Daffodil International University

Department of Software Engineering

Faculty of Science & Information Technology

Midterm Examination, Fall 2024

Course Code: MAT 101; Course Title: Mathematics I

Sections & Teachers: A, B, C, D, E,F (SB) ,G, H, I, J, K, L (SKI), M, N (NA),O, P, Q, Retake (GRS),

Time: 1 Hour 30 Mins **Marks: 25**

Answer ALL Questions

[The figures in the right margin indicate the full marks and corresponding course outcomes. All portions of each question must be answered sequentially.]

.	*a)*	**Explain** the followings with example: i) Domain and range of function. ii) Periodic function. iii) Even function. iv) One-one function.	[04]	CLO-1 *Level-C2*
	b)	$f(x) = \begin{cases} 2x+3; & x<0 \\ x^2; & 0 \le x \le 2 \\ \sqrt{x-2} & x>2 \end{cases}$ is a piecewise function. i) **Identify** the domain and range of the given function. ii) **Graphically** show the function.	[05]	
2.	*a)*	A function is given by $f(x) = \|x\|$.**Examine** the continuity and differentiability of $f(x)$ at $x = 0$.	[06]	CLO-2 *Level-C3*
3.	*a)*	**Differentiate :** i) $tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\sin^{-1}\frac{2x}{1+x^2}$ ii) $y = \frac{cosx-sinx}{\sqrt{1-sin2x}}$ with respect to x.	[04]	CLO-2 *Level-C3*
	b)	If $y = cos\{\ln(1+x)\}$,then **show** that $(1+x)^2y_{n+2} + (2n+1)(1+x)y_{n+1} + (n^2+1)y_n = 0.$	[06]	

Solution of Batch 43 Fall 2024 – Mid Exam

1.(a)

(i) Domain – The set of all possible input values for which the function is defined. Example: For $f(x) = \frac{1}{x-2}$, the domain is $D_f = \mathbb{R} - \{2\}$.

Range – The set of all possible output values produced by the function's inputs. Example: For $f(x) = x^2$, the range is $R_f = [0, \infty)$.

(ii) Periodic function – (Not in the syllabus)

(iii) Even function – (Not in the ")

(iii) One-one function – A function is one-one if every element in domain A maps to unique element/output in the codomain function. Formally, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
Example: $f(x) = 3x-1$ is one-one function.

1 (b)

(i) Domain and range of $f(x)$

Domain – For $f(x)$, all the real number can define each piecewise functions. So, Domain of $f(x)$, $D_f = \mathbb{R}$. or (∞, ∞).

Range –

1. For $x<0$: $f(x) = 2x+3$, as $x \to -\infty$, $f(x) = -\infty$

2. For $0 \leq x \leq 2$: $f(x) = x^2$. at $x=0$, $f(x)=0$, at $x=2$, $f(x) = 4$. $R[0,4]$

3. For $x>2$: $f(x) = \sqrt{x-2}$, at, $x \to 2^+$, $f(x) \to 0^+$, as $x = \infty$, $f(x) = \infty$

$\therefore$ Range of $f(x)$, $R_f = \mathbb{R}$ or $(-\infty, \infty)$ (Ans.)

(ii) Graph sketching

2.(a) Given, $f(x) = |x|$, at $x=0$, $f(0) = |0| = 0$.

$$LHL = \lim_{x \to 0^-} f(x) = \lim_{x \to 0} (-x) = 0$$

$$RHL = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} |x| = 0$$

$\therefore$ The function is continuous as $f(0) = LHL = RHL$

So, this function is differentiable at $x=0$

Now,

$$Lf'(0) = \lim_{h \to 0^-} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-(h)}{-h}$$

$$= 1$$

$$Rf'(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

$$= 1.$$

$\therefore f(0) \neq Lf' \neq Rf'$

$\therefore$ The function isn't differentiable at $x=0$.

(Ans.)

3.(a) (i) $y_1 = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

Let, $x = \tan\theta$, $\sqrt{1+x^2} = \sec\theta$

$\therefore y_1 = \tan^{-1}\left(\dfrac{\sec\theta - 1}{\tan\theta}\right)$

$= \tan^{-1}\left(\dfrac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\dfrac{\cancel{2}\sin^{\cancel{2}}\frac{\theta}{2}}{\cancel{2}\cancel{\sin\frac{\theta}{2}}\cdot\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$= \dfrac{\theta}{2}$

$= \dfrac{1}{2}\tan^{-1}x$

Now, $z = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \sin^{-1}(\sin 2\theta)$

$= 2\theta$

$= 2\tan^{-1}x$

$\therefore \dfrac{dy}{dz} = \dfrac{\frac{dy}{dx}}{\frac{dz}{dx}} = \dfrac{\frac{d}{dx}\left(\frac{1}{2}\tan^{-1}x\right)}{\frac{d}{dx}\left(2\tan^{-1}x\right)} = \dfrac{\frac{1}{2}\cdot\cancel{\frac{1}{1+x^2}}}{2\cdot\cancel{\frac{1}{1+x^2}}} = \dfrac{1}{4}$

(Ans.)

(ii) given, $y = \frac{\cos x - \sin x}{\sqrt{1-\sin 2x}}$

$\Rightarrow y = \frac{\cos x - \sin x}{\sqrt{1-2\sin x \cdot \cos x}}$

$= \frac{\cos x - \sin x}{|\sin x - \cos x|}$ $\Big|$ $(a-b)^2$ is implemented here.

$= \pm 1.$

$\therefore \frac{dy}{dx} = 0$ (Ans.)

3.(b) If, $y = \cos[\ln(1+x)]$

$\Rightarrow y_1 = -\sin[\ln(1+x)] \cdot \frac{1}{1+x}$

$\Rightarrow y_2 = -\left(\frac{1}{1+x}\right)^2 \cdot \cos[\ln(1+x)] \cdot$

$\Rightarrow y_3 = \left(\frac{1}{1+x}\right)^3 \cdot \sin[\ln(1+x)]$

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Now, $(y_1)^2 = \left[-\sin[\ln(1+x)\frac{1}{1+x}\right]^2$

$\Rightarrow (1+x)^2 y_1^2 = \sin^2[\ln(1+x)]$

$\Rightarrow (1+x)^2 y_1^2 = 1-\cos^2[\ln(1+x)]$

$\Rightarrow (1+x)^2 y_1^2 = 1-y^2$

$\Rightarrow (1+x)^2 y_1^2 + y^2 = 1$

$\Rightarrow (1+x)^2 (2y_1y_2) + y_1^2 \cdot 2(1+x) + 2yy_1 = 0$ [Differentiate]

$\Rightarrow (1+x)^2 y_2 + (1+x)y_1 + y = 0$ [Divided by $2y_1$]

If, equation (1) is differentiated n times,

$$\frac{d^n}{dx^n}[(1+x^2)y_2]+\frac{d^n}{dx^n}[(1+x)y_1]+\frac{d^n}{dx^n}[y]=0$$

For, $\frac{d^n}{dx^n}[(1+x^2)y_2]$; $u=(1+x)^2,\ v=y_2$

$u_1=2(1+x)$

$u_2=2$

$u_3=0.$

$$\therefore [(1+x)^2 y_{n+2}]_n = \binom{n}{0}(1+x)^2 y_{n+2} + \binom{n}{1}2(1+x)y_{n+1} + \binom{n}{2}2\cdot y_n$$

$$= (1+x)^2 y_{n+2} + 2n(1+x)y_{n+1} + n(n-1)y_n$$

For, $\frac{d^n}{dx^n}[(1+x)y_1]$; $u=(1+x),\ v=y_1$

$u_1=1$

$u_2=0$

$$\therefore [(1+x)y_1]_n = \binom{n}{0}(1+x)y_{n+1} + \binom{n}{1}1\cdot y_n = (1+x)y_{n+1}+ny_n$$

and For $\frac{d^n}{dx^n}[y] = y_n$

So, $[(1+x)^2 y_{n+2} + 2n(1+x)y_{n+1} + n(n-1)y_n] + [(1+x)y_{n+1}+ny_n] + [y_n] = 0$

$$\therefore (1+x)^2 y_{n+2} + (2n+1)(1+x)y_{n+1} + (n(n-1)+n+1)y_n = 0$$

$$\Rightarrow (1+x)^2 y_{n+2} + (2n+1)(1+x)y_{n+1} + (n^2+1)y_n = 0$$

(Ans)

Daffodil International University

Faculty of Science & Information Technology

Department of Software Engineering

Midterm Examination, Summer 2025

Course Code: MAT101

Course Title: Mathematics I

Batch: 44; Section: A-L

Teachers Initial: MMH, DMMK, MIA, GRS

Time: 1 Hour 30 Minutes **Marks: 25**

Answer ALL Questions

[The figures in the right margin indicate the full marks and corresponding course outcomes. All portions of each question must be answered sequentially]

1.	**Explain** the following with a suitable example **a.** Relation **b.** Bijective Function **c.** Homogeneous Function	3×1=3	CLO-1 L-2
2.	**a. Compute** the domain and range of the following function $$f(x) = \frac{2x^2+3x+17}{x^2-x-2}$$	3	
	b. Express the following function graphically and write a comment on its domain and range from the graph $$f(x) = \lvert x-2\rvert - \lvert x+3\rvert + \lvert x\rvert$$	4	
3.	**a. Calculate** the non-zero value of k that makes the following function continuous at $x=0$, then also determine whether the function is differentiable or not at the same point by using the non-zero value of k $$f(x) = \begin{cases} \dfrac{\tan kx}{x}, & x<0 \\ 3x+k^2, & x \geq 0 \end{cases}$$	5	CLO-2 L-3
	b. If $u = \tan^{-1}\left(\frac{y+x}{\sqrt{y}+\sqrt{x}}\right)$ then **show** that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u$	4	
	c. Compute the derivative of $x^{\sin(\ln(\tan^{-1}\sqrt{ax}))}$ with respect to $a^{bx}\sin^m(rx) + a^m$	3	
	d. If $y = \ln(a^n x + b^n)$ then **calculate** the *n-th* derivative (y_n) of y	3	

Solution of Batch 44 Summer 2025 – Mid Term

1.(a) Relation – A relation from set A to set B is any subset of $A \times B$.

Example:

$A = \{1, 2\}$, $B = \{a, b\}$, Relation $= \{(1,a), (2,b)\}$.

(b) Bijective function –

A function is bijective if it is both injective and surjective.

Example:

$f: \{1,2,3\} \to \{a,b,c\}$ with $f(1)=a$, $f(2)=b$, $f(3)=c$.

(c) Homogeneous function – A function $f(x_1, \ldots, x_n)$ is homonogeneous of degree k if for all t inputs $f(tx_1, \ldots, tx_n) = t^k f(x_1, \ldots, x_n)$.

Examples:

$f(x,y) = 3x^2y$ is homogeneous of degree 3.

[Not for mid]

2.(a) Domain and range of $f(x)$:

Here, $f(x) = \dfrac{2x^2+3x+17}{x^2-x-2}$

$= \dfrac{2x^2+3x+17}{x^2-2x+x-2} = \dfrac{2x^2+3x+17}{x(x-2)+1(x-2)} = \dfrac{2x^2+3x+17}{(x-2)(x+1)}$

So, $f(x)$ is not defined for $x = -1, 2$ [$x+1=0 \Rightarrow x=-1$ & $x=2$]

$\therefore$ domain of $f(x) = \mathbb{R} - \{-1, 2\}$.

Let, $y = f(x) = \dfrac{2x^2+3x+17}{x^2-x-2}$

$\Rightarrow x^2y - xy - 2y = 2x^2+3x+17$

$\Rightarrow x^2(y-2) + x(-y-3) + (-2y-17) = 0$

$\therefore x = \dfrac{-(-y-3) \pm \sqrt{(-y-3)^2 - 4(y-2)(-2y-17)}}{2(y-2)}$

So x is not defined for $y = 2$.

$\therefore$ range of $f(x) = \mathbb{R} - \{2\}$. (Ans)

2.(b) Graph Sketch (Not in the mid syllabus)
(Piecewise ~~formula~~ will be applied here)

3.(a) continuity and differentiability for $f(n)$ —

$$f(n) = \begin{cases} \frac{\tan(kn)}{n}, & n < 0 \\ 3n + k^n, & n \geq 0 \end{cases}$$

Continuity:

$LHL = \lim_{n \to 0^-} f(n)$

$= \lim_{n \to 0} \frac{\tan(kn)}{n}$

$= k \cdot \lim_{n \to 0} \frac{\tan n}{n}$

$= k$

$RHL = \lim_{n \to 0^+} f(n)$

$= \lim_{n \to 0} (3n + k^n)$

$= k^n$

and, at, $n = 0$, $f(0) = 3 \cdot 0 + k^n = k^n$

$\therefore f(0) = RHL \neq LHL$

As, the function is not continuous at $n=0$, so it's not differentiable.

Now, $k^n = k$

$\Rightarrow k^n - k = 0$

$\Rightarrow k(k-1) = 0$

$\Rightarrow k = 0, 1$

$\therefore k = 1$ (Ans.)

3.b) Not in the Mid Syllabus

3.(c) Given $y = x^{\sin(\ln(\tan^{-1}\sqrt{ax}))}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left\{x^{\sin(\ln(\tan^{-1}\sqrt{ax}))}\right\}$$

$$= x^{\sin(\ln(\tan^{-1}\sqrt{ax}))}\cdot\frac{d}{dx}\left[\sin(\ln(\tan^{-1}x))\cdot\ln x\right]$$

$$= x^{\sin(\ln(\tan^{-1}\sqrt{ax}))}\left[\sin(\ln(\tan^{-1}\sqrt{ax}))\cdot\frac{1}{x} + \ln x\cdot\left(\cos(\ln(\tan^{-1}\sqrt{ax}))\cdot\frac{1}{\tan^{-1}\sqrt{ax}}\cdot\left(\frac{1}{1+(\sqrt{ax})^2}\cdot\frac{a}{2\sqrt{ax}}\right.\right.\right.$$

$$= x^{\sin(\ln(\tan^{-1}\sqrt{ax}))}\left[\frac{\sin(\ln(\tan^{-1}\sqrt{ax}))}{x} + \ln x\left(\cos(\ln(\tan^{-1}\sqrt{ax})\cdot\frac{1}{\tan^{-1}\sqrt{ax}}\cdot\frac{a}{2\sqrt{ax}(1+ax)}\right)\right]$$

Now, $u = a^{bx}\sin^m(rx) + a^m$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}\left(a^{bx}\sin^m(rx)\right) + \frac{d}{dx}(a^m)$$

$$= \left[a^{bx}\cdot b\cdot\ln a\cdot\sin^m(rx) + a^{bx}\cdot m\cdot\sin^{m-1}(rx)\cdot r\cdot\cos(rx)\right]$$

$$= a^{bx}\cdot\sin^m(rx)\left[b\cdot\ln a + mr\cot(rx)\right]$$

$\sin^{m-1}(rx) = \sin^m(rx)\cdot\frac{1}{\sin(rx)}$

$$\therefore \frac{dy}{du} = \frac{x^{\sin(\ln(\tan^{-1}\sqrt{ax}))}\left[\frac{\sin(\ln(\tan^{-1}\sqrt{ax}}{x} + \ln x(\cos(\ln(\tan^{-1}\sqrt{ax}\cdot\frac{a}{\tan^{-1}\sqrt{ax}\cdot 2\sqrt{}}\right.}{a^{bx}\cdot\sin^m(rx)\left[b\cdot\ln a + mr\cot(rx)\right]}$$

(Ans.)

3.(d) $y = \ln(a^n x^n + b^n)$

$$\Rightarrow y_1 = \frac{1}{a^n x^n + b^n} \cdot a^n n x^{n-1} = n a^n \cdot \frac{x^{n-1}}{a^n x^n + b^n}$$

$$\Rightarrow y_2 = n a^n \left(\frac{(a^n x^n + b^n)(n-1) \cdot x^{n-2} - x^{n-1}(n a^n x^{n-1})}{(a^n x^n + b^n)^2} \right)$$

$$= n a^n x^{n-2} \frac{(a^n x^n + b^n)(n-1) - n a^n x^n}{(a^n x^n + b^n)^2}$$

$$y_n = a^n (n-1)! \, (a^n x^n + b^n)^{-n} \cdot (-1)^{n-1}$$ [General solve for $\ln(ax+b)$]

(Ans.)