Chapter 1

Spin one-half

1.0.1 Conventions

Spinors:

$$u = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \tag{1.0.1}$$

$$\begin{aligned} \eta &= \left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w \\ \xi &= \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2m} \left(1 - \frac{3\mathbf{p}^2}{8m^2}\right) w \end{aligned}$$

structures (in the useful represenation)

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \tag{1.0.2}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{1.0.3}$$

$$\sigma^{\mu\nu} = i\frac{1}{2}[\gamma^\mu, \gamma^\nu] \tag{1.0.4}$$

$$[\gamma^0, \gamma^i] = 2\gamma^0\gamma^i = 2\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{1.0.5}$$

$$\gamma^i\gamma^j = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_i\sigma_j & 0 \\ 0 & -\sigma_i\sigma_j \end{pmatrix} \tag{1.0.6}$$

$$[\gamma^i, \gamma^j] = \begin{pmatrix} [\sigma_j, \sigma_i] & 0 \\ 0 & [\sigma_j, \sigma_i] \end{pmatrix} = i\epsilon_{jik}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} = -i\epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \tag{1.0.7}$$

1.0.2 QED calculations

First form:

$$(p + p')^\mu \bar{u}u = (p + p')^\mu \left(\eta^\dagger\eta - \xi^\dagger\xi\right) \tag{1.0.8}$$

$$= (p + p')^\mu \left\{ w^\dagger \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right)\left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w - w^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{2m}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\right) w \right\} \tag{1.0.9}$$

$$= (p + p')^\mu w^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{8m^2} - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2}\right) w \tag{1.0.10}$$

Second form: For the term $\bar{u}\gamma^\mu u$ it'll be necessary to treat the spatial/time-like indices separately.

time-like

$$\bar{u}\gamma^0 u = u^\dagger u \tag{1.0.11}$$

$$= \eta^\dagger\eta + \xi^\dagger\xi \tag{1.0.12}$$

$$= w^\dagger \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right)\left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w + w^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{2m}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\right) w \tag{1.0.13}$$

$$= w^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{8m^2} + \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2}\right) w \tag{1.0.14}$$

spatial

$$\bar{u}\gamma^i u = u^\dagger \gamma^0 \gamma^i u \tag{1.0.15}$$

$$= \bar{u}^\dagger \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} u \tag{1.0.16}$$

$$= \eta^\dagger \sigma_i \xi + \xi^\dagger \sigma_i \eta \tag{1.0.17}$$

$$= w^\dagger \left\{ \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right) \sigma_k \left(1 - \frac{3\mathbf{p}^2}{8m^2}\right) - \left(1 - \frac{3\mathbf{p}'^2}{8m^2}\right) \sigma_k \left(1 - \frac{\mathbf{p}^2}{8m^2}\right) - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_k\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2} \right\} \tag{1.0.18}$$

Third type (tensor)

$$\bar{u}\frac{i}{2m} q_j \sigma^{ij} u = \frac{i\epsilon_{ijk}q_j}{2m} \bar{u} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} u \tag{1.0.19}$$

$$\frac{i\epsilon_{ijk}q_j}{2m} \left(\eta^\dagger \sigma_k \eta - \xi^\dagger \sigma_k \xi\right) \tag{1.0.20}$$

$$\frac{i\epsilon_{ijk}q_j}{2m} w^\dagger \left\{ \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right) \sigma_k \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right) - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_k\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2} w \right\} \tag{1.0.21}$$

Need triple sigma identity

$$\sigma_a\sigma_b\sigma_c = \sigma_a(\delta_{bc} + i\epsilon_{bcd}\sigma_d) = \sigma_a\delta_{bc} - \sigma_b\delta_{ca} + \sigma_c\delta_{ab} + i\epsilon_{abc} \tag{1.0.22}$$

Then using above

$$\bar{u}\frac{i}{2m}q_j\sigma^{ij}u = \frac{i\epsilon_{ijk}q_j}{2m}w^\dagger\left\{\sigma_k\left(1-\frac{\mathbf{p}'^2+\mathbf{p}^2}{8m^2}\right)-\frac{\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{p}')p_k-\sigma_k\mathbf{p}\cdot\mathbf{p}'+i\epsilon_{akc}q_ap_c}{4m^2}\right\}w \tag{1.0.23}$$

The 'time-like' part of the tensor term

$$\bar{u}\frac{i}{2m}q_j\sigma^{0j}u = -\frac{q_j}{2m}\bar{u}\gamma^0\gamma^j u \tag{1.0.24}$$

$$= -\frac{q_j}{2m}u^\dagger\gamma^j u \tag{1.0.25}$$

$$= -\frac{q_j}{2m}\left(\eta^\dagger\sigma_j\chi-\chi^\dagger\sigma_j\eta\right) \tag{1.0.26}$$

$$= -\frac{q_j}{2m}w^\dagger\left\{\left(1-\frac{\mathbf{p}'^2}{8m^2}\right)\frac{\sigma_j\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\left(1-\frac{3\mathbf{p}^2}{8m^2}\right)-\left(1-\frac{3\mathbf{p}'^2}{8m^2}\right)\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_j}{2m}\left(1-\frac{\mathbf{p}^2}{8m^2}\right)\right\}w \tag{1.0.27}$$

Dropping terms quadratic in q, all p' can be written just as p.

$$\approx -\frac{q_j}{2m}w^\dagger\left\{\frac{\sigma_j\boldsymbol{\sigma}\cdot\mathbf{p}-\boldsymbol{\sigma}\cdot\mathbf{p}\sigma_j}{2m}\left(1-\frac{\mathbf{p}^2}{2m^2}\right)\right\}w \tag{1.0.28}$$

$$= \frac{q_j}{2m}w^\dagger\left\{\frac{i\epsilon_{ijk}\sigma_kp_i}{2m}\left(1-\frac{\mathbf{p}^2}{2m^2}\right)\right\}w \tag{1.0.29}$$

$$= w^\dagger\left\{\frac{i\epsilon_{ijk}p_iq_j\sigma_k}{4m^2}\left(1-\frac{\mathbf{p}^2}{2m^2}\right)\right\}w \tag{1.0.30}$$

1.1 Fouldy-Wouthyusen approach