

Chapter 4: Credit Scoring — Probabilistic Decision Systems and Risk Ranking

1 Credit Scoring as a Decision Problem

Credit scoring is the process of estimating the likelihood that a borrower will fail to meet contractual repayment obligations. Unlike lending contracts, which define cash flows, credit scoring defines *decisions* under uncertainty. The output of a credit scoring system is not truth, but a probabilistic assessment used to rank borrowers by relative risk.

Formally, credit scoring maps observed borrower attributes to a probability of default. This probability is then compared against policy thresholds to determine approval, pricing, and exposure limits.

2 Borrower State and Feature Representation

Let each borrower be represented by a feature vector:

$$\mathbf{x} = (x_1, x_2, \dots, x_d)$$

where features may include financial history, transactional behavior, stability indicators, and external signals. These features are treated as realizations of random variables drawn from an underlying population distribution.

The objective is to estimate the conditional probability:

$$P(D = 1 | \mathbf{x})$$

where D is a binary random variable indicating default.

3 Default as a Random Variable

Define the default indicator:

$$D = \begin{cases} 1, & \text{if default occurs} \\ 0, & \text{otherwise} \end{cases}$$

The unconditional probability of default is:

$$PD = P(D = 1)$$

while the conditional probability given borrower characteristics is:

$$PD(\mathbf{x}) = P(D = 1 | \mathbf{x})$$

Credit scoring seeks to approximate this conditional probability.

4 Logistic Regression as a Credit Model

A common parametric approach models default probability using logistic regression:

$$PD(\mathbf{x}) = \frac{1}{1 + e^{-z}}, \quad z = \beta_0 + \sum_{i=1}^d \beta_i x_i$$

This formulation ensures that predicted probabilities lie in the interval $(0, 1)$. The coefficients β_i quantify the marginal effect of each feature on default risk in log-odds space.

5 Odds, Log-Odds, and Score Transformation

The odds of default are defined as:

$$\text{Odds} = \frac{PD}{1 - PD}$$

Taking the logarithm yields the log-odds:

$$\log \left(\frac{PD}{1 - PD} \right) = z$$

Credit scores are often constructed as linear transformations of log-odds:

$$Score = A - B \cdot \log\left(\frac{PD}{1 - PD}\right)$$

where A and B are scaling constants chosen to meet operational constraints.

6 Ranking Versus Prediction

Credit scoring systems are evaluated primarily on their ability to *rank* borrowers, not to predict exact default probabilities.

Given two borrowers i and j , a scoring system is useful if:

$$PD_i > PD_j \Rightarrow \text{Borrower } i \text{ is riskier than } j$$

Absolute calibration errors are often less damaging than ranking errors.

7 Decision Thresholds and Cutoffs

Decisions are made by comparing predicted default probabilities to a threshold τ :

$$\text{Approve if } PD(\mathbf{x}) < \tau$$

Changing τ directly controls approval rate and portfolio risk. Small changes in τ can produce large changes in aggregate outcomes.

8 Expected Loss Integration

Credit scoring outputs are integrated with exposure and recovery assumptions via the expected loss framework:

$$EL(\mathbf{x}) = PD(\mathbf{x}) \times LGD \times EAD$$

Decision rules are often expressed as constraints on expected loss rather than on probability alone.

9 Confusion Matrix and Classification Errors

Predictions generate four possible outcomes:

- True Positive (TP): Default correctly predicted
- False Positive (FP): Non-default incorrectly rejected
- True Negative (TN): Non-default correctly approved
- False Negative (FN): Default incorrectly approved

These outcomes define the error structure of the system.

10 Loss Functions and Threshold Optimization

Let:

- C_{fp} = cost of rejecting a good borrower
- C_{fn} = cost of approving a bad borrower

The expected decision loss is:

$$L(\tau) = FP(\tau) \cdot C_{fp} + FN(\tau) \cdot C_{fn}$$

The optimal threshold satisfies:

$$\tau^* = \arg \min_{\tau} L(\tau)$$

11 Model Performance Metrics

11.1 Receiver Operating Characteristic

The ROC curve plots true positive rate against false positive rate across thresholds.

The Area Under the Curve (AUC) measures ranking quality:

$$0.5 \leq AUC \leq 1$$

11.2 Gini Coefficient

The Gini coefficient is derived from AUC:

$$G = 2 \cdot AUC - 1$$

Higher values indicate stronger discriminatory power.

11.3 Kolmogorov–Smirnov Statistic

The KS statistic measures the maximum separation between cumulative distributions:

$$KS = \max_x |CDF_{good}(x) - CDF_{bad}(x)|$$

12 Population Stability and Concept Drift

Credit models assume stable data distributions. Let $P_t(\mathbf{x})$ denote the feature distribution at time t . Population drift occurs when:

$$P_t(\mathbf{x}) \neq P_{t+k}(\mathbf{x})$$

Drift degrades model validity even if in-sample performance remains strong.

13 Reject Inference Problem

Observed default labels exist only for approved borrowers. Let Y denote default outcomes. Then:

$$P(Y \mid \mathbf{x}, \text{Approved}) \neq P(Y \mid \mathbf{x})$$

This selection bias complicates model estimation and calibration.

14 Score Calibration

Calibration aligns predicted probabilities with observed default frequencies.

For a score band S :

$$PD_{observed}(S) \approx PD_{predicted}(S)$$

Poor calibration leads to systematic under- or overestimation of risk.

15 Summary

Credit scoring systems transform borrower information into probabilistic risk rankings. Their mathematical foundation rests on conditional probability estimation, loss minimization, and stability analysis. Effectiveness depends more on ranking accuracy and decision integration than on precise probability prediction.