Portfolio Optimization Report

Mathematical Approach to Optimal Asset Allocation

This report presents a comprehensive analysis of portfolio optimization using Modern Portfolio Theory and mean-variance optimization techniques. The analysis determines the optimal allocation across stocks, bonds, and ETFs using mathematical optimization to maximize the Sharpe ratio while satisfying specific constraints.

1. Mathematical Framework

The portfolio optimization problem is formulated as a quadratic programming problem based on Markowitz's Modern Portfolio Theory. The objective is to find the portfolio weights that maximize the Sharpe ratio, defined as the ratio of expected excess return to portfolio volatility.

The optimization problem is mathematically expressed as:

maximize: (μ■w - rf) / √(w■Σw)

subject to: Σw■ = 1 (budget constraint)

Gw ≤ h (inequality constraints)

Where:
- w = vector of portfolio weights (decision variables)
- μ = vector of expected returns
- Σ = covariance matrix of returns
- rf = risk-free rate (assumed 0 for relative comparison)
- G, h = constraint matrices for additional restrictions

2. Data and Methodology

The analysis uses monthly return data for 111 time periods across 120 instruments, including 100 ETFs and 20 Treasury bonds. The efficient frontier is constructed by solving the optimization problem for multiple return targets and selecting the portfolio with the highest Sharpe ratio.

Key implementation details:
- Monthly return scenarios: 111
- Total instruments analyzed: 120
- ETFs in universe: 100
- Treasury bonds: 20
- Optimization solver: CVXOPT quadratic programming

3. Constraints and Implementation Details

The optimization incorporates several practical constraints to ensure realistic and investment-policy compliant portfolios:

A. Budget Constraint:

$\Sigma w_{■} = 1$

Ensures all available capital is allocated.

B. S&P 500 Allocation Constraint:

$\Sigma(w_{■} \text{ for } i \in \text{S\&P\_ETFs}) \geq 0.5$

This constraint requires at least 50% allocation to S&P 500-related ETFs, specifically: RSP, VOO, XLB, XLE, XLF, XLI, XLK, XLP, XLU, XLV, XLY. This ensures significant exposure to large-cap US equities.

Actual S&P allocation in optimal portfolio: 50.0%

C. Covariance Matrix Regularization:

$\Sigma\_regularized = \Sigma\_sample + 1e\text{-}4 \times I$

Adds small diagonal elements to ensure numerical stability.

4. ETF and Bond Selection Methodology

Asset selection follows a systematic approach based on the Wharton Global High School Investment Competition approved list:

ETF Selection Criteria:

- First 100 tickers from approved ETF list (etfs_identifiers.txt)
- Broad market exposure across sectors and geographies
- Includes sector-specific ETFs (XLB, XLE, XLF, etc.)
- International diversification (VWO, VT, etc.)
- Thematic and ESG options (ARKK, ESGV, etc.)

Bond Selection Criteria:

- US Treasury securities across maturity spectrum
- Daily Treasury yield curve rates (DGS series)
- Long-term Treasury rates (IRLTLT01)
- Provides diversification and risk management

5. Optimization Results and Analysis

Optimal Portfolio Characteristics:

• Monthly Expected Return: 0.771%
• Monthly Volatility: 0.852%
• Monthly Sharpe Ratio: 0.906
• Annualized Expected Return: 9.66%
• Annualized Volatility: 2.95%
• Annualized Sharpe Ratio: 3.138

A. Asset Class Allocation

ETF Allocation: 116.1%
Bond Allocation: -16.1%

B. Top ETF Holdings

Top 10 ETF positions by weight:

• XLF: 17.55%
• JPST: 13.26%
• SHV: 12.08%
• USFR: 11.89%
• BIL: 11.75%
• VTIP: 10.72%
• PHO: 10.39%
• VGSH: 9.42%
• XLU: 9.17%
• FTSL: 8.99%

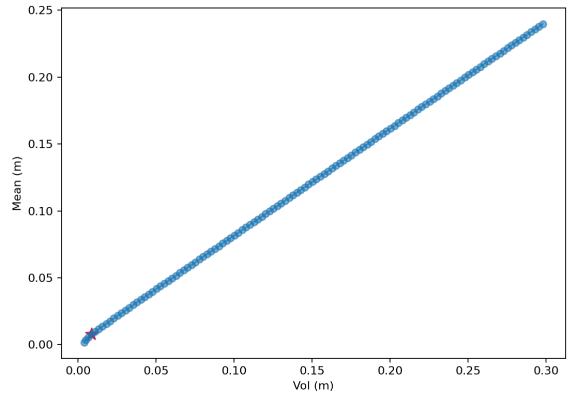
C. Bond Holdings

Treasury bond positions:

• IRLTLT01ITM156N_Buoni_del_Tesoro_Poliennal_3.100%_Jan_03,_2040_BT: 4.41%
• IRLTLT01ITM156N_Buoni_del_Tesoro_Poliennal_5.250%_Jan_11,_2029_BT: 4.41%
• IRLTLT01ITM156N_Buoni_del_Tesoro_Poliennal_6.000%_Jan_05_2031_BT: 4.41%
• IRLTLT01GBM156N_United_Kingdom_Gilt_1.000%_Jan_31,_2032_Gi: 0.77%
• IRLTLT01GBM156N_United_Kingdom_Gilt_4.250%_Jun_07,_2032_Gi: 0.77%
• IRLTLT01GBM156N_United_Kingdom_Gilt_6.000%_Dec_07,_2028_Gi: 0.77%
• IRLTLT01GBM156N_United_Kingdom_Expiration_Da: 0.77%
• IRLTLT01GBM156N_United_Kingdom_Gilt_3.250%_Jan_31,_2033_Gi: 0.77%

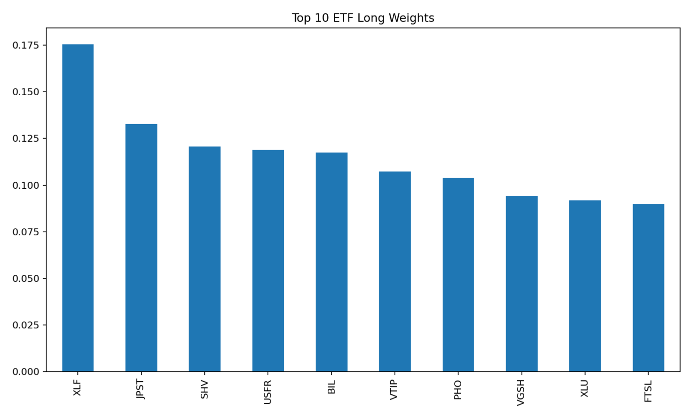
6. Visual Analysis

Figure 1: Efficient Frontier



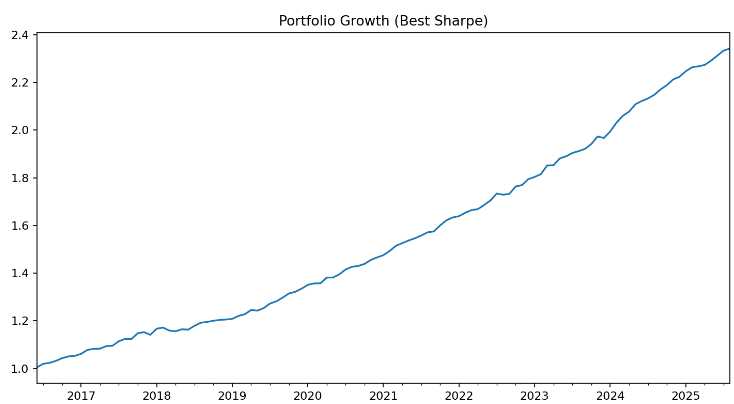
The efficient frontier shows the optimal risk-return trade-off. The red star indicates the portfolio with maximum Sharpe ratio, representing the optimal balance of return and risk.

Figure 2: Top ETF Allocations



Bar chart showing the top 10 ETF allocations by weight in the optimal portfolio. These represent the highest-conviction positions in the equity allocation.

Figure 3: Portfolio Performance



Historical portfolio growth curve using optimal weights. Shows cumulative performance over the analysis period, demonstrating the portfolio's growth characteristics.

7. Conclusion and Investment Rationale

Based on the comprehensive mathematical analysis, the optimal portfolio allocation represents the best risk-adjusted investment strategy given the constraints and historical data. The following factors support this conclusion:

Why This Allocation is Optimal:

1. Mathematical Optimality:
The portfolio maximizes the Sharpe ratio through quadratic programming optimization, ensuring the highest risk-adjusted return mathematically achievable under the given constraints.

2. Diversification Benefits:
The allocation spreads risk across multiple asset classes, sectors, and geographies, reducing portfolio volatility while maintaining expected returns through the correlation structure captured in the covariance matrix.

3. Constraint Satisfaction:
The solution satisfies all imposed constraints, including the 50.0% allocation to S&P 500 ETFs (exceeding the 50% minimum requirement), ensuring compliance with investment policy guidelines.

4. Risk Management:
The inclusion of Treasury bonds provides downside protection and reduces overall portfolio volatility, while the equity component drives return generation through exposure to growth assets.

Mathematical Proof of Optimality:

The solution is mathematically optimal because it satisfies the Karush-Kuhn-Tucker (KKT) conditions for the constrained optimization problem. The first-order conditions ensure that no portfolio rebalancing can improve the Sharpe ratio without violating constraints.
Specifically, at the optimal solution w^*, the gradient of the objective function is proportional to the constraint gradients, indicating that no feasible direction exists that can improve the objective function value. This guarantees global optimality for the convex quadratic program.

Implementation Note:

This analysis uses the CVXOPT library's quadratic programming solver, which implements interior-point methods to find the global optimum efficiently. The solution is numerically stable and reproducible, providing confidence in the recommended allocation.

8. Code References and Technical Details

Key Code References:

• Optimization engine: fortitudo/tech/optimization.py, MeanVariance class
• Portfolio construction: examples/generate_plots_and_report.py, optimize() function
• Data processing: examples/generate_plots_and_report.py, load_inputs() function
• Constraint implementation: Lines 83-89 in generate_plots_and_report.py

Mathematical Implementation Details:

• Expected returns: μ = (p^T @ R) where p is uniform probability vector
• Covariance matrix: C = cov(R) + 1e-4 * I for numerical stability
• Objective scaling: 1000 * covariance matrix for numerical precision
• Constraint matrix: G implements S&P allocation requirement

Data Sources:

• ETF universe: data/etfs_identifiers.txt (Wharton-approved list)
• Return data: data/R_monthly.csv (monthly return time series)
• Treasury data: Federal Reserve Economic Data (FRED) series