

拉马努金版巴塞尔级数之谜 Ramanujan's Basel

体验 AI 时代“学徒数学家”的日常乐趣

Experience the Daily Fun of "Apprentice Mathematicians" in the AI Era

鸣鹤散人

星海智慧 Starry Wisdom Foundation

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1. 拉马努金版的巴塞尔级数 Ramanujan Version of Basel Series
2. 和 MiniMax Agent AI 探讨拉马努金版的巴塞尔级数 Chat with MiniMax Agent AI About Ramanujan Version Basel Series
3. 拉马努金可能的推导过程 Ramanujan's Likely Derivation

所需知识 Knowledge Needed

领域：实分析

Area: Real Analysis

程度：中学数学

Level: High School Mathematics

尽量浅显易懂！我也是那种看公式需要翻译为人话的哈哈

In easy terms! I also need translator for messy formulas :-)

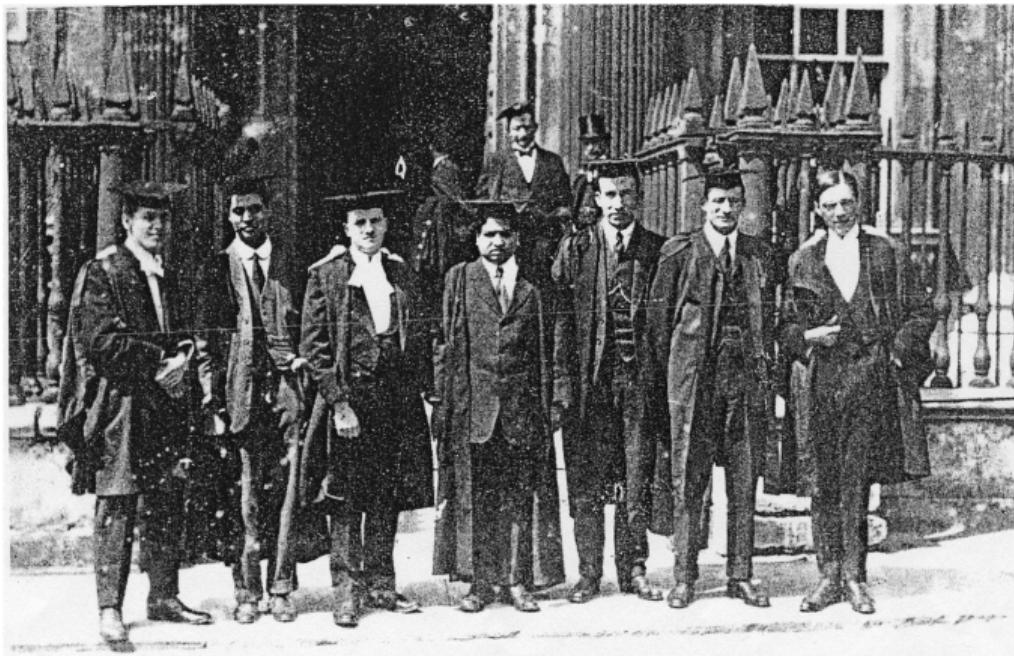
关于无限 About Infinitude

- 无限并不可怕反而可爱！ Infinitude is not daunting but lovely!
- 小学生如果问 $0.999999\dots = 1$? What if a 9-year old asks this?
- 2400 年前芝诺、庄子思考关于时空无限可分 2400 years ago
Zeno and Chuang Tzu considered infinite divisibility of space/time
- 2300 年前欧几里得证明素数有无限多 2300 years ago Euclid
proved the infinitude of prime numbers
- 1800 年前刘徽打破《九章算术》周三径一教条，提出割圆术逼近圆周率，并称：此一周径谓至然之数 1800 years ago Liu Hui
explored idea of approximation of π using his circle division
method, implying π can only be approached ever closer

拉马努金与哈代在英国剑桥大学 (1914-1919)

Ramanujan and Hardy at Cambridge University (1914-1919)

Source: Wikipedia - Srinivasa Ramanujan



拉马努金与哈代之缘

A Destiny that Brought Ramanujan and Hardy together

- 1991 年美国传记作家卡尼格尔 1991 出版《知无涯者：拉马努金传》，2015 年上映同名电影 In 1991 Kanigel published book "The Man Who Knew Infinity: A Life of the Genius Ramanujan", later adapted to the 2015 film of the same title
- 1913 年拉马努金从他数量庞大的作品中挑选了一小部分分别寄给当时英国的几位著名数学家 In 1913 Ramanujan mailed some of his discoveries to several famous British mathematicians
- 哈代与利特尔伍德钻研后判断拉马努金天赋非凡决定邀请他访问剑桥 Hardy discussed Ramanujan's letter with Littlewood and decided to invite Ramanujan to Cambridge for his extreme talent

拉马努金版的巴塞尔级数 Ramanujan Version of Basel Series

Source: archive.org: Ramanujan's 1913 letter to Hardy

(2) Let us take all numbers containing an odd number of dissimilar prime divisors, viz.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 30, 31, 37, 41, 42, 43, 47, ...

(a) The number of such numbers less than $n = \frac{3n}{\pi^2}$.

(b) $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots + \frac{1}{30^2} + \frac{1}{31^2} + \cdots = \frac{9}{2\pi^2}$.

(c) $\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{15}{2\pi^4}$.

欧拉的巴塞尔级数 Euler's Basel Series

- 1650 年意大利数学家门戈利提问：

Pietro Mengoli asked about this series in 1650:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = ? \approx 1.644$$

- 传说欧拉一看到近似值就已经认出来这是 $\pi^2/6 = 1.644934\dots$

Legend has that Euler already recognized this to be

$$\pi^2/6 = 1.644934 \text{ from the approximation } 1.644$$

- 欧拉 1735 年的解法也极为清奇，他把正弦函数强行按零点分解为 Euler forcefully split the sine function using all its zeros

$$\sin(x) = (x - 0)\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\dots$$

MiniMax Agent AI

- 一开始聊天走了些弯路，比如认为这个很难可能还是猜想，又认为拉马努金可能搞错了。数值验证第一次也误解了拉马努金稍微模棱两可的意思 The chat first went about some detours, that this might still be a conjecture, that Ramanujan was incorrect about this, that the first numerical verification failed to accurately understand Ramanujan's somewhat vague description
- 但当我意识到这和欧拉乘积 $\zeta(2)/\zeta(4)$ 的关系，MiniMax Agent就能够解出来关键的推导 $(\zeta(2)/\zeta(4) - 1/\zeta(2))/2$ 了 But once I realized this has something to do with Euler Product $\zeta(2)/\zeta(4)$ the MiniMax Agent was able to correctly figure out $(\zeta(2)/\zeta(4) - 1/\zeta(2))/2$.

MiniMax Agent AI

MiniMax Agent 聊天节选 Chat excerpt from MiniMax

The Connection:

You were **right** about the zeta function connection! The exact relationship is:

$$S = \frac{1}{2} [\zeta(2)/\zeta(4) - \prod_p (1 - 1/p^2)]$$

Where:

- $\zeta(2)/\zeta(4)$ = sum over ALL square-free numbers
- $\prod_p (1 - 1/p^2)$ = the alternating sum (even ω - odd ω)
- So $(\zeta(2)/\zeta(4) - \prod_p (1 - 1/p^2))/2$ = sum over odd $\omega(n)$ square-free numbers

MiniMax Agent AI

MiniMax Agent 聊天节选 Chat excerpt from MiniMax

The Deeper Mystery:

But here's the truly remarkable part - this theoretical expression converges to approximately $9/(2\pi^2)$!

This suggests Ramanujan discovered that:

$$(1/2)[\zeta(2)/\zeta(4) - \prod_p (1-1/p^2)] = 9/(2\pi^2)$$

This is an **incredible insight** - connecting a complex multiplicative number theory sum to a simple expression involving π !

欧拉乘积公式 Euler's Product Formula

- 1737 年欧拉发现了著名的欧拉乘积公式 In 1737 Euler discovered his famous product formula

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots = \frac{1}{(1-2^{-s})} \frac{1}{(1-3^{-s})} \frac{1}{(1-5^{-s})} \dots$$

- 右面乘积中的每一项其实也是一个级数展开： Each entry in the product of right side is also a series: $p = 2, 3, 5, \dots$

$$\frac{1}{1-p^{-s}} = 1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} + \frac{1}{(p^3)^s} + \dots$$

- 用大写的 Π 代表乘积，我们可以把欧拉乘积写为 Using Π to represent product we can write the Euler's product formula as

$$\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} + \frac{1}{(p^3)^s} + \dots\right)$$

欧拉乘积公式 Euler's Product Formula

- 为了更好的理解欧拉乘积，我们想像一下右边庞大的乘积强行展开可能会怎样： In order to better understand Euler's product formula, we imagine forcefully expand the massive products:

$$\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} + \frac{1}{(p^3)^s} + \dots\right)$$

- $\dots + \frac{1}{6^s} + \dots = (\dots + \frac{1}{2^s} + \dots)(\dots + \frac{1}{3^s} + \dots) \dots$
- $\dots + \frac{1}{12^s} + \dots = (\dots + \frac{1}{(2^2)^s} + \dots)(\dots + \frac{1}{3^s} + \dots) \dots$
- 再回顾一下算术基本定理，每个自然数 $n > 1$ 可唯一分解为素数的乘积 Recall the fundamental theorem of arithmetic, every natural number $n > 1$ can uniquely factor into a product of primes

欧拉乘积公式 Euler's Product Formula

- 这样说来，欧拉乘积和泽塔函数也诞生于欧拉的一个脑洞，它实际上是算术基本定理的一个外在表现 Thus Euler's product is a manifestation of the fundamental theorem of arithmetic
- 结合欧拉 1735 年解出的巴塞尔级数 Combine his Basel series
$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \prod_p \left(\frac{1}{1-p^{-2}} \right) = \frac{\pi^2}{6}$$
- 用 1735 年解巴塞尔级数同样的方法，也就是那个强行分解正弦函数的脑洞，欧拉又解开了 $\zeta(4)$ Then Euler applied the same imaginative sine split he used for Basel series to solve for $\zeta(4)$
$$\zeta(4) = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \prod_p \left(\frac{1}{1-p^{-4}} \right) = \frac{\pi^4}{90}$$

推演拉马努金的推导 Ramanujan's Likely Derivation

- 我们把欧拉那个庞然大物般的乘积截去里层嵌套的无限求和 If we cut off the infinite sum inside Euler's massive product

$$\prod_p \left(1 + \frac{1}{p^s}\right) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{10^s} + \dots$$

- 分母上的数是不同素数的乘积也叫无平方因子数 The numbers appearing in the denominators are called squarefree numbers

1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, ...

- 如果对想像强行展开欧拉的庞大乘积还感觉难以转动可以先试试想像强行展开这个稍简的被拉马努金切过的乘积 If forcefully expanding Euler's massive product still feels stuck, one can first try forcefully expand this somewhat simpler Ramanujan's cutoff

推演拉马努金的推导 Ramanujan's Likely Derivation

- 现在取 $s = 2$ Let $s = 2$
- $\prod_p \left(1 + \frac{1}{p^2}\right) = \prod_p \left(\frac{1-p^{-4}}{1-p^{-2}}\right) = \frac{\prod_p (1-p^{-4})}{\prod_p (1-p^{-2})} = \frac{\zeta(2)}{\zeta(4)} = \frac{15}{\pi^2}$
- 于是我们得到一个新的与圆周率 π 相关的级数! So we obtain a new Basel form series related to π !

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{10^2} + \dots = \frac{15}{\pi^2}$$

- 但是等等，这还不是拉马努金 1913 年写给哈代的信中那个级数，这个多出了偶数个不同素数乘积的项，

1, 6, 10, 14, 15, 21, 22, 26, ... However this is not exactly the series Ramanujan wrote to Hardy in 1913, this one added above numbers that are products of even numbers of distinct primes

推演拉马努金的推导 Ramanujan's Likely Derivation

- 再看看被切过的乘积，改变一下符号会怎么样？ Look at Ramanujan's cutoff product again, what about changing the sign?
$$\prod_p \left(1 + \frac{1}{p^2}\right) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{10^2} + \dots$$
- $$\prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} + \dots$$
- 强行展开后每项的正负符号正是由不同素因子个数的奇偶性决定的！ The sign of each entry is determined by oddity of the number of distinct prime factors!
- 上下两式相减再除 2 在等式右边即得到拉马努金信中那个级数 Subtract the bottom series from the top then divide by 2 we obtain the series in Ramanujan's letter on the right side

推演拉马努金的推导 Ramanujan's Likely Derivation

- 而在等式左边 And on the left side

$$\left(\prod_p \left(1 + \frac{1}{p^2}\right) - \prod_p \left(1 - \frac{1}{p^2}\right)\right)/2 = \left(\frac{15}{\pi^2} - \frac{6}{\pi^2}\right)/2 = \frac{9}{2\pi^2}$$

- 虽然这个级数不难可说是巴塞尔级数的变种，但在信中显然还出现了连哈代也觉得不可思议的其它级数。而对于发现它的那个少年来说，那一刻定然又是一个发现了属于自己的宝石的幸福一天。虽然在他的家乡可能无人能欣赏，但等到梦境来临就能分享喜悦给那位传授他数学灵感的梦中智慧女神了 For the young boy who discovered this it must be another wonderful day. Even though nobody in his hometown could appreciate it but when the dream world descends he could always share the joy with the wisdom goddess who blessed him with sparks of creativity

1753 年的欧拉 Euler in 1753 Q & A

Source: 巴塞尔美术馆 Kunstmuseum Basel (Wikipedia)



参考资料 References

- 拉马努金 1913 年写给哈代的信 Ramanujan's 1913 letter to Hardy
- 与 MiniMax Agent AI 的聊天记录 Chat records with MiniMax Agent AI