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Chapter 1

Nonrelativistic Quantum Electrodynamics

1.1 Effective Field Theories

The most generally powerful approach to nonrelativistic bond state theories is the use of the techniques of effective field theory. In this chapter we develop the theory of such an approach.

1.1.1 Brief background

First, a brief background of the development of effective field theories.

The first workable relativistic theory of quantum mechanics came from Dirac. By realising that a relativistic equation for fermions necessitated a four component spinor, he was able to write down what is now known as Dirac's equation. This led to the prediction of the electron's antiparticle the positron.

The relativistic theory necessitated the introduction of an infinite degrees of freedom. Thus, wave functions had to be promoted to fields, and the first quantum field theory arose. It was highly successful at predicting leading order quantities, but when attempting to use perturbation theory within this context, a number of infinities arose.

To resolve this matter, QED and renormalization were developed in the late 1940s. The

key was in changing the parametrization from of the theory from "bare" values, to measured values at some particular energy scale. So to completely define the theory one needed to write down not only the Lagrangian but also the renormalization point. The coupling constants of the theory would depend upon this. With this approach QED was enormously successful, producing very accurate predictions to such quantities as the anomalous magnetic moment.

There was much frustrating work to formulate a theory of the other interactions along the same lines. During this work (such as that by Gell-man and Low) with these types of theories it became clear that there was structure to how the coupling constants behaved as the renormalization point changed.

While working through the Ising model, Wilson, who had a background working with QFTs, realised there were applications of these renormalisation ideas to critical phenomena. Later, working with fixed-source meson theory, had an insight that led to effective field theory. Working with a hierarchy of momentum "slices" and a process of eliminating energy scales, he found that although an infinite number of interactions was generated, the effects at each iteration were bounded – only a finite number of terms were necessary for any particular level of precision. A successful theory with arbitrarily many constants still worked, as long as there was a clear hierarchy of terms.

An effective field theory was not renormalizable in the way QED was, but as Wilson found the proliferation of terms could be contained such that only a finite number mattered for a particular calculation. The constraint is that an effective field theory has application only well below some energy scale. But, the very feature that it is formulated at a particular scale makes it in many ways the natural theory to use at that scale.

1.1.2 A Worked example of Renormalization

Let us review how the process of renormalization works when a theory *is* renormalizable.

The QED Lagrangian defined at some cut-off is:

$$\mathcal{L}_0 = \bar{\Psi}\left(i\partial\cdot\gamma - e_0 A\cdot\gamma - m_0\right)\Psi - \frac{1}{4}F^{\mu\nu})F^{\mu\nu} \tag{1.1.1}$$

and additionally the cut-off regulator Λ_0.

In calculating a process without a cut-off, all intermediate states must be summed over. When kinematics do not strictly dictate the intermediate momenta of some particles, this means integrating over an infinite range. This is exactly the case with an internal loop (as it described in the language of Feynman diagrams), and is what caused the infinities that plauged initial attempts to formulate relativistic quantum mechanics.

By introducing a cut-off, the integrals only take place over a bounded domain of momenta, and are thus finite. The cost is that physical quantities should not depend upon this arbitrary cut-off Λ_0. Still, the parameters can be fixed by comparing with experiment. This theory will then produce results correct up to some terms of $\mathcal{O}(1/\Lambda_0^2)$.

There are two parameters: the mass m_0 and the bare charge e_0. These parameters are determined from experiment. Two processes can be calculated (such as electron-electron scattering and the electron scattering off some external field), compared to the experimental measurements, and thus the parameters fixed.

Handed this theory with a high cut-off Λ_0, it is possible to reformulate it in terms of a new, lower cut-off Λ. The new theory will hold valid for processes where the external momenta are much less than Λ.

By introducing this lower cut-off, high energy virtual processes are eliminated from the theory. Rather than being explicitly included, their effects will implicitly be included by corrections to the parameters of the theory.

These corrections can be calculated from the original theory. Before, loop-integrals over momenta ran from 0 to the old cut-off Λ_0. In the new theory, they will run from 0 to Λ. Clearly the difference between the two calculations will be an integral from Λ to Λ_0. Importantly, because Λ is taken to be greater than the energy of any process considered, so will the loop momentum in that sector of the integral.

First consider the calculation of the electron vertex. Call the original value T, which will have contributions from several diagrams. On such contribution is from the one loop diagram, where a photon is exchanged between the fermion line before and after the interaction. Call this contribution $T^{(a)}$.

In the original $\mathcal{L}_0$ theory the contribution would be

$$T(a)(k>0) = -e_0^3 \int_0^{\Lambda_0} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \left\{ \bar{u}(p')\gamma^\mu \frac{1}{(p'-k)\cdot\gamma - m_0} A_{\text{ext}}(p'-p)\cdot\gamma \frac{1}{(p-k)\cdot\gamma - m_0}\gamma_\mu u(p) \right\} \tag{1.1.2}$$

When the momenta are cut-off at the lower point Λ, the part of the old integral missing from the new calculation will be

$$T(a)(k>\Lambda) = -e_0^3 \int_\Lambda^{\Lambda_0} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \left\{ \bar{u}(p')\gamma^\mu \frac{1}{(p'-k)\cdot\gamma - m_0} A_{\text{ext}}(p'-p)\cdot\gamma \frac{1}{(p-k)\cdot\gamma - m_0}\gamma_\mu u(p) \right\} \tag{1.1.3}$$

Remember that Λ is chosen to be a great deal greater than p, p' or m_0, and this then holds for k over the entire range of the integral. Then, if corrections of the type p/Λ are discarded, the integral can be greatly simplified. The approximation used is:

$$\frac{1}{(p'-k)\cdot\gamma - m_0} \approx -\frac{1}{k\cdot\gamma} \tag{1.1.4}$$

Of course for any four vector a, it holds that $(a\cdot\gamma)^2 = a^2$, so

$$-\frac{1}{k\cdot\gamma} = -\frac{k\cdot\gamma}{k^2} \tag{1.1.5}$$

Then

$$\begin{aligned} T^{(a)}(k>\Lambda) &\approx -e_0^3 \int_\Lambda^{\Lambda_0} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \left\{ \bar{u}(p')\gamma^\mu \frac{k\cdot\gamma}{k^2} A_{\text{ext}}(p'-p)\cdot\gamma \frac{k\cdot\gamma}{k^2}\gamma_\mu u(p) \right\} \\ &\approx -e_0^3 \bar{u}(p') A_{\text{ext}}(p'-p)\cdot\gamma u(p) \int_\Lambda^{\Lambda_0} \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} \end{aligned}$$

There are other one electron scattering diagrams. When the above analysis is applied to them all, the total difference is found to be of the form:

$$T^(k>\Lambda) = -ie_0 c_0 (\Lambda/\Lambda_0)\bar{u}(p') A_{\text{ext}}(p'-p)\cdot\gamma u(p) \tag{1.1.6}$$

This is the piece of the electron vertex structure that we are missing if we calculate using the original Lagrangian with the lower cut-off. The correct results will be obtained if we

incorporate into the Lagrangian a new term:

$$\delta\mathcal{L} = -e_0 c_0(\Lambda/\Lambda_0)\bar{u}(p')A_{\text{ext}}(p'-p)\cdot\gamma u(p) \tag{1.1.7}$$

What about the nature of this constant c_0? Consider the above calculation of $T^{(a)}(k > \Lambda)$. It had, like the total correction, had a structure of some constant terms times $\bar{u}(p')A_{\text{ext}}(p'-p)\cdot\gamma u(p)$. The structure of the constant term came from the integral over a function of only k. Since there were no scales involved in this integration, other than the limits of integration, the result must be some function of Λ and Λ_0. And because the integral is dimensionless, it must actually be a function of their ratio Λ/Λ_0. The same logic goes through when the other terms are computed. The final result is that

$$c_0 = -\frac{\alpha_0}{6\pi}\log(\Lambda/\Lambda_0) \tag{1.1.8}$$

That the result of these integrals involves only the limits is contingent upon the approximation made earlier, that the scales p, p' and m_0 are all small compared to Λ. If corrections of that order are important, then there will be additional terms with structures including these momenta and mass. This can be accomplished by, instead of completely neglecting these terms, doing Taylor expansion in terms of p/k, m_0/k and so forth. We'll return to this later.

Going back to the correction $\delta\mathcal{L}$, note that it has almost the same form as a term in the original $\mathcal{L}_0$, the difference being an explicit dependence on the cut-offs Λ, Λ_0. (Although of course, e_0 itself depended on comparing measurements to calculations in the $\mathcal{L}_0$ theory, so it really *was* dependent on Λ_0.) Rather than interpret it as new interaction, then, it can be seen as change in the strength of e_0.

$$-\bar{\Psi}e_0 A\cdot\gamma\Psi \to -\bar{\Psi}e_0[1 + c_0(\Lambda/\Lambda_0)]A\cdot\gamma\Psi \tag{1.1.9}$$

As long as all other scales are considered to small to Λ to enter the calculations, it isn't possible for truly new terms to enter, only corrections to the already existing terms. That means that all that can happen is an adjustment of the existing coupling constants e_0 and

m_0.

There are indeed corrections to m_0, coming from the electron self-energy. An additional correction term is required of the form

$$\delta\mathcal{L} = -m_0\tilde{c}_0(\Lambda/\Lambda_0)\bar{\Psi}\Psi \tag{1.1.10}$$

The Lagrangian valid with the new cut-off Λ can be written in terms of the old as:

$$\mathcal{L}_\Lambda = \bar{\Psi}\left(i\partial\cdot\gamma - e_\Lambda A\cdot\gamma - m_\Lambda\right)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{1.1.11}$$

where the constants are

$$\begin{aligned} e_\Lambda &= e_0(1+c_0) \\ m_\Lambda &= m_0(1+\tilde{c}_0) \end{aligned}$$

Above only two processes were considered, which produced corrections to known interactions. In principle corrections will also arise from other processes, such as electron-electron scattering.

But, consider the form of such cross-sections. They must have a spinor structure something like $\bar{u}u\bar{u}u$ or $\bar{u}\gamma^\mu u\bar{u}\gamma_\mu u$, with an accompanying factor. Certainly it involves four fermion fields. The key point is that the scattering amplitude with four external lines must be dimensionless. Since u has mass dimension $1/2$, d_0 must have dimension -2.

So to write a dimensionless factor d_0 analogous to c_0 above, there must be an additional factor of mass dimension -2. The relevent scale is Λ, so this factor must be $1/\Lambda^2$. (Again, we follow the earlier logic that no other mass scales can enter, being negligible to the highly virtual loop momentum of the correction terms.) So here the correction would need to look something like

$$d_0(\Lambda/\Lambda_0)\frac{1}{\Lambda^2}\bar{u}u\bar{u}u \tag{1.1.12}$$

The actual value of the spinors structure only involves the low energy momenta of the

theory, so it must be suppressed by the larger factor $1/\Lambda^2$. Therefore, these four fermion terms enter at a smaller order than the terms already discussed.

Nonrenormalizable cut-off theories

In the above, corrections small compared to the scale Λ were ignored, terms of order $\mathcal{O}(p/\Lambda)$, $\mathcal{O}(m/\Lambda)$ and so forth. If such terms are important, they may be calculated in the same general manner as outlined for the $\log \Lambda/\Lambda_0$ corrections. However, the logic above that prevented new terms from being introduced now fails, so new types of interaction are to be expected.

There are two key points where terms of this nature were discarded. The first was in the highly virtual loop integrals, where all scales smaller than k were neglected. The second was in considering the types of processes which might introduce corrections to the Lagrangian, where (for example) terms involving four fermions were subjected to dimensional analysis and found to be suppressed by $1/\Lambda^2$.

Going back to the loop integrals, now instead of simply discarding all terms involving p, p', or m_0, a Taylor expansion in p/Λ m_0/Λ (and so on) can be performed. As an example, instead of approximating $1/\{(p-k)\cdot\gamma) - m_0\}$ as $-1/k\cdot\gamma$

$$\begin{aligned}\frac{1}{(p-k)\cdot\gamma) - m_0} &\approx -\frac{1}{k\cdot\gamma(1 - p\cdot\gamma/k\cdot\gamma + m_0/k\cdot\gamma)} \\ &\approx -\frac{k\cdot\gamma}{k^2} + \frac{p\cdot\gamma}{k^2} - \frac{m_0}{k^2}\end{aligned}$$

Systematically using such expansions, the high energy part of the loop calculations unaccounted for by the new cut-off theory can be found. The general form is

$$T(k>\Lambda) = -iec_0\bar{u}(A_{\text{ext}}\cdot\gamma)u - \frac{ie_0m_0c_1}{\Lambda^2}\bar{u}(A^\mu_{\text{ext}}\sigma_{\mu\nu}(p-p')^\nu)u - \frac{ie_0m_0c_2}{\Lambda^2}(p-p')^2\bar{u}(A_{\text{ext}}\cdot\gamma)u \tag{1.1.13}$$

The form of these corrections can in some cases be found from explicit calculation. In such cases, the calculation will also fix the coefficients in the Lagrangian. However, often it

is either difficult or impossible to explicitly "integrate out" high momentum contributions to the theory. One can already see above how much more tedious the integrations over intermediate momenta will become when polynomial terms in p and so forth are included. There will be a proliferation of new terms to account for in the loop integrals, and if pushed to the next order of correction an unpleasant combinatorical explosion will occur. The hope is to use NRQED to *simplify* the calculation.

And there also exist theories where explicit calculation is simply impossible, and the use of an effective Lagrangian is not just a convenience but a necessity. In low energy QCD it is impossible to work with a perturbative expansion of Feynman diagrams. Because of the strength of the coupling, such series do not converge.

In either case, there is an alternate method. For any particular calculation and given level of precision there will be a finite number of terms in the Lagrangian that contribute. The existence of terms is limited by two factors:

- First, there are direct constraints on the form of the Lagrangian: current conservation, Lorentz invariance, chiral symmetry and so on, and these will apply to each term separately
- Second, only terms of up to a particular order are kept

It is this second point that ensures that the number of terms is finite. Typically each building block that might be used to construct a term comes with at least one power of mass dimension. And the greater the mass dimension of the term, the stronger the suppression by $1/\Lambda$. Building blocks of order unity do exist, such as spin space operators, but there will be a finite basis for such.

Once the form of all possible terms is catalogued, how then to fix the coefficients before each? It is important that, no matter what, physical predictions obtained from either the effective theory or the high energy theory must coincide. So if the same physical process is calculated using both theories, then demanding equality of the two results will determine the coefficients of the low energy theory in terms of the original.

Still, that in principle the coefficients can be obtained by integrating out high momentum loops can still tell us something of their behavior. When all other energy scales were ignored,

the constant c_0 could depend only on Λ/Λ_0. When they are included, it may additionally depend on m_0/Λ. It will never depend upon the momentum p, because such terms are instead included as new interactions with separate coefficients.

So c_0 will be the same coefficient calculated earlier, but with additional corrections of order $\mathcal{O}(m_0^2/\Lambda^2)$. The Lagrangian must also be augmented by the new interactions, so there is an additional correction of the form

$$\delta\mathcal{L} = \frac{e_0 m_0 c_1}{\Lambda^2}\bar{\Psi}F^{\mu\nu}\sigma_{\mu\nu}\Psi + \frac{e_0 m_0 c_2}{\Lambda^2}\bar{\Psi}i\partial_\mu F^{\mu\nu}\sigma_{\mu\nu}\partial_\nu\Psi \tag{1.1.14}$$

(In writing down the terms in the Lagrangian, momentum become derivatives of the fields, so for example $q = p' - p$ becomes a derivative of A_{ext}.)

In addition to these new higher order corrections to the already calculated quantities, there will be contributions to the Lagrangian of new processes. For instance, electron-electron scattering enters at the $\mathcal{O}(p^2/m^2)$. But of course it's not the case that all processes now enter — a process with 6 external legs would be suppressed by $1/\Lambda^4$ and not enter at the currently considered order.

These four fermion terms come from the contributions of other process than simple scattering off an external field. Rather, they come from integrating out the high momentum modes of processes such as electron-electron scattering. The loop diagram corrections to such processes involve in the high-energy theory, like the other loops mentioned, integrals over momentum higher than the cut-off. These intermediate states are highly virtual.

The uncertainty principle says that such high energy virtual states are allowed only if they exist for a correspondingly short amount of time. The result is that in the low energy theory they may be treated as effectively instantaneous interactions, appearing as local contact terms in the Lagrangian. This is how new multiple particle interactions arise in an effective theory – the high energy process becomes a new local interaction.

For electron-electron scattering at the order discussed, such terms would be the likes of

$$\delta\mathcal{L}_{4-\text{fermion}} = d_1\frac{e_0^2}{\Lambda^2}(\Psi^\dagger\Psi)^2 + d_2\frac{e_0^2}{\Lambda^2}(\Psi^\dagger\gamma\Psi)\cdot(\Psi^\dagger\gamma\Psi) \tag{1.1.15}$$

These coefficients would be fixed by calculating a process like electron-electron scattering in both theories. However, one would first have to fix the constants c_i. For in the new theory with a low cut-off there will be contributions to the scattering not only from contact terms, but also from tree level diagrams involving two 2-fermion vertices.

1.2 Nonrelativistic Quantum Electrodynamics

Above we explored how, by changing the cut-off in QED, new nonrenormalizable terms appeared, and a new effective theory emerged. It was suitable only for calculations below the new cut-off, which was chosen to be well above the scale of any other momenta or energy in the theory. However, one particularly fruitful use of this technique is the formulation of a theory of nonrelativistic quantum electrodynamics (NRQED).

NRQED is most useful when working with bound state systems. There are many important corrections to bound state energy levels that come from high energy physics. The more precisely one measures these energy levels, the more information is gained about the high energy theory. But only if the predictions of that theory have been worked out with the necessary precision.

While in principle the full QED theory can be used to accomplish this, it is unweildy and unsuited for the task. Most typically the systems studied are loosely bound and non-relativistic. While calculating non-relativistic scattering in QED might not be too bad, the situation is different for a bound state. What spoils everything is the existence of new energy scales.

One such scale is the inverse of the Bohr radius. If μ is the reduced mass of the system and Ze the charge of the center, then

$$p \sim Z\mu\alpha = \frac{1}{r_{\text{Bohr}}} \tag{1.2.1}$$

is the typical momentum scale of the bound system.

In a QED scattering calculation, the order of a term may be addressed as the number of loops in a diagram. Thus we talk about tree level diagrams, one-loop diagrams, two-loop diagrams and so on, with the understanding that each loop carries with it a suppressing

factor of α. In this bound state system the typical momentum scale may enter in a way that exactly cancels the loop factor. So to calculate contributions of order α an infinite number of diagrams must in fact be summed.

There are techniques of doing this, but clearly it becomes trickier to easily sort out what diagrams contribute at a particular order. The matter is made worse by the existence of a third distinct scale, the kinetic energy of each particle:

$$E = \frac{(Z\mu\alpha)^2}{m_i} \tag{1.2.2}$$

further complicating the process of calculating the contributions of a given order.

Well, QED is a high energy theory, while calculations should be easiest in a low energy theory formulated for this nonrelativistic regime. What is desired is a theory that lives at the appropriate scale, thus avoiding the business of summing infinite numbers of diagrams, but never-the-less incorporating all the relevant effects of high energy physics.

Of course, an effective field theory has exactly these characteristics. The procedure is as follows. First write down the most general Lagrangian that obeys the symmetries of the theory. Of course here, in going from the high energy theory to the effective nonrelativistic, Lorentz symmetry is no longer required.

Once the form of the Lagrangian is fixed, the same physical process may be calculated in QED and NRQED. There is no problem in performing the QED calculation in this step because we don't have to choose a bound-state calculation – the scattering of free particles will suffice. The idea here is to find the simplest calculation that will fix the coefficients.

It is these coefficients that then contain all the information about the high energy theory. Bound state calculations may then be performed using the NRQED Lagrangian and diagrams, with the desired result: high energy physics is included, but we have a workable theory in the nonrelativistic regime. Instead of figuring out how to find a correct perturbation series in α, the theory uses expansions in terms of the other two nonrelativistic (and thus small) scales; basically expanding in terms of the velocity v as well as α.

Before we examine how the process works explicitly for NRQED, there is an alternate approach that should be examined first.

1.2.1 The Foldy-Wouthuyesen-Tani Transformation

As sketched in the previous section, to work with bound state problems it is simplest to have a nonrelativistic Lagrangian to work with. After all, in regular quantum mechanics it isn't so hard to calculate the levels of hydrogen. It is using a relativistic theory to crack a nonrelativistic nut that causes problems.

If the goal is to derive a nonrelativistic Lagrangian or Hamiltonian for the system, one approach is to start from the Dirac equation and find a way to express it nonrelativistically.

One such approach is the Foldy-Wouthuyesen-Tani transformation. Consider the Dirac Lagrangian for an electron:

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi} \left(i\partial \cdot \gamma - eA \cdot \gamma - m \right) \Psi \tag{1.2.3}$$

If the system is nonrelativistic, then one of the important consequences is that electrons and positrons behave pretty much as independent particles. A representation can be chosen such that the upper and lower components of the bispinors Ψ are roughly equivalent to the electron and positron spinors. There will be some mixing, and it is the FWT transformation that finds a form for the bispinors in such a way that all the operators above become diagonal. Then, the upper and lower components are completely separated, and one can easily treat them as separate fields.

Technically this is impossible when an external field A_{ext} is present. However, a basis where the mixing between "particle" and "anti-particle" is arbitrarily small may be found perturbatively. The result will be an expansion in terms of $|\mathbf{p}|/m$, that relates the original upper and lower components to the desired "Schrodinger-like" spinors.

The second order result will be, for a Dirac bispinor u with upper and lower components ϕ and ξ:

$$u = \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} (1 - \frac{p^2}{8m^2})\phi_s \\ \frac{\sigma \cdot p}{2m}(1 - \frac{3p^2}{8m^2})\phi_s \end{pmatrix}$$

An important feature here is that the lower component is suppressed compared to the upper, by a factor of $\sigma \cdot p/2m$. Conservation of probability demands that, for a Schroginer-like

wave function, $\int \phi_S^\dagger \phi_S = 1$. By demanding equality with the relativistic probability density $\bar{u}\gamma^0 u$ the above form can be derived, without performing the actual transformation.

Once this is accomplished, the Lagrangian may be rewritten directly in terms of, say, the electron spinor. The relativistic bispinor has been replaced by a two-component nonrelativistic spinor. Additionally, all the other terms can be rewritten in terms of Galilean three vectors and scalars instead of Lorentz 4-vectors. The result is a manifestly nonrelativistic expression.

If an external electric field is considered acting on a single charged particle of mass m and charge e is considered, then to the second order

$$H = \frac{\mathbf{p}^2}{2m} + e\Phi - \frac{\mathbf{p}^4}{8m^3} - \frac{1}{4m^2}\boldsymbol{\sigma}\cdot\mathbf{E}\times\mathbf{p} - \frac{1}{8m^2}\boldsymbol{\nabla}\cdot\mathbf{E} \tag{1.2.4}$$

In writing the Hamiltonian, of course derivatives become momentum operators instead. Clearly, the higher order terms have relativistic origins — $\mathbf{p}^4/8m^3$, for instance, is the first relativistic correction to the kinetic energy $\mathbf{p}^2/2m$.

From this starting point ordinary Rayleigh-Schrodinger perturbation theory may be used. Because the theory is explicitly nonrelativistic, it will avoid all the scale problems with QED.

However, this isn't as powerful as an effective field theory. The idea here was to discard high energy processes rather than to incorporate them. For instance, the process

$$e^-e^+ \to \gamma \to e^-e^+ \tag{1.2.5}$$

involves a fundamentally relativistic intermediate photon that this formulation hasn't incorporated, and so relativistic corrections to the photon propagator will matter. If instead the techniques of effective field theory are employed, no process with such relativistic internal momenta need be considered. Instead, the effects of this process will be incorporated as four fermion contact terms.

As the uncertainty principle tells us, the higher the energy of the intermediate state, the shorter the length of the time that the state can exist. So a process with a relativistic internal

state as above should happen instantaneously, acting just like a local contact interaction between four fermions.

When higher-order perturbation theory is attempted with the above Hamiltonian, it also will fail. The terms diverge, producing infinities. This of course has the same root cause.

So, if high precision is important and all high energy processes must be accounted for, this method isn't accurate enough. It does, of course, predict the leading order coefficients of NRQED.

1.2.2 An NRQED Calculation

Let us now consider the effective field theory approach to finding, from the relativistic theory, a nonrelativistic Lagrangian suitable for the bound-state problems already mentioned. As an example, consider a hydrogen-like muonium system with an electron and a muon. Formally the idea is to introduce into QED a cut-off at the nonrelativistic energy, λ. This cut-off is somewhere about the energy of the electron mass m_e. The higher energy states are removed, leaving an explicitly nonrelativistic Lagrangian.

As normal for an effective field theory, this Lagrangian will have an inifinte number of terms, but they can be arranged in a hierarchy that allows any particular calculation to be performed with only a finite number. This hierarchy can be treated as an expansion over the large mass scale of the system, $1/m$. In the example calculation, terms up to order $1/m^3$ will be kept.

First a Lagrangian is written that contains all the possible terms that might contribute to the process considered. Lorentz symmetry is no longer required, but the following constraints on the forms of terms exist:

- Galilean invariance
- Invariance under spatial reflection
- Invariance under time reversal
- Gauge invariance

- Hermiticity
- Locality

Once all the terms are catalogued, their coefficients must be determined. The idea is to consider some particular physical process. For the two formulations to be consistent, they must produce the same predictions for any such process. So something like a scattering amplitude can be calculated in both theories, and the result compared.

1.2.3 Comparison of QED and NRQED

The simplest process that will give the required information can be used. So any assumption that still distinguishes between needed terms in the NRQED Lagrangian may be used. In comparing scattering diagrams, it is often most useful to perform the calculation at threshold. Likewise, the frame of reference which most readily simplifies the calculation should be used. Any physical assumption must be applied to each calculation in the same way, of course – it must be the same physical process!

However, given the assumption of gauge invariance, different gauges may be chosen for each calculation. Since the physical measurement will be gauge invariant, there is nothing inconsistent about this. And indeed the most convenient gauges for a relativistic and nonrelativistic calculation often differ.

Remember that the goal is to use NRQED to calculate bound state energies. To this end, it is necessary that the theory has poles in the complex plain and can thus be analytically continued to include off-shell bound states. To ensure this, it is necessary to demand that all external particles be on mass-shell when fixing the coefficients. For the same reason, it is necessary to perform all the intermediate calculations with a finite photon mass. **Why? I don't really follow this point very well.**

So that the processes we compare have the same meaning, it is necessary for the S-matrix to have the same normalisation in both theories. Because of this, the normal relativistic normalisation of the QED spinor cannot be used. It is conventional to set $\bar{u}u = 2m$, but this will make it hard to find a sensible relation between the spinors of QED and NRQED. Instead, the normalisation $u^\dagger u = 1$ will be used.

Given that all these conditions are met, the process of comparing physical results will fix, without ambiguity, the coefficients of NRQED.

1.2.4 Construction of NRQED Lagrangian

In order to perform any comparison with QED, it is first necessary to catalogue all the terms that could arise in the Lagrangian. Let us work first with only the two-fermion terms.

What are the building blocks at our disposal? Each term will have two electron fields, possibly the external electromagnetic field, and derivatives of either type of field. There can also be mixing between the spin-space of the fermion fields.

The first constraint to apply is gauge-invariance. If only gauge invariant combinations are allowed, then instead of any mixture of derivatives and fields, only the long derivative $D = \partial + ieA$ is allowed, along with E, B and their derivatives.

The operators that mix spin for the spin-1/2 electron are just the Pauli spin matrices σ_i. Together with the identity they form a basis for the space of Hermitian spin-operators, so no quadratic terms in spin can appear. Those can be directly reduced with the identity $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}S_k$.

Many of these "blocks" are written as three-vectors. For Galilean invariance to hold, there can be no dangling indices — all terms must be contracted, either with each other or the anti-symmetric tensor ϵ_{ijk}.

Consider now the symmetries of each type of term.

- The electric field is Hermitian, odd under parity and even under time reversal.
- The magnetic field is Hermitian, even under parity and odd under time reversal
- The long derivative is anti-Hermitian, odd under parity, and even under time reversal
- Spin is Hermitian, even under parity, odd under time reversal.

Given a collection of these blocks, then as long as the set as a whole isn't anti-Hermitian we can always simply add the Hermitian conjugate to form a Hermitian term. This does reduce the number of independent terms. Likewise, to make a term invariant under time-reversal a factor of i can be appended. However, there is no way to fix a term that is odd

under parity, so that is the one hard constraint. There can be no term $\mathbf{B} \cdot \mathbf{D}$, for instance, because it is not invariant under parity.

So far these constraints still allow for an infinite number of terms. But the greater the number of fields, or derivatives of fields, that exist in a term, the more strongly it will be suppressed. Each additional power of energy will carry a factor of $1/m$ to ensure that the total dimension of the term is allowed.

To figure out how far down the hierarchy of terms we need to go for a particular calculation, we also need to establish the individual order of magnitude of each term and "building block". This will allow us to eliminate all but a finite number of terms from consideration. The example case is an electron in an atomic system, so A_0 is the Coulomb potential, and $\mathbf{B}$ comes from interaction with the nucleus, in this case a muon.

.

$$\begin{aligned} \partial &\sim mv \\ \partial_t &\sim mv^2 \\ eA_0 &\sim mv^2 \\ e\mathbf{A} &\sim mv^3 \\ e\mathbf{E} &\sim m^2v^3 \\ e\mathbf{B} &\sim m^2v^4 \end{aligned}$$

The magnitudes of the fields can be readily derived from their atomic origin. The spin operator is order unity (when we have $\hbar = 1$.)

For the Coulomb potential,

$$e\Phi \sim \frac{Z\alpha}{r} \tag{1.2.6}$$

For an atom, $1/r \sim mv = m(Z\alpha)$. So the order of $e\Phi$ is mv^2.

The electric field is the first derivative of Φ, so

$$eE_i \sim \frac{Z\alpha}{r^3} r_i \sim m^2(Z\alpha)^3 = m^2v^3 \tag{1.2.7}$$

There are two sources for a magnetic field acting on the electron. The first is the spin-orbit coupling, a result of a spinning particle moving in an electric field. There is also a spin-spin coupling between the spins of the nucleus and electron, but here it is the weaker of the two effects.

The order of the magnetic field can be found by considering a simple nonrelativistic approximation. Consider a shift to the frame of the electron. It then sees the nucleus moving along at speed v. This predicts a magnetic field of

$$B = q\frac{\mathbf{v} \times \mathbf{r}}{r^3} = \mathbf{v} \times \mathbf{E} \tag{1.2.8}$$

Now, this is not truly accurate, because relativistic effects are ignored. However, they do not change the overall order, so B must go as m^2v^4.

Since B is the derivative of $\mathbf{A}$, A is of order mv^3.

To be consistent with the Schrodinger equation, $\partial_t \sim mv^2$.

When acting on the fermion wave functions, the spatial derivative will bring down a factor of p. Upon the electromagnetic fields, it gives an overall $1/r$. In both these cases, ∂ can be considered to be of order mv

Knowing both the order and symmetries of each building block, the NRQED Lagrangian can be explicitly constructed.

It is the suppressing powers of $1/m$ that keep the number of terms to be considered finite. A term with six derivatives like D^6 need not be considered, because it must appear in $\mathcal{L}_{NRQED}$ as $\mathbf{D}^6/32m^5$. So the first step is to consider what combination of such terms *are* allowed, if terms of higher order than mv^4 are discarded. The only allowed spin structure is σ.

The leading order terms appear at mv^2. Since E and B are already too high order, the only available terms are A_0, ∂_0, and D_i. Without spin, A_0 and ∂_0 appear by themselves. A single power of D_i has nothing to contract with, and two powers may only be contracted with each other. So the spinless terms are

$$eA_0,\ \partial_0,\ \frac{\mathbf{D}^2}{2m} \tag{1.2.9}$$

A single power of D contracted with σ is disallowed by parity, and something like $(\boldsymbol{\sigma}\cdot\mathbf{D})^2$ is redundant, since quadratic terms in σ reduce. Only linear combinations of σ need be considered. None of the other ingredients have vector indices, so no spin terms appear at this order.

The next order of terms are mv^3. This could be a single power of E or three powers of D. However, since E and D are both odd under parity, those terms are not allowed, and nothing arises at this order.

The only one left is a single power of B. Since the index needs to be contracted with something, it must be the order unity spin matrix. Thus, the only term allowed at this order is $\boldsymbol{\sigma}\cdot\mathbf{B}$:

$$\frac{e}{m}\boldsymbol{\sigma}\cdot\mathbf{B} \tag{1.2.10}$$

At the next order, something like $\mathbf{B}\cdot\mathbf{D}$ is forbidden by parity.

At order mv^4, there could be four powers of D, or a single power of D accompanied by E. With four powers of D, the only way to contract them is as D^4. With E and D there are two ways to contract them: either with δ_{ij} or with $\sigma_k\epsilon_{ijk}$. In any case the resulting term must be Hermitian and invariant under time reversal, so the two terms are

$$\frac{\mathbf{E}\cdot\mathbf{D}-\mathbf{D}\cdot\mathbf{E}}{4m^2},\ \frac{i\epsilon_{ijk}(E_iD_j-D_iE_j)}{4m^2} \tag{1.2.11}$$

Finally at order $1/m^3$ there are more terms involving B. One power of B and two powers of D is allowed. Because there are an odd number of indices, and contracting ϵ_{ijk} with two powers of D is redundant (producing an $\mathcal{O}(mv^6)$ term B^2), all the terms will involve σ_i.

The allowed combinations are

$$\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B}+\mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3},\ \frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3},\ \frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D})+(\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3} \tag{1.2.12}$$

Of course in the special case where derivatives of B vanish (i.e. B commutes with D) then the first two terms become indistinguishable. That is not the case here.

Now we have the Lagrangian, including terms up to order $1/m^3$.

What will the overall structure of the Lagrangian be? There will be two fermion terms,

four-fermion terms, and photon terms. The photon terms are taken as just that of QED at leading order: $\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$.

Now examine the two-fermion part in more detail. It will include kinetic terms that are not renormalized and thus need no coefficients. The leading order terms will go as $1/m$, and there will then be corrections coming from additional terms.

Call the coefficients in this two-fermion Lagrangian c_i. We can expect that the leading order coefficients of these two-fermion terms should replicate exactly the results of doing a Foldy-Wouthuysen-Tani transformation. So for convenience they can be written such that they are, if they exist at this order, equal to 1. So such coefficients will have the form

$$c_i = 1 + \mathcal{O}(\alpha) = 1 + c_i^{(1)}\alpha + c_i^{(2)}\alpha^2 + \dots \tag{1.2.13}$$

If we consider terms up to $1/m^3$, then the two-fermion part of the Lagrangian is:

$$\begin{aligned}\mathcal{L}_{NRQED} =& \Psi^\dagger \Bigg\{ iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^2} + c_F\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + c_D\frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2} \\ &+ c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}}{8m^2} + c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3} \\ &+ c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3} \Bigg\}\Psi\end{aligned} \tag{1.2.14}$$

In addition to the two-fermion terms, at the order $1/m^2$ four fermion contact terms are allowed. Since the QED Lagrangian has no exact four-fermion contact terms, these are terms that arise from the removal of four-fermion diagrams involving high momenta loops. In contrast with the two-fermion Lagrangian, label the coefficients of such terms d_i.

These coefficients will have a more complex structure than c_i. Any process involving two distinct fermion fields will have a richer set of energy scales and parameters to draw from.

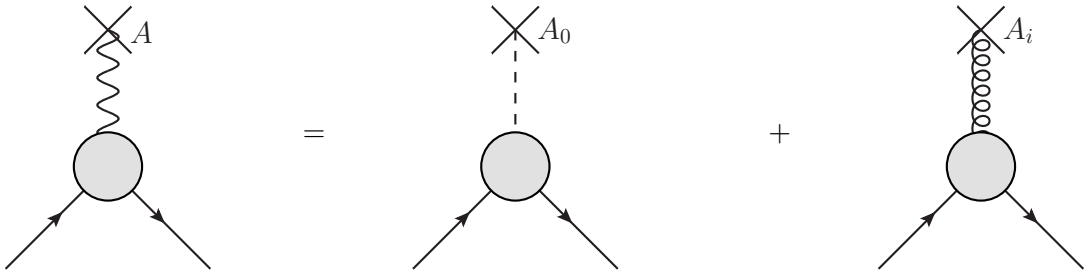
If the additional fermion has mass M and spinor χ, then the contact Lagrangian is:

$$\begin{aligned}\mathcal{L}_{\text{contact}} =& d_1 \frac{1}{mM}(\Psi^\dagger \boldsymbol{\sigma}\Psi)\cdot(\chi^\dagger\boldsymbol{\sigma}\chi) + d_2\frac{1}{mM}(\Psi^\dagger\Psi)(\chi^\dagger\chi) \\ &+ d_3\frac{1}{mM}(\Psi^\dagger\boldsymbol{\sigma}\chi)\cdot(\chi^\dagger\boldsymbol{\sigma}\Psi) + d_4\frac{1}{mM}(\Psi^\dagger\chi)(\chi^\dagger\Psi)\end{aligned} \tag{1.2.15}$$

The terms with coefficients d_3 and d_4 only enter if the additional fermion is actually the anti-particle of the original.

1.2.5 Determination of coefficients

To determine the coefficients of the two-fermion piece of the NRQED Lagrangian, it will suffice to calculate the scattering of the electron off an external field. Every coefficient that needs to be fixed accompanies at least one term with a single power of the photon field. First the scattering amplitude will be calculated in QED, then compared to the NRQED terms. Schematically the equivalence can be written as:



A general one-photon vertex in QED may be expressed in terms of form factors $F_1(q^2)$ and $F_2(q^2)$. These encode all the information about radiative corrections, so the QED calculation here can conveniently be expressed with such factors.

The amplitude of scattering off a static vector potential is

$$e\bar{u}(p')\left[-\boldsymbol{\gamma}\cdot\mathbf{A}(\mathbf{q})F_1 + \frac{i}{2m}\sigma^{ij}A^i q^j F_2\right]u(p) \tag{1.2.16}$$

or in terms of Pauli spinors instead, up to $1/m^3$

$$
\begin{aligned}
=&F_1\phi_S^\dagger\left[-\frac{e}{2m}(\mathbf{p}'+\mathbf{p})\cdot\mathbf{A}-\frac{ie}{2m}\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A}+\frac{ie}{8m^3}(\mathbf{p}'^2+\mathbf{p}^2)\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A}\right]\phi_S\\
&+F_2\phi_S^\dagger\left[-\frac{ie}{2m}\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A}+\frac{ie}{16m^3}(\mathbf{p}'^2+\mathbf{p}^2)\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A}+\frac{ie}{8m^3}(\boldsymbol{\sigma}\cdot\mathbf{p}')(\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A})(\boldsymbol{\sigma}\cdot\mathbf{p})\right]\phi_S
\end{aligned}
\tag{1.2.17}
$$

For scattering off a static potential A_0

$$
e\bar{u}(p')\left[\gamma^0A^0F_1-\frac{i}{2m}\sigma^{0j}A^0q^jF_2\right]u(\mathbf{p}) \tag{1.2.18}
$$

Again expressing in terms of Pauli spinors

$$
\begin{aligned}
=&F_1\phi_S^\dagger\left[eA^0-\frac{e}{8m^2}\mathbf{q}^2+\frac{ie}{4m^2}\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p})A^0\right]\phi_S\\
&+F_2\phi_S^\dagger\left[-\frac{e}{8m^2}\mathbf{q}^2+\frac{ie}{4m^2}\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p})A^0\right]\phi_S
\end{aligned}
\tag{1.2.19}
$$

So the electron-external field scattering, as calculated in QED, has been expressed in terms of mostly nonrelativistic quantities. There are still the form factors to expand, which depend on $q^2=(q-0)^2-\mathbf{q}^2$. But q_0 is of lower order than $\mathbf{q}$, so the leading order corrections will involve only $\mathbf{q}^2$. They are

$$
\begin{aligned}
F_1(q^2) &= 1-\frac{\alpha}{3\pi}\left[\frac{\mathbf{q}^2}{m^2}\left(\ln\frac{m}{\lambda}\right)\right]\\
F_2(q^3) &= a_e-\frac{\alpha}{\pi}\frac{\mathbf{q}^2}{12m^2}
\end{aligned}
$$

They contain an explicit dependence on the finite photon mass λ.

The first part of the calculation of scattering from the NRQED Lagrangian is straightforward — only tree level vertices enter the calculation, so it can be read directly off the Lagrangian. However, this would fix the coefficients as having a direct dependence on the photon mass, rather than, as one might expect, the value of the cut-off. For instance, the

coefficient of the Darwin term would be found to be

$$c_D^{QED} = 1 + \frac{\alpha}{\pi}\frac{8}{3}\left[\ln\left(\frac{m}{\lambda}\right) - \frac{3}{8}\right] \tag{1.2.20}$$

But of course, in calculating from NRQED at this level of precision perturbation theory terms will also enter. This introduces a further renormalization of the coefficients, in a similar manner to the production of counterterms in QED. Without further correction these perturbations would spoil the agreement of QED and NRQED. The solution is to introduce an additional term in the NRQED Lagrangian that simply subtracts off the unwanted term.

To again use the Darwin term as an example, it was found that in the absence of perturbation theory the value for c_D labelled c_D^{QED} would cause QED and NRQED to predict the same result. However, when perturbations are taken into account there will be an *additional* contribution, of the same form of the Darwin term, with some coefficient we can call c_D^{NRQED}. So the actual value of c_D should be adjusted in order to bring the two theories back into agreement:

$$c_D = c_D^{QED} - c_D^{NRQED} \tag{1.2.21}$$

The particular value found for c_D^{NRQED} is

$$c_D^{NRQED} = -\frac{\alpha}{\pi}\frac{8}{3}\left[\ln\left(\frac{\lambda}{2\Lambda}\right) + \frac{5}{6}\right] \tag{1.2.22}$$

Which means

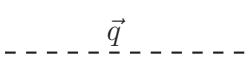
$$c_D = 1 + \frac{\alpha}{\pi}\frac{8}{3}\left[\ln\left(\frac{m}{\lambda}\right) - \frac{3}{8} + \frac{5}{6}\right] \tag{1.2.23}$$

With the result that the final correction to this coefficient depends upon the cut-off, just as with the regular renormalization theory. The same idea goes through to each other coefficient.

1.2.6 Vertices

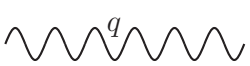
To formally establish rules for the various vertices written above, it is necessary to strip off the external fields, and write derivatives of such fields in terms of their momentum. The coulomb potential A_0 becomes a Coulomb photon, so (for example) $\boldsymbol{\nabla} \cdot \mathbf{E}$ becomes the Darwin vertex, consisting of two fermion legs and a Coulomb photon, and a vertex with a factor of $-(e/8m^2)\mathbf{q}^2$.

Coulomb photon propagator



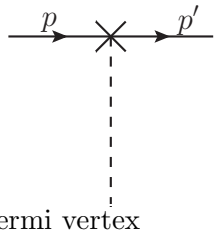
$$= \quad \frac{1}{\mathbf{q}^2+\lambda^2}$$

Transverse photon propagator



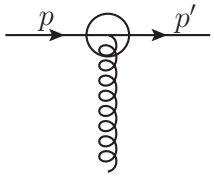
$$= \quad \frac{\delta^{ij}-\frac{q^i q^j}{\mathbf{q}^2-\lambda^2}}{(q^0)^2-\mathbf{q}^2-\lambda^2+ie}$$

Darwin vertex



$$= \quad \frac{-e}{8m^2}\left|\mathbf{p}'-\mathbf{p}\right|^2$$

Fermi vertex



$$= \quad \frac{ie}{2m}(\mathbf{p}'-\mathbf{p})\times\boldsymbol{\sigma}$$

Chapter 2

Spin one-half

2.1 Structure of spin one-half theory

The goal is to obtain a nonrelativistic theory from the relativistic. To that end it'll help to have a clear understanding of the structure of the two theories.

2.1.1 Relativistic framework for spin one-half

In the relativistic theory, the electrons are part of a fermion field that also includes the positron anti-particle. Since both are spin-1/2, each has two spin orientations. There are, then, a total of four degrees of freedom.

The relativistic Lagrangian for the fermion fields is

$$\mathcal{L} \quad = \quad \bar{\Psi}(i\partial \cdot \gamma - eA \cdot \gamma - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

The fermion fields are Ψ, the photon field is A. m is the particle's mass, and e the electron's charge. The γ matrices mix the different components.

It will be convenient to work in a representation which already suggests the nonrelativistic behavior. At low momenta, it should be expected that the free electron and positron fields act approximately as independent fields. This is exactly the case for the Dirac representation. In the rest frame, a free particle can be said to be definitively an electron or positron, and in the Dirac representation these correspond to the upper and lower parts of

the bispinor.

In this representation, the gamma matrices are written

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1.1}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{2.1.2}$$

The gamma matrices by themselves do not form a complete basis for this space. To find such a basis products of the matrices can be considered. It will make sense to consider combinations such that bilinears are Hermitian.

There is the identity — such bilinears transform as a scalar.

Symmetric combinations are not considered because $\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$. The antisymmetric combinations are explicitly

$$\begin{aligned} [\gamma^0, \gamma^i] &= 2\gamma^0\gamma^i = \begin{pmatrix} 0 & 2\sigma_i \\ 2\sigma_i & 0 \end{pmatrix} \\ [\gamma^i, \gamma^j] &= \begin{pmatrix} -2i\epsilon_{ijk}\sigma_k & 0 \\ 0 & -2i\epsilon_{ijk}\sigma_k \end{pmatrix} \end{aligned}$$

Using these a tensor like structure arises:

$$\sigma^{\mu\nu} = i\frac{1}{2}[\gamma^\mu, \gamma^\nu] \tag{2.1.3}$$

The specific form in the Dirac representation is

$$\sigma^{ij} = \frac{i}{2}[\gamma^i, \gamma^j] = \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \tag{2.1.4}$$

$$\sigma^{0i} = i\gamma^0\gamma^i = \begin{pmatrix} 0 & i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \tag{2.1.5}$$

The product of each gamma matrix in turn gives a pseudo-scalar:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.1.6}$$

The product of γ^5 and γ^μ gives a pseudo-vector

$$\gamma^5\gamma^\mu \tag{2.1.7}$$

$$u = \begin{pmatrix} \eta \\ \chi \end{pmatrix} \tag{2.1.8}$$

2.1.2 Nonrelativistic framework for spin one-half

$$\begin{aligned} \eta &= \left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w \\ \chi &= \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\left(1 - \frac{3\mathbf{p}^2}{8m^2}\right) w \end{aligned}$$

2.2 Foldy-Wouthyusen approach

The goal is to derive a nonrelativistic Hamiltonian or Lagrangian starting from relativistic theory. (Having obtained one, we can easily obtain the other, of course.) One method is to take the relativistic equations of motion and use them to obtain a Schrodinger like equation.

The starting point is the relativistic equations of motion, which can come from the Lagrangian of the relativistic theory. Those equations can then be written in terms of the noncovariant quantities that appear in the nonrelativistic theory. In doing so, the energy of the particle will now explicitly appear.

The relativistic theory will contain not only the particle of interest (the electron) but also its anti-particle (the positron.) The nonrelativistic theory should contain only the electron.

Before obtaining an expression for the nonrelativistic Hamiltonian it will be necessary to somehow disentangle the two fields. This is impossible in the general case, but as long as the energy and momenta in question are nonrelativistic, can be accomplished to any desired order.

Formally this is accomplished by the Foldy-Wouthyusen transformation, the result of which is that all operators are diagonal, the coupling between the particle and anti-particle suppressed to which ever order is desired. However, practically the same result can be obtained by examining the normalisation of the two theory's particles. By demanding that the Schrodinger like wave functions are appropriately normalized, the relationship between relativistic and nonrelativistic spinors can be established.

The result of this procedure will be an equation for the energy of the electron, accurate at some order in the nonrelativistic expansion. However, it will not perfectly replicate the predictions of the high energy theory. Unlike the process of NRQED, it does not truly incorporate the high energy sector of the theory.

2.2.1 Equations of motion

To reestablish the problem considered, the system to be examined is an electron placed in a loosely bound system with another charged particle, subject to an infinitesimal and constant magnetic field. There will be, because of the bound system, an electric field acting on the electron as well as the external magnetic field. When recoil effects are ignored, the electric field can just be taken as given.

The corrections to the gyromagnetic ratio of the electron are to be established. There are two small scales that appear in the problem, the velocity v of the electron and the infinitesimally small magnetic field eB. The precision desired requires terms of up to order mv^4 and $(e/m)Bv^2$.

The starting point will be the relativistic Lagrangian of Dirac. However, remember that the technique to be used simply ignores behavior introduced by the high energy sector of the theory, even if it might effect the low energy behavior. One such effect is corrections to the free gyromagnetic ratio of the electron, which first arise when one loop diagrams are considered. Without such corrections the g-factor will be exactly $g = 2$.

Knowing that there will actually be bound-state corrections proportional to $g-2$, it is necessary to somehow include this anomalous term. The way to do so is to introduce a new local interaction into the Lagrangian, coming from the high energy radiative corrections which dress the electron vertex. The Lagrangian to be used is, then

$$\mathcal{L} \quad = \quad \bar{\Psi}(\not{D} - m)\Psi + \frac{1}{2}\mu'\bar{\Psi}\sigma^{\mu\nu}F_{\mu\nu}\Psi$$

μ' is the correction to the classical moment $\mu_0 = \frac{e}{2m}$, and is equal to $(g-2)/2\mu_0$. D is the long derivative $\partial + ieA$.

From this Lagrangian the equations of motion of the particle may be obtained from the Euler-Lagrange method. The Euler-Lagrange equation is

$$\frac{\partial\mathcal{L}}{\partial\bar{\Psi}} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\Psi})} = 0 \tag{2.2.1}$$

In the Lagrangian above, we can consider that all differential operators act only on the right field Ψ. (This freedom of choice comes from being able to rewrite the Lagrangian through integration by parts, without changing its physical meaning.) So the second term in the Euler-Lagrange equation can be ignored, and after differentiating with respect to $\bar{\Psi}$ the following equation is obtained:

$$(i\not{D} - m + \frac{1}{2}\mu'\sigma^{\mu\nu}F_{\mu\nu})\Psi = 0 \tag{2.2.2}$$

Writing explicitly in terms of the γ matrices, this is

$$\left((i\partial_\mu - eA_\mu)\gamma^\mu - m + i\mu'\frac{1}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right)\Psi = 0 \tag{2.2.3}$$

This equation of motion is invariant under Lorentz transformations. It is written in terms of the Dirac bispinor Ψ, external fields A_μ and $F_{\mu\nu}$, and the gamma matrices. To apply in to a nonrelativistic problem, the very first step will be to rewrite it in terms of the sorts of quantities that appear in that domain: three-vectors and scalars.

The scalars that appear will be $\partial_0 = \partial_t$ and $A_0 = \Phi$. The external fields $\mathbf{E}$ and $\mathbf{B}$

will appear explicitly, while the vector field $\mathbf{A}$ will appear in the gauge-invariant operator $\boldsymbol{\pi} = \mathbf{p} - e\mathbf{A}$. The gamma matrices can be written in terms of the Pauli spin matrices $\boldsymbol{\sigma}$. Finally, the bispinor will be written in terms of its upper and lower components.

Of the terms that appear in (2.2.3), all except the last are trivial to write in this manner. To deal with that last term, the antisymmetric tensor $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ needs to be written explicitly.

Using the antisymmetry of $\sigma^{\mu\nu} \equiv$, and that we deal with time-independent fields:

$$\begin{aligned} F_{\mu\nu}\sigma^{\mu\nu} &= F_i\sigma^{ij} - F_{0i}\sigma^{0i} - F_{i0}\sigma^{i0} + F_{00}\sigma^{00} \\ &= F_{ij}\sigma^{ij} - 2F_{0i}\sigma^{0i} \\ &= 2\partial_i A_j \sigma^{ij} - 2\partial_i \Phi \sigma^{0i} \\ &= -2i \begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{B} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\mathbf{B} \end{pmatrix} - 2 \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{E} \\ \boldsymbol{\sigma}\cdot\mathbf{E} & 0 \end{pmatrix} \end{aligned}$$

So far the discussion has been in position space. To work out the nonrelativistic form it will be easier to talk about the equations of motion directly in terms of energy and momentum. So replace $i\partial_t$ with p_0, and $i\partial_i$ with p_i. Likewise, replace the gauge invariant derivative $iD_i = \pi_i = p_i - eA_i$.

A solution of definite momentum p to the equation is written in terms of upper and lower components

$$u = \begin{pmatrix} \eta \\ \chi \end{pmatrix} \tag{2.2.4}$$

With these considerations, the (2.2.3) can be rewritten acting explicitly on the bispinor.

$$\left\{ \begin{pmatrix} p_0 - e\Phi - m & 0 \\ 0 & -p_0 + e\Phi - m \end{pmatrix} + \begin{pmatrix} 0 & -\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \\ \boldsymbol{\sigma}\cdot\boldsymbol{\pi} & 0 \end{pmatrix} + \mu' \left[\begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{B} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\mathbf{B} \end{pmatrix} - i \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{E} \\ \boldsymbol{\sigma}\cdot\mathbf{E} & 0 \end{pmatrix} \right] \right\} \begin{pmatrix} \eta \\ \chi \end{pmatrix} = 0 \tag{2.2.5}$$

This gives rise to exact coupled equations for η and χ. So far this is in principle the same as the relativistic equation, only the form in which it is written is non covariant.

2.2.2 Nonrelativistic limit

The particle under consideration is a nonrelativistic electron. Roughly, the expectation is that η corresponds to the electron field and χ to that of the positron. The off-diagonal terms in the equation above represent some sort of mixing between the electron and positron: the electron wave function still has some small positron component, that decreases as momentum is decreased. The off diagonal component that does *not* vanish at 0 momentum is proportional to μ', the term introduced to account for a high-energy process.

The upshot is that although the equation above is really a set of coupled equations for η and χ, χ will be small compared to η — the very leading order diagonal term will indicate that $\chi \sim \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\eta$.

Because the off diagonal terms are small, the set of coupled equations may be solved perturbatively. The particular quantity of interest is the nonrelativistic energy of the particle $\epsilon = p_0 - m$. For a free particle, this would be

$$\epsilon = p_0 - m \approx \frac{\mathbf{p}^2}{2m} + \frac{\mathbf{p}^4}{8m^3} + \mathcal{O}\left(\frac{\mathbf{p}^6}{m^5}\right) \tag{2.2.6}$$

In order to perform this perturbative analysis the order of various terms needs to be established. It's evident that at leading order $\epsilon \sim mv^2$. From earlier analysis, $\Phi \sim mv^2$, $\pi \sim mv$, and $\mathbf{E} \sim m^2v^3$.

First, find an expression for χ in terms of η. The second of the set of equations represented by (2.2.5) is

$$(-p_0 + e\Phi - m)\chi + \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\eta + \mu'(\boldsymbol{\sigma}\cdot\mathbf{B}\,\chi - i\boldsymbol{\sigma}\cdot\mathbf{E}\,\eta) \tag{2.2.7}$$

Writing $p_0 = \epsilon + m$, and grouping terms, the result is that

$$\left(\epsilon + 2m - \mu'\boldsymbol{\sigma}\cdot\mathbf{B}\right)\chi = (\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - i\boldsymbol{\sigma}\cdot\mathbf{E}\,)\,\eta \tag{2.2.8}$$

It is necessary now to approximate χ in terms of η. Because ϵ and $|B|$ are smaller than m,

and only second order terms are needed for the final result:

$$\chi \approx \frac{1}{2m}\left(1 - \frac{\epsilon - e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B}}{2m}\right)(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - i\mu'\boldsymbol{\sigma}\cdot\mathbf{E})\phi \tag{2.2.9}$$

With this expression χ may be eliminated from the first of the set of equations (at least at the necessary order). The resulting equation will only involve η, and so may be used to solve for energy ϵ of η.

The original equation is

$$(p_0 - e\Phi - m)\eta - \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\chi + \mu'(\boldsymbol{\sigma}\cdot\mathbf{B}\,\eta - i\boldsymbol{\sigma}\cdot\mathbf{E}\,\chi) \tag{2.2.10}$$

So again using $p_0 - m = \epsilon$

$$\epsilon\eta = (e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B})\eta + (\boldsymbol{\sigma}\cdot\mathbf{E} + \boldsymbol{\sigma}\cdot\boldsymbol{\pi})\chi \tag{2.2.11}$$

Now the expression for χ in terms of η may be used.

$$\epsilon\eta \approx (e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B})\eta + (\boldsymbol{\sigma}\cdot\mathbf{E} + \boldsymbol{\sigma}\cdot\boldsymbol{\pi})\frac{1}{2m}\left(1 - \frac{\epsilon - e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B}}{2m}\right)(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - i\mu'\boldsymbol{\sigma}\cdot\mathbf{E})\eta \tag{2.2.12}$$

Writing the $1/m$ and $1/m^2$ terms separately:

$$\begin{aligned}\epsilon\eta \approx \Bigg\{ & e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} + i\mu'\boldsymbol{\sigma}\cdot\mathbf{E})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - i\mu'\boldsymbol{\sigma}\cdot\mathbf{E})}{2m} \\ & + \frac{1}{4m^2}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} + i\mu'\boldsymbol{\sigma}\cdot\mathbf{E})(\mu'\boldsymbol{\sigma}\cdot\mathbf{B} - [\epsilon - e\Phi])(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - i\mu'\boldsymbol{\sigma}\cdot\mathbf{E})\Bigg\}\eta\end{aligned} \tag{2.2.13}$$

Several of the terms are of too high order to consider. A term with both E and B, for instance, will be of higher order than $(e/m)\,|B|\,v^2$. Likewise, a term of $E\Phi$ or $E\epsilon$ is also too small. Dropping all such:

$$\epsilon\eta \approx \left\{e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2 - i\mu'[\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,,\boldsymbol{\sigma}\cdot\mathbf{E}]}{2m} + \frac{1}{4m^2}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,(\mu'\boldsymbol{\sigma}\cdot\mathbf{B} - [\epsilon - e\Phi])\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right\}\eta \tag{2.2.14}$$

This is an expression for the energy ϵ of the particle, in terms of operators. This will yield the nonrelativistic Hamiltonian. There is still some manipulation required, though,

because the right hand side also contains ϵ. But since the leading order terms don't, it may be perturbatively solved for. (The above expression could be simplified somewhat, using the properties of σ matrices for instance, but for now it is more convenient to write it compactly.)

To that end, the Hamiltonian can be split into leading order and second order terms. The leading order will be of mv^2 and $(e/m)B$, while the next order will be suppressed by an additional factor of v^2. Because the magnetic field is infinitesimally small no B^2 terms are needed. Since the leading order term in H is $\mathcal{O}(mv^2)$, this suggests we split it into two parts: $H = H_0 + H_1 + \mathcal{O}(mv^6, (e/m)Bv^4)$, where H_1 consists of only second order terms.

$$\begin{aligned} \hat{H}_0 &= e\Phi - \mu' \boldsymbol{\sigma}\cdot\mathbf{B} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{2m} \\ \hat{H}_1 &= -\frac{i\mu'}{2m}[\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,, \boldsymbol{\sigma}\cdot\mathbf{E}] + \frac{1}{4m^2}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,(\mu'\boldsymbol{\sigma}\cdot\mathbf{B} - [\epsilon - e\Phi])\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \end{aligned}$$

H_1 contains ϵ, along with other terms of total order mv^2. So to eliminate ϵ from H_1 it'll only be necessary to find it to leading order.

$$\begin{aligned} \epsilon\eta &= \left(\hat{H}_0 + \mathcal{O}(mv^4)\right)\eta \\ &\approx \left(e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{2m}\right)\eta \end{aligned}$$

The operators on the right hand side, operating on η, produce ϵ. The combination actually needed is $\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\epsilon\boldsymbol{\sigma}\cdot\boldsymbol{\pi}$. To that end, start with $\boldsymbol{\sigma}\cdot\boldsymbol{\pi}^{\,2}\epsilon$ and use commutation relations.

$$\begin{aligned} (\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2(\epsilon - e\Phi)\eta &= (\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\left(\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{2m} - \mu'\boldsymbol{\sigma}\cdot\mathbf{B}\right)\eta \\ \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,(\epsilon - e\Phi)\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\eta &= \left(\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^4}{2m} - \mu'(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\boldsymbol{\sigma}\cdot\mathbf{B} - \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,[e\Phi, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}]\right)\eta \end{aligned}$$

With this ϵ is eliminated, leaving:

$$\hat{H}_1 = -\frac{i\mu'}{2m}[\boldsymbol{\sigma}\boldsymbol{\pi}\,, \boldsymbol{\sigma}\mathbf{E}] + \frac{1}{4m^2}\left(\mu'\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\boldsymbol{\sigma}\cdot\mathbf{B}\,\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^4}{2m} + \mu'(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\boldsymbol{\sigma}\cdot\mathbf{B} + \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,[e\Phi, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}]\right) \tag{2.2.15}$$

Some terms couple A and B; they can be dropped. Some simplification of the structures

involving σ matrices can be done. To start with, simplify $(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2$.

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2 = \sigma_i\sigma_j\pi_i\pi_j = \pi^2 - i\epsilon_{ijk}\pi_i\pi_j\sigma_k \tag{2.2.16}$$

Since $\mathbf{p}\times\mathbf{p} = \mathbf{A}\times\mathbf{A} = 0$, from $\boldsymbol{\pi}\times\boldsymbol{\pi}$ only the cross terms survive:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2 = \pi^2 - ie\epsilon_{ijk}(p_iA_j - A_ip_j) = \pi^2 - e\boldsymbol{\sigma}\cdot\mathbf{B} \tag{2.2.17}$$

Looking at the terms $\mu'\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\boldsymbol{\sigma}\cdot\mathbf{B}\,\boldsymbol{\sigma}\cdot\boldsymbol{\pi} + \mu'(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\boldsymbol{\sigma}\cdot\mathbf{B}$, they contain an anticommutator involving B and p. Because the magnetic field is assumed to be constant, p and B commute, so:

$$\begin{aligned}\{\boldsymbol{\sigma}\cdot\mathbf{B}\,,\boldsymbol{\sigma}\cdot\mathbf{p}\,\} &= B_ip_j\{\sigma_i,\sigma_j\}\\ &= 2\mathbf{B}\cdot\mathbf{p}\end{aligned}$$

The commutator of Φ and a derivative operator should give the electric field E:

$$[\Phi,\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,] = [\Phi,p_i]\sigma_i = -iE_i\sigma_i = -i\boldsymbol{\sigma}\cdot\mathbf{E} \tag{2.2.18}$$

Using these identities, the Hamiltonian can be expressed as:

$$\hat{H}_0 = e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{\pi^2}{2m} - \frac{e}{2m}\boldsymbol{\sigma}\cdot\mathbf{B} \tag{2.2.19}$$

$$\hat{H}_1 = -\frac{\pi^4}{8m^3} + e\frac{p^2}{4m^3}\boldsymbol{\sigma}\cdot\mathbf{B} - i\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\boldsymbol{\sigma}\cdot\mathbf{E}}{4m^2} + \mu'\left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}\,\mathbf{B}\cdot\mathbf{p}}{2m^2} - \frac{i[\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,,\boldsymbol{\sigma}\cdot\mathbf{E}\,]}{2m}\right) \tag{2.2.20}$$

There are still some simplifications that can be made to terms quadratic in σ, but it'll be more convenient for now to keep H written as is.

2.2.3 Foldy-Wouthyusen Transform

To find a complete description of a single nonrelativistic particle in normal quantum mechanics, we must work in a basis where the lower component χ is truly negligible, at least at the desired order. While there exists a formal technique for finding this Foldy-Wouthyusen

transformation, for our purposes we can simply demand that the wave function after transformation $\phi_S = (1+\Delta)\eta$ obeys $\langle \phi_S, \phi_S \rangle = 1$. This follows from the necessity of probability conservation.

To find the necessary transformation, we can use the relativistic current density, and demand it equal that of the nonrelativistic Schrodinger-like wave functions. Using the expression for χ in terms of η:

$$\begin{aligned} \int d^3x(\eta^\dagger\eta + \chi^\dagger\chi) &= \int d^3x \left[\eta^\dagger\phi + \left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\pi}}{2m}\phi\right)^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\pi}}{2m}\eta\right)\right] \\ &= \int d^3x\eta^\dagger \left[1 + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{4m^2}\right]\eta \end{aligned}$$

Since we know that $\langle \phi_S, \phi_S \rangle = 1$ this shows that if $\eta = \left(1 - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{8m^2}\right)\phi_S$, the current conservation works out correctly.

We now need to find the form of $\hat{H}$ after this transformation. For now work with the general form:

$$\epsilon\eta = (\hat{H}_0 + \hat{H}_1)\eta \tag{2.2.21}$$

After changing to the Schrodinger like wave functions, this becomes:

$$\epsilon\left(1 - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{8m^2}\right)\phi_S = (\hat{H}_0 + \hat{H}_1)\left(1 - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{8m^2}\right)\phi_S \tag{2.2.22}$$

To the order needed the inverse of $1 + \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,^2/8m^2$ is just $1 - \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,^2/8m^2$, so eliminating that on the left hand side gives:

$$\epsilon\phi_S = (1 + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{8m^2})(\hat{H}_0 + \hat{H}_1)(1 - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{8m^2})\phi_S \tag{2.2.23}$$

Since H_1 is already second order, these further corrections don't involve it directly. Expressing the result as a commutator:

$$\epsilon\phi_S = \left(\hat{H}_0 + \frac{1}{8m^2}[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \hat{H}_0] + \hat{H}_1\right)\phi_S \tag{2.2.24}$$

So under the FW transformation, the leading order term is unchanged, and the second

order term is:

$$\hat{H}_1 \rightarrow \hat{H}_1' = \hat{H}_1 + \frac{1}{8m^2}[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \hat{H}_0] \tag{2.2.25}$$

The final step is to simplify the commutator $[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \hat{H}_0]$.

$$[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \hat{H}_0] = [(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{2m}] \tag{2.2.26}$$

Obviously $(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2$ commutes with itself, so that term vanishes. Since $\boldsymbol{\sigma}\cdot\mathbf{B}$ is constant, that commutator will also disappear. Writing $\boldsymbol{\sigma}\cdot\boldsymbol{\pi}$ as shown earlier:

$$[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \mu'\boldsymbol{\sigma}\cdot\mathbf{B}\,] = [\pi^2 - e\boldsymbol{\sigma}\cdot\mathbf{B}\,, \mu'\boldsymbol{\sigma}\cdot\mathbf{B}\,] = 0$$

The non-trivial part is the commutation of the derivative operators with the electric potential Φ.

$$\begin{aligned}
[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \hat{H}_0] &= [(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, e\Phi] \\
&= e(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,[\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,, \Phi] + [\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,, \Phi]\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,) \\
&= ie(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\boldsymbol{\sigma}\cdot\mathbf{E} + \boldsymbol{\sigma}\cdot\mathbf{E}\,\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,)
\end{aligned}$$

So, writing down the new H_1':

$$\begin{aligned}
\hat{H}_1' &= \hat{H}_1 + \frac{ie}{8m^2}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\boldsymbol{\sigma}\cdot\mathbf{E} + \boldsymbol{\sigma}\cdot\mathbf{E}\,\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,) \\
&= -\frac{\pi^4}{8m^3} + e\frac{p^2}{4m^3}\boldsymbol{\sigma}\cdot\mathbf{B} - \frac{ie}{8m^2}[\boldsymbol{\sigma}\cdot\mathbf{E}\,, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,] + \mu'\left(\frac{(\boldsymbol{\sigma}\cdot\mathbf{p}\,)(B\cdot p)}{2m^2} - \frac{i}{2m}[\boldsymbol{\sigma}\cdot\mathbf{E}\,, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,]\right)
\end{aligned}$$

Nonrelativistic Hamiltonian

From the relativistic equations of motion, a nonrelativistic Hamiltonian was defined in terms of Schrodinger-like wave functions that conserve probability. The final result may now be written down. For comparison with other work, rather than writing in terms of μ', it will be written in terms of the gyromagnetic ratio g. Because terms proportional to $g-2$ may enter seperately, it is convenient to express all g dependent terms as linear combinations of

g and $g-2$.

So the entire Hamiltonian, using $\mu' = \frac{g-2}{2}\mu_0 = \frac{g-2}{2}\frac{e}{2m}$, is:

$$\begin{aligned}
H &= e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} - (1+\frac{g-2}{2})\frac{e}{2m}\boldsymbol{\sigma}\cdot\mathbf{B} \\
&\quad + e\frac{p^2}{4m^3}\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{ie}{8m^2}(1+(g-2))[\boldsymbol{\sigma}\cdot\mathbf{E}\,,\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,] + (g-2)\frac{e}{2m}\frac{(\boldsymbol{\sigma}\cdot\mathbf{p}\,)(\mathbf{B}\cdot\mathbf{p})}{4m^2} \\
&= e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} \\
&\quad - \frac{e}{2m}\left\{\frac{g}{2}\boldsymbol{\sigma}\cdot\mathbf{B} - \frac{p^2}{2m^2}\boldsymbol{\sigma}\cdot\mathbf{B} - (g-2)\frac{(\boldsymbol{\sigma}\cdot\mathbf{p}\,)(\mathbf{B}\cdot\mathbf{p})}{4m^2} + (g-1)\boldsymbol{\sigma}\cdot(\mathbf{E}\times\boldsymbol{\pi})\right\} \\
&= e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} \\
&\quad - \frac{e}{2m}\left\{\frac{g}{2}\left(1-\frac{p^2}{2m^2}\right)\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{g-2}{2}\frac{p^2}{2m^2}\boldsymbol{\sigma}\cdot\mathbf{B} - \frac{g-2}{2}\frac{(\boldsymbol{\sigma}\cdot\mathbf{p}\,)(\mathbf{B}\cdot\mathbf{p})}{2m^2} + \left(\frac{g}{2}+\frac{g-2}{2}\right)\boldsymbol{\sigma}\cdot(\mathbf{E}\times\boldsymbol{\pi})\right\}
\end{aligned}$$

2.3 Method of NRQED

Instead of starting from the relativistic equations of motion and rewriting them in the form of a Schrodinger like equation, the method of NRQED can be employed. The nonrelativistic Lagrangian can be written in the most general way, but ignoring all possible terms that will have too small a contribution to calculations. The coefficients before each term will be unknown, but there is a straightforward method of fixing them.

Within the realm of NRQEDs validity, it should produce the same predictions as QED. The same physical process can be calculated in both frameworks, and then the results compared. The calculation in QED will then fix the coefficients in NRQED. In comparing the two calculations it will be necessary to write the QED results nonrelativisitcally, in the same manner as was done for the equation of motion.

For spin one-half, up to the order desired, the form of the NRQED Lagrangian involving

two fermion fields has been derived in the previous chapter. It is

$$\begin{aligned}\mathcal{L}_{NRQED} =&\Psi^\dagger\Bigg\{iD_0+\frac{\mathbf{D}^2}{2m}+\frac{\mathbf{D}^4}{8m^2}+c_F\frac{e}{m}\mathbf{S}\cdot\mathbf{B}+c_D\frac{e(\mathbf{D}\cdot\mathbf{E}-\mathbf{E}\cdot\mathbf{D})}{8m^2}\\&+c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D}}{8m^2}+c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B}+\mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3}-c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3}\\&+c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D})+(\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3}\Bigg\}\Psi\end{aligned} \tag{2.3.1}$$

Each term contains two fermion fields, and zero, one or two powers of the photon field. Terms with a different number of photon fields, of course, correspond to different physical processes. But because gauge invariance is a necessary feature of the theory, the coefficients of many terms involving the photon field are constrained to be the same as that of terms with a smaller number of such fields. For instance, the gauge invariant term $\mathbf{D}^2/(2m)$ is:

$$\frac{\mathbf{D}^2}{2m}=\frac{\boldsymbol{\nabla}^2-ie(\boldsymbol{\nabla}\cdot\mathbf{A}+e\mathbf{A}\cdot\boldsymbol{\nabla})-e^2\mathbf{A}^2}{2m} \tag{2.3.2}$$

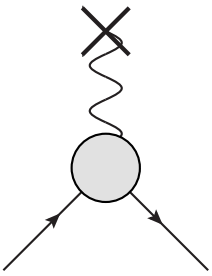
The first term above, containing $\boldsymbol{\nabla}^2$, is a purely kinetic term. The other two represent interactions with one or two photon fields. To fix these coefficients, two physical processes could be calculated and compared with the QED results, but in the end they must have the same coefficient as guaranteed by gauge invariance.

Obviously not every term goes this same way. The term with $\mathbf{S}\cdot\mathbf{B}$, for instance, is in and of itself gauge-invariant, and so there is no option but to calculate it from a process involving a single photon. For the necessary terms appearing in the above Lagrangian, it would suffice to consider just the one photon processes. However, since in principle some of the interesting coefficients could be calculated from two photon diagrams, this also will be done.

To fix these terms, some physical process must be chosen. For the one photon terms, scattering off an external field will be calculated. For the two photon terms, Compton scattering will be used.

2.3.1 Calculation of electron scattering off external field in QED

To fix all the terms in the NRQED Lagrangian which have a single power of the photon field, it suffices to calculate the scattering of an electron off an external field, as represented (in QED) by the diagram



The leading order contributions (in QED) to this interaction come from the fundamental electron vertex. There are radiative corrections to this process, starting at the one loop order. In principle, such calculations (up to some desired order in α) should be included in the QED calculation.

However, it turns out that the actual *form* of the interaction is highly constrained by the symmetries of the theory. No matter the source of contributions to the vertex, their effects can be incorporated into two coefficients or form factors. The NRQED coefficients can then be written in terms of these form factors, which can later be calculated to whatever precision is necessary.

The first symmetry that constrains the interaction is that it must be invariant under Lorentz transformations. Since every term involves the photon field A_μ and has external fermion legs, then the interaction must be proportional to the general form:

$$A_\mu \bar{u}(p')\Gamma^\mu(p',p)u(p) \tag{2.3.3}$$

and $\bar{u}(p')\Gamma^\mu(p',p)u(p)$ must transform as a Lorentz vector. If it did not, the whole would not be Lorentz invariant. To leading order $\Gamma^\mu = \gamma^\mu$, since the fundamental vertex is just that. The corrections, whatever the exact details of the processes which produce them, can only depend on the momenta p and p', in addition to constants m and e, and such structures as may act upon the spinors.

A basis for such structures is known, with well defined Lorentz transformations. That

gives scalar, vector, tensor, pseudo-vector, and pseudoscalar terms. From the momenta in the problem can be constructed scalar and vector quantities. What symmetries control the allowed terms?

In addition to proper Lorentz invariance, it is necessary that the interaction as a whole preserve parity. Since A is a vector, the term $\bar{u}\Gamma^\mu u$ must also behave as a vector under parity transformations. And there is no Lorentz invariant combination of external momenta that can be written that is not even under parity. Because of this, there can be no contribution from the pseudovector and pseudoscalar bilinears.

The vector bilinear $\bar{u}\gamma^\mu u$ already has the correct transformation properties. From the scalar bilinear $\bar{u}u$, a vector can be constructed by adjoining a single power of the momentum. (As, for example, $p^\mu \bar{u}u$.) And after contracting one index of the tensor with a momentum vector, it will also behave as a vector.

The momenta terms available are p and p'. However, the terms constructed from them are not independent, because each must separately obey current conservation. So whatever terms go into Γ^μ, $q_\mu \bar{u}\Gamma^\mu u = 0$. The unique scalar term will be

$$\frac{p^\mu + p'^\mu}{2m}\bar{u}u \tag{2.3.4}$$

which vanishes because $(p' + p) \cdot (p' - p) = p'^2 - p^2 = m^2 - m^2 = 0$.

The unique tensor term allowed will be

$$\frac{q_\nu}{2m}\bar{u}\sigma^{\mu\nu}u \tag{2.3.5}$$

which conserves current because $\sigma^{\mu\nu}$ is antisymmetric. So from these considerations, the general form of the vertex will have three terms, each with a momentum dependant coefficient:

$$\bar{u}\Gamma^\mu u = c_1(p,p')\frac{p^\mu + p'^\mu}{2m}\bar{u}u + c_2(p,p')\bar{u}\gamma^\mu u + c_3(p,p')\frac{q_\mu}{2m}\bar{u}\sigma^{\mu\nu}u \tag{2.3.6}$$

However, there is one more consideration. From the Dirac equation can be derived the

Gordon identity, which relates these three terms:

$$\bar{u}\gamma^{\mu}u = \bar{u}\left(\frac{p^{\mu}+p'^{\mu}}{2m} + \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\right)u \tag{2.3.7}$$

With this, any one of the terms in (2.3.6) can be rewritten as some combination of the other two, and its coefficient effectively absorbed into the other two. To write down the most general form of the vertex, one need only choose two of the three terms. Which two to choose might depend on the nature of the calculation; at any rate, there are always three paths to go down.

In the calculation of scattering off an external field an agnostic approach will be taken at first. The scattering amplitude is some combination of the three bilinears. The goal is to rewrite the scattering amplitude in nonrelativistic language. So, each bilinear in turn will be rewritten in this manner.

The vertex can be written in each of three ways. The form factors will be defined with respect to the particular combination of γ^{μ} and $\sigma^{\mu\nu}$ terms.

$$\Gamma^{\mu} = \gamma^{\mu}F_1(q^2) + i\frac{\sigma^{\mu\nu}q_{\nu}}{2m}F_2(q^2) \tag{2.3.8}$$

Using the Gordon identity to write this in terms of the scalar and vector bilinears, the result is

$$\Gamma^{\mu} = \gamma^{\mu}[F_1(q^2) + F_2(q^2)] - \frac{p^{\mu}+p'^{\mu}}{2m}F_2(q^2) \tag{2.3.9}$$

And writing in terms of the scalar and tensor:

$$\Gamma^{\mu} = \frac{p^{\mu}+p'^{\mu}}{2m}F_1(q^2) + i\frac{\sigma^{\mu\nu}q_{\nu}}{2m}[F_1(q^2) + F_2(q^2)] \tag{2.3.10}$$

The momentum dependence can be uniquely written in terms of q^2. The factors must be Lorentz invariant, so only such combinations of momenta can be considered. From the momenta p and p', only three such quantities can be constructed. But since the external fermions are on mass-shell, $p^2 = p'^2 = m^2$. That leaves only $p \cdot p' = m^2 + p \cdot q$, and this can

be related to q^2 by noting that since $p^2 = p'^2$,

$$p^2 = p^2 + 2p \cdot q + q^2 \rightarrow q^2 = -\frac{1}{2} p \cdot q \tag{2.3.11}$$

Given that there exist these three ways to write the vertex, there are three ways to perform the calculation. The scattering amplitude will have the form

$$iM = A_\mu \bar{u}(p') \Gamma^\mu(p', p) u(p) \tag{2.3.12}$$

and no matter which way Γ^μ is written, a nonrelativistic expansion in terms of ϕ_S will be needed. For now an agnostic approach will be taken, and the expressions for each of the three possible bilinears found.

Nonrelativistic expressions for the bilinears

To compare with the NRQED scattering amplitude, everything needs to be written with consistent language. We start with the relativistic bilinears, each of which behaves like a four vector, and appears in the amplitude dotted with the external field A. Any dot products $a \cdot b$ should instead be written as $a_0 b_0 - \mathbf{a} \cdot \mathbf{b}$. To this end, it might be necessary to treat the spatial and time-like components of the four-vector bilinears separately.

The bispinors u will be first written in terms of upper and lower components η, χ, and then in terms of the nonrelativistic wave spinors w. The relationship between the two sets are

$$\begin{aligned} \eta(p) &= \left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w(p) \\ \chi(p) &= \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2m} \left(1 - \frac{3\mathbf{p}^2}{8m^2}\right) w(p) \end{aligned}$$

In writing in terms of the upper and lower components, the explicit expressions of the γ matrices, as well as $\sigma^{\mu\nu}$ will be needed.

Of course the above expressions for the spinors in terms of w is approximate. In the NRQED Lagrangian, the terms we wish to fix involve A_0 with up to two powers of mo-

mentum (such as $\boldsymbol{\nabla}\cdot\mathbf{E}$), and A_i with up to three (as in $\boldsymbol{\sigma}\cdot\mathbf{B}\,\mathbf{p}^2$). Because only the case of a constant magnetic field is needed, in any term which will explicitly contain B higher derivatives may be ignored. Throwing away unneeded terms, whatever is left can be used to calculate the scattering amplitude and compare with NRQED.

This same general procedure will be followed for each of the bilinears.

Scalar bilinear: Start with the first bilinear, a scalar coupled with a momentum four-vector. Rewrite it in terms of η and χ.

$$(p+p')^\mu \bar{u}u = (p+p')^\mu \left(\eta^\dagger\eta - \chi^\dagger\chi\right) \tag{2.3.13}$$

Now express in terms of w

$$= (p+p')^\mu \left\{ w^\dagger \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right)\left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w - w^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{2m}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\right) w \right\} \tag{2.3.14}$$

And finally drop terms beyond the order that we need.

$$= (p+p')^\mu w^\dagger \left(1 - \frac{\mathbf{p}^2+\mathbf{p}'^2}{8m^2} - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2}\right) w \tag{2.3.15}$$

Simplifying $\boldsymbol{\sigma}\cdot\mathbf{p}'\,\boldsymbol{\sigma}\cdot\mathbf{p} = \mathbf{p}\cdot\mathbf{p}' + i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}$.

$$(p+p')^\mu \bar{u}u = (p+p')^\mu w^\dagger \left(1 - \frac{\mathbf{p}^2+2\mathbf{p}\cdot\mathbf{p}'+\mathbf{p}'}{8m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w \tag{2.3.16}$$

The spatial part is straightforward, but the time-like part is

$$(p+p')^0 \bar{u}u = (p+p')_0 w^\dagger \left(1 - \frac{\mathbf{p}^2+2\mathbf{p}\cdot\mathbf{p}'+\mathbf{p}'^2}{8m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w \tag{2.3.17}$$

Approximating the relativistic energies gives $p_0 = m + \mathbf{p}^2/(2m)$. So

$$p_0 + p_0' \approx 2m + \frac{\mathbf{p}^2+\mathbf{p}'^2}{2m} = 2m\left(1 + \frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right) \tag{2.3.18}$$

Then using this correction to the leading order term

$$(p+p')^0\bar{u}u = 2mw^\dagger\left(1+\frac{\mathbf{p}^2-2\mathbf{p}\cdot\mathbf{p}'+\mathbf{p}'}{8m^2}-\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right)w \tag{2.3.19}$$

$$(p+p')^0\bar{u}u \approx 2mw^\dagger\left(1+\frac{\mathbf{q}^2}{8m^2}-\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right)w \tag{2.3.20}$$

Vector bilinear: For the term $\bar{u}\gamma^\mu u$ it'll be necessary to treat the spatial/time-like indices separately, since they have different spinor structure.

The time-like part is:

$$\bar{u}\gamma^0 u = u^\dagger u \tag{2.3.21}$$

which in terms of η and χ is just

$$= \eta^\dagger\eta + \chi^\dagger\chi \tag{2.3.22}$$

Then rewritten with w

$$= w^\dagger\left(1-\frac{\mathbf{p}'^2}{8m^2}\right)\left(1-\frac{\mathbf{p}^2}{8m^2}\right)w + w^\dagger\left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{2m}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\right)w \tag{2.3.23}$$

Which is, at the order needed

$$= w^\dagger\left(1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{8m^2}+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2}\right)w \tag{2.3.24}$$

Simplifying the last term, using $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$, gives

$$\bar{u}\gamma^0 u = w^\dagger\left(1-\frac{\mathbf{q}^2}{8m^2}+\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right)w \tag{2.3.25}$$

The spatial part is

$$\bar{u}\gamma^i u = u^\dagger\gamma^0\gamma^i u \tag{2.3.26}$$

writing the matrices explicitly

$$= \bar{u}^\dagger\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}u \tag{2.3.27}$$

which in terms of spinors is

$$= \eta^\dagger \sigma_i \chi + \chi^\dagger \sigma_i \eta \tag{2.3.28}$$

Replacing the spinors with w gives

$$= w^\dagger \left\{ \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right) \frac{\sigma_i \boldsymbol{\sigma}\cdot\mathbf{p}}{2m} \left(1 - \frac{3\mathbf{p}^2}{8m^2}\right) + \left(1 - \frac{3\mathbf{p}'^2}{8m^2}\right) \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\,\sigma_i}{2m} \left(1 - \frac{\mathbf{p}^2}{8m^2}\right) \right\} w \tag{2.3.29}$$

Using $\mathbf{p}'^2 = \mathbf{p}^2$ gives

$$= w^\dagger \left\{ \frac{\sigma_i \boldsymbol{\sigma}\cdot\mathbf{p} + \boldsymbol{\sigma}\cdot\mathbf{p}'\,\sigma_i}{2m} \left(1 - \frac{\mathbf{p}^2}{2m^2}\right) \right\} w \tag{2.3.30}$$

Then $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$

$$= w^\dagger \left\{ \frac{p_i + \epsilon_{ijk} p_j \sigma_k + p'_i + \epsilon_{jik} p'_j \sigma_k}{2m} \left(1 - \frac{\mathbf{p}^2}{2m^2}\right) \right\} w \tag{2.3.31}$$

So finally

$$\bar{u}\gamma^i u = w^\dagger \left\{ \frac{p_i + p'_i - \epsilon_{ijk} q_j \sigma_k}{2m} \left(1 - \frac{\mathbf{p}^2}{2m^2}\right) \right\} w \tag{2.3.32}$$

Tensor term: The tensor term is subject to an additional simplification. Because the process under considering is elastic scattering off an external static field, terms involving q_0 can be dropped. Under this approximation $q_\nu \sigma^{\mu\nu} \approx q_j \sigma^{\mu j}$.

Dealing first with the case where $\mu = i$, write the tensor structure explicitly as a matrix:

$$\bar{u} \frac{i}{2m} q_j \sigma^{ij} u = \frac{i\epsilon_{ijk} q_j}{2m} \bar{u} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} u \tag{2.3.33}$$

Write in terms of spinors

$$= \frac{i\epsilon_{ijk} q_j}{2m} \left(\eta^\dagger \sigma_k \eta - \chi^\dagger \sigma_k \chi \right) \tag{2.3.34}$$

And then in terms of w

$$= \frac{i\epsilon_{ijk} q_j}{2m} w^\dagger \left\{ \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right) \sigma_k \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right) - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_k\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2} w \right\} \tag{2.3.35}$$

There now appears a term with a triple product of σ matrices. That can be simplified with

the following expression:

$$\sigma_a\sigma_b\sigma_c = \sigma_a(\delta_{bc} + i\epsilon_{bcd}\sigma_d) = \sigma_a\delta_{bc} - \sigma_b\delta_{ca} + \sigma_c\delta_{ab} + i\epsilon_{abc} \tag{2.3.36}$$

Using that identity,

$$\bar{u}\frac{i}{2m}q_j\sigma^{ij}u = \frac{i\epsilon_{ijk}q_j}{2m}w^\dagger\left\{\sigma_k\left(1 - \frac{\mathbf{p}'^2 + \mathbf{p}^2}{8m^2}\right) - \frac{\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{p}')p_k - \sigma_k\mathbf{p}\cdot\mathbf{p}' + i\epsilon_{akc}q_ap_c}{4m^2}\right\}w \tag{2.3.37}$$

This can be further simplified by combining the like terms $\sigma_k(\mathbf{p}'^2+\mathbf{p}^2)-2\sigma_k\mathbf{p}\cdot\mathbf{p}' = \sigma_k\mathbf{q}^2$.

$$\bar{u}\frac{i}{2m}q_j\sigma^{ij}u = \frac{i\epsilon_{ijk}q_j}{2m}w^\dagger\left\{\sigma_k\left(1 - \frac{\mathbf{q}^2}{8m^2}\right) - \frac{\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{p}')p_k + i\epsilon_{akc}q_ap_c}{4m^2}\right\}w \tag{2.3.38}$$

Now it is necessary to consider exactly what derivatives of the field A_i are to be kept. The assumption is that B is constant and so $\partial_i B_j = 0$. Contracted with A_i above, $\epsilon_{ijk}A_iq_j \sim B_k$. So besides the leading factor, no terms with q are needed. Applying this simplification,

$$\bar{u}\frac{i}{2m}q_j\sigma^{ij}u = \frac{i\epsilon_{ijk}q_j}{2m}w^\dagger\left\{\sigma_k - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}p_k}{2m^2}\right\}w \tag{2.3.39}$$

The case with $\mu = 0$ goes as follows.

$$\bar{u}\frac{i}{2m}q_j\sigma^{0j}u = -\frac{q_j}{2m}\bar{u}\gamma^0\gamma^ju = -\frac{q_j}{2m}u^\dagger\gamma^ju \tag{2.3.40}$$

Explicitly in matrix form

$$= -\frac{q_j}{2m}u^\dagger\begin{pmatrix}0 & \sigma_j\\ \sigma_j & 0\end{pmatrix}u \tag{2.3.41}$$

Written in terms of spinors, and Galilean three-vector $q^j = -q_j$.

$$= \frac{q^j}{2m}\left(\eta^\dagger\sigma_j\chi - \chi^\dagger\sigma_j\eta\right) \tag{2.3.42}$$

And then in terms of w:

$$= \frac{q^j}{2m} w^\dagger \left\{ \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right) \frac{\sigma_j \boldsymbol{\sigma}\cdot\mathbf{p}}{2m} \left(1 - \frac{3\mathbf{p}^2}{8m^2}\right) - \left(1 - \frac{3\mathbf{p}'^2}{8m^2}\right) \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_j}{2m} \left(1 - \frac{\mathbf{p}^2}{8m^2}\right) \right\} w \tag{2.3.43}$$

This bilinear is contracted with A_0, and so actually, only terms involving two additional powers of momenta need be kept.

$$= \frac{q^j}{2m} w^\dagger \left\{ \frac{\sigma_j \boldsymbol{\sigma}\cdot\mathbf{p}}{2m} - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_j}{2m} \right\} w \tag{2.3.44}$$

or simplifying using the commutator of σ matrices

$$= \frac{q^j}{2m} w^\dagger \left\{ \frac{2i\epsilon_{jik} p_i \sigma_k - q_i - i\epsilon_{ijk} q_j \sigma_k}{2m} \right\} w \tag{2.3.45}$$

One term above vanishes because of symmetry

$$\bar{u} \frac{i}{2m} q_j \sigma^{0j} u = -w^\dagger \left\{ \frac{\mathbf{q}^2}{4m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{2m^2} \right\} w \tag{2.3.46}$$

Full vertex Now that the three bilinears have been calculated, the complete scattering amplitude can be written down. Each of the three forms should prove equivalent. To simplify comparison, the coupling to A_0 and A_i can be treated separately.

The first has $\Gamma^\mu = \gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)$. Using (2.3.25) and (2.3.46)

$$A_0 \bar{u} \Gamma^0 u = A_0 F_1 w^\dagger \left(1 - \frac{\mathbf{q}^2}{8m^2} + \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w - A_0 F_2 w^\dagger \left\{ \frac{\mathbf{q}^2}{4m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{2m^2} \right\} w \tag{2.3.47}$$

The second is $\Gamma^\mu = \gamma^\mu [F_1(q^2) + F_2(q^2)] - \frac{p^\mu + p'^\mu}{2m} F_2(q^2)$. Using (2.3.25) and (2.3.20).

$$A_0 \bar{u} \Gamma^0 u = A_0 (F_1 + F_2) w^\dagger \left(1 - \frac{\mathbf{q}^2}{8m^2} + \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w - A_0 F_2 w^\dagger \left(1 + \frac{\mathbf{q}^2}{8m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w \tag{2.3.48}$$

The third combination is $\Gamma^\mu = \frac{p^\mu + p'^\mu}{2m} F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m} [F_1(q^2) + F_2(q^2)]$. Using (2.3.20) and

(2.3.46)

$$A_0\bar{u}\Gamma^0 u = F_1 w^\dagger \left(1 + \frac{\mathbf{q}^2}{8m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w - [F_1+F_2] w^\dagger \left\{\frac{\mathbf{q}^2}{4m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{2m^2}\right\} w \quad (2.3.49)$$

Taking any of these three results and collecting like terms gives the result:

$$A_0 w^\dagger \left(F_1 + [F_1+2F_2]\left[\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2} - \frac{\mathbf{q}^2}{8m^2}\right]\right) w \quad (2.3.50)$$

So the calculations are at least consistant with the Gordon identity.

Now turning to the coupling with $\mathbf{A}$. The first has $\Gamma^\mu = \gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)$. Using (2.3.32) and (2.3.39)

$$\begin{aligned} A_i\bar{u}\gamma^i u =& F_1 w^\dagger \left\{\frac{p_i + p'_i - \epsilon_{ijk}q_j\sigma_k}{2m}\left(1 - \frac{\mathbf{p}^2}{2m^2}\right)\right\} w \\ &+ F_2\frac{i\epsilon_{ijk}q_j}{2m} w^\dagger \left\{\sigma_k\left(1 - \frac{\mathbf{q}^2}{8m^2}\right) - \frac{\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{p}')p_k + i\epsilon_{akc}q_a p_c}{4m^2}\right\} w \end{aligned} \quad (2.3.51)$$

The second is $\Gamma^\mu = \gamma^\mu[F_1(q^2)+F_2(q^2)] - \frac{p^\mu+p'^\mu}{2m}F_2(q^2)$. Using (2.3.32) and (2.3.16):

$$\begin{aligned} A_i\bar{u}\gamma^i u =& A_i[F_1+F_2] w^\dagger \left\{\frac{p_i + p'_i - \epsilon_{ijk}q_j\sigma_k}{2m}\left(1 - \frac{\mathbf{p}^2}{2m^2}\right)\right\} w \\ &- F_2\frac{(p+p')^i}{2m} w^\dagger \left(1 - \frac{\mathbf{p}^2 + 2\mathbf{p}\cdot\mathbf{p}' + \mathbf{p}'}{8m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w \end{aligned} \quad (2.3.52)$$

The third combination is $\Gamma^\mu = \frac{p^\mu+p'^\mu}{2m}F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m}[F_1(q^2)+F_2(q^2)]$. Using (2.3.16) and (2.3.39)

$$\begin{aligned} A_i\bar{u}\gamma^i u =& A_i F_1 \frac{p^i + p'^i}{2m} w^\dagger \left(1 - \frac{\mathbf{p}^2 + 2\mathbf{p}\cdot\mathbf{p}' + \mathbf{p}'}{8m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w \\ &+ A_i(F_1+F_2)\frac{i\epsilon_{ijk}q_j}{2m} w^\dagger \left\{\sigma_k - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}p_k}{2m^2}\right\} w \end{aligned} \quad (2.3.53)$$

2.3.2 Calculation of Compton scattering in QED

The relevant terms in the NRQED Lagrangian can all be fixed by the previous calculation of scattering off an external field, because even though there are terms involving two powers of the photon field, the requirement of gauge invariance means they share a coefficient with

one photon terms. However, for reasons of self consistency it would be good to check that the coefficients really do work out the same if calculated independently.

The easiest process to calculate involving two photons will be Compton scattering.

2.4 The two-photon vertex of NRQED

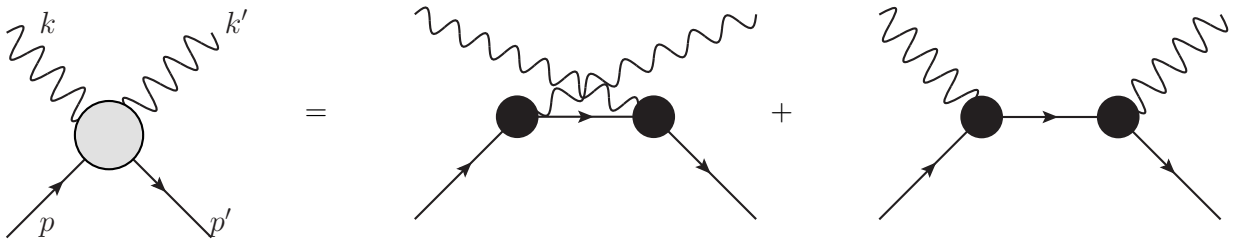
In the NRQED Lagrangian, in addition to the terms involving the fermions interaction with a single photon, there are terms which represent the interaction of a fermion with two photons. At the order needed, all such terms are fixed by gauge invariance. There are terms, such as those involving $\mathbf{E}^2$, that would are by themselves gauge invariant, but these occur at too high an order. (The order of such a term would be $E^2/m^3 \sim mv^6$.)

So though the coefficients of concern are all fixed by considering just the one-photon interactions, they could also be fixed from considering two-photon interactions. Since it *is* possible, it makes sense to do so, as a check of consistency. In this section, the coefficients of two-photon terms in the NRQED Lagrangian will be fixed from QED calculations.

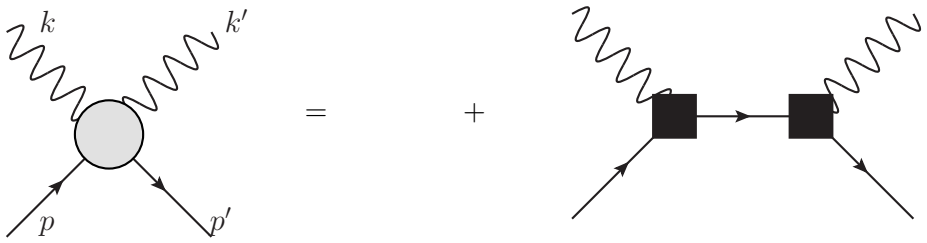
As before, this will involve calculating some physical process in both QED and NRQED, and comparing the result. The simplest two photon process to consider is Compton scattering. By calculating Compton scattering in each theory, the coefficients desired will be obtained.

This is not quite as straightforward as in the case of the one-photon scattering, for the following reason: while the one-photon scattering is a local interaction in both QED and NRQED, Compton scattering will involve some mix of local and non-local diagrams. In QED, there are of course no local interactions between a fermion and two photons. The situation is most readily stated diagrammatically.

In QED, the leading order diagrams contributing to Compton scattering are:



While in NRQED, the following diagrams contribute to the scattering:



In each set of diagrams, the vertices represent the *total* electron vertex. For QED this is determined, as before, by the form factors, and for NRQED it is determined by the calculations of the previous section.

Since the two amplitudes must be equal, in principle the process is this: First calculate the scattering amplitude in QED. Then, calculate the contribution to the scattering amplitude coming from the tree diagrams I and II above. Whatever discrepancy remains must be the value of the local two-photon vertex III.

The process of subtracting the one set of diagrams from the other could be slightly complicated, but luckily it turns out there is a simpler path. By considering the physical origin of the local terms in NRQED, it will be possible to split the QED diagrams into local and non-local parts, where the latter can be shown to be equal to the non-local diagrams in NRQED. Then, comparing the two scattering processes becomes much easier.

Z diagrams

The high energy theory (QED) doesn't contain any two-photon vertices, while the low energy theory (NRQED) does. This is a general feature of effective field theories, that new types of local interactions arise. The high energy theory can have intermediate states that are highly virtual, while the low energy theory doesn't. Instead, as according to the uncertainty principle, intermediate states with extremely high energy can be considered to occur almost instantaneously, giving rise in the effective theory to local interactions.

How does the local two-photon interaction arise in NRQED? Of course there are an infinite number of contributions, but we'll consider just the leading order contributions. These will come from the tree level two photon diagrams as shown above. Compare the

tree-level diagrams in the two theories: in addition to the vertices being different, so are the propagators. The propagator in QED represents some admixture of the electron and positron field, while in NRQED it is only the electron. In both QED and NRQED, a process is calculated as the sum of a series of diagrams, representing an expansion in perturbation theory. However, there is a difference btween the two in the nature of perturbation theory employed.

In QED, at each vertex both energy and momentum is conserved. But intermediate particles may be off mass-shell; that is, it is no longer the case that for a particle of four-momentum p and mass m that $p^2 = m^2$.

In NRQED, the old Rayleigh-Schrodinger perturbation theory is used. All intermediate particles are on mass-shell. But at the vertices (when represented diagrammatically), although momenta is conserved, energy is not.

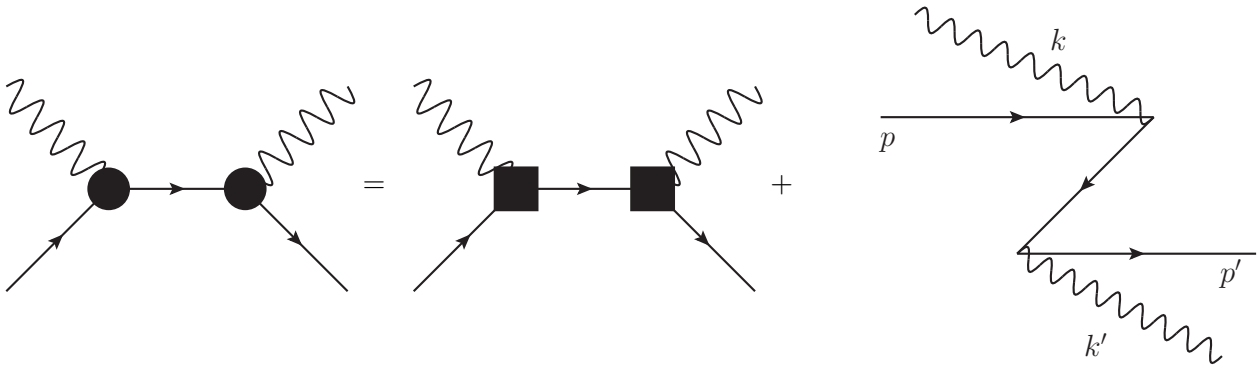
Remember that in old perturbation theory, corrections have the rough form

$$\Delta = \Sigma_{\text{int}} \frac{\langle \text{out}| V |\text{int}\rangle \langle \text{int}| V |\text{in}\rangle}{E_{\text{in}} - E_{\text{int}}} \tag{2.4.1}$$

where the sum is over intermediate states, which can have differing energy from the initial state.

The trick, then is to take the relativistic tree-level diagrams of QED and rewrite them in the language of Rayleigh-Schrodinger before trying to compare them to NRQED. In NRQED, only intermediate states involving electrons can be considered, but in QED intermediate states identified with positrons will appear as well. It is *these* processes, involving a large violation of energy conservation, that will appear as contact terms in NRQED.

There are two diagrams in QED, and both can be dealt with in the same general way. Consider the uncrossed diagram:



There are two tree level processes that can be considered in the old time-ordered perturbation theory. The first corresponds to an incoming electron, which first absorbs a photon and then emits one. The second, more complicated process, involves the creation of intermediate positron. While a free electron travels along, an incoming photon decays into an electron and positron. Then, the positron annihilates the incoming electron and emits the outgoing photon. Because of the shape of this diagram, it is called a "Z diagram."

In the Z-diagrams, the electrons and the photons are external, so the sum over the intermediate states is specifically the intermediate states of the positrons. Likewise, the other diagrams are written as a sum over intermediate electron states. But because of the rules of Rayleigh-Schrodinger perturbation theory, all these states are on mass shell. And since the momenta here is fixed, the sum over intermediate states is a sum over spin states.

So the original QED diagram should somehow split into two terms, one involving a sum over electron states and the other a sum over anti-particles.

Call the intermediate momentum $\mathbf{q}$. The initial energy will be q_0, the intermediate energy will be that of the on mass shell particle, $E_q = \sqrt{\mathbf{q}^2 + m^2}$.

$$\longrightarrow \quad = \quad i\frac{\not{q}-m}{q^2-m^2} = i\frac{1}{\sqrt{\mathbf{q}^2+m^2}}\left(\frac{\Sigma\bar{u}(\mathbf{q})u(\mathbf{q})}{q_0-\sqrt{\mathbf{q}^2+m^2}} + \frac{\Sigma\bar{v}(-\mathbf{q})v(-\mathbf{q})}{q_0+\sqrt{\mathbf{q}^2+m^2}}\right)$$

This identity can be technically reproduced as follows. First write the denominator of the propagator as

$$i\frac{\not{q}-m}{q^2-m^2} = i\frac{\not{q}-m}{q_0^2-(\mathbf{q}^2+m^2)} \tag{2.4.2}$$

This could be factored into

$$q_0^2 - (\mathbf{q}^2 + m^2) = (q_0 + \sqrt{\mathbf{q}^2 + m^2})(q_0 - \sqrt{\mathbf{q}^2 + m^2}) \tag{2.4.3}$$

So it implies poles at $q_0 = \pm\sqrt{\mathbf{q}^2 + m^2}$. There is one unique way of factoring the original propagator into the two poles:

$$\frac{1}{2\sqrt{\mathbf{q}^2 + m^2}} \left(\frac{\gamma^0\sqrt{\mathbf{q}^2 + m^2} - \gamma \cdot \mathbf{q} + m}{q_0 - \sqrt{\mathbf{q}^2 + m^2}} - \frac{\gamma^0\sqrt{\mathbf{q}^2 + m^2} + \gamma \cdot \mathbf{q} - m}{q_0 + \sqrt{\mathbf{q}^2 + m^2}} \right) \tag{2.4.4}$$

The two numerators can be exactly equated to sums over polarisation states of electron and positron spinors:

$$\begin{aligned} \Sigma u(\mathbf{p})\bar{u}(\mathbf{p}) &= \gamma \cdot p + m \\ \Sigma v(\mathbf{p})\bar{v}(\mathbf{p}) &= \gamma \cdot p - m \end{aligned}$$

These relations hold for particles which are on mass-shell. That is exactly the case here. But then, there is an assumption that the quantity p_0 above is the on mass-shell energy, $\sqrt{\mathbf{p}^2 + m^2}$.

So, noting that particles with momentum $\pm\mathbf{q}$ have the same energy $\sqrt{\mathbf{q}^2 + m^2}$ and that q_0 is the off-mass shell energy from the relativistic diagram, the propagator can be rewritten:

$$\frac{1}{2\sqrt{\mathbf{q}^2 + m^2}} \left(\frac{\Sigma u(\mathbf{q})\bar{u}(\mathbf{q})}{q_0 - \sqrt{\mathbf{q}^2 + m^2}} - \frac{\Sigma v(-\mathbf{q})\bar{v}(-\mathbf{q})}{q_0 + \sqrt{\mathbf{q}^2 + m^2}} \right) \tag{2.4.5}$$

The numerators have now been put into exactly the forms expected for the regular- and Z-diagrams of old perturbation theory.

As mentioned above this can be understood in the context of old perturbation theory as sum over intermediate states in the numerator, with the expected form of the denominator being $E_{\text{in}} - E_{\text{int}}$. It can be shown that the denominators have exactly this form.

Now consider the first, regular diagram. The initial energy is q_0, since in relativistic theory the total energy at the vertex is conserved. The intermediate energy is the on-mass

shell energy of the electron: $\sqrt{\mathbf{q}^2 + m^2}$. Thus the denominator of $q_0 - \sqrt{\mathbf{q}^2 + m^2}$ is that expected.

Now consider the Z-diagram. The initial energy is still q_0. The intermediate energy is more complicated: there are two photons, two electrons, and a positron present. The total combined energy is:

$$E_{\text{int}} = p_0 + p_0' + k_0 + k_0' + \sqrt{\mathbf{q}^2 + m^2} = 2q_0 + \sqrt{\mathbf{q}^2 + m^2} \tag{2.4.6}$$

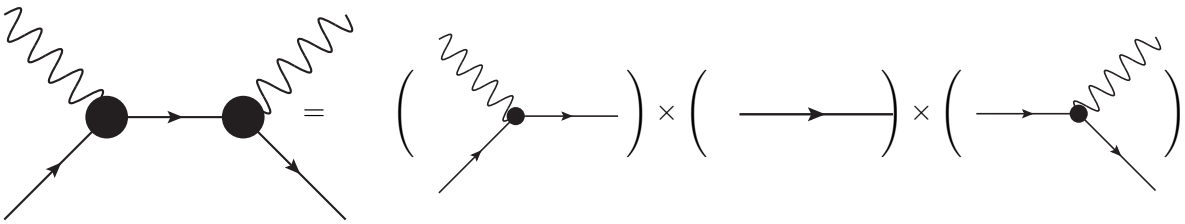
Then the difference $E_{\text{in}} - E_{\text{int}} = -q_0 - \sqrt{\mathbf{q}^2 + m^2}$, as expected.

So rewriting the propagator into these two separate terms can be understood as how the fully relativistic process appears in old perturbation theory. In a relativistic theory, intermediate states involving positrons must be accounted for. For higher order QED diagrams more complicated processes will appear, but at the tree level only those discussed above are involved. No approximations are involved — the identity for the propagator is exact. The form is more convenient for discussing nonrelativistic energies, but is still equivalent to the relativistic diagrams.

Relation between NRQED and old perturbation theory

Now it is necessary to show how the diagrams in old perturbation theory relate to those of NRQED.

The normal diagrams can be easily interpreted as the product of vertices and the relativistic Rayleigh-Schrodinger propagator:



Where Recall that the total NRQED one-photon vertex was derived by comparing to

the QED vertex, so necessarily

$$\bar{u}\Gamma^0 u = \phi^\dagger V^0 \phi,\ \bar{u}\Gamma^i u = \phi^\dagger V^i \phi \tag{2.4.7}$$

The propagator $1/(q_0 - \sqrt{m^2 + \mathbf{q}^2})$ is equal to the total propagator in NRQED (including relativistic corrections). So it really is the case that the two sets of diagrams are equivalent. Whether the vertices are written in terms of NRQED or QED, the ultimate expression will be the same.

It then follows that the contributions to the local contact term in NRQED come (at tree level) exactly from the Z-diagrams. To calculate the contact terms to the order (in the nonrelativistic expansion) needed, the Z-diagrams need to be approximated just as the one-photon vertex diagrams were in the previous section. Then the NRQED coefficients can be obtained by comparing the two calculations.

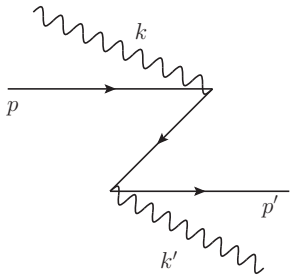
At nonrelativistic energies, it would be expected that the sum over intermediate states now does not resemble a propagator. Because only terms involving up to one power of momentum need be kept, the square root term becomes simply m. q_0, the total incoming energy, will involve both the electron and photon energy. Depending on whether the crossed or uncrossed diagram is considered, it will be either $q_0 = p_0 + k_0$ or $q_0 = p_0 - k_0'$. In either case, it will be m at the leading order, with a first order correction due to the photon energy.

$$\frac{\Sigma \bar{v}(-\mathbf{q}) v(-\mathbf{q})}{q_0 + \sqrt{\mathbf{q}^2 + m^2}} \approx \frac{\Sigma \bar{v}(-\mathbf{q}) v(-\mathbf{q})}{2m + (q_0 - m)} \tag{2.4.8}$$

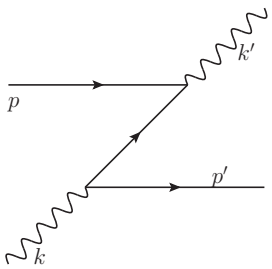
Since $q_0 - m << m$, this can then be written as

$$\approx \left(1 - \frac{q_0 - m}{2m}\right) \frac{\Sigma \bar{v}(-\mathbf{q}) v(-\mathbf{q})}{2m} \tag{2.4.9}$$

In this approximation, the two Z diagrams become



$$= \quad \Sigma_{\text{spin}} \left(1 - \tfrac{k_0}{2m}\right) \tfrac{1}{2m} \bar{u}(\mathbf{p}')\gamma^\mu \epsilon^*_\mu(k')v_s(-\mathbf{p}-\mathbf{k})\bar{v}_s(-\mathbf{p}-\mathbf{k})\Gamma^\nu \epsilon_\nu(k)u(\mathbf{p})$$



$$= \quad \Sigma_{\text{spin}} \left(1 + \tfrac{k'_0}{2m}\right) \tfrac{1}{2m} \bar{u}(\mathbf{p}')\gamma^\mu \epsilon_\mu(k)v(\mathbf{k}'-\mathbf{p})_s\bar{v}_s(\mathbf{k}'-\mathbf{p})\Gamma^\nu \epsilon^*_\nu(k')u(\mathbf{p})$$

While the sums over intermediate states could also be expanded, it'll be easiest to calculate these in the above form. The vertices will be the sum of particle-antiparticle bilinears, which can be calculated separately.

2.4.1 Nonrelativistic expressions for Z diagrams

Looking at the equations for the Z diagrams, they are both the product of two types of terms to calculate:

$$\bar{u}(\mathbf{p}')\Gamma^\mu(q)v(\boldsymbol{\ell}) \text{ and } \bar{u}(\boldsymbol{\ell})\Gamma^\mu(q)v(\mathbf{p}) \tag{2.4.10}$$

Here ℓ is the intermediate momentum of the positron, and q is the momentum of the photon going into the vertex (either k or $-k'$). The form of Γ is

$$\Gamma^\mu(q) = F_1\gamma^\mu + F_2\frac{q_\nu\sigma^{\mu\nu}}{2m} \tag{2.4.11}$$

To compare the Z diagrams to the contact terms of NRQED, first express the bilinears in the vertices of (2.4.10) in terms of the nonrelativistic quantities. This can be done for each of the two terms in Γ^μ separately. The bispinors u will be replaced by the spinor ϕ, as before. Now, the bispinor v will be replaced by a spinor for a positron, which shall be called

χ. In doing the expansion only terms up to $\mathcal{O}(1/m)$ need be kept.

To calculate the vector like bilinears, treat the spatial and time-like components separately. First $\mu = 0$:

$$\begin{aligned}
\bar{u}(p')\gamma^0 v(\ell) &= \begin{pmatrix} \varphi^\dagger & \frac{\boldsymbol{\sigma}\mathbf{p}'}{2m}\varphi^\dagger \end{pmatrix} \begin{pmatrix} \frac{\boldsymbol{\sigma}\mathbf{q}}{2m}\chi \\ \chi \end{pmatrix} \\
&= \phi^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\ell + \boldsymbol{\sigma}\cdot\mathbf{p}'}{2m} \right) \chi \\
\bar{v}\ell\gamma^0 u(p) &= \begin{pmatrix} \frac{\boldsymbol{\sigma}\ell}{2m}\chi^\dagger & \chi^\dagger \end{pmatrix} \begin{pmatrix} \varphi \\ \frac{\boldsymbol{\sigma}\mathbf{p}}{2m}\varphi \end{pmatrix} \\
&= \chi^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\ell + \boldsymbol{\sigma}\cdot\mathbf{p}}{2m} \right) \phi
\end{aligned}$$

Then $\mu = i$:

$$\begin{aligned}
\bar{u}(p')\gamma^i v(\ell) &= \begin{pmatrix} \varphi^\dagger & \frac{\boldsymbol{\sigma}\mathbf{p}'}{2m}\varphi^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} \frac{\boldsymbol{\sigma}\ell}{2m}\chi \\ \chi \end{pmatrix} \\
&= \phi^\dagger \sigma_i \chi \\
\bar{v}(\ell)\gamma^i u(p) &= \begin{pmatrix} \frac{\boldsymbol{\sigma}\ell}{2m}\chi^\dagger & \chi^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \frac{\boldsymbol{\sigma}\mathbf{p}}{2m}\varphi \end{pmatrix} \\
&= \chi^\dagger \sigma_i \phi
\end{aligned}$$

In the tensor terms a factor of momentum q_ν appears. The spatial part, q_j is "naturally raised" so $q_j = -(\mathbf{q})_j$. As before the two types of indices should be treated separately. For $\mu = 0$:

$$
\begin{aligned}
\frac{iq_\nu}{2m}\bar{u}(p')\sigma^{0\nu}v(\ell) &= \frac{iq_j}{2m}\bar{u}(p')\sigma^{0j}v(\ell) \\
&= -\frac{q_j}{2m}\begin{pmatrix}\varphi^\dagger & \frac{\boldsymbol{\sigma}\mathbf{p}'}{2m}\varphi^\dagger\end{pmatrix}\begin{pmatrix}0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix}\frac{\boldsymbol{\sigma}\ell}{2m}\chi \\ \chi\end{pmatrix} \\
&= \phi^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{q}}{2m}\chi \\
\frac{iq_\nu}{2m}\bar{v}(\ell)\sigma^{0\nu}u(p) &= \frac{iq_j}{2m}\bar{v}(\ell)\sigma^{0j}u(p) \\
&= -\frac{q_j}{2m}\bar{v}\ell\begin{pmatrix}0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix}\varphi \\ \frac{\boldsymbol{\sigma}\mathbf{p}}{2m}\varphi\end{pmatrix} \\
&= -\chi^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{q}}{2m}\phi
\end{aligned}
$$

Then for $\mu = i$

$$
\frac{iq_\nu}{2m}\bar{u}(p')\sigma^{i\nu}v(\ell) = \frac{iq_0}{2m}\bar{u}(p')\sigma^{i0}v(\ell) + \frac{iq_j}{2m}\bar{u}(p')\sigma^{ij}v(\ell) \tag{2.4.12}
$$

$$
= -\frac{iq_0}{2m}\begin{pmatrix}\varphi^\dagger & \frac{\boldsymbol{\sigma}\mathbf{p}'}{2m}\varphi^\dagger\end{pmatrix}\begin{pmatrix}0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix}\frac{\boldsymbol{\sigma}\ell}{2m}\chi \\ \chi\end{pmatrix} + \frac{iq_j}{2m}\epsilon_{ijk}\begin{pmatrix}\varphi^\dagger & \frac{\boldsymbol{\sigma}\mathbf{p}'}{2m}\varphi^\dagger\end{pmatrix}\begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}\begin{pmatrix}\frac{\boldsymbol{\sigma}\ell}{2m}\chi \\ \chi\end{pmatrix}
$$

The second term will be of order $\mathcal{O}(1/m^2)$ and so can be neglected. So

$$
\frac{iq_\nu}{2m}\bar{u}(p')\sigma^{i\nu}v(\ell) = \phi^\dagger\frac{q_0\sigma_i}{2m}\chi \tag{2.4.13}
$$

The complementary term is

$$
\frac{iq_\nu}{2m}\bar{v}(\ell)\sigma^{i\nu}u(p) = \frac{iq_0}{2m}\bar{v}(\ell)\sigma^{i0}v(\ell) + \frac{iq_j}{2m}\bar{u}(p')\sigma^{ij}u(p) \tag{2.4.14}
$$

Again, the second term with σ^{ij} has the same general structure and will be of order $1/m^2$.

So

$$\begin{aligned}\frac{iq_\nu}{2m}\bar{v}(\ell)\sigma^{i\nu}u(p) &= \frac{iq_0}{2m}\begin{pmatrix}\frac{\boldsymbol{\sigma}\boldsymbol{\ell}}{2m}\chi^\dagger & \chi^\dagger\end{pmatrix}\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix}\varphi\\ \frac{\boldsymbol{\sigma}\mathbf{p}}{2m}\varphi\end{pmatrix}\\ &= -\chi^\dagger\frac{q_0\sigma^i}{2m}\phi\end{aligned}$$

Now the total vertices $\bar{u}\Gamma^\mu v$ can be expressed nonrelativistically:

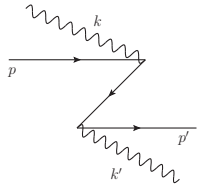
$$\bar{u}(p')\Gamma^0 v(\ell) = \phi^\dagger(p')\left(F_1\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\ell}+\boldsymbol{\sigma}\cdot\mathbf{p}'}{2m}+F_2\frac{\boldsymbol{\sigma}\cdot\mathbf{q}}{2m}\right)\chi(\ell) \tag{2.4.15}$$

$$\bar{v}(\ell)\Gamma^0 v(p) = \chi^\dagger(p')\left(F_1\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\ell}+\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}+F_2\frac{\boldsymbol{\sigma}\cdot\mathbf{q}}{2m}\right)\phi(\ell) \tag{2.4.16}$$

$$\bar{u}(p')\Gamma^i v(\ell) = \phi^\dagger\left(F_1+F_2\frac{q_0}{2m}\right)\sigma^i\chi \tag{2.4.17}$$

$$\bar{v}(\ell)\Gamma^i v(p) = \chi^\dagger\left(F_1-F_2\frac{q_0}{2m}\right)\sigma^i\phi \tag{2.4.18}$$

Returning to the Z diagrams, the structures Γ^μ appear contracted with the photon polarization. Because a physical process is being calculated, the result should not depend on the gauge chosen. So it will be easiest to choose the gauge where the photons are transverse and $\epsilon_0 = 0$. Then $\Gamma\cdot\epsilon = -\boldsymbol{\Gamma}\cdot\boldsymbol{\epsilon}$.



$$\begin{aligned}= \Sigma_{\text{spin}}\left(1-\frac{k_0}{2m}\right)\frac{1}{2m}&\left[\phi^\dagger\left(F_1+F_2\frac{-k_0'}{2m}\right)\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(k')\chi\right]\\ &\times\left[\epsilon_j(k)\chi^\dagger\left(F_1-F_2\frac{k_0}{2m}\right)\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}(k)\phi\right]\end{aligned} \tag{2.4.19}$$

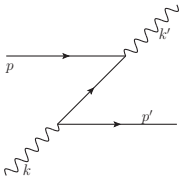
The sum over intermediate spin spates just becomes the identity: $\Sigma_{\text{spin}}\chi^\dagger\chi = 1$.

$$= \frac{1}{2m}\left(1-\frac{k_0}{2m}\right)\phi^\dagger\left[\left(F_1-F_2\frac{k_0'}{2m}\right)\left(F_1-F_2\frac{k_0}{2m}\right)\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(k')\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}(k)\right]\phi \tag{2.4.20}$$

Since only terms up to order $1/m$ are needed, this can be simplified. Also, in this approximation $k_0' \approx k_0$, as the total conservation of energy implies $k_0' - k_0 = p_0 - p_0' \sim \mathbf{p}^2/m^2$. So for convenience everything will be written in terms of k_0.

$$= \frac{1}{2m}\phi^\dagger \left[\left(F_1^2 - F_1[F_1 + 2F_2]\frac{k_0}{2m} \right) \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(k')\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}(k) \right] \phi \tag{2.4.21}$$

The other Z diagram, coming from the crossed diagram is:



$$= \quad \Sigma_{\text{spin}} \left(1 + \tfrac{k_0'}{2m}\right) \tfrac{1}{2m} \left[\phi^\dagger \left(F_1 + F_2 \tfrac{k_0}{2m} \right) \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^(k)\chi \right] \left[\epsilon_j(k)\chi^\dagger \left(F_1 + F_2 \tfrac{k_0'}{2m} \right) \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(k')\phi \right]$$

After summing over spin states

$$= \frac{1}{2m}\left(1 + \frac{k_0'}{2m}\right)\phi^\dagger \left[\left(F_1 + F_2\frac{k_0}{2m} \right) \left(F_1 + F_2\frac{k_0'}{2m} \right) \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^(k)\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(k') \right] \phi \tag{2.4.22}$$

And then applying the same simplifications as before

$$= \frac{1}{2m}\phi^\dagger \left[\left(F_1^2 + F_1[F_1 + 2F_2]\frac{k_0}{2m} \right) \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^(k)\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(k') \right] \phi \tag{2.4.23}$$

The local interaction comes from the sum of the two diagrams. Adding them together,

$$\begin{aligned}
\text{(Sum of Z diagrams)} &= \frac{1}{2m}\phi^\dagger \Bigg[\left(F_1^2 + F_1[F_1 + 2F_2]\frac{k_0}{2m} \right) \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^(k)\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(k') \\
&\quad + \left(F_1^2 - F_1[F_1 + 2F_2]\frac{k_0}{2m} \right) \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(k')\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}(k) \Bigg] \phi \\
&= \frac{F_1}{2m}\phi^\dagger \left[F_1\{\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}, \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*\} + (F_1 + 2F_2)\frac{k_0}{2m}[\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}, \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*] \right] \phi \\
&= \frac{F_1}{m}\phi^\dagger \left[F_1 \boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}^* + (F_1 + 2F_2)\frac{k_0}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\times\boldsymbol{\epsilon}^* \right] \phi
\end{aligned}$$

It now remains to calculate the same amplitude in NRQED and compare.

2.4.2 Compton scattering in NRQED

We want to calculate the Compton scattering in the nonrelativistic theory. We choose the gauge such that the photon polarisations obey $\epsilon_0 = 0$. In general we should consider both terms arising from two-photon vertices, and those from tree level diagrams of two one-photon vertices. However, because of the approach we take in calculating the process from the relativistic Lagrangian, we only need the former terms. That is, we know how to separate contact terms (which arise from some combination of Z-diagrams in the relativistic theory) from the rest.

Further, ultimately we only care about a few of these terms, ignoring terms which have both $\mathbf{B}$ and $\mathbf{A}$.

The remaining terms we're interested in are:

$$\mathcal{L}_{A^2} = \Psi^\dagger(-\frac{e^2\mathbf{A}^2}{2m} - e^2\frac{\{\boldsymbol{\nabla}^2, \mathbf{A}^2\}}{8m^3} - e^2\frac{\{\nabla_i, A_i\}\{\nabla_j, A_j\}}{8m^3} + c_S^2\frac{e^2\mathbf{S}\cdot(\mathbf{A}\times\mathbf{E} - \mathbf{E}\times\mathbf{A})}{8m^2})\Psi$$

The process considered has an incoming photon with momentum k and polarisation $\epsilon(k)$, and an outgoing photon with momentum k' and polarisation $\epsilon^*(k')$. The charged particle has incoming momentum p and outgoing $p' = p + k - k'$.

As in the single-photon calculation, it is simply a matter of reading terms off the Lagrangian. To find the scattering amplitude, replace Ψ with ϕ_S, and replace $\mathbf{A}$ with photon polarisations ϵ and ϵ^*. In the gauge chosen, $\mathbf{E}(k) = -\partial_0\epsilon = ik_0\boldsymbol{\epsilon}(k) = -ik_0'\boldsymbol{\epsilon}^*(k')$.

Contracted with the photon of momentum k we get $\mathbf{E} \to ik_0\epsilon$, while with the photon of momentum k' we have $\mathbf{E} \to ik_0'\epsilon'$. Both processes must be considered in calculating the scattering, so:

$$\mathbf{A}\times\mathbf{E} = -\mathbf{A}\times(\partial_0\mathbf{A}) \to -i(k_0'\epsilon\times\epsilon - k_0\epsilon\times\epsilon) = i(k_0' + k_0)\epsilon\times\epsilon^*$$

And

$$\mathbf{E}\times\mathbf{A} = -(\partial_0\mathbf{A})\times\mathbf{A} \to -i(k_0\epsilon\times\epsilon^* - k_0'\epsilon^*\times\epsilon) = -i(k_0' + k_0)\epsilon\times\epsilon^*$$

So from the term in the Lagrangian

$$c_S \Psi^\dagger \frac{e^2 \mathbf{S} \cdot (\mathbf{A} \times \mathbf{E} - \mathbf{E} \times \mathbf{A})}{8m^2}) \Psi$$

we get in the scattering amplitude

$$-ic_S \phi^\dagger \Big(\frac{e^2}{4m^2} i(k_0 + k_0') \epsilon \times \epsilon^* \Big) \phi = c_S \frac{e^2}{4m^2} \phi^\dagger \Big((k_0 + k_0') \epsilon \times \epsilon^* \Big) \phi$$

Using the approximation $k_0 \approx k_0'$, that part of the amplitude now becomes

$$c_S \frac{e^2}{2m^2} \phi^\dagger \Big(k_0 \epsilon \times \epsilon^* \Big) \phi \tag{2.4.24}$$

This is the only part of the amplitude we wanted to compare for reasons of consistency.

2.4.3 Comparison of QED and NRQED Compton scattering

Now that the amplitude has been calculated in both theories, the coefficient of NRQED c_S can be fixed. Using that $F_1 = e$ and $F_2 = e\frac{g-2}{2}$, $F_1 + 2F_2 = e^2(g-1)$. So the final result is that

$$c_S = g - 1 \tag{2.4.25}$$

Chapter 3

Spin-one particles

3.1 Spin-one Equations of Motion

The ultimate goal is to investigate the g-factor of bound particles of arbitrary spin. In the previous section, two methods were reviewed for a spin-half particle: deriving the Hamiltonian from the equations of motion, and calculating the NRQED Lagrangian by comparing scattering diagrams.

Both of these methods can be employed anytime a Lagrangian is known for the particle. The standard model contains charged spin-one particles, the W bosons. So as a next step to investigating the behavior of general spin particles, both of the methods used for spin-half can be applied to the W.

3.1.1 Lagrangian for the W boson

The Lagrangian describing the interaction of the W boson with the photon field is

$$\mathcal{L} = -\frac{1}{2}(D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) + m^2 W^{\mu\dagger} W_\mu - ieW^{\mu\dagger} W^\nu F_{\mu\nu} \qquad (3.1.1)$$

This Lagrangian is part of a renormalizable theory. If the free g-factor is calculated at tree level, it will be found that $g = 2$. Phenomenologically, behavior when $g \neq 2$ is

interesting, so the Lagrangian can be modified to

$$\mathcal{L} = -\frac{1}{2}(D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) + m^2 W^{\mu\dagger} W_\mu - i[g-1]eW^{\mu\dagger} W^\nu F_{\mu\nu} \quad (3.1.2)$$

When $g = 2$ the two Lagrangians are identical. The contribution to the free g-factor comes from two separate terms, but modifying the other term would necessitate also modifying the kinetic terms of the Lagrangian.

In this Lagrangian, W_μ is the charged boson field of mass m, and A_μ is the massless photon field. Both are vector fields, but W has three degrees complex of freedom and A has two real degrees of freedom. The electromagnetic field-strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

3.1.2 Equations of motion

The goal is to obtain, from the relativistic Lagrangian, a nonrelativistic Schrodinger-like equation that gives the nonrelativistic Hamiltonian. As before, with spin one-half, the path will be to first find the relativistic equations of motion, solve them for the relativistic energies of the particles, and then consider how the one-particle arises.

If we call the six component bispinor Υ, then the relativistic Hamiltonian will have the form $i\partial_0 \Upsilon = \hat{H}\Upsilon$, or explicitly writing the upper and lower components of the bispinor and the block structure of the Hamiltonian:

$$i\partial_0 \begin{pmatrix} \Upsilon_u \\ \Upsilon_\ell \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \Upsilon_u \\ \Upsilon_\ell \end{pmatrix} \quad (3.1.3)$$

Seeking an equation of this form, first the equations of motion are obtained using the Euler-Lagrange method:

$$\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial[\partial_\mu W^{\dagger\alpha}]} = 0 \quad (3.1.4)$$

It is easiest to use integration by parts to have all derivative operators act to the right, upon W. The equation of motion obtained is

$$m^2 W_\alpha - ie[g-1]W^\nu F_{\alpha\nu} + D_\mu (D^\mu W_\alpha - D_\alpha W^\mu) = 0 \quad (3.1.5)$$

This is a set of four second order equations for W_μ. But (3.1.3) is a first order equation. Second order equations may be transformed into first order by introducing additional fields. So introduce the field $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$. Being antisymmetric, it has 6 components, in addition to the four components of W_μ. The resulting set of first order equations is

$$W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu \tag{3.1.6}$$

$$m^2 W_\alpha - ie[g-1]W^\nu F_{\alpha\nu} + D^\mu W_{\mu\alpha} = 0 \tag{3.1.7}$$

In addition to being first order, the Schrodinger like equation for a spin-one particle should involve a three component spinor. It is obtained from a relativistic equation for a bispinor, with the upper and lower components each having three components. So the ten total degrees of freedom above need to be reduced somehow.

On inspection, it turns out that only six of the components are dynamic – the other four occur only in equations which do not involve the time derivative. W_0 is one such field, the other three are $W_{ij} = D_i W_j - D_j W_i$. These four fields need not appear in any Schrodinger-like equation expressing the time evoluation of the fields, and so the necessary 6 components are obtained.

The six components that remain are W_i and $W_{0i} = -W_{i0}$. Eventually these need to be arranged in a bispinor, but note that W_{i0} has a different mass dimension and different Hermiticity than W_i. So define instead the field

$$\eta_i = -\frac{i}{m} W_{0i} \tag{3.1.8}$$

This is the quantity that will appear directly in the bispinor. Note that in the momentum space and in the rest frame, that $\eta_i = -W_i$.

Now, express the equations of motion only in terms of η_i and W_i, eliminating the other fields. (As defined, $W_{0i} = im\eta_i$.) Three of the nondynamic fields can be replaced directly with

$$W_{ij} = D_i W_j - D_j W_i \tag{3.1.9}$$

To find W_0, consider the second equation of motion with $\alpha = 0$, and solve for W_0.

$$m^2 W_0 - ie[g-1]W^\nu F_{0\nu} + D^\mu W_{\mu 0} = 0 \tag{3.1.10}$$

$$m^2 W_0 - ie[g-1]W^j F_{0j} + D^j W_{j0} = 0 \tag{3.1.11}$$

$$W_0 = \frac{1}{m^2}\left(ie[g-1]W^j F_{0j} + D^j im\eta_j\right) \tag{3.1.12}$$

The remaining equations of motion all somehow involve the time derivative. From

$$W_{0i} = D_0 W_i - D_i W_0 \tag{3.1.13}$$

is obtained

$$im\eta_i = D_0 W_i - D_i \frac{1}{m^2}\left(ie[g-1]W^j F_{0j} + D^j im\eta_j\right) \tag{3.1.14}$$

And from the (3.1.7) equation of motion with $\alpha = i$:

$$m^2 W_i - ie[g-1]W^\nu F_{i\nu} + D^\mu W_{\mu i} = 0 \tag{3.1.15}$$

$$m^2 W_i - ie[g-1]\left(W^0 F_{i0} + W^j F_{ij}\right) + D^0 W_{0i} + D^j W_{ji} = 0 \tag{3.1.16}$$

$$m^2 W_i - ie[g-1]\left(\frac{1}{m^2}\left(ie[g-1]W^j F_{0j} + D^j im\eta_j\right) F_{i0} + W^j F_{ij}\right) + D^0 im\eta_i + D^j\left(D_i W_j - D_j W_i\right) = 0 \tag{3.1.17}$$

These can be solved for the quantities $D_0\eta_i$ and $D_0 W_i$.

To descend to the lower energy theory, it will be most useful to write everything in terms of three vectors. Spatial vectors should be written with their indices naturally raised, while the derviative operator $\mathbf{D}$ is naturally lowered. The components of $F_{\mu\nu}$ in terms of three vectors are:

$$F_{0i} = -E_i = E^i,\ F_{ij} = -\epsilon_{ijk}B^k \tag{3.1.18}$$

Then (3.1.17) becomes

$$m^2 W_i - \frac{e^2[g-1]^2}{m^2} W^j E^j E^i - \frac{e[g-1]}{m} D_j \eta^j E^i + ie[g-1]\epsilon_{ijk} B^k W^j + D^0(im\eta_i) + D^j(D_i W_j - D_j W_i) = 0 \tag{3.1.19}$$

Or, writing all vector indices in the "natural" position, and writing contractions between such "natural" vectors as dot products,

$$-m^2 W_i - \frac{e^2[g-1]^2}{m^2} E^i \mathbf{E}\cdot\mathbf{W} - \frac{e[g-1]}{m} E^i \mathbf{D}\cdot\boldsymbol{\eta} + ie[g-1]\epsilon_{ijk} B^k W^j - D^0(im\eta^i) + D_i \mathbf{D}\cdot\mathbf{W} - \mathbf{D}^2 W^i = 0 \tag{3.1.20}$$

And likewise with (3.1.14)

$$im\eta_i = D_0 W_i - D_i \frac{1}{m^2}\left(ie[g-1]W^j E^j + D^j im\eta_j\right) \tag{3.1.21}$$

which with natural indices is

$$-im\eta^i = -D_0 W^i - D_i \frac{1}{m^2}\left(ie[g-1]\mathbf{E}\cdot\mathbf{W} + im\mathbf{D}\cdot\boldsymbol{\eta}\right) \tag{3.1.22}$$

Next, solve for the quantities $iD_0\eta^i$ and $iD_0 W^i$.

$$iD_0 W^i = -m\eta^i + D_i \frac{1}{m^2}\left(e[g-1]\mathbf{E}\cdot\mathbf{W} + m\mathbf{D}\cdot\boldsymbol{\eta}\right) \tag{3.1.23}$$

$$\begin{aligned} iD_0\eta^i = &- mW_i - \frac{e^2[g-1]^2}{m^3} E^i \mathbf{E}\cdot\mathbf{W} - \frac{e[g-1]}{m^2} E^i \mathbf{D}\cdot\boldsymbol{\eta} \\ &+ \frac{ie[g-1]}{m}\epsilon_{ijk} B^k W^j + \frac{1}{m}\left(D_i \mathbf{D}\cdot\mathbf{W} - \mathbf{D}^2 W^i\right) \end{aligned} \tag{3.1.24}$$

Here are the desired equations for the dynamics of the η and W. However, they are still not in the desired form (3.1.3), because it is not written in terms of operators acting directly upon the fields. This is because the different components of the field are mixed. These different components correspond to different spin states of the particle, so their mixing can be written as the action of spin space operators on the fields. To explicitly disentangle the equations in this manner, it will be useful to first derive a couple of identities relating to these spin operators.

3.1.3 Spin identities

The spin matrix for a spin one particle can represented as:

$$(S^k)_{ij} = -i\epsilon_{ijk} \tag{3.1.25}$$

which leads to the following identities:

$$(\mathbf{a} \times \mathbf{v})_i = -i(\mathbf{S} \cdot \mathbf{a})_{ij} v_j \tag{3.1.26}$$

$$a_i(\mathbf{b} \cdot \mathbf{v}) = \{\mathbf{a} \cdot \mathbf{b}\, \delta_{ij} - (S^k S^\ell)_{ij} a^\ell b^k\} v_j \tag{3.1.27}$$

Using these identities

$$iD^0\mathbf{W} = -\frac{1}{m^2}\{\mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\}\mathbf{W} + \frac{1}{m}\{\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2\}\boldsymbol{\eta} + m\boldsymbol{\eta} \tag{3.1.28}$$

and for the other equation

$$iD_0\boldsymbol{\eta} = m\mathbf{W} + \frac{e^2\lambda^2}{m^3}\{\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\}\mathbf{W} - \frac{1}{m^2}\{\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\}\boldsymbol{\eta} + \frac{e\lambda}{m}(\mathbf{S}\cdot\mathbf{B})\mathbf{W} - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2\mathbf{W} \tag{3.1.29}$$

$$iD_0\boldsymbol{\eta} = \left(m + \frac{e^2\lambda^2}{m^3}\{\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\} - \frac{e\lambda}{m}(\mathbf{S}\cdot\mathbf{B}) - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2\right)\mathbf{W} + \frac{1}{m^2}\{\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\}\boldsymbol{\eta} \tag{3.1.30}$$

Now that the equations are in the correct form, we can write them as follows:

$$iD_0 \begin{pmatrix} W \\ \eta \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} W \\ \eta \end{pmatrix} \tag{3.1.31}$$

Where the components of M are

$$\begin{aligned}
M_{11} &= [g-1]\frac{e}{m^2}\left[\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot\mathbf{E}\times\mathbf{D}\right] \\
M_{12} &= m - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2 - [g-1]\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + [g-1]^2\frac{e^2}{m^3}\left[\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\right] \\
M_{21} &= m - \frac{1}{m}\left[\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2 + e\mathbf{S}\cdot\mathbf{B}\right] \\
M_{22} &= -[g-1]\frac{e}{m^2}\left[\mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) + i\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}\right]
\end{aligned}$$

This leads to a relativistic Hamiltonian, but not one of quite the right form. The η and W do not correspond to the particle and antiparticle. To see this consider the rest frame where $D_i = 0$, then the two components have equal magnitude. A transformation to a representation where the upper and lower components correspond to particle-antiparticle will be necessary. Since it is not yet in the final desired form, call the bispinor

$$\Upsilon' = \begin{pmatrix} \eta \\ W \end{pmatrix} \tag{3.1.32}$$

Also, it is clear that the matrix on the right hand side is not Hermitian in the traditional sense, and so the Hamiltonian would not be either. The necessary condition is that the Hamiltonian is Hermitian *with respect to the inner product.* To discover what the inner product should look like, the electromagnetic current density of the charged particle will be investigated. Before transforming representation, this matter of Hermiticity will be dealt with.

3.1.4 Current density

The inner product will take two wave functions and map them onto a scalar. To be interpreted as a probability, the inner product must be Lorentz invariant. In standard quantum mechanics the inner product is $\langle \xi, \chi \rangle = \int d^3x\, \xi^\dagger \chi$. For the spinors W and η defined above, this will not be a Lorentz invariant quantity — to discover such a quantity, consider electromagnetic current density. Since $Q = \int d^3x\, j_0$ is invariant, the form of j_0 it will suggest the correct form for the inner product.

The conserved current can be derived from the Lagrangian (3.1.2). The electromagnetic current corresponds to the transformation $W_i \to e^{i\alpha} W_i$, which in infinitesimal form is:

$$W_\mu \to W_\mu + i\alpha W_\mu,\ W_\mu^\dagger \to W_\mu^\dagger - i\alpha W_\mu^\dagger \tag{3.1.33}$$

The 4-current density will be:

$$j^\sigma = -i\frac{\partial \mathcal{L}}{\partial W_{\mu,\sigma}} W_\mu + i\frac{\partial \mathcal{L}}{\partial W_{\mu,\sigma}^\dagger} W_\mu^\dagger \tag{3.1.34}$$

Only one term in the Lagrangian contains derivatives of the field:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial W_{\alpha,\sigma}} &= \frac{\partial}{\partial W_{\alpha,\sigma}} \left\{ -\frac{1}{2}(D_\mu W_\nu - D_\nu W_\mu)^\dagger (D^\mu W^\nu - D^\nu W^\mu) \right\} \\
&= -\frac{1}{2}(D_\mu W_\nu - D_\nu W_\mu)^\dagger (g_{\sigma\mu} g_{\alpha\nu} - g_{\sigma\nu} g_{\alpha\mu}) \\
&= -(D_\alpha W_\sigma - D_\sigma W_\alpha)^\dagger
\end{aligned}$$

Likewise:

$$\frac{\partial \mathcal{L}}{\partial W_{\alpha,\sigma}^\dagger} = -(D_\alpha W_\sigma - D_\sigma W_\alpha) \tag{3.1.35}$$

As before, define $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$; then the 4-current and charge density are:

$$\begin{aligned}
j_\sigma &= iW_{\sigma\mu}^\dagger W^\mu - iW_{\sigma\mu} W^{\dagger\mu} \\
j_0 &= iW_{0\mu}^\dagger W^\mu - iW_{0\mu} W^{\dagger\mu} \\
&= iW_{0i}^\dagger W^i - iW_{0i} W^{\dagger i}
\end{aligned}$$

Where the last equality follows from the antisymmetry of $W_{\mu\nu}$.

Now, previously was defined $\eta_i = -iW_{i0}/m$. In terms of this quantity, the current density is

$$j_0 = m(\eta_i^\dagger W^i + \eta_i W^{\dagger i}) \tag{3.1.36}$$

Now the Hermiticity of the matrix above can be investigated in light of the inner product this suggests.

3.1.5 Hermiticity and the inner product

As already mentioned, it is clear from inspection that the matrix in (3.1.43) is not Hermitian in the standard sense. (Of the operators in use, the only one which is not self adjoint is $D_i^\dagger = -D_i$.) Noticing that

$$[\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot\mathbf{E}\times\mathbf{D}]^\dagger = -[\mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) + i\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}] \quad (3.1.37)$$

it can be seen that H would have the general form

$$H = \begin{pmatrix} A & B \\ C & A^\dagger \end{pmatrix} \quad (3.1.38)$$

where the off diagonal blocks are Hermitian in the normal sense: $B^\dagger = B$ and $C^\dagger = C$.

An operator is defined as Hermitian with respect to a particular inner product. One generalisation of the normal inner product from quantum mechanics is found by allowing a weight matrix Ω inserted between the wave functions:

$$\langle \xi, \chi \rangle = \int d^3x\, \xi^\dagger \Omega \chi \quad (3.1.39)$$

In the usual product of quantum mechanics Ω would be the identity matrix, but here it differs. Previously the conserved charge was derived. Writing (3.1.36) in the form $\Upsilon'^\dagger \Omega \Upsilon'$

$$j_0 = m[\eta^\dagger W + W^\dagger \eta] = m \begin{pmatrix} \eta^\dagger & W^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ W \end{pmatrix} \quad (3.1.40)$$

so if the weight is defined as $\Omega = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, the inner product $\langle \Upsilon', \Upsilon \rangle$ will be necessarily be conserved.

An operator H is Hermitian with respect to this inner product if

$$\langle H\xi, \chi \rangle = \langle \xi, H\chi \rangle \rightarrow \int d^3x \xi^\dagger H^\dagger \Omega \chi = \int d^3x \xi^\dagger \Omega H \chi \quad (3.1.41)$$

For this equality to hold, it is sufficient for $H^\dagger \Omega = \Omega H$. With Ω as above, and $H = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$

this condition reduces to

$$\begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \tag{3.1.42}$$

The matrix in (3.1.43) fulfills exactly this requirement, and so is Hermitian with respect to this particular inner product.

3.1.6 Non-relativistic Hamiltonian

Now that the issue of hermiticity has been dealt with, it is time to find a non-relativistic Hamiltonian from the equation (3.1.43). First, write it explicitly as a Hamiltonian, replacing the derivative operator with $\boldsymbol{\pi} = \mathbf{p} - e\mathbf{A}$, so that $\mathbf{D} = -i\boldsymbol{\pi}$. Then if

$$H'\Upsilon' = \begin{pmatrix} H'_{11} & H'_{12} \\ H'_{21} & H'_{22} \end{pmatrix} \begin{pmatrix} W \\ \eta \end{pmatrix} \tag{3.1.43}$$

Then

$$\begin{aligned}
H'_{11} &= e\Phi + [g-1]\frac{e}{m^2}\left[-i\mathbf{E}\cdot\boldsymbol{\pi} + i(\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\boldsymbol{\pi}) + \mathbf{S}\cdot\mathbf{E}\times\pi\right] \\
H'_{12} &= m + \frac{1}{m}(\mathbf{S}\cdot\boldsymbol{\pi})^2 - [g-1]\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + [g-1]^2\frac{e^2}{m^3}\left[\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\right] \\
H'_{21} &= m + \frac{1}{m}\left[\boldsymbol{\pi}^2 - (\mathbf{S}\cdot\boldsymbol{\pi})^2 - e\mathbf{S}\cdot\mathbf{B}\right] \\
H'_{22} &= e\Phi - [g-1]\frac{e}{m^2}\left[-i\boldsymbol{\pi}\cdot\mathbf{E} + i(\mathbf{S}\cdot\boldsymbol{\pi})(\mathbf{S}\cdot\mathbf{E}) + \mathbf{S}\cdot\boldsymbol{\pi}\times\mathbf{E}\right]
\end{aligned}$$

That equation is written in a representation where the particle and anti-particle states are not separated. Even in the nonrelativistic limit, it strongly couples η and W, because the off-diagonal terms are of $\mathcal{O}(m)$. To descend to a nonrelativistic picture, it will be most convenient if the lower component of Υ is, in that limit, only weakly coupled to the upper.

In the rest frame of the particle state (which should have $E = m$), $W = \eta$. Then, if the lower component was defined to be $\Upsilon_\ell = \eta - W$, it would vanish in the rest frame and be small compared to Υ_u in any nonrelativistic frame. The orthogonal upper component would then be $\Upsilon_u = \eta + W$. For an anti-particle state in the rest frame ($E = -m$, $W = -\eta$) and the upper component vanishes.

Implementing this transformation as a unitary transformation requires the matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3.1.44}$$

(This transformation will transform the current to $j_0 = m(\Upsilon_u^\dagger \Upsilon_u - \Upsilon_\ell^\dagger \Upsilon_\ell)$ and the weight M, used in the inner product, to $M \to M' = U^\dagger M U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ The transformed Hamiltonian will of course be Hermitian with respect to the transformed inner product.)

After implementing this transformation, $H' \to H = U^\dagger H' U$. The Hamiltonian still contains off-diagonal elements, so the Schrodinger-like equation represents a pair of coupled equations for the upper and lower components of $\boldsymbol{\Upsilon}$. But the off-diagonal terms are small, so Υ_ℓ can be considered small compared to Υ_u. Solving for Υ_ℓ in terms of Υ_u and the block components of H:

$$\begin{aligned} E\Upsilon_\ell &= H_{21}\Upsilon_u + H_{22}\Upsilon_\ell \\ \Upsilon_\ell &= (E - H_{22})^{-1} H_{21} \Upsilon_u \end{aligned}$$

This gives the exact formula:

$$E\Upsilon_u = \left(H_{11} + H_{12}[E - H_{22}]^{-1} H_{21}\right) \Upsilon_u$$

However, we only need corrections to the magnetic moment of order v^2. This means we only need the Hamiltonian to at most order mv^4 or $(e/m)Bv^2$. Examining the leading order terms of the matrix H, the diagonal elements are order m while the off-diagonal elements are order mv^2 or $(e/m)B$. To leading order the term $[E - H_{22}]^{-1} = \frac{1}{2m}$. So we'll need H_{11} to $\mathcal{O}(v^4, (e/m)Bv^2)$, and H_{12}, H_{21}, and $[E - H_{22}]^{-1}$ each to only the leading order.

$$E\Upsilon_u = \left(H_{11} + \frac{1}{2m} H_{12} H_{21} + \mathcal{O}(mv^6)\right) \Upsilon_u$$

The needed terms of H are, after performing the transformation:

$$\begin{aligned}
H_{11} &= H'_{11} + H'_{12} + H'_{21} + H'_{22} \\
&\approx m + e\Phi + \frac{\pi^2}{2m} - \frac{g}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} \\
&\quad -i(g-1)\frac{e}{2m^2}\left[\mathbf{E}\cdot\boldsymbol{\pi} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\boldsymbol{\pi}) + i\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi} - \boldsymbol{\pi}\cdot\mathbf{E} + (\mathbf{S}\cdot\boldsymbol{\pi})(\mathbf{S}\cdot\mathbf{E}) - i\mathbf{S}\cdot\boldsymbol{\pi}\times\mathbf{E}\right] \\
H_{12} &= H'_{11} - H'_{12} + H'_{21} - H'_{22} \\
&\approx -\frac{\pi^2}{2m} + \frac{1}{m}(\mathbf{S}\cdot\boldsymbol{\pi})^2 - \frac{g-2}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} \\
H_{21} &= H'_{11} + H'_{12} - H'_{21} - H'_{22} \\
&\approx \frac{\pi^2}{2m} - \frac{1}{m}(\mathbf{S}\cdot\boldsymbol{\pi})^2 + \frac{g-2}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B}
\end{aligned}$$

The product $H_{12}H_{21}$ is calculated in the appendix. To first order in the magnetic field strength the result (**??**) is:

$$\frac{1}{2m}H_{12}H_{21} = -\frac{1}{2m^3}\left(\frac{\boldsymbol{\pi}^4}{4} - e\mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - \frac{g-2}{2}e(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\right) \tag{3.1.45}$$

So finally, the direct expression for $E\Upsilon_u$ is:

$$\begin{aligned}
E\Upsilon_u &= \Big\{m + e\Phi + \frac{\pi^2}{2m} - \frac{g}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} - \frac{\pi^4}{8m^3} + \frac{e\mathbf{p}^2(\mathbf{S}\cdot\mathbf{B})}{2m^3} + (g-2)\frac{e}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \\
&\quad +(g-1)\frac{ie}{2m^2}\left[\mathbf{E}\cdot\boldsymbol{\pi} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\boldsymbol{\pi}) + i\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi} - \boldsymbol{\pi}\cdot\mathbf{E} + (\mathbf{S}\cdot\boldsymbol{\pi})(\mathbf{S}\cdot\mathbf{E}) - i\mathbf{S}\cdot\boldsymbol{\pi}\times\mathbf{E}\right]\Big\}\Upsilon_u
\end{aligned}$$

The complicated expression in square brackets can be cleaned up a bit:

$$
\begin{aligned}
\mathbf{E}\cdot\boldsymbol{\pi}-\boldsymbol{\pi}\cdot\mathbf{E} &= [E_i,\pi_i] \\
&= [E_i,-i\partial_i] \\
&= i(\partial_i E_i) \\
(\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\boldsymbol{\pi})-(\mathbf{S}\cdot\boldsymbol{\pi})(\mathbf{S}\cdot\mathbf{E}) &= S_iS_jE_i\pi_j - S_iS_j\pi_iE_j \\
&= (S_iS_j)(E_i\pi_j - E_j\pi_i - [\pi_i,E_j]) \\
&= [S_i,S_j](E_i\pi_j) - (S_iS_j)(-i\nabla_i E_j) \\
&= (i\epsilon_{ijk}S_k)E_j\pi_i - (S_iS_j)(-i\nabla_i E_j) \\
&= i\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi} + iS_iS_j\nabla_i E_j) \\
(\mathbf{E}\times\boldsymbol{\pi}-\boldsymbol{\pi}\times\mathbf{E})_k &= \epsilon_{ijk}(E_i\pi_j - \pi_iE_j) \\
&= \epsilon_{ijk}(E_i\pi_j + \pi_jE_i) \\
&= \epsilon_{ijk}(2E_i\pi_j + [\pi_i,E_j]) \\
&= 2\epsilon_{ijk}E_i\pi_j \\
&= 2(\mathbf{E}\times\boldsymbol{\pi})_k
\end{aligned}
$$

Using these identities and collecting terms, and then writing everything in terms of g, $g-2$

$$
\begin{aligned}
E\Upsilon_u &= \left\{ m + e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} - \frac{e}{m}\mathbf{S}\cdot\mathbf{B}\left(\frac{g}{2} - \frac{p^2}{2m^2}\right) + (g-2)\frac{e}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \right. \\
&\quad \left. -(g-1)\frac{e}{2m^2}\left[\nabla\cdot\mathbf{E} - S_iS_j\nabla_iE_j + \mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi}\right] \right\}\Upsilon_u \\
&= \left\{ m + e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} - \frac{g}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\left(1 - \frac{p^2}{2m^2}\right) - \frac{g-2}{2}\frac{e}{m}\frac{p^2}{2m^2}\mathbf{S}\cdot\mathbf{B} + (g-2)\frac{e}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \right. \\
&\quad \left. -\left(\frac{g}{2} + \frac{g-2}{2}\right)\frac{e}{2m^2}\left[\nabla\cdot\mathbf{E} - S_iS_j\nabla_iE_j + \mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi}\right] \right\}\Upsilon_u
\end{aligned}
$$

This is a Hamiltonian for the upper component of the bispinor, as desired. In the spin-half case, the Foldy-Wouthyusen transformation was necessary, going to a representation where all the physics up to the desired order is contained in the single spinor equation. However, to the needed order, it can be shown that the Hamiltonian above is correct.

3.1.7 Schrodinger-like wave functions

It is necessary to establish a connection between the upper component of the spinor Υ_u and the Schrodinger-like wave function ϕ_S. Because the lower component of the bispinor is small but nonzero, it is not necessarily true that the above equation accurately captures the physics. As in spin-half, transformation to a basis where the lower component is truly negligible might be necessary.

However, here it can be shown that such a transformation will have no effect at the desired order. The transformation U would have the form:: $U = e^{iS}$ where S is Hermitian. Because the Hamiltonian is diagonal at leading order, S must be small, and the transformation will affect Υ_u as

$$\Upsilon_u \to \phi_S = (1 + \Delta)\Upsilon_u$$

where Δ is some small operator. The probability density must be unaffected by this change, so: On the one hand, with $\Upsilon_\ell = \epsilon\Upsilon_u$, $\epsilon \sim \mathcal{O}(v^2)$

$$\begin{aligned} \int d^3x(\Upsilon_u^\dagger\Upsilon_u - \Upsilon^\dagger{}_\ell\Upsilon_\ell) &= \int d^3x(\Upsilon_u^\dagger\Upsilon_u - (\epsilon\Upsilon_u)^\dagger\epsilon\Upsilon_u) \\ &= \int d^3x\Upsilon_u^\dagger(1 + \mathcal{O}(v^4))\Upsilon_u \end{aligned}$$

And on the other hand:

$$\int d^3x\phi_S^\dagger\phi_S = \int d^3x([1 + \Delta]\Upsilon_u)^\dagger(1 + \Delta)\Upsilon_u$$

Comparing the two, it can be seen that Δ must be no larger than $\mathcal{O}(v^4)$. Now considering

the new equation for ϕ_S:

$$\begin{aligned}
(E-m)\Upsilon_u &= \left(H_{11}+\frac{1}{2m}H_{12}H_{21}\right)\Upsilon_u \\
(E-m)(1-\Delta)\phi_S &= \left(H_{11}+\frac{1}{2m}H_{12}H_{21}\right)(1-\Delta)\phi_S \\
(E-m)\phi_S &= (1+\Delta)\left(H_{11}+\frac{1}{2m}H_{12}H_{21}\right)(1-\Delta)\phi_S
\end{aligned}$$

Since $H \sim \mathcal{O}(mv^2)$ and $\Delta \sim \mathcal{O}(v^4)$, to $\mathcal{O}(v^4)$, ϕ_S obeys exactly the same equation as Υ_u:

$$(E-m)\phi_S = \left(H_{11}+\frac{1}{2m}H_{12}H_{21}\right)\phi_S$$

Because it obeys the same equation, it then follows that the Hamiltonian for ϕ_S is just that already found for Υ_u:

$$\begin{aligned}
E\phi_S &= \Big\{m+e\Phi+\frac{\pi^2}{2m}-\frac{\pi^4}{8m^3}-\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\left(\frac{g}{2}-\frac{p^2}{2m^2}\right)+(g-2)\frac{e}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \\
&\quad -(g-1)\frac{e}{2m^2}\left[\nabla\cdot\mathbf{E}-S_iS_j\nabla_iE_j+\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi}\right]\Big\}\Upsilon_u \\
&= \Big\{m+e\Phi+\frac{\pi^2}{2m}-\frac{\pi^4}{8m^3}-\frac{g}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\left(1-\frac{p^2}{2m^2}\right)-\frac{g-2}{2}\frac{e}{m}\frac{p^2}{2m^2}\mathbf{S}\cdot\mathbf{B}+(g-2)\frac{e}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \\
&\quad -\left(\frac{g}{2}+\frac{g-2}{2}\right)\frac{e}{2m^2}\left[\nabla\cdot\mathbf{E}-S_iS_j\nabla_iE_j+\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi}\right]\Big\}\phi_S
\end{aligned}$$

Chapter 4

Diagrammatic approach to spin-one particles

4.1 Introduction

In the previous section, a nonrelativistic Hamiltonian was derived from a relativistic Lagrangian of a charged spin one particle, by solving the equations of motion for the energies of the fields. As in the case of a spin one-half particle, there is an alternate approach. The most general NRQED Lagrangian for a spin one particle interacting with an electromagnetic field can be written down. Then, by considering the same physical processes in the NRQED theory and the initial relativistic spin one theory, the coefficients in the NRQED Lagrangian may be fixed.

As for spin one-half, the most appropriate physical processes to consider are scattering off an external magnetic field and Compton scattering. (The latter is not strictly necessary, but is a check of the consistency of the calculation.)

First the NRQED Lagrangian for a spin one particle will be developed. Then, the scattering processes will be calculated from this Lagrangian. Next the same processes will be calculated in the relativistic theory, and a connection between the relativistic polarisations and nonrelativistic spinors established. Finally the two calculations will be compared, and thus the NRQED Lagrangian determined.

4.2 Nonrelativistic Lagrangian for spin one

For spin half, the NRQED Lagrangian (2.3.1) has been constructed. All of the terms that exist in that Lagrangian can occur in the spin one Lagrangian as well. So what must be established is what type of new terms arise in the higher spin theory.

All the "building blocks" used in constructing the spin half theory may be used: the fields $\mathbf{E}$ and $\mathbf{B}$, the gauge invariant derivative $\mathbf{D}$, and the spin operator $\mathbf{S}$. In moving from spin half to spin one, the only new terms allowed are those that are quadratic in S. In the spin half theory, the fields had two components, and so together with the identity, the set S_i completely spanned the space of Hermitian spin operators. In spin one the nonrelativistic fields have three components, and so the basis for spin operators is larger.

To ensure that new terms introduced are completely independent of those specific to spin half, combinations of S matrices which vanish for spin half should be considered. The appropriate combination is the symmetric, traceless structure $\overline{S}_{ij} \equiv S_iS_j+S_jS_i-(2/3)S^2\delta_{ij}$, which for spin one is $S_iS_j + S_jS_i - (4/3)\delta_{ij}$.

So, any new term will be some combination of $\overline{S}_{ij}$ and E, B, and D. $\overline{S}_{ij}B_iD_j$ and $\overline{S}_{ij}B_iE_j$ are banned by parity, and terms quadratic in B are not considered. Terms quadratic in E are also too high order. And $\overline{S}_{ij}D_iD_j$ is not allowed because it would spoil the structure of the kinetic term.

So the only new term that arises involves $\overline{S}_{ij}$, E, and D. The Hermitian combination of these of the proper dimension is

$$\frac{e}{8m^2}\ \overline{S}_{ij}(E_iD_j - D_jE_i) = \frac{e\overline{S}_{ij}\partial_iE_j}{8m^2} \tag{4.2.1}$$

This structure is related to the quadrupole moment, so label the coefficient c_Q. Then, the complete spin one Lagrangian will be

$$\begin{aligned}\mathcal{L}_{NRQED} \quad = \quad & \Psi^\dagger\{i(\partial_0 + ieA_0) + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^3} + c_F e\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2} + c_Q\frac{e\overline{S}_{ij}(D_iE_j - E_iD_j)}{8m^2} \\ & + c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D})}{8m^2} + c_{W_1}\frac{e[\mathbf{D}^2(\mathbf{S}\cdot\mathbf{B}) + (\mathbf{S}\cdot\mathbf{B})\mathbf{D}^2]}{8m^3} - c_{W_2}\frac{eD^i(\mathbf{S}\cdot\mathbf{B})D^i}{4m^3} + c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})}{8m^3}\}\Psi\end{aligned}$$

4.2.1 Scattering off external field in NRQED

With the Lagrangian in hand, the amplitude for the charged particle scattering off an external field can be calculated. At the tree level (which is the only level contributing to the final calculation) the process involves just a single photon. So consider those terms in the NRQED Lagrangian which contain one power of the external field.

This calculation is largely the same as that for spin one-half, with the only change being the addition of the quadrupole term. Augmented by this term, the Lagrangian (**??**) becomes

$$\begin{aligned}\mathcal{L}_A &= \Psi^\dagger(-eA_0 - ie\frac{\{\nabla_i, A_i\}}{2m} - ie\frac{\{\boldsymbol{\nabla}^2, \{\nabla_i, A_i\}\}}{8m^3} + c_F e\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{e(\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{E}\cdot\boldsymbol{\nabla})}{8m^2} + c_Q\frac{eQ_{ij}(\nabla_i E_j - E_i\nabla_j)}{8m^2} \\ &\quad + c_S^1\frac{ie\mathbf{S}\cdot(\boldsymbol{\nabla}\times\mathbf{E} - \mathbf{E}\times\boldsymbol{\nabla})}{8m^2} + c_{W_1}\frac{e[\boldsymbol{\nabla}^2(\mathbf{S}\cdot\mathbf{B}) + (\mathbf{S}\cdot\mathbf{B})\boldsymbol{\nabla}^2]}{8m^3} - c_{W_2}\frac{e\nabla^i(\mathbf{S}\cdot\mathbf{B})\nabla^i}{4m^3} + c_{p'p}\frac{e[(\mathbf{S}\cdot\boldsymbol{\nabla})(\mathbf{B}\cdot\boldsymbol{\nabla}) + (}{8m^3}\end{aligned}$$

The process under consideration is this: scattering off an external field, with incoming momentum $\mathbf{p}$, outgoing $\mathbf{p'}$, and $\mathbf{q} = \mathbf{p'} - \mathbf{p}$. There is one diagram associated with each term above, but the total amplitude is just going to be the sum of all these one-photon vertices. These of course can just be read off directly from the Lagrangian: replace the fields Ψ with the spinors ϕ, and any operator $\boldsymbol{\nabla}$ acting will become $i\mathbf{p}$ if it acts on the right, $i\mathbf{p'}$ if it is to the left.

Some expressions involving $\boldsymbol{\nabla}$ and $\mathbf{E}$ can be simplified. Because Q_{ij} is symmetric:

$$Q_{ij}(\nabla_i E_j - E_i\nabla_j) = Q_{ij}[\nabla_i, E_j] = Q_{ij}(\partial_i E_j)$$

And because $E_i = -\partial_i\Phi$

$$\boldsymbol{\nabla}\times\mathbf{E} - \mathbf{E}\times\boldsymbol{\nabla} = -2\mathbf{E}\times\boldsymbol{\nabla}$$

And also use that

$$\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{E}\cdot\boldsymbol{\nabla} = (\partial_i E_i)$$

Now the scattering amplitude for scattering off the external field can be written down.

Before any assumptions about the particular process are made, it is:

$$\begin{aligned} iM &= ie\phi^\dagger\Bigg(-A_0 + \frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')}{2m} - \frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')\mathbf{p}^2 + \mathbf{p}'^2\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')}{8m^3} \\ &\quad + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2} + c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2} + c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2} \\ &\quad - c_{W_1}\frac{(\mathbf{S}\cdot\mathbf{B})(\mathbf{p}^2+\mathbf{p}'^2)}{8m^3} + c_{W_2}\frac{(\mathbf{S}\cdot\mathbf{B})(\mathbf{p}\cdot\mathbf{p}')}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p}')(\mathbf{B}\cdot\mathbf{p}) + (\mathbf{B}\cdot\mathbf{p}')(\mathbf{S}\cdot\mathbf{p})}{8m^3}\Bigg)\phi \end{aligned}$$

The above can be simplified somewhat. The gauge can be chosen such that $\nabla_i A_i = 0$. If elastic scattering is specified then kinematics dictate that $\mathbf{p}'^2 = \mathbf{p}^2$. Finally, if $\mathbf{B}$ is constant, the c_W terms become indistinguishable, since $[\nabla_i, B_j] = 0$. (It is only this last assumption that costs us any information.) Then the scattering amplitude, as calculated from $\mathcal{L}_{NRQED}$, is:

$$\begin{aligned} iM &= ie\phi^\dagger\Bigg(-A_0 + \frac{\mathbf{A}\cdot\mathbf{p}}{m} - \frac{(\mathbf{A}\cdot\mathbf{p})\mathbf{p}^2}{2m^3} + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2} + c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2} \\ &\quad + c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2} - (c_{W_1} - c_{W_2})\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3}\Bigg)\phi \end{aligned}$$

Calculating the scattering amplitude for the same process in the relativistic theory will fix the unknown coefficients. Then the NRQED Lagrangian can be used in other calculations, such as for the bound g-factor.

4.2.2 Compton scattering in NRQED

While the previous calculation, together with gauge invariance, is enough to fix all relevant coefficients, for consistency the two photon process of Compton scattering can also be considered in NRQED.

The gauge is chosen such that the photon polarisations obey $\epsilon_0 = 0$. In general terms arising from two-photon vertices, and those from tree level diagrams of two one-photon vertices should both be considered. However, as with the similar calculation for spin half, only the contact like terms consisting of the two-photon vertices are needed. The QED calculation will be organised in such a way that nonlocal and local terms are separated.

From the full Lagrangian (4.2), consider only those terms which involve two powers of the photon field. Note that both $(\mathbf{A}\cdot\mathbf{E}-\mathbf{E}\cdot\mathbf{A})=0$ and $\overline{S}_{ij}(A_iE_j-E_iA_j)=0$ by symmetry. The remaining terms of interest are:

$$\mathcal{L}_{A^2} \quad = \quad \Psi^\dagger(-\frac{e^2\mathbf{A}^2}{2m}-e^2\frac{\{\boldsymbol{\nabla}^2,\mathbf{A}^2\}}{8m^3}-e^2\frac{\{\nabla_i,A_i\}\{\nabla_j,A_j\}}{8m^3}+c_S\frac{e^2\mathbf{S}\cdot(\mathbf{A}\times\mathbf{E}-\mathbf{E}\times\mathbf{A})}{8m^2})\Psi$$

There are several terms quadratic in $\mathbf{A}$ — these are tied to the kinetic term by gauge invariance, and thus have no associated constants to be determined. The sole parameter is c_S. It can be seen that the two-photon Lagrangian for spin one is exactly the same as that for spin one-half.

So the scattering amplitude will be that previously calculated as (2.4.24)

$$iM = c_S\frac{e^2}{4m^2}\phi^\dagger\Big((k_0-k_0')\epsilon\times\epsilon'\Big)\phi \tag{4.2.2}$$

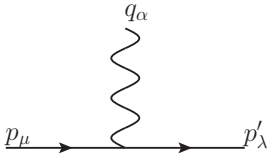
4.3 Feynman rules in the relativistic theory

To determine the coefficients of the NRQED theory, these scattering amplitudes must be compared to that of QED, as calculated from the Lagrangian (3.1.2). There are two types of interaction vertices which arise from this Lagrangian, representing interaction of the charged particle with one or two photons.

The one photon terms, involving one power of A are

$$\mathcal{L}_A = -\frac{ie}{2}(A^\mu W^\nu - A^\nu W^\mu)^\dagger(D_\mu W_\nu - D_\nu W_\mu) - \frac{ie}{2}(D^\mu W^\nu - D^\nu W^\mu)^\dagger(A_\mu W_\nu - A_\nu W_\mu) + ie[g-2]W^{\dagger\mu}W^\nu F_{\mu\nu} \tag{4.3.1}$$

So the diagram is



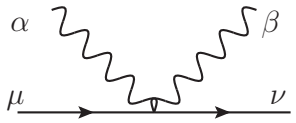
$$= -ie[g^{\mu\lambda}(p+p')^\alpha - g^{\lambda\alpha}(p'+[g-1]q)^\mu - g^{\alpha\mu}(p-[g-1]q)^\lambda]$$

The part of the Lagrangian containing only the two photon terms:

$$\mathcal{L}_{A^2} = \frac{e^2}{2}(A^\mu W^\nu - A^\nu W^\mu)^\dagger (A_\mu W_\nu - A_\nu W_\mu)$$

$$= \frac{e^2}{2}\{2(W^\dagger \cdot W)(A^\dagger \cdot A) - 2(W^\dagger \cdot A)(A^\dagger \cdot W)\}$$

For the two-photon vertex, the term in the Lagrangian can be contracted two ways with the external photon field, so the corresponding diagram is:



$$= -ie^2(2g^{\mu\nu}g^{\alpha\beta} - g^{\mu\beta}g^{\nu\alpha} - g^{\nu\beta}g^{\mu\alpha})$$

In addition to the vertices, there are rules for external legs. External charged particle legs are replaced with the charged particle polarisation ω_μ, while photon legs are replaced by ϵ_μ. The polarisations ω have three degrees of freedom, and obey the equation

$$p \cdot \omega(p) = 0 \tag{4.3.2}$$

4.4 Relation between ω and ϕ_S

The amplitudes calculated in QED will involve the polarisations ω, while those of NRQED will involve ϕ_S. Both represent the three degrees of freedom of a charged spin one particle, so a connection can be established between the two.

To determine how to write ω in terms of ϕ_S, the current density of the two theories can be compared. The current density (3.1.36) for the relativistic theory was calculated in the previous chapter, while that of the nonrelativistic theory is just $\phi_S^\dagger \phi_S$.

Now consider the current density in the case of $q = 0$. The identity $\hat{\partial}^\dagger = -\hat{\partial}$ is used.

$$\begin{aligned} j_0 &= i\left\{(\partial_0 W_\nu - \partial_\nu W_0)^\dagger W^\nu - (\partial_0 W_\nu - \partial_\nu W_0) W^{\dagger\nu}\right\} \\ &= i\left\{(\partial_0 W_i - \partial_i W_0)^\dagger W^i - (\partial_0 W_i - \partial_i W_0) W^{\dagger i}\right\} \\ &= i\left\{W_i^\dagger \partial_0^\dagger W^i - W_0^\dagger \partial_i^\dagger W^i - W^{\dagger i}\partial_0 W_i + W^{\dagger i}\partial_i W_0\right\} \\ &= i\left\{-W_i^\dagger \partial_0 W^i + W_0^\dagger \partial_i W^i - W^{\dagger i}\partial_0 W_i + W^{\dagger i}\partial_i W_0\right\} \\ &= i\left\{-2W_i^\dagger \partial_0 W^i + W_0^\dagger \partial_i W^i + W^{\dagger i}\partial_i W_0\right\} \end{aligned}$$

To express in terms of charged particle polarisations ω and momentum p:

$$\langle j_0(p)\rangle = -2\omega_i^\dagger p_0 \omega^i + \omega_0^\dagger p_i \omega^i + \omega^{\dagger i} p_i \omega_0$$

$$= +2p_0 \boldsymbol{\omega}^\dagger \cdot \boldsymbol{\omega} - \omega_0^\dagger \mathbf{p}\cdot\boldsymbol{\omega} - \boldsymbol{\omega}^\dagger \cdot \mathbf{p}\omega_0$$

ω has four components but only three degrees of freedom. The most sensible approach is to eliminate ω_0 by using (4.3.2). Then $\omega_0 = \frac{\boldsymbol{\omega}\cdot\mathbf{p}}{p_0}$, and the current density is

$$\langle j_0\rangle = 2p_0\boldsymbol{\omega}^\dagger\cdot\boldsymbol{\omega} - 2\frac{(\boldsymbol{\omega}^\dagger\cdot\mathbf{p})(\mathbf{p}\cdot\boldsymbol{\omega})}{p_0}$$

Here the various components of ω are mixed up. As in the previous chapter, they can be disentangled by introducing spin matrices $\mathbf{S}$. Using the same identities as before,

$$j_0 = 2p_0\omega_i^\dagger\left\{\delta_{ij} - \frac{\mathbf{p}^2\delta_{ij} - (\mathbf{S}\cdot\mathbf{p})^2_{ij}}{p_0^2}\right\}\omega_j$$

Then by demanding $j_0 = \phi^\dagger\phi$, a relation between ϕ and $\boldsymbol{\omega}$ is found.

$$2p_0\omega_i^\dagger\left\{\delta_{ij} - \frac{\mathbf{p}^2\delta_{ij} - (\mathbf{S}\cdot\mathbf{p})^2_{ij}}{p_0^2}\right\}\omega_j = \phi^\dagger\phi$$

$$\boldsymbol{\omega} = \left\{2p_0\left(1 - \frac{\mathbf{p}^2 - (\mathbf{S}\cdot\mathbf{p})^2}{p_0^2}\right)\right\}^{-\frac{1}{2}}\phi$$

To the order needed, this is

$$\begin{aligned}\boldsymbol{\omega} &= \frac{1}{\sqrt{2p_0}}\left(1+\frac{\mathbf{p}^2-(\mathbf{S}\cdot\mathbf{p})^2}{2m^2}\right)\phi \\ &= \frac{1}{\sqrt{2m}}\left(1+\frac{\mathbf{p}^2}{4m^2}-\frac{(\mathbf{S}\cdot\mathbf{p})^2}{2m^2}\right)\phi\end{aligned} \tag{4.4.1}$$

4.5 Scattering off an external field in the relativistic theory

The first step will be to do the calculations necessary to fix the one-photon terms in the NRQED Lagrangian. This is done by considering the process of a single charged particle scattering off an external field, and calculating the amplitude in the relativistic theory.

The one photon diagram, for incoming momentum p, outgoing p' and photon momentum $q = p' - p$ is

$$ie\left[g^{\mu\nu}(p+p')^\lambda - g^{\nu\lambda}([g-1]q+p')^\mu + g^{\lambda\mu}([g-1]q-p)^\nu\right] \tag{4.5.1}$$

Contracted with external polarizations $\omega_\mu(p)$, $\omega^*_\nu(p')$ and external field $A_\lambda(q)$, this becomes

$$ie\ \omega_\mu(p)\omega^*_\nu(p')\left[g^{\mu\nu}(p+p')\cdot A - A^\nu([g-1]q+p')^\mu + A^\mu([g-1]q-p)^\nu\right] \tag{4.5.2}$$

The W polarizations are subject to the condition that $k\cdot\omega(k)=0$. This can be used to simplify the above expression, since then $p'^\mu\omega_\mu(p) = q^\mu\omega_\mu(p)$ and $p^\nu\omega^*_\nu(p') = -q^\nu\omega^*_\nu(p')$. With this, the vertex becomes

$$ie\ \omega_\mu(p)\omega^*_\nu(p')\left[g^{\mu\nu}(p+p')\cdot A + g(q^\nu A^\mu - q^\mu A^\nu)\right] \tag{4.5.3}$$

There are two terms here, one proportional to g and one that has no g dependence. Call the first term, which doesn't depend on g

$$M_q = ie\ \omega_\mu(p)\omega^*_\nu(p')g^{\mu\nu}(p+p')\cdot A \tag{4.5.4}$$

and call the second

$$M_g = ieg\ \omega_\mu(p)\omega^*_\nu(p')(q^\nu A^\mu - q^\mu A^\nu) \tag{4.5.5}$$

4.5.1 Nonrelativistic expression for M_q

Having written the amplitude as the sum of two terms, they must be cast in a form that can readily be compared with the NRQED result. So, in terms of operators acting between $\phi^\dagger_S$ and ϕ_S, and involving Galilean vectors and scalars. The general strategy for each will be the same: first write in terms of ω_i and then in terms of ϕ_S. Because the different components of ω_i will be mixed up, as before it will be necessary to introduce spin matrices to express them as operators sandwiched between $\omega^\dagger$ and ω. Finally, derivatives of A_μ should be written in terms of $\mathbf{E}$ and $\mathbf{B}$ where possible.

The first term is

$$\begin{aligned}
M_q &= ie\omega_\mu(p)\omega^*_\nu(p')g^{\mu\nu}(p+p')\cdot A \\
&= ie(p+p')\cdot A(\omega_0(p)\omega^*_0(p') - \omega_i(p)\omega_i(p)) \\
&= ie[(p+p')\cdot A]\omega^*_j(p')\left(\frac{p'_j p_i}{p_0^2} - \delta_{ij}\right)\omega_i(p) \\
&= ie[(p+p')\cdot A]\boldsymbol{\omega}^\dagger(p')\left(\frac{\mathbf{p}\cdot\mathbf{p}' - (\mathbf{S}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p}')}{p_0^2} - 1\right)\boldsymbol{\omega}(p)
\end{aligned}$$

Since it is assumed that $q_0 = 0$, it follows that $p'_0 = p_0$. In terms of the wave functions ϕ_S the above becomes

$$M_q \approx -ie\frac{(p+p')\cdot A}{2p_0}\phi^\dagger_S\left(1 + \frac{\mathbf{p}'^2 - (\mathbf{S}\cdot\mathbf{p}')^2}{2m^2}\right)\left(1 - \frac{\mathbf{p}\cdot\mathbf{p}' - (\mathbf{S}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p}')}{m^2}\right)\left(1 + \frac{\mathbf{p}^2 - (\mathbf{S}\cdot\mathbf{p})^2}{2m^2}\right)\phi_S \tag{4.5.6}$$

Simplifying this to the order needed

$$\begin{aligned}
M_q &\approx -ie\frac{(p+p')\cdot A}{2p_0}\phi^\dagger\left(1 + \frac{1}{2m^2}\left\{\mathbf{q}\cdot\mathbf{p}' - (\mathbf{S}\cdot\mathbf{q})(\mathbf{S}\cdot\mathbf{p}') - \mathbf{p}\cdot\mathbf{q} + (\mathbf{S}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{q})\right\}\right)\phi \\
&= -ie\frac{(p+p')\cdot A}{2p_0}\phi^\dagger\left(1 + \frac{1}{2m^2}\left\{\mathbf{q}^2 + [\mathbf{S}\cdot\mathbf{p}, \mathbf{S}\cdot\mathbf{q}] - (\mathbf{S}\cdot\mathbf{q})^2\right\}\right)\phi \\
&= -ie\frac{(p+p')\cdot A}{2p_0}\phi^\dagger\left(1 + \frac{1}{2m^2}\left\{\mathbf{q}^2 + i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q} - (\mathbf{S}\cdot\mathbf{q})^2\right\}\right)\phi
\end{aligned}$$

The outside factor involves a Lorentz dot product, and should be expressed in terms of Galilean quantities. Again using that $p_0 = p_0'$:

$$\frac{(p+p')\cdot A}{2p_0} = \frac{2p_0}{2p_0}A_0 - \frac{(\mathbf{p}+\mathbf{p}')\cdot\mathbf{A}}{2p_0} = A_0 - \frac{(\mathbf{p}+\mathbf{p}')\cdot\mathbf{A}}{2p_0} \tag{4.5.7}$$

Applying this to M_q, the result is:

$$M_q = -ie\phi^\dagger(p')\left(A_0 + \frac{A_0}{2m^2}(\mathbf{q}^2 + i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q} - (\mathbf{S}\cdot\mathbf{q})^2) - \frac{(\mathbf{p}+\mathbf{p}')\cdot\mathbf{A}}{2p_0} - \frac{(\mathbf{p}+\mathbf{p}')\cdot\mathbf{A}}{2m}\frac{i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q}}{2m^2}\right)\phi(p) \tag{4.5.8}$$

The NRQED result for the amplitude was written in position terms of E and B, so it is necessary to express this result in the same sense. A Fourier transform dictates that the transferred momentum $\mathbf{q}$ becomes a derivative on the external field. The prescription is $\mathbf{q} \to -i\boldsymbol{\nabla}$. The gauge has been chosen such that E depends on A_0 terms only, and B only upon A_i. So it makes sense to consider such terms separately.

Transformation of A_0 terms in M_q

The second order term involving A_0 has both first and second derivatives of the potential. The electric field is $E_i = -\partial_i A_0$ and so $q_i A_0 \to iE_i$. Then $q_i q_j A_0 = \partial_j E_i = \partial_i E_j$. Applying this to the higher order terms coupled to A_0:

$$A_0(\mathbf{q}^2 + i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q} - (\mathbf{S}\cdot\mathbf{q})^2) \to \boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - S_iS_j\boldsymbol{\nabla}_i E_j \tag{4.5.9}$$

Transformation of A terms in M_q

The only term with derivatives of $\mathbf{A}$ has the form $(\mathbf{p}+\mathbf{p}')\cdot\mathbf{A}(\mathbf{S}\cdot\mathbf{p}\times i\mathbf{q})$. The relation between this and the field B is nonobvious. First, it is convenient to develop an identity for $iq_i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{A}$. Using $\epsilon_{ijk}B_k = i(q_iA_j - q_jA_i)$:

$$(p+p')_i\epsilon_{ijk}B_k = (p+p')_i\,(iq_iA_j - iq_jA_i) \tag{4.5.10}$$

There are additional conditions coming from the fact that this is elastic scattering. Since in this case $(\mathbf{p}' + \mathbf{p}) \cdot \mathbf{q} = \mathbf{p}'^2 - \mathbf{p}^2 = 0$, the $(p + p')_i q_i$ term vanishes, leaving

$$(p + p')_i \epsilon_{ijk} B_k = -i(p + p')_i q_j A_i = -iq_j(\mathbf{p} + \mathbf{p}') \cdot \mathbf{A} \tag{4.5.11}$$

This gives the identity

$$iq_j(\mathbf{p} + \mathbf{p}') \cdot \mathbf{A} = -\epsilon_{ijk}(p + p')_i B_k \tag{4.5.12}$$

This identity helps reduce the more complicated term:

$$\begin{aligned}
(\mathbf{p} + \mathbf{p}') \cdot \mathbf{A}(\mathbf{S} \cdot \mathbf{p} \times i\mathbf{q}) &= S_i p_j \epsilon_{ijk}[iq_k(\mathbf{p} + \mathbf{p}') \cdot \mathbf{A}] \\
&= -S_i p_j \epsilon_{ijk}[\epsilon_{\ell km}(p + p')_\ell B_m] \\
&= S_i p_j \epsilon_{ijk}[\epsilon_{\ell mk}(p + p')_\ell B_m] \\
&= S_i p_j (p + p')_\ell B_m[\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}] \\
&= \mathbf{S} \cdot (\mathbf{p} + \mathbf{p}')\mathbf{B} \cdot \mathbf{p} - \mathbf{S} \cdot \mathbf{B}(\mathbf{p} + \mathbf{p}') \cdot \mathbf{p}
\end{aligned}$$

For the case of a constant $\mathbf{B}$, any terms of the type $q_i B_j$ vanish. So the above reduces to

$$(\mathbf{p} + \mathbf{p}') \cdot \mathbf{A}(\mathbf{S} \cdot \mathbf{p} \times i\mathbf{q}) = 2\{(\mathbf{S} \cdot \mathbf{p})(\mathbf{B} \cdot \mathbf{p}) - \mathbf{p}^2 \mathbf{S} \cdot \mathbf{B}\} \tag{4.5.13}$$

Total contribution of from M_q

So the first term in the vertex produces the following contribution to the scattering amplitude.

$$\begin{aligned}
M_q = ie\omega_\mu(p)\omega^*_\nu(p')g^{\mu\nu}(p + p') \cdot A \rightarrow -ie\phi^\dagger(p')\Big(A_0 + \frac{1}{2m^2}\{\boldsymbol{\nabla} \cdot \mathbf{E} - \mathbf{S} \cdot \mathbf{p} \times \mathbf{E} - S_i S_j \boldsymbol{\nabla}_i E_j\} \\
- \frac{\mathbf{p} \cdot \mathbf{A}}{p_0} + \frac{1}{2m^3}\{\mathbf{p}^2 \mathbf{S} \cdot \mathbf{B} - (\mathbf{S} \cdot \mathbf{p})(\mathbf{B} \cdot \mathbf{p})\}\Big)\phi(p)
\end{aligned} \tag{4.5.14}$$

4.5.2 Nonrelativistic expression for M_g

The second term looks like the external polarisations contracted with something like $F_{\mu\nu}$:

$$M_g = ie\ \omega_\mu(p)\omega^*_\nu(p')[g](q^\nu A^\mu - q^\mu A^\nu) \tag{4.5.15}$$

General tensor type term

Before calculating the specific term in question, consider the general type of term $\omega^*_\nu(p')\omega_\mu(p)u^\nu v^\mu$. This should be expressed as a matrix element between the vector part of the polarization ω.

$$\begin{aligned}
\omega^*_\nu(p')\omega_\mu(p)u^\nu v^\mu &= \omega'_0 u_0 v_0 \omega_0 - (\omega'\cdot\mathbf{u})v_0\omega_0 - \omega'_0 u_0(\mathbf{v}\cdot\boldsymbol{\omega}) + (\boldsymbol{\omega'}\cdot\mathbf{u})(\mathbf{v}\cdot\boldsymbol{\omega}) \\
&= \frac{\boldsymbol{\omega'}\cdot\mathbf{p}'u_0}{p'_0}\frac{\boldsymbol{\omega}\cdot\mathbf{p}v_0}{p_0} - (\omega'\cdot\mathbf{u})\frac{\boldsymbol{\omega}\cdot\mathbf{p}v_0}{p_0} - \frac{\boldsymbol{\omega'}\cdot\mathbf{p}'u_0}{p'_0}(\mathbf{v}\cdot\boldsymbol{\omega}) + (\boldsymbol{\omega'}\cdot\mathbf{u})(\mathbf{v}\cdot\boldsymbol{\omega}) \\
&= \boldsymbol{\omega}'_j\left(\frac{u_0v_0}{p_0p'_0}p'_jp_i - \frac{v_0}{p_0}u_jp_i - \frac{u_0}{p'_0}p'_jv_i + u_jv_i\right)\omega_i \\
&= \boldsymbol{\omega'}^\dagger\left(\frac{u_0v_0}{p_0p'_0}[\mathbf{p}\cdot\mathbf{p}' - (\mathbf{p}\cdot\mathbf{S})(\mathbf{p}'\cdot\mathbf{S})] - \frac{v_0}{p_0}[\mathbf{p}\cdot\mathbf{u} - (\mathbf{p}\cdot\mathbf{S})(\mathbf{u}\cdot\mathbf{S})]\right. \\
&\quad \left. -\frac{u_0}{p'_0}[\mathbf{v}\cdot\mathbf{p}' - (\mathbf{v}\cdot\mathbf{S})(\mathbf{p}'\cdot\mathbf{S})] + [\mathbf{v}\cdot\mathbf{u} - (\mathbf{v}\cdot\mathbf{S})(\mathbf{u}\cdot\mathbf{S})]\right)\boldsymbol{\omega}
\end{aligned}$$

M_g term

Now that identity can be used to calculate M_g with $q_0 = 0$ (and thus $p'_0 = p_0$).

$$\begin{aligned}
M_g &= +ieg\boldsymbol{\omega'}^\dagger\left(-\frac{A_0}{p_0}\{\mathbf{p}\cdot\mathbf{q} - (\mathbf{p}\cdot\mathbf{S})(\mathbf{q}\cdot\mathbf{S})\} + \frac{A_0}{p_0}\{\mathbf{q}\cdot\mathbf{p}' - (\mathbf{q}\cdot\mathbf{S})(\mathbf{p}'\cdot\mathbf{S})\}\right. \\
&\quad \left. +\{\mathbf{q}\cdot\mathbf{A} - (\mathbf{A}\cdot\mathbf{S})(\mathbf{q}\cdot\mathbf{S})\} - \{\mathbf{A}\cdot\mathbf{q} - (\mathbf{q}\cdot\mathbf{S})(\mathbf{A}\cdot\mathbf{S})\}\right)\boldsymbol{\omega} \\
&= +ieg\boldsymbol{\omega'}^\dagger\left\{\frac{A_0}{p_0}\Big(\mathbf{q}^2 - (\mathbf{q}\cdot\mathbf{S})^2 + [\mathbf{p}\cdot\mathbf{S}, \mathbf{q}\cdot\mathbf{S}]\Big) - [\mathbf{A}\cdot\mathbf{S}, \mathbf{q}\cdot\mathbf{S}]\right\}\boldsymbol{\omega}
\end{aligned}$$

Next replace ω with ϕ_S and rewrite in terms of E and B. Again, it makes sense to treat terms with A_0 and A_i separately.

Transformation of A_0 terms in M_g

The first term coupled to A_0 is already second order, so only the first order approximation for ω is needed: $\omega \approx \frac{1}{\sqrt{2m}}\phi$

$$M_g^E = ieg\phi^\dagger(\mathbf{p}')\frac{A_0}{2m^2}\left(\mathbf{q}^2 - (\mathbf{q}\cdot\mathbf{S})^2 + i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q}\right)\phi(\mathbf{p}) \tag{4.5.16}$$

This is exactly the same structure that arose in working with M_q. From (4.5.9) it is:

$$M_g^E = ieg\frac{1}{2m^2}\phi^\dagger(\mathbf{p}')(\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - S_iS_j\boldsymbol{\nabla}_iE_j)\phi(\mathbf{p}) \tag{4.5.17}$$

Transformation of A terms in M_g

The second term is $M_g^B = -ieg\boldsymbol{\omega}^\dagger[\mathbf{A}\cdot\mathbf{S}, \mathbf{q}\cdot\mathbf{S}]\boldsymbol{\omega}$. The commutator is:

$$[\mathbf{A}\cdot\mathbf{S}, \mathbf{q}\cdot\mathbf{S}] = A_iq_j[S_i, S_j] = A_iq_j(i\epsilon_{ijk}S_k) = -\mathbf{S}\cdot i\mathbf{q}\times\mathbf{A} \tag{4.5.18}$$

In terms of B this yields

$$\begin{aligned} -\mathbf{S}\cdot i\mathbf{q}\times\mathbf{A} &\rightarrow -\mathbf{S}\cdot\boldsymbol{\nabla}\times\mathbf{A} \\ &= -\mathbf{S}\cdot\mathbf{B} \end{aligned}$$

So the whole thing is $ieg\boldsymbol{\omega}^\dagger(\mathbf{S}\cdot\mathbf{B})\boldsymbol{\omega}$. This is first order so corrections to this term, coming from the expression for ω in terms of ϕ, are needed.

$$M_g^B = ieg\boldsymbol{\omega}^\dagger(\mathbf{S}\cdot\mathbf{B})\boldsymbol{\omega} \rightarrow ieg\frac{1}{2p_0}\phi^\dagger\left(1 + \frac{\mathbf{p}'^2 - [\mathbf{S}\cdot\mathbf{p}']^2}{2m^2}\right)\mathbf{S}\cdot\mathbf{B}\left(1 + \frac{\mathbf{p}^2 - [\mathbf{S}\cdot\mathbf{p}]^2}{2m^2}\right)\phi \tag{4.5.19}$$

Since the B field is constant, all terms with q vanish, so the above reduces to

$$M_g^B = ieg\frac{1}{2p_0}\phi^\dagger\left(\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{p^2}{2m^2} + \frac{1}{2m^2}\left\{\mathbf{S}\cdot\mathbf{B}, \frac{\mathbf{p}^2}{2} - [\mathbf{S}\cdot\mathbf{p}]^2\right\}_+\right)\phi \tag{4.5.20}$$

Expanding $\frac{1}{2p_0} = \frac{1}{2m}(1 - \frac{\mathbf{p}^2}{2m^2})$ eliminates the second term:

$$M_g^B = ieg\frac{1}{2m}\phi^\dagger\left(\mathbf{S}\cdot\mathbf{B} + \frac{1}{2m^2}\left\{\mathbf{S}\cdot\mathbf{B}, \frac{\mathbf{p}^2}{2} - [\mathbf{S}\cdot\mathbf{p}]^2\right\}_+\right)\phi \tag{4.5.21}$$

This anticommutator actually occurred as part of the previous calculation of the spin-one Hamiltonian. From (**??**):

$$\frac{1}{2m^2}\left\{\frac{\mathbf{p}^2}{2} - (\mathbf{S}\cdot\mathbf{p})^2, \mathbf{S}\cdot\mathbf{B}\right\} = -\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{2m^2} \tag{4.5.22}$$

Using that, the part of M_g which contains $\mathbf{A}$ reduces to

$$M_g^B = ieg\frac{1}{2m}\phi^\dagger\left(\mathbf{S}\cdot\mathbf{B} - \frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{2m^2}\right)\phi \tag{4.5.23}$$

Total contribution of M_g

Now the total contribution of the second g-dependent term to the scattering amplitude is:

$$M_g = ieg\frac{1}{2m}\phi^\dagger\Big(\mathbf{S}\cdot\mathbf{B} - \frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{2m^2} + \frac{1}{m}\left\{\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - S_iS_j\boldsymbol{\nabla}_iE_j\right\}\Big)\phi \tag{4.5.24}$$

4.5.3 All terms together

Having expressed each of the terms, the total scattering amplitude, as calculated in the relativistic theory, is found simply by adding them together. (Elastic scattering off an external field, with constant $\mathbf{B}$.)

First consider the terms related to the electric field/potential:

$$M^E = -ie\phi^\dagger\left(A_0 - (g-1)\frac{1}{2m^2}\{\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - S_iS_j\boldsymbol{\nabla}_iE_j\}\right)\phi \tag{4.5.25}$$

The terms with bare $\mathbf{A}$:

$$M^A = ie\phi^\dagger\left(\frac{\mathbf{p}\cdot\mathbf{A}}{p_0}\right)\phi \approx ie\phi^\dagger\left(\frac{\mathbf{p}\cdot\mathbf{A}}{m} - \frac{\mathbf{p}\cdot\mathbf{A}\mathbf{p}^2}{2m^3}\right)\phi \tag{4.5.26}$$

Then the terms with B in them:

$$M^B = i\frac{e}{2m}\phi^\dagger\left(g\mathbf{S}\cdot\mathbf{B} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{g-2}{4m^2}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\right)\phi \tag{4.5.27}$$

So to the desired order, the amplitude for scattering off an external field, as calculated from relativistic theory, is

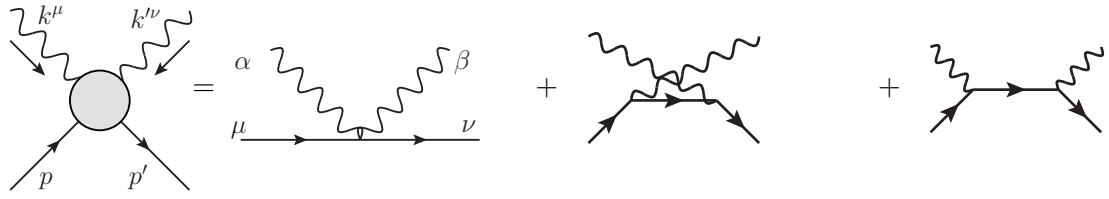
$$\begin{aligned} iM_{REL} = &-ie\phi^\dagger\Big(A_0 - \frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{\mathbf{p}\cdot\mathbf{A}\mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3}\{\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - S_iS_j\boldsymbol{\nabla}_iE_j\} \\ &- g\frac{1}{2m}\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\Big)\phi \end{aligned} \tag{4.5.28}$$

4.6 Compton scattering in the relativistic theory

The previous calculation is enough to fix all the coefficients of the NRQED lagrangian. Just like for spin one-half, however, calculating the the coefficients of some two-photon terms will help check the consistency of the approach. And as before, the two photon coefficients may be fixed by calculating the amplitude for Compton scattering in the relativistic theory.

In the spin one-half case, there was only one type of electromagnetic vertex, representing the interaction of a single photon with an electron line. So the process of Compton scattering (at the leading orders) was determined entirely by diagrams composed of this vertex. In the spin one Lagrangian, however, there are also four particle vertices involving two photons.

So here there are two types of contributions, one from the fundamental two-photon vertex and the other from the combination of two one-photon vertices. Two-photon contact terms in the Lagrangian will have an additional symmetry factor of 1/2 compared to the expression for the vertex.



To define conventions for this calculation, say that the incoming charged particle has momentum p, and the outgoing, momentum p'. Define both photon momenta k and k' going *into* the vertex or vertices. Charged particle polarisations are ω, while photon polarisations

are ϵ.

Once scattering amplitude has been calculated in terms of relativistic quantities, it must be compared to the NRQED vertices. The process is the same as for the calculation of the one-photon process. First, write all terms "sandwiched" between $\omega^\dagger$ and ω. Some terms will already be proportional to $\boldsymbol{\omega}^\dagger \cdot \boldsymbol{\omega}$. But there will be others of the form $[\boldsymbol{\omega}^\dagger \cdot \mathbf{u}][\mathbf{v} \cdot \boldsymbol{\omega}]$. To deal with these the spin matrices $\mathbf{S}$ are introduced, with the identity:

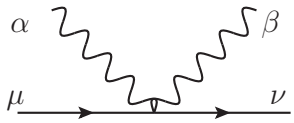
$$(\boldsymbol{\omega}^\dagger \cdot \mathbf{u})(\mathbf{v} \cdot \boldsymbol{\omega}) = \omega_a^\dagger \left(\delta_{ab}\mathbf{u} \cdot \mathbf{v} - v_i u_j \{S_i S_j\}_{ab}\right) \omega_b$$

The charged particle polarisations must be replaced by the Schrodinger-like wave functions ϕ. Consistent with the calculations for the one-photon terms, the prescription is:

$$\boldsymbol{\omega}(p) \to \frac{1}{\sqrt{2m}} \left(1 + \frac{\mathbf{p}^2}{4m^2} - \frac{[\mathbf{S} \cdot \mathbf{p}]^2}{2m^2}\right) \phi_S(\mathbf{p})$$

4.6.1 Two-photon vertex

The relativistic two-photon vertex is



$$= \qquad -ie^2(2g^{\mu\nu}g^{\alpha\beta} - g^{\mu\beta}g^{\nu\alpha} - g^{\mu\alpha}g^{\nu\beta})$$

The contribution of this diagram to Compton scattering is, contracting with the photon and charged particle polarisations:

$$M_{\text{contact}} = -ie^2(2[\omega^\dagger \cdot \omega][\epsilon' \cdot \epsilon] - [\omega^\dagger \cdot \epsilon][\omega \cdot \epsilon'] - [\omega^\dagger \cdot \epsilon'][\omega \cdot \epsilon]) \tag{4.6.1}$$

In calculating a physical process, the result will not depend upon the gauge. To simplify the calculation, choose a gauge where $\epsilon_0 = 0$ and use the identity $\omega_0 = \frac{\boldsymbol{\omega} \cdot \mathbf{p}}{p_0}$. Then the above reduces to

$$M_{\text{contact}} = -2ie^2(\boldsymbol{\omega}^\dagger \cdot \boldsymbol{\omega})(\epsilon' \cdot \epsilon) + ie^2(\boldsymbol{\omega}^\dagger \cdot \epsilon)(\epsilon' \cdot \boldsymbol{\omega}) + ie^2(\boldsymbol{\omega}^\dagger \cdot \epsilon')(\epsilon \cdot \boldsymbol{\omega}) \tag{4.6.2}$$

4.6.2 Terms arising from two vertex diagrams

The diagrams with two vertices contribute not just to the point-like interaction, but also the part of the scattering amplitude corresponding to two vertex diagrams in NRQED. By decomposing the relativistic propagator, the local and nonlocal terms can be separated. This is exactly analogous to what was done for the spin one-half case, but the structure of both the propagator and the sums over intermediate states differ.

Dealing with the relativistic propagator

The relativistic propagator will be decomposed into the sum of two pole terms and a constant term. The numerators of the two pole terms are usefully expressed as a sum over intermediate on-shell states i.e. a sum over polarisations. If a indexes the three orthogonal polarisation states, and ℓ is some on-shell momentum, then this sum is

$$\sum_a \omega_a^\mu(\ell)\omega_a^\nu(\ell) = \frac{\ell^\mu \ell^\nu}{m^2} - g^{\mu\nu} \tag{4.6.3}$$

It's convenient to define for some arbitrary (*not* necessarily on mass shell) momentum k

$$G^{\mu\nu}(k) = g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2} \tag{4.6.4}$$

The idea is that the two parts of the propagator with poles represent two types of processses, one involving intermediate particles, the other intermediate anti-particles. These particles are to be considered on-shell, so it is useful to define Lorentz four vectors which represent the momentum of on-shell particles and anti-particles. For the particles, let $q^{(+)} = (\sqrt{\mathbf{q}^2 + m^2}, \mathbf{q})$. And for the antiparticles, let $q^{(-)} = (-\sqrt{\mathbf{q}^2 + m^2}, \mathbf{q})$. Also define $E_q = \sqrt{\mathbf{q}^2 + m^2}$.

The relativistic propagator to be decomposed is

$$i\frac{\frac{q^\mu q^\nu}{m^2} - g^{\mu\nu}}{q^2 - m^2} = \frac{-iG^{\mu\nu}(q)}{q^2 - m^2} \tag{4.6.5}$$

Here q is no on mass-shell. This will be written as the sum of two pole terms, having

as their numerator a sum over intermediate on mass-shell states. In comparison to the spin one-half case, there will be an additional constant which for now can be called $x^{\mu\nu}$.

$$\frac{-iG^{\mu\nu}(q)}{q^2-m^2} = \frac{1}{2\sqrt{m^2+\mathbf{q}^2}}\left(\frac{-iG^{\mu\nu}(q^{(+)})}{q_0-\sqrt{m^2+\mathbf{q}^2}} - \frac{-iG^{\mu\nu}(q^{(-)})}{q_0+\sqrt{m^2+\mathbf{q}^2}}\right) + ix^{\mu\nu}$$

Explicit calculation shows that $x^{\mu\nu} = 1/m^2$ when $\mu = \nu = 0$, but is zero otherwise. So $x^{\mu\nu} = g^{\mu 0}g^{\nu 0}/m^2$, and the relativistic propagator can be written as

$$\frac{-iG^{\mu\nu}(q)}{q^2-m^2} = \frac{i}{2\sqrt{m^2+\mathbf{q}^2}}\left(\frac{-G^{\mu\nu}(q^{(+)})}{q_0-\sqrt{m^2+\mathbf{q}^2}} - \frac{-G^{\mu\nu}(q^{(-)})}{q_0+\sqrt{m^2+\mathbf{q}^2}}\right) + i\frac{g^{\mu 0}g^{\nu 0}}{m^2}$$

The reason for this decomposition is that only the part of the propagator which gives rise to the point-like interaction is needed. Other contributions to the scattering amplitude in the nonrelativistic theory come from diagrams involving two one-photon vertices. At the necessary order these are exactly the $1/(q_0 - \sqrt{m^2+\mathbf{q}^2})$ terms. So the point-like interactions can be obtained by simply dropping these terms. After this, rewriting the amplitude in terms of nonrelativistic quantities will still be necessary.

From the full propagator, define the needed part $P^{\mu\nu}$ as outlined above.

$$P^{\mu\nu} = \frac{i}{2\sqrt{m^2+\mathbf{q}^2}}\frac{G^{\mu\nu}(q^{(-)})}{q_0+\sqrt{m^2+\mathbf{q}^2}} + i\frac{g^{\mu 0}g^{\nu 0}}{m^2} \tag{4.6.6}$$

This is just the propagator without the nonlocal terms.

All that is needed for the final result are the leading order terms, and those with one additional power of momentum. So $E_q \approx m$. q_0 is the off-mass shell energy of the propagator, but in this nonrelativistic scenario it will be close to m. So write $q_0 + E_q \approx 2m + [q_0 - m]$ and then use that $[q_0 - m]$ is small.

$$P^{\mu\nu} \approx \frac{1}{2m}\frac{iG^{\mu\nu}(q^{(-)})}{2m+[q_0-m]} + i\frac{g^{\mu 0}g^{\nu 0}}{m^2} \approx \left(1-\frac{[q_0-m]}{2m}\right)\frac{iG^{\mu\nu}(q^{(-)})}{4m^2} + i\frac{g^{\mu 0}g^{\nu 0}}{m^2}$$

Consider now $G^{\mu\nu}(q^{(-)}) = g^{\mu\nu} - \frac{q^{(-)\mu}q^{(-)\nu}}{m^2}$. Using ${q^{(-)}}_0 \approx -m$ it follows that

$$G^{00} = (1 - \frac{q^{(-)0}q^{(-)0}}{m^2} \approx 0$$

$$G^{0i} = G^{i0} = -\frac{q^{(-)0} q^{(-)i}}{m^2} \approx \frac{q^i}{m}$$

$$G^{ij} = \delta^{ij} - \frac{q^{(-)i} q^{(-)j}}{m^2} \approx \delta^{ij}$$

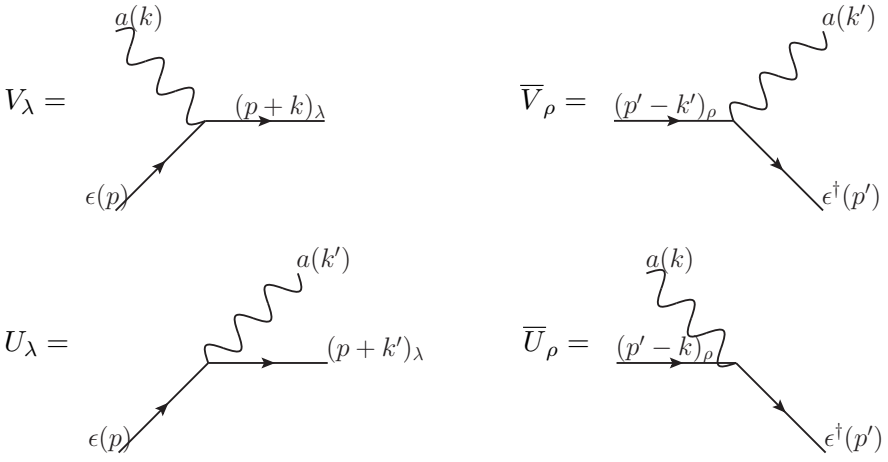
So the different components fo $P^{\mu\nu}$ are, to the order needed.

$$\begin{aligned} P^{00} &= \frac{i}{m^2} \\ P^{0i} &= P^{i0} = i\frac{q^i}{4m^3} \\ P^{ij} &= -\left(1 - \frac{q_0 - m}{2m}\right)\frac{i\delta^{ij}}{4m^2} \end{aligned}$$

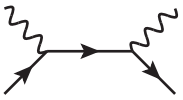
Vertex calculations

There are two tree diagarms, crossed and uncrossed. Each consists of a propagator and two vertices contracted with external fields. In the previous section the propagator was considered; the result is that it may simply be replaced by the above quantities $P^{\mu\nu}$ to compare with the local vertices of NRQED. Now the part of the calculation involving the vertices is considered.

Use V and $\overline{V}$ to represent the vertices of the uncrossed diagram, and U, $\overline{U}$ for the crossed. These can be defined in terms of vertex diagrams, the external lines contracted with polarisations.



With the above definitions it is clear that the point-like interactions, taken from the two diagrams, will be

 $\rightarrow \overline{V}_\rho P^{\rho\lambda}(p+k)V_\lambda$  $\rightarrow \overline{U}_\rho P^{\rho\lambda}(p+k')U_\lambda$

Only terms one past leading order are needed to fix the coefficient of the $\mathbf{S}\cdot\mathbf{E}\times\mathbf{A}$ term in NRQED. So tabulate the first two orders of each vertex for time and spatial components separately.

The basic vertex inside V_λ is, with incoming particle momentum p^μ, incoming photon momentum k^α and outgoing $(p+k)^\lambda$:

$$ie[g^{\mu\lambda}(2p+k)^\alpha - g^{\lambda\alpha}(p+[g]k)^\mu - g^{\alpha\mu}(p-[g-1]k)^\lambda]$$

Contracted with external charged particle and photon polarizations ω_μ and $\epsilon_\alpha(k)$ gives:

$$V_\lambda = ie[\omega^\lambda(2p+k)\cdot\epsilon - \epsilon^\lambda(p+[g]k)\cdot\omega - \epsilon\cdot\omega(p-[g-1]k)^\lambda]$$

The other three vertices can be obtained simply by substituting the appropriate momenta and polarisations. For instance, the similar vertex for outgoing momentum $(p+k)^\rho = (p'-k')^\rho$, photon k', and final momentum p' is

$$\overline{V}_\rho = ie[\omega^{\dagger\rho}(2p'-k')\cdot\epsilon' - \epsilon^\rho(p'-[g]k')\cdot\omega^\dagger - \epsilon'\cdot\omega^\dagger(p'+[g-1]k')^\rho]$$

So $\overline{V}_\rho$ can be obtained by just substituting $\omega^\dagger$ for ω, p' for p, and $-k'$ for k. With this in mind, only the calculation for V_λ need be explicitly performed, since all four have the same form.

Calculation of V_λ

In calculating the two photon vertex, the gauge was chosen such that $\epsilon_0 = 0$. This will help simplify the calculation. Also, only the first two orders are needed.

The leading order terms must have only the time-component of the charged particles momentum: the on mass-shell $p_0 = p_0' \approx m$. Corrections to p_0 do not enter at the necessary

level of approximation for the two photon diagrams.

The next to leading order terms have exactly one power of external momentum $\mathbf{p}$ or photon momentum $\mathbf{k}$, $\mathbf{k}'$ or k_0, k_0'. Any term containing ω_0 is of at least this order, since $\omega^0(p) \approx \frac{\boldsymbol{\omega}\cdot\mathbf{p}}{m}$.

With these considerations, the vertex V can now be explicitly calculated, treating V_0 and V_i separately.

The V_0 term Consider first the time-component of V. It has three parts — label them T_0^N.

$$V_0 = ie[\underbrace{\omega^0(2p+k)\cdot\epsilon}_{I} - \underbrace{\epsilon^0(p+[g]k)\cdot\omega}_{II} - \underbrace{\epsilon\cdot\omega(p-[g-1]k)^0}_{III}] \tag{4.6.7}$$

Up to some outer factor the first term is then $\omega^0(2p+k)\cdot\epsilon$. Since $\omega^0 = \frac{\boldsymbol{\omega}\cdot\mathbf{p}}{m}$ this becomes

$$T_0^I = \omega^0(2p+k)\cdot\epsilon = \frac{\boldsymbol{\omega}\cdot\mathbf{p}}{m}(2p+k)\cdot\epsilon = 2\frac{\boldsymbol{\omega}\cdot\mathbf{p}}{m}p_0\epsilon_0 + \mathcal{O}(\frac{p^2}{m}) \tag{4.6.8}$$

The spatial component of ϵ only appears with an additional power of momentum and so is too high order, and the ϵ_0 term vanishes in the chosen gauge. So this term doesn't actually contribute at all to the result:

$$T_0^I = \omega^0(2p+k)\cdot\epsilon \sim \mathcal{O}(\frac{p^2}{m}) \tag{4.6.9}$$

The second term goes as $-\epsilon^0(p+[g]k)\cdot\omega$. Obviously with the choice of gauge this doesn't contribute either.

$$T_0^{II} = 0 \tag{4.6.10}$$

The final term here is $-\epsilon\cdot\omega(p-[g-1]k)^0$ Expanding the dot product and again using the choice of gauge

$$T_0^{III} = -\epsilon\cdot\omega(p-[g-1]k)^0 \approx -(\epsilon_0\omega_0 - \epsilon\cdot\boldsymbol{\omega})(m-[g-1]k_0) = (m-[g-1]k_0)\epsilon\cdot\boldsymbol{\omega} \tag{4.6.11}$$

That is the only contribution to the $\lambda = 0$ vertex, so the total is just

$$V_0 = ie(m - [g-1]k_0)\epsilon \cdot \boldsymbol{\omega} \tag{4.6.12}$$

The V_i term Now look at the spatial component, which again has three parts which can be labelled T_i^N.

$$V_i = ie[\underbrace{\omega^i(2p+k)\cdot\epsilon}_{I} - \underbrace{\epsilon^i(p+[g]k)\cdot\omega}_{II} - \underbrace{\epsilon\cdot\omega(p-[g-1]k)^i}_{III}] \tag{4.6.13}$$

The first term is simply

$$T_i^I = \omega^i(2p+k)\cdot\epsilon = \omega^i([2m+k_0]\epsilon_0 - [2\mathbf{p}+\mathbf{k}]\cdot\epsilon]) \tag{4.6.14}$$

Which after applying the gauge conditions becomes

$$T_i^I = \omega^i(2p+k)\cdot\epsilon = -2(\mathbf{p}\cdot\epsilon)\omega^i \tag{4.6.15}$$

For the second term $-\epsilon^i(p+2k)\cdot\omega$, again use that $p\cdot\omega = 0$ so

$$T_i^{II} = -\epsilon^i(p+[g]k)\cdot\omega \approx g\epsilon^i\mathbf{k}\cdot\boldsymbol{\omega} \tag{4.6.16}$$

And in the third term $-\epsilon\cdot\omega(p-[g-1]k)^i$, the ϵ_0 term is of too high order (irregardless of the gauge), leaving

$$T_i^{III} = -\epsilon\cdot\omega(p-[g-1]k)^i \approx \epsilon\cdot\boldsymbol{\omega}(p-[g-1]k)^i \tag{4.6.17}$$

All three terms together give

$$V_i = ie\left([g]\mathbf{k}\cdot\boldsymbol{\omega}\epsilon^i - 2\mathbf{p}\cdot\epsilon\omega^i + \epsilon\cdot\boldsymbol{\omega}[p-[g-1]k]^i\right) \tag{4.6.18}$$

Tabulation of results So of course parallel calculations for the other vertices could be done, as previously mentioned it is easier to note that they have the same general for. To

obtain $\overline{V}$ the recipe is that $p \to p'$, $k \to -k'$, $\omega \to \omega^\dagger$. Similar considerations work for U, where the only difference is that because of the crossed lines, k and k' are swapped.

If V is considered as a function of p, k and ω: $V_\mu(p,k,\omega)$, then

$$\begin{aligned}
\overline{V}_\mu &= V_\mu(p',-k',\omega^\dagger) \\
U_\mu &= V_\mu(p,k',\omega) \\
\overline{U}_\mu &= V_\mu(p',-k,\omega^\dagger)
\end{aligned}$$

Only the first two orders are needed, so for each type of vertex write $X^{(1)}$ to indicate a leading order terms, and $X^{(2)}$ to indicate next to leading order terms.

The vertex V was explicitly calculated, the results are

$$\begin{aligned}
V_0^{(1)} &= iem\epsilon\cdot\boldsymbol{\omega} \\
V_0^{(2)} &= -ie[g-1]k_0\epsilon\cdot\boldsymbol{\omega} \\
V_i^{(1)} &= 0 \\
V_i^{(2)} &= ie\left([g]\mathbf{k}\cdot\boldsymbol{\omega}A^i - 2\mathbf{p}\cdot\epsilon\omega^i + \epsilon\cdot\boldsymbol{\omega}[p-[g-1]k]^i\right)
\end{aligned}$$

$\overline{V}$ is obtained from V as explained above:

$$\begin{aligned}
\overline{V}_0^{(1)} &= iem\epsilon'\cdot\boldsymbol{\omega}^\dagger \\
\overline{V}_0^{(2)} &= ie[g-1]k_0'\epsilon'\cdot\boldsymbol{\omega}^\dagger \\
\overline{V}_i^{(1)} &= 0 \\
\overline{V}_i^{(2)} &= ie(-[g]\mathbf{k}'\cdot\boldsymbol{\omega}^\dagger\epsilon'^i - 2\mathbf{p}'\cdot\epsilon'\omega^{\dagger^i} + \epsilon'\cdot\boldsymbol{\omega}^\dagger[p'+[g-1]k']^i)
\end{aligned}$$

Using U to denote the vertices of the crossed diagram, the first is:

$$\begin{aligned}
U_0^{(1)} &= iem\epsilon' \cdot \boldsymbol{\omega} \\
U_0^{(2)} &= -ie[g-1]k_0'\epsilon' \cdot \boldsymbol{\omega} \\
U_i^{(1)} &= 0 \\
U_i^{(2)} &= ie\left([g]\mathbf{k}' \cdot \boldsymbol{\omega}\epsilon'^i - 2\mathbf{p} \cdot \epsilon'\omega^i + \epsilon' \cdot \boldsymbol{\omega}[p - [g-1]k']^i\right)
\end{aligned}$$

And for the second:

$$\begin{aligned}
\overline{U}_0^{(1)} &= iem\epsilon \cdot \boldsymbol{\omega}^\dagger \\
\overline{U}_0^{(2)} &= ie[g-1]k_0\epsilon \cdot \boldsymbol{\omega}^\dagger \\
\overline{U}_i^{(1)} &= 0 \\
\overline{U}_i^{(2)} &= ie(-[g]\mathbf{k} \cdot \boldsymbol{\omega}^\dagger\epsilon^i - 2\mathbf{p}' \cdot \epsilon\omega^{\dagger i} + \epsilon \cdot \boldsymbol{\omega}^\dagger[p' + [g-1]k]^i)
\end{aligned}$$

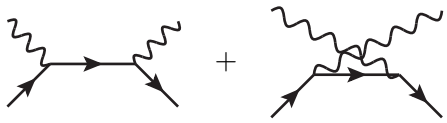
In addition to the vertices, the final result will involve the propagator terms which were, as previously derived:

$$\begin{aligned}
P^{00} &= \frac{i}{m^2} \\
P^{0i} &= P^{i0} = i\frac{q^i}{4m^3} \\
P^{ij} &= -\left(1 - \frac{q_0 - m}{2m}\right)\frac{i\delta^{ij}}{4m^2}
\end{aligned}$$

Above, $q = p + k$ for the uncrossed diagrams, and $q = p + k'$ for crossed. It turns out that for this particular calculation, only P_{00} will contribute. And since at the needed order only the pure contact term contributes, the momentum dependence of the propagator doesn't show up in the final result.

From the above pieces can be calculated the point-like terms which arise from the two-vertex diagram. Including both crossed and uncrossed diagrams, the contact part that

arises is:



$$\rightarrow \overline{V}_\rho P^{\rho\lambda}(p+k)V_\lambda + \overline{U}_\rho P^{\rho\lambda}(p+k')U_\lambda$$

4.6.3 Two vertex contribution

Each piece has been calculated, it is now just a matter of putting it all together.

Leading order

The leading order term from uncrossed diagrams will be $V_\lambda^{(1)} P^{\lambda\rho} \overline{V}_\rho^{(1)}$, or

$$\begin{aligned} V_{\lambda=0}^{(1)} P^{00} \overline{V}_0^{(1)} + V_i^{(1)} P^{ij} \overline{V}_j^{(1)} &= -e^2 m^2 \boldsymbol{\omega}^\dagger \cdot \epsilon' \frac{i}{m^2} \epsilon \cdot \boldsymbol{\omega} + 0 \\ &= -ie^2 (\boldsymbol{\omega}^\dagger \cdot \epsilon')(\epsilon \cdot \boldsymbol{\omega}) \end{aligned}$$

The leading order from the crossed diagrams is

$$U_0^{(1)} P^{00} \overline{U}_0^{(1)} + U_i^{(1)} P^{ij} \overline{U}_j^{(1)} = -ie^2 (\boldsymbol{\omega}^\dagger \cdot \epsilon)(\epsilon' \cdot \boldsymbol{\omega})$$

So the total contribution is

$$eq : S1 : trees 0 - ie^2 (\boldsymbol{\omega}^\dagger \cdot \epsilon)(\epsilon' \cdot \boldsymbol{\omega}) - ie^2 (\boldsymbol{\omega}^\dagger \cdot \epsilon')(\epsilon \cdot \boldsymbol{\omega}) \tag{4.6.19}$$

Contributions to $\mathbf{E} \times \mathbf{A}$ terms

The particular coefficient in the NRQED Lagrangian to check contains $k_0 \mathbf{A}(k)$.

Looking above, from the uncrossed diagrams only the second order terms $V_0^{(2)}$ will contribute to this coefficient.

$$V_{\lambda=0}^{(1)} P^{00} \overline{V}_{\rho=0}^{(2)} + V_{\lambda=2}^{(1)} P^{00} \overline{V}_{\rho=0}^{(1)} \tag{4.6.20}$$

So the uncrossed contribution is

$$-e^2\left([m\epsilon\cdot\boldsymbol{\omega}]\frac{i}{m^2}[(g-1)k_0'\epsilon'\cdot\boldsymbol{\omega}^\dagger]-[(g-1)k_0\epsilon\cdot\boldsymbol{\omega}]\frac{i}{m^2}[m\epsilon'\cdot\boldsymbol{\omega}^\dagger]\right)=-ie^2(g-1)\frac{k_0'-k_0}{m}[\epsilon'\cdot\boldsymbol{\omega}^\dagger][\epsilon\cdot\boldsymbol{\omega}]$$

The contribution from the crossed diagrams is

$$U^{(1)}_{\lambda=0}P^{00}\overline{U}^{(2)}_{\rho=0}+U^{(1)}_{\lambda=2}P^{00}\overline{U}^{(1)}_{\rho=0}$$

$$=-e^2\left([m\epsilon'\cdot\boldsymbol{\omega}]\frac{i}{m^2}[(g-1)k_0\epsilon\cdot\boldsymbol{\omega}^\dagger]-[(g-1)k_0'\epsilon'\cdot\boldsymbol{\omega}]\frac{i}{m^2}[m\epsilon\cdot\boldsymbol{\omega}^\dagger]\right)=-ie^2(g-1)\frac{k_0-k_0'}{m}[\epsilon\cdot\boldsymbol{\omega}^\dagger][\epsilon'\cdot\boldsymbol{\omega}]$$

So the total contribution is

$$-ie^2(g-1)\frac{k_0-k_0'}{m}\left([\epsilon\cdot\boldsymbol{\omega}^\dagger][\epsilon'\cdot\boldsymbol{\omega}]-[\epsilon'\cdot\boldsymbol{\omega}^\dagger][\epsilon\cdot\boldsymbol{\omega}]\right)\tag{4.6.21}$$

This can put this in the form of a matrix element using the previously derived identity

$$(\mathbf{W}^\dagger\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{W})=W_a^\dagger\left[\mathbf{u}\cdot\mathbf{v}\delta_{ab}-(\mathbf{S}_{ac}\cdot\mathbf{u})(\mathbf{S}_{cb}\cdot\mathbf{v})\right]W_b$$

Understanding that the wave functions ω are contracted only with the spin structure, the result can be written

$$-i(g-1)e^2\frac{k_0-k_0'}{m}\omega^\dagger\left(\epsilon'\cdot\epsilon-(\mathbf{S}\cdot\epsilon')(\mathbf{S}\cdot\epsilon)-\epsilon\cdot\epsilon'+\mathbf{S}\cdot\epsilon\mathbf{S}\cdot\epsilon'\right)\omega$$

Which after simplification is

$$M_{\text{trees}}=-ie^2(g-1)\frac{k_0-k_0'}{m}\omega^\dagger\left([\mathbf{S}\cdot\epsilon,\mathbf{S}\cdot\epsilon']\right)\omega=e^2(g-1)\frac{k_0-k_0'}{m}\omega^\dagger\left(\mathbf{S}\cdot\epsilon\times\epsilon'\right)\omega\tag{4.6.22}$$

4.6.4 Sum of two-photon vertex and two-vertex tree diagrams

Now the scattering amplitude, less terms of nonlocal origin, can be written to the needed order. (One power of $\frac{p}{m}$ past leading order.)

First consider just the leading order terms. The contribution from the fundamental two-photon vertex is:

$$-2ie^2\boldsymbol{\omega}^\dagger \cdot \boldsymbol{\omega}\epsilon' \cdot \epsilon + ie^2(\boldsymbol{\omega}^\dagger \cdot \epsilon)(\epsilon' \cdot \boldsymbol{\omega}) + ie^2(\boldsymbol{\omega}^\dagger \cdot \epsilon')(\epsilon \cdot \boldsymbol{\omega})$$

The contribution from leading order contact terms of the two-vertex diagrams is:

$$-ie^2(\boldsymbol{\omega}^\dagger \cdot \epsilon)(\epsilon' \cdot \boldsymbol{\omega}) - ie^2(\boldsymbol{\omega}^\dagger \cdot \epsilon')(\epsilon \cdot \boldsymbol{\omega})$$

The sum of leading order contributions to the scattering is then just

$$-2ie^2\boldsymbol{\omega}^\dagger \cdot \boldsymbol{\omega}\epsilon' \cdot \epsilon$$

which in terms of the spinors ϕ would be

$$-i\frac{e^2}{m}\phi^\dagger(\epsilon \cdot \epsilon')\phi$$

Then consider terms which implicitly will fix those parts of the NRQED Lagrangian involving $\mathbf{E} \times \mathbf{A}$. The fundamental vertex contains no terms corresponding to $\mathbf{E}$, so the only term is from the two-vertex diagrams:

$$e^2(g-1)\frac{k_0 - k_0'}{m}\omega^\dagger \left(\mathbf{S} \cdot \epsilon \times \epsilon'\right)\omega$$

Which in terms of ϕ is just

$$\frac{e^2}{2m^2}(g-1)(k_0 - k_0')\phi^\dagger \left(\mathbf{S} \cdot \epsilon \times \epsilon'\right)\phi$$

4.7 Determination of NRQED coefficients

4.7.1 Comparison with relativistic result (one photon)

Having calculated the same process in both the relativistic theory and in our NRQED effective theory, the two amplitudes can be compared, and the coefficients fixed.

The NRQED amplitude for scattering off an external field is

$$\begin{aligned} iM &= ie\phi^\dagger\Bigg(-A_0 + \frac{\mathbf{A}\cdot\mathbf{p}}{m} - \frac{(\mathbf{A}\cdot\mathbf{p})\mathbf{p}^2}{2m^3} + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2} + c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2} \\ &\quad + c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2} - (c_{W_1}-c_{W_2})\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3}\Bigg)\phi \end{aligned}$$

While the relativistic amplitude was

$$iM_{REL} = -ie\phi^\dagger\Big(A_0 - \frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{\mathbf{p}\cdot\mathbf{A}\mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3}\{\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - S_iS_j\boldsymbol{\nabla}_iE_j\} - g\frac{1}{2m}\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\Big)$$

(4.7.1)

The term $\boldsymbol{\nabla}\cdot\mathbf{E} - S_iS_j\boldsymbol{\nabla}_iE_j$ should be rewritten using the quadrupole moment tensor $Q_{ij} = \frac{1}{2}(S_iS_j + S_jS_i - \frac{2}{3}\mathbf{S}^2)$.

Remember that ∇_iE_j is actually symmetric under exchange of i and j. Then

$$S_iS_j\nabla_iE_j = \frac{1}{2}(S_iS_j + S_jS_i) = (Q_{ij} + \frac{1}{3}\mathbf{S}^2\delta_{ij})\nabla_iE_j$$

$$= Q_{ij}\nabla_iE_j + \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{E}$$

Written using this identity, the relativistic amplitude is

$$iM_{REL} = -ie\phi^\dagger\Big(A_0 - \frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{\mathbf{p}\cdot\mathbf{A}\mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3}\{\frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - Q_{ij}\boldsymbol{\nabla}_iE_j\} - g\frac{1}{2m}\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\Big)$$

Comparing the two, the coefficients are:

$$\begin{aligned} c_F &= g \\ c_D &= \frac{4(g-1)}{3} \\ c_Q &= -4(g-1) \\ c_S^1 &= 2(g-1) \\ (c_{W_1} - c_{W_2}) &= 2 \\ c_{p'p} &= (g-2) \end{aligned}$$

4.7.2 Comparison with relativistic result (two photon)

From the relativistic theory, the local contributions to Compton scattering were calculated.

$$iM_{REL} = \frac{e^2}{2m^2}(g-1)(k_0 - k_0')\phi^\dagger \left(\mathbf{S}\cdot\epsilon\times\epsilon'\right)\phi$$

So comparing the two results, we can fix the coefficient.

$$c_s^2 = 2(g-1)$$

4.7.3 Final Lagrangian

Now we can write down what the NRQED Lagrangian looks like (for constant $\mathbf{B}$.)

$$\begin{aligned}\mathcal{L}_{NRQED} &= \Psi^\dagger\{i(\partial_0 + ieA_0) + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^3} + eg\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + e(g-1)\frac{(\partial_i E_i)}{6m^2} - e(g-1)\frac{Q_{ij}(\partial_i E_j)}{2m^2} \\ &\quad +(g-1)\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D})}{4m^2} - e\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{2m^3} + (g-2)\frac{e(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D})}{4m^3}\}\Psi\end{aligned}$$

Hamiltonian

Or if we write the Hamiltonian instead (using $\mathbf{D} = i(\mathbf{p} - e\mathbf{A})$)

$$\begin{aligned}\mathcal{H}_{NRQED} &= \Psi^\dagger\{\frac{(\mathbf{p}-e\mathbf{A})^2}{2m} - \frac{(\mathbf{p}-e\mathbf{A})^4}{8m^3} + eA_0 - eg\frac{\mathbf{S}\cdot\mathbf{B}}{2m} - e(g-1)\frac{(\partial_i E_i)}{6m^2} + e(g-1)\frac{Q_{ij}(\partial_i E_j)}{2m^2} \\ &\quad +(g-1)\frac{e\mathbf{S}\cdot(\mathbf{p}\times\mathbf{E} - \mathbf{E}\times\mathbf{p})}{4m^2} - (g-1)\frac{e^2\mathbf{S}\cdot\mathbf{A}\times\mathbf{E}}{2m^2} + e\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{2m^3} + (g-2)\frac{e(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3}\}\Psi\end{aligned}$$

We can construct an effective, nonrelativistic Lagrangian for a charged particle interacting with an electromagnetic field.

Chapter 5

NRQED calculation

5.1 Constructing the NRQED Lagrangian

We want to construct an effective Lagrangian in the nonrelativistic limit. Our goal is to calculate the leading order corrections to the g-factor, which are corrections of order α^2. To this end, we need terms in the effective nonrelativistic Lagrangian which are equivalent corrections.

5.1.1 Constraints on the form of the Lagrangian

The Lagrangian is constrained to obey several symmetries. It must be invariant under the symmetries of parity and time reversal. It must also be invariant under Galilean transformations. The Lagrangian must also be Hermitian, and gauge invariant.

What are the gauge invariant building blocks we can use to construct this Lagrangian? We have the fields $\mathbf{E}$ and $\mathbf{B}$, the spin operators $\mathbf{S}$, and the long derivative $\mathbf{D} = \partial - ie\mathbf{A}$. The fields should always be accompanied by the charge e of the particle.

When considering the case of higher spin particles, we might consider terms quadratic and above in spin operators. For a particle of spin s, there must be $(2s+1)^2$ independent hermitian operators. We can span this set of operators by considering products of up to $2s$ spin matrices which are symmetric and traceless in every vector index. For example, for spin-1 we have quadratic, in addition to I and S_i, five independent structures of the form $S_iS_j + S_jS_i + \delta_{ij}\mathbf{S}$. This idea is pursued further in a section below.

We also have the scalar D_0, however, we need only include a single such term because we insist on having only one power of the time derivative.

Energy Scales

The effective Lagrangian contains an infinite number of terms. But these terms may be organised by the order of their contribution to scattering. Then, up to a particular order there are only a finite number of terms — and for the first couple of orders this finite number is quite small and manageable.

Consider again the particular physical situation of interest: a loosely bound state system of two charged particles, each with arbitrary spin, placed in an infinitesimal magnetic field. It is crucial that the system is loosely bound, so that the entire system is nonrelativistic. For now it is assumed that recoil effects are ignored.

Then what are the energy scales of this particular problem? In addition to the mass of the particle, there are two: one is related to the external magnetic field, an energy scale of $e/m\,|\mathbf{B}|$. The other is related to the kinetic energy or momentum of the particle. There is an electric field present as well, representing the interaction between the bound state particles. But by the quantum virial theorem the Coulomb potential is proportional to the kinetic energy, and so is not an independent energy scale. In general kinetic energy and momentum might be considered seprate energy scales, but in the nonrelativistic regime are just mv and $(1/2)mv^2$.

Both of these energy scales are nonrelativistic — that is, small compared to the mass of the particle. So the effective Lagrangian represents an expansion in the small quantities e/m^2B and $p/m = v$. However, the calculation of the gyromagnetic ratio involves only linear terms in the magnetic field. (The g-factor is defined by the linear response to an infinitesimal magnetic field.) So no terms quadratic in B are needed.

The goal is to calculate corrections to the g-factor of up to $\mathcal{O}(\alpha^2)$. In the bound state system $v \sim Z\alpha$, so in the Lagrangian should definitely be included terms of up to order $(e/m)Bv^2$. It will also be necessary to consider terms of order mv^4, for they *might* effect the calculation of the g-factor through perturbation theory.

What is the exact order of each term? The magnetic field defines its own scale, but the

scale of operators D, A_0, and E come from the particular bound state.

The term involving the Coulomb potential, $eA_0 = Ze^2/r^2$ is of order mv^2. The first derivative $\partial_i \Phi = E_i$ is of order mv^3. The operator $\boldsymbol{\nabla}$ may then be said to be of order mv.

Discrete Constraints

All the building blocks have been chosen so that their transformation under rotation is clear. If a term has an index, it transforms like a 3-vector. (Considering this nonrelativistic theory, there are no upper or lower indices.) The Lagrangian must also be symmetric under the discrete symmetries of refleciton and time reversal, and terms must be Hermitian.

To construct terms that obey these symmetries, it is necessary to know how the building blocks themselves transform under spatial reflections and time reversal, as well as under complex conjugation.

- Under spatial reflections, **E** and **D** are vectors, and so are odd under P. While **B** and **S** are pseudovectors, even under P.

- E and D are even under time reversal, while spin and the magnetic field are odd. Also, under time reversal imaginary constant i is also odd.

- Under complex conjugation E, B and S are Hermitian, while D_i, D_0 and (obviously) i change sign.

For reference, all these properties are tabulated below:

	Order	P	T	†
eE_i	m^2v^3	-	+	+
eB_i	m^2v^2	+	-	+
D_i	mv	-	+	-
D_0	mv^2	+	-	-
S_i	1	+	-	+
i	1	+	-	-

5.1.2 Properties of D_i

Before going any further, note that it would be possible to write the entire Lagrangian without explicitly writing B. Instead we could write antisymmetric combinations of D_i.

Consider a term containing $D_i D_j \epsilon_{ijk}$. First write them using $\mathbf{D} = \boldsymbol{\nabla} - ie\mathbf{A}$:

$$(\mathbf{D} \times \mathbf{D}) = D_i D_j \epsilon_{ijk} = (\boldsymbol{\nabla}_i - ieA_i)(\boldsymbol{\nabla}_j - ieA_j)\epsilon_{ijk} \tag{5.1.1}$$

Or in full as

$$= (\boldsymbol{\nabla}_i \boldsymbol{\nabla}_j - ie\boldsymbol{\nabla}_i A_j - ie\boldsymbol{\nabla}_j A_i - e^2 A_i A_j)\epsilon_{ijk} \tag{5.1.2}$$

The antisymmetric tensor will kill the first and last terms, leaving

$$= -ie(\boldsymbol{\nabla}_i A_j + \boldsymbol{\nabla}_j A_i)\epsilon_{ijk} \tag{5.1.3}$$

We can use put this in the form of a commutator by switching indices in the second term: $\boldsymbol{\nabla}_j A_i \epsilon_{ijk} = -\boldsymbol{\nabla}_i A_j \epsilon_{ijk}$

$$= -ie[\boldsymbol{\nabla}_i, A_j]\epsilon_{ijk} \tag{5.1.4}$$

and of course the commutator between $\boldsymbol{\nabla}$ and any field will just be the derivative of that field.

$$= -ie\partial_i A_j \epsilon_{ijk} \tag{5.1.5}$$

Finally, from Maxwell's equations this is just $-ieB_k$, so

$$(\mathbf{D} \times \mathbf{D})_k = -ieB_k \tag{5.1.6}$$

Going the other way, the commutator $[D_i, B_i]$ can be shown to vanish:

$$[D_i, B_i] = D_i(\epsilon_{ijk} D_j D_k) - (\epsilon_{ijk} D_j D_k)D_i = \epsilon_{ijk}(D_i D_j D_k - D_j D_k D_i) \tag{5.1.7}$$

Taking the second term and taking an even permutation of the indices: $i \to k \to j \to i$ we

show that

$$[D_i, B_i] = \epsilon_{ijk}(D_i D_j D_k - D_i D_j D_k) = 0 \tag{5.1.8}$$

5.1.3 Properties of spin operators

In formulating NRQED for general spin particles, we need to consider all the possible operators might show up in the Lagrangian. The state-space of a spin-s particle is the direct product of it's spin-state and all the other state information. Because the spaces are orthogonal, we can treat separately operators in the two spaces. The operators and fields which exist in position space are the same for a particle of any spin, but unsurpsingly the operators allowed in spin space do depend upon the spin of the particle. As the spin of the particle is increased, and thus its spin degrees of freedom rise, there are more ways to mix these components, and thus a greater number of spin operators to consider.

For a particular representation, we can always write a bilinear as the spin operators and other operators acting between two spinors:

$$\Psi^\dagger \mathcal{O}_S \mathcal{O}_X \Psi \tag{5.1.9}$$

The two types of operators will always commute, since they act on orthogonal spaces, so it doesn't matter what order they're written in. All such bilinears must be Galilean invariant, but individual operators might not be. The non-spin operators we consider, such as $\mathbf{D}$, $\mathbf{B}$ or contractions with the tensor ϵ_{ijk} are already all written as 3-vectors or (in combination) as higher rank tensors. Therefore, it will be most convenient to write spin-operators in the same way, so that writing Galilean-invariant combinations of the two types of operators is done just by contracting indices.

Even though the number of spin opeartors does depend upon the spin of the particle, it is still possible to proceed in such a way that the same notation may be used no matter the spin. There are a few requirements:

- We write all high spin operators in terms of combinations of S_i, since these have universal properties regardless of the representation they are written in.

- If an operator exists and is non-zero in the representation of spin-s, it also exists in spin-$s+1$
- All operators introduced to account for the additional degrees of freedom in higher spin representations vanish when written in a lower spin theory. (As an example of the last point, the operator $S_iS_j + S_jS_i - \delta_{ij}S^2$ is needed to account for the degrees of freedom in a spin-1 theory, but vanishes in spin-1/2.
-)

If these requirements are met a consistent spin-agnostic notation can be adopted. Now we attempt to construct operators that meet these conditions.

The spinors Ψ are written with $2s+1$ independent components. The spin operators will be matrices acting on these components, and so for a spin-s particle will be $(2s+1)\times(2s+1)$ matrices. The combined operator $\mathcal{O}_S\mathcal{O}_X$ must be Hermitian, but without loss of generality we can require any $\mathcal{O}_S$, $\mathcal{O}_X$ to be Hermitian separately. So there is the additional constraint that the matrices be Hermitian, and this means a total of $(2s+1)^2$ degrees of freedom.

For spin-0 there is only one component to the spinor, so the only possible matrix is equivalent to the identity.

For spin-1/2 we have, in addition to the identity matrix, the spin matrices $s_i = \frac{1}{2}\sigma_i$. This is four independent matrices, and since the space has $(2s+1)^2 = 4$ degrees of freedom, exactly spans the space of all spin-operators. If we try to construct terms which are bilinear in spin matrices, they just reduce through the identity $\sigma_i\sigma_j = \delta_{ij} + \epsilon_{ijk}\sigma_k$, which we can already construct through combinations of the four operators we already have. Since those four operators form a basis for the space, independent bilinears were forbidden even without an explicit form for the equation.

What about spin-1? We need 9 independent operators to span the space. All the operators that exist in spin-1/2 will work here as well, though the spin matrices will have a different representation. That leaves 5 operators to construct. It is natural to try to construct these from bilinear combinations of spin matrices. Naively S_iS_j would itself be 9 independent structures, but clearly some of these are expressible in terms of the lower order operators. (By the order of a spin operator we mean its greatest degree in S_i)

Regardless of their representation, the spin operators always fulfill certain identities based on their Lie group. Namely

$$S_i S_j \delta_{ij} \sim I,\ [S_i, S_j] = \epsilon_{ijk} S_k \tag{5.1.10}$$

and it is these identities which allow certain combinations of $S_i S_j$ to be related to lower order operators.

If instead of general spin bilinears we consider only combinations which are

- Symmetric in i, j
- Traceless

then such a structure will be independent of the set of operators $\{I, S_1, S_2, S_3\}$. Because it is symmetric no combination may be related using the commutator, and because it is traceless there is no combination that reduces due to the other identity.

This conditions form a set of 4 constraints, so from the original 9 degrees of freedom possessed by combinations of $S_i S_j$ are left only 5. Together with the 4 lower order operators this is exactly enough to span the space.

We can explicitly write this symmetric, traceless structure as

$$S_i S_j + S_j S_i - \delta_{ij} S^2 \tag{5.1.11}$$

Having explored how the procedure works for spin-1, move on to consider the general spin case. The idea is to proceed inductively using the same rough attack as for the case of spin-1. In addition to all the "lower order" operators which were used for lower spin representations introduce new operators which are of higher degree in the spin matrices and guaranteed to be independent of the lower spin operators.

So suppose that for a spin-$s{-}1$ particle we have a set of operators written as $\bar{S}^0, \bar{S}^1, \ldots \bar{S}^{(s-1)}$, where a structure $\bar{S}^n$ carries n Galilean indices and is symmetric and traceless between any pair of indices, that is:

$$\bar{S}^n_{..i..j..} = \bar{S}^n_{..j..i..},\ \delta_{ij} \bar{S}^n_{..i..j..} = 0 \tag{5.1.12}$$

(From above, $\bar{S}^0 = I$, $\bar{S}^1_i = S_i$, and $\bar{S}^2 = S_iS_j + S_jS_i + \delta_{ij}S^2$.)

The objects $\bar{S}^n$ are built as follows: start with all combinations involving the product of exactly n spin matrices. (There are 3^n such structures.) Form them into combinations which are symmetric in all indices. Each index has three possible values, so we can label each structure by how many indices are equal to 1 and 2. If a is the number of indices equal to 1, and b the number of indices equal to 2, then for a given a there are $n+1-a$ possible choices for b. The total number of symmetric structures is then

$$\Sigma_{a=0}^{n}(n-a+1) = \frac{1}{2}(n+1)(n+2) \tag{5.1.13}$$

We want to apply the additional constraint that the $\bar{S}^n$ be traceless in all indices. This will involve subtracting all the lower order structures which result when the trace of the completely symmetric combinations is taken.

It introduces an additional constraint on $\overline{S}^n$ for each pair of indices, and there are $n(n-1)/2$ distinct pairs of indices. The total degrees of freedom left are

$$\frac{1}{2}(n+1)(n+2) - \frac{1}{2}n(n-1) = \frac{1}{2}\left(n^2+3n+2-n^2+n\right) = 2n+1 \tag{5.1.14}$$

In combination with the lower order spin operators, this is exactly the number of independent operators we need to span the space. Combined with the lower order operators this is a complete basis, so we know we haven't missed any terms. Because they are constructed to be independent from all the lower order operators, it must necessarily be true that they will vanish in lower spin representations.

Using this notation we can write down terms in the Lagrangian that are valid for particles of any spin. By writing all spin operators in terms of S_i they are representation agnostic, and by construction they will vanish for low spin particles where they do not "fit".

5.1.4 Composition of position space operators

There are several symmetries that terms in the Lagrangian must preserve: spatial reflection, time reversal, gauge invariance, and Hermiticity. Gauge invariance is taken care of by only

considering gauge-invariant operators.

All of the possible spin-space structures preserve spatial parity. If the term in the Lagrangian is to obey such a symmetry, then any allowed collection of position-space operators must be themselves invariant under reflection.

Each term must also be invariant under time reversal. The spin operator S_i flips sign under time reversal. Thus, $\bar{S}_{ij}$ is even, and $\bar{S}_{ijk}$ odd, under the same transformation. E_i and D_i are even, while B_i is odd. Any set of operators with definite behavior under time reversal, though, can be made invariant by including an extra factor of i. So even if some term ABC is odd, $iABC$ will be even.

Finally, any sequence of operators can be made Hermitian by simply adding the Hermitian conjugate; if we wish to include a term ABC in the Lagrangian, we add $C^\dagger B^\dagger A^\dagger$. Since all the operator s under consideration are either Hermitian or anti-Hermitian, this means that exactly one of $ABC \pm CBA$ will be allowed. The spin space operators are all Hermitian of themselves, and commute with all other operators, so only the position space operators are nontrivial.

Thus for any possible set of operators, first it is determined if such a set is allowed by considerations of parity. Then, whether an additional factor of i is needed is determined by examining the properties under time reversal. Finally, the allowed Hermitian terms are enumerated.

Of the "building blocks" for the general Lagrangian, those odd in parity are also odd in v. And those even in parity are of an order even in v. So only terms of an order even in v can possibly be allowed under parity, and at the same time every such term will have the correct symmetry properties under spatial reflection.

First consider terms of leading order mv^2. What collections of position space operators exist? E is already too high of an order, leaving only that with two powers of D:

$$\text{Sets of } \mathcal{O}(mv^2) \text{ combinations} = \{\frac{1}{m} D_i D_j\} \tag{5.1.15}$$

This is in addition to the single permitted term containing $e\Phi$ which is not, by itself, gauge-invariant.

Next consider terms of order mv^4. We could have up to a single power of E, or only the long derivative operators:

$$\text{Set of } \mathcal{O}(mv^2) \text{ combinations} = \{\frac{1}{m^3} D_i D_j D_k D_\ell,\ \frac{1}{m^2} e E_i D_j\} \tag{5.1.16}$$

The leading order term in B is just $\frac{e}{m} B_i$.

The order $\frac{e}{m} B_i v^2$ terms are drawn from:

$$\text{Set of } \mathcal{O}(\frac{e}{m} B v^2) \text{ combinations} = \{\frac{e}{m^3} D_i D_j B_k\} \tag{5.1.17}$$

Contraction of terms

In this nonrelativistic theory the Lagrangian need not be Lorentz invariant, but must still be Galilean invariant. All the operators we consider transform as 3-vectors or higher order 3-tensors. To form allowed terms, all indices must be somehow contracted.

Above, all relevant sets of position space operators are considered for each order of term. The greatest number of indices free was four. These must be contracted with order unity structures, which as well as the spin operators with up to four indices (S_i, $\overline{S}_{ij}$, $\overline{S}_{ijk}$,$\overline{S}_{ijk\ell}$) include δ_{ij} and the completely antisymmetric tensor ϵ_{ijk}.

- The only structure with one index is just S_i.
- With two indices, there are the simple structures $\overline{S}_{ij}$ and δ_{ij}, and also $S_i \epsilon_{ijk}$.
- With three indices, there are ϵ_{ijk} and $\overline{S}_{ijk}$ as well as $\delta_{ij} S_k$ or $\overline{S}_{ij} \epsilon_{jk\ell}$.
- The only position space operator with four indices is just $D_i D_j D_k D_\ell$. While there are a fairly large number of order unity structures with four indices, any with spin operators are forbidden because of the kinetic nature of the term. So only $\delta_{ij}\delta_{k\ell}$ need be considered.

Above are categorized the possible sets of position space operators for each order, and the ways of contracting them by number of indices. The next step is to write all Hermitian combinations we can form from these operators. This will form a complete catalogue of allowed terms in the Lagrangian.

Some terms which on the surface appear distinct might in fact be identical. One identity of this nature is

$$D_i D_j \epsilon_{ijk} = -ieB_k \tag{5.1.18}$$

Which means that any term where two long derivative terms are contracted with the anti-symmetric tensor will be equivalent to some already catalogued term involving B.

Similarly,

$$[D_i, D_j] = -ie(\partial_i A_j - \partial_j A_i) = -ieF_{ij} = -ie\epsilon_{ijk}B_k \tag{5.1.19}$$

so terms which appear distinct because of the ordering of two long derivative operators are really the same, up to some difference that is absorbed into a term involving B.

Finally, while in the general case $[D_i, B_j] \neq 0$, it is always true that $[D_i, B_j] = 0$.

So many apparently distinct terms may be rewritten and absorbed as the combination of several already accounted for terms. As an example, it is not necessary to write a term such as $D_i D_j D_i D_j$ in the Lagrangian. It can be rewritten as

$$\begin{aligned} D_i D_j D_i D_j &= D_i D_i D_j D_j + D_i [D_i, D_j] D_j \\ &= \mathbf{D}^4 - ieD_i \epsilon_{ijk} B_k D_j \\ &= \mathbf{D}^4 - ie\epsilon_{ijk}(D_i D_j B_k + [B_k, D_j]) \\ &= \mathbf{D}^4 - e^2 B^2 + ie\epsilon_{ijk} D_i (\partial_j B_k) \end{aligned}$$

The term $D_i D_j$ has two indices. There are three ways to contract it: with δ_{ij}, $\overline{S}_{ij}$, or $S_k \epsilon_{ijk}$. Since kinetic terms with spin are not considered, that only leaves one possible term: $\mathbf{D}^2$. This term is Hermitian in and of itself. It has mass dimension two, so the term as it appears in the Lagrangian will be:

$$D_i D_j \rightarrow \frac{\mathbf{D^2}}{2m} \tag{5.1.20}$$

Terms from the set $E_i D_j$ also have two indices. It can be contracted with all three of δ_{ij}, $\overline{S}_{ij}$, or $iS_k \epsilon_{ijk}$. E_i, δ_{ij} and $\overline{S}_{ij}$ are Hermitian, while D_j and $iS_k \epsilon_{ijk}$ are anti-Hermitian. It is convenient to use that $[D_j, E_i] = \boldsymbol{\nabla}_j E_i$.

The Hermitian combination with δ_{ij} is

$$(D_iE_j - E_jD_i)\delta_{ij} \rightarrow \frac{\boldsymbol{\nabla}\cdot\mathbf{E}}{4m^2} = \frac{\boldsymbol{\nabla}\cdot\mathbf{E}}{4m^2} \tag{5.1.21}$$

The Hermitian combination with $\overline{S}_{ij}$ is

$$(D_iE_j - E_jD_i)\overline{S}_{ij} \rightarrow \frac{\overline{S}_{ij}\partial_i E_j}{4m^2} \tag{5.1.22}$$

The Hermitian combination with $iS_k\epsilon_{ijk}$ is

$$(D_iE_j + E_jD_i) \rightarrow iS_k\epsilon_{ijk} \rightarrow \frac{i\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} + \mathbf{E}\times\mathbf{D})}{4m^2} \tag{5.1.23}$$

A single power of the magnetic field, B_i may only be contracted with S_i. So the term is

$$\frac{e}{m}\mathbf{S}\cdot\mathbf{B} \tag{5.1.24}$$

The second order terms involving the magnetic field are drawn from the set $D_iD_jB_k$. There are four order-unity structures which have three indices, but only two of those need be considered. This is because $\mathbf{D}\times\mathbf{D} \sim \mathbf{B}$ and $\mathbf{D}\times\mathbf{B} = 0$. If is not assumed that $[D_i, B_j] = 0$ then the allowed Hermitian combinations will be with $D_iD_jB_k + B_kD_jD_i$ and $D_iB_jD_k$.

While $[D_i, D_j] \neq 0$, it does produce another term proportional to B and thus, smaller than considered. So for instance, $D_iD_jB_i$ need not be considered separate from $D_jD_iB_i$.

$$\begin{aligned}
(D_iD_jB_k + B_kD_jD_i)\delta_{ij}S_k &\rightarrow \frac{e}{m}\frac{\mathbf{D}^2(\mathbf{S}\cdot\mathbf{B}) + (\mathbf{S}\cdot\mathbf{B})\mathbf{D}^2}{4m^2} \\
(D_iD_jB_k + B_kD_jD_i)\delta_{jk}S_i &\rightarrow \frac{e}{m}\frac{(\mathbf{S}\cdot\mathbf{D})(\mathbf{D}\cdot\mathbf{B}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})}{4m^2} \\
D_iB_jD_k\delta_{ij}S_k &\rightarrow \frac{e}{m}\frac{(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{D}\cdot\mathbf{B})(\mathbf{S}\cdot\mathbf{B})}{4m^2}
\end{aligned}$$

$$
\begin{aligned}
(D_iD_jB_k + B_kD_jD_i)\overline{S}_{ijk} &\rightarrow \frac{e}{m}\frac{\overline{S}_{ijk}(D_iD_jB_k + B_kD_jD_i)}{4m^2} \\
D_iB_jD_k\overline{S}_{ijk} &\rightarrow \frac{e}{m}\frac{\overline{S}_{ijk}D_iB_jD_k}{4m^2}
\end{aligned}
$$

Finally, there is only one way to contract $D_iD_jD_kD_\ell$, which is as $\mathbf{D}^4$. Again, variations such as $D_i\mathbf{D}^2D_i$ need not be considered because they just reproduce already considered terms with B.

$$
D_iD_jD_kD_\ell\delta_{ij}\delta_{k\ell} \rightarrow \frac{\mathbf{D}^4}{8m^3}
$$

Having organised all the operators into all allowed combinations, we can write down the different orders of the Lagrangian.

The leading order terms in the Lagrangian, of order mv^2 and $(e/m)B$, are:

$$
\mathcal{L}_{mv^2} = \Psi^\dagger\left\{iD_0 + \frac{\mathbf{D}^2}{2m} + c_F\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\right\}\Psi \tag{5.1.25}
$$

The part of the Lagrangian of order mv^4 is:

$$
\mathcal{L}_{mv^4} = \Psi^\dagger\left\{\frac{\mathbf{D}^4}{8m^2} + c_D\frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2} + c_Q\frac{eQ_{ij}(D_iE_j - E_iD_j)}{8m^2} + c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}}{8m^2}\right\}\Psi \tag{5.1.26}
$$

And the terms which are corrections to the magnetic field terms:

$$
\begin{aligned}
\mathcal{L}_{Bv^2} = \Psi^\dagger\Bigg\{ & c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3} + c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3} \\
& + c_{T_1}\frac{e\bar{S}_{ijk}(D_iD_jB_k + B_kD_jD_i)}{8m^3} + c_{T_2}\frac{e\bar{S}_{ijk}D_iB_jD_k}{8m^3}\Bigg\}\Psi
\end{aligned} \tag{5.1.27}
$$

Full Lagrangian

Combining all of the above, the full Lagrangian we consider is then:

$$\mathcal{L}_{NRQED} = \Psi^\dagger \Bigg\{ iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^2} + c_F \frac{e}{m}\mathbf{S}\cdot\mathbf{B} + c_D \frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2} + c_Q \frac{eQ_{ij}(D_iE_j - E_iD_j)}{8m^2}$$
$$+ c_S \frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}}{8m^2} + c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3}$$
$$+ c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3} + c_{T_1}\frac{e\bar{S}_{ijk}(D_iD_jB_k + B_kD_jD_i)}{8m^3} + c_{T_2}\frac{e\bar{S}_{ijk}D_iB_jD_k}{8m^3} \Bigg\}\Psi \tag{5.1.28}$$

One of the features of this Lagrangian is that every coefficient is fixed by the one-photon interaction. Although some terms might represent two-photon interactions, they are terms like $\mathbf{S}\cdot\mathbf{A}\times\mathbf{E}$, whose coefficient is fixed by the gauge-invariant term $\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}$. This in turn means that we can calculate the corrections to the g-factor by considering only one-photon interactions.

5.2 Scattering off external field in NRQED

We now have established the general form of the Lagrangian for NRQED of arbitrary spin. In order to fix the coefficients, it will be necessary to compare the predictions of NRQED with some other source, such as an exact relativistic theory.

5.2.1 One-photon Lagrangian

The simplest (and most important for our purposes) set of coefficients to fix are those involved in the process of the charged particle scattering off an external field. We can write down those terms in the NRQED Lagrangian which have one power of the external field. This set of terms will not, by themselves, be gauge invariant. However, we had previously taken care to group terms in a gauge invariant way. So first we must untangle from these gauge invariant terms (involving one or more powers of $\mathbf{D}$), those which involve a single field.

First take the kinetic terms. $\mathbf{D} = \boldsymbol{\nabla} - ie\mathbf{A}$, so

$$\mathbf{D}^2 = (\nabla_i - ieA_i)(\nabla_i - ieA_i) \tag{5.2.1}$$

It is a mixture of terms with two, one, or no powers of the external field A. If just the terms with one power of $\mathbf{A}$ are included, what remains is

$$-ie(A_i\nabla_i + A_i\nabla_i = -ie\{A_i, \nabla_i\} \tag{5.2.2}$$

So from $\mathbf{D}^2/2m$ we can write $-ie\{\nabla_i, A_i\}/2m$ in $\mathcal{L}_A$.

The second kinetic term is $\mathbf{D}^4$:

$$\mathbf{D}^4 = (\nabla_i - ieA_i)(\nabla_i - ieA_i)(\nabla_j - ieA_j)(\nabla_j - ieA_j) \tag{5.2.3}$$

or

$$\mathbf{D}^4 = (\boldsymbol{\nabla}^2 - ie\{A_i, \nabla_i\} - e^2\mathbf{A}^2)(\boldsymbol{\nabla}^2 - ie\{A_j, \nabla_j\} - e^2\mathbf{A}^2) \tag{5.2.4}$$

Keeping again only the terms with a single power of A, what remains may be expressed as the double anti-commutator

$$-ie\{\boldsymbol{\nabla}^2, \{\nabla_i, A_i\}\} \tag{5.2.5}$$

There are then several terms involving one or more powers of D combined with either E or B. Since only terms with a single power of the field are of interest, from all such $\mathbf{D}$ only the part involving $\boldsymbol{\nabla}$ is kept.

So finally, writing all such terms from the original Lagrangian involved in scattering off an external field leaves:

$$\begin{aligned}\mathcal{L}_A &= \Psi^\dagger(-eA_0 - ie\frac{\{\nabla_i, A_i\}}{2m} - ie\frac{\{\boldsymbol{\nabla}^2, \{\nabla_i, A_i\}\}}{8m^3} + c_F e\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{e(\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{E}\cdot\boldsymbol{\nabla})}{8m^2} + c_Q\frac{eQ_{ij}(\nabla_i E_j - E_i\nabla_j)}{8m^2} \\ &\quad + c_S^1\frac{ie\mathbf{S}\cdot(\boldsymbol{\nabla}\times\mathbf{E} - \mathbf{E}\times\boldsymbol{\nabla})}{8m^2} + c_{W_1}\frac{e[\boldsymbol{\nabla}^2(\mathbf{S}\cdot\mathbf{B}) + (\mathbf{S}\cdot\mathbf{B})\boldsymbol{\nabla}^2]}{8m^3} - c_{W_2}\frac{e\nabla^i(\mathbf{S}\cdot\mathbf{B})\nabla^i}{4m^3} + c_{p'p}\frac{e[(\mathbf{S}\cdot\boldsymbol{\nabla})(\mathbf{B}\cdot\boldsymbol{\nabla}) + (}{8m^3}\end{aligned}$$

5.2.2 Calculation

Now we have the Lagrangian that will account for all interactions with a single photon. We want to calculate from this a particular process: scattering off an external field, with incoming momentum $\mathbf{p}$, outgoing $\mathbf{p}'$, and $\mathbf{q} = \mathbf{p}' - \mathbf{p}$. There is one diagram associated with each term above, but the total amplitude is just going to be the sum of all these one-photon vertices. These of course can just be read off directly from the Lagrangian.

It is necessary to switch to the language of momentum space. The recipe is this: replace the fields Ψ with the spinors ϕ, and any operator $\boldsymbol{\nabla}$ acting will become $i\mathbf{p}$ if it acts on the right, $i\mathbf{p}'$ if it is to the left. Go through term by term.

The terms originating from D^2 are:

$$-ie\frac{\{\nabla_i, A_i\}}{2m} \tag{5.2.6}$$

which in position space become

$$e\frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')}{2m} \tag{5.2.7}$$

While the terms arising from D^4

$$-ie\frac{\{\boldsymbol{\nabla}^2, \{\nabla_i, A_i\}\}}{8m^3} \tag{5.2.8}$$

become

$$-e\frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')(\mathbf{p}'^2+\mathbf{p}^2)}{8m^3} \tag{5.2.9}$$

We can simplify some expressions involving $\boldsymbol{\nabla}$ and $\mathbf{E}$: Because Q_{ij} is symmetric:

$$Q_{ij}(\nabla_i E_j - E_i\nabla_j) = Q_{ij}[\nabla_i, E_j] = Q_{ij}(\partial_i E_j)$$

And because $E_i = -\partial_i\Phi$

$$\boldsymbol{\nabla}\times\mathbf{E} - \mathbf{E}\times\boldsymbol{\nabla} = -2\mathbf{E}\times\boldsymbol{\nabla}$$

And also use that

$$\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{E}\cdot\boldsymbol{\nabla} = (\partial_i E_i)$$

Now we can write down the scattering amplitude for scattering off the external field,

before we apply any assumptions about the particular process.

$$\begin{aligned} iM &= ie\phi_S^\dagger\Bigg(-A_0 + \frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')}{2m} - \frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')\mathbf{p}^2 + \mathbf{p}'^2\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')}{8m^3} \\ &\quad + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2} + c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2} + c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2} \\ &\quad - c_{W_1}\frac{(\mathbf{S}\cdot\mathbf{B})(\mathbf{p}^2+\mathbf{p}'^2)}{8m^3} + c_{W_2}\frac{(\mathbf{S}\cdot\mathbf{B})(\mathbf{p}\cdot\mathbf{p}')}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p}')(\mathbf{B}\cdot\mathbf{p}) + (\mathbf{B}\cdot\mathbf{p}')(\mathbf{S}\cdot\mathbf{p})}{8m^3}\Bigg)\phi_S \end{aligned}$$

The above can be simplified somewhat. We choose our gauge such that $\nabla_i A_i = 0$. If we specify elastic scattering then kinematics dictate that $\mathbf{p}'^2 = \mathbf{p}^2$. Finally, if we consider $\mathbf{B}$ constant, the c_W terms become indistinguishable, since $[\nabla_i, B_j] = 0$. (It is only this last assumption that costs us any information.) Then the scattering amplitude, as calculated from $\mathcal{L}_{NRQED}$, is:

$$\begin{aligned} iM =& ie\phi_S^\dagger\Bigg(-A_0 + \frac{\mathbf{A}\cdot\mathbf{p}}{m} - \frac{(\mathbf{A}\cdot\mathbf{p})\mathbf{p}^2}{2m^3} + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2} + c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2} \\ &+ c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2} - (c_{W_1} - c_{W_2})\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3}\Bigg)\phi_S \end{aligned} \tag{5.2.10}$$

This may be compared with a relativistic calculation to fix the coefficients. (Although of course it does not fix c_{W_1} and c_{W_2} separately.)

5.2.3 Two photon scattering in NRQED

The calculation above suffices to fix all coefficients in the NRQED Lagrangian needed to calculate corrections to the g-factor. While there is a contribution that comes from a term with two electromagnetic fields, the coefficient of that term is still fixed by just considering the one-photon interaction, due to the necessity of the overall gauge-invariance of the term.

Still, we *can* calculate the coefficient separately from an exact theory, so it would be good to be able to compare the results to make sure no surprises occur. To that end it will be necessary to calculate a two-photon process, Compton scattering.

So, we want to calculate the Compton scattering in the nonrelativistic theory. We choose the gauge such that the photon polarisations obey $\epsilon_0 = 0$. In general we should consider

both terms arising from two-photon vertices, and those from tree level diagrams of two one-photon vertices. However, because of the approach we take in calculating the process from the relativistic Lagrangian, we only need the former terms. That is, we know how to separate contact terms (which arise from some combination of Z-diagrams and two-photon vertices in the relativistic theory) from the rest.

Further, ultimately we only care about a few of these terms, ignoring terms which have both $\mathbf{B}$ and $\mathbf{A}$.

Note that both $(\mathbf{A}\cdot\mathbf{E}-\mathbf{E}\cdot\mathbf{A})=0$ and $Q_{ij}(A_iE_j-E_iA_j)=0$ by symmetry.
The remaining terms we're interested in are:

$$\mathcal{L}_{A^2} \quad = \quad \Psi^\dagger(-\frac{e^2\mathbf{A}^2}{2m}-e^2\frac{\{\boldsymbol{\nabla}^2,\mathbf{A}^2\}}{8m^3}-e^2\frac{\{\nabla_i,A_i\}\{\nabla_j,A_j\}}{8m^3}+c_S^2\frac{e^2\mathbf{S}\cdot(\mathbf{A}\times\mathbf{E}-\mathbf{E}\times\mathbf{A})}{8m^2})\Psi$$

The process we consider has an incoming photon with momentum k and polarisation ϵ, and an outgoing photon with momentum $-k'$ and polarisation ϵ'. The charged particle has incoming momentum p and outgoing $p'=p+k+k'$.

As in the single-photon calculation, it is simply a matter of reading terms off the Lagrangian. If we want the scattering amplitude, we replace Ψ with ϕ, and replace $\mathbf{A}$ with photon polarisations ϵ and ϵ'. In the gauge chosen, $\mathbf{E}(k)=-\partial_0\epsilon=ik_0\epsilon(k)$.

Contracted with the photon of momentum k we get $\mathbf{E}\to ik_0\epsilon$, while with the photon of momentum k' we have $\mathbf{E}\to ik_0'\epsilon'$. Both processes must be considered in calculating the scattering, so:

$$\mathbf{A}\times\mathbf{E}=-\mathbf{A}\times(\partial_0\mathbf{A})\to -i(k_0'\epsilon\times\epsilon'+k_0\epsilon'\times\epsilon)=i(k_0'-k_0)\epsilon\times\epsilon'$$

And

$$\mathbf{E}\times\mathbf{A}=-(\partial_0\mathbf{A})\times\mathbf{A}\to -i(k_0\epsilon\times\epsilon'+k_0'\epsilon'\times\epsilon)=-i(k_0'-k_0)\epsilon\times\epsilon'$$

So from the term in the Lagrangian

$$c_S\Psi^\dagger\frac{e^2\mathbf{S}\cdot(\mathbf{A}\times\mathbf{E}-\mathbf{E}\times\mathbf{A})}{8m^2})\Psi$$

we get in the scattering amplitude

$$-ic_S\phi^\dagger\Big(\frac{e^2}{4m^2}i(k_0-k_0')\epsilon\times\epsilon'\Big)\phi = c_S\frac{e^2}{4m^2}\phi^\dagger\Big((k_0-k_0')\epsilon\times\epsilon'\Big)\phi$$

This is the only part of the amplitude we wanted to compare for reasons of consistency.

Chapter 6

General Spin Formalism

Our ultimate goal is to calculate corrections to the g-factor of a loosely bound charged particle of arbitrary spin. Our strategy is to obtain an effective Lagrangian in the nonrelativistic limit.

We first consider features of a general-spin formalism in both the relativistic and nonrelativistic cases, and the connection between the wave functions of the free particles. Then we consider how constraints of the relativistic theory let us calculate scattering off an external field. Comparing this result to that done with an effective NRQED Lagrangian, we can obtain the coefficients of that Lagrangian for particles of general spin.

6.0.4 Spinors for general-spin charged particles

Relativistic bispinors

First we need to work out a formalism that will apply to the general spin case. We want to represent the spin state of the particles by an object that looks like a generalization of the Dirac bispinor.

It is easiest to start with the Dirac basis, where the upper and lower components of the bispinor are objects of opposite helicity, each transforming as an object of spin 1/2.

To that end define an object

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{6.0.1}$$

that we wish to have the appropriate properties. Each component should transform as a particle of spin s, but with opposite helicity. Under reflection the upper and lower components transform into each other.

Representations of the proper Lorentz group are spinors which are separately symmetric in dotted and undotted indices. If ξ is an object with p undotted and q dotted indices

$$\xi = \{\xi^{\alpha_1 \ldots \alpha_p}_{\dot{\beta}_1 \ldots \dot{\beta}_q}\} \tag{6.0.2}$$

Then this can be a representation of a particle of spin $s = (p+q)/2$.

We have some free choice in how to partition the dotted/undotted indices, and we cannot choose exactly the same scheme for all spin as long as both types of indices are present. However, we can make separately consistent choices for integral and half-integral spin. For integral spin we can say $p = q = s$, while for the half-integral case we'll choose $p = s + \frac{1}{2}$, $q = s - \frac{1}{2}$.

We want the ξ and η to transform as objects of opposite helicity. Under reflection they will transform into each other. So

$$\eta = \{\eta^{\beta_1 \ldots \beta_q}_{\dot{\alpha}_1 \ldots \dot{\alpha}_p}\} \tag{6.0.3}$$

In the rest frame of the particle, they will have clearly defined and identical properties under rotation. The rest frame spinors are equivalent to rank $2s$ nonrelativistic spinors. So the bispinor in the rest frame looks like

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_0 \\ \xi_0 \end{pmatrix} \tag{6.0.4}$$

where

$$\xi_0 = \{(\xi_0)_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q}\} \tag{6.0.5}$$

and all indices are symmetric.

We can obtain the spinors in an arbitrary frame by boosting from the rest frame. The upper and lower components we have defined to have opposite helicity, and so will act in opposite ways under boost:

$$\xi = \exp\left(\frac{\mathbf{\Sigma}\cdot\phi}{2}\right)\xi_0, \qquad \eta = \exp\left(-\frac{\mathbf{\Sigma}\cdot\phi}{2}\right)\xi_0 \tag{6.0.6}$$

What form should the operator $\mathbf{\Sigma}$ have? Under an infinitesimal boost by a rapidity ϕ, a spinor with a single undotted index is transformed as

$$\xi_\alpha \to \xi'_\alpha = \left(\delta_{\alpha\beta} + \frac{\boldsymbol{\phi}\cdot\boldsymbol{\sigma}_{\alpha\beta}}{2}\right)\xi_\beta$$

while one with a dotted index will transform as

$$\xi_{\dot\alpha} \to \xi'_{\dot\alpha} = \left(\delta_{\dot\alpha\dot\beta} - \frac{\boldsymbol{\phi}\cdot\boldsymbol{\sigma}_{\dot\alpha\dot\beta}}{2}\right)\xi_{\dot\beta}$$

The infinitesimal transformation of a higher spin object with the first p indices undotted and the last q dotted would then be

$$\xi \to \xi' = \left(1 + \sum_{a=0}^{p}\frac{\boldsymbol{\sigma}_a\cdot\boldsymbol{\phi}}{2} - \sum_{a=p+1}^{p+q}\frac{\boldsymbol{\sigma}_a\cdot\boldsymbol{\phi}}{2}\right)\xi$$

where a denotes which spinor index of ξ is operated on.

If we define

$$\mathbf{\Sigma} = \sum_{a=0}^{p}\boldsymbol{\sigma}_a - \sum_{a=p+1}^{p+q}\boldsymbol{\sigma}_a \tag{6.0.7}$$

Then the infinitesimal transformations would be

$$\xi \to \xi' = \left(1 + \frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi$$

$$\eta \to \eta' = \left(1 - \frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\eta$$

So the exact transformation should be

$$\xi \to \xi' = \exp\left(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi$$

$$\eta \to \eta' = \exp\left(-\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\eta$$

Therefore, the bispinor of some particle boosted by $\boldsymbol{\phi}$ from rest will be

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \exp\left(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi_0 \\ \exp\left(\frac{-\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi_0 \end{pmatrix} \tag{6.0.8}$$

In dealing with the relativistic theory, we'll want a basis that separates the particle and antiparticle parts of the wave function. If we want the upper component to be the particle, then in the rest frame the lower component will vanish, and for low momentum will be small compared to the upper component. The unitary transformation which accomplishes this is

$$\Psi' = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

$$\phi = \frac{1}{\sqrt{2}}(\xi + \eta)$$

$$\chi = \frac{1}{\sqrt{2}}(\eta - \xi)$$

Which is equivalent to

$$\Psi' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\Psi$$

Then,

$$\phi = \cosh\left(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi_0 \tag{6.0.9}$$

$$\chi = \sinh\left(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi_0 \tag{6.0.10}$$

Spinors for nonrelativistic theory

We also need to discuss the nonrelativistic, single-particle theory. This is much simpler: the spin state of particles in this theory is represented by symmetric spinors with $2s + 1$ undotted indices. The only operators we need to consider acting on this space are spin matrices and products of spin matrices.

6.0.5 Electromagnetic Interaction

Knowing how the wave functions themselves behave, we want to see what that tells us about possible electromagnetic interaction. Interaction with a single electromagnetic photon should take the form

$$M = A_\mu j^\mu$$

where j^μ is the electromagnetic current.

The electromagnetic current must be built out of the particle's momenta and bilinears of the charged particle fields in such a way that they have the correct Lorentz properties. We must also demand current conservation: the equation $q_\mu j^\mu = 0$ must hold. Above we already have shown that, in the case of general spin, there exist only two such bilinears, a scalar and a tensor.

There will be two permissible terms in the current. We could consider a scalar bilinear coupled with a single power of external momenta. In order to fulfill the current conservation requirement, it should be

$$\frac{p^\mu + p'^\mu}{2m} \bar{\Psi}^\dagger \Psi$$

This will obey current conservation because $q = p' - p$, and $(p + p') \cdot (p' - p) = p^2 - p'^2 = 0$

We can also consider a tensor term contracted with a power of momenta. To fulfill current conservation, we can demand that the tensor bilinear be antisymmetric, and contract it with q:

$$\frac{q_\nu}{2m} \bar{\Psi}^\dagger \Sigma^{\mu\nu} \Psi$$

We don't need to worry about higher order tensor bilinears: they will necessitate too

many powers of the external momenta.

So the most general current would look like

$$j^\mu = F_e \frac{p^\mu + p'^\mu}{2m} \bar{\Psi}^\dagger \Psi + F_m \frac{q_\nu}{2m} \bar{\Psi}^\dagger \Sigma^{\mu\nu} \Psi \tag{6.0.11}$$

In general the form factors might have quite complicated dependence on q, but these corrections will be too small compared to the type of result we're interested in. At leading order F_e will just be the electric charge of the particle in question, and F_m will, as we'll see after connecting this result to the nonrelativistic limit, be related to the particle's g-factor. So to the order we need, we can write the current as

$$j^\mu = e \frac{p^\mu + p'^\mu}{2m} \bar{\Psi}^\dagger \Psi + eg \frac{q_\nu}{2m} \bar{\Psi}^\dagger \Sigma^{\mu\nu} \Psi \tag{6.0.12}$$

This captures the essence of the interaction between a charged particle of general-spin and a single photon.

Connection between the spinors of the two theories

Knowing something of how the relativistic theory behaves, we can find the connection between the relativistic and nonrelativistic spinors. In the rest frame, there are two independent bispinors which represent particle and antiparticle states:

$$\Psi = \begin{pmatrix} \xi_0 \\ 0 \end{pmatrix}$$

or

$$\Psi = \begin{pmatrix} 0 \\ \xi_0 \end{pmatrix}$$

However, when we consider a particle with zero momentum it is not the case that the upper component of the bispinor can be directly associated with the Schrodinger like wavefunction of the particle — for instance, it would not be correctly normalized, for there is some mixing with the lower component.

We can obtain a relation between ξ_0 and the Schrodinger amplitude ϕ_s by considering the current density at zero momentum transfer. For ϕ_s it will be $j_0 = \phi_s^\dagger \phi$. For the relativistic theory we have, as calculated above:

$$j^0 = F_e \frac{p^0 + p'^0}{2m} \bar{\Psi}^\dagger \Psi + F_m \frac{q_\nu}{2m} \bar{\Psi}^\dagger T^{0\nu} \Psi$$

At $q = 0$ the expression simplifies

$$j^0(q=0) = F_e \frac{p_0}{m} \bar{\Psi}^\dagger \Psi$$

$$= F_e \frac{p_0}{m} (\phi^\dagger \phi - \chi^\dagger \chi)$$

ϕ and χ are both related to the rest frame spinor ξ_0. So we can write instead

$$j^0 = F_e \frac{p_0}{m} \xi_0^\dagger \left\{ \cosh^2(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}) - \sinh^2(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}) \right\} \xi_0 = F_e \frac{p_0}{m} \xi_0^\dagger \xi_0$$

where the last equality follows from the hyperbolic trig identity.

If we demand that the two current densities be equal to each other, we find

$$\frac{p_0}{m} \xi_0^\dagger \xi_0 = \phi_s^\dagger \phi_s$$

Approximating

$$\left(1 + \frac{\mathbf{p}^2}{2m}\right) \xi_0^\dagger \xi_0 = \phi_s^\dagger \phi_s$$

This will hold to the necessary order if we identify

$$\xi_0 = \left(1 - \frac{\mathbf{p}^2}{4m}\right) \phi_s$$

To write the relativistic bispinors in terms of ϕ_s we will also need approximations to $\cosh(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2})$ and $\sinh(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2})$. We only need the rapidity to the leading order: $\boldsymbol{\phi} \approx \mathbf{v} \approx \frac{\mathbf{p}}{m}$.

$$\cosh(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}) \approx 1 + \frac{1}{2}\left(\frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\right)^2$$

$$\sinh(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}) \approx \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}$$

The the two bispinor components are

$$\begin{aligned}
\phi &\approx \left(1 + \left[\frac{1}{2}\frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\right]^2\right)\xi_0 \\
&\approx \left(1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m}\right)\phi_S \qquad (6.0.13)\\
\chi &\approx \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\xi_0 \\
&\approx \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\phi_S \qquad (6.0.14)
\end{aligned}$$

6.0.6 Bilinears in terms of nonrelativistic theory

The next step is to express the relativistic bilinears, built out of the bispinors Ψ, in terms of the Schrodinger like wave functions.

We have above written the bispinors in terms of ϕ_s, so we can use those identities to express the bilinears in the same manner.

Scalar bilinear

$$\begin{aligned}
\bar{\Psi}^\dagger(p')\Psi(p) &= \phi^\dagger\phi - \chi^\dagger\chi \\
&= \phi_s^\dagger\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p}')^2}{8m^2} - \frac{\mathbf{p}'^2}{4m^2}\right]\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m^2}\right]\phi_s - \phi_s^\dagger\left[\frac{(\mathbf{\Sigma}\cdot\mathbf{p}')(\mathbf{\Sigma}\cdot\mathbf{p})}{4m^2}\right]\phi_s \\
&= \phi_s^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{(\mathbf{\Sigma}\cdot\mathbf{p}')^2 + (\mathbf{\Sigma}\cdot\mathbf{p})^2 - 2(\mathbf{\Sigma}\cdot\mathbf{p}')(\mathbf{\Sigma}\cdot\mathbf{p})\right\}\right)\phi_s \\
&= \phi_s^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{[\mathbf{\Sigma}\cdot\mathbf{p}, \mathbf{\Sigma}\cdot\mathbf{q}] + (\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_s \\
&= \phi_s^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{[4i\epsilon_{ijk}p_iq_jS_k + (\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_s
\end{aligned}$$

Tensor ij component

In calculating the nonrelativistic limit of the antisymmetric tensor bilinear, we will treat the $0i$ and the ij components seperately. First let us consider $\bar{\Psi}\Sigma_{ij}\Psi$.

$$\begin{aligned}
\bar{\Psi}\Sigma_{ij}\Psi &= \bar{\Psi}(-2\epsilon_{ijk}S_k)\Psi \\
&= -2i\epsilon_{ijk}(\phi^\dagger S_k\phi - \chi^\dagger S_k\chi) \\
&= -2i\epsilon_{ijk}\Big(\phi_S^\dagger\left[1+\frac{(\boldsymbol{\Sigma}\cdot\mathbf{p}')^2}{8m^2}-\frac{\mathbf{p}'^2}{4m^2}\right]S_k\left[1+\frac{(\boldsymbol{\Sigma}\cdot\mathbf{p})^2}{8m^2}-\frac{\mathbf{p}^2}{4m^2}\right]\phi_S - \phi_S^\dagger\frac{(\boldsymbol{\Sigma}\cdot\mathbf{p}')S_k(\boldsymbol{\Sigma}\cdot\mathbf{p})}{4m^2}\phi_S\Big) \\
&= -2i\epsilon_{ijk}\phi_S^\dagger\left\{S_k\left(1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right)+\frac{1}{8m^2}\Big[(\boldsymbol{\Sigma}\cdot\mathbf{p}')^2S_k+S_k(\boldsymbol{\Sigma}\cdot\mathbf{p})^2-2(\boldsymbol{\Sigma}\cdot\mathbf{p}')S_k(\boldsymbol{\Sigma}\cdot\mathbf{p})\Big]\right\}\phi_S
\end{aligned}$$

We want to write the terms in square brackets explicitly in terms of $\mathbf{p}$ and $\mathbf{q}$.

$$\begin{aligned}
(\boldsymbol{\Sigma}\cdot\mathbf{p}')^2S_k+S_k(\boldsymbol{\Sigma}\cdot\mathbf{p})^2-2(\boldsymbol{\Sigma}\cdot\mathbf{p}')S_k(\boldsymbol{\Sigma}\cdot\mathbf{p}) &= (\boldsymbol{\Sigma}\cdot\mathbf{p})^2S_k+S_k(\boldsymbol{\Sigma}\cdot\mathbf{p})-2(\boldsymbol{\Sigma}\cdot\mathbf{p})S_k(\boldsymbol{\Sigma}\cdot\mathbf{p}) \\
&\quad +\{(\boldsymbol{\Sigma}\cdot\mathbf{p})(\boldsymbol{\Sigma}\cdot\mathbf{q})+(\boldsymbol{\Sigma}\cdot\mathbf{q})(\boldsymbol{\Sigma}\cdot\mathbf{p})\}S_k \\
&\quad -2(\boldsymbol{\Sigma}\cdot\mathbf{q})S_k(\boldsymbol{\Sigma}\cdot\mathbf{p})+(\boldsymbol{\Sigma}\cdot\mathbf{q})^2S_k
\end{aligned}$$

We can express many of these terms as commutators

$$= \boldsymbol{\Sigma}\cdot\mathbf{p}[\boldsymbol{\Sigma}\cdot\mathbf{p},S_k]+[S_k,\boldsymbol{\Sigma}\cdot\mathbf{p}]\boldsymbol{\Sigma}\cdot\mathbf{p}+2\boldsymbol{\Sigma}\cdot\mathbf{q}[\boldsymbol{\Sigma}\cdot\mathbf{p},S_k]-[\boldsymbol{\Sigma}\cdot\mathbf{q},S_k]\boldsymbol{\Sigma}\cdot\mathbf{p}+(\boldsymbol{\Sigma}\cdot\mathbf{q})^2S_k$$

$$= i\epsilon_{ijk}p_j\{(\boldsymbol{\Sigma}\cdot\mathbf{p})\Sigma_i-\Sigma_i(\boldsymbol{\Sigma}\cdot\mathbf{p})\}+2i\epsilon_{ijk}\{(\boldsymbol{\Sigma}\cdot\mathbf{q})\Sigma_ip_j-\Sigma_i(\boldsymbol{\Sigma}\cdot\mathbf{p})q_j)\}+(\boldsymbol{\Sigma}\cdot\mathbf{q})^2S_k$$

$$= i\epsilon_{ijk}p_j[(\boldsymbol{\Sigma}\cdot\mathbf{p}),\Sigma_i]+2i\epsilon_{ijk}\{(\boldsymbol{\Sigma}\cdot\mathbf{q})\Sigma_ip_j-\Sigma_i(\boldsymbol{\Sigma}\cdot\mathbf{p})q_j)\}+(\boldsymbol{\Sigma}\cdot\mathbf{q})^2S_k$$

$$= 4(\mathbf{p}^2S_k-(\mathbf{S}\cdot\mathbf{p})p_k)+2i\epsilon_{ijk}\{(\boldsymbol{\Sigma}\cdot\mathbf{q})\Sigma_ip_j-\Sigma_i(\boldsymbol{\Sigma}\cdot\mathbf{p})q_j)\}+(\boldsymbol{\Sigma}\cdot\mathbf{q})^2S_k$$

Thus the whole bilinear is

$$-2i\epsilon_{ijk}\phi_S^\dagger\left\{S_k\left(1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right)+\frac{1}{8m^2}\Big[4(\mathbf{p}^2S_k-(\mathbf{S}\cdot\mathbf{p})p_k)+2i\epsilon_{\ell mk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_\ell p_m-\Sigma_\ell(\mathbf{\Sigma}\cdot\mathbf{p})q_m)\}+(\mathbf{\Sigma}\cdot\mathbf{q})^2S_k\Big]\right\} \tag{6.0.15}$$

Tensor Σ_{0i} component

We calculate $\bar{\Psi}\Sigma_{0i}\Psi$.

$$\bar{\Psi}\Sigma_{0i}\Psi=\bar{\Psi}\begin{pmatrix}0 & \Sigma_i\\ \Sigma_i & 0\end{pmatrix}\Psi$$

$$=\phi^\dagger\Sigma_i\chi-\chi^\dagger\Sigma_i\phi$$

We'll only need ϕ and χ to first order here.

$$=\phi_S^\dagger\left(\frac{\Sigma_i\Sigma_jp_j-\Sigma_j\Sigma_ip'j}{2m}\right)\phi_S$$

Using $p'=p+q$ the terms involving only p can be simplified using the commutator of Σ matrices.

$$=\phi_S^\dagger\left(\frac{4i\epsilon_{ijk}p_jS_k-\Sigma_j\Sigma_iq_j}{2m}\right)\phi$$

6.0.7 Current in terms of nonrelativistic wave functions

We derived the four-current (6.0.11) above; in nonrelativistic notation it is:

$$j_0=F_e\frac{p_0+p_0'}{2m}\bar{\Psi}^\dagger\Psi-F_m\frac{q_j}{2m}\bar{\Psi}^\dagger\Sigma^{0j}\Psi \tag{6.0.16}$$

$$j_i=F_e\frac{p_i+p_i'}{2m}\bar{\Psi}^\dagger\Psi-F_m\frac{q_j}{2m}\bar{\Psi}^\dagger\Sigma^{ij}\Psi+F_m\frac{q_0}{2m}\bar{\Psi}^\dagger\Sigma^{i0}\Psi \tag{6.0.17}$$

We have expressions for the bilinears in terms of the nonrelativistic wave functions ϕ_S,

so it is fairly straight forward to apply them here. The calculation of j_0 is straightforward:

$$\begin{aligned}
F_e\frac{p_0+p_0'}{2m}\bar{\Psi}^\dagger\Psi &= F_e\left(1+\frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right)\phi_S^\dagger\left(1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}+\frac{1}{8m^2}\left\{4i\epsilon_{ijk}p_iq_jS_k+(\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_S \\
&\approx F_e\phi_S^\dagger\left(1+\frac{1}{8m^2}\left\{4i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q}+(\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_S \\
F_m\frac{q_j}{2m}\bar{\Psi}^\dagger\Sigma^{0j}\Psi &= F_m\frac{q_i}{2m}\phi_S^\dagger\left(\frac{4i\epsilon_{ijk}p_jS_k-\Sigma_j\Sigma_iq_j}{2m}\right)\phi_S \\
&= F_m\phi_S^\dagger\left(\frac{4i\mathbf{S}\cdot\mathbf{q}\times\mathbf{p}-(\mathbf{\Sigma}\cdot\mathbf{q})^2}{4m^2}\right)\phi_S
\end{aligned}$$

It turns out that both terms here have the same form, so combining them we get

$$j_0=\phi_S^\dagger\left(F_e+\frac{F_e+2F_m}{8m^2}\left\{4i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q}+(\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_S \tag{6.0.18}$$

To calculate j_i we want to first simplify things by considering the constraints of our particular problem. The term with Σ_ij can be simplified by dropping terms with more than one power of q; these will turn into derivatives of the magnetic field, and our problem concerns only a constant field. Further, we need only calculate elastic scattering, and so $q_0=0$. With those simplifications

$$\bar{\Psi}\Sigma_{ij}\Psi\approx-2i\epsilon_{ijk}\phi_S^\dagger\left\{S_k\left(1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right)+\frac{\mathbf{p}^2S_k-(\mathbf{S}\cdot\mathbf{p})p_k}{2m^2}\right\}\phi_S$$

$$\begin{aligned}
F_e\frac{p_i+p_i'}{2m}\bar{\Psi}^\dagger\Psi &= F_e\frac{p_i+p_i'}{2m}\phi_S^\dagger\left(1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}+\frac{1}{8m^2}\left\{4i\epsilon_{\ell jk}p_\ell q_jS_k+(\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_S \\
&\approx F_e\frac{p_i+p_i'}{2m}\phi_S^\dagger\left(1+\frac{1}{8m^2}\left\{4i\epsilon_{\ell jk}p_\ell q_jS_k\right\}\right)\phi_S \\
F_m\frac{q_j}{2m}\bar{\Psi}^\dagger\Sigma^{ij}\Psi &= -F_m\frac{i\epsilon_{ijk}q_j}{m}\phi_S^\dagger\left\{S_k\left(1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right)+\frac{\mathbf{p}^2S_k-(\mathbf{S}\cdot\mathbf{p})p_k}{2m^2}\right\}\phi_S
\end{aligned}$$

So the full spatial part of the current is

$$j_i = \phi_S^\dagger \left\{ F_e \frac{p_i + p_i'}{2m} \left(1 + \frac{i\epsilon_{\ell jk} p_\ell q_j S_k}{2m^2} \right) + F_m \frac{i\epsilon_{ijk} q_j}{m} \left(S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S}\cdot\mathbf{p}) p_k}{2m^2} \right) \right\} \phi_S \tag{6.0.19}$$

6.0.8 Scattering off external field

To compare to the NRQED Lagrangian, we want to calculate scattering off an external field for an arbitrary spin particle. We already have the current, so the scattering is just

$$M = e j_\mu A^\mu = e j_0 A_0 - e\mathbf{j}\cdot\mathbf{A}$$

Above we have expressions for both j_0 (6.0.18) and $\mathbf{j}$ (6.0.19). So we can write down the parts of the amplitude directly:

$$\begin{aligned} e j_0 A_0 =& e A_0 \phi_S^\dagger \left(F_e + \frac{F_e + 2F_m}{8m^2} \left\{ 4i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q} + (\mathbf{\Sigma}\cdot\mathbf{q})^2 \right\} \right) \phi_S \\ e\mathbf{j}\cdot\mathbf{A} =& A_i \phi_S^\dagger \left\{ F_e \frac{p_i + p_i'}{2m} \left(1 + \frac{i\epsilon_{\ell jk} p_\ell q_j S_k}{2m^2} \right) + F_m \frac{i\epsilon_{ijk} q_j}{m} \left(S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S}\cdot\mathbf{p}) p_k}{2m^2} \right) \right\} \phi_S \end{aligned} \tag{6.0.20}$$

As much as possible we want to express the result in terms of gauge invariant quantities $\mathbf{B}$ and $\mathbf{E}$. We write the relations between these fields and A_μ in position space and the equivalent equation in momentum space.

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \to i\mathbf{q}\times\mathbf{A}$$

$$\mathbf{E} = -\boldsymbol{\nabla} A_0 \to -i\mathbf{q} A_0$$

There is one term above that can only be put into gauge-invariant form by considering the kinematic constraints of elastic scattering. If the scattering is elastic, we have $\mathbf{q}\cdot(\mathbf{p}' +$

$\mathbf{p}) = \mathbf{p}'^2 - \mathbf{p}^2 = 0$. We can use this identity on the term $q_j(p'_i + p_i)A_i$ as follows:

$$\begin{aligned}
\epsilon_{ijk}B_k &= \partial_i A_j - \partial_j A_i = i(q_i A_j - q_j A_i) \\
(p_i + p'_i)\epsilon B_k &= i(p_i + p'_i)(q_i A_j - q_j A_i) \\
&= -i(p_i + p'_i)q_j A_i
\end{aligned}$$

So we have the identity

$$i(p_i + p'_i)q_j A_i = -\epsilon_{ijk}B_k(p_i + p'_i) \tag{6.0.21}$$

Now we can write each term involving q in terms of position space quantities.

$$\begin{aligned}
i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q}A_0 &= -\mathbf{S}\cdot\mathbf{p}\times\mathbf{E} \\
(\boldsymbol{\Sigma}\cdot\mathbf{q})^2 A_0 &= \Sigma_i\Sigma_j q_i q_j A_0 \\
&= \Sigma_i\Sigma_j\partial_i E_j \\
i\epsilon_{ijk}A_i q_j &= -i(\mathbf{q}\times\mathbf{A})_k \\
&= = -B_k \\
A_i(p_i + p'_i)i\epsilon_{\ell jk}p_\ell q_j S_k &= \epsilon_{\ell jk}p_\ell S_k i(p_i + p'_i)q_j A_i \\
&= -\epsilon_{\ell jk}p_\ell S_k\{\epsilon_{ijm}B_m(p_i + p'_i)\} \\
&= -(\delta_{\ell i}\delta_{km} - \delta\ell m\delta_{ik})p_\ell S_k\{\epsilon_{ijm}B_m(p_i + p'_i)\} \\
&= 2\{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p}) - (\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2\}
\end{aligned}$$

Using these

$$ej_0 A_0 = e\phi_S^\dagger\left\{A_0 + \frac{1-2F_2}{8m^2}\left(4\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \Sigma_i\Sigma_j\partial_i E_j\right)\right\}$$

$$e\mathbf{j}\cdot\mathbf{A} = e\phi_S^\dagger\left\{\frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p}) - (\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2}{m^2} - F_m\left(\frac{\mathbf{S}\cdot\mathbf{B}}{m}\left\{1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right\} + \frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p}) - (\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2}{2m^2}\right.\right.$$

$$= e\phi_S^\dagger \left\{ \frac{\mathbf{p}\cdot\mathbf{A}}{m} + [1-2F_m]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{F_m}{m}\mathbf{S}\cdot\mathbf{B} \right\}$$

From this we can see that our F_m is actually $g/2$, so in such terms

$$ej_0 A_0 = e\phi_S^\dagger \left\{ A_0 - \frac{g-1}{2m^2}\left(\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \frac{1}{4}\Sigma_i\Sigma_j\partial_i E_j \right) \right\}$$

$$e\mathbf{j}\cdot\mathbf{A} = e\phi_S^\dagger \left\{ \frac{\mathbf{p}\cdot\mathbf{A}}{m} - [g-1]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{g}{2m}\mathbf{S}\cdot\mathbf{B} \right\}$$

So the entire scattering process is

$$e\phi_S^\dagger \left\{ A_0 - \frac{g-1}{2m^2}\left(\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \frac{1}{4}\Sigma_i\Sigma_j\partial_i E_j \right) \frac{\mathbf{p}\cdot\mathbf{A}}{m} - [g-1]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{g}{2m}\mathbf{S}\cdot\mathbf{B} \right\} \phi_S \tag{6.0.22}$$

6.0.9 Comparison with relativistic result

Having calculated the same process in both the relativistic theory and in our NRQED effective theory, we can compare the two amplitudes and fix the coefficients.

The NRQED amplitude (5.2.10) is

$$\begin{aligned} iM &= ie\phi^\dagger \Bigg(-A_0 + \frac{\mathbf{A}\cdot\mathbf{p}}{m} - \frac{(\mathbf{A}\cdot\mathbf{p})\mathbf{p}^2}{2m^3} + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2} + c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2} \\ &\quad + c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2} - (c_{W_1}-c_{W_2})\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3} \Bigg)\phi \end{aligned}$$

While the relativistic amplitude was

$$iM_{REL} = -ie\phi^\dagger \Big(A_0 - \frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{\mathbf{p}\cdot\mathbf{A}\mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3}\{\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - S_iS_j\boldsymbol{\nabla}_i E_j\} - g\frac{1}{2m}\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \Big)$$

We should rewrite term $\boldsymbol{\nabla}\cdot\mathbf{E} - S_iS_j\boldsymbol{\nabla}_i E_j$ using the quadropole moment tensor $Q_{ij} = \frac{1}{2}(S_iS_j + S_jS_i - \frac{2}{3}\mathbf{S}^2)$.

Remember that $\nabla_i E_j$ is actually symmetric under exchange of i and j. Then we can

write

$$S_iS_j\nabla_iE_j = \frac{1}{2}(S_iS_j + S_jS_i) = (Q_{ij} + \frac{1}{3}\mathbf{S}^2\delta_{ij})\nabla_iE_j$$

$$= Q_ij\nabla_iE_j + \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{E}$$

Written using this identity, the relativistic amplitude is

$$iM_{REL} = -ie\phi^\dagger\Big(A_0 - \frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{\mathbf{p}\cdot\mathbf{A}\mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3}\{\frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - Q_{ij}\boldsymbol{\nabla}_iE_j\} - g\frac{1}{2m}\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3}(\mathbf{S}\cdot\mathbf{p}$$

Comparing the two, the coefficients are:

$$\begin{aligned}
c_F &= g \\
c_D &= \frac{4(g-1)}{3} \\
c_Q &= -4(g-1) \\
c_S^1 &= 2(g-1) \\
(c_{W_1} - c_{W_2}) &= 2 \\
c_{p'p} &= (g-2)
\end{aligned}$$

6.1 Allowed terms

We wish to evaluate the magnetic moment of a bound particle. What terms of the NRQED Hamiltonian do we need in order to do so?

We treat the Hamiltonian as an exact part plus a perturbation:

$$H = H_0 + V; H_0 = H_0 = e\Phi + \frac{p^2}{2m}$$

To correctly apply perturbation theory V should be small compared to H_0, and this will be true here. We have two energy scales: the kinetic energy of the particle mv^2 and a scale associated with the magnetic field $\frac{eB}{m}$. The electric potential is $e\Phi \sim mv^2$ and derivatives thereof are $\frac{eE}{m} \sim mv^3$, $\frac{e\partial_i\partial_j\Phi}{m^2} \sim mv^4$.

We want to calculate first order corrections to the magnetic moment, which will be of order $\frac{eB}{m}v^2$.

For first order perturbation theory, we will then need terms explicitly of order $\frac{eB}{m}$, $\frac{eB}{m}v^2$.

We must also consider second order perturbation theory, whose contributions look like

$$\Delta\epsilon = \frac{\langle 0| V |n\rangle \langle n| V |0\rangle}{E_n - E_0}$$

The denominator is order mv^2, so we need terms in the numerator of up to $m\frac{B}{m}v^4$. These terms arise as the product of two matrix elements, one of which is leading order in $\frac{B}{m}$, the other the leading order terms which do not contain B: mv^4.

We can thus conceptually divide all the needed terms in the potential into three categories

$$\begin{aligned} V_I &\sim \frac{B}{m} \\ V_{II} &\sim mv^4 \\ V_{III} &\sim \frac{B}{m}v^2 \end{aligned}$$

With an exhaustive list of allowed terms in the Hamiltonian up to this order, we can examine what terms contribute to the magnetic moment.

$$V_I = a_1 \frac{e}{m} \mathbf{S} \cdot \mathbf{B} + \frac{e}{m} (\mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p})$$

$$V_{II} = \frac{p^4}{8m^3} + a_2 \frac{e}{m^2} \left[\{S_i, S_j\} - \frac{2}{3} S^2 \delta_{ij} \right] \partial_i E_j + a_3 \frac{e}{m^2} S_i E_j p_k \epsilon_{ijk}$$

$$\begin{aligned} V_{III} &= e \frac{p^2(\mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p}) + (\mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p}) p^2}{8m^3} + a_3 \frac{e^2}{m^2} S_i E_j A_k \epsilon_{ijk} \\ &\quad + a_4 \frac{e}{m} \mathbf{S} \cdot \mathbf{B} \frac{p^2}{2m} + a_5 \frac{e}{m} \frac{(\mathbf{S} \cdot \mathbf{p})(\mathbf{B} \cdot \mathbf{p})}{2m^2} \end{aligned}$$

To evaluate the matrix elements of these operators we will need to abandon generality and consider our specific case, where $\mathbf{E}$ is the Coulomb field.

$$\mathbf{E} = -\frac{Ze}{4\pi} \frac{\mathbf{r}}{r^3}$$

For a constant $\mathbf{B}$, we also chose a particularly convenient form for $\mathbf{A}$:

$$\mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{r}$$

We can take the direction of the magnetic field to be along the z axis, so that the term $\mathbf{S} \cdot \mathbf{B}$ is diagonal in spin space.

First order perturbation

The first order perturbation term to the ground state energy is

$$\Delta_0^{(0)} = \langle n | \, V \, | n \rangle$$

$$H_{S\cdot B} = -\frac{e}{m}\left\{g\mathbf{S}\cdot\mathbf{B}\left(1-\frac{p^2}{2m^2}\right)+(g-2)\mathbf{S}\cdot\mathbf{B}\frac{p^2}{2m^2}-(g-2)\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{2m^2}+\frac{e}{m}\left(\frac{g}{2}+\frac{g-2}{2}\right)\mathbf{S}\cdot\mathbf{E}\times\mathbf{A}\right\}$$

With our definition of E and B, $\mathbf{S}\cdot\mathbf{E}\times\mathbf{A} = -\frac{Ze}{8\pi}\frac{1}{r^3}\mathbf{S}\cdot\mathbf{r}\times[\mathbf{B}\times\mathbf{r}]$.

We take the matrix elements of these terms in the ground state. The spherical symmetry of the unperturbed state dictates that $\left\langle\frac{r_i r_j}{r^3}\right\rangle = \delta_{ij}\frac{1}{3}\left\langle\frac{1}{r}\right\rangle$. Using these identities, we find that:

$$\begin{aligned}
\left\langle\mathbf{S}\cdot\mathbf{B}p^2\right\rangle &= \mathbf{S}\cdot\mathbf{B}\left\langle p^2\right\rangle \\
\left\langle\frac{1}{r^3}\mathbf{S}\cdot\mathbf{r}\times(\mathbf{B}\times\mathbf{r})\right\rangle &= \left\langle\frac{1}{r^3}S_k R_i B_l r_m \epsilon_{jlm}\epsilon_{ijk}\right\rangle \\
&= \left\langle\frac{1}{r}\mathbf{S}\cdot\mathbf{B}-\frac{1}{r^3}B_i S_j r_i r_j\right\rangle \\
&= \mathbf{S}\cdot\mathbf{B}\left\langle\frac{1}{r}\right\rangle - B_i S_j\left\langle\frac{r_i r_j}{r^3}\right\rangle \\
&= \mathbf{S}\cdot\mathbf{B}(\left\langle\frac{1}{r}\right\rangle-\frac{1}{3}\left\langle\frac{1}{r}\right\rangle) \\
&= \frac{2}{3}\mathbf{S}\cdot\mathbf{B}\left\langle\frac{1}{r}\right\rangle \\
\langle(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\rangle &= \langle S_i B_j p_i p_j\rangle \\
&= S_i B_j\left\langle p_i p_j\right\rangle \\
&= S_i B_j\delta_{ij}\frac{\left\langle p^2\right\rangle}{3} \\
&= \mathbf{S}\cdot\mathbf{B}\frac{\left\langle p^2\right\rangle}{3}
\end{aligned}$$

For the ground state, $\left\langle\frac{1}{r}\right\rangle = mZ\alpha$, $\left\langle p^2\right\rangle = (mZ\alpha)^2$. So

$$\begin{aligned}
\langle H_{S\cdot B}\rangle &= -\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\left(1-\frac{\left\langle p^2\right\rangle}{2m^2}+\frac{Ze^2}{6m}\left\langle\frac{1}{r}\right\rangle\right) \\
&= -\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\left(1-\frac{(Z\alpha)^2}{2}+\frac{(Z\alpha)^2}{6}\right) \\
&= -\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\left(1-\frac{(Z\alpha)^2}{3}\right)
\end{aligned}$$

Second order perturbation

We want the second order correction to the ground state energy level. First we consider matrix elements between the ground level and excited levels, and show that none contribute to the magnetic moment. Then we consider degenerate levels with differing spin, but it will turn out that such matrix elements also vanish.

Given that we throw away terms quadratic in the magnetic field, and of order greater than mv^4 or $\frac{B}{m}v^2$, the only terms contributing to second order in perturbation theory are

$$\Delta^{(1)}E_0 = -2\Sigma_{m\neq 0}\frac{\langle 0|\, V_I\, |m\rangle \langle m|\, V_{III}\, |0\rangle}{E_m - E_0}$$

Now, since the term $\mathbf{S}\cdot\mathbf{B}$ contains only constants and spin space operators, it clearly doesn't connect different energy levels. So $\langle 0|\, \mathbf{S}\cdot\mathbf{B}\, |m\rangle = 0$.

The term $\langle 0|\, \mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p}\, |m\rangle$ does contain position space operators, but will vanish due to symmetry considerations.

$$\begin{aligned}
\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p} &= p_i(r_j B_k \epsilon_{ijk}) + (r_j B_k \epsilon_{ijk})p_i \\
&= [p_i, r_j]B_k\epsilon ijk + 2B_k r_j p_i \epsilon_{ijk} \\
&= 2B_k r_j p_i \epsilon_{ijk}
\end{aligned}$$

However, $r_j p_i \epsilon_{ijk}$ is proportional to the angular momentum operator L_k. When acting on the spherically symmetric ground state, this operator must vanish.

Therefore, for any excited state m, $\langle m|\, V_I\, |0\rangle$ vanishes. Now we must consider the matrix element between two degenerate ground level states of differing spin.

6.1.1 Degenerate

Clearly any term not involving spin operators will not connect states of differing spin. Likewise, with our convention that we label spin states by their projection along the direction of the magnetic field, the operator $\mathbf{S}\cdot\mathbf{B} = S_3 B_3$ is also diagonal in spin space.

The remaining terms involving spin operators are:

$$-\frac{e}{2m^2}(S_i E_j (p_k - eA_k)\epsilon_{ijk} - S_i S_j \partial_i E_j)$$

By the same reasoning used above, $E_j p_k \epsilon_{ijk}$ is proportional to L_i, which will vanish when acting on the ground state.

Now consider the term containing $S_i E_j A_k \epsilon_{ijk}$. Using the specific gauge chosen, and the particular form of $\mathbf{E}$, it will be proportional to

$$\begin{aligned} S_i \left(\frac{r_j}{r^3}\right)(r_l B_m \epsilon_{lmk})\epsilon_{ijk} &= \frac{1}{r^3} S_i r_j r_l B_m (\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi}) \\ &= \frac{1}{r^3}(S_i r_i r_j B_j - S_i B_i r_j r_j) \end{aligned}$$

Between two ground state wave functions $\left\langle \frac{r_i r_j}{r^3} \right\rangle = \delta_{ij}\frac{1}{3}\left\langle \frac{1}{r} \right\rangle$. So after averaging over position space (but not yet calculating the spin part) the above reduces to

$$\frac{1}{r}(S_i \delta_{ij} B_j - S_i B_i \delta_{jj}) = -2\mathbf{S}\cdot\mathbf{B}\left\langle \frac{1}{r} \right\rangle$$

Which is again diagonal in spin space.

The last term to consider contains the position space operator $\partial_i E_j$. The expectation value of this in the ground state must be something proportional to δ_{ij}. Then the operator $S_i S_j$ in spin space will be proportional to S^2. So this term too is diagonal in spin space.

Term by term we have shown that the potential does not connect any two ground level states of different spin. Thus, degenerate perturbation theory is not needed.

So the result of first order perturbation theory stands as our final result:

$$\langle H_{S\cdot B}\rangle = -\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\left(1 - \frac{(Z\alpha)^2}{3}\right)$$