Chapter 1

The interaction Hamiltonian

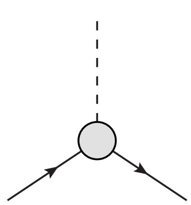
1.1 Calculation of Breit potential

We want to calculate the interaction potential between two charged particles, at next to leading order. We derive this potential from the one-photon exchange. We already have a Lagrangian for general-spin NRQED that is accurate to the order needed. From this we can obtain the one photon vertex. Working in the Coulomb gauge, the photon propagator is such that $D_{0i} = 0$. So we can think of it as splitting into two parts: represent D_{00} by a dashed line, and D_{ij} by a wavy one. Then terms with A_0 and A_i represent two types of vertices.

1.1.1 Feynman rules

The Coulomb vertex comes from the terms out of the general NRQED Lagrangian involving A_0:

$$\mathcal{L}_{A_0} \quad = \quad \Psi^\dagger\big(- eA_0 + c_D \frac{e(\boldsymbol{\nabla} \cdot \mathbf{E} - \mathbf{E} \cdot \boldsymbol{\nabla})}{8m^2} + c_Q \frac{eQ_{ij}(\nabla_i E_j - E_i \nabla_j)}{8m^2} + c_S^1 \frac{ie\mathbf{S} \cdot (\boldsymbol{\nabla} \times \mathbf{E} - \mathbf{E} \times \boldsymbol{\nabla})}{8m^2}\big)\Psi$$

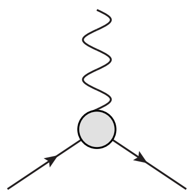


$$= \quad V_0 \quad = \quad ie\Big(1 + \tfrac{1}{8m^2}\big[c_D \mathbf{q}^2 + c_Q Q_{ij} q_i q_j - 2ic_S \mathbf{S} \cdot \mathbf{p} \times \mathbf{q}\big]\Big)$$

The part of the Lagrangian representing interaction with a transverse photon is

$$\mathcal{L}_{\mathbf{A}} = \Psi^\dagger\big(- ie\frac{\{\nabla_i, A_i\}}{2m} + c_F e\frac{\mathbf{S}\cdot\mathbf{B}}{2m}\big)\Psi$$

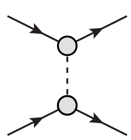
So the transverse vertex is



$$= \quad V_i \quad = \quad i\tfrac{e}{2m}\Big(\mathbf{p}+\mathbf{p}' + c_F i\mathbf{S}\times\mathbf{q}\Big)_i$$

1.1.2 Calculation of interaction

The two particles will have Lagrangians, and thus vertices, of the same general form. However, the coefficients will differ, so denote the coefficients of the second particle by d rather than c. Call the particles 1 and 2, and denote which field operators act on with these numerals. Then the interaction diagrams will be

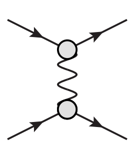


$$= \quad V_0^1 V_0^2 D_{00} \quad = \quad \left[-\frac{1}{\mathbf{q}^2}\right] ie_2\phi_2^\dagger\left\{1 - \frac{1}{8m_2^2}\left(d_D\mathbf{q}^2 + d_Q Q_{2ij}q_iq_j + 2id_S\mathbf{S}_2\cdot\mathbf{p}_2\times\mathbf{q}\right)\right\}\phi_2$$
$$\times ie_1\phi_1^\dagger\left\{1 - \frac{1}{8m_1^2}\left(c_D\mathbf{q}^2 + c_Q Q_{1ij}q_iq_j - 2ic_S\mathbf{S}_1\cdot\mathbf{p}_1\times\mathbf{q}\right)\right\}\phi_1 \tag{1.1.1}$$

The expansion to the order needed is trivial

$$= e_1e_2\frac{1}{\mathbf{q}^2}(\phi_1^\dagger\phi_2^\dagger\phi_2\phi_1)\left[1 - \frac{1}{8m_2^2}\left(d_D\mathbf{q}^2 + d_Q Q_{2ij}q_iq_j + 2ic_S\mathbf{S}_2\cdot\mathbf{p}_2\times\mathbf{q}\right) - \frac{1}{8m_1^2}\left(c_D\mathbf{q}^2 + c_Q Q_{1ij}q_iq_j - 2ic_S\mathbf{S}_1\cdot\mathbf{p}_1\times\right.\right. \tag{1.1.2}$$

The exchange of a transverse photon is given by



$$= \quad V_i^1 V_j^2 D_{ij} \quad = \quad \left[\frac{ie_1}{2m_1}\phi_1^\dagger\left\{(2p_1+q)^i - c_F i\epsilon_{k\ell i}q_kS_{1\ell}\right\}\phi_1\right]\left[\frac{1}{\mathbf{q}^2}\left(\delta_{ij} - \frac{q_iq_j}{\mathbf{q}^2}\right)\right]$$
$$\times\left[\frac{ie_2}{2m_2}\phi_2^\dagger\left\{(2p_2-q)^j + d_F i\epsilon_{mnj}q_mS_{2n}\right\}\phi_1\right]$$

This is a mess of terms that will simplify. Consider first the inner terms with just δ_{ij}. Some terms drop out because $q_iq_j\epsilon_{ijk} = 0$.

$$\delta_{ij}\left[(2p_1+q)^i - ic_F(\epsilon_{k\ell i}q_kS_{1\ell})\right]\left[(2p_2-q)^j + id_F(\epsilon_{mnj}q_mS_{2n})\right] \tag{1.1.3}$$

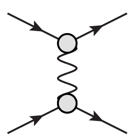
$$
\begin{aligned}
=&(2\mathbf{p_1}+\mathbf{q})\cdot(2\mathbf{p_2}-\mathbf{q})+2id_F q_m S_{2n} p_j \epsilon_{mnj} - 2ic_F q_k S_{1\ell} p_k \epsilon_{k\ell i}) \\
&+(c_F d_F) q_k q_m S_{1\ell} S_{2n} (\delta_{km}\delta_{\ell n} - \delta_{kn}\delta_{\ell m}) \\
=&(2\mathbf{p_1}+\mathbf{q})\cdot(2\mathbf{p_2}-\mathbf{q})+2i\mathbf{q}\cdot(d_F \mathbf{S_2}\times\mathbf{p_1} - c_F \mathbf{S_1}\times\mathbf{p_2}) \\
&+ c_F d_F\big(\mathbf{q}^2 \mathbf{S_1}\cdot\mathbf{S_2} - (\mathbf{S_1}\cdot\mathbf{q})(\mathbf{S_2}\cdot\mathbf{q})\big)
\end{aligned}
\tag{1.1.4}
$$

Now consider the contraction with $q_i q_j$. This is much simpler, as $\mathbf{q}\times\mathbf{q}=0$.

$$
q_i q_j \left[(2p_1+q)^i - ic_F \epsilon_{k\ell i} q_k S_{1\ell}\right]\left[(2p_2-q)^j + id_F \epsilon_{mnj} q_m S_{2n}\right]
$$

$$
= (2\mathbf{p_1}+\mathbf{q})\cdot\mathbf{q}(2\mathbf{p_2}-\mathbf{q})\cdot\mathbf{q} = 4(\mathbf{p_1}\cdot\mathbf{q})(\mathbf{p_2}\cdot\mathbf{q}) + \mathbf{q}^2(2\mathbf{p_2}\cdot\mathbf{q} - 2\mathbf{p_1}\cdot\mathbf{q} - \mathbf{q}^2)
$$

So



$$
= \quad V_i^1 V_j^2 D_{ij} \quad = \quad -\frac{e_1 e_2}{4m_1 m_2}\frac{1}{\mathbf{q}^2}(\phi_1^\dagger \phi_2^\dagger \phi_2 \phi_1)\Bigg\{4\left(\mathbf{p_1}\cdot\mathbf{p_2} - \frac{(\mathbf{p_1}\cdot\mathbf{q})(\mathbf{p_2}\cdot\mathbf{q})}{\mathbf{q}^2}\right)
$$

$$
+2i\mathbf{q}\cdot(d_F\mathbf{S_2}\times\mathbf{p_1} - c_F\mathbf{S_1}\times\mathbf{p_2}) + c_F d_F\big(\mathbf{q}^2\mathbf{S_1}\cdot\mathbf{S_2} - (\mathbf{S_1}\cdot\mathbf{q})(\mathbf{S_2}\cdot\mathbf{q})\big)\Bigg\}
$$

The total interaction is then

$$
\begin{aligned}
M \quad = \quad & e_1 e_2 \frac{1}{\mathbf{q}^2}(\phi_1^\dagger \phi_2^\dagger \phi_2 \phi_1)\Big\{1 - \frac{1}{8m_2^2}\left(d_D \mathbf{q}^2 + d_Q Q_{2ij} q_i q_j + 2id_S \mathbf{S}_2\cdot\mathbf{p}_2\times\mathbf{q}\right) \\
& -\frac{1}{8m_1^2}\left(c_D\mathbf{q}^2 + c_Q Q_{1ij} q_i q_j - 2ic_S \mathbf{S}_1\cdot\mathbf{p}_1\times\mathbf{q}\right) - \frac{1}{m_1 m_2}\left(\mathbf{p_1}\cdot\mathbf{p_2} - \frac{(\mathbf{p_1}\cdot\mathbf{q})(\mathbf{p_2}\cdot\mathbf{q})}{\mathbf{q}^2}\right) \\
& -\frac{i}{2m_1 m_2}\mathbf{q}\cdot(d_F\mathbf{S_2}\times\mathbf{p_1} - c_F\mathbf{S_1}\times\mathbf{p_2}) - \frac{c_F d_F}{4m_1 m_2}\big(\mathbf{q}^2\mathbf{S_1}\cdot\mathbf{S_2} - (\mathbf{S_1}\cdot\mathbf{q})(\mathbf{S_2}\cdot\mathbf{q})\big)\Big\}
\end{aligned}
$$

If we distribute the $1/\mathbf{q}^2$ we get

$$
\begin{aligned}
M \quad = \quad & e_1 e_2(\phi_1^\dagger \phi_2^\dagger \phi_2 \phi_1)\Big\{\frac{1}{\mathbf{q}^2} - \frac{1}{8m_2^2}\left(d_D + d_Q Q_{2ij}\frac{q_i q_j}{\mathbf{q}^2} + 2id_S\frac{\mathbf{q}\cdot\mathbf{S_2}\times\mathbf{p_2}}{\mathbf{q}^2}\right) \\
& -\frac{1}{8m_1^2}\left(c_D + c_Q Q_{1ij}\frac{q_i q_j}{\mathbf{q}^2} - 2ic_S\frac{\mathbf{q}\cdot\mathbf{S_1}\times\mathbf{p_1}}{\mathbf{q}^2}\right) - \frac{1}{m_1 m_2}\left(\frac{\mathbf{p_1}\cdot\mathbf{p_2}}{\mathbf{q}^2} - \frac{(\mathbf{p_1}\cdot\mathbf{q})(\mathbf{p_2}\cdot\mathbf{q})}{\mathbf{q}^4}\right) \\
& -\frac{i}{2m_1 m_2}\frac{\mathbf{q}\cdot(d_F\mathbf{S_2}\times\mathbf{p_1} - c_F\mathbf{S_1}\times\mathbf{p_2})}{\mathbf{q}^2} - \frac{c_F d_F}{4m_1 m_2}\big(\mathbf{S_1}\cdot\mathbf{S_2} - \frac{(\mathbf{S_1}\cdot\mathbf{q})(\mathbf{S_2}\cdot\mathbf{q})}{\mathbf{q}^2}\big)\Big\}
\end{aligned}
$$

1.1.3 Position space potential

Define $U(\mathbf{p_1}, \mathbf{p_2}, \mathbf{q})$ by

$$M = (\phi_1^\dagger \phi_2^\dagger \phi_2 \phi_1) U(\mathbf{p_1}, \mathbf{p_2}, \mathbf{q})$$

so

$$\begin{aligned}
U(\mathbf{p_1}, \mathbf{p_2}, \mathbf{q}) \quad &= \quad e_1 e_2 \Big\{ \frac{1}{\mathbf{q}^2} - \frac{1}{8m_2^2}\left(d_D + d_Q Q_{2ij} \frac{q_i q_j}{\mathbf{q}^2} + 2 i d_S \frac{\mathbf{q} \cdot \mathbf{S_2} \times \mathbf{p_2}}{\mathbf{q}^2} \right) \\
&\quad - \frac{1}{8m_1^2}\left(c_D + c_Q Q_{1ij} \frac{q_i q_j}{\mathbf{q}^2} - 2 i c_S \frac{\mathbf{q} \cdot \mathbf{S_1} \times \mathbf{p_1}}{\mathbf{q}^2} \right) - \frac{1}{m_1 m_2}\left(\frac{\mathbf{p_1} \cdot \mathbf{p_2}}{\mathbf{q}^2} - \frac{(\mathbf{p_1} \cdot \mathbf{q})(\mathbf{p_2} \cdot \mathbf{q})}{\mathbf{q}^4} \right) \\
&\quad - \frac{i}{2m_1 m_2} \frac{\mathbf{q} \cdot (d_F \mathbf{S_2} \times \mathbf{p_1} - c_F \mathbf{S_1} \times \mathbf{p_2})}{\mathbf{q}^2} - \frac{c_F d_F}{4 m_1 m_2} (\mathbf{S_1} \cdot \mathbf{S_2} - \frac{(\mathbf{S_1} \cdot \mathbf{q})(\mathbf{S_2} \cdot \mathbf{q})}{\mathbf{q}^2}) \Big\}
\end{aligned}$$

We want to transform this expression into position space. Opening up Berestetskii/Pitaevskii we have the following transformations.

$$\begin{aligned}
\int e^{i\mathbf{q}\cdot\mathbf{r}} \frac{d^3 q}{(2\pi)^3} \quad &= \quad \delta^{(3)}(\mathbf{r}) \\
\int e^{i\mathbf{q}\cdot\mathbf{r}} \frac{d^3 q}{(2\pi)^3} \frac{1}{\mathbf{q}^2} \quad &= \quad \frac{1}{4\pi r} \\
\int e^{i\mathbf{q}\cdot\mathbf{r}} \frac{d^3 q}{(2\pi)^3} \frac{q_i}{\mathbf{q}^2} \quad &= \quad \frac{i r_i}{4\pi r^3} \\
\int e^{i\mathbf{q}\cdot\mathbf{r}} \frac{d^3 q}{(2\pi)^3} \frac{q_i q_j}{\mathbf{q}^4} \quad &= \quad \frac{1}{8\pi r}\left(\delta_{ij} - \frac{r_i r_j}{r^2} \right) \\
\int e^{i\mathbf{q}\cdot\mathbf{r}} \frac{d^3 q}{(2\pi)^3} \frac{q_i q_j}{\mathbf{q}^2} \quad &= \quad \frac{1}{4\pi r^3}\left(\delta_{ij} - 3\frac{r_i r_j}{r^2} \right) + \frac{1}{3}\delta_{ij}\delta^{(3)}(\mathbf{r})
\end{aligned}$$

The quadrupole moment Q_{ij} is symmetric and traceless. This means

$$Q_{ij}(\delta_{ij} - 3\frac{r_i r_j}{r^2}) = -3\frac{Q_{ij} r_i r_j}{r^2}$$

Now applying the identities above, we find the Fourier transform of U.

$$
\begin{aligned}
\overline{U}(\mathbf{p_1},\mathbf{p_2},\mathbf{r}) \;=\; & e_1 e_2\Bigg[\frac{1}{4\pi r}-\frac{1}{8m_2^2}\left(d_D\delta(\mathbf{r})-3d_Q\frac{Q_{2ij}r_ir_j}{4\pi r^5}-d_S\frac{\mathbf{r}\cdot\mathbf{S_2}\times\mathbf{p_2}}{2\pi r^3}\right)\\
&-\frac{1}{8m_1^2}\left(\delta(\mathbf{r})-3c_Q\frac{Q_{1ij}r_ir_j}{4\pi r^5}+c_S\frac{\mathbf{r}\cdot\mathbf{S_1}\times\mathbf{p_1}}{2\pi r^3}\right)\\
&-\frac{1}{m_1m_2}\left(\frac{\mathbf{p_1}\cdot\mathbf{p_2}}{4\pi r}-\frac{1}{8\pi r}\left\{\mathbf{p_1}\cdot\mathbf{p_2}-\frac{(\mathbf{p_1}\cdot\mathbf{r})(\mathbf{p_2}\cdot\mathbf{r})}{r^2}\right\}\right)\\
&+\frac{1}{2m_1m_2}\frac{\mathbf{r}\cdot(d_F\mathbf{S_2}\times\mathbf{p_1}-c_F\mathbf{S_1}\times\mathbf{p_2})}{4\pi r^3}\\
&-\frac{c_Fd_F}{4m_1m_2}\left(\mathbf{S_1}\cdot\mathbf{S_2}\delta(\mathbf{r})-\frac{1}{4\pi r^3}\left\{\mathbf{S_1}\cdot\mathbf{S_2}-3\frac{(\mathbf{S_1}\cdot\mathbf{r})(\mathbf{S_2}\cdot\mathbf{r})}{r^2}\right\}-\mathbf{S_1}\cdot\mathbf{S_2}\frac{1}{3}\delta(\mathbf{r})\right)\Bigg]
\end{aligned}
$$

Or combining the few like terms

$$
\begin{aligned}
\overline{U}(\mathbf{p_1},\mathbf{p_2},\mathbf{r}) \;=\; & e_1 e_2\Bigg[\frac{1}{4\pi r}-\frac{1}{8m_2^2}\left(d_D\delta(\mathbf{r})-3d_Q\frac{Q_{2ij}r_ir_j}{4\pi r^5}-d_S\frac{\mathbf{r}\cdot\mathbf{S_2}\times\mathbf{p_2}}{2\pi r^3}\right)\\
&-\frac{1}{8m_1^2}\left(c_D\delta(\mathbf{r})-3c_Q\frac{Q_{1ij}r_ir_j}{4\pi r^5}+c_S\frac{\mathbf{r}\cdot\mathbf{S_1}\times\mathbf{p_1}}{2\pi r^3}\right)\\
&-\frac{1}{m_1m_2}\left(\frac{\mathbf{p_1}\cdot\mathbf{p_2}}{8\pi r}+\frac{(\mathbf{p_1}\cdot\mathbf{r})(\mathbf{p_2}\cdot\mathbf{r})}{8\pi r^3}\right)\\
&+\frac{1}{2m_1m_2}\frac{\mathbf{r}\cdot(d_F\mathbf{S_2}\times\mathbf{p_1}-c_F\mathbf{S_1}\times\mathbf{p_2})}{4\pi r^3}\\
&-\frac{c_Fd_F}{4m_1m_2}\left(\frac{2}{3}\mathbf{S_1}\cdot\mathbf{S_2}\delta(\mathbf{r})-\frac{1}{4\pi r^3}\left\{\mathbf{S_1}\cdot\mathbf{S_2}-3\frac{(\mathbf{S_1}\cdot\mathbf{r})(\mathbf{S_2}\cdot\mathbf{r})}{r^2}\right\}\right)\Bigg]
\end{aligned}
$$

This represents the interaction in the absence of any external magnetic field. However, to derive the g-factor that is exactly the needed interaction. These additional terms will come from diagrams which involve both the exchange of a photon and interaction with the potential $\mathbf{A}$.

1.2 Photon exchange in an external field

1.2.1 Feynman rules

We want to derive the interaction potential when there is an external magnetic field present. We derive this from diagrams which involve not just the exchange of one photon, but also an additional interaction with the external field.

Because we consider the point-like interaction we don't need to consider diagrams which contain either charged-particle's propagator. So the interaction we're interested in will necessarily involve two-photon vertices. The part of the NRQED Lagrangian with such terms is:

$$\mathcal{L}_{A^2} = \Psi^\dagger\Big(-\frac{e^2\mathbf{A}^2}{2m} - e^2\frac{\{\boldsymbol{\nabla}^2, \mathbf{A}^2\}}{8m^3} - e^2\frac{\{\nabla_i, A_i\}\{\nabla_j, A_j\}}{8m^3} + c_S\frac{e^2\mathbf{S}\cdot(\mathbf{A}\times\mathbf{E} - \mathbf{E}\times\mathbf{A})}{8m^2}\Big)\Psi$$

Organise the terms in the two-photon Lagrangian so that they look like:

$$\mathcal{L}_{A^2} = A_0 A_i \Psi^\dagger\Psi W_i(\mathbf{p}, q) + A_i A_j \Psi^\dagger\Psi W_{ij}(\mathbf{p}, q) + A_0 A_0 \Psi^\dagger\Psi W_0$$

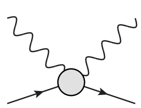
It'll turn out we want only the very first leading order terms of W_i and W_{ij}. (These W shouldn't be confused with the fields in the spin-one calculation.) The leading order term in W_{ij} is

$$W_{ij} = -\frac{e^2\delta_{ij}}{2m}$$

The leading order term in W_i comes from the $\mathbf{A}\times\mathbf{E}$ term. $\mathbf{E} = -\partial_0\mathbf{A} - \boldsymbol{\nabla}A_0$. So

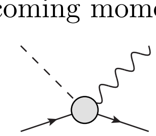
$$W_i = -ic_S\frac{e^2}{4m^2}\epsilon_{ijk}q_j S_k$$

From these we can write down the diagrams. The vertex with two transverse photons carrying indices i and j is associated with a symmetry factor of 1/2, so



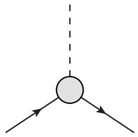
$$= 2W_{ij} = -i\frac{e^2\delta_{ij}}{m}$$

The vertex one transverse line (index i and momentum k) and one Coulomb photon (incoming momentum q):

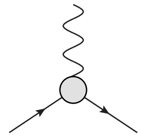


$$= W_i = c_S\frac{e^2}{4m^2}\epsilon_{ijk}q_j S_k$$

We also need the one-photon vertices which were derived in the previous section, but only to leading order. These are, dropping those higher order terms:



$$= \quad ie$$



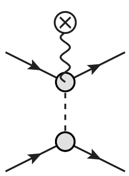
$$= \quad i\tfrac{e}{2m}\Big(\mathbf{p} + \mathbf{p}' + c_F i\mathbf{S}\times\mathbf{q}\Big)_i$$

The Coulomb propagator is just as it was before.

1.2.2 Interaction diagrams

There are four diagrams that capture the type of interaction we're looking for. The particles can exchanged either a transverse or Coulomb photon, and either particle can interact with the external magnetic field. In each of the diagrams, let q be the momentum of the exchanged photon, directed towards the vertex of particle one. Let k be the momenta of the photon interacting with the external field.

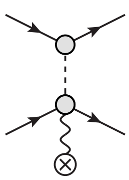
First, the two diagrams with Coulomb exchange:



$$= \quad \phi_2^\dagger\Big(ie_2\Big)\phi_2\Big(-\tfrac{1}{\mathbf{q}^2}\Big)\phi_1^\dagger\Big(c_S\tfrac{e_1^2}{4m_1^2}\mathbf{S_1}\cdot\mathbf{A}\times\mathbf{q}\Big)\phi_1$$

$$= -ie_1e_2(\phi_2^\dagger\phi_1^\dagger\phi_1\phi_2)c_s\frac{e_1}{4m_1^2}\frac{\mathbf{S_1}\cdot\mathbf{A}\times\mathbf{q}}{\mathbf{q}^2}$$

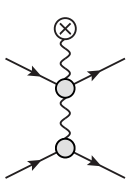
The diagram representing the second particle's interaction:



$$= \quad -\phi_2^\dagger\Big(d_S\frac{e_2^2}{4m_2^2}\mathbf{S_2}\cdot\mathbf{A}\times\mathbf{q}\Big)\phi_2\Big(-\frac{1}{\mathbf{q}^2}\Big)\phi_1^\dagger\Big(ie_1\Big)\phi_1$$

$$= +ie_1e_2(\phi_2^\dagger\phi_1^\dagger\phi_1\phi_2)d_S\frac{e_2}{4m_2^2}\frac{\mathbf{S_2}\cdot\mathbf{A}\times\mathbf{q}}{\mathbf{q}^2}$$

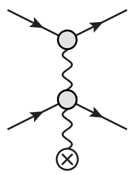
Then, diagrams with transverse exchange:



$$= \quad \phi_2^\dagger\Big(i\frac{e_2}{2m_2}[2p_2 - q - d_F i\mathbf{S_2}\times\mathbf{q}]_i\Big)\phi_2\Big(\frac{\delta_{ij}}{\mathbf{q}^2} - \frac{q_iq_j}{\mathbf{q}^4}\Big)\phi_1^\dagger\Big(\frac{-ie_1^2A_j}{m_1}\Big)\phi_1$$

$$= e_1e_2(\phi_2^\dagger\phi_1^\dagger\phi_1\phi_2)\frac{e_1}{2m_1m_2}\Big(\frac{(2\mathbf{p}_2 - \mathbf{q})\cdot\mathbf{A}}{\mathbf{q}^2} - id_F\frac{\mathbf{S}_2\cdot\mathbf{q}\times\mathbf{A}}{\mathbf{q}^2} - \frac{(2\mathbf{p}_2 - \mathbf{q})\cdot\mathbf{q}\mathbf{A}\cdot\mathbf{q}}{\mathbf{q}^4}\Big)$$

$$= e_1e_2(\phi_2^\dagger\phi_1^\dagger\phi_1\phi_2)\frac{e_1}{2m_1m_2}\Big(\frac{2\mathbf{p}_2\cdot\mathbf{A}}{\mathbf{q}^2} - id_F\frac{\mathbf{S}_2\cdot\mathbf{q}\times\mathbf{A}}{\mathbf{q}^2} - \frac{(2\mathbf{p}_2\cdot\mathbf{q})(\mathbf{A}\cdot\mathbf{q})}{\mathbf{q}^4}\Big)$$



$$= \quad \phi_2^\dagger\Big(\frac{-ie_2^2A_j}{m_2}\Big)\phi_2\Big(\frac{\delta_{ij}}{\mathbf{q}^2}-\frac{q_iq_j}{\mathbf{q}^4}\Big)\phi_1^\dagger\Big(i\frac{e_1}{2m_1}[2p_1+q+c_F i\mathbf{S_1}\times\mathbf{q}]_i\Big)\phi_1$$

$$= e_1e_2(\phi_2^\dagger\phi_1^\dagger\phi_1\phi_2)\frac{e_2}{2m_1m_2}\Big(\frac{(2\mathbf{p}_1+\mathbf{q})\cdot\mathbf{A}}{\mathbf{q}^2} + ic_F\frac{\mathbf{S}_1\cdot\mathbf{q}\times\mathbf{A}}{\mathbf{q}^2} - \frac{(2\mathbf{p}_1+\mathbf{q})\cdot\mathbf{q}\mathbf{A}\cdot\mathbf{q}}{\mathbf{q}^4}\Big)$$

$$= e_1e_2(\phi_2^\dagger\phi_1^\dagger\phi_1\phi_2)\frac{e_2}{2m_1m_2}\Big(\frac{2\mathbf{p}_1\cdot\mathbf{A}}{\mathbf{q}^2} + ic_F\frac{\mathbf{S}_1\cdot\mathbf{q}\times\mathbf{A}}{\mathbf{q}^2} - \frac{(2\mathbf{p}_1\cdot\mathbf{q})(\mathbf{A}\cdot\mathbf{q})}{\mathbf{q}^4}\Big)$$

The total contribution to the interaction potential from these diagrams would then be

$$\begin{aligned} U_2(\mathbf{p_1},\mathbf{p_2},\mathbf{q}) \quad = \quad & e_1e_2\Big\{-ic_s\frac{e_1}{4m_1^2}\frac{\mathbf{S_1}\cdot\mathbf{A}\times\mathbf{q}}{\mathbf{q}^2} + id_S\frac{e_2}{4m_2^2}\frac{\mathbf{S_2}\cdot\mathbf{A}\times\mathbf{q}}{\mathbf{q}^2} \\ & +\frac{e_1}{m_1m_2}\Big(\frac{\mathbf{p}_2\cdot\mathbf{A}}{\mathbf{q}^2} - id_F\frac{\mathbf{S}_2\cdot\mathbf{q}\times\mathbf{A}}{2\mathbf{q}^2} - \frac{(\mathbf{p}_2\cdot\mathbf{q})(\mathbf{A}\cdot\mathbf{q})}{\mathbf{q}^4}\Big) \\ & +\frac{e_2}{m_1m_2}\Big(\frac{\mathbf{p}_1\cdot\mathbf{A}}{\mathbf{q}^2} + ic_F\frac{\mathbf{S}_1\cdot\mathbf{q}\times\mathbf{A}}{2\mathbf{q}^2} - \frac{(\mathbf{p}_1\cdot\mathbf{q})(\mathbf{A}\cdot\mathbf{q})}{\mathbf{q}^4}\Big)\Big\} \end{aligned}$$

None of these terms are gauge-invariant by themselves. So we'd expect the above to agree with what we'd get if we took the terms in the potential without $\mathbf{A}$ and restored gauge invariance. That potential was:

$$\begin{aligned} U_1(\mathbf{p_1},\mathbf{p_2},\mathbf{q}) \quad = \quad & e_1e_2\Big\{\frac{1}{\mathbf{q}^2} - \frac{1}{8m_2^2}\left(d_D + d_Q Q_{2ij}\frac{q_iq_j}{\mathbf{q}^2} + 2id_S\frac{\mathbf{q}\cdot\mathbf{S_2}\times\mathbf{p_2}}{\mathbf{q}^2}\right) \\ & -\frac{1}{8m_1^2}\left(c_D + c_Q Q_{1ij}\frac{q_iq_j}{\mathbf{q}^2} - 2ic_S\frac{\mathbf{q}\cdot\mathbf{S_1}\times\mathbf{p_1}}{\mathbf{q}^2}\right) - \frac{1}{m_1m_2}\left(\frac{\mathbf{p_1}\cdot\mathbf{p_2}}{\mathbf{q}^2} - \frac{(\mathbf{p_1}\cdot\mathbf{q})(\mathbf{p_2}\cdot\mathbf{q})}{\mathbf{q}^4}\right) \\ & -\frac{i}{2m_1m_2}\frac{\mathbf{q}\cdot(d_F\mathbf{S_2}\times\mathbf{p_1} - c_F\mathbf{S_1}\times\mathbf{p_2})}{\mathbf{q}^2} - \frac{c_Fd_F}{4m_1m_2}(\mathbf{S_1}\cdot\mathbf{S_2} - \frac{(\mathbf{S_1}\cdot\mathbf{q})(\mathbf{S_2}\cdot\mathbf{q})}{\mathbf{q}^2})\Big\} \end{aligned}$$

Picking out only terms with the momentum operators $\mathbf{p}_1$, $\mathbf{p}_2$, which we'll call U_p, we find:

$$
\begin{aligned}
U_p(\mathbf{p_1}, \mathbf{p_2}, \mathbf{q}) \quad = \quad & e_1 e_2 \Big\{ \frac{1}{4m_1^2} \left(ic_S \frac{\mathbf{q} \cdot \mathbf{S_1} \times \mathbf{p_1}}{\mathbf{q}^2} \right) - \frac{1}{4m_2^2} \left(id_S \frac{\mathbf{q} \cdot \mathbf{S_2} \times \mathbf{p_2}}{\mathbf{q}^2} \right) \\
& - \frac{1}{m_1 m_2} \left(\frac{\mathbf{p_1} \cdot \mathbf{p_2}}{\mathbf{q}^2} - \frac{(\mathbf{p_1} \cdot \mathbf{q})(\mathbf{p_2} \cdot \mathbf{q})}{\mathbf{q}^4} \right) - \frac{i}{2m_1 m_2} \frac{\mathbf{q} \cdot (d_F \mathbf{S_2} \times \mathbf{p_1} - c_F \mathbf{S_1} \times \mathbf{p_2})}{\mathbf{q}^2} \Big\}
\end{aligned}
$$

If we substitute $\mathbf{p} \to \mathbf{p} - e\mathbf{A}$ we restore exactly (ignoring A^2 terms) the terms in U_2. So the position space potential in the presence of this external field may also be obtained by just the procedure of minimal substitution.

1.3 Interaction Hamiltonian

1.4 Separation of motion

In calculating the interaction above, the potential depends only upon the relative displacement $\mathbf{r}$ between the two particles. The free Hamiltonians H_1 and H_2 depend upon the particular position and momentum of the two. A transformation is needed which splits the Hamiltonian into two parts – one which corresponding to the motion of a single composite particle, the other corresponding to the internal degrees of freedom of the system.

The initial free Lagrangian (in the asbsence of some external field) is of the form

$$L = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_1 \dot{r}_2^2 - V(\mathbf{r_1} - \mathbf{r_2}) \tag{1.4.1}$$

The appropriate transformation is obtained by defining

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad\qquad \mathbf{R} = \mu_1 \mathbf{r}_1 + \mu_2 \mathbf{r}_2 \tag{1.4.2}$$

where $\mu_i = m_i / M_{\text{total}}$.

Under transformation to such coordinates, L becomes

$$L = \frac{1}{2}(m_1 + m_2) \dot{R}^2 + \frac{1}{2} m_{\text{red}} \dot{r}^2 - V(\mathbf{r}) \tag{1.4.3}$$

Since the potential V depends only upon $\mathbf{r}$, this produces two fully separate sets of terms – one depending on $\mathbf{R}$, the other on $\mathbf{r}$.

Such separation is spoiled when an external field is present. Then the Lagrangian has the form

$$L = \frac{1}{2}m_1\dot{r}_1^2 + e_1\mathbf{A}(\mathbf{r}_1)\cdot\dot{\mathbf{r}}_1 + \frac{1}{2}m_1\dot{r}_2^2 + e_1\mathbf{A}(\mathbf{r}_2)\cdot\dot{\mathbf{r}}_2 - V(\mathbf{r_1} - \mathbf{r_2}) \tag{1.4.4}$$

If the same procedure as before is attempted, it will not fully separate the Lagrangian:

$$\begin{aligned} L = {} & \frac{1}{2}(m_1+m_2)\dot{R}^2 + \frac{1}{2}m_{\mathrm{red}}\dot{r}^2 + (e_1+e_2)\mathbf{A}(\mathbf{R})\cdot\dot{\mathbf{R}} \\ & + (e_1\mu_1 - e_2\mu_2)(\mathbf{A}(\mathbf{r})\cdot\dot{\mathbf{R}} + \mathbf{A}(\mathbf{R})\cdot\dot{\mathbf{r}}) + (e_1\mu_2^2 + e_2\mu_1^2)\mathbf{A}(\mathbf{r})\cdot\mathbf{r} - V(\mathbf{r}) \end{aligned} \tag{1.4.5}$$

Even after the transformation, this includes terms which mix $\mathbf{r}$ and $\mathbf{R}$.

It can also be seen that the coordinate $\mathbf{R}$ does not describe the motion of a particle by considering the pseudomomentum in the Hamiltonian picture. The Hamiltonian corresponding to (1.4.5) is

$$H = \tag{1.4.6}$$

Moving in an external field, the momentum of a charged particle is no longer conserved. However, there is a new conserved quantity: the pseudo-momentum $\mathbf{K} = \mathbf{P} + e\mathbf{A}(\mathbf{R})$. After a transformation that truly separates the internal and external degrees of freedom, the pseudo-momentum should be conserved and thus commute with the Hamiltonian.

Bibliography