Chapter 1

Spin one-half

1.0.1 Conventions

Spinors:

$$u = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \tag{1.0.1}$$

$$\begin{aligned} \eta &= \left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w \\ \xi &= \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2m} \left(1 - \frac{3\mathbf{p}^2}{8m^2}\right) w \end{aligned}$$

structures (in the useful represenation)

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \tag{1.0.2}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{1.0.3}$$

$$\sigma^{\mu\nu} = i\frac{1}{2}[\gamma^\mu, \gamma^\nu] \tag{1.0.4}$$

$$[\gamma^0, \gamma^i] = 2\gamma^0\gamma^i = 2\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{1.0.5}$$

$$\gamma^i\gamma^j = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_i\sigma_j & 0 \\ 0 & -\sigma_i\sigma_j \end{pmatrix} \tag{1.0.6}$$

$$[\gamma^i, \gamma^j] = \begin{pmatrix} [\sigma_j, \sigma_i] & 0 \\ 0 & [\sigma_j, \sigma_i] \end{pmatrix} = i\epsilon_{jik}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} = -i\epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \tag{1.0.7}$$

1.0.2 QED calculations

First form:

$$(p+p')^\mu \bar{u}u = (p+p')^\mu \left(\eta^\dagger\eta - \xi^\dagger\xi\right) \tag{1.0.8}$$

$$= (p+p')^\mu \left\{ w^\dagger \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right)\left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w - w^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{2m}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\right) w \right\} \tag{1.0.9}$$

$$= (p+p')^\mu w^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{8m^2} - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2}\right) w \tag{1.0.10}$$

Second form: For the term $\bar{u}\gamma^\mu u$ it'll be necessary to treat the spatial/time-like indices separately.

time-like

$$\bar{u}\gamma^0 u = u^\dagger u \tag{1.0.11}$$

$$= \eta^\dagger\eta + \xi^\dagger\xi \tag{1.0.12}$$

$$= w^\dagger \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right)\left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w + w^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{2m}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\right) w \tag{1.0.13}$$

$$= w^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{8m^2} + \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2}\right) w \tag{1.0.14}$$

spatial

$$\bar{u}\gamma^i u = u^\dagger \gamma^0 \gamma^i u \tag{1.0.15}$$

$$= \bar{u}^\dagger \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} u \tag{1.0.16}$$

$$= \eta^\dagger \sigma_i \xi + \xi^\dagger \sigma_i \eta \tag{1.0.17}$$

$$= w^\dagger \left\{ \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right) \sigma_k \left(1 - \frac{3\mathbf{p}^2}{8m^2}\right) - \left(1 - \frac{3\mathbf{p}'^2}{8m^2}\right) \sigma_k \left(1 - \frac{\mathbf{p}^2}{8m^2}\right) - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_k\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2} \right\} \tag{1.0.18}$$

Third type (tensor)

$$\bar{u}\frac{i}{2m} q_j \sigma^{ij} u = \frac{i\epsilon_{ijk}q_j}{2m} \bar{u} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} u \tag{1.0.19}$$

$$\frac{i\epsilon_{ijk}q_j}{2m} \left(\eta^\dagger \sigma_k \eta - \xi^\dagger \sigma_k \xi\right) \tag{1.0.20}$$

$$\frac{i\epsilon_{ijk}q_j}{2m} w^\dagger \left\{ \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right) \sigma_k \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right) - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_k\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2} w \right\} \tag{1.0.21}$$

Need triple sigma identity

$$\sigma_a\sigma_b\sigma_c = \sigma_a(\delta_{bc} + i\epsilon_{bcd}\sigma_d) = \sigma_a\delta_{bc} - \sigma_b\delta_{ca} + \sigma_c\delta_{ab} + i\epsilon_{abc} \tag{1.0.22}$$

Then using above

$$\bar{u}\frac{i}{2m}q_j\sigma^{ij}u = \frac{i\epsilon_{ijk}q_j}{2m}w^\dagger\left\{\sigma_k\left(1-\frac{\mathbf{p}'^2+\mathbf{p}^2}{8m^2}\right)-\frac{\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{p}')p_k-\sigma_k\mathbf{p}\cdot\mathbf{p}'+i\epsilon_{akc}q_ap_c}{4m^2}\right\}w \tag{1.0.23}$$

The 'time-like' part of the tensor term

$$\bar{u}\frac{i}{2m}q_j\sigma^{0j}u = -\frac{q_j}{2m}\bar{u}\gamma^0\gamma^j u \tag{1.0.24}$$

$$= -\frac{q_j}{2m}u^\dagger\gamma^j u \tag{1.0.25}$$

$$= -\frac{q_j}{2m}\left(\eta^\dagger\sigma_j\chi-\chi^\dagger\sigma_j\eta\right) \tag{1.0.26}$$

$$= -\frac{q_j}{2m}w^\dagger\left\{\left(1-\frac{\mathbf{p}'^2}{8m^2}\right)\frac{\sigma_j\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\left(1-\frac{3\mathbf{p}^2}{8m^2}\right)-\left(1-\frac{3\mathbf{p}'^2}{8m^2}\right)\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_j}{2m}\left(1-\frac{\mathbf{p}^2}{8m^2}\right)\right\}w \tag{1.0.27}$$

Dropping terms quadratic in q, all p' can be written just as p.

$$\approx -\frac{q_j}{2m}w^\dagger\left\{\frac{\sigma_j\boldsymbol{\sigma}\cdot\mathbf{p}-\boldsymbol{\sigma}\cdot\mathbf{p}\sigma_j}{2m}\left(1-\frac{\mathbf{p}^2}{2m^2}\right)\right\}w \tag{1.0.28}$$

$$= \frac{q_j}{2m}w^\dagger\left\{\frac{i\epsilon_{ijk}\sigma_kp_i}{2m}\left(1-\frac{\mathbf{p}^2}{2m^2}\right)\right\}w \tag{1.0.29}$$

$$= w^\dagger\left\{\frac{i\epsilon_{ijk}p_iq_j\sigma_k}{4m^2}\left(1-\frac{\mathbf{p}^2}{2m^2}\right)\right\}w \tag{1.0.30}$$

1.1 Fouldy-Wouthyusen approach

1.2 Equations of motion

The regular Dirac Lagrangian can be modified by including a term responsible for the anomalous magnetic moment. The added term is $\mu'\bar{\Psi}\sigma^{\mu\nu}F_{\mu\nu}\Psi$, where $\mu' = \frac{g-2}{2}\mu_0$

The Lagrangian then is

$$\mathcal{L} \;=\; \bar{\Psi}(p - m)\Psi + \frac{1}{2}\mu'\bar{\Psi}\sigma^{\mu\nu}F_{\mu\nu}\Psi$$

The Euler-Lagrange equations give us

$$\left((p_\mu - eA_\mu)\gamma^\mu - m + i\mu'\frac{1}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}\right)\Psi = 0$$

From this equation of motion we'll derive exact relations between the upper and lower components of Ψ.

First we'll replace $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ with terms involving the electric and magnetic fields. We use the antisymmetry of $\sigma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$, and that we deal with time-independent fields.

$$
\begin{aligned}
[\gamma^0,\gamma^i] &= \begin{pmatrix} 0 & 2\sigma_i \\ 2\sigma_i & 0 \end{pmatrix} \\
[\gamma^i,\gamma^j] &= \begin{pmatrix} -2i\epsilon_{ijk}\sigma_k & 0 \\ 0 & -2i\epsilon_{ijk}\sigma_k \end{pmatrix} \\
F_{\mu\nu}\sigma^{\mu\nu} &= F_i\sigma^{ij} - F_{0i}\sigma^{0i} - F_{i0}\sigma^{i0} + F_{00}\sigma^{00} \\
&= F_{ij}\sigma^{ij} - 2F_{0i}\sigma^{0i} \\
&= 2\partial_i A_j \sigma^{ij} - 2\partial_i \Phi \sigma^{0i} \\
&= -2i \begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{B} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\mathbf{B} \end{pmatrix} - 2 \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{E} \\ \boldsymbol{\sigma}\cdot\mathbf{E} & 0 \end{pmatrix}
\end{aligned}
$$

Considering $\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ as a bispinor

$$
\left\{ \begin{pmatrix} p_0 - e\Phi - m & 0 \\ 0 & -p_0 + e\Phi - m \end{pmatrix} + \begin{pmatrix} 0 & -\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \\ \boldsymbol{\sigma}\cdot\boldsymbol{\pi} & 0 \end{pmatrix} + \mu' \left[\begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{B} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\mathbf{B} \end{pmatrix} - i \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{E} \\ \boldsymbol{\sigma}\cdot\mathbf{E} & 0 \end{pmatrix} \right] \right\} \begin{pmatrix} \phi \\ \chi \end{pmatrix}
$$

This gives rise to exact coupled equations for ϕ and χ.

1.2.1 NR limit

We're interested in the nonrelativistic limit, in which we can treat the amplitude of χ as much smaller than ϕ. So we'll solve the coupled equations for ϕ to the desired order in $\frac{p}{m}$. Here we're interested in finding the nonrelativistic Hamiltonian to $\mathcal{O}(mv^4)$. We can also throw away terms proportional to μ'^2, as well as terms nonlinear in the magnetic field. In taking the relativistic limit we'll use the nonrelativistic energy $\epsilon = p_0 - m$.

Considering the order of various types of terms, we can see that $\epsilon \sim mv^2$, $\Phi \sim mv^2$, $\pi \sim mv$, and $\mathbf{E} \sim m^2v^3$.

Under these approximations we first find χ.

$$\begin{aligned}(\epsilon + 2m - \mu' \boldsymbol{\sigma}\cdot\mathbf{B})\,\chi &= (\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - i\boldsymbol{\sigma}\cdot\mathbf{E})\,\phi \\ \chi &\approx \frac{1}{2m}\left(1 - \frac{\epsilon - e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B}}{2m}\right)(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - i\mu'\boldsymbol{\sigma}\cdot\mathbf{E})\phi\end{aligned}$$

Then, solving for $\epsilon\phi$,

$$\begin{aligned}\epsilon\phi &= (e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B})\phi + (\boldsymbol{\sigma}\cdot\mathbf{E} + \boldsymbol{\sigma}\cdot\boldsymbol{\pi})\chi \\ &\approx \Big\{e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{}{2m} \\ &\quad + \frac{1}{4m^2}(\mu'\boldsymbol{\sigma}\cdot\mathbf{B} - [\epsilon - e\Phi])\Big\}\,\phi \\ &= \left\{e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2 - i\mu'[\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,,\boldsymbol{\sigma}\cdot\mathbf{E}]}{2m} + \frac{1}{4m^2}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,(\mu'\boldsymbol{\sigma}\cdot\mathbf{B} - [\epsilon - e\Phi])\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right\}\phi\end{aligned}$$

We have an expression for the kinetic energy ϵ of the particle, in terms of operators. This will give us the nonrelativistic Hamiltonian. Since the leading order term in H is $\mathcal{O}(mv^2)$, this suggests we split it into two parts: $H = H_0 + H_1 + \mathcal{O}(mv^6)$, where H_1 consists of only $\mathcal{O}(mv^4)$ terms. As written our expression for H_1 contains ϵ, so we'll have to solve for it perturbatively.

Our two terms are

$$\begin{aligned}\hat{H}_0 &= e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{2m} \\ \hat{H}_1 &= -\frac{i\mu'}{2m}[\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,,\boldsymbol{\sigma}\cdot\mathbf{E}] + \frac{1}{4m^2}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,(\mu'\boldsymbol{\sigma}\cdot\mathbf{B} - [\epsilon - e\Phi])\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\end{aligned}$$

To eliminate ϵ from H_1

$$
\begin{aligned}
\epsilon\phi &= \left(\hat{H}_0 + \mathcal{O}(mv^4)\right)\phi \\
&= \left(e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{2m}\right)\phi \\
(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2(\epsilon - e\Phi)\phi &= (\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\left(\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{2m} - \mu'\boldsymbol{\sigma}\cdot\mathbf{B}\right) \\
\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,(\epsilon - e\Phi)\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\phi &= \left(\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^4}{2m} - \mu'(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\boldsymbol{\sigma}\cdot\mathbf{B} - \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,[e\Phi, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}]\right)\phi
\end{aligned}
$$

So we can now write down H_1

$$
\hat{H}_1 = -\frac{i\mu'}{2m}[\boldsymbol{\sigma}\boldsymbol{\pi}\,, \boldsymbol{\sigma}\mathbf{E}] + \frac{1}{4m^2}\left(\mu'\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\boldsymbol{\sigma}\cdot\mathbf{B}\,\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^4}{2m} + \mu'(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\boldsymbol{\sigma}\cdot\mathbf{B} + \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,[e\Phi, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}]\right)
$$

To simplify, we'll eliminate terms of second order in the magnetic field. We'll also use the following identities:

$$
\begin{aligned}
\{\boldsymbol{\sigma}\cdot\mathbf{B}\,, \boldsymbol{\sigma}\cdot\mathbf{p}\} &= 2\mathbf{B}\cdot\mathbf{p} \\
(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2 &= \pi^2 - e\boldsymbol{\sigma}\cdot\mathbf{B}
\end{aligned}
$$

$$
[\Phi, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}] = -i\boldsymbol{\sigma}\cdot\mathbf{E}
$$

Then

$$
\hat{H}_0 = e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{\pi^2}{2m} - \frac{e}{2m}\boldsymbol{\sigma}\cdot\mathbf{B}
$$

$$
\hat{H}_1 = -\frac{\pi^4}{8m^3} + e\frac{p^2}{4m^3}\boldsymbol{\sigma}\cdot\mathbf{B} - i\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\boldsymbol{\sigma}\cdot\mathbf{E}}{4m^2} + \mu'\left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}\,\mathbf{B}\cdot\mathbf{p}}{2m^2} - \frac{i[\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,, \boldsymbol{\sigma}\cdot\mathbf{E}]}{2m}\right)
$$

1.3 FW Transform

To find a complete description of a single nonrelativistic particle in normal quantum mechanics, we must work in a basis where the lower component χ is truly negligible, at least at the desired order. While there exists a formal technique for finding this Fouldy-Wouthyusen transformation, for our purposes we can simply demand that the wave function after transformation $\phi_s = (1+\Delta)\phi$ obeys the normalization $<\phi_s, \phi_s>= 1$.

To find the necessary transformation, we can use our expression for χ in terms of ϕ to find the normalization

$$\begin{aligned}\int d^3x(\phi^\dagger\phi + \chi^\dagger\chi) &= \int d^3x \left[\phi^\dagger\phi + \left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\pi}}{2m}\phi\right)^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\pi}}{2m}\phi\right)\right] \\ &= \int d^3x\phi^\dagger \left[1 + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{4m^2}\right]\phi\end{aligned}$$

Since we know that $<\Psi, \Psi>= 1$ this shows that if $\phi = \left(1 - \frac{(\boldsymbol{\sigma}\boldsymbol{\pi})^2}{8m^2}\right)\phi$, the Schrodinger wave functions are properly normalized.

We now need to find the form of $\hat{H}$ after this transformation:

$$\begin{aligned}\epsilon\phi &= (\hat{H}_0 + \hat{H}_1)\phi \\ \epsilon(1 - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{8m^2})\phi_S &= (\hat{H}_0 + \hat{H}_1)(1 - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{8m^2})\phi_S \\ \epsilon\phi_S &= (1 + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{8m^2})(\hat{H}_0 + \hat{H}_1)(1 - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{8m^2})\phi_S \\ \epsilon\phi_S &= \left(\hat{H}_0 + \frac{1}{8m^2}[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \hat{H}_0] + \hat{H}_1\right)\phi_S\end{aligned}$$

So under the FW transformation, the leading order term is unchanged, and the second order term is:

$$\hat{H}_1 \to \hat{H}_1' = \hat{H}_1 + \frac{1}{8m^2}[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \hat{H}_0]$$

$[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \hat{H}_0]$ can be simplified.

$$[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \hat{H}_0] = [(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{2m}]$$

Obviously $(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2$ commutes with itself, and since $\boldsymbol{\sigma}\cdot\mathbf{B}$ is constant, that commutator also vanishes:

$$[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \mu'\boldsymbol{\sigma}\cdot\mathbf{B}] = [\pi^2 - e\boldsymbol{\sigma}\cdot\mathbf{B}, \mu'\boldsymbol{\sigma}\cdot\mathbf{B}] = 0$$

Leaving:

$$\begin{aligned}
[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \hat{H}_0] &= [(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, e\Phi] \\
&= e(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}[\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \Phi] + [\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \Phi]\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) \\
&= ie(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\boldsymbol{\sigma}\cdot\mathbf{E} + \boldsymbol{\sigma}\cdot\mathbf{E}\,\boldsymbol{\sigma}\cdot\boldsymbol{\pi})
\end{aligned}$$

So,

$$\begin{aligned}
\hat{H}_1' &= \hat{H}_1 + \frac{ie}{8m^2}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\boldsymbol{\sigma}\cdot\mathbf{E} + \boldsymbol{\sigma}\cdot\mathbf{E}\,\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) \\
&= -\frac{\pi^4}{8m^3} + e\frac{p^2}{4m^3}\boldsymbol{\sigma}\cdot\mathbf{B} - \frac{ie}{8m^2}[\boldsymbol{\sigma}\cdot\mathbf{E}, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}] + \mu'\left(\frac{(\boldsymbol{\sigma}\cdot\mathbf{p})(B\cdot p)}{2m^2} - \frac{i}{2m}[\boldsymbol{\sigma}\cdot\mathbf{E}, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}]\right)
\end{aligned}$$

NR Hamiltonian

Now we'll write down the entire Hamiltonian, using $\mu' = \frac{g-2}{2}\mu_0 = \frac{g-2}{2}\frac{e}{2m}$.

$$\begin{aligned}
H &= e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} - (1 + \frac{g-2}{2})\frac{e}{2m}\boldsymbol{\sigma}\cdot\mathbf{B} \\
&\quad + e\frac{p^2}{4m^3}\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{ie}{8m^2}(1 + (g-2))[\boldsymbol{\sigma}\cdot\mathbf{E}, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}] + (g-2)\frac{e}{2m}\frac{(\boldsymbol{\sigma}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^2} \\
&= e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} \\
&\quad - \frac{e}{2m}\left\{\frac{g}{2}\boldsymbol{\sigma}\cdot\mathbf{B} - \frac{p^2}{2m^2}\boldsymbol{\sigma}\cdot\mathbf{B} - (g-2)\frac{(\boldsymbol{\sigma}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^2} + (g-1)\boldsymbol{\sigma}\cdot(\mathbf{E}\times\boldsymbol{\pi})\right\} \\
&= e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} \\
&\quad - \frac{e}{2m}\left\{\frac{g}{2}\left(1 - \frac{p^2}{2m^2}\right)\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{g-2}{2}\frac{p^2}{2m^2}\boldsymbol{\sigma}\cdot\mathbf{B} - \frac{g-2}{2}\frac{(\boldsymbol{\sigma}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{2m^2} + \left(\frac{g}{2} + \frac{g-2}{2}\right)\boldsymbol{\sigma}\cdot(\mathbf{E}\times\boldsymbol{\pi})\right\}
\end{aligned}$$