AI 数学的突破进展 Breakthrough in AI Math

"亚里士多德" AI 独立解决部分埃尔德什 124 问题
Aristotle AI Independently Solved Part of Erdős Problem 124

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纲要 Overview

1. **突破 The Breakthrough**

2. **背景知识 Background Knowledge**

3. **埃尔德什 124 号猜想第一部分 Erdős Conjecture 124 First Part**

所需知识 Knowledge Needed

领域：初等数论
Area：Elementary Number Theory

等级：中学数学
Level：High School Mathematics

尽量浅显易懂！我也是那种看公式需要翻译为人话的哈哈
In easy terms! I also need translator for messy formulas :-)

突破 The Breakthrough

- 11 月 29 日，"亚里士多德" AI 独立解决部分埃尔德什 124 问题 On November 29, researcher from Harmonic announced that AI 'aristotle' independently solved the first part of Erdős problem 124

- 其证明已为著名数学家在 erdosproblems.com 论坛讨论所确认 The proof has been confirmed through the discussions at the erdosproblems.com forum by well known mathematicians

- 虽然看上去像是 AI 摘到了冷门的低垂果实，但仍然标志着科学研究即将进入新的 AI 合作时代 Although it looks like AI has grabbed a low hanging fruit that was neglected for decades, still it marks the new age of scientific research with AI collaboration

突破 The Breakthrough

- 该猜想由已故著名数学家保罗·埃尔德什 (1913-1996) 等数学家于 1996 年提出 The conjecture was proposed by Paul Erdős(1913-1996) et al in 1996
- 该证明属于初等且简洁容易理解 The proof is elementary, short and easy to understand
- 在这里我们做一详细介绍供大家讨论 Here we give a more detailed introduction for public discussion
- 时间仓促有不足之处欢迎批评指正 Time was limited in my study, any correction and suggestion are very welcome

加法数论 Additive Number Theory

- 124 号猜想是一个关于从自然数的若干子集中挑选数字相加以表达任意自然数的问题 The conjecture 124 is about picking number from a list of subsets of the natural numbers to add up, in order to represent any natural number
- 这类问题的描述往往都相当初等易于理解 Typically this type of problems are elementarily presented and easy to understand
- 以下举个最著名的这类猜想 Let's list one of the most famous conjectures of this type

哥德巴赫猜想 Goldbach's Conjecture

- 哥德巴赫猜想是在 1742 年由哥德巴赫在与欧拉的通信中首次提出 The Goldbach's conjecture was first mentioned in 1742 in a letter written to Euler by Goldbach

- 基本表述为任意不小于 6 的偶数可以表达为两个素数 (质数) 之和 The basic form is that any even number greater than or equal to 6 can be represented by the sum of two primes

- 上世纪五十至七十年代中国数学家陈景润等对此猜想作出过突破性贡献但并未完全解决 Chinese mathematician such as Chen Jingrun made significant progress on the conjecture from 1950's through 1970's but still couldn't completely solve it

哥德巴赫猜想 Goldbach's Conjecture

- 一个等价变化版的哥德巴赫猜想可表述为，大于 1 的任意自然数可表达为不超过三个素数之和 An equivalent variation form of Goldbach's conjecture states that any natural number greater than 1 can be expressed as the sum of no more than three primes

- 如果用 $\mathcal{P}$ 代表所有素数的集合 $\{2, 3, 5, 7, 11, 13, 17, 19, ...\}$，可以把变化版的哥德巴赫猜想标记为 $\mathcal{P} + \mathcal{P} + \mathcal{P} \approx \mathcal{N}$

- 这里指从每个集合里选不超过一个数字然后加起来，也就是说选的时候可以跳过某些集合 Here we select no more than one number from each of the sets, meaning you can choose to skip some of the sets

埃尔德什 124 号猜想第一部分 Erdős Conjecture 124 First Part

- 我们以 $P(d, 0)$ 代表 d 的不同幂的和的集合 We let $P(d, 0)$ represent the set of numbers that can be represented as sum of distinct power of d

- 比如，$d = 2$ 的不同幂相加就是计算机行业所广泛运用的二进制表达，所以 $P(2, 0) = \mathcal{N}$. Sum of distinct powers of $d = 2$ is the binary representation widely used in computer industry, so $P(2, 0) = \mathcal{N}$.

- 再比如，$d = 3$ 的不同幂相加就得到集合 Sum of distinct power of 3 form this set: $\{1, 3, 4, 9, 10, 12, 13, 27, ...\}$

埃尔德什 124 号猜想第一部分 Erdős Conjecture 124 First Part

- 现在我们随便找些这样的集合，问题是他们加起来是否就可以覆盖了自然数？Now let's list a few such sets, the question is can they add up to cover the natural numbers?
- 写的稍微准确点就是说 Written slightly more precise:
  $P(d_1, 0) + P(d_2, 0) + ... + P(d_r, 0) \approx \mathcal{N}$?
- 猜想是说是否 $d_1, d_2, ..., d_r$ 只要满足以下条件就成立了 The conjecture ask whether if the following conditions are met then the conjecture is true:
  $3 \leq d_1 < d_2 < ... < d_r$ and $\frac{1}{d_1-1} + \frac{1}{d_2-1} + ... + \frac{1}{d_r-1} \geq 1$.

- 试了一下 $\{d_1, d_2, ..., d_r\} = \{3, 4, 5\}$ 满足以上条件 Tried $\{d_1, d_2, ..., d_r\} = \{3, 4, 5\}$ satisfy the above mentioned conditions

- $P(3, 0) = \{1, 3, 4, 9, 10, 12, 13, 27, ...\}$

- $P(4, 0) = \{1, 4, 5, 16, 17, 20, 21, ...\}$

- $P(5, 0) = \{1, 5, 6, 25, ...\}$

- 试试看是否不超过 25 的数都可以表达为某种选择的和，注意最多每个集合选一个，可以跳过不选 Try to see if numbers less than 25 can all be represented by sum of some selection, remember at most one number from each set, can skip sets

AI 的证明 AI's proof

- 首先我们把不同幂的集合排序合并在一起成为序列并允许重复
  First let's sort sets of powers and then merge them into a
  sequence that allows for repetition
- 仍以 $\{d_1, d_2, ..., d_r\} = \{3, 4, 5\}$ 为例, 生成幂序列:
  $\{1, 1, 1, 3, 4, 5, 9, 16, 25, 27, ...\}$
- 根据前面的不等式条件这个序列满足一个性质即每一项都不超
  过前面所有项之和加一 From the inequality condition this
  sequence satisfy a property that any element is not more than the
  sum of all previous element plus 1

AI 的证明 AI's proof

- 这里用到中学基础多项式求和 $1 + d + d^2 + ...d^k = \frac{d^{k+1}-1}{d-1}$ Here we need to use the polynomial sum from high school
- 前面所有项之和可反拆成单个幂级数之和，而序列中的下一项正是所有单个幂级数下一项中最小的一个 The sum of all previous numbers in the sequence can be split back into the sum of individual series of powers of d_i, and the next number in the sequence is the minimum among all the next number in the individual series

AI 的证明 AI's proof

- 得到这个序列的这一重要属性，此等序列称为完全序列，根据已知定理任意自然数可表达为其一子序列的和（数学归纳法）
  This property of the sequence is known as the complete sequence and a known elementary theorem tells us that any natural number can be represented as the sum of a subsequence
- 再把该子序列拆分回单个幂级数即得到所需的求和表达 Now split such subsequence back into the sum of individual series of powers we derive the required representation

tsaf 诠释的证明 Proof Summarized by tsaf

Aristotle's solution is as follows. It is surprisingly easy.

Let (a_n) be the sequence of powers of d_i (sorted, with multiplicity). For example, if $d_1 = 2$ and $d_2 = 3$, then the sequences is: $1, 1, 2, 3, 4, 8, 9, 16, 27, \ldots$.

We want to show that every positive integer is a subsequence sum. This is equivalent to $a_{n+1} - 1 \le (a_1 + \cdots + a_n)$. The RHS is $\sum_{i=1}^{k} (d_i^{e_{i,n}} - 1)/(d_i - 1)$, where $e_{i,n}$ is the first power of d_i that has not ocurred in the first n terms. This is bounded below by $\min_i (d_i^{e_{i,n}} - 1)$. However, $a_{n+1} = \min_i d_i^{e_{i,n}}$. Done.
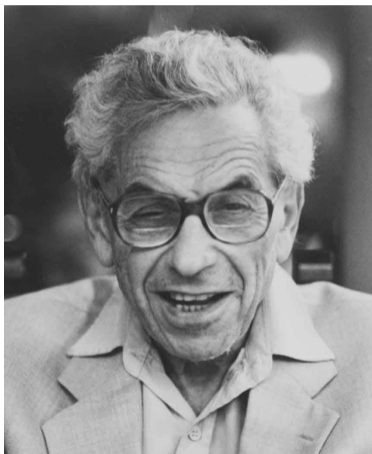
Note, there is some ambiguity in the definition of $e_{i,n}$. In the example $d_1 = 2, d_2 = 3$, we can decide arbitrarily that a_1 is a power of 2 and a_2 is a power of 3, so $e_{2,1} = 0$ but $e_{2,2} = 1$.

保罗・埃尔德什 Paul Erdős (1913-1996) Q & A

Source: Rutgers University (https://sites.math.rutgers.edu/~sg1108/People/Math/Erdos)

参考资料 References

- 埃尔德什问题论坛 erdosproblems forum：
  https://www.erdosproblems.com/forum/thread/124