0.1 The NRQED Lagrangian

We want to construct an effective Lagrangian in the nonrelativistic limit. Our goal is to calculate the leading order corrections to the g-factor, which are corrections of order α^2. To this end, we need terms in the effective nonrelativistic Lagrangian which are equivalent corrections.

0.1.1 position space operators

	Order	P	T	†
eE_i	m^2v^3	-	+	+
eB_i	m^2v^2	+	-	+
D_i	mv	-	+	-
D_0	mv	+	-	-
S_i	1	+	-	+
i	1	+	-	-

There are several symmetries that terms in the Lagrangian must preserve: spatial reflection, time reversal, gauge invariance, and Hermiticity. Gauge invariance is taken care of by only considering gauge-invariant operators.

All of the possible spin-space structures preserve spatial parity. If the term in the Lagrangian is to obey such a symmetry, then any allowed collection of position-space operators must be themselves invariant under reflection.

Each term must also be invariant under time reversal. The spin operator S_i flips sign under time reversal. Thus, $\bar{S}_{ij}$ is even, and $\bar{S}_{ijk}$ odd, under the same transformation. E_i and D_i are even, while B_i is odd. Any set of operators with definite behavior under time reversal, though, can be made invariant by including an extra factor of i. So even if some term ABC is odd, $iABC$ will be even.

Finally, any sequence of operators can be made Hermitian by simply adding the Hermitian conjugate; if we wish to include a term ABC in the Lagrangian, we add $C^\dagger B^\dagger A^\dagger$. Since all the operator s under consideration are either Hermitian or anti-Hermitian, this means that exactly one of $ABC \pm CBA$ will be allowed. The spin space operators are all Hermitian of themselves, and commute with all other operators, so only the position space operators are nontrivial.

Thus for any possible set of operators, first it is determined if such a set is allowed by considerations of parity. Then, whether an additional factor of i is needed is determined by examining the properties under time reversal. Finally, the allowed Hermitian terms are enumerated.

Of the "building blocks" for the general Lagrangian, those odd in parity are also odd in v. And those even in parity are of an order even in v. So only terms of an order even in v can possibly be allowed under parity, and at the same time every such term will have the correct symmetry properties under spatial reflection.

First consider terms of leading order mv^2. What collections of position space operators exist? E is already too high of an order, leaving only that with two powers of D:

$$\text{Sets of } \mathcal{O}(mv^2) \text{ combinations} = \{\frac{1}{m} D_i D_j\} \tag{0.1}$$

This is in addition to the single permitted term containing $e\Phi$ which is not, by itself, gauge-invariant.

Next consider terms of order mv^4. We could have up to a single power of E, or only the long derivative operators:

$$\text{Set of } \mathcal{O}(mv^2) \text{ combinations} = \{\frac{1}{m^3} D_i D_j D_k D_\ell,\ \frac{1}{m^2} eE_i D_j\} \tag{0.2}$$

The leading order term in B is just $\frac{e}{m} B_i$.

The order $\frac{e}{m} B_i v^2$ terms are drawn from:

$$\text{Set of } \mathcal{O}(\frac{e}{m} Bv^2) \text{ combinations} = \{\frac{e}{m^3} D_i D_j B_k\} \tag{0.3}$$

0.1.2 Contraction of terms

In this nonrelativistic theory the Lagrangian need not be Lorentz invariant, but must still be Galilean invariant. All the operators we consider transform as 3-vectors or higher order 3-tensors. To form allowed terms, all indices must be somehow contracted.

Above, all relevant sets of position space operators are considered for each order of term. The greatest number of indices free was four. These must be contracted with order unity structures, which as well as the spin operators with up to four indices (S_i, $\bar{S}_{ij}$, $\bar{S}_{ijk}$,$\bar{S}_{ijk\ell}$) include δ_{ij} and the completely antisymmetric tensor ϵ_{ijk}.

- The only structure with one index is just S_i.
- With two indices, there are the simple structures $\bar{S}_{ij}$ and δ_{ij}, and also $S_i\epsilon_{ijk}$.
- With three indices, there are ϵ_{ijk} and $\bar{S}_{ijk}$ as well as $\delta_{ij}S_k$ or $\bar{S}_{ij}\epsilon_{jk\ell}$.
- The only position space operator with four indices is just $D_iD_jD_kD_\ell$. While there are a fairly large number of order unity structures with four indices, any with spin operators are forbidden because of the kinetic nature of the term. So only $\delta_{ij}\delta_{k\ell}$ need be considered.

Above are categorized the possible sets of position space operators for each order, and the ways of contracting them by number of indices. The next step is to write all Hermitian combinations we can form from these operators. This will form a complete catalogue of allowed terms in the Lagrangian.

The term D_iD_j has two indices. There are three ways to contract it: with δ_{ij}, $\bar{S}_{ij}$, or $S_k\epsilon_{ijk}$. Since kinetic terms with spin are not considered, that only leaves one possible term: $\mathbf{D}^2$. This term is Hermitian in and of itself. It has mass dimension two, so the term as it appears in the Lagrangian will be:

$$D_iD_j \rightarrow \frac{\mathbf{D^2}}{2m} \tag{0.4}$$

Terms from the set E_iD_j also have two indices. It can be contracted with all three of δ_{ij}, $\bar{S}_{ij}$, or $iS_k\epsilon_{ijk}$. E_i, δ_{ij} and $\bar{S}_{ij}$ are Hermitian, while D_j and $iS_k\epsilon_{ijk}$ are anti-Hermitian. It is convenient to use that $[D_j, E_i] = \boldsymbol{\nabla}_jE_i$.

The Hermitian combination with δ_{ij} is

$$(D_iE_j - E_jD_i)\delta_{ij} \rightarrow \frac{\boldsymbol{\nabla}\cdot\mathbf{E}}{4m^2} = \frac{\boldsymbol{\nabla}\cdot\mathbf{E}}{4m^2} \tag{0.5}$$

The Hermitian combination with $\bar{S}_{ij}$ is

$$(D_iE_j - E_jD_i)\bar{S}_{ij} \rightarrow \frac{\bar{S}_{ij}\partial_iE_j}{4m^2} \tag{0.6}$$

The Hermitian combination with $iS_k\epsilon_{ijk}$ is

$$(D_iE_j + E_jD_i) \rightarrow iS_k\epsilon_{ijk} \rightarrow \frac{i\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} + \mathbf{E}\times\mathbf{D})}{4m^2} \tag{0.7}$$

A single power of the magnetic field, B_i may only be contracted with S_i. So the term is

$$\frac{e}{m}\mathbf{S}\cdot\mathbf{B} \tag{0.8}$$

The second order terms involving the magnetic field are drawn from the set $D_iD_jB_k$. There are four order-unity structures which have three indices, but only two of those need be considered. This is because $\mathbf{D}\times\mathbf{D} \sim \mathbf{B}$ and $\mathbf{D}\times\mathbf{B} = 0$. If is not assumed that $[D_i, B_j] = 0$ then the allowed Hermitian combinations will be with $D_iD_jB_k + B_kD_jD_i$ and $D_iB_jD_k$.

While $[D_i, D_j] \neq 0$, it does produce another term proportional to B and thus, smaller than considered. So for instance, $D_iD_jB_i$ need not be considered separate from $D_jD_iB_i$.

$$
\begin{aligned}
(D_iD_jB_k + B_kD_jD_i)\delta_{ij}S_k &\rightarrow \frac{e}{m}\frac{\mathbf{D}^2(\mathbf{S}\cdot\mathbf{B}) + (\mathbf{S}\cdot\mathbf{B})\mathbf{D}^2}{4m^2} \\
(D_iD_jB_k + B_kD_jD_i)\delta_{jk}S_i &\rightarrow \frac{e}{m}\frac{(\mathbf{S}\cdot\mathbf{D})(\mathbf{D}\cdot\mathbf{B}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})}{4m^2} \\
D_iB_jD_k\delta_{ij}S_k &\rightarrow \frac{e}{m}\frac{(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{D}\cdot\mathbf{B})(\mathbf{S}\cdot\mathbf{B})}{4m^2}
\end{aligned}
$$

$$
\begin{aligned}
(D_iD_jB_k + B_kD_jD_i)\bar{S}_{ijk} &\rightarrow \frac{e}{m}\frac{\bar{S}_{ijk}(D_iD_jB_k + B_kD_jD_i)}{4m^2} \\
D_iB_jD_k\bar{S}_{ijk} &\rightarrow \frac{e}{m}\frac{\bar{S}_{ijk}D_iB_jD_k}{4m^2}
\end{aligned}
$$

Finally, there is only one way to contract $D_iD_jD_kD_\ell$, which is as $\mathbf{D}^4$. Again, variations such as $D_i\mathbf{D}^2D_i$ need not be considered because they just reproduce already considered terms with B.

$$
D_iD_jD_kD_\ell\delta_{ij}\delta_{k\ell} \rightarrow \frac{\mathbf{D}^4}{8m^3}
$$

$$
\mathcal{L}_{NRQED} = \Psi^\dagger\left\{iD_0 + \frac{\mathbf{D}^2}{2m} + c_F\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\right\}\Psi \tag{0.9}
$$

$$
\mathcal{L}_{mv^4} = \Psi^\dagger\left\{\frac{\mathbf{D}^4}{8m^2} + c_D\frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2} + c_Q\frac{eQ_{ij}(D_iE_j - E_iD_j)}{8m^2} + c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}}{8m^2}\right\}\Psi \tag{0.10}
$$

$$
\begin{aligned}
\mathcal{L}_{Bv^2} =& \Psi^\dagger\left\{c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3} + c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3}\right. \\
&\left. + c_{T_1}\frac{e\bar{S}_{ijk}(D_iD_jB_k + B_kD_jD_i)}{8m^3} + c_{T_2}\frac{e\bar{S}_{ijk}D_iB_jD_k}{8m^3}\right\}\Psi
\end{aligned} \tag{0.11}
$$

0.1.3 Full Lagrangian

The full Lagrangian we consider is then:

$$
\begin{aligned}
\mathcal{L}_{NRQED} =& \Psi^\dagger\left\{iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^2} + c_F\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + c_D\frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2} + c_Q\frac{eQ_{ij}(D_iE_j - E_iD_j)}{8m^2}\right. \\
&+ c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}}{8m^2} + c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3} \\
&\left. + c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3} + c_{T_1}\frac{e\bar{S}_{ijk}(D_iD_jB_k + B_kD_jD_i)}{8m^3} + c_{T_2}\frac{e\bar{S}_{ijk}D_iB_jD_k}{8m^3}\right\}\Psi
\end{aligned} \tag{0.12}
$$

One of the features of this Lagrangian is that every coefficient is fixed by the one-photon interaction. Although some terms might represent two-photon interactions, they are terms like $\mathbf{S}\cdot\mathbf{A}\times\mathbf{E}$, whose coefficient is fixed by the gauge-invariant term $\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}$. This in turn means that we can calculate the corrections to the g-factor by considering only one-photon interactions.

0.1.4 Order of terms

We consider constant, infinitesimal external magnetic fields, so we need only consider terms linear in $\mathbf{B}$.

The velocity of the particles in our bound state system will be $v \sim \alpha$.

The electric field we consider is the Coulomb field, so $e\Phi \sim mZ\alpha^2 \sim mv^2$, and $eE \sim m^2v^3$.

Each derivative of the electric field will add an additional factor of mv, so the operator $\mathbf{D}$ can be taken to be of this order.

We need to keep terms up to order mv^4 and $\frac{B}{m}v^2$ in order to calculate the g-factor to the necessary precision. We include mv^4 terms so we can be sure that there are no effects entering from second-order perturbation theory.

0.1.5 Constraints on the form of the Lagrangian

The Lagrangian is constrained to obey several symmetries. It must be invariant under the symmetries of parity and time reversal. It must also be invariant under Galilean transformations. The Lagrangian must also be Hermitian, and gauge invariant.

What are the gauge invariant building blocks we can use to construct this Lagrangian? We have the external fields $\mathbf{E}$ and $\mathbf{B}$, the spin operators $\mathbf{S}$, and the long derivative $\mathbf{D} = \partial - ie\mathbf{A}$. The fields should always be accompanied by the charge e of the particle.

When considering the case of higher spin particles, we might consider terms quadratic and above in spin operators. For a particle of spin s, there must be $(2s+1)^2$ independent hermitian operators. We can span this set of operators by considering products of up to $2s$ spin matrices which are symmetric and traceless in every vector index. For example, for spin-1 we have quadratic, in addition to I and S_i, five independent structures of the form $S_iS_j + S_jS_i + \delta_{ij}\mathbf{S}$.

We also have the scalar D_0, however, we need only include a single such term because we insist on having only one power of the time derivative.

To consider possible terms, we need to know how each of the above behave under the discrete transformations and Hermitian conjugate. The signs under these transformations are listed in the table below. (Also included is the imaginary number i.)

	Order	P	T	†
eE_i	m^2v^3	-	+	+
eB_i	m^2v^2	+	-	+
D_i	mv	-	+	-
D_0	mv	+	-	-
S_i	1	+	-	+
i	1	+	-	-

In formulating NRQED for general spin particles, we need to consider all the possible operators might show up in the Lagrangian. The state-space of a spin-s particle is the direct product of it's spin-state and all the other state information. We want to write down all possible operators which act only on the spin space.

For a particular representation, we can always write a bilinear as the spin operators and other operators acting between two spinors:

$$\Psi^\dagger \mathcal{O}_S \mathcal{O}_X \Psi \tag{0.13}$$

The two types of operators will always commute, since they act on orthogonal spaces, so it doesn't matter what order they're written in. All such bilinears must be Galilean invariant, but individual operators might not be. The non-spin operators we consider, such as $\mathbf{D}$, $\mathbf{B}$ or contractions with the tensor ϵ_{ijk} are already all written as 3-vectors or (in combination) as higher rank tensors. Therefore, it will be most convenient to write spin-operators in the same way, so that writing Galilean-invariant combinations of the two types of operators is done just by contracting indices.

The spinors Ψ are written with $2s+1$ independent components. The spin operators will be matrices acting on these components, and so for a spin-s particle will be $(2s+1)\times(2s+1)$ matrices. The combined operator $\mathcal{O}_S\mathcal{O}_X$ must be Hermitian, but without loss of generality we can require any $\mathcal{O}_S$, $\mathcal{O}_X$ to be Hermitian separately. So there is the additional constraint that the matrices be Hermitian, and this means a total of $(2s+1)^2$ degrees of freedom.

For spin-0 there is only one component to the spinor, so the only possible matrix is equivalent to the identity.

For spin-1/2 we have, in addition to the identity matrix, the spin matrices $s_i = \frac{1}{2}\sigma_i$. This is four independent matrices, and since the space has $(2s+1)^2 = 4$ degrees of freedom, exactly spans the space of all spin-operators. If we try to construct terms which are bilinear in spin matrices, they just reduce through the identity $\sigma_i\sigma_j = \delta_{ij} + \epsilon_{ijk}\sigma_k$, which we can already construct through combinations of the four operators we already have. Since those four operators form a basis for the space, independent bilinears were forbidden even without an explicit form for the equation.

What about spin-1? We need 9 independent operators to span the space. All the operators that exist in spin-1/2 will work here as well, though the spin matrices will have a different representation. That leaves 5 operators to construct. It is natural to try to construct these from bilinear combinations of spin matrices. Naively S_iS_j would itself be 9 independent structures, but clearly some of these are expressible in terms of the lower order operators.

Regardless of their representation, the spin operators always fulfill certain identities based on their Lie group. Namely

$$S_iS_j\delta_{ij} \sim I,\ [S_i, S_j] = \epsilon_{ijk}S_k \tag{0.14}$$

and it is these identities which allow certain combinations of S_iS_j to be related to lower order operators.

If instead of general spin bilinears we consider only combinations which are

- Symmetric in i, j
- Traceless

then such a structure will be independent of the set of operators $\{I, S_1, S_2, S_3\}$. This is a set of 4 constraints, so from the original 9 degrees of freedom possessed by combinations of S_iS_j are left only 5. Together with the 4 lower order operators this is exactly enough to span the space.

We can explicitly write this symmetric, traceless structure as

$$S_iS_j + S_jS_i + \delta_{ij}S^2 \tag{0.15}$$

Finally consider the general spin case, proceeding inductively using the same rough idea as for the case of spin-1. We construct all the possible operators from the set of all "lower order" operators which were used for lower spin particles, and by introducing new operators which are of higher degree in the spin matrices.

So suppose that for a spin-$s-1$ particle we have a set of operators written as $\bar{S}^0, \bar{S}^1, \ldots \bar{S}^{(s-1)}$, where a structure $\bar{S}^n$ carries n Galilean indices and is symmetric and traceless between any pair of indices, that is:

$$\bar{S}^n_{..i..j..} = \bar{S}^n_{..j..i..},\ \delta_{ij}\bar{S}^n_{..i..j..} = 0 \tag{0.16}$$

(From above, $\bar{S}^0 = I$, $\bar{S}^1_i = S_i$, and $\bar{S}^2 = S_iS_j + S_jS_i + \delta_{ij}S^2$.)

The objects $\bar{S}^n$ are built as follows: start with all combinations involving the product of exactly n spin matrices. (There are 3^n such structures.) Form them into combinations which are symmetric in all indices. Each index has three possible values, so we can label each structure by how many indices are equal to 1 and 2. If a is the number of indices equal to 1, and b the number of indices equal to 2, then for a given a there are $n+1-a$ possible choices for b. The total number of symmetric structures is then

$$\Sigma^n_{a=0}(n-a+1) = \frac{1}{2}(n+1)(n+2) \tag{0.17}$$

We have the additional constraint that the $\bar{S}^n$ be traceless in all indices. This is an additional constrain for each pair of indices, and there are $n(n-1)/2$ pairs of indices. The total degrees of freedom left are

$$\frac{1}{2}(n+1)(n+2) - \frac{1}{2}n(n-1) = \frac{1}{2}\left(n^2+3n+2-n^2+n\right) = 2n+1 \tag{0.18}$$

In combination with the lower order spin operators, this is exactly the number of independent operators we need to span the space.