AI 数学的突破进展 Breakthrough in AI Math

“亚里士多德”AI 再度发力 - 埃尔德什 #1026 问题

”Aristotle” AI Once Again Flexes Muscles - Erdős Problem #1026

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16 DEC 2025
POCY 13 CHNM 10
Version 13.10.27

纲要 Overview

1. 再度突破 Another Breakthrough
2. 背景知识 Background Knowledge
3. 埃尔德什 1026 号猜想 Erdős Conjecture #1026

所需知识 Knowledge Needed

领域：组合数学

Area：Combinatorics

等级：中学数学

Level：High School Mathematics

尽量浅显易懂！我也是那种看公式需要翻译为人话的哈哈

In easy terms! I also need translator for messy formulas :-)

再度突破 Another Breakthrough

- 12 月 7 日，“亚里士多德”AI 率先对埃尔德什 1026 问题作出重大突破 On December 7, Boris Alexeev from Harmonic announced that AI 'aristotle' made a significant breakthrough on Erdős problem #1026
- 该部分猜想经专家们讨论由 Stijn Cambie 于 9 月 13 日定型 The part of the conjecture was proposed by Stijn Cambie on September 13 after discussions at erdosproblems.com by experts
- 此为继 11 月 29 日突破部分 124 猜想后短短九天亚里士多德 AI 再次对开放数学猜想作出突破 This is another breakthrough by 'Aristotle' AI within 9 days after the November 29 breakthrough on Erdős #124

突破 The Breakthrough

- 一天后 12 月 8 日陶哲轩发文宣布推广形式猜想 (任意 n) 告解决 A day later on December 8 Terrence Tao announced in his article that the generalized case (any n) has been solved
- 12 月 12 日 Boris Alexeev 发布 AI 对推广猜想任意 n 的形式化 LEAN 语言证明 On December 12 Boris Alexeev published AI's proof on the general case for any n
- 在这里我们做一详细介绍供大家讨论 Here we give a more detailed introduction for public discussion
- 时间仓促有不足之处欢迎批评指正 Time was limited in my study, any correction and suggestion are very welcome

组合数学 Combinatorics

- 1026 号猜想是一个关于从一任意有限序列中挑选单调子序列使其和为最大的问题 The conjecture 1026 is about maximizing the sum of monotone subsequences of any finite sequence
- 1935 年埃尔德什和塞克雷斯证明了如果一个不同数字组成的序列有至少 $(r-1)(s-1)+1$ 项，那么要么存在长度为 r 的递增子序列，要么存在长度为 s 的递减子序列。此后这一定理称为埃尔德什-塞克雷斯定理。In 1935, Erdős–Szekeres proved for a sequence of distinct numbers of at least $(r-1)(s-1)+1$ entries, either there exists a monotone increasing subsequence of length r, or there exists a monotone decreasing subsequence of length s. This is known as the Erdős–Szekeres Theorem since

埃尔德什-塞克雷斯定理 Erdős–Szekeres Theorem

- 定理里给出的序列长度是不能再缩短的，可以通过以下例子理解：$r = s = 4$, 序列 $\{3, 2, 1,\ 6, 5, 4,\ 9, 8, 7\}$ 长度为 9，无法找到长度为 4 的单调子序列 The length of the sequence given in the theorem cannot be further shortened. This can be illustrated with a simple example: $r = s = 4$, sequence is $\{3, 2, 1,\ 6, 5, 4,\ 9, 8, 7\}$. The sequence is of length 9 but there is no length 4 monotone subsequence.
- 1959 年塞登伯格给出了一个极为简洁漂亮的证明 In 1959 Seidenberg gave a very short and beautiful proof of the Erdős–Szekeres Theorem

塞登伯格的 1959 证明 Seidenberg's 1959 Proof

- 我们以 $\{x_1, x_2, \ldots, x_n\}$ 代表长度为 n 的序列 Let $\{x_1, x_2, \ldots, x_n\}$ represent a sequence of length n
- 我们给序列中每一项 x_i 打上两个长度标签 (E_i, D_i)，分别代表**以 x_i 结束**的最长递增子序列和最长递减子序列的长度 We attach two length labels (E_i, D_i) to each x_i, representing the lengths of the longest increasing subsequence and longest decreasing subsequence **ending at x_i**
- 分析任意两项 x_i 和 x_j，$i < j$，发现它们的标签不能全同, 即可运用鸽巢原理 Analyze any two entries x_i and x_j, realizing their label pairs cannot be the same, then apply the pigeonhole principle

埃尔德什 1026 号猜想 Erdős Conjecture #1026

- 陶哲轩给出了一个博弈形态的等价解读，更为通俗有趣 In his December 8 article Terrence Tao gave a game theoretic equivalent view of the problem which is more fun
- 爱丽丝有一些相同的硬币她可选择分配为小堆硬币排列成序列 Alice has a number of same type coins that she can choose to split into a sequence of smaller piles
- 鲍勃可从中选择单调不减或单调不增子序列小堆拿走 Bob can then take a monotone (can stay equal) subsequence of the coin
- 鲍勃的博弈目标是拿走尽量多个数的硬币而爱丽丝则是要保留尽量多 Both Alice and Bob's aim to keep as many coins as possible

埃尔德什 1026 号猜想 Erdős Conjecture #1026

- 我们拿前面给的例子试一下比如爱丽丝将硬币分为小堆序列为：Let's take the previous example sequence, say Alice split the coins into this sequence of small piles: $\{3, 2, 1,\ 6, 5, 4,\ 9, 8, 7\}$
- 这个例子里鲍勃可以拿走超过一半的硬币 In this example Bob could take more than half of the coins
- 如果爱丽丝有很多很多硬币情况发生了变化，鲍勃只能拿到约三分之一了 $\{1003, 1002, 1001,\ 1006, 1005, 1004,\ 1009, 1008, 1007\}$ If Alice has a lot more coins then the situation changes, Bob could only expect to get about $1/3$ of the coins

埃尔德什 1026 号猜想 Erdős Conjecture #1026

- 也就是说序列长度为平方数 $n = k^2$ 时，爱丽丝有个近似于平均分堆的策略，即运用埃尔德什-塞克雷斯定理里无法缩短的反例构造，可以限制鲍勃的最大收益逼近于总量的 $1/k$。This demonstrates that, when the length of the sequence is a square $n = k^2$, Alice has a strategy to split the pile nearly evenly and apply the counter-example in the Erdős–Szekeres theorem that the length of the sequence cannot be shortened
- 由此可提出猜想当序列长度为平方数 $n = k^2$ 时，鲍勃的最大化收益至少可达到总量的 $1/k$。From this Cambie proposed conjecture for the case of $n = k^2$ that Bob's max gain is at least $1/k$ of the total amount

埃尔德什 1026 号猜想 Erdős Conjecture #1026

- 现在把 Cambie 的猜想形式写的正式一些：$n = k^2$，$\{x_1, x_2, ..., x_n\}$ 为自然数的序列，则存在一单调不减或单调不增的子序列其和至少为序列总和的 $1/k$。Now we write Cambie's conjectured form in a more formal way: $n = k^2$, $\{x_1, x_2, ..., x_n\}$ is a sequence of natural numbers, then there exists a non-decreasing or a non-increasing subsequence which sums to at least $1/k$ of the total sum of the sequence.
- 这里我们简化只考虑自然数，这并不影响 AI 证明的要点。Here we simply consider natural numbers only, this does not affect the gist of AI's proof.

AI 的证明 AI's proof

- 我们用 S_i 代表以 x_i 结束的所有单调不减子序列之和的最大值，T_i 代表以 x_i 结束的所有单调不增子序列之和的最大值 Let S_i represent the largest sum among the sums of every non-decreasing subsequences ending at x_i, T_i represent the largest sum among the sums of every non-increasing subsequences ending at x_i
- 对每一个 x_i，构造一个以 $(S_i - x_i, T_i - x_i)$，(S_i, T_i) 为对角坐标的正方形 Q_i - For each x_i, construct a square Q_i with the diagonal corners located at coordinates $(S_i - x_i, T_i - x_i)$ and (S_i, T_i)
- 这些正方形两两互不重叠！两种情况分析：$x_i \leq x_j$, $x_i \geq x_j$ Analysis of both cases shows these squares are all disjoint from each other!

AI 的证明 AI's proof

- 而这些正方形 Q_i 又包含在以 $(0,0)$, $(max\{S_i\}, max\{T_i\})$ 为对角坐标的长方形内 Also these squares Q_i are all contained inside the rectangle with diagonal corners located at coordinates $(0,0)$, $(max\{S_i\}, max\{T_i\})$
- 得到关于面积的关系：$max\{S_i\} \times max\{T_i\} \geq x_1^2 + x_2^2 + \ldots + x_n^2$, area relationship between squares Q_i and the containing rectangle
- 又由平方平均和算术平均的关系：then from the inequality between square average and arithmetic average:

$$\sqrt{\frac{x_1^2+x_2^2+\ldots+x_n^2}{n}} \geq \frac{x_1+x_2+\ldots+x_n}{n}$$

AI 的证明 AI's proof

- 简单的推导一下：Some simple derivation:
- $max\{S_i\} \times max\{T_i\} \geq x_1^2 + x_2^2 + \ldots + x_n^2$
  $\geq \frac{(x_1+x_2+\ldots+x_n)^2}{n} = (\frac{x_1+x_2+\ldots+x_n}{k})^2$
- 于是: $max\{S_i\} \times max\{T_i\} \geq (\frac{x_1+x_2+\ldots+x_n}{k})^2$
- 所以 $max\{S_i\}, max\{T_i\}$ 中至少有一个不小于$\frac{x_1+x_2+\ldots+x_n}{k}$
  So at least one of $max\{S_i\}, max\{T_i\}$ is no less than $\frac{x_1+x_2+\ldots+x_n}{k}$

陶哲轩的示例 Example Illustrated by Terrence Tao

Source: 陶哲轩 Terrence Tao "The story of Erdős problem #1026"

This idea of using packing to prove Erdős-Szekeres type results goes back to a 1959 paper of Seidenberg, although it was a discrete rectangle-packing argument that was not phrased in such an elegantly geometric form. It is possible that Aristotle was "aware" of the Seidenberg argument via its training data, as it had incorporated a version of this argument in its proof.

Here is an illustration of the above argument using the AlphaEvolve-provided example

$$[99998, 99997, 116305, 117032, 116304,$$

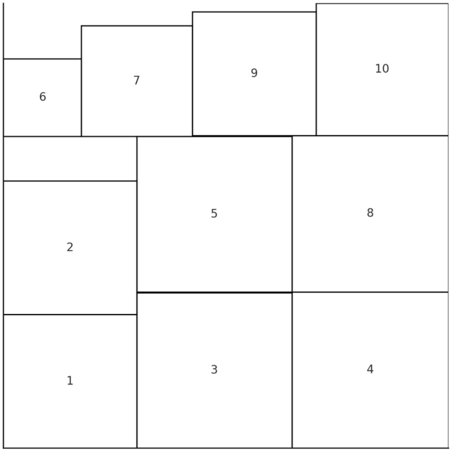
$$58370, 83179, 117030, 92705, 99080]$$

for $n = 10$ to convert it to a square packing (image produced by ChatGPT Pro):

Square packing for the n=10 Seidenberg construction

陶哲轩的图示 Square Packing Illustrated by Terrence Tao

Source: 陶哲轩 Terrence Tao "The story of Erdős problem #1026"



Lawrence Wu 诠释的证明 Proof Interpreted by Lawrence Wu

Source: 埃尔德什问题论坛 `https://www.erdosproblems.com/forum/thread/1026`

Aristotle's square-packing argument in a bit more detail:

We follow the approach of Seidenberg (1959) in proving Erdős-Szekeres. Let S_i be the maximal sum over all increasing subsequences ending in x_i, and T_i be the maximal sum over all decreasing subsequences ending in x_i. Now consider the squares $(S_i - x_i, T_i - x_i), (S_i, T_i)$. These are disjoint and contained in the rectangle $(0,0), (\max_i S_i, \max_i T_i)$. Hence, $(\max_i S_i)(\max_i T_i) \geq \sum_i x_i^2 \geq \frac{1}{k^2}$ as desired.

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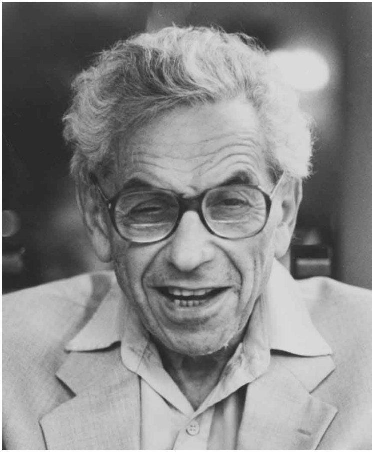
保罗·埃尔德什 Paul Erdős (1913-1996) Q & A

Source: Rutgers University (https://sites.math.rutgers.edu/~sg1108/People/Math/Erdos)



参考资料 References

- 埃尔德什问题论坛 erdosproblems forum：
  `https://www.erdosproblems.com/forum/thread/1026`
- 陶哲轩 Terrence Tao "The story of Erdős problem #1026":
  `https://terrytao.wordpress.com/2025/12/08/the-story-of-erdos-problem-126/`