# Chapter 3: Investing — Portfolio Theory, Risk, Compounding, and Market Uncertainty

## 1 Investing as a Stochastic Wealth Process

Investing is the allocation of capital across uncertain assets with the objective of increasing purchasing power over time. Unlike payments or lending, investing does not involve contractual cash flows. Instead, outcomes emerge from market dynamics, making investing inherently probabilistic and path-dependent.

Let $W_t$ denote wealth at time $t$. Wealth evolves according to the stochastic process:

$$W_{t+1} = W_t(1 + R_{t+1})$$

where $R_{t+1}$ is a random variable representing the return during period $t+1$. The sequence $\{R_t\}$ is generally neither independent nor identically distributed.

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## 2 Return Definitions and Measurement

For a single asset, the simple return over one period is defined as:

$$R_t = \frac{P_t - P_{t-1} + D_t}{P_{t-1}}$$

where:

- $P_t$ is the asset price at time $t$
- $D_t$ is the cash distribution during the period

For analytical convenience, log-returns are often used:

$$r_t = \ln(1 + R_t)$$

Log-returns are additive over time:

$$\sum_{t=1}^{n} r_t = \ln\left(\frac{P_n}{P_0}\right)$$

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## 3 Expected Return and Probability Distributions

Let $R$ be a discrete random variable with outcomes $\{R_1, R_2, \ldots, R_n\}$ and probabilities $\{p_1, p_2, \ldots, p_n\}$. The expected return is:

$$E[R] = \sum_{i=1}^{n} p_i R_i$$

Expected return represents the center of the return distribution, not a guaranteed outcome. Two assets may have identical expected returns while exhibiting vastly different risk characteristics.

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## 4 Risk as Dispersion of Returns

Risk is quantified as the dispersion of returns around the expected value.

The variance of returns is defined as:

$$\sigma^2 = E[(R - E[R])^2]$$

and the standard deviation $\sigma$ serves as the primary measure of volatility.

Higher variance implies greater uncertainty in future wealth trajectories.

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## 5 Compounding and Exponential Growth

Compounding is the dominant force in long-term investing. For an initial investment $P$ growing at a constant rate $r$ over $n$ periods:

$$FV = P(1 + r)^n$$

Compounding introduces exponential growth, making early returns disproportionately important relative to later returns.

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# 6 Sequence Risk and Path Dependency

Wealth outcomes depend on the sequence of returns, not merely their average.

Given returns $\{R_1, R_2, \ldots, R_n\}$:

$$W_n = W_0 \prod_{t=1}^{n} (1 + R_t)$$

Negative returns early in the sequence reduce the base on which future gains compound, a phenomenon known as sequence risk.

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# 7 Portfolio Construction

Consider a portfolio of $N$ assets with weights $w_i$ satisfying:

$$\sum_{i=1}^{N} w_i = 1$$

The portfolio return is:

$$R_p = \sum_{i=1}^{N} w_i R_i$$

The expected portfolio return is therefore:

$$E[R_p] = \sum_{i=1}^{N} w_i E[R_i]$$

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# 8 Portfolio Risk and Covariance Structure

Portfolio variance incorporates asset correlations:

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i \neq j} w_i w_j \sigma_{ij}$$

where covariance is defined as:

$$\sigma_{ij} = E[(R_i - E[R_i])(R_j - E[R_j])]$$

If asset correlations $\rho_{ij} < 1$, diversification reduces total portfolio risk.

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## 9 Diversification Limits

As the number of assets increases, idiosyncratic risk diminishes, but systematic risk remains:

$$\lim_{N \to \infty} \sigma_p^2 = \sigma_{systematic}^2$$

Diversification cannot eliminate market-wide shocks.

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## 10 Mean–Variance Optimization

The optimal portfolio minimizes variance for a target expected return $R^*$:

$$\min_{w} \ \sigma_p^2 \quad \text{subject to} \quad E[R_p] = R^*, \quad \sum w_i = 1$$

This optimization defines the efficient frontier.

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## 11 Risk-Free Asset and Capital Market Line

Introducing a risk-free asset with return $R_f$ modifies the feasible set of portfolios. Any combination of the risk-free asset and a risky portfolio lies on a straight line.

The optimal risky portfolio maximizes the Sharpe ratio:

$$S = \frac{E[R_p] - R_f}{\sigma_p}$$

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## 12 Factor Models and Systematic Risk

Asset returns can be decomposed using factor models:

$$R_i = \alpha_i + \beta_i R_m + \epsilon_i$$

where:

- $R_m$ is the market return
- $\beta_i$ measures market sensitivity
- $\epsilon_i$ is idiosyncratic noise

Only systematic risk commands a risk premium.

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## 13 Periodic Investment and Annuity Mathematics

For periodic investments of amount $P$ each period:

$$FV = P \cdot \frac{(1+r)^n - 1}{r} \cdot (1+r)$$

This formulation explains the effectiveness of disciplined periodic investing under volatile markets.

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## 14 Drawdowns and Recovery Asymmetry

Drawdown is defined as:

$$D = \frac{W_{peak} - W_{trough}}{W_{peak}}$$

A drawdown of $x\%$ requires a recovery gain of:

$$\frac{x}{1-x}$$

which grows non-linearly as $x$ increases.

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## 15 Inflation and Real Wealth Preservation

Nominal returns must be adjusted for inflation $\pi$:

$$1 + R_{real} = \frac{1 + R_{nominal}}{1 + \pi}$$

Sustained real returns are required to preserve purchasing power.

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## 16 Summary

Investing is a stochastic, path-dependent process governed by compounding, risk dispersion, and correlation structure. Mathematical frameworks such as variance analysis, portfolio optimization, and factor decomposition provide formal tools for understanding wealth evolution under uncertainty.