0.1 The two-photon vertex of NRQED

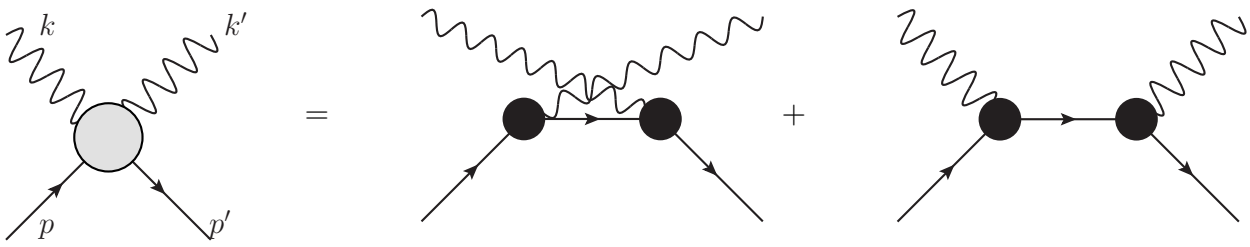
In the NRQED Lagrangian, in addition to the terms involving the fermions interaction with a single photon, there are terms which represent the interaction of a fermion with two photons. At the order needed, all such terms are fixed by gauge invariance. There are terms, such as those involving $\mathbf{E}^2$, that would are by themselves gauge invariant, but these occur at too high an order. (The order of such a term would be $E^2/m^3 \sim mv^6$.)

So though the coefficients of concern are all fixed by considering just the one-photon interactions, they could also be fixed from considering two-photon interactions. Since it *is* possible, it makes sense to do so, as a check of consistency. In this section, the coefficients of two-photon terms in the NRQED Lagrangian will be fixed from QED calculations.

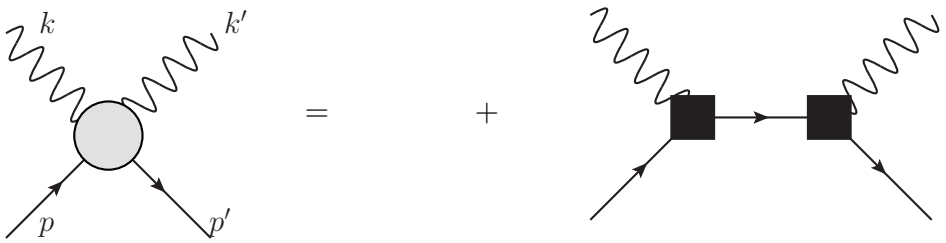
As before, this will involve calculating some physical process in both QED and NRQED, and comparing the result. The simplest two photon process to consider is Compton scattering. By calculating Compton scattering in each theory, the coefficients desired will be obtained.

This is not quite as straightforward as in the case of the one-photon scattering, for the following reason: while the one-photon scattering is a local interaction in both QED and NRQED, Compton scattering will involve some mix of local and non-local diagrams. In QED, there are of course no local interactions between a fermion and two photons. The situation is most readily stated diagrammatically.

In QED, the leading order diagrams contributing to Compton scattering are:



While in NRQED, the following diagrams contribute to the scattering:



In each set of diagrams, the vertices represent the *total* electron vertex. For QED this is determined, as before, by the form factors, and for NRQED it is determined by the calculations of the previous section.

Since the two amplitudes must be equal, in principle the process is this: First calculate the scattering amplitude in QED. Then, calculate the contribution to the scattering amplitude coming from the tree diagrams I and II above. Whatever discrepancy remains must be the value of the local two-photon vertex III.

The process of subtracting the one set of diagrams from the other could be slightly complicated, but luckily it turns out there is a simpler path. By considering the physical origin of the local terms in NRQED, it will be possible to split the QED diagrams into local and non-local parts, where the latter can be shown to be equal to the non-local diagrams in NRQED. Then, comparing the two scattering processes becomes much easier.

Z diagrams

The high energy theory (QED) doesn't contain any two-photon vertices, while the low energy theory (NRQED) does. This is a general feature of effective field theories, that new types of local interactions arise. The high energy theory can have intermediate states that are highly virtual, while the low energy theory doesn't. Instead, as according to the uncertainty principle, intermediate states with extremely high energy can be considered to occur almost instantaneously, giving rise in the effective theory to local interactions.

How does the local two-photon interaction arise in NRQED? Of course there are an infinite number of contributions, but we'll consider just the leading order contributions. These will come from the tree level two photon diagrams as shown above. Compare the

tree-level diagrams in the two theories: in addition to the vertices being different, so are the propagators. The propagator in QED represents some admixture of the electron and positron field, while in NRQED it is only the electron. In both QED and NRQED, a process is calculated as the sum of a series of diagrams, representing an expansion in perturbation theory. However, there is a difference btween the two in the nature of perturbation theory employed.

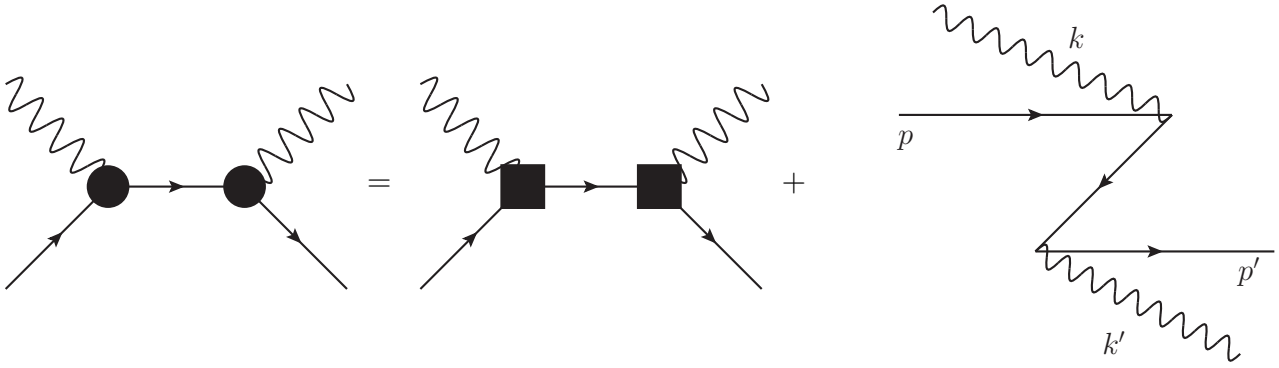
In QED, at each vertex both energy and momentum is conserved. But intermediate particles may be off mass-shell; that is it is no longer the case that for a particle of four-momentum p and mass m that $p^2 = m^2$.

In NRQED, the old Rayleigh-Schrodinger perturbation theory is used. All intermediate particles are on mass-shell. But at the vertices (when represented diagrammatically), although momenta is conserved, energy is not.

$$\Delta = \Sigma_{\text{int}} \frac{\langle \text{out}| V |\text{int}\rangle \langle \text{int}| V |\text{in}\rangle}{E_{\text{in}} - E_{\text{int}}} \tag{0.1.1}$$

The trick, then is to take the relativistic tree-level diagrams of QED and rewrite them in the language of Rayleigh-Schrodinger before trying to compare them to NRQED. In NRQED, only intermediate states involving electrons can be considered, but in QED intermediate states identified with positrons will appear as well. It is *these* processes, involving a large violation of energy conservation, that will appear as contact terms in NRQED.

There are two diagrams in QED, and both can be dealt with in the same general way. First consider the uncrossed diagram:



There are two tree level processes that can be considered in the old time-ordered perturbation theory. The first corresponds to an incoming electron, which first absorbs a photon and then emits one. The second, more complicated process, involves the creation of intermediate positron. While a free electron travels along, an incoming photon decays into an electron and positron. Then, the positron annihilates the incoming electron and emits the outgoing photon. Because of the shape of this diagram, it is called a "Z diagram."

In the Z-diagrams, the electrons and the photons are external, so the sum over the intermediate states is specifically the intermediate states of the positrons. Likewise, the other diagrams are written as a sum over intermediate electron states. But because of the rules of Rayleigh-Schrodinger perturbation theory, all these states are on mass shell. And since the momenta here is fixed, the sum over intermediate states is a sum over spin states.

So the original QED diagram should somehow split into two terms, one involving a sum over electron states and the other a sum over anti-particles.

Call the intermediate momentum $\mathbf{q}$. The initial energy will be q_0, the intermediate energy will be that of the on mass shell particle, $E_q = \sqrt{\mathbf{q}^2 + m^2}$.

$= \quad i\frac{\not{q}-m}{q^2-m^2} = i\frac{1}{\sqrt{\mathbf{q}^2+m^2}}\left(\frac{\Sigma\bar{u}(\mathbf{q})u(\mathbf{q})}{q_0-\sqrt{\mathbf{q}^2+m^2}} + \frac{\Sigma\bar{v}(-\mathbf{q})v(-\mathbf{q})}{q_0+\sqrt{\mathbf{q}^2+m^2}}\right)$

This identity can be technically reproduced as follows. First write the denominator of the propagator as

$$i\frac{\not{q}-m}{q^2-m^2} = i\frac{\not{q}-m}{q_0^2-(\mathbf{q}^2+m^2)} \tag{0.1.2}$$

This could be factored into

$$q_0^2-(\mathbf{q}^2+m^2) = (q_0+\sqrt{\mathbf{q}^2+m^2})(q_0-\sqrt{\mathbf{q}^2+m^2}) \tag{0.1.3}$$

So it implies poles at $q_0 = \pm\sqrt{\mathbf{q}^2+m^2}$. There is one unique way of factoring the original propagator into the two poles:

$$\frac{1}{2\sqrt{\mathbf{q}^2+m^2}}\left(\frac{\gamma^0\sqrt{\mathbf{q}^2+m^2}-\gamma\cdot\mathbf{q}+m}{q_0-\sqrt{\mathbf{q}^2+m^2}} - \frac{\gamma^0\sqrt{\mathbf{q}^2+m^2}+\gamma\cdot\mathbf{q}-m}{q_0+\sqrt{\mathbf{q}^2+m^2}}\right) \tag{0.1.4}$$

The two numerators can be exactly equated to sums over polarisation states of electron and positron spinors:

$$\begin{aligned} \Sigma u(\mathbf{p})\bar{u}(\mathbf{p}) &= \gamma \cdot p + m \\ \Sigma v(\mathbf{p})\bar{v}(\mathbf{p}) &= \gamma \cdot p - m \end{aligned}$$

These relations hold for particles which are on mass-shell. That is exactly the case here. But then, there is an assumption that the quantity p_0 above is the on mass-shell energy, $\sqrt{\mathbf{p}^2 + m^2}$.

So, noting that particles with momentum $\pm\mathbf{q}$ have the same energy $\sqrt{\mathbf{q}^2 + m^2}$ and that q_0 is the off-mass shell energy from the relativistic diagram, the propagator can be rewritten:

$$\frac{1}{2\sqrt{\mathbf{q}^2 + m^2}} \left(\frac{\Sigma u(\mathbf{q})\bar{u}(\mathbf{q})}{q_0 - \sqrt{\mathbf{q}^2 + m^2}} - \frac{\Sigma v(-\mathbf{q})\bar{v}(-\mathbf{q})}{q_0 + \sqrt{\mathbf{q}^2 + m^2}} \right) \tag{0.1.5}$$

The numerators have now been put into exactly the forms expected for the regular- and Z-diagrams of old perturbation theory. The denominators also correspond to the expected form of $E_{\text{in}} - E_{\text{int}}$.

First consider the regular diagram. The initial energy is q_0, since in relativistic theory the total energy at the vertex is conserved. The intermediate energy is the on-mass shell energy of the electron: $\sqrt{\mathbf{q}^2 + m^2}$. Thus the denominator of $q_0 - \sqrt{\mathbf{q}^2 + m^2}$ is that expected.

Now consider the Z-diagram. The initial energy is still q_0. The intermediate energy is more complicated: there are two photons, two electrons, and a positron present. The total combined energy is:

$$E_{\text{int}} = p_0 + p_0' + k_0 + k_0' + \sqrt{\mathbf{q}^2 + m^2} = 2q_0 + \sqrt{\mathbf{q}^2 + m^2} \tag{0.1.6}$$

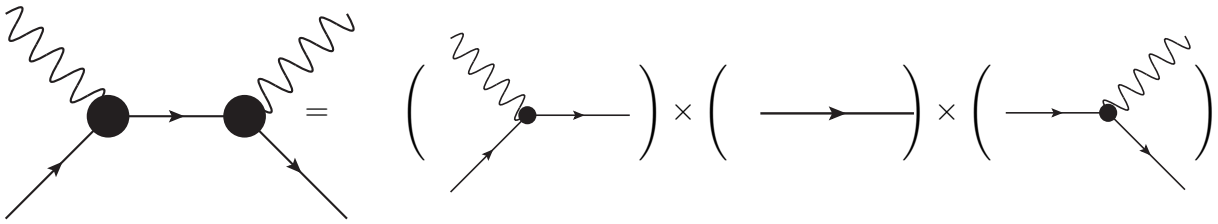
Then the difference $E_{\text{in}} - E_{\text{int}} = -q_0 - \sqrt{\mathbf{q}^2 + m^2}$. This correspons to the denominator found above, with the overall negative factor providing the relative minus sign.

So rewriting the propagator into these two separate terms can be understood as how the fully relativistic process appears in old perturbation theory. In a relativistic theory, intermediate states involving positrons must be accounted for. For higher order QED diagrams more complicated processes will appear, but at the tree level only those discussed above are involved. No approximations are involved — the identity for the propagator is exact. The form is more convenient for discussing nonrelativistic energies, but is still equivalent to the relativistic diagrams.

Relation between NRQED and old perturbation theory

Now it is necessary to show how the diagrams in old perturbation theory relate to those of NRQED.

The normal diagrams can be easily interpreted as the product of vertices and the relativistic Rayleigh-Schrodinger propagator:



Where Recall that the total NRQED one-photon vertex was derived by comparing to the QED vertex, so necessarily

$$\bar{u}\Gamma^0 u = \phi^\dagger V^0 \phi,\ \bar{u}\Gamma^i u = \phi^\dagger V^i \phi \tag{0.1.7}$$

The propagator $1/(q_0 - \sqrt{m^2 + \mathbf{q}^2})$ is equal to the total propagator in NRQED (including relativistic corrections). So it really is the case that the two sets of diagrams are equivalent. Whether the vertices are written in terms of NRQED or QED, the ultimate expression will be the same.

It then follows that the contributions to the local contact term in NRQED come (at

tree level) exactly from the Z-diagrams. To calculate the contact terms to the order (in the nonrelativistic expansion) needed, the Z-diagrams need to be approximated just as the one-photon vertex diagrams were in the previous section. Then the NRQED coefficients can be obtained by comparing the two calculations.

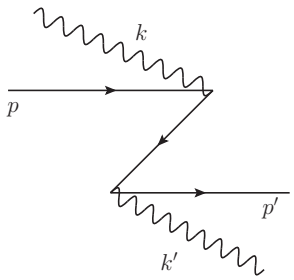
At nonrelativistic energies, it would be expected that the sum over intermediate states now does not resemble a propagator. Because only terms involving up to one power of momentum need be kept, the square root term becomes simply m. q_0, the total incoming energy, will involve both the electron and photon energy. Depending on whether the crossed or uncrossed diagram is considered, it will be either $q_0 = p_0 + k_0$ or $q_0 = p_0 - k'_0$. In either case, it will be m at the leading order, with a first order correction due to the photon energy.

$$\frac{\Sigma \bar{v}(-\mathbf{q}) v(-\mathbf{q})}{q_0 + \sqrt{\mathbf{q}^2 + m^2}} \approx \frac{\Sigma \bar{v}(-\mathbf{q}) v(-\mathbf{q})}{2m + (q_0 - m)} \tag{0.1.8}$$

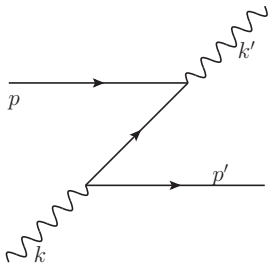
Since $q_0 - m << m$, this can then be written as

$$\approx \left(1 - \frac{q_0 - m}{2m}\right) \frac{\Sigma \bar{v}(-\mathbf{q}) v(-\mathbf{q})}{2m} \tag{0.1.9}$$

In this approximation, the two Z diagrams become



$$= \quad \Sigma_{\text{spin}} \left(1 - \tfrac{k_0}{2m}\right) \tfrac{1}{2m} \bar{u}(\mathbf{p}') \gamma^\mu \epsilon^*_\mu(k') v_s(-\mathbf{p} - \mathbf{k}) \bar{v}_s(-\mathbf{p} - \mathbf{k}) \Gamma^\nu \epsilon_\nu(k) u(\mathbf{p})$$



$$= \quad \Sigma_{\text{spin}} \left(1 + \tfrac{k'_0}{2m}\right) \tfrac{1}{2m} \bar{u}(\mathbf{p}') \gamma^\mu \epsilon_\mu(k) v(\mathbf{k}' - \mathbf{p})_s \bar{v}_s(\mathbf{k}' - \mathbf{p}) \Gamma^\nu \epsilon^*_\nu(k') u(\mathbf{p})$$

While the sums over intermediate states could also be expanded, it'll be easiest to calculate these in the above form. The vertices will be the sum of particle-antiparticle bilinears, which can be calculated separately.

Calculation of uv bilinears

The bulk of the calculation has been reduced to finding the product of particle-antiparticle vertices. As in the one-photon calculation, the vertices can be broken down into the sum of bilinears. To compare to NRQED, it will be necessary to find the nonrelativistic expressions for these bilinears, just as before. However, only terms of the leading order and suppressed by a single power of momenta need be kept, since the *product* of two bilinears is the final result.

To calculate Compton scattering, the assumption of elasticity will no longer be applicable. So it can no longer be assumed that the photon energy is zero. And in fact conservation of energy is explicitly violated at the vertices.

The intermediate positron states will be labelled χ (and $\chi^\dagger$), and the incoming/outgoing electron states ϕ (and $\phi^\dagger$). The incoming momentum is p, the outgoing p', and the intermediate momentum (posessed by the anti-particle) will be called q.

There are two types of vertices, coming from $\bar{v}\Gamma^\mu u$ and $\bar{u}\Gamma^\mu v$. The general form of these must of course be related.

$$\bar{v}(\mathbf{q})\Gamma^\mu u(\mathbf{p}) = v^\dagger(\mathbf{q})\gamma^0\Gamma^\mu u(\mathbf{p}) = \left[u^\dagger(\mathbf{p})\Gamma^{\mu\dagger}\gamma^0 v(\mathbf{q})\right]^\dagger \tag{0.1.10}$$

Using that $\gamma^0\gamma^0 = I$.

$$= \left[\bar{u}(\mathbf{p})\gamma^0\Gamma^{\mu\dagger}\gamma^0 v(\mathbf{q})\right]^\dagger \tag{0.1.11}$$

But all the bilinears are exactly defined such that the bilinears are Hermitian.

First start by calculating the scalar like bilinears.

Scalar bilinears The scalar bilinear is straightforward:

$$\begin{aligned}
\bar{u}(p')v(q) &= \begin{pmatrix} \varphi^\dagger & \frac{\boldsymbol{\sigma}\mathbf{p}'}{2m}\varphi^\dagger \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\boldsymbol{\sigma}\mathbf{q}}{2m}\chi \\ \chi \end{pmatrix} \\
&= \phi^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{q} - \boldsymbol{\sigma}\cdot\mathbf{p}'}{2m} \right) \chi \\
\bar{v}(q)u(p) &= \begin{pmatrix} \frac{\boldsymbol{\sigma}\mathbf{q}}{2m}\chi^\dagger & \chi^\dagger \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \varphi \\ \frac{\boldsymbol{\sigma}\mathbf{p}}{2m}\varphi \end{pmatrix} \\
&= \chi^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{q} - \boldsymbol{\sigma}\cdot\mathbf{p}}{2m} \right) \phi
\end{aligned}$$

Vector bilinears To calculate the vector like bilinears, treat the spatial and time-like components separately. First the time-like:

$$\begin{aligned}
\bar{u}(p')\gamma^0 v(q) &= \begin{pmatrix} \varphi^\dagger & \frac{\boldsymbol{\sigma}\mathbf{p}'}{2m}\varphi^\dagger \end{pmatrix} \begin{pmatrix} \frac{\boldsymbol{\sigma}\mathbf{q}}{2m}\chi \\ \chi \end{pmatrix} \\
&= \phi^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{q} + \boldsymbol{\sigma}\cdot\mathbf{p}'}{2m} \right) \chi \\
\bar{v}q\gamma^0 u(p) &= \begin{pmatrix} \frac{\boldsymbol{\sigma}\mathbf{q}}{2m}\chi^\dagger & \chi^\dagger \end{pmatrix} \begin{pmatrix} \varphi \\ \frac{\boldsymbol{\sigma}\mathbf{p}}{2m}\varphi \end{pmatrix} \\
&= \chi^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{q} + \boldsymbol{\sigma}\cdot\mathbf{p}}{2m} \right) \phi
\end{aligned}$$

Then the spatial:

$$\begin{aligned}
\bar{u}(p')\gamma^i v(q) &= \begin{pmatrix} \varphi^\dagger & \frac{\boldsymbol{\sigma}\mathbf{p}'}{2m}\varphi^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} \frac{\boldsymbol{\sigma}\mathbf{q}}{2m}\chi \\ \chi \end{pmatrix} \\
&= \phi^\dagger \sigma_i \chi \\
\bar{v}(q)\gamma^i u(p) &= \begin{pmatrix} \frac{\boldsymbol{\sigma}\mathbf{q}}{2m}\chi^\dagger & \chi^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \frac{\boldsymbol{\sigma}\mathbf{p}}{2m}\varphi \end{pmatrix} \\
&= \chi^\dagger \sigma_i \phi
\end{aligned}$$

Tensor bilinears First consider the part of the tensor bilinear with one spatial and one time-like index:

$$\begin{aligned}
\bar{u}(p')\sigma^{0i} v(q) &= \begin{pmatrix} \varphi^\dagger & \frac{\boldsymbol{\sigma}\mathbf{p}'}{2m}\varphi^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} \frac{\boldsymbol{\sigma}\mathbf{q}}{2m}\chi \\ \chi \end{pmatrix} \\
&= \phi^\dagger \sigma_i \chi \\
\bar{v}(q)\sigma^{0i} u(p) &= \begin{pmatrix} \frac{\boldsymbol{\sigma}\mathbf{q}}{2m}\chi^\dagger & \chi^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \frac{\boldsymbol{\sigma}\mathbf{p}}{2m}\varphi \end{pmatrix} \\
&= -\chi^\dagger \sigma_i \phi
\end{aligned}$$

Then the part with two spatial indices:

$$\begin{aligned}
\bar{u}(p')\sigma^{ij} v(q) &= -i\epsilon_{ijk} \begin{pmatrix} \varphi^\dagger & \frac{\boldsymbol{\sigma}\mathbf{p}'}{2m}\varphi^\dagger \end{pmatrix} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \begin{pmatrix} \frac{\boldsymbol{\sigma}\mathbf{q}}{2m}\chi \\ \chi \end{pmatrix} \\
&= -i\epsilon_{ijk}\phi^\dagger \left(\frac{\sigma_k \boldsymbol{\sigma}\cdot\mathbf{q} + \boldsymbol{\sigma}\cdot\mathbf{p}'\,\sigma_k}{2m} \right) \chi \\
\bar{v}(q)\sigma^{ij} u(p) &= -i\epsilon_{ijk} \begin{pmatrix} \frac{\boldsymbol{\sigma}\mathbf{q}}{2m}\chi^\dagger & \chi^\dagger \end{pmatrix} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \begin{pmatrix} \varphi \\ \frac{\boldsymbol{\sigma}\mathbf{p}}{2m}\varphi \end{pmatrix} \\
&= -i\epsilon_{ijk}\chi^\dagger \left(\frac{\sigma_k \boldsymbol{\sigma}\cdot\mathbf{p} + \boldsymbol{\sigma}\cdot\mathbf{q}\,\sigma_k}{2m} \right) \phi
\end{aligned}$$