GU TECH, AL GHAZALI UNIVERSITY

Fall-2024 Department of Computer Science
Midterm Exam
12^{th} December 2024, 09:00am – 11:00am

SOLUTION PAPER

Course Code: CS103	**Course Name:** Discrete Structures
Instructor Name: Dr. M. Shahzad	
Student Roll No:	**Section:**

Instructions:

- Return the question paper and make sure to keep it inside your answer sheet.
- Read each question completely before answering it. There are **Three questions and two pages (front plus back)**.
- In case of any ambiguity, you may make assumption. However, your assumption should not contradict any statement in the question paper.
- Do not write anything on the question paper (except your ID and group).

Total Time: 2 Hours **Max Points**: 25

CLO 1: Understand the key concepts of Discrete Structures such as propositional logics, Sets, Permutations, Relations, Graphs and Trees etc.

Q1a: For the universe of all integers, let p(x), q(x), r(x), s(x) and t(x) be the following statements:
[5 points, 20 mins]

p(x): x is even
q(x) : x is (exactly) divisible by 4
r(x) : x is (exactly) divisible by 5
s(x) : x > 0 (positive integer)
t(x) : x is a perfect square

Write the following statements in symbolic form:

(i) At lease one integer is even.

Symbolic Form:

$\exists x\, p(x)$

Explanation: There exists at least one integer x such that x is even.

(ii) There exists a positive integer that is even.

Symbolic Form:

$\exists x(s(x) \wedge p(x))$

Explanation: There exists an integer x such that x>0 and x is even.

(iii) If x is even, then x is not divisible by 5.

Symbolic Form:

$\forall x(p(x)\rightarrow\neg r(x))$

Explanation: For all integers x, if x is even, then x is not divisible by 5.

(iv) There exists an even integer divisible by 5.

Symbolic Form:

$\exists x(p(x)\wedge r(x))$

Explanation: There exists an integer x such that x is even and x is divisible by 5.

(v) if x is even and x is a perfect square, then x is divisible by 4.

Symbolic Form:

$\forall x((p(x)\wedge t(x))\rightarrow q(x))$

Explanation: For all integers x, if x is even and x is a perfect square, then x is divisible by 4.

Q1b. Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people. **[5 points, 20 mins]**

(i) $\forall x(C(x) \rightarrow F(x))$

Translation: For all people, if someone is a comedian, then they are funny.

Explanation: This means that being funny is a necessary condition for being a comedian.

(ii) $\forall x(C(x) \wedge F(x))$

Translation: Everyone is a comedian and funny.

Explanation: This means that every person in the domain is both a comedian and funny.

(iii) $\exists x(C(x) \rightarrow F(x))$

Translation: There exists at least one person such that if they are a comedian, then they are funny.

Explanation: This is a weaker condition than (i); it requires that for at least one person, being funny is a necessary condition for being a comedian.

(iv) $\exists x(C(x) \wedge F(x))$

Translation: There exists at least one person who is both a comedian and funny.

Explanation: This asserts that there is at least one individual in the domain who satisfies both properties.

(v) $\exists x(C(x) \vee F(x))$

Translation: There exists at least one person who is either a comedian or is funny.

Explanation: This statement asserts that there is at least one person in the domain who satisfies either the condition of being a comedian or being funny.

Q1c. Perform the following tasks on each of these statements: **[5 points, 20 mins]**

- *There is a chemistry major in CS103.*
- *No chemistry major knows Python.*

(i) Negate it in English. Do not use the phrase "It is not the case that"

There is no chemistry major in CS103

(ii) Express it using quantifiers (define appropriate predicates and UoD)

Let:

- C(x): "x is a chemistry major."
- S(x): "x is enrolled in CS103."
- Domain: All students in the class.

Quantified Expression:

$\exists x\,(C(x) \land S(x))$

There exists at least one student xxx who is a chemistry major and is enrolled in CS103.

(iii) Negate the quantified expression, with no ¬ symbol left of a quantifier

Negation:

$\neg(\exists x\,(C(x) \land S(x))) \equiv \forall x\,\neg(C(x) \land S(x))$

Using De Morgan's law:

$\forall x\,(\neg C(x) \lor \neg S(x))$

For every student x, either x is not a chemistry major or x is not enrolled in CS103.

(iv) Translate the negation into simple English

Negation in English:
Every student is either not a chemistry major or is not enrolled in CS103.

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Q2a. Write the contrapositive, converse and inverse of the statement "If P is a square, then P is a rectangle." Are all the statements true? **[3 points, 10 mins]**

Original Statement:
"If P is a square, then P is a rectangle."
This can be written as:
P is a square $\rightarrow$ P is a rectangle.

Contrapositive:

The contrapositive negates and switches the hypothesis and conclusion:
P is not a rectangle →P is not a square.
Truth of the Contrapositive:
This is logically equivalent to the original statement, so it is **true**.

Converse:
The converse switches the hypothesis and conclusion:
P is a rectangle →P is a square.
Truth of the Converse:
This is **false** because not all rectangles are squares. A rectangle is a broader category, and a square is a special type of rectangle.

Inverse:
The inverse negates both the hypothesis and conclusion:
P is not a square →P is not a rectangle
Truth of the Inverse:
This is **false** because a non-square figure (e.g., a generic rectangle) can still be a rectangle.

Summary of Truth Values:

- Original Statement: **True**
- Contrapositive: **True**
- Converse: **False**
- Inverse: **False**

Q2b. Use a Truth table to determine if the statements [P → (Q ∨ R)] ≡ [¬R → (P → Q)] is true.
[1 point, 15 mins]

P	Q	R	$Q \vee R$	$P \to (Q \vee R)$	$\neg R$	$P \to Q$	$\neg R \to (P \to Q)$
T	T	T	T	T	F	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	F	F	T
T	F	F	F	F	T	F	F
F	T	T	T	T	F	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	F	T	T
F	F	F	F	T	T	T	T

Conclusion:

The column for P→(Q∨R) matches the column for ¬R→(P→Q) in all rows.

Thus, [P→(Q∨R)] ≡ [¬R→(P→Q)] is true (the statements are logically equivalent).

Q2c. Show the following equivalences using logical equivalence laws. **[2 points, 20 mins]**

(i) $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

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\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F} \\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction} \\
&\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

(ii) $\neg (p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

The biconditional $p \leftrightarrow q$ is equivalent to:
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
Thus,
$\neg(p \leftrightarrow q) \equiv \neg((p \wedge q) \vee (\neg p \wedge \neg q))$

Step 2: Apply De Morgan's Laws
Using De Morgan's laws:
$\neg((p \wedge q) \vee (\neg p \wedge \neg q)) \equiv \neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q)$
Simplify each term:
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Substitute back:
$\neg(p \leftrightarrow q) \equiv (\neg p \vee \neg q) \wedge (p \vee q)$

Step 3: Expand p↔¬qp \leftrightarrow \neg qp↔¬q
The statement $p \leftrightarrow \neg q$ expands as:
$p \leftrightarrow \neg q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

Step 4: Verify Equivalence
To prove equivalence, simplify both sides step by step:
Left Side:
$\neg(p \leftrightarrow q) \equiv (\neg p \vee \neg q) \wedge (p \vee q)$
Expand using the distributive law:
$(\neg p \vee \neg q) \wedge (p \vee q) \equiv (\neg p \wedge p) \vee (\neg p \wedge q) \vee (\neg q \wedge p) \vee (\neg q \wedge q)$
Simplify terms:

- $\neg p \wedge p \equiv \text{False}$
- $\neg q \wedge q \equiv \text{False}$
  Thus:

$(\neg p \wedge q) \vee (\neg q \wedge p)$
Right Side:
Expand $p \leftrightarrow \neg q$
$(p \wedge \neg q) \vee (\neg p \wedge q)$
This matches the simplified form of the left side.

Conclusion:
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$
The two statements are logically equivalent.

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Q3. Construct a truth table for each of these compound propositions. **[5 points, 10 mins]**

- $(p \oplus q) \land (p \oplus \neg q)$
- $\neg p \leftrightarrow q$