What does the Shuttle coordinate system look like to us in 26-100 if we arrange for the shuttle to go by in such a way that at $t = 0$, $t' = x = x' = 0$? The x' axis is all those points where $t' = 0$, so to see what the x' axis looks like in the 26-100 frame, put $t' = 0$ into the Lorentz transformations Eq. 12:

$$x = \frac{1}{\sqrt{1 - u^2/c^2}} x'$$

$$t = \frac{1}{\sqrt{1 - u^2/c^2}} \frac{ux'}{c}.$$

Eliminating x' gives $x = c^2 t/u$ which is the equation of the x' axis in the 26-100 coordinate system. The t' axis is the locus of all points such that $x' = 0$ and putting $x' = 0$ into the Lorentz equations and eliminating t' gives
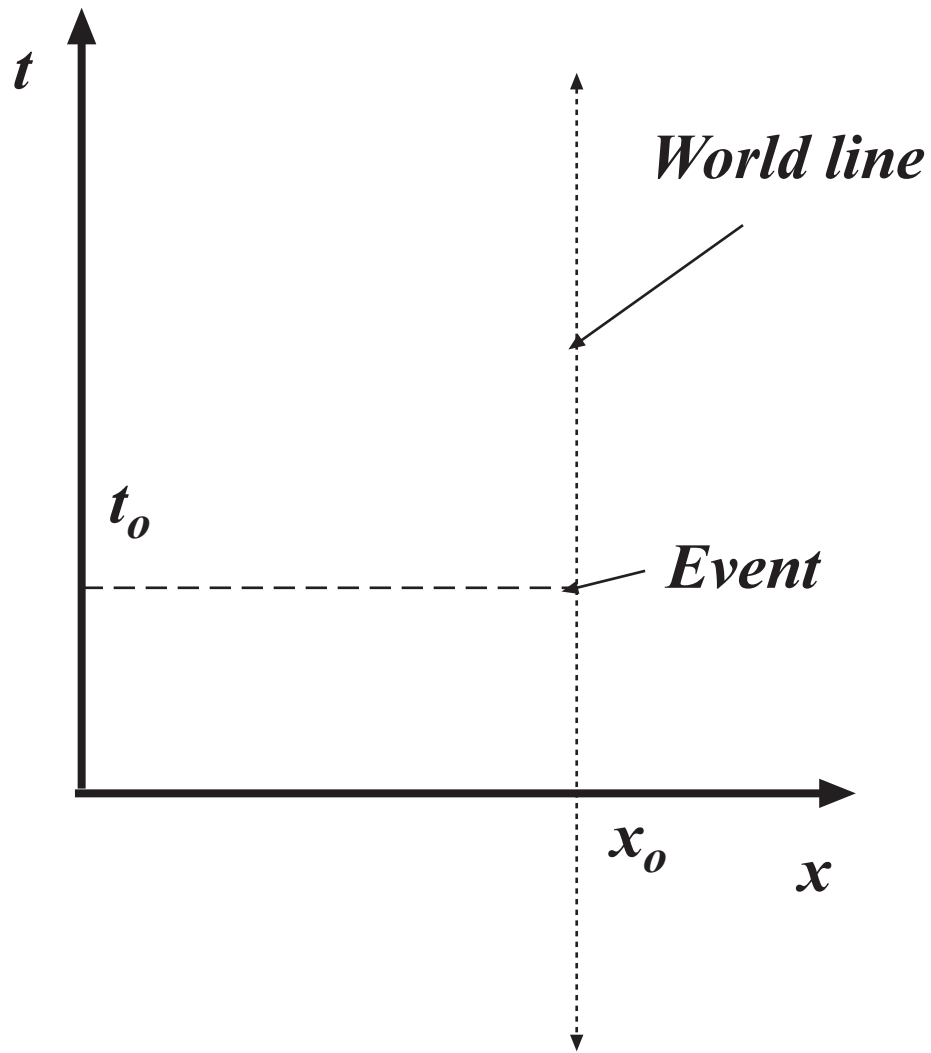
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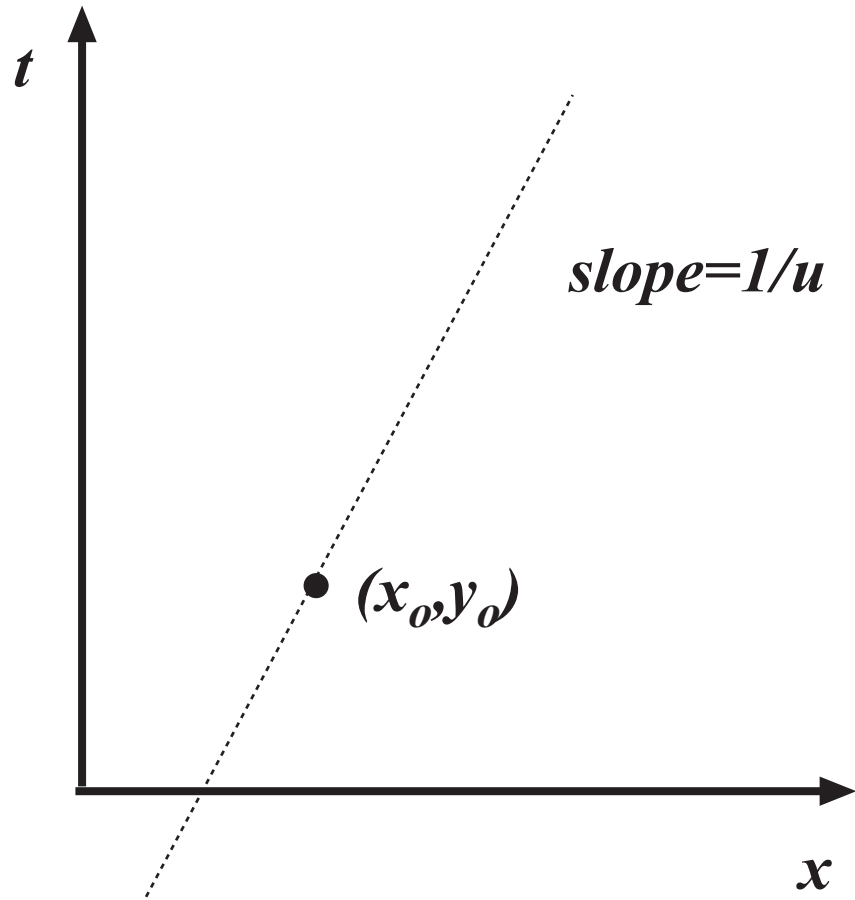Figure 1: Coordinate system for the 26-100 frame.

Figure 2: 26-100 showing the world line for a particle moving with velocity u passing through (x_o, t_o).
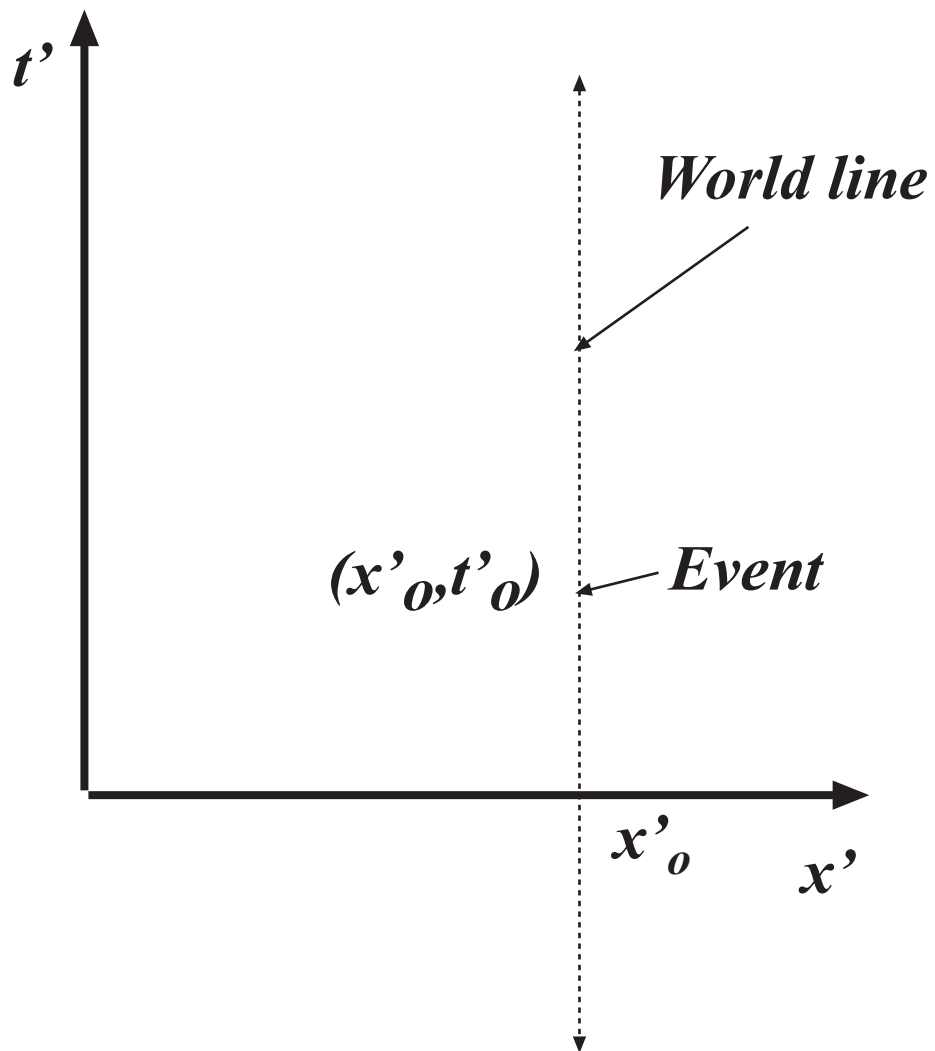
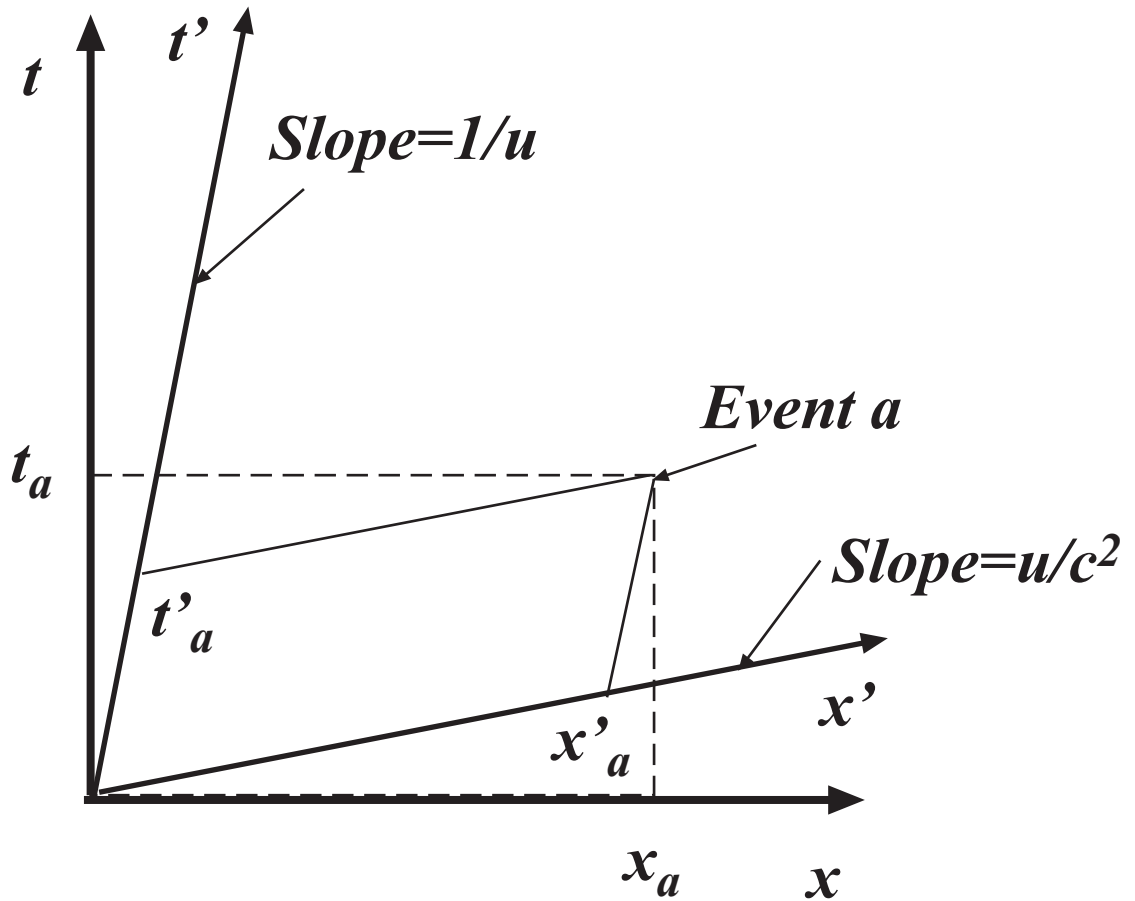Figure 3: World line for a particle at rest with respect to the Shuttle. Both move with velocity u in the 26-100 frame.

Figure 4: Coordinates of Event a in both S and S' frames.

$t = x/u$ for the equation of the t' axis in the 26-100 rest frame, shown in Fig. 4

The event a is located at a unique point in space-time which is described by (x_a, t_a) in the 26-100 frame and (x'_a, t'_a) in the Shuttle frame. The coordinates (x_a, t_a) and (x'_a, t'_a) are related by the Lorentz transformation. Be sure you can answer the following questions:

- What happens to the x' and t' axis as viewed from the 26-100 frame as u goes to zero?

- What happens to the x' and t' axis as viewed from the 26-100 frame as u goes to c, the velocity of light?

The Lorentz Contraction

Suppose there is a 2×4 [2] located of length L located in the Shuttle with the left end located at $x' = 0$ and the right end located at $x' = L$. At time $t' = 0$, the Shuttle Mission Specialist in Wood Measuring measures the length of the 2×4. He is aided by a light on each end of the 2×4 which flash at the time of the measurement, $t' = 0$. The lights define two events, one at $(0,0)$ and the other at $(L,0)$ in the Shuttle rest frame. The measurement events are shown in Fig. 5. We define measuring the length as the difference in spatial coordinates of two simultaneous events.

In the 26-100 frame, we see the Shuttle come roaring through 26-100 at velocity u and at $t = t' = 0$ the origin of the Shuttle coordinate system corresponds to the origin of the 26-100 coordinate system. We can use the Lorentz transformation to determine the coordinates of the measurement events on board the Shuttle. The left end measurement event is $(0,0)$ in the Shuttle frame and putting into the Lorentz transformation gives $(0,0)$ in the 26-100 frame as well. The right end measurement event is $(L,0)$ in the Shuttle frame and the Lorentz transformation to the 26-100 frame gives

$$(\frac{L}{\sqrt{1 - u^2/c^2}}, \frac{uL}{\sqrt{1 - u^2/c^2}}).$$

Notice that it appears to us in the 26-100 frame that the measurement of the right end appears to have been made later than the measurement of the left end while on the Shuttle they were both made at $t' = 0$. Diagrammatically, the situation in the 26-100 frame appears as in Fig. 6.

In the 26-100 frame, we measure the length of the 2×4 at $t = 0$. The left end coincides with the origin at $t = 0$. The right end follows along a world line

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[2] A "2×4" is a US standard piece of wood 1 1/2 by 3 1/2 inches and several feet long.
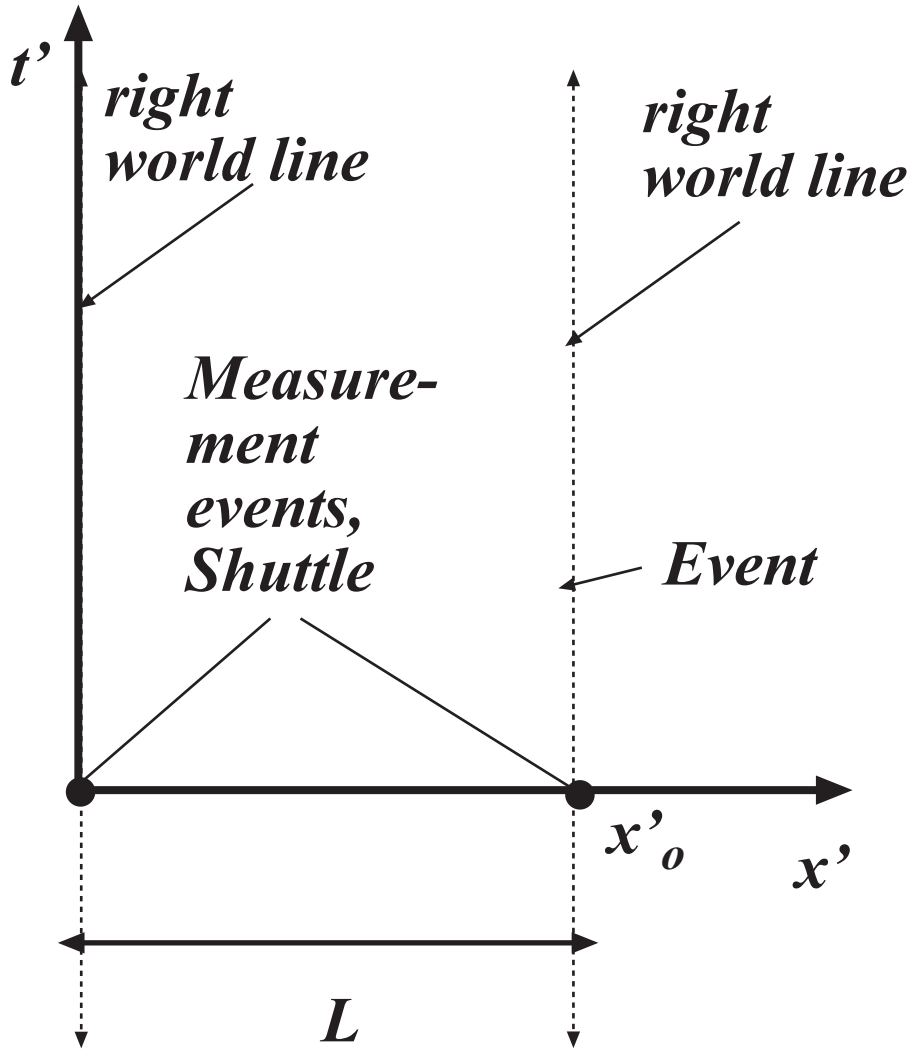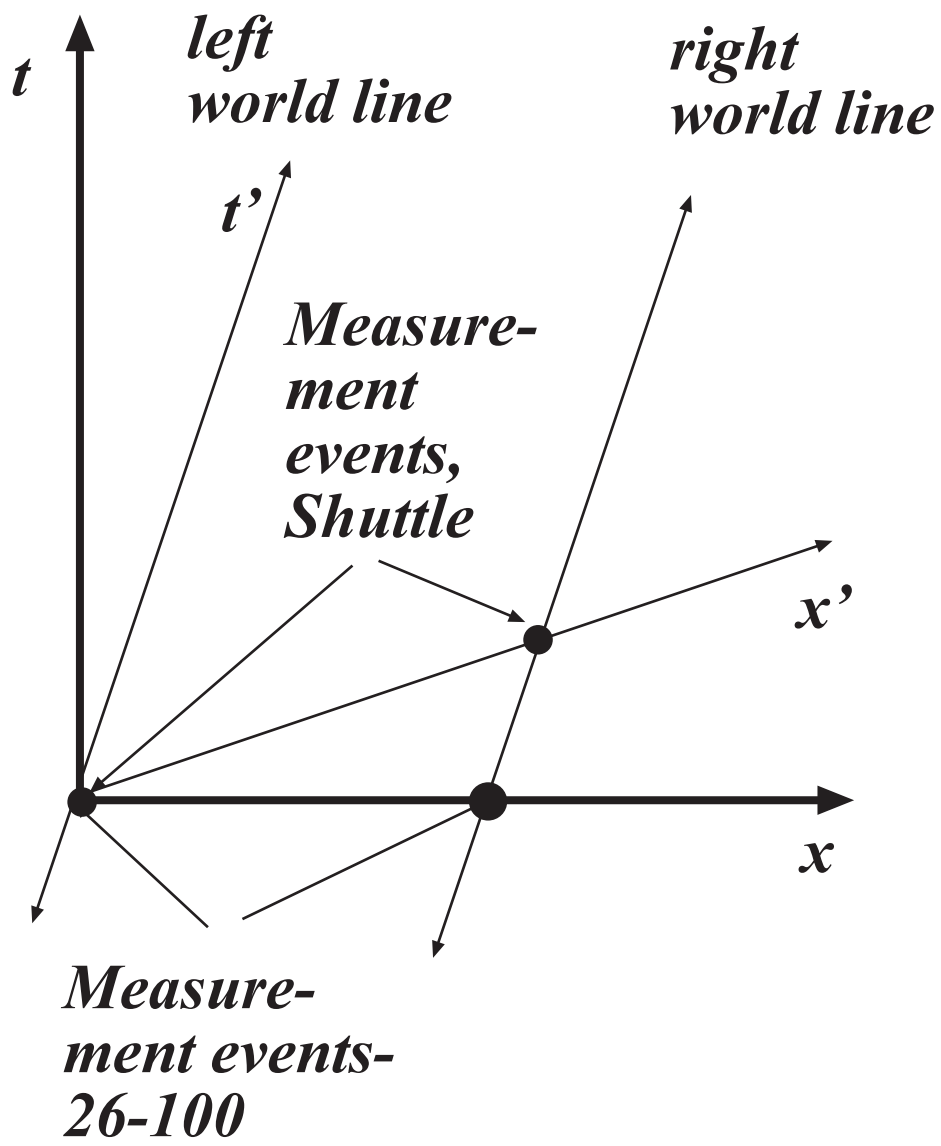
Figure 5: Measurement of 2×4 in the 26-100 frame.

Figure 6: Measurement in the Shuttle frame.

with slope $1/u$ and crosses through the point $(L/\sqrt{1-u^2/c^2}, uL/\sqrt{1-u^2/c^2})$, so the world line for the right hand end in the 26-100 coordinates is $t = x/u - L\sqrt{1-u^2/c^2}/u$. If we in the 26-100 frame measure the length of the 2×4 at $t = 0$, we find the length to be $L\sqrt{1-u^2/c^2}$, which is *shorter* by a factor $\sqrt{1-u^2/c^2}$ than the length in the Shuttle frame, in which the 2×4 is at rest.

Some numbers: the Shuttle orbits the earth every 90 minutes at a distance of 148 million km, which gives an average velocity of $u = 7 \times 10^5$ cm/s. This gives a contraction of $L(1 - 2 \times 10^{-10})$, so a 1 meter long 2×4 would appear in the 26-100 frame to be only shorter, about 5 atomic radii! The point here is that $c = 3 \times 10^{10}$cm/s is a very large compared with velocities we experience in everyday life. You should be sure you can carry through this argument for a 2×4 at rest in the 26-100 frame and measured by the Shuttle astronauts.

Time Dilation

Suppose now the Shuttle commander has been approved to take a 1 hour nap by Mission Control. He sets his alarm clock (situated at $x' = 0$) to go off at $t' = 0$ and goes to sleep. When the clock goes off, Mission Control approves 5 minutes additional rest (the commander being worn out from 2×4 measurements), so he hits the snooze button. Then at t'_{snooze}=300 s, the alarm goes off again. The situation in the Shuttle frame is shown in Fig. 7.

As before, the 26-100 coordinates and the Shuttle coordinates $x = x' = 0$ correspond at $t = t' = 0$. We do we see in 26-100? Since the coordinates correspond at $t = t' = 0$, in 26-100 we see the first alarm at $(0,0)$. Applying the Lorentz Transformation to the coordinates of the second alarm in the Shuttle rest frame gives $(ut_{snooze}/\sqrt{1-u^2/c^2}, t_{snooze}/\sqrt{1-u^2/c^2})$. Viewed from our frame, this says the commander slept *longer* by a factor $1/\sqrt{1-u^2/c^2}$ of than he did in the Shuttle rest frame. In the 26-100 frame, the alarm clock describes a trajectory along and our diagram is shown in Fig. 8.

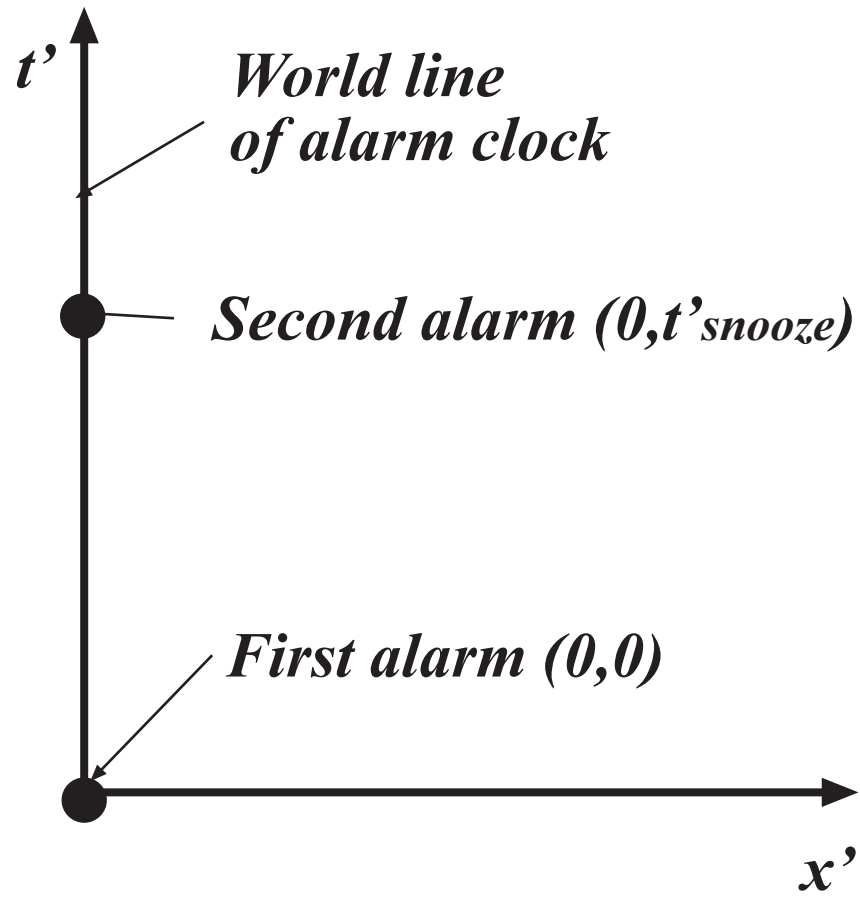This is the Lorentz time dilation. Putting in the same numbers as before,

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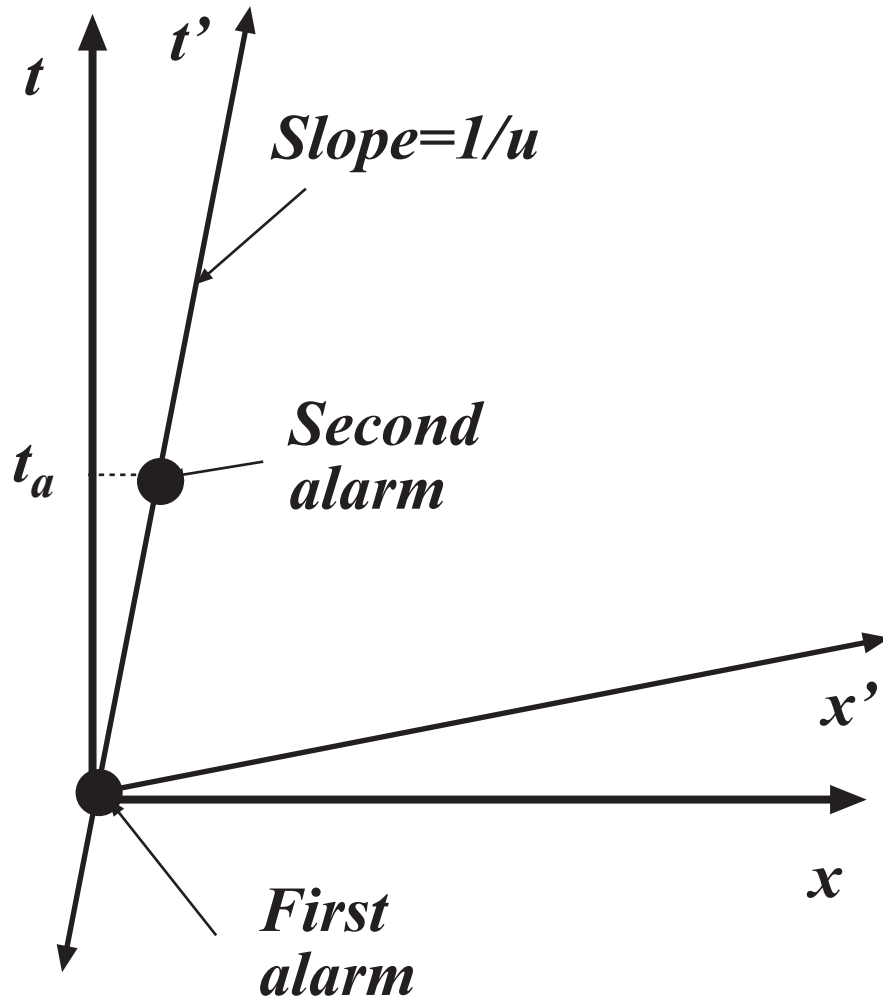Figure 7: Events of the snooze interval in the Shuttle frame.

Figure 8: Alarm clock snooze in the 26-100 frame.

we find the Shuttle commander appears to get about 1 nanosecond $(10^{-9}s)$ more sleep when viewed from the 26-100 frame.

Addition of velocities

Suppose an object moves with velocity v' along the x axis of the Shuttle frame. What is its velocity in the 26-100 frame? With the Galilean transformation (which we know is not correct), we would say the velocity in the 26-100 frame would be $v = v' + u$. If $v' = c$, then $v = c + u > c$, which contradicts Einstein's postulate that c is the same in all inertial frames. How do we add velocities using the Lorentz transformation? Take the total derivative of each of the equations:

$$x = \frac{1}{\sqrt{1 - u^2/c^2}}(x' + ut') \quad \rightarrow \quad dx = \frac{1}{\sqrt{1 - u^2/c^2}}(dx' + udt')$$

$$t\frac{1}{\sqrt{1 - u^2/c^2}}(t' + \frac{ux'}{c^2}) \quad \rightarrow \quad dt = \frac{1}{\sqrt{1 - u^2/c^2}}(dt' + \frac{udx'}{c^2})$$

then

$$\begin{aligned} v \;&=\; \frac{dx}{dt} = \frac{dx' + udt'}{dt' + \frac{udx'}{c^2}} = \frac{(dx' + udt')/dt'}{(dt' + \frac{udx'}{c^2})/dt'} \\ &=\; \frac{v' + u}{1 + \frac{uv'}{c^2}}. \end{aligned}$$

Notice that for $u, v' << c$, we recover the expression from the Galilean transformation. If $v' = c$, then

$$v = \frac{c + u}{1 + \frac{uc}{c^2}} = \frac{c + u}{\frac{1}{c}(c + u)} = c$$

which shows the Lorentz Transformation has the required property: *The velocity of light, c, is the same in all inertial frames.*

Finally, suppose an object moves in the S frame with velocity $\vec{v} = v_x\hat{x} + v_y\hat{y}$. We already know how to find v'_x from the previous paragraph. What about

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v'_y?

$$v'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma dt + \frac{\gamma u}{c^2} dx}$$

$$= \frac{dy/dt}{\gamma(1 + \frac{u dx/dt}{c^2})} = \frac{v_y}{\gamma(1 + \frac{u v_x}{c^2})}.$$

Here we have introduced some new notation: $\gamma = 1/\sqrt{1 - u^2/c^2}$. Also, since u/c appears frequently, pros usually write $\beta = u/c$ and $\gamma = 1/\sqrt{1 - \beta^2}$. If there are several different velocities in the problem, one writes $\beta_v = v/c$, $\gamma_v = 1/\sqrt{1 - \beta_v^2}$, etc.

Invariant interval

The Lorentz transformations 9 amount to a change of coordinates in Minkowski space; the space of position and time $(\vec{x}, t)$ (also referred to as spacetime). In Euclidean space, we can rotate the coordinate system in some way, yet the distance between to points $d = |\vec{x}_1 - \vec{x}_2|$ remains the same. Similarly, the Lorentz transformations leave the *interval* between two events inchanged. The interval between two events is just

$$s = \sqrt{c^2(t_1 - t_2)^2 - (\vec{x}_1 - \vec{x}_2)^2}. \tag{13}$$

We can prove the invariance of the interval in the following way: start with Eq. 13 measured in S. Then in S', the interval will be

$$s'^2 = c^2(t'_1 - t'_2) - (x'_1 - x'_2)^2 - (y'_1 - y'_2)^2 - (z'_1 - z'_2)^2.$$

Now substitute the Lorentz transformation Eq. 9

$$s'^2 = c^2((\gamma t_1 + \gamma\beta x_1/c) - (\gamma t_2 + \gamma\beta x_2/c))^2$$
$$-((\gamma x_1 + \gamma\beta ct_1) - (\gamma x_2 + \gamma\beta ct_2))^2$$
$$-(y_1 - y_2)^2 - (z_1 - z_2)^2$$

and expanding and collecting terms gives

$$s'^2 = c^2(t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2 = s.$$

So s, the interval between two events, is the same in all inertial frames. In spacetime, s fills the same role as distance does in Euclidian space: they are both invariant quantities under coordinate transformations.