0.1 Exact equations of motion for spin-1

$$\mathcal{L} = -\frac{1}{2}(D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) + m^2 W^{\mu\dagger} W_\mu - i\lambda e W^{\mu\dagger} W^\nu F_{\mu\nu} \tag{0.1}$$

where as usual D is the long derivative $D^\mu = \partial^\mu + ieA^\mu$.

Obtain the equations of motion from the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial[\partial_\mu W^{\dagger\alpha}]} = 0$$

Or equivalently,

$$\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} - D_\mu \frac{\partial \mathcal{L}}{\partial[D_\mu W^{\dagger\alpha}]} = 0$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} &= \frac{\partial}{\partial W^{\dagger\alpha}} \left(m^2 W^{\mu\dagger} W_\mu - i\lambda e W^{\mu\dagger} W^\nu F_{\mu\nu} \right) \\
&= m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial[D_\gamma W^{\dagger\alpha}]} &= -\frac{1}{2} \frac{\partial}{\partial[D_\gamma W^{\dagger\alpha}]} (D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) \\
&= -\frac{1}{2}(g^{\mu\gamma} g^\nu_\alpha - g^{\nu\gamma} g^\mu_\alpha)(D_\mu W_\nu - D_\nu W_\mu) \\
&= D_\alpha W^\gamma - D^\gamma W_\alpha
\end{aligned}$$

So the complete equation from Euler-Lagrange is

$$m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} + D_\mu (D^\mu W_\alpha - D_\alpha W^\mu) = 0 \tag{0.2}$$

This is a set of four coupled second order equations for the field W. We rewrite as a set of first order equations by introducing a field $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$. So (0.2) becomes

$$m^2 W_\alpha - ie\lambda W^\mu F_{\alpha\mu} + D^\mu W_{\mu\alpha} = 0 \tag{0.3}$$

$W^{\mu\nu}$ is antisymmetric and so has six degrees of freedom, corresponding to six independent fields. Together with W^μ this represents a total of ten fields. However, upon examination only some of these fields are dynamic. The fields W^{0i} and W^i appear in the equations with time derivatives, while the fields W^{ij} and W^0 never do. So it is only necessary to consider the former six fields. So that these six fields all have the same dimension, we will define $\frac{W^{i0}}{m} = i\eta^i$.

We will now eliminate the extraneous fields and solve for $iD_0 W^i$, $iD_0 \eta^i$.

$$m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} + D_\mu (D^\mu W_\alpha - D_\alpha W^\mu) = 0 \tag{0.4}$$

Define $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$. Then

$$m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} + D^\mu W_{\mu\alpha} = 0 \tag{0.5}$$

To get the exact Hamiltonian of a bispinor, we can eliminate nondynamic fields.

First consider (0.5) with $\alpha = 0$.

$$m^2 W_0 - ie\lambda W^\nu F_{0\nu} + D^\mu W_{\mu 0} \tag{0.6}$$

$$m^2 W_0 - ie\lambda W^j F_{0j} + D^j W_{j0} \tag{0.7}$$

Solve this for W_0

$$W_0 = \frac{1}{m^2}\left(ie\lambda W^j F_{0j} - D^j W_{j0}\right) \tag{0.8}$$

Now, consider (0.5) with $\alpha = i$

$$m^2 W_i - ie\lambda W^\mu F_{i\mu} + D^\mu W_{\mu i} = 0 \tag{0.9}$$

$$m^2 W_i - ie\lambda W^0 F_{i0} + D^0 W_{0i} - ie\lambda W^j F_{ij} + D^j W_{ji} = 0 \tag{0.10}$$

Using (0.8) we can replace W_0

$$m^2 W_i - \frac{ie\lambda}{m^2}\left(ie\lambda W^\nu F_{0\nu} - D^j W_{j0}\right) F_{i0} + D^0 W_{0i} - ie\lambda W^j F_{ij} + D^j W_{ji} = 0 \tag{0.11}$$

Using $W_{ji} = D_j W_i - D_i W_j$

$$m^2 W_i - \frac{ie\lambda}{m^2}\left(ie\lambda W^\nu F_{0\nu} - D^j W_{j0}\right) F_{i0} + D^0 W_{0i} - ie\lambda W^j F_{ij} + D^j (D_j W_i - D_i W_j) = 0 \tag{0.12}$$

Solve this for $D^0 W_{0i}$:

$$D^0 W_{0i} = -m^2 W_i + \frac{ie\lambda}{m^2}\left(ie\lambda W^\nu F_{0\nu} - D^j W_{j0}\right) F_{i0} + ie\lambda W^j F_{ij} - D^j (D_j W_i - D_i W_j) \tag{0.13}$$

To get a similar equation for W_i, consider

$$W_{i0} = D_i W_0 - D_0 W_i \tag{0.14}$$

Then

$$D^0 W_i = D_i W_0 - W_{i0} \tag{0.15}$$

$$D^0 W_i = D_i \frac{1}{m^2}\left(ie\lambda W^j F_{0j} - D^j W_{j0}\right) - W_{i0} \tag{0.16}$$

We now have equations that tell us the time evolution of a total of six fields: W_i and W_{i0}. We want to treat these as the components of some sort of bispinor. To that end, first define $\eta_i = -i/m W_{i0}$ so that we have a pair of fields with the same mass dimension and hermiticity. (Since $W_{i0} = D_i W_0 - D_0 W_i$ would pick up another minus sign under complex conjugation compared to W_i.) Going the other way $W_{i0} = im\eta_i$.

Since eventually we want a nonrelativistic expression, we should write the spatial components of four vectors as three vectors. To that end also write the components of the tensor $F_{\mu\nu}$ in terms of three vectors.

$$F_{0i} = E^i, \; F_{ij} = -\epsilon_{ijk} B^k$$

Regular spatial vectors are "naturally raised" while D_i is "naturally lowered". Then we can rewrite (0.13).

$$-imD_0 \eta_i = -m^2 W_i - \frac{ie\lambda}{m^2}\left(ie\lambda W^j E^j - D^j W_{j0}\right) E^i + ie\lambda W^j \epsilon_{ijk} B^k - D^j (D_j W_i - D_i W_j) \tag{0.17}$$

$$iD_0 \eta^i = mW^i + \frac{ie\lambda}{m^3}\left(ie\lambda W^j E^j - D^j W_{j0}\right) E^i - \frac{ie\lambda}{m} W^j \epsilon_{ijk} B^k + \frac{1}{m} D_j (D_j W^i - D_i W^j) \tag{0.18}$$

And likewise (0.16) becomes

$$D^0 W_i = D_i \frac{1}{m^2}\left(-ie\lambda W^j E^j - imD^j \eta_j\right) - im\eta_i \tag{0.19}$$

$$iD^0 W^i = -iD_i \frac{1}{m^2}\left(-ie\lambda W^j E^j - imD_j \eta^j\right) + m\eta^i \tag{0.20}$$

$$iD^0 W^i = -\frac{1}{m^2} D_i e\lambda \mathbf{W} \cdot \mathbf{E} - \frac{1}{m} D_i \mathbf{D} \cdot \eta + m\eta^i \tag{0.21}$$

0.1.1 Spin identities

To obtain some Schrodinger like equation for a bispinor, we need to express the time derivative of the bispinor in terms of other operators which can be interpreted as the Hamiltonian: $i\partial_0\Psi = \hat{H}\Psi$. We have expressions for the time derivatives of W_i and η_i, but some mixing of the indices is involved. To write the Hamiltonian in a block form we can introduce spin matrices whose action will mix the components of the fields.

The spin matrix for a spin one particle can represented as:

$$(S^k)_{ij} = -i\epsilon_{ijk} \tag{0.22}$$

which leads to the following identities:

$$(\mathbf{a}\times\mathbf{v})_i = -i(\mathbf{S}\cdot\mathbf{a})_{ij}v_j \tag{0.23}$$

$$\{\mathbf{a}\times(\mathbf{a}\times\mathbf{v})\}_i = -(\{\mathbf{S}\cdot\mathbf{a}\}^2)_{ij}v_j \tag{0.24}$$

$$a_i(\mathbf{b}\cdot\mathbf{v}) = \{\mathbf{a}\cdot\mathbf{b}\,\delta_{ij} - (S^kS^\ell)_{ij}a^\ell b^k\}v_j \tag{0.25}$$

Using these identities

$$iD^0\mathbf{W} = -\frac{e\lambda}{m^2}\mathbf{D}(\mathbf{W}\cdot\mathbf{E}) + \frac{1}{m}\mathbf{D}(\mathbf{D}\cdot\boldsymbol{\eta}) + m\boldsymbol{\eta} \tag{0.26}$$

$$iD^0\mathbf{W} = -\frac{1}{m^2}\{\mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\}\mathbf{W} + \frac{1}{m}\{\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2\}\boldsymbol{\eta} + m\boldsymbol{\eta} \tag{0.27}$$

and for the other equation

$$iD_0\eta^i = mW^i - \frac{ie\lambda}{m^3}\left(ie\lambda W^jE^j - D^jW_{j0}\right)E^i + \frac{ie\lambda}{m}W^j\epsilon_{ijk}B^k - \frac{1}{m}D_j(D_jW^i - D_iW^j) \tag{0.28}$$

$$iD_0\eta^i = mW^i - \frac{ie\lambda}{m^3}\left(ie\lambda W^jE^j - D^jW_{j0}\right)E^i + \frac{ie\lambda}{m}W^j\epsilon_{ijk}B^k - \frac{1}{m}D_j(D_jW^i - D_iW^j) \tag{0.29}$$

$$iD_0\boldsymbol{\eta} = m\mathbf{W} + \frac{e^2\lambda^2}{m^3}\mathbf{E}(\mathbf{W}\cdot\mathbf{E}) - \frac{1}{m^2}\mathbf{E}(\mathbf{D}\cdot\boldsymbol{\eta}) + \frac{ie\lambda}{m}(\mathbf{W}\times\mathbf{B}) + \frac{1}{m}\mathbf{D}\times(\mathbf{D}\times\mathbf{W}) \tag{0.30}$$

$$iD_0\boldsymbol{\eta} = m\mathbf{W} + \frac{e^2\lambda^2}{m^3}\{\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\}\mathbf{W} - \frac{1}{m^2}\{\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\}\boldsymbol{\eta} + \frac{e\lambda}{m}(\mathbf{S}\cdot\mathbf{B})\mathbf{W} - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2\mathbf{W} \tag{0.31}$$

$$iD_0\boldsymbol{\eta} = \left(m + \frac{e^2\lambda^2}{m^3}\{\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\} - \frac{e\lambda}{m}(\mathbf{S}\cdot\mathbf{B}) - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2\right)\mathbf{W} + \frac{1}{m^2}\{\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\}\boldsymbol{\eta} \tag{0.32}$$

Now that the equations are in the correct form, we can write them as follows:

$$iD_0\begin{pmatrix} W \\ \eta \end{pmatrix} = \begin{pmatrix} \lambda\frac{e}{m^2}\left[\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot\mathbf{E}\times\mathbf{D}\right] & m - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2 - \lambda\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + \lambda^2\frac{e^2}{m^3}\left[\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\right] \\ m - \frac{1}{m}\left[\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2 + e\mathbf{S}\cdot\mathbf{B}\right] & -\lambda\frac{e}{m^2}\left[\mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) + i\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}\right] \end{pmatrix} \tag{0.33}$$

0.1.2 Current

We can also derive the conserved current from the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}(D \times W)^\dagger \cdot (D \times W) + m_w^2 W^\dagger W - \lambda ie W^{\dagger\mu} W^\nu F_{\mu\nu}$$

Where

$$D^\mu = \partial^\mu - ieA^\mu, \qquad D \times W = D^\mu W^\nu - D^\nu W^\mu$$

We want the conserved current corresponding to the transformation $W_i \to e^{i\alpha} W_i$, which in infinitesimal form is:

$$W_\mu \to W_\mu + i\alpha W_\mu,\ W_\mu^\dagger \to W_\mu^\dagger - i\alpha W_\mu^\dagger$$

The 4-current density will be:

$$j^\sigma = -i\frac{\partial \mathcal{L}}{\partial W_{\mu,\sigma}} W_\mu + i\frac{\partial \mathcal{L}}{\partial W^\dagger_{\mu,\sigma}} W_\mu^\dagger$$

Only one term contains derivatives of the field:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial W_{\alpha,\sigma}} &= \frac{\partial}{\partial W_{\alpha,\sigma}}\left\{-\frac{1}{2}(D_\mu W_\nu - D_\nu W_\mu)^\dagger (D^\mu W^\nu - D^\nu W^\mu)\right\} \\ &= -\frac{1}{2}(D_\mu W_\nu - D_\nu W_\mu)^\dagger (g_{\sigma\mu} g_{\alpha\nu} - g_{\sigma\nu} g_{\alpha\mu}) \\ &= -(D_\alpha W_\sigma - D_\sigma W_\alpha)^\dagger\end{aligned}$$

Likewise:

$$\frac{\partial \mathcal{L}}{\partial W^\dagger_{\alpha,\sigma}} = -(D_\alpha W_\sigma - D_\sigma W_\alpha)$$

If we define $W_{\mu\nu} = D^\mu W^\nu - D^\nu W^\mu$ then the 4-current and charge density are:

$$\begin{aligned}j_\sigma &= iW^\dagger_{\sigma\mu} W^\mu - iW_{\sigma\mu} W^{\dagger\mu} \\ j_0 &= iW^\dagger_{0\mu} W^\mu - iW_{0\mu} W^{\dagger\mu} \\ &= iW^\dagger_{0i} W^i - iW_{0i} W^{\dagger i}\end{aligned}$$

Where the last equality follows from the antisymmetry of $W_{\mu\nu}$.

Now, we defined the fields $\eta_i = -i\frac{W_{i0}}{m}$. In terms of these fields, $j_0 = m(\eta_i^\dagger W^i + \eta_i W^{\dagger i})$.

We can do the same to find the vector part of the current.

$$\begin{aligned}j_i &= iW^\dagger_{i\mu} W^\mu - iW_{i\mu} W^{\dagger\mu} \\ &= iW_j^\dagger W_{ij} + iW^\dagger_{i0} W_0 + c.c.\end{aligned}$$

We have $W_{ij} = D_i W_j - D_j W_i$. Using the identities developed in the appendix, we can obtain

$$D_j W_i = D_i W_j - D_k (S_i S_k)_{ja} W_a$$

Then

$$W_{ij} = D_k (S_i S_k)_{ja} W_a$$

In the absence of an electric field E, $W_0 = \frac{i}{m} D_j \eta_j$, with $W_{i0} = im\eta_i$.

$$W_{i0}^\dagger W_0 = -\eta_i^\dagger D_j \eta_j$$

Again we introduce spin matrices to get the equation in the desired form, and obtain

$$W_{i0}^\dagger W_0 = -\eta_j^\dagger D_k (\delta_{ik} - S_k S_i) \eta_j$$

This leads to

$$j_i = iW_j^\dagger D_k (S_i S_k W)_j - i\eta_j^\dagger D_k ([\delta_{ik} - S_k S_i]\eta)_j + c.c.$$

Writing this in terms of the bispinor $\begin{pmatrix} \eta \\ W \end{pmatrix}$, the expression for the current is

$$j_i = \frac{i}{2} \begin{pmatrix} \eta^\dagger & W^\dagger \end{pmatrix} \left[(\{S_i, S_j\} - \delta_{ij}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - ([S_i, S_j] + \delta_{ij}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] D_j \begin{pmatrix} \eta \\ W \end{pmatrix} + c.c. \tag{0.34}$$