0.1 Constructing the NRQED Lagrangian

We want to construct an effective Lagrangian in the nonrelativistic limit. Our goal is to calculate the leading order corrections to the g-factor, which are corrections of order α^2. To this end, we need terms in the effective nonrelativistic Lagrangian which are equivalent corrections.

0.1.1 Constraints on the form of the Lagrangian

The Lagrangian is constrained to obey several symmetries. It must be invariant under the symmetries of parity and time reversal. It must also be invariant under Galilean transformations. The Lagrangian must also be Hermitian, and gauge invariant.

What are the gauge invariant building blocks we can use to construct this Lagrangian? We have the fields $\mathbf{E}$ and $\mathbf{B}$, the spin operators $\mathbf{S}$, and the long derivative $\mathbf{D} = \partial - ie\mathbf{A}$. The fields should always be accompanied by the charge e of the particle.

When considering the case of higher spin particles, we might consider terms quadratic and above in spin operators. For a particle of spin s, there must be $(2s + 1)^2$ independent hermitian operators. We can span this set of operators by considering products of up to $2s$ spin matrices which are symmetric and traceless in every vector index. For example, for spin-1 we have quadratic, in addition to I and S_i, five independent structures of the form $S_i S_j + S_j S_i + \delta_{ij}\mathbf{S}$. This idea is pursued further in a section below.

We also have the scalar D_0, however, we need only include a single such term because we insist on having only one power of the time derivative.

Energy Scales

The effective Lagrangian contains an infinite number of terms. But these terms may be organised by the order of their contribution to scattering. Then, up to a particular order there are only a finite number of terms — and for the first couple of orders this finite number is quite small and manageable.

Consider again the particular physical situation of interest: a loosely bound state system of two charged particles, each with arbitrary spin, placed in an infinitesimal magnetic field. It is crucial that the system is loosely bound, so that the entire system is nonrelativistic. For now it is assumed that recoil effects are ignored.

Then what are the energy scales of this particular problem? In addition to the mass of the particle, there are two: one is related to the external magnetic field, an energy scale of $e/m\,|\mathbf{B}|$. The other is related to the kinetic energy or momentum of the particle. There is an electric field present as well, representing the interaction between the bound state particles. But by the quantum virial theorem the Coulomb potential is proportional to the kinetic energy, and so is not an independent energy scale. In general kinetic energy and momentum might be considered seprate energy scales, but in the nonrelativistic regime are just mv and $(1/2)mv^2$.

Both of these energy scales are nonrelativistic — that is, small compared to the mass of the particle. So the effective Lagrangian represents an expansion in the small quantities $e/m^2 B$ and $p/m = v$. However, the calculation of the gyromagnetic ratio involves only linear terms in the magnetic field. (The g-factor is defined by the linear response to an infinitesimal magnetic field.) So no terms quadratic in B are needed.

The goal is to calculate corrections to the g-factor of up to $\mathcal{O}(\alpha^2)$. In the bound state system $v \sim Z\alpha$, so in the Lagrangian should definitely be included terms of up to order $(e/m)Bv^2$. It will also be necessary to consider terms of order mv^4, for they *might* effect the calculation of the g-factor through perturbation theory.

What is the exact order of each term? The magnetic field defines its own scale, but the scale of operators D, A_0, and E come from the particular bound state.

The term involving the Coulomb potential, $eA_0 = Ze^2/r^2$ is of order mv^2. The first derivative $\partial_i \Phi = E_i$ is of order mv^3. The operator $\boldsymbol{\nabla}$ may then be said to be of order mv.

Discrete Constraints

All the building blocks have been chosen so that their transformation under rotation is clear. If a term has an index, it transforms like a 3-vector. (Considering this nonrelativistic theory, there are no upper or lower indices.) The Lagrangian must also be symmetric under the discrete symmetries of refleciton and time reversal, and terms must be Hermitian.

To construct terms that obey these symmetries, it is necessary to know how the building blocks themselves transform under spatial reflections and time reversal, as well as under complex conjugation.

- Under spatial reflections, **E** and **D** are vectors, and so are odd under P. While **B** and **S** are pseudovectors, even under P.

- E and D are even under time reversal, while spin and the magnetic field are odd. Also, under time reversal imaginary constant i is also odd.

- Under complex conjugation E, B and S are Hermitian, while D_i, D_0 and (obviously) i change sign.

For reference, all these properties are tabulated below:

	Order	P	T	†
eE_i	m^2v^3	-	+	+
eB_i	m^2v^2	+	-	+
D_i	mv	-	+	-
D_0	mv^2	+	-	-
S_i	1	+	-	+
i	1	+	-	-

0.1.2 Properties of D_i

Before going any further, note that it would be possible to write the entire Lagrangian without explicitly writing B. Instead we could write antisymmetric combinations of D_i.

Consider a term containing $D_iD_j\epsilon_{ijk}$. First write them using $\mathbf{D} = \boldsymbol{\nabla} - ie\mathbf{A}$:

$$(\mathbf{D} \times \mathbf{D}) = D_iD_j\epsilon_{ijk} = (\boldsymbol{\nabla}_i - ieA_i)(\boldsymbol{\nabla}_j - ieA_j)\epsilon_{ijk} \tag{0.1.1}$$

Or in full as

$$= (\boldsymbol{\nabla}_i\boldsymbol{\nabla}_j - ie\boldsymbol{\nabla}_iA_j - ie\boldsymbol{\nabla}_jA_i - e^2A_iA_j)\epsilon_{ijk} \tag{0.1.2}$$

The antisymmetric tensor will kill the first and last terms, leaving

$$= -ie(\boldsymbol{\nabla}_iA_j + \boldsymbol{\nabla}_jA_i)\epsilon_{ijk} \tag{0.1.3}$$

We can use put this in the form of a commutator by switching indices in the second term: $\boldsymbol{\nabla}_jA_i\epsilon_{ijk} = -\boldsymbol{\nabla}_iA_j\epsilon_{ijk}$

$$= -ie[\boldsymbol{\nabla}_i, A_j]\epsilon_{ijk} \tag{0.1.4}$$

and of course the commutator between $\boldsymbol{\nabla}$ and any field will just be the derivative of that field.

$$= -ie\partial_iA_j\epsilon_{ijk} \tag{0.1.5}$$

Finally, from Maxwell's equations this is just $-ieB_k$, so

$$(\mathbf{D} \times \mathbf{D})_k = -ieB_k \tag{0.1.6}$$

Going the other way, the commutator $[D_i, B_i]$ can be shown to vanish:

$$[D_i, B_i] = D_i(\epsilon_{ijk}D_jD_k) - (\epsilon_{ijk}D_jD_k)D_i = \epsilon_{ijk}(D_iD_jD_k - D_jD_kD_i) \tag{0.1.7}$$

Taking the second term and taking an even permutation of the indices: $i \to k \to j \to i$ we show that

$$[D_i, B_i] = \epsilon_{ijk}(D_iD_jD_k - D_iD_jD_k) = 0 \tag{0.1.8}$$

0.1.3   Properties of spin operators

In formulating NRQED for general spin particles, we need to consider all the possible operators might show up in the Lagrangian. The state-space of a spin-s particle is the direct product of it's spin-state and all the other state information. Because the spaces are orthogonal, we can treat separately operators in the two spaces. The operators and fields which exist in position space are the same for a particle of any spin, but unsurpsingly the operators allowed in spin space do depend upon the spin of the particle. As the spin of the particle is increased, and thus its spin degrees of freedom rise, there are more ways to mix these components, and thus a greater number of spin operators to consider.

For a particular representation, we can always write a bilinear as the spin operators and other operators acting between two spinors:

$$\Psi^\dagger \mathcal{O}_S \mathcal{O}_X \Psi \tag{0.1.9}$$

The two types of operators will always commute, since they act on orthogonal spaces, so it doesn't matter what order they're written in. All such bilinears must be Galilean invariant, but individual operators might not be. The non-spin operators we consider, such as $\mathbf{D}$, $\mathbf{B}$ or contractions with the tensor ϵ_{ijk} are already all written as 3-vectors or (in combination) as higher rank tensors. Therefore, it will be most convenient to write spin-operators in the same way, so that writing Galilean-invariant combinations of the two types of operators is done just by contracting indices.

Even though the number of spin opeartors does depend upon the spin of the particle, it is still possible to proceed in such a way that the same notation may be used no matter the spin. There are a few requirements:

- We write all high spin operators in terms of combinations of S_i, since these have universal properties regardless of the representation they are written in.

- If an operator exists and is non-zero in the representation of spin-s, it also exists in spin-$s+1$

- All operators introduced to account for the additional degrees of freedom in higher spin representations vanish when written in a lower spin theory. (As an example of the last point, the operator $S_i S_j + S_j S_i - \delta_{ij} S^2$ is needed to account for the degrees of freedom in a spin-1 theory, but vanishes in spin-1/2.

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If these requirements are met a consistent spin-agnostic notation can be adopted. Now we attempt to construct operators that meet these conditions.

The spinors Ψ are written with $2s+1$ independent components. The spin operators will be matrices acting on these components, and so for a spin-s particle will be $(2s+1) \times (2s+1)$ matrices. The combined operator $\mathcal{O}_S \mathcal{O}_X$ must be Hermitian, but without loss of generality we can require any $\mathcal{O}_S$, $\mathcal{O}_X$ to be Hermitian separately. So there is the additional constraint that the matrices be Hermitian, and this means a total of $(2s+1)^2$ degrees of freedom.

For spin-0 there is only one component to the spinor, so the only possible matrix is equivalent to the identity.

For spin-1/2 we have, in addition to the identity matrix, the spin matrices $s_i = \frac{1}{2}\sigma_i$. This is four independent matrices, and since the space has $(2s+1)^2 = 4$ degrees of freedom, exactly spans the space of all spin-operators. If we try to construct terms which are bilinear in spin matrices, they just reduce through the identity $\sigma_i \sigma_j = \delta_{ij} + \epsilon_{ijk}\sigma_k$, which we can already construct through combinations of the four operators we already have. Since those four operators form a basis for the space, independent bilinears were forbidden even without an explicit form for the equation.

What about spin-1? We need 9 independent operators to span the space. All the operators that exist in spin-1/2 will work here as well, though the spin matrices will have a different representation. That leaves 5 operators to construct. It is natural to try to construct these from bilinear combinations of spin matrices. Naively $S_i S_j$ would itself be 9 independent structures, but clearly some of these are expressible in terms of the lower order operators. (By the order of a spin operator we mean its greatest degree in S_i)

Regardless of their representation, the spin operators always fulfill certain identities based on their Lie group. Namely

$$S_i S_j \delta_{ij} \sim I, \ [S_i, S_j] = \epsilon_{ijk} S_k \tag{0.1.10}$$

and it is these identities which allow certain combinations of $S_i S_j$ to be related to lower order operators.

If instead of general spin bilinears we consider only combinations which are

- Symmetric in i, j

- Traceless

then such a structure will be independent of the set of operators $\{I, S_1, S_2, S_3\}$. Because it is symmetric no combination may be related using the commutator, and because it is traceless there is no combination that reduces due to the other identity.

This conditions form a set of 4 constraints, so from the original 9 degrees of freedom possessed by combinations of $S_i S_j$ are left only 5. Together with the 4 lower order operators this is exactly enough to span the space.

We can explicitly write this symmetric, traceless structure as

$$S_i S_j + S_j S_i - \delta_{ij} S^2 \tag{0.1.11}$$

Having explored how the procedure works for spin-1, move on to consider the general spin case. The idea is to proceed inductively using the same rough attack as for the case of spin-1. In addition to all the "lower order" operators which were used for lower spin representations introduce new operators which are of higher degree in the spin matrices and guaranteed to be independent of the lower spin operators.

So suppose that for a spin-$s-1$ particle we have a set of operators written as $\bar{S}^0, \bar{S}^1, \ldots \bar{S}^{(s-1)}$, where a structure $\bar{S}^n$ carries n Galilean indices and is symmetric and traceless between any pair of indices, that is:

$$\bar{S}^n_{..i..j..} = \bar{S}^n_{..j..i..}, \ \delta_{ij} \bar{S}^n_{..i..j..} = 0 \tag{0.1.12}$$

(From above, $\bar{S}^0 = I$, $\bar{S}^1_i = S_i$, and $\bar{S}^2 = S_i S_j + S_j S_i + \delta_{ij} S^2$.)

The objects $\bar{S}^n$ are built as follows: start with all combinations involving the product of exactly n spin matrices. (There are 3^n such structures.) Form them into combinations which are symmetric in all indices. Each index has three possible values, so we can label each structure by how many indices are equal to 1 and 2. If a is the number of indices equal to 1, and b the number of indices equal to 2, then for a given a there are $n + 1 - a$ possible choices for b. The total number of symmetric structures is then

$$\Sigma^n_{a=0}(n - a + 1) = \frac{1}{2}(n+1)(n+2) \tag{0.1.13}$$

We want to apply the additional constraint that the $\bar{S}^n$ be traceless in all indices. This will involve subtracting all the lower order structures which result when the trace of the completely symmetric combinations is taken.

It introduces an additional constraint on $\bar{S}^n$ for each pair of indices, and there are $n(n-1)/2$ distinct pairs of indices. The total degrees of freedom left are

$$\frac{1}{2}(n+1)(n+2) - \frac{1}{2}n(n-1) = \frac{1}{2}\left(n^2 + 3n + 2 - n^2 + n\right) = 2n + 1 \tag{0.1.14}$$

In combination with the lower order spin operators, this is exactly the number of independent operators we need to span the space. Combined with the lower order operators this is a complete basis, so we know we haven't missed any terms. Because they are constructed to be independent from all the lower order operators, it must necessarily be true that they will vanish in lower spin representations.

Using this notation we can write down terms in the Lagrangian that are valid for particles of any spin. By writing all spin operators in terms of S_i they are representation agnostic, and by construction they will vanish for low spin particles where they do not "fit".

0.1.4 Composition of position space operators

There are several symmetries that terms in the Lagrangian must preserve: spatial reflection, time reversal, gauge invariance, and Hermiticity. Gauge invariance is taken care of by only considering gauge-invariant operators.

All of the possible spin-space structures preserve spatial parity. If the term in the Lagrangian is to obey such a symmetry, then any allowed collection of position-space operators must be themselves invariant under reflection.

Each term must also be invariant under time reversal. The spin operator S_i flips sign under time reversal. Thus, $\bar{S}_{ij}$ is even, and $\bar{S}_{ijk}$ odd, under the same transformation. E_i and D_i are even, while B_i is odd. Any set of operators with definite behavior under time reversal, though, can be made invariant by including an extra factor of i. So even if some term ABC is odd, $iABC$ will be even.

Finally, any sequence of operators can be made Hermitian by simply adding the Hermitian conjugate; if we wish to include a term ABC in the Lagrangian, we add $C^\dagger B^\dagger A^\dagger$. Since all the operator s under consideration are either Hermitian or anti-Hermitian, this means that exactly one of $ABC \pm CBA$ will be allowed. The spin space operators are all Hermitian of themselves, and commute with all other operators, so only the position space operators are nontrivial.

Thus for any possible set of operators, first it is determined if such a set is allowed by considerations of parity. Then, whether an additional factor of i is needed is determined by examining the properties under time reversal. Finally, the allowed Hermitian terms are enumerated.

Of the "building blocks" for the general Lagrangian, those odd in parity are also odd in v. And those even in parity are of an order even in v. So only terms of an order even in v can possibly be allowed under parity, and at the same time every such term will have the correct symmetry properties under spatial reflection.

First consider terms of leading order mv^2. What collections of position space operators exist? E is already too high of an order, leaving only that with two powers of D:

$$\text{Sets of } \mathcal{O}(mv^2) \text{ combinations} = \{\frac{1}{m}D_i D_j\} \tag{0.1.15}$$

This is in addition to the single permitted term containing $e\Phi$ which is not, by itself, gauge-invariant.

Next consider terms of order mv^4. We could have up to a single power of E, or only the long derivative operators:

$$\text{Set of } \mathcal{O}(mv^2) \text{ combinations} = \{\frac{1}{m^3}D_i D_j D_k D_\ell, \ \frac{1}{m^2}eE_i D_j\} \tag{0.1.16}$$

The leading order term in B is just $\frac{e}{m}B_i$.

The order $\frac{e}{m}B_i v^2$ terms are drawn from:

$$\text{Set of } \mathcal{O}(\frac{e}{m}Bv^2) \text{ combinations} = \{\frac{e}{m^3}D_i D_j B_k\} \tag{0.1.17}$$

Contraction of terms

In this nonrelativistic theory the Lagrangian need not be Lorentz invariant, but must still be Galilean invariant. All the operators we consider transform as 3-vectors or higher order 3-tensors. To form allowed terms, all indices must be somehow contracted.

Above, all relevant sets of position space operators are considered for each order of term. The greatest number of indices free was four. These must be contracted with order unity structures, which as well as the spin operators with up to four indices (S_i, $\bar{S}_{ij}$, $\bar{S}_{ijk}$, $\bar{S}_{ijk\ell}$) include δ_{ij} and the completely antisymmetric tensor ϵ_{ijk}.

- The only structure with one index is just S_i.

- With two indices, there are the simple structures $\bar{S}_{ij}$ and δ_{ij}, and also $S_i\epsilon_{ijk}$.

- With three indices, there are ϵ_{ijk} and $\bar{S}_{ijk}$ as well as $\delta_{ij}S_k$ or $\bar{S}_{ij}\epsilon_{jk\ell}$.

- The only position space operator with four indices is just $D_iD_jD_kD_\ell$. While there are a fairly large number of order unity structures with four indices, any with spin operators are forbidden because of the kinetic nature of the term. So only $\delta_{ij}\delta_{k\ell}$ need be considered.

Above are categorized the possible sets of position space operators for each order, and the ways of contracting them by number of indices. The next step is to write all Hermitian combinations we can form from these operators. This will form a complete catalogue of allowed terms in the Lagrangian.

Some terms which on the surface appear distinct might in fact be identical. One identity of this nature is

$$D_iD_j\epsilon_{ijk} = -ieB_k \tag{0.1.18}$$

Which means that any term where two long derivative terms are contracted with the antisymmetric tensor will be equivalent to some already catalogued term involving B.

Similarly,

$$[D_i, D_j] = -ie(\partial_i A_j - \partial_j A_i) = -ieF_{ij} = -ie\epsilon_{ijk}B_k \tag{0.1.19}$$

so terms which appear distinct because of the ordering of two long derivative operators are really the same, up to some difference that is absorbed into a term involving B.

Finally, while in the general case $[D_i, B_j] \neq 0$, it is always true that $[D_i, B_j] = 0$.

So many apparently distinct terms may be rewritten and absorbed as the combination of several already accounted for terms. As an example, it is not necessary to write a term such as $D_iD_jD_iD_j$ in the Lagrangian. It can be rewritten as

$$\begin{aligned} D_iD_jD_iD_j &= D_iD_iD_jD_j + D_i[D_i, D_j]D_j \\ &= \mathbf{D}^4 - ieD_i\epsilon_{ijk}B_kD_j \\ &= \mathbf{D}^4 - ie\epsilon_{ijk}(D_iD_jB_k + [B_k, D_j]) \\ &= \mathbf{D}^4 - e^2B^2 + ie\epsilon_{ijk}D_i(\partial_j B_k) \end{aligned}$$

The term D_iD_j has two indices. There are three ways to contract it: with δ_{ij}, $\bar{S}_{ij}$, or $S_k\epsilon_{ijk}$. Since kinetic terms with spin are not considered, that only leaves one possible term: $\mathbf{D}^2$. This term is Hermitian in and of itself. It has mass dimension two, so the term as it appears in the Lagrangian will be:

$$D_iD_j \to \frac{\mathbf{D}^2}{2m} \tag{0.1.20}$$

Terms from the set E_iD_j also have two indices. It can be contracted with all three of δ_{ij}, $\bar{S}_{ij}$, or $iS_k\epsilon_{ijk}$. E_i, δ_{ij} and $\bar{S}_{ij}$ are Hermitian, while D_j and $iS_k\epsilon_{ijk}$ are anti-Hermitian. It is convenient to use that $[D_j, E_i] = \boldsymbol{\nabla}_j E_i$.

The Hermitian combination with δ_{ij} is

$$(D_iE_j - E_jD_i)\delta_{ij} \to \frac{\boldsymbol{\nabla}\cdot\mathbf{E}}{4m^2} = \frac{\boldsymbol{\nabla}\cdot\mathbf{E}}{4m^2} \tag{0.1.21}$$

The Hermitian combination with $\bar{S}_{ij}$ is

$$(D_iE_j - E_jD_i)\bar{S}_{ij} \to \frac{\bar{S}_{ij}\partial_i E_j}{4m^2} \tag{0.1.22}$$

The Hermitian combination with $iS_k\epsilon_{ijk}$ is

$$(D_iE_j + E_jD_i) \to iS_k\epsilon_{ijk} \to \frac{i\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} + \mathbf{E}\times\mathbf{D})}{4m^2} \tag{0.1.23}$$

A single power of the magnetic field, B_i may only be contracted with S_i. So the term is

$$\frac{e}{m}\mathbf{S}\cdot\mathbf{B} \tag{0.1.24}$$

The second order terms involving the magnetic field are drawn from the set $D_iD_jB_k$. There are four order-unity structures which have three indices, but only two of those need be considered. This is because $\mathbf{D} \times \mathbf{D} \sim \mathbf{B}$ and $\mathbf{D} \times \mathbf{B} = 0$. If is not assumed that $[D_i, B_j] = 0$ then the allowed Hermitian combinations will be with $D_iD_jB_k + B_kD_jD_i$ and $D_iB_jD_k$.

While $[D_i, D_j] \neq 0$, it does produce another term proportional to B and thus, smaller than considered. So for instance, $D_iD_jB_i$ need not be considered separate from $D_jD_iB_i$.

$$(D_iD_jB_k + B_kD_jD_i)\delta_{ij}S_k \quad \rightarrow \quad \frac{e}{m}\frac{\mathbf{D}^2(\mathbf{S}\cdot\mathbf{B}) + (\mathbf{S}\cdot\mathbf{B})\mathbf{D}^2}{4m^2}$$

$$(D_iD_jB_k + B_kD_jD_i)\delta_{jk}S_i \quad \rightarrow \quad \frac{e}{m}\frac{(\mathbf{S}\cdot\mathbf{D})(\mathbf{D}\cdot\mathbf{B}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})}{4m^2}$$

$$D_iB_jD_k\delta_{ij}S_k \quad \rightarrow \quad \frac{e}{m}\frac{(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{D}\cdot\mathbf{B})(\mathbf{S}\cdot\mathbf{B})}{4m^2}$$

$$(D_iD_jB_k + B_kD_jD_i)\bar{S}_{ijk} \quad \rightarrow \quad \frac{e}{m}\frac{\bar{S}_{ijk}(D_iD_jB_k + B_kD_jD_i)}{4m^2}$$

$$D_iB_jD_k\bar{S}_{ijk} \quad \rightarrow \quad \frac{e}{m}\frac{\bar{S}_{ijk}D_iB_jD_k}{4m^2}$$

Finally, there is only one way to contract $D_iD_jD_kD_\ell$, which is as $\mathbf{D}^4$. Again, variations such as $D_i\mathbf{D}^2D_i$ need not be considered because they just reproduce already considered terms with B.

$$D_iD_jD_kD_\ell\delta_{ij}\delta_{k\ell} \quad \rightarrow \quad \frac{\mathbf{D}^4}{8m^3}$$

Having organised all the operators into all allowed combinations, we can write down the different orders of the Lagrangian.

The leading order terms in the Lagrangian, of order mv^2 and $(e/m)B$, are:

$$\mathcal{L}_{mv^2} = \Psi^\dagger\left\{iD_0 + \frac{\mathbf{D}^2}{2m} + c_F\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\right\}\Psi \tag{0.1.25}$$

The part of the Lagrangian of order mv^4 is:

$$\mathcal{L}_{mv^4} = \Psi^\dagger\left\{\frac{\mathbf{D}^4}{8m^2} + c_D\frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2} + c_Q\frac{eQ_{ij}(D_iE_j - E_iD_j)}{8m^2} + c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D})}{8m^2}\right\}\Psi \tag{0.1.26}$$

And the terms which are corrections to the magnetic field terms:

$$\mathcal{L}_{Bv^2} = \Psi^\dagger\left\{c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3} + c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3}\right.$$

$$\left. + c_{T1}\frac{e\bar{S}_{ijk}(D_iD_jB_k + B_kD_jD_i)}{8m^3} + c_{T2}\frac{e\bar{S}_{ijk}D_iB_jD_k}{8m^3}\right\}\Psi$$

$$\tag{0.1.27}$$

Full Lagrangian

Combining all of the above, the full Lagrangian we consider is then:

$$\mathcal{L}_{NRQED} = \Psi^\dagger \left\{ iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^2} + c_F \frac{e}{m} \mathbf{S}\cdot\mathbf{B} + c_D \frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2} + c_Q \frac{eQ_{ij}(D_i E_j - E_i D_j)}{8m^2} \right.$$

$$+ c_S \frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D})}{8m^2} + c_{W1} \frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2} \frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3}$$

$$\left. + c_{p'p} \frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3} + c_{T_1} \frac{e\bar{S}_{ijk}(D_i D_j B_k + B_k D_j D_i)}{8m^3} + c_{T_2} \frac{e\bar{S}_{ijk}D_i B_j D_k}{8m^3} \right\} \Psi$$

$$(0.1.28)$$

One of the features of this Lagrangian is that every coefficient is fixed by the one-photon interaction. Although some terms might represent two-photon interactions, they are terms like $\mathbf{S}\cdot\mathbf{A}\times\mathbf{E}$, whose coefficient is fixed by the gauge-invariant term $\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}$. This in turn means that we can calculate the corrections to the g-factor by considering only one-photon interactions.