Contents

1 Introduction

This is an introductory section.

1.1 A part of the intro

more stuff

2 NRQED

We can construct an effective, nonrelativistic Lagrangian for a charged particle interacting with an electromagnetic field.

2.1 The NRQED Lagrangian

We want to construct an effective Lagrangian in the nonrelativistic limit. Our goal is to calculate the leading order corrections to the g-factor, which are corrections of order α^2. To this end, we need terms in the effective nonrelativistic Lagrangian which are equivalent corrections.

2.1.1 Order of terms

We consider constant, infinitesimal external magnetic fields, so we need only consider terms linear in $\mathbf{B}$.

The velocity of the particles in our bound state system will be $v \sim \alpha$.

The electric field we consider is the Coulomb field, so $e\Phi \sim mZ\alpha^2 \sim mv^2$, and $eE \sim m^2v^3$.

Each derivative of the electric field will add an additional factor of mv, so the operator $\mathbf{D}$ can be taken to be of this order.

We need to keep terms up to order mv^4 and $\frac{B}{m}v^2$ in order to calculate the g-factor to the necessary precision. We include mv^4 terms so we can be sure that there are no effects entering from second-order perturbation theory.

2.1.2 Constraints on the form of the Lagrangian

The Lagrangian is constrained to obey several symmetries. It must be invariant under the symmetries of parity and time reversal. It must also be invariant under Galilean transformations. The Lagrangian must also be Hermitian, and gauge invariant.

What are the gauge invariant building blocks we can use to construct this Lagrangian? We have the external fields $\mathbf{E}$ and $\mathbf{B}$, the spin operators $\mathbf{S}$, and the long derivative $\mathbf{D} = \partial - ie\mathbf{A}$. The fields should always be accompanied by the charge e of the particle.

When considering the case of higher spin particles, we might consider terms quadratic and above in spin operators. For a particle of spin s, there must be $(2s+1)^2$ independent hermitian operators. We can span this set of operators by considering products of up to $2s$ spin matrices which are symmetric and traceless in every vector index. For example, for spin-1 we have quadratic, in addition to I and S_i, five independent structures of the form $S_iS_j + S_jS_i + \delta_{ij}\mathbf{S}$.

We also have the scalar D_0, however, we need only include a single such term because we insist on having only one power of the time derivative.

To consider possible terms, we need to know how each of the above behave under the discrete transformations and Hermitian conjugate. The signs under these transformations are listed in the table below. (Also included is the imaginary number i.)

	Order	P	T	†
eE_i	m^2v^3	-	+	+
eB_i	m^2v^2	+	-	+
D_i	mv	-	+	-
D_0	mv	+	-	-
S_i	1	+	-	+
i	1	+	-	-

2.1.3 Allowed terms

Our strategy in cataloguing terms will be to first list all the combinations of E, B and D which might be allowed at a particular order, to consider the various ways of contracting these vectors, and finally to eliminate terms which do not obey the proper symmetries. We can always make a particular combination

Hermitian, and get the proper behavior under time reversal by adding a factor of i, but parity will kill several terms. Note that of the structures we can contract with, all are even under parity.

We can also insist that the Lagrangian have the expected form in the absence of external fields, which eliminates terms like $\bar{S}_{ij} D_i D_j$. The leading order terms should be of order mv^2 or $\frac{eB}{m}$. Combinations of the correct order are:

- The single D_0 term. To have the correct transformation properties this should be iD_0.
- The kinetic $\mathbf{D}^2$ term, which must be simply $\frac{\mathbf{D}^2}{2m}$
- A term with a single power of B_i. The only way to contract this is with the spin matrix, so the term will have the form $\frac{e}{m}\mathbf{S} \cdot \mathbf{B}$

All these terms are Hermitian in themselves.

So, the allowed terms at this order are:

$$iD_0, \frac{\mathbf{D}^2}{m}, \frac{e}{m}\mathbf{S} \cdot \mathbf{B}$$

The first two terms have their coefficients fixed, while we wish to honestly calculate the factor before the last.

$$\mathcal{L}_{NRQED} = \Psi^\dagger \left\{ iD_0 + \frac{\mathbf{D}^2}{2m} + c_F \frac{e}{m}\mathbf{S} \cdot \mathbf{B} \right\} \Psi \tag{2.1}$$

Are there any terms of order mv^3 or $\frac{B}{m}v$ allowed? Possible combinations are:

- Three powers of D: $D_i D_j D_k$. However, this is odd under parity, and so not allowed.
- A term with both the derivative and magnetic field: $D_i B_j$. Again, this is odd under parity and so forbidden.
- A single power of E. Again, odd under parity.

So, all such terms are foribdden by consideration of parity.

Next we consider terms of order mv^4.

- Four powers of D: fixed by the kinetic term to be $\frac{\mathbf{D}^2}{8m^3}$
- One power of E_i and one of D_j. This combination is even under parity and odd under Hermitian conjugate. There are three ways of contracting these two fields.
  - With the delta function. The allowed Hermitian term is then $\delta_{ij}(D_i E_j - E_j D_i)$.
  - With the combination $i\epsilon_{ijk} S_k$. The allowed term is $i\epsilon_{ijk}(D_i E_j S_k + S_k E_j D_i)$.
  - With the quadratic spin structure Q_{ij}: $Q_{ij}(D_i E_j - E_j D_i)$.

In the Lagrangian we'll write these as:

$$\mathcal{L}_{mv^4} = \Psi^\dagger \left\{ \frac{\mathbf{D}^4}{8m^2} + c_D \frac{e(\mathbf{D} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{D})}{8m^2} + c_Q \frac{eQ_{ij}(D_i E_j - E_i D_j)}{8m^2} + c_S \frac{ie\mathbf{S} \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D})}{8m^2} \right\} \Psi \tag{2.2}$$

Terms of order $\frac{B}{m}v^2$. The only allowed combination is $D_i D_j B_k$. We can contract two indices with each other and the third with a spin matrix in three different ways:

- $(\mathbf{S} \cdot \mathbf{B})\mathbf{D}^2 + \mathbf{D}^2(\mathbf{S} \cdot \mathbf{B})$
- $S_i D_j B_i D_j$

- $S_i(D_iB_jD_j + D_jB_jD_i)$

We can also contract all indices with a cubic spin structure:

- $\bar{S}_{ijk}(D_iD_jB_k + B_kD_jD_i)$
- $\bar{S}_{ijk}D_iB_jD_k$

In the Lagrangian we'll write these as:

$$\begin{aligned}\mathcal{L}_{Bv^2} =&\Psi^\dagger\Bigg\{c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B}+\mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3} + c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D})+(\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3}\\ &+ c_{T_1}\frac{e\bar{S}_{ijk}(D_iD_jB_k+B_kD_jD_i)}{8m^3} + c_{T_2}\frac{e\bar{S}_{ijk}D_iB_jD_k}{8m^3}\Bigg\}\Psi\end{aligned} \tag{2.3}$$

2.1.4 Full Lagrangian

The full Lagrangian we consider is then:

$$\begin{aligned}\mathcal{L}_{NRQED} =&\Psi^\dagger\Bigg\{iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^2} + c_F\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + c_D\frac{e(\mathbf{D}\cdot\mathbf{E}-\mathbf{E}\cdot\mathbf{D})}{8m^2} + c_Q\frac{eQ_{ij}(D_iE_j-E_iD_j)}{8m^2}\\ &+ c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D}}{8m^2} + c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B}+\mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3}\\ &+ c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D})+(\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3} + c_{T_1}\frac{e\bar{S}_{ijk}(D_iD_jB_k+B_kD_jD_i)}{8m^3} + c_{T_2}\frac{e\bar{S}_{ijk}D_iB_jD_k}{8m^3}\Bigg\}\Psi\end{aligned} \tag{2.4}$$

One of the features of this Lagrangian is that every coefficient is fixed by the one-photon interaction. Although some terms might represent two-photon interactions, they are terms like $\mathbf{S}\cdot\mathbf{A}\times\mathbf{E}$, whose coefficient is fixed by the gauge-invariant term $\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}$. This in turn means that we can calculate the corrections to the g-factor by considering only one-photon interactions.

2.2 Scattering off external field in NRQED

We can write down those terms in the NRQED Lagrangian which have one power of the external field. This set of terms will not, by themselves, be gauge invariant. If, for example, a coefficient exists before a term with both one and two powers of A, we'll want to make sure we get the same result in both calculations. We'll add a superscript to the coefficient to keep track of this: so below we write c_S^1.

In writing the expansion of terms like $\mathbf{D}^4$ it is convenient to use anticommutators.

$$\begin{aligned}\mathcal{L}_A &= \Psi^\dagger(-eA_0 - ie\frac{\{\nabla_i, A_i\}}{2m} - ie\frac{\{\boldsymbol{\nabla}^2,\{\nabla_i,A_i\}\}}{8m^3} + c_Fe\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{e(\boldsymbol{\nabla}\cdot\mathbf{E}-\mathbf{E}\cdot\boldsymbol{\nabla})}{8m^2} + c_Q\frac{eQ_{ij}(\nabla_iE_j-E_i\nabla_j)}{8m^2}\\ &+c_S^1\frac{ie\mathbf{S}\cdot(\boldsymbol{\nabla}\times\mathbf{E}-\mathbf{E}\times\boldsymbol{\nabla})}{8m^2} + c_{W_1}\frac{e[\boldsymbol{\nabla}^2(\mathbf{S}\cdot\mathbf{B})+(\mathbf{S}\cdot\mathbf{B})\boldsymbol{\nabla}^2]}{8m^3} - c_{W_2}\frac{e\nabla^i(\mathbf{S}\cdot\mathbf{B})\nabla^i}{4m^3} + c_{p'p}\frac{e[(\mathbf{S}\cdot\boldsymbol{\nabla})(\mathbf{B}\cdot\boldsymbol{\nabla})+(\mathbf{B}\cdot\boldsymbol{\nabla})(\mathbf{S}\cdot\boldsymbol{\nabla}}{8m^3}\end{aligned}$$

We want to calculate from this a particular process: scattering off an external field, with incoming momentum $\mathbf{p}$, outgoing $\mathbf{p}'$, and $\mathbf{q} = \mathbf{p}' - \mathbf{p}$. There is one diagram associated with each term above, but the total amplitude is just going to be the sum of all these one-photon vertices. These of course can just be read off directly from the Lagrangian: we replace the fields Ψ with the spinors ϕ, and any operator $\boldsymbol{\nabla}$ acting will become $i\mathbf{p}$ if it acts on the right, $i\mathbf{p}'$ if it is to the left.

We can simplify some expressions involving $\boldsymbol{\nabla}$ and $\mathbf{E}$: Because Q_{ij} is symmetric:

$$Q_{ij}(\nabla_iE_j - E_i\nabla_j) = Q_{ij}[\nabla_i, E_j] = Q_{ij}(\partial_iE_j)$$

And because $E_i = -\partial_i \Phi$

$$\boldsymbol{\nabla} \times \mathbf{E} - \mathbf{E} \times \boldsymbol{\nabla} = -2\mathbf{E} \times \boldsymbol{\nabla}$$

And also use that

$$\boldsymbol{\nabla} \cdot \mathbf{E} - \mathbf{E} \cdot \boldsymbol{\nabla} = (\partial_i E_i)$$

Now we can write down the scattering amplitude for scattering off the external field, before we apply any assumptions about the particular process.

$$\begin{aligned} iM &= ie\phi^\dagger \Bigg(- A_0 + \frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')}{2m} - \frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')\mathbf{p}^2 + \mathbf{p}'^2\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')}{8m^3} \\ &\quad + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2} + c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2} + c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2} \\ &\quad - c_{W_1}\frac{(\mathbf{S}\cdot\mathbf{B})(\mathbf{p}^2+\mathbf{p}'^2)}{8m^3} + c_{W_2}\frac{(\mathbf{S}\cdot\mathbf{B})(\mathbf{p}\cdot\mathbf{p}')}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p}')(\mathbf{B}\cdot\mathbf{p}) + (\mathbf{B}\cdot\mathbf{p}')(\mathbf{S}\cdot\mathbf{p})}{8m^3}\Bigg)\phi \end{aligned}$$

The above can be simplified somewhat. We choose our gauge such that $\nabla_i A_i = 0$. If we specify elastic scattering then kinematics dictate that $\mathbf{p}'^2 = \mathbf{p}^2$. Finally, if we consider $\mathbf{B}$ constant, the c_W terms become indistinguishable, since $[\nabla_i, B_j] = 0$. (It is only this last assumption that costs us any information.) Then the scattering amplitude, as calculated from $\mathcal{L}_{NRQED}$, is:

$$\begin{aligned} iM =& ie\phi^\dagger \Bigg(- A_0 + \frac{\mathbf{A}\cdot\mathbf{p}}{m} - \frac{(\mathbf{A}\cdot\mathbf{p})\mathbf{p}^2}{2m^3} + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2} + c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2} \\ & + c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2} - (c_{W_1} - c_{W_2})\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3}\Bigg)\phi \end{aligned} \tag{2.5}$$

3 General Spin Formalism

Our ultimate goal is to calculate corrections to the g-factor of a loosely bound charged particle of arbitrary spin. Our strategy is to obtain an effective Lagrangian in the nonrelativistic limit.

We first consider features of a general-spin formalism in both the relativistic and nonrelativistic cases, and the connection between the wave functions of the free particles. Then we consider how constraints of the relativistic theory let us calculate scattering off an external field. Comparing this result to that done with an effective NRQED Lagrangian, we can obtain the coefficients of that Lagrangian for particles of general spin.

3.1 Spinors for general-spin charged particles

3.1.1 Relativistic bispinors

First we need to work out a formalism that will apply to the general spin case. We want to represent the spin state of the particles by an object that looks like a generalization of the Dirac bispinor.

It is easiest to start with the Dirac basis, where the upper and lower components of the bispinor are objects of opposite helicity, each transforming as an object of spin 1/2.

To that end define an object

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix}\xi \\ \eta\end{pmatrix} \tag{3.1}$$

that we wish to have the appropriate properties. Each component should transform as a particle of spin s, but with opposite helicity. Under reflection the upper and lower components transform into each other.

The irreducible representations of the proper Lorentz group are spinors which are separately symmetric in dotted and undotted indices. The spin of the particle will be half the total number of indices. So if ξ is an object with p undotted and q dotted indices

$$\xi = \{\xi^{\alpha_1\ldots\alpha_p}_{\dot\beta_1\ldots\dot\beta_q}\} \tag{3.2}$$

Then this is a representation of a particle of spin $s = (p+q)/2$.

We have some free choice in how to partition the dotted/undotted indices, and we cannot choose exactly the same scheme for all spin as long as both types of indices are present. However, we can make separately consistent choices for integral and half-integral spin. For integral spin we can say $p = q = s$, while for the half-integral case we'll choose $p = s + \frac{1}{2}$, $q = s - \frac{1}{2}$.

We want the ξ and η to transform as objects of opposite helicity. Under reflection they will transform into each other. So

$$\eta = \{\eta^{\beta_1\ldots\beta_q}_{\dot\alpha_1\ldots\dot\alpha_p}\} \tag{3.3}$$

In the rest frame of the particle, they will have clearly defined and identical properties under rotation. The rest frame spinors are equivalent to rank $2s$ nonrelativistic spinors. So the bispinor in the rest frame looks like

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix}\xi_0 \\ \xi_0\end{pmatrix} \tag{3.4}$$

where

$$\xi_0 = \{(\xi_0)_{\alpha_1\ldots\alpha_p\beta_1\ldots\beta_q}\} \tag{3.5}$$

and all indices are symmetric.

We can obtain the spinors in an arbitrary frame by boosting from the rest frame. The upper and lower components we have defined to have opposite helicity, and so will act in opposite ways under boost:

$$\xi = \exp\left(\frac{\boldsymbol{\Sigma}\cdot\phi}{2}\right)\xi_0, \qquad \eta = \exp\left(-\frac{\boldsymbol{\Sigma}\cdot\phi}{2}\right)\xi_0 \tag{3.6}$$

What form should the operator $\mathbf{\Sigma}$ have? Under an infinitesimal boost by a rapidity ϕ, a spinor with a single undotted index is transformed as

$$\xi_\alpha \to \xi'_\alpha = \left(\delta_{\alpha\beta} + \frac{\boldsymbol{\phi}\cdot\boldsymbol{\sigma}_{\alpha\beta}}{2}\right)\xi_\beta \tag{3.7}$$

while one with a dotted index will transform as

$$\xi_{\dot\alpha} \to \xi'_{\dot\alpha} = \left(\delta_{\dot\alpha\dot\beta} - \frac{\boldsymbol{\phi}\cdot\boldsymbol{\sigma}_{\dot\alpha\dot\beta}}{2}\right)\xi_{\dot\beta} \tag{3.8}$$

The infinitesimal transformation of a higher spin object with the first p indices undotted and the last q dotted would then be

$$\xi \to \xi' = \left(1 + \sum_{a=0}^{p}\frac{\boldsymbol{\sigma}_a\cdot\boldsymbol{\phi}}{2} - \sum_{a=p+1}^{p+q}\frac{\boldsymbol{\sigma}_a\cdot\boldsymbol{\phi}}{2}\right)\xi \tag{3.9}$$

where a denotes which spinor index of ξ is operated on.

If we define

$$\mathbf{\Sigma} = \sum_{a=0}^{p}\boldsymbol{\sigma}_a - \sum_{a=p+1}^{p+q}\boldsymbol{\sigma}_a \tag{3.10}$$

Then the infinitesimal transformations would be

$$\xi \to \xi' = \left(1 + \frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi \tag{3.11}$$

$$\eta \to \eta' = \left(1 - \frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\eta \tag{3.12}$$

So the exact transformation should be

$$\xi \to \xi' = \exp\left(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi \tag{3.13}$$

$$\eta \to \eta' = \exp\left(-\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\eta \tag{3.14}$$

Therefore, the bispinor of some particle boosted by $\boldsymbol{\phi}$ from rest will be

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \exp\left(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi_0 \\ \exp\left(\frac{-\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi_0 \end{pmatrix} \tag{3.15}$$

In dealing with the relativistic theory, we'll want a basis that separates the particle and antiparticle parts of the wave function. If we want the upper component to be the particle, then in the rest frame the lower component will vanish, and for low momentum will be small compared to the upper component. The unitary transformation which accomplishes this is

$$\Psi' = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

$$\phi = \frac{1}{\sqrt{2}}(\xi + \eta)$$

$$\chi = \frac{1}{\sqrt{2}}(\eta - \xi)$$

Which is equivalent to

$$\Psi' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\Psi$$

Then,

$$\phi = \cosh\left(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi_0$$

$$\chi = \sinh\left(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi_0$$

3.1.2 Spinors for nonrelativistic theory

We also need to discuss the nonrelativistic, single-particle theory. This is much simpler: the spin state of particles in this theory is represented by symmetric spinors with $2s+1$ undotted indices. The only operators we need to consider acting on this space are spin matrices and products of spin matrices.

3.1.3 Connection between the spinors of the two theories

In the rest frame, there are two independent bispinors which represent particle and antiparticle states:

$$\Psi = \begin{pmatrix}\xi_0\\ 0\end{pmatrix}$$

or

$$\Psi = \begin{pmatrix}0\\ \xi_0\end{pmatrix}$$

However, when we consider a particle with zero momentum it is not the case that the upper component of the bispinor can be directly associated with the Schrodinger like wave-function of the particle — for instance, it would not be correctly normalized, for there is some mixing with the lower component.

We can obtain a relation between ξ_0 and the Schrodinger amplitude ϕ_s by considering the current density at zero momentum transfer. For ϕ_s it will be $j_0 = e\phi_s^\dagger\phi$. For the relativistic theory we have, as calculated above:

$$j^0 = e\frac{p^0+p'^0}{2m}\bar{\Psi}^\dagger\Psi + F_m\frac{q_\nu}{2m}\bar{\Psi}^\dagger T^{0\nu}\Psi$$

At $q=0$ the expression simplifies

$$j^0(q=0) = e\frac{p_0}{m}\bar{\Psi}^\dagger\Psi$$

$$= e\frac{p_0}{m}(\phi^\dagger\phi - \chi^\dagger\chi)$$

ϕ and χ are both related to the rest frame spinor ξ_0. So we can write instead

$$j^0 = e\frac{p_0}{m}\xi_0^\dagger\left\{\cosh^2(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}) - \sinh^2(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2})\right\}\xi_0 = e\frac{p_0}{m}\xi_0^\dagger\xi_0$$

where the last equality follows from the hyperbolic trig identity.

If we demand that the two current densities be equal to each other, we find

$$\frac{p_0}{m}\xi_0^\dagger\xi_0 = \phi_s^\dagger\phi_s$$

Approximating

$$\left(1+\frac{\mathbf{p}^2}{2m}\right)\xi_0^\dagger\xi_0 = \phi_s^\dagger\phi_s$$

This will hold to the necessary order if we identify

$$\xi_0 = \left(1-\frac{\mathbf{p}^2}{4m}\right)\phi_s$$

To write the relativistic bispinors in terms of ϕ_s we will also need approximations to $\cosh(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2})$ and $\sinh(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2})$. We only need the rapidity to the leading order: $\boldsymbol{\phi} \approx \mathbf{v} \approx \frac{\mathbf{p}}{m}$.

$$\cosh(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}) \approx 1 + \frac{1}{2}\left(\frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\right)^2$$

$$\sinh(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}) \approx \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}$$

The the two bispinor components are

$$\begin{aligned}\phi &\approx \left(1 + \left[\frac{1}{2}\frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\right]^2\right)\xi_0 \\ &\approx \left(1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m}\right)\phi_S \end{aligned}\tag{3.16}$$

$$\begin{aligned}\chi &\approx \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\xi_0 \\ &\approx \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\phi_S \end{aligned}\tag{3.17}$$

3.2 Electromagnetic Interaction

Knowing how the wave functions themselves behave, we want to see what that tells us about possible electromagnetic interaction. Interaction with a single electromagnetic photon should take the form

$$M = A_\mu j^\mu$$

where j^μ is the electromagnetic current.

The electromagnetic current must be built out of the particle's momenta and bilinears of the charged particle fields in such a way that they have the correct Lorentz properties. We must also demand current conservation: the equation $q_\mu j^\mu = 0$ must hold. Above we already have shown that, in the case of general spin, there exist only two such bilinears, a scalar and a tensor.

There will be two permissible terms in the current. We could consider a scalar bilinear coupled with a single power of external momenta. In order to fulfill the current conservation requirement, it should be

$$\frac{p^\mu + p'^\mu}{2m}\bar{\Psi}^\dagger\Psi$$

This will obey current conservation because $q = p' - p$, and $(p + p')\cdot(p' - p) = p^2 - p'^2 = 0$

We can also consider a tensor term contracted with a power of momenta. To fulfill current conservation, we can demand that the tensor bilinear be antisymmetric, and contract it with q:

$$\frac{q_\nu}{2m}\bar{\Psi}^\dagger\Sigma^{\mu\nu}\Psi$$

We don't need to worry about higher order tensor bilinears: they will necessitate too many powers of the external momenta.

So the most general current would look like

$$j^\mu = F_e\frac{p^\mu + p'^\mu}{2m}\bar{\Psi}^\dagger\Psi + F_m\frac{q_\nu}{2m}\bar{\Psi}^\dagger\Sigma^{\mu\nu}\Psi \tag{3.18}$$

In general the form factors might have quite complicated dependence on q, but these corrections will be too small compared to the type of result we're interested in. At leading order F_e will just be the electric charge of the particle in question, and F_m will, as we'll see after connecting this result to the nonrelativistic limit, be related to the particle's g-factor. So to the order we need, we can write the current as

$$j^\mu = e\frac{p^\mu + p'^\mu}{2m}\bar{\Psi}^\dagger\Psi + eg\frac{q_\nu}{2m}\bar{\Psi}^\dagger\Sigma^{\mu\nu}\Psi \tag{3.19}$$

This captures the essence of the interaction between a charged particle of general-spin and a single photon.

3.3 Bilinears in terms of nonrelativistic theory

The next step is to express the relativistic bilinears, built out of the bispinors Ψ, in terms of the Schrodinger like wave functions.

We have above written the bispinors in terms of ϕ_s, so we can use those identities to express the bilinears in the same manner.

3.3.1 Scalar bilinear

$$\begin{aligned}
\bar{\Psi}^\dagger(p')\Psi(p) &= \phi^\dagger\phi - \chi^\dagger\chi \\
&= \phi_s^\dagger\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p}')^2}{8m^2} - \frac{\mathbf{p}'^2}{4m^2}\right]\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m^2}\right]\phi_s - \phi_s^\dagger\left[\frac{(\mathbf{\Sigma}\cdot\mathbf{p}')(\mathbf{\Sigma}\cdot\mathbf{p})}{4m^2}\right]\phi_s \\
&= \phi_s^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{(\mathbf{\Sigma}\cdot\mathbf{p}')^2 + (\mathbf{\Sigma}\cdot\mathbf{p})^2 - 2(\mathbf{\Sigma}\cdot\mathbf{p}')(\mathbf{\Sigma}\cdot\mathbf{p})\right\}\right)\phi_s \\
&= \phi_s^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{[\mathbf{\Sigma}\cdot\mathbf{p}, \mathbf{\Sigma}\cdot\mathbf{q}] + (\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_s \\
&= \phi_s^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{[4i\epsilon_{ijk}p_i q_j S_k + (\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_s
\end{aligned}$$

3.3.2 Tensor ij component

In calculating the nonrelativistic limit of the antisymmetric tensor bilinear, we will treat the $0i$ and the ij components seperately. First let us consider $\bar{\Psi}\Sigma_{ij}\Psi$.

$$\begin{aligned}
\bar{\Psi}\Sigma_{ij}\Psi &= \bar{\Psi}(-2\epsilon_{ijk}S_k)\Psi \\
&= -2i\epsilon_{ijk}(\phi^\dagger S_k\phi - \chi^\dagger S_k\chi) \\
&= -2i\epsilon_{ijk}\Big(\phi_S^\dagger\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p}')^2}{8m^2} - \frac{\mathbf{p}'^2}{4m^2}\right]S_k\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m^2}\right]\phi_S - \phi_S^\dagger\frac{(\mathbf{\Sigma}\cdot\mathbf{p}')S_k(\mathbf{\Sigma}\cdot\mathbf{p})}{4m^2}\phi_S\Big) \\
&= -2i\epsilon_{ijk}\phi_S^\dagger\left\{S_k\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{1}{8m^2}\Big[(\mathbf{\Sigma}\cdot\mathbf{p}')^2 S_k + S_k(\mathbf{\Sigma}\cdot\mathbf{p})^2 - 2(\mathbf{\Sigma}\cdot\mathbf{p}')S_k(\mathbf{\Sigma}\cdot\mathbf{p})\Big]\right\}\phi_S
\end{aligned}$$

We want to write the terms in square brackets explicitly in terms of $\mathbf{p}$ and $\mathbf{q}$.

$$\begin{aligned}
(\mathbf{\Sigma}\cdot\mathbf{p}')^2 S_k + S_k(\mathbf{\Sigma}\cdot\mathbf{p})^2 - 2(\mathbf{\Sigma}\cdot\mathbf{p}')S_k(\mathbf{\Sigma}\cdot\mathbf{p}) &= (\mathbf{\Sigma}\cdot\mathbf{p})^2 S_k + S_k(\mathbf{\Sigma}\cdot\mathbf{p}) - 2(\mathbf{\Sigma}\cdot\mathbf{p})S_k(\mathbf{\Sigma}\cdot\mathbf{p}) \\
&\quad + \{(\mathbf{\Sigma}\cdot\mathbf{p})(\mathbf{\Sigma}\cdot\mathbf{q}) + (\mathbf{\Sigma}\cdot\mathbf{q})(\mathbf{\Sigma}\cdot\mathbf{p})\}S_k \\
&\quad - 2(\mathbf{\Sigma}\cdot\mathbf{q})S_k(\mathbf{\Sigma}\cdot\mathbf{p}) + (\mathbf{\Sigma}\cdot\mathbf{q})^2 S_k
\end{aligned}$$

We can express many of these terms as commutators

$$= \mathbf{\Sigma}\cdot\mathbf{p}[\mathbf{\Sigma}\cdot\mathbf{p}, S_k] + [S_k, \mathbf{\Sigma}\cdot\mathbf{p}]\mathbf{\Sigma}\cdot\mathbf{p} + 2\mathbf{\Sigma}\cdot\mathbf{q}[\mathbf{\Sigma}\cdot\mathbf{p}, S_k] - [\mathbf{\Sigma}\cdot\mathbf{q}, S_k]\mathbf{\Sigma}\cdot\mathbf{p} + (\mathbf{\Sigma}\cdot\mathbf{q})^2 S_k$$

$$= i\epsilon_{ijk}p_j\{(\mathbf{\Sigma}\cdot\mathbf{p})\Sigma_i - \Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})\} + 2i\epsilon_{ijk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_i p_j - \Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})q_j)\} + (\mathbf{\Sigma}\cdot\mathbf{q})^2 S_k$$

$$= i\epsilon_{ijk}p_j[(\mathbf{\Sigma}\cdot\mathbf{p}), \Sigma_i] + 2i\epsilon_{ijk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_i p_j - \Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})q_j)\} + (\mathbf{\Sigma}\cdot\mathbf{q})^2 S_k$$

$$= 4(\mathbf{p}^2 S_k - (\mathbf{S}\cdot\mathbf{p})p_k) + 2i\epsilon_{ijk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_i p_j - \Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})q_j)\} + (\mathbf{\Sigma}\cdot\mathbf{q})^2 S_k$$

Thus the whole bilinear is

$$-2i\epsilon_{ijk}\phi_S^\dagger \left\{ S_k\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{1}{8m^2}\Big[4(\mathbf{p}^2 S_k - (\mathbf{S}\cdot\mathbf{p})p_k) + 2i\epsilon_{\ell mk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_\ell p_m - \Sigma_\ell(\mathbf{\Sigma}\cdot\mathbf{p})q_m)\} + (\mathbf{\Sigma}\cdot\mathbf{q})^2 S_k\Big]\right\}\phi_S \tag{3.20}$$

3.3.3 Tensor Σ_{0i} component

We calculate $\bar{\Psi}\Sigma_{0i}\Psi$.

$$\bar{\Psi}\Sigma_{0i}\Psi = \bar{\Psi}\begin{pmatrix} 0 & \Sigma_i \\ \Sigma_i & 0 \end{pmatrix}\Psi$$

$$= \phi^\dagger\Sigma_i\chi - \chi^\dagger\Sigma_i\phi$$

We'll only need ϕ and χ to first order here.

$$= \phi_S^\dagger\left(\frac{\Sigma_i\Sigma_j p_j - \Sigma_j\Sigma_i p'j}{2m}\right)\phi_S$$

Using $p' = p + q$ the terms involving only p can be simplified using the commutator of Σ matrices.

$$= \phi_S^\dagger\left(\frac{4i\epsilon_{ijk}p_j S_k - \Sigma_j\Sigma_i q_j}{2m}\right)\phi$$

3.4 Current in terms of nonrelativistic wave functions

We derived the four-current (3.18) above; in nonrelativistic notation it is:

$$j_0 = F_e\frac{p_0 + p_0'}{2m}\bar{\Psi}^\dagger\Psi - F_m\frac{q_j}{2m}\bar{\Psi}^\dagger\Sigma^{0j}\Psi \tag{3.21}$$

$$j_i = F_e\frac{p_i + p_i'}{2m}\bar{\Psi}^\dagger\Psi - F_m\frac{q_j}{2m}\bar{\Psi}^\dagger\Sigma^{ij}\Psi + F_m\frac{q_0}{2m}\bar{\Psi}^\dagger\Sigma^{i0}\Psi \tag{3.22}$$

We have expressions for the bilinears in terms of the nonrelativistic wave functions ϕ_S, so it is fairly straight forward to apply them here. The calculation of j_0 is straightforward:

$$\begin{aligned}
F_e\frac{p_0 + p_0'}{2m}\bar{\Psi}^\dagger\Psi &= F_e\left(1 + \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right)\phi_S^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{4i\epsilon_{ijk}p_i q_j S_k + (\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_S \\
&\approx F_e\phi_S^\dagger\left(1 + \frac{1}{8m^2}\left\{4i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q} + (\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_S \\
F_m\frac{q_j}{2m}\bar{\Psi}^\dagger\Sigma^{0j}\Psi &= F_m\frac{q_i}{2m}\phi_S^\dagger\left(\frac{4i\epsilon_{ijk}p_j S_k - \Sigma_j\Sigma_i q_j}{2m}\right)\phi_S \\
&= F_m\phi_S^\dagger\left(\frac{4i\mathbf{S}\cdot\mathbf{q}\times\mathbf{p} - (\mathbf{\Sigma}\cdot\mathbf{q})^2}{4m^2}\right)\phi_S
\end{aligned}$$

It turns out that both terms here have the same form, so combining them we get

$$j_0 = \phi_S^\dagger \left(F_e + \frac{F_e + 2F_m}{8m^2} \left\{ 4i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q} + (\mathbf{\Sigma}\cdot\mathbf{q})^2 \right\} \right) \phi_S \tag{3.23}$$

To calculate j_i we want to first simplify things by considering the constraints of our particular problem. The term with $\Sigma_i j$ can be simplified by dropping terms with more than one power of q; these will turn into derivatives of the magnetic field, and our problem concerns only a constant field. Further, we need only calculate elastic scattering, and so $q_0 = 0$. With those simplifications

$$\bar{\Psi}\Sigma_{ij}\Psi \approx -2i\epsilon_{ijk}\phi_S^\dagger \left\{ S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S}\cdot\mathbf{p})p_k}{2m^2} \right\} \phi_S$$

$$\begin{aligned} F_e \frac{p_i + p_i'}{2m} \bar{\Psi}^\dagger\Psi &= F_e \frac{p_i + p_i'}{2m} \phi_S^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2} \left\{ 4i\epsilon_{\ell jk} p_\ell q_j S_k + (\mathbf{\Sigma}\cdot\mathbf{q})^2 \right\} \right) \phi_S \\ &\approx F_e \frac{p_i + p_i'}{2m} \phi_S^\dagger \left(1 + \frac{1}{8m^2} \left\{ 4i\epsilon_{\ell jk} p_\ell q_j S_k \right\} \right) \phi_S \\ F_m \frac{q_j}{2m} \bar{\Psi}^\dagger \Sigma^{ij}\Psi &= -F_m \frac{i\epsilon_{ijk} q_j}{m} \phi_S^\dagger \left\{ S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S}\cdot\mathbf{p})p_k}{2m^2} \right\} \phi_S \end{aligned}$$

So the full spatial part of the current is

$$j_i = \phi_S^\dagger \left\{ F_e \frac{p_i + p_i'}{2m} \left(1 + \frac{i\epsilon_{\ell jk} p_\ell q_j S_k}{2m^2} \right) + F_m \frac{i\epsilon_{ijk} q_j}{m} \left(S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S}\cdot\mathbf{p})p_k}{2m^2} \right) \right\} \phi_S \tag{3.24}$$

3.5 Scattering off external field

To compare to the NRQED Lagrangian, we want to calculate scattering off an external field for an arbitrary spin particle. We already have the current, so the scattering is just

$$M = e j_\mu A^\mu = e j_0 A_0 - e\mathbf{j}\cdot\mathbf{A}$$

Above we have expressions for both j_0 (3.23) and $\mathbf{j}$ (3.24). So we can write down the parts of the amplitude directly:

$$\begin{aligned} e j_0 A_0 =& e A_0 \phi_S^\dagger \left(F_e + \frac{F_e + 2F_m}{8m^2} \left\{ 4i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q} + (\mathbf{\Sigma}\cdot\mathbf{q})^2 \right\} \right) \phi_S \\ e\mathbf{j}\cdot\mathbf{A} =& A_i \phi_S^\dagger \left\{ F_e \frac{p_i + p_i'}{2m} \left(1 + \frac{i\epsilon_{\ell jk} p_\ell q_j S_k}{2m^2} \right) + F_m \frac{i\epsilon_{ijk} q_j}{m} \left(S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S}\cdot\mathbf{p})p_k}{2m^2} \right) \right\} \phi_S \end{aligned} \tag{3.25}$$

As much as possible we want to express the result in terms of gauge invariant quantities $\mathbf{B}$ and $\mathbf{E}$. We write the relations between these fields and A_μ in position space and the equivalent equation in momentum space.

$$\begin{aligned} \mathbf{B} = \mathbf{\nabla}\times\mathbf{A} \rightarrow i\mathbf{q}\times\mathbf{A} \\ \mathbf{E} = -\mathbf{\nabla}A_0 \rightarrow -i\mathbf{q}A_0 \end{aligned}$$

There is one term above that can only be put into gauge-invariant form by considering the kinematic constraints of elastic scattering. If the scattering is elastic, we have $\mathbf{q}\cdot(\mathbf{p}' + \mathbf{p}) = \mathbf{p}'^2 - \mathbf{p}^2 = 0$. We can use this identity on the term $q_j(p_i' + p_i)A_i$ as follows:

$$\begin{aligned} \epsilon_{ijk} B_k &= \partial_i A_j - \partial_j A_i = i(q_i A_j - q_j A_i) \\ (p_i + p_i')\epsilon B_k &= i(p_i + p_i')(q_i A_j - q_j A_i) \\ &= -i(p_i + p_i') q_j A_i \end{aligned}$$

So we have the identity

$$i(p_i + p_i')q_j A_i = -\epsilon_{ijk} B_k (p_i + p_i') \tag{3.26}$$

Now we can write each term involving q in terms of position space quantities.

$$\begin{aligned}
i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q}A_0 &= -\mathbf{S}\cdot\mathbf{p}\times\mathbf{E} \\
(\boldsymbol{\Sigma}\cdot\mathbf{q})^2 A_0 &= \Sigma_i\Sigma_j q_i q_j A_0 \\
&= \Sigma_i\Sigma_j \partial_i E_j \\
i\epsilon_{ijk}A_i q_j &= -i(\mathbf{q}\times\mathbf{A})_k \\
&= = -B_k \\
A_i(p_i+p_i')i\epsilon_{\ell jk}p_\ell q_j S_k &= \epsilon_{\ell jk}p_\ell S_k i(p_i+p_i')q_j A_i \\
&= -\epsilon_{\ell jk}p_\ell S_k\{\epsilon_{ijm}B_m(p_i+p_i')\} \\
&= -(\delta_{\ell i}\delta_{km} - \delta\ell m\delta_{ik})p_\ell S_k\{\epsilon_{ijm}B_m(p_i+p_i')\} \\
&= 2\{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p}) - (\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2\}
\end{aligned}$$

Using these

$$ej_0A_0 = e\phi_S^\dagger\left\{A_0 + \frac{1-2F_2}{8m^2}\left(4\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \Sigma_i\Sigma_j\partial_i E_j\right)\right\}$$

$$e\mathbf{j}{\cdot}\mathbf{A} = e\phi_S^\dagger\left\{\frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p}) - (\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2}{m^2} - F_m\left(\frac{\mathbf{S}\cdot\mathbf{B}}{m}\left\{1 - \frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right\} + \frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p}) - (\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2}{2m^2}\right)\right\}\phi$$

$$= e\phi_S^\dagger\left\{\frac{\mathbf{p}\cdot\mathbf{A}}{m} + [1-2F_m]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{F_m}{m}\mathbf{S}\cdot\mathbf{B}\right\}$$

From this we can see that our F_m is actually $g/2$, so in such terms

$$ej_0A_0 = e\phi_S^\dagger\left\{A_0 - \frac{g-1}{2m^2}\left(\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \frac{1}{4}\Sigma_i\Sigma_j\partial_i E_j\right)\right\}$$

$$e\mathbf{j}\cdot\mathbf{A} = e\phi_S^\dagger\left\{\frac{\mathbf{p}\cdot\mathbf{A}}{m} - [g-1]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{g}{2m}\mathbf{S}\cdot\mathbf{B}\right\}$$

So the entire scattering process is

$$e\phi_S^\dagger\left\{A_0 - \frac{g-1}{2m^2}\left(\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \frac{1}{4}\Sigma_i\Sigma_j\partial_i E_j\right)\frac{\mathbf{p}\cdot\mathbf{A}}{m} - [g-1]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{g}{2m}\mathbf{S}\cdot\mathbf{B}\right\}\phi_S \tag{3.27}$$

3.6 Comparison with relativistic result

Having calculated the same process in both the relativistic theory and in our NRQED effective theory, we can compare the two amplitudes and fix the coefficients.

The NRQED amplitude (2.5) is

$$\begin{aligned}
iM &= ie\phi^\dagger\Bigg(-A_0 + \frac{\mathbf{A}\cdot\mathbf{p}}{m} - \frac{(\mathbf{A}\cdot\mathbf{p})\mathbf{p}^2}{2m^3} + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2} + c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2} \\
&\quad + c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2} - (c_{W_1} - c_{W_2})\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3}\Bigg)\phi
\end{aligned}$$

While the relativistic amplitude was

$$iM_{REL} = -ie\phi^\dagger \Big(A_0 - \frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{\mathbf{p}\cdot\mathbf{A}\mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3}\{\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - S_iS_j\boldsymbol{\nabla}_iE_j\} - g\frac{1}{2m}\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\Big)\phi$$

We should rewrite term $\boldsymbol{\nabla}\cdot\mathbf{E} - S_iS_j\boldsymbol{\nabla}_iE_j$ using the quadropole moment tensor $Q_{ij} = \frac{1}{2}(S_iS_j + S_jS_i - \frac{2}{3}\mathbf{S}^2)$.

Remember that ∇_iE_j is actually symmetric under exchange of i and j. Then we can write

$$S_iS_j\nabla_iE_j = \frac{1}{2}(S_iS_j + S_jS_i) = (Q_{ij} + \frac{1}{3}\mathbf{S}^2\delta_{ij})\nabla_iE_j$$

$$= Q_ij\nabla_iE_j + \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{E}$$

Written using this identity, the relativistic amplitude is

$$iM_{REL} = -ie\phi^\dagger \Big(A_0 - \frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{\mathbf{p}\cdot\mathbf{A}\mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3}\{\frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - Q_{ij}\boldsymbol{\nabla}_iE_j\} - g\frac{1}{2m}\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\Big)\phi$$

Comparing the two, the coefficients are:

$$\begin{aligned} c_F &= g \\ c_D &= \frac{4(g-1)}{3} \\ c_Q &= -4(g-1) \\ c_S^1 &= 2(g-1) \\ (c_{W_1} - c_{W_2}) &= 2 \\ c_{p'p} &= (g-2) \end{aligned}$$