Chapter 1

Arbitrary spin: Derivation of nonrelativistic Hamiltonian

Above, two particular relativistic theories have been considered: those sectors of QED involving spin half and spin one particles. A nonrelativistic theory for each was derived in two ways, each starting from the relativistic Lagrangian; once by solving the equations of motion, the other by comparing scattering amplitudes so as to determine NRQED coefficients.

Although the two nonrelativistic Lagrangians do differ, it turns out that all terms contributing to the bound g-factor coincide, at least at the second order in α^2. This offers the hope that this remains true for all charged particles, no matter their spin.

The obvious obstacle is that in the previous chapters a specific, known relativistic Lagrangian was the starting point. However, a key point is that, in the NRQED approach, only the one-photon vertex for the charged particle was necessary to fix all relevant NRQED coefficients. If this holds true for the case of general spin, then only that portion of a relativistic Lagrangian representing the one photon interaction need be considered. And there are enough constraints on this interaction to render the problem quite tractable.

First, the form of the nonrelativistic Lagrangian and fields will be constructed for general spin. Then the relativistic fields will be treated similarly. Finally, by considering physical results from both theories, the coefficients will be fixed for the general spin Lagrangian.

1.1 The nonrelativistic Lagrangian

Spinors for nonrelativistic theory

The first question is how to represent the fields in a nonrelativistic theory. A single particle of spin s has $2s + 1$ degrees of freedom. A regular spinor of rank $2s$ has the correct transformation properties under rotation, and by considering only such spinors which are symmetric in all indices, the necessary number of degrees of freedom is obtained.

Building the NRQED Lagrangian

Previously, NRQED Lagrangians were developed for spin half and spin one. They included all the terms that could arise at the required order for such theories. As discussed in developing the Lagrangian for spin one, new terms will arise because the increasing degrees of spin freedom allow a larger set of spin operators. So to describe the NREQED Lagrangian for general spin, a complete description of spin space operators are needed.

Properties of spin operators

In formulating NRQED for general spin particles, we need to consider all the possible operators might show up in the Lagrangian. The state-space of a spin-s particle is the direct product of it's spin-state and all the other state information. Because the spaces are orthogonal, we can treat separately operators in the two spaces. The operators and fields which exist in position space are the same for a particle of any spin, but unsurpsingly the operators allowed in spin space do depend upon the spin of the particle. As the spin of the particle is increased, and thus its spin degrees of freedom rise, there are more ways to mix these components, and thus a greater number of spin operators to consider.

For a particular representation, we can always write a bilinear as the spin operators and other operators acting between two spinors:

$$\Psi^\dagger \mathcal{O}_S \mathcal{O}_X \Psi \tag{1.1.1}$$

The two types of operators will always commute, since they act on orthogonal spaces, so it doesn't matter what order they're written in. All such bilinears must be Galilean invariant, but individual operators might not be. The non-spin operators we consider, such as $\mathbf{D}$, $\mathbf{B}$ or contractions with the tensor ϵ_{ijk} are already all written as 3-vectors or (in combination) as higher rank tensors. Therefore, it will be most convenient to write spin-operators with well defined properties under Galilean transformations. In that way, writing Galilean-invariant combinations of the two types of operators is done just by contracting indices.

Even though the number of spin operators does depend upon the spin of the particle, it is still possible to proceed in such a way that the same notation may be used no matter the spin. There are a few requirements:

- We write all high spin operators in terms of combinations of S_i, since these have universal properties regardless of the representation they are written in.
- If an operator exists and is non-zero in the representation of spin-s, it also exists in spin-$s+1$
- All operators introduced to account for the additional degrees of freedom in higher spin representations vanish when written in a lower spin theory. (As an example of the last point, the operator $S_iS_j + S_jS_i - \delta_{ij}S^2$ is needed to account for the degrees of freedom in a spin-1 theory, but vanishes in spin-1/2.)

If these requirements are met a consistent spin-agnostic notation can be adopted. Now we attempt to construct operators that meet these conditions.

The spinors Ψ are written with $2s+1$ independent components. The spin operators will be isomorphic to matrices acting on these components, which for a spin-s particle would be $(2s+1)\times(2s+1)$ matrices. The combined operator $\mathcal{O}_S\mathcal{O}_X$ must be Hermitian, but without loss of generality we can require any $\mathcal{O}_S$, $\mathcal{O}_X$ to be Hermitian separately. So there is the additional constraint that these matrices be Hermitian, and this means a total of $(2s+1)^2$ degrees of freedom.

For spin-0 there is only one component to the spinor, so the only possible operator is equivalent to the identity.

For spin-1/2 we have, in addition to the identity, the spin matrices $s_i = \frac{1}{2}\sigma_i$. This is four independent matrices, and since the space has $(2s+1)^2 = 4$ degrees of freedom, exactly spans the space of all spin-operators. If we try to construct terms which are bilinear in spin matrices, they just reduce through the identity $\sigma_i \sigma_j = \delta_{ij} + \epsilon_{ijk}\sigma_k$, which we can already construct through combinations of the four operators we already have. Since those four operators form a basis for the space, independent bilinears were forbidden even without an explicit form for the equation.

What about spin-1? We need 9 independent operators to span the space. All the operators that exist in spin-1/2 will work here as well, though the spin matrices will have a different representation. That leaves 5 operators to construct. It is natural to try to construct these from bilinear combinations of spin matrices. Naively $S_i S_j$ would itself be 9 independent structures, but clearly some of these are expressible in terms of the lower order operators. (By the order of a spin operator we mean its greatest degree in S_i)

Regardless of their representation, the spin operators always fulfill certain identities based on their Lie group. Namely

$$S_i S_j \delta_{ij} \sim I, \ [S_i, S_j] = \epsilon_{ijk} S_k \tag{1.1.2}$$

and it is these identities which allow certain combinations of $S_i S_j$ to be related to lower order operators.

If instead of general spin bilinears we consider only combinations which are

- Symmetric in i, j
- Traceless

then such a structure will be independent of the set of operators $\{I, S_1, S_2, S_3\}$. Because it is symmetric no combination may be related using the commutator, and because it is traceless there is no combination that reduces due to the other identity.

This conditions form a set of 4 constraints, so from the original 9 degrees of freedom possessed by combinations of $S_i S_j$ are left only 5. Together with the 4 lower order operators this is exactly enough to span the space.

We can explicitly write this symmetric, traceless structure as

$$S_iS_j + S_jS_i - \delta_{ij}S^2 \tag{1.1.3}$$

Having explored how the procedure works for spin-1, move on to consider the general spin case. The idea is to proceed inductively using the same rough attack as for the case of spin-1. In addition to all the "lower order" operators which were used for lower spin representations introduce new operators which are of higher degree in the spin matrices and guaranteed to be independent of the lower spin operators.

So suppose that for a spin-$s-1$ particle we have a set of operators written as $\bar{S}^0, \bar{S}^1, \ldots \bar{S}^{(s-1)}$, where a structure $\bar{S}^n$ carries n Galilean indices and is symmetric and traceless between any pair of indices, that is:

$$\bar{S}^n_{..i..j..} = \bar{S}^n_{..j..i..}, \;\delta_{ij}\bar{S}^n_{..i..j..} = 0 \tag{1.1.4}$$

(From above, $\bar{S}^0 = I$, $\bar{S}^1_i = S_i$, and $\bar{S}^2 = S_iS_j + S_jS_i + \delta_{ij}S^2$.)

The objects $\bar{S}^n$ are built as follows: start with all combinations involving the product of exactly n spin matrices. (There are 3^n such structures.) Form them into combinations which are symmetric in all indices. Each index has three possible values, so we can label each structure by how many indices are equal to 1 and 2. If a is the number of indices equal to 1, and b the number of indices equal to 2, then for a given a there are $n+1-a$ possible choices for b. The total number of symmetric structures is then

$$\sum_{a=0}^{n}(n-a+1) = \frac{1}{2}(n+1)(n+2) \tag{1.1.5}$$

We want to apply the additional constraint that the $\bar{S}^n$ be traceless in all indices. This will involve subtracting all the lower order structures which result when the trace of the completely symmetric combinations is taken.

It introduces an additional constraint on $\overline{S}^n$ for each pair of indices, and there are $n(n-1)/2$ distinct pairs of indices. The total degrees of freedom left are

$$\frac{1}{2}(n+1)(n+2) - \frac{1}{2}n(n-1) = \frac{1}{2}\left(n^2+3n+2-n^2+n\right) = 2n+1 \tag{1.1.6}$$

In combination with the lower order spin operators, this is exactly the number of independent operators we need to span the space. Combined with the lower order operators this is a complete basis, so we know we haven't missed any terms. Because they are constructed to be independent from all the lower order operators, it must necessarily be true that they will vanish in lower spin representations.

Using this notation we can write down terms in the Lagrangian that are valid for particles of any spin. By writing all spin operators in terms of S_i they are representation agnostic, and by construction they will vanish for low spin particles where they do not "fit".

New terms arising at higher spin

With the general set of spin operators in hand, what new terms can occur in the Lagrangian involving them? Although a large number of such operators exist for high spin, only a small number need be considered for the NRQED Lagrangian. That is because only position space operators up to a certain order are considered.

The different combinations of position space operators were catalogued in describing the spin half Lagrangian. The first new spin structure to consider is $\overline{S}_{ijk}$, which has three indices. The only allowed set of position space operators that can be contracted with this operator is $\mathbf{B}, \mathbf{D}, \mathbf{D}$. Considering Hermitian combinations, two new terms can arise:

$$c_{T_1}\frac{e\overline{S}_{ijk}D_iB_jD_k}{8m^3} + c_{T_2}\frac{e\overline{S}_{ijk}(D_iD_jB_k + B_iD_jD_i)}{8m^2}. \tag{1.1.7}$$

Only one combination of position space operators existed at the needed order, $D_iD_jD_kD_\ell$. This cannot be coupled to $\overline{S}_{ijk\ell}$ because it would spoil the kinetic term. So the only new terms are those in (1.1.7). Adding these to the Lagrangian (**??**) found for spin one, the result is

$$\begin{aligned}\mathcal{L}_{NRQED} =&\psi^\dagger\Bigg\{iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^2} + c_F\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + c_D\frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2} + c_Q\frac{eQ_{ij}(D_iE_j - E_iD_j)}{8m^2} \\ &+ c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D})}{8m^2} + c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3} \\ &+ c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3} + c_{T_1}\frac{e\bar{S}_{ijk}(D_iD_jB_k + B_kD_jD_i)}{8m^3} + c_{T_2}\frac{e\bar{S}_{ijk}D_iB_jD_k}{8m^3}\Bigg\}\psi.\end{aligned} \tag{1.1.8}$$

Scattering off an external field in NRQED

To calculate scattering off an external field, that part of the Lagrangian involving only a single power of A is needed. The terms which arise for spin greater than one are the only modifications from (**??**)

$$\begin{aligned}\mathcal{L}_A =&\psi^\dagger(-eA_0 - ie\frac{\{\nabla_i, A_i\}}{2m} - ie\frac{\{\boldsymbol{\nabla}^2, \{\nabla_i, A_i\}\}}{8m^3} + c_F e\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{e(\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{E}\cdot\boldsymbol{\nabla})}{8m^2} \\ &+ c_Q\frac{eQ_{ij}(\nabla_i E_j - E_i\nabla_j)}{8m^2} + c_S^1\frac{ie\mathbf{S}\cdot(\boldsymbol{\nabla}\times\mathbf{E} - \mathbf{E}\times\boldsymbol{\nabla})}{8m^2} + c_{W_1}\frac{e[\boldsymbol{\nabla}^2(\mathbf{S}\cdot\mathbf{B}) + (\mathbf{S}\cdot\mathbf{B})\boldsymbol{\nabla}^2]}{8m^3} \\ &- c_{W_2}\frac{e\nabla^i(\mathbf{S}\cdot\mathbf{B})\nabla^i}{4m^3} + c_{p'p}\frac{e[(\mathbf{S}\cdot\boldsymbol{\nabla})(\mathbf{B}\cdot\boldsymbol{\nabla}) + (\mathbf{B}\cdot\boldsymbol{\nabla})(\mathbf{S}\cdot\boldsymbol{\nabla})}{8m^3} \\ &+ c_{T_1}\frac{e\bar{S}_{ijk}(D_iD_jB_k + B_kD_jD_i)}{8m^3} + c_{T_2}\frac{e\bar{S}_{ijk}D_iB_jD_k}{8m^3})\psi.\end{aligned} \tag{1.1.9}$$

The calculation of the scattering amplitude goes just as before, with a couple of extra terms. The result is

$$\begin{aligned}iM_{\mathrm{NR}} = ie\phi_S^\dagger\Bigg(&-A_0 + \frac{\mathbf{A}\cdot\mathbf{p}}{m} - \frac{(\mathbf{A}\cdot\mathbf{p})\mathbf{p}^2}{2m^3} + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2} + c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2} + c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2} \\ &-(c_{W_1} - c_{W_2})\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3} + (c_{T_1} + c_{T_2})\frac{e\bar{S}_{ijk}D_iB_jD_k}{8m^3}\Bigg)\phi_S.\end{aligned} \tag{1.1.10}$$

1.2 Relativistic theory for general spin particles

Relativistic bispinors

To work out a relativistic theory that describes particles of arbitrary spin, the first step will be to consider how the fields should be represented. The tactic here will be to represent the spin state by an object that looks like a generalization of the Dirac bispinor.

It is easiest to start with the chiral basis, where the upper and lower components of the bispinor are objects of opposite helicity, each transforming as an object of spin s.

To that end define an object

$$\Upsilon = \frac{1}{\sqrt{2}}\begin{pmatrix}\xi \\ \eta\end{pmatrix}, \tag{1.2.1}$$

the components of which will have the desired properties. Each component should transform as a particle of spin s, but with opposite helicity. Under reflection the upper and lower components transform into each other.

Spinors which are separately symmetric in dotted and undotted indices are representations of the proper Lorentz group. If ξ is an object with p undotted and q dotted indices

$$\xi = \{\xi^{\alpha_1 \ldots \alpha_p}_{\dot{\beta}_1 \ldots \dot{\beta}_q}\}, \tag{1.2.2}$$

then this can be a representation of a particle of spin $s = (p+q)/2$.

There is some free choice in how to partition the dotted/undotted indices — the same scheme won't work for all spin as long as both types of indices are present. However, separately consistent choices can be made for integral and half-integral spin. For integral spin let $p = q = s$, while for the half-integral case choose $p = s + \frac{1}{2}$, $q = s - \frac{1}{2}$.

ξ and η should transform as objects of opposite helicity. Under reflection they will transform into each other. So the choices made for ξ then dictate that

$$\eta = \{\eta^{\beta_1 \ldots \beta_q}_{\dot{\alpha}_1 \ldots \dot{\alpha}_p}\}. \tag{1.2.3}$$

In the rest frame of the particle, there is no helicity. Both objects will have clearly defined and identical properties under rotation. The rest frame spinors are equivalent to rank $2s$ nonrelativistic spinors. So the bispinor in the rest frame looks like

$$\Upsilon = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_0 \\ \xi_0 \end{pmatrix}, \tag{1.2.4}$$

where

$$\xi_0 = \{(\xi_0)_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q}\}, \tag{1.2.5}$$

and is symmetric in all indices.

We can obtain the spinors in an arbitrary frame by boosting from the rest frame. The upper and lower components we have defined to have opposite helicity, and so will act in opposite ways under boost by some rapidity ϕ:

$$\xi = \exp\left(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}\right)\xi_0, \qquad \eta = \exp\left(-\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}\right)\xi_0. \tag{1.2.6}$$

What form should the operator $\mathbf{\Sigma}$ have? Under an infinitesimal boost by a rapidity ϕ, a spinor with a single undotted index is transformed as

$$\xi_\alpha \to \xi'_\alpha = \left(\delta_{\alpha\beta} + \frac{\boldsymbol{\phi}\cdot\boldsymbol{\sigma}_{\alpha\beta}}{2}\right)\xi_\beta,$$

while one with a dotted index will transform as

$$\xi_{\dot\alpha} \to \xi'_{\dot\alpha} = \left(\delta_{\dot\alpha\dot\beta} - \frac{\boldsymbol{\phi}\cdot\boldsymbol{\sigma}_{\dot\alpha\dot\beta}}{2}\right)\xi_{\dot\beta}.$$

The infinitesimal transformation of a higher spin object with the first p indices undotted and the last q dotted would then be

$$\xi \to \xi' = \left(1 + \sum_{a=0}^{p}\frac{\boldsymbol{\sigma}_a\cdot\boldsymbol{\phi}}{2} - \sum_{a=p+1}^{p+q}\frac{\boldsymbol{\sigma}_a\cdot\boldsymbol{\phi}}{2}\right)\xi,$$

where a denotes which spinor index of ξ is operated on.

So if $\mathbf{\Sigma}$ is defined as

$$\mathbf{\Sigma} = \sum_{a=0}^{p}\boldsymbol{\sigma}_a - \sum_{a=p+1}^{p+q}\boldsymbol{\sigma}_a, \tag{1.2.7}$$

then the infinitesimal transformations would be

$$\xi \to \xi' = \left(1 + \frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi,$$

$$\eta \to \eta' = \left(1 - \frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\eta.$$

That is satisfied if the exact transformation are

$$\xi \to \xi' = \exp\left(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi,$$

$$\eta \to \eta' = \exp\left(-\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\eta.$$

So the bispinor of some particle boosted by $\boldsymbol{\phi}$ from rest will be

$$\Upsilon = \frac{1}{\sqrt{2}}\begin{pmatrix}\exp\left(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi_0\\ \exp\left(\frac{-\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi_0\end{pmatrix}. \tag{1.2.8}$$

The helical basis was a sensible one in which to examine the transformation properties of the bispinors. But in descending to the nonrelativistic theory, a basis that separates the particle and antiparticle parts of the field will be more convenient. If the upper component is supposed to be the particle, then in the rest frame the lower component will vanish, and for low momentum will be small compared to the upper component.

Define the new bispinor as

$$\Upsilon' = \begin{pmatrix} \phi \\ \chi \end{pmatrix}.$$

The components of this bispinor are, in terms of the old components ξ and η

$$\phi = \frac{1}{\sqrt{2}}(\xi + \eta),$$

$$\chi = \frac{1}{\sqrt{2}}(\eta - \xi).$$

And Υ' can obtained by a unitary transformation:

$$\Upsilon' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Upsilon.$$

In terms of the original rest frame spinors, the components of Υ' are

$$\phi = \cosh\left(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi_0, \tag{1.2.9}$$

$$\chi = \sinh\left(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi_0. \tag{1.2.10}$$

Properties of Σ matrices

The matrices Σ_i act on wave functions with $p+q$ indices. If we treat the two sets of indices as two seperate spaces, with spins $\frac{p}{2}, \frac{q}{2}$ respectively, we can write it as the sum of those spaces spin operators: $\Sigma_i = 2(S_i^P - S_i^Q)$. We can then write the total spin operator as $S_i^I = S_i^P + S_i^Q$. (The case of spin one-half is degenerate: $\Sigma_i = 2S_i = \sigma_i$.)

Because the two spaces are orthogonal, $[S_i^P, S_j^Q] = 0$. Using that and the above definitions, the algebra of these matrices is found.

$$[S_i^I, \Sigma_j] = i\epsilon_{ijk}\Sigma_k \tag{1.2.11}$$

$$[\Sigma_i, \Sigma_j] = 4i\epsilon_{ijk}S_k^I \tag{1.2.12}$$

Wave functions of definite spin are also eigenfunctions of Σ^2 and $\Sigma \cdot S^I$. This can be shown by calculating various scalar quantities:

$$\begin{aligned}
S^{P2} &= \frac{p}{2}(\frac{p}{2}+1) \\
S^{Q2} &= \frac{q}{2}(\frac{q}{2}+1) \\
(S^I)^2 &= \frac{p+q}{2}(\frac{p+q}{2}+1) \\
&= \frac{p}{2}(\frac{p}{2}+1)+\frac{q}{2}(\frac{q}{2}+1)+2\frac{pq}{4} \\
&= S^{P2}+2\frac{pq}{4}+S^{Q2} \\
S^P \cdot S^Q &= \frac{pq}{4} \\
\Sigma^2 &= 4(S^P - S^Q)^2 \\
&= 4(S^{P2}+S^{Q2}-2S^P \cdot S^Q) \\
&= 4\left(\frac{p}{2}(\frac{p}{2}+1)+\frac{q}{2}(\frac{q}{2}+1)-\frac{pq}{2}\right) \\
&= p^2+q^2-2pq+2(p+q) \\
&= (p-q)^2+2(p+q) \\
\Sigma \cdot S &= 2(S^P - S_Q)\cdot(S^P+S_Q) \\
&= 2(S^P)^2-2(S^Q)^2 \\
&= \frac{1}{2}(p^2-q^2+p-q)
\end{aligned}$$

Expressing these in terms of $I = \frac{p+q}{2}$ and $\Delta = p - q$ we can write

$$S^2 = I(I+1) \tag{1.2.13}$$

$$\Sigma^2 = 4I + \Delta \tag{1.2.14}$$

$$\Sigma \cdot S = \frac{\Delta}{2}(2I+1) \tag{1.2.15}$$

It will eventually be necessary to take the matrix element of terms containing $\Sigma_i\Sigma_j$ in the nonrelativistic theory. These must be expressible in terms of the spin operator's matrix element, because that operator spans that space. Considering the traceless and symmetric structure, it must be proportional to the similar structure composed of spin matrices.

$$\left\langle \Sigma_i\Sigma_j + \Sigma_j\Sigma_i - \frac{2}{3}\delta_{ij}\mathbf{\Sigma}^2 \right\rangle = \lambda \left\langle \left\{ I_iI_j + I_jI_i - \frac{2}{3}\delta_{ij}\mathbf{I}^2 \right\} \right\rangle \tag{1.2.16}$$

To determine the constant λ, consider the zz component of the tensor structure acting on a wave function with all spin projected in the z direction. Then $I_3 \to \frac{p+q}{2} = I$, and $\Sigma_3 \to p - q = \Delta$, giving

$$\begin{aligned} 2\Delta^2 - \frac{2}{3}(4I + \Delta) &= \lambda \left\{ 2I^2 - \frac{2}{3}(I^2 + I) \right\} \\ &= \lambda \frac{2}{3} I(2I - 1) \end{aligned}$$

For integer spin $\Delta = 0$, giving

$$\begin{aligned} -\frac{8}{3}I &= \lambda_1 \frac{2}{3} I(2I - 1) \\ \lambda_1 &= -\frac{4}{2I - 1} \end{aligned}$$

For half spin we have $\Delta = 1$

$$\begin{aligned} \frac{4}{3}(1 - 2I) &= \lambda_{\frac{1}{2}} \frac{2}{3} I(2I - 1) \\ \lambda_{\frac{1}{2}} &= -\frac{4}{2I} \end{aligned}$$

Bilinears in the relativistic theory

Recall that in the case of spin one-half it was possible to catalogue a complete set of field bilinears with definite Lorentz transformation properties. The structure of the electron vertex was necessarily expressible in terms of those bilinears. These bilinears were built out of the 4×4 γ^μ matrices, which were themselves built out of 2×2 σ matrices.

What is the generalisation of these bilinears for the case of general spin? The fields have been written as bispinors, with upper and lower components spinors with transformation properties defined above. Whereas in the case of spin one-half $\{\mathcal{I}, \sigma_i\}$ formed a basis for operators acting on these spinors, here more complicated quantities built out of S and Σ are allowed.

It is still possible to write down a set of bilinears with definite transformation properties. The greater the spin (and thus the degrees of freedom of the spinors) the larger their number and rank. However, for the purposes of constructing the electromagnetic current and then calculating its nonrelativistic limit, only bilinears of up to rank 2 tensors need be considered.

In the appendix **??** the transformations of various bilinears are worked out. There are two necessary for constructing the electromagnetic current; the simple scalar $\bar{\Upsilon}\Psi$ and the antisymmetric tensor $\bar{\Upsilon}\Sigma_{\mu\nu}\Upsilon$.

The tensor structure is defined by

$$\Sigma_{ij} = -2i\epsilon_{ijk} \begin{pmatrix} S_k & 0 \\ 0 & S_k \end{pmatrix} \tag{1.2.17}$$

$$\Sigma_{0i} = \begin{pmatrix} 0 & \Sigma_i \\ \Sigma_i & 0 \end{pmatrix} \tag{1.2.18}$$

For the case of spin one-half, where $2S_i = \Sigma_i = \sigma_i$, this reduces to the familiar antisymmetric tensor $\sigma_{\mu\nu} = [\Gamma_\mu, \Gamma_\nu]$.

Electromagnetic Interaction

Knowing how the fields themselves behave, what form might the electromagnetic interaction take? In general, it can be written

$$M = eA_\mu j^\mu$$

where j^μ is the electromagnetic current.

The electromagnetic current must be built out of the particle's momenta and bilinears of the charged particle fields in such a way that they have the correct Lorentz properties. It must also obey current conservation: the equation $q_\mu j^\mu = 0$ must hold. Above it was shown that, in the case of general spin, there exist two such bilinears, a scalar and a tensor. A vector bilinear could not be constructed.

As higher spins are considered, higher order bilinears will appear. But necessarily, they will be coupled to a greater number of powers of the external momenta, and so suppressed in the nonrelativistic limit.

So there will be just two relevant terms in the current that transform as Lorentz vectors. One is a scalar bilinear coupled with a single power of external momenta. In order to fulfill the current conservation requirement, the exact combination will be

$$\frac{p^\mu + p'^\mu}{2m}\bar{\Upsilon}^\dagger\Upsilon$$

This obeys current conservation because $q = p' - p$, and $(p + p') \cdot (p' - p) = p^2 - p'^2 = 0$

The other type of term will be a tensor term contracted with a power of momenta. To fulfill current conservation, the tensor must be antisymmetric, and contracted with q:

$$\frac{q_\nu}{2m}\bar{\Upsilon}^\dagger\Sigma^{\mu\nu}\Upsilon$$

There is no need to consider higher order tensor bilinears; they will necessitate additional powers of the external momenta.

So the most general current would look like

$$j^\mu = F_e\frac{p^\mu + p'^\mu}{2m}\bar{\Upsilon}^\dagger\Upsilon + F_m\frac{q_\nu}{2m}\bar{\Upsilon}^\dagger\Sigma^{\mu\nu}\Upsilon \tag{1.2.19}$$

The form factors might have quite complicated dependence on q, but these corrections will be too small. They will be suppressed beyond the order of the calculation. At leading order F_e will just be the electric charge of the particle in question, and F_m will, as will be seen after calculating the nonrelativistic limit, be related to the particle's g-factor. So to the order needed, the current can be written

$$j^\mu = e\frac{p^\mu + p'^\mu}{2m}\bar{\Upsilon}^\dagger\Upsilon + eg\frac{q_\nu}{2m}\bar{\Upsilon}^\dagger\Sigma^{\mu\nu}\Upsilon \tag{1.2.20}$$

This captures the essence of the interaction between a charged particle of general-spin and a single photon.

1.3 Comparison of scattering amplitudes

Connection between the spinors of the two theories

Having sufficient knowledge of the relativistic theory, the approach of NRQED can be applied. In NRQED, the amplitude for scattering off an external field has been calculated. To compare the same process in this relativistic theory, first a way to write the nonrealtivistic spinors in terms of the relativistic ones is needed.

In the rest frame, there are two independent bispinors which represent particle and antiparticle states. The particle state is represented by

$$\Upsilon = \begin{pmatrix} \xi_0 \\ 0 \end{pmatrix} \tag{1.3.1}$$

the antiparticle by

$$\Upsilon = \begin{pmatrix} 0 \\ \xi_0 \end{pmatrix} \tag{1.3.2}$$

However, when considering a particle with non-zero momentum it is not the case that the upper component of the bispinor can be directly associated with the Schrodinger like wave-function of the particle — probability would not be properly conserved, for there is some mixing with the lower component, and thus a nonzero chance of the particle being in such a state.

To obtain a relation between ξ_0 and the Schrodinger amplitude ϕ_s, consider the current density at zero momentum transfer. For ϕ_S it will be $j_0 = \phi_S^\dagger \phi_S$. For the relativistic theory, as calculated above:

$$j^0 = F_e \frac{p^0 + p'^0}{2m} \bar{\Upsilon}^\dagger \Upsilon + F_m \frac{q_\nu}{2m} \bar{\Upsilon}^\dagger T^{0\nu} \Upsilon \tag{1.3.3}$$

At $q = 0$ the expression simplifies

$$j^0(q=0) = F_e \frac{p_0}{m} \bar{\Upsilon}^\dagger \Upsilon = F_e \frac{p_0}{m} (\phi^\dagger \phi - \chi^\dagger \chi) \tag{1.3.4}$$

ϕ and χ are both related to the rest frame spinor ξ_0. In such terms, j^0 is given by

$$j^0 = F_e \frac{p_0}{m} \xi_0^\dagger \left\{ \cosh^2(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}) - \sinh^2(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}) \right\} \xi_0 = F_e \frac{p_0}{m} \xi_0^\dagger \xi_0 \tag{1.3.5}$$

where the last equality follows from the hyperbolic trig identity.

Demanding that the two current densities be equal to each other, the result is that

$$\frac{p_0}{m} \xi_0^\dagger \xi_0 = \phi_s^\dagger \phi_s \tag{1.3.6}$$

As throughout the calculation, only corrections of order $1/m^2$ are needed. So appoximating

$$\left(1 + \frac{\mathbf{p}^2}{2m}\right) \xi_0^\dagger \xi_0 = \phi_s^\dagger \phi_s \tag{1.3.7}$$

This will hold to the necessary order if

$$\xi_0 = \left(1 - \frac{\mathbf{p}^2}{4m}\right)\phi_s \tag{1.3.8}$$

That relates the the Schrodinger like wave functions to the quantities ξ_0. To write the relativistic bispinors in terms of ϕ_S, approximations for $\cosh(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2})$ and $\sinh(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2})$ are also needed. The rapidity is needed only to the leading order: $\boldsymbol{\phi} \approx \mathbf{v} \approx \frac{\mathbf{p}}{m}$.

$$\cosh(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}) \approx 1 + \frac{1}{2}\left(\frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\right)^2 \tag{1.3.9}$$

$$\sinh(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}) \approx \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m} \tag{1.3.10}$$

Using this, the two bispinor components are

$$\begin{aligned}
\phi &\approx \left(1 + \left[\frac{1}{2}\frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\right]^2\right)\xi_0 \\
&\approx \left(1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m}\right)\phi_S && (1.3.11) \\
\chi &\approx \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\xi_0 \\
&\approx \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\phi_S && (1.3.12)
\end{aligned}$$

Bilinears in terms of nonrelativistic theory

The next step is to express the relativistic bilinears, built out of the bispinors Υ, in terms of the Schrodinger like wave functions.

Above the bispinors were written terms of ϕ_s, so those identities can be used to express the bilinears in the same manner.

Scalar bilinear

Calculating the scalar bilinear is just straightforward substitution and expansion. To express the commutator of Σ operators, the identity (1.2.12) is needed.

$$\begin{aligned}
\bar{\Upsilon}^\dagger(p')\Upsilon(p) &= \phi^\dagger\phi - \chi^\dagger\chi \\
&= \phi_s^\dagger\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p}')^2}{8m^2} - \frac{\mathbf{p}'^2}{4m^2}\right]\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m^2}\right]\phi_s - \phi_s^\dagger\left[\frac{(\mathbf{\Sigma}\cdot\mathbf{p}')(\mathbf{\Sigma}\cdot\mathbf{p})}{4m^2}\right]\phi_s \\
&= \phi_s^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{(\mathbf{\Sigma}\cdot\mathbf{p}')^2 + (\mathbf{\Sigma}\cdot\mathbf{p})^2 - 2(\mathbf{\Sigma}\cdot\mathbf{p}')(\mathbf{\Sigma}\cdot\mathbf{p})\right\}\right)\phi_s \\
&= \phi_s^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{[\mathbf{\Sigma}\cdot\mathbf{p}, \mathbf{\Sigma}\cdot\mathbf{q}] + (\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_s \\
&= \phi_s^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{[4i\epsilon_{ijk}p_iq_jS_k + (\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_s
\end{aligned}$$

Tensor ij component

In calculating the nonrelativistic limit of the antisymmetric tensor bilinear, the $0i$ and the ij components will be treated separately. First consider $\bar{\Upsilon}\Sigma_{ij}\Upsilon$. The value of Σ_{ij} itself was written in (1.2.17)

$$\begin{aligned}
\bar{\Upsilon}\Sigma_{ij}\Upsilon &= \bar{\Upsilon}(-2\epsilon_{ijk}S_k)\Upsilon \\
&= -2i\epsilon_{ijk}(\phi^\dagger S_k\phi - \chi^\dagger S_k\chi) \\
&= -2i\epsilon_{ijk}\left(\phi_S^\dagger\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p}')^2}{8m^2} - \frac{\mathbf{p}'^2}{4m^2}\right]S_k\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m^2}\right]\phi_S - \phi_S^\dagger\frac{(\mathbf{\Sigma}\cdot\mathbf{p}')S_k(\mathbf{\Sigma}\cdot\mathbf{p})}{4m^2}\phi_S\right) \\
&= -2i\epsilon_{ijk}\phi_S^\dagger\left\{S_k\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{1}{8m^2}\underbrace{\left[(\mathbf{\Sigma}\cdot\mathbf{p}')^2S_k + S_k(\mathbf{\Sigma}\cdot\mathbf{p})^2 - 2(\mathbf{\Sigma}\cdot\mathbf{p}')S_k(\mathbf{\Sigma}\cdot\mathbf{p})\right]}_{T_k}\right\}\phi_S
\end{aligned}$$

Call the term in square brackets T_k. It should be written explicitly in terms of $\mathbf{p}$ and $\mathbf{q}$.

$$\begin{aligned}
T_k &= (\mathbf{\Sigma}\cdot\mathbf{p})^2S_k + S_k(\mathbf{\Sigma}\cdot\mathbf{p}) - 2(\mathbf{\Sigma}\cdot\mathbf{p})S_k(\mathbf{\Sigma}\cdot\mathbf{p}) + \{(\mathbf{\Sigma}\cdot\mathbf{p})(\mathbf{\Sigma}\cdot\mathbf{q}) \\
&\quad +(\mathbf{\Sigma}\cdot\mathbf{q})(\mathbf{\Sigma}\cdot\mathbf{p})\}S_k - 2(\mathbf{\Sigma}\cdot\mathbf{q})S_k(\mathbf{\Sigma}\cdot\mathbf{p}) + (\mathbf{\Sigma}\cdot\mathbf{q})^2S_k
\end{aligned}$$

Many of these terms may be expressed as commutators, and then these commutators simplified.

$$\begin{aligned}
T_k &= \mathbf{\Sigma}\cdot\mathbf{p}[\mathbf{\Sigma}\cdot\mathbf{p}, S_k] + [S_k, \mathbf{\Sigma}\cdot\mathbf{p}]\mathbf{\Sigma}\cdot\mathbf{p} + 2\mathbf{\Sigma}\cdot\mathbf{q}[\mathbf{\Sigma}\cdot\mathbf{p}, S_k] - [\mathbf{\Sigma}\cdot\mathbf{q}, S_k]\mathbf{\Sigma}\cdot\mathbf{p} + (\mathbf{\Sigma}\cdot\mathbf{q})^2S_k \\
&= i\epsilon_{ijk}p_j\{(\mathbf{\Sigma}\cdot\mathbf{p})\Sigma_i - \Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})\} + 2i\epsilon_{ijk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_ip_j - \Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})q_j)\} + (\mathbf{\Sigma}\cdot\mathbf{q})^2S_k \\
&= i\epsilon_{ijk}p_j[(\mathbf{\Sigma}\cdot\mathbf{p}), \Sigma_i] + 2i\epsilon_{ijk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_ip_j - \Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})q_j)\} + (\mathbf{\Sigma}\cdot\mathbf{q})^2S_k \\
&= 4(\mathbf{p}^2S_k - (\mathbf{S}\cdot\mathbf{p})p_k) + 2i\epsilon_{ijk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_ip_j - \Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})q_j)\} + (\mathbf{\Sigma}\cdot\mathbf{q})^2S_k
\end{aligned}$$

Thus the whole bilinear is

$$\bar{\Upsilon}\Sigma_{ij}\Upsilon = -2i\epsilon_{ijk}\phi_S^\dagger\Bigg\{S_k\left(1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right)+\frac{1}{8m^2}\Big[4(\mathbf{p}^2 S_k-(\mathbf{S}\cdot\mathbf{p})p_k) \\ +2i\epsilon_{\ell mk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_\ell p_m-\Sigma_\ell(\mathbf{\Sigma}\cdot\mathbf{p})q_m)\}+(\mathbf{\Sigma}\cdot\mathbf{q})^2 S_k\Big]\Bigg\}\phi_S \tag{1.3.13}$$

Tensor Σ_{0i} component

Now calculate the $0i$ component, $\bar{\Upsilon}\Sigma_{0i}\Upsilon$.

$$\bar{\Upsilon}\Sigma_{0i}\Upsilon = \bar{\Upsilon}\begin{pmatrix} 0 & \Sigma_i \\ \Sigma_i & 0 \end{pmatrix}\Upsilon \tag{1.3.14}$$

$$= \phi^\dagger\Sigma_i\chi - \chi^\dagger\Sigma_i\phi \tag{1.3.15}$$

Because this tensor structure is coupled to q_i/m, ϕ and χ are needed only to first order here.

$$\bar{\Upsilon}\Sigma_{0i}\Upsilon = \phi_S^\dagger\left(\frac{\Sigma_i\Sigma_j p_j - \Sigma_j\Sigma_i p'_j}{2m}\right)\phi_S \tag{1.3.16}$$

Using $p' = p + q$ the terms involving only p can be simplified using the commutator of Σ matrices.

$$\bar{\Upsilon}\Sigma_{0i}\Upsilon = \phi_S^\dagger\left(\frac{4i\epsilon_{ijk}p_j S_k - \Sigma_j\Sigma_i q_j}{2m}\right)\phi_S \tag{1.3.17}$$

Current in terms of nonrelativistic wave functions

The four-current (1.2.19) was derived above; in Galilean form it is

$$j_0 = F_e\frac{p_0+p'_0}{2m}\bar{\Upsilon}^\dagger\Upsilon - F_m\frac{q_j}{2m}\bar{\Upsilon}^\dagger\Sigma^{0j}\Upsilon \tag{1.3.18}$$

$$j_i = F_e\frac{p_i+p'_i}{2m}\bar{\Upsilon}^\dagger\Upsilon - F_m\frac{q_j}{2m}\bar{\Upsilon}^\dagger\Sigma^{ij}\Upsilon + F_m\frac{q_0}{2m}\bar{\Upsilon}^\dagger\Sigma^{i0}\Upsilon \tag{1.3.19}$$

Expressions for the bilinears in terms of the nonrelativistic wave functions ϕ_S have been calcaluted, so it is fairly straight forward to apply them here. The calculation of j_0 goes:

$$
\begin{aligned}
F_e \frac{p_0 + p_0'}{2m} \bar{\Upsilon}^\dagger \Upsilon &= F_e \left(1 + \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) \phi_S^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2} \left\{4i\epsilon_{ijk} p_i q_j S_k + (\mathbf{\Sigma} \cdot \mathbf{q})^2\right\}\right) \phi_S \\
&\approx F_e \phi_S^\dagger \left(1 + \frac{1}{8m^2} \left\{4i\mathbf{S} \cdot \mathbf{p} \times \mathbf{q} + (\mathbf{\Sigma} \cdot \mathbf{q})^2\right\}\right) \phi_S \\
F_m \frac{q_j}{2m} \bar{\Upsilon}^\dagger \Sigma^{0j} \Upsilon &= F_m \frac{q_i}{2m} \phi_S^\dagger \left(\frac{4i\epsilon_{ijk} p_j S_k - \Sigma_j \Sigma_i q_j}{2m}\right) \phi_S \\
&= F_m \phi_S^\dagger \left(\frac{4i\mathbf{S} \cdot \mathbf{q} \times \mathbf{p} - (\mathbf{\Sigma} \cdot \mathbf{q})^2}{4m^2}\right) \phi_S
\end{aligned}
$$

It turns out that both terms here have the same form, so combining them,

$$
j_0 = \phi_S^\dagger \left(F_e + \frac{F_e + 2F_m}{8m^2} \left\{4i\mathbf{S} \cdot \mathbf{p} \times \mathbf{q} + (\mathbf{\Sigma} \cdot \mathbf{q})^2\right\}\right) \phi_S \tag{1.3.20}
$$

To calculate j_i it helps to first simplify things by considering the constraints of this particular problem. The term with $\Sigma_i j$ can be simplified by dropping terms with more than one power of q; these will turn into derivatives of the magnetic field, and this problem concerns only a constant field. Further, only elastic scattering is considered, and so $q_0 = 0$. With those simplifications

$$
\bar{\Upsilon} \Sigma_{ij} \Upsilon \approx -2i\epsilon_{ijk} \phi_S^\dagger \left\{ S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p}) p_k}{2m^2} \right\} \phi_S \tag{1.3.21}
$$

$$
\begin{aligned}
F_e \frac{p_i + p_i'}{2m} \bar{\Upsilon}^\dagger \Upsilon &= F_e \frac{p_i + p_i'}{2m} \phi_S^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2} \left\{4i\epsilon_{\ell jk} p_\ell q_j S_k + (\mathbf{\Sigma} \cdot \mathbf{q})^2\right\}\right) \phi_S \\
&\approx F_e \frac{p_i + p_i'}{2m} \phi_S^\dagger \left(1 + \frac{1}{8m^2} \left\{4i\epsilon_{\ell jk} p_\ell q_j S_k\right\}\right) \phi_S \\
F_m \frac{q_j}{2m} \bar{\Upsilon}^\dagger \Sigma^{ij} \Upsilon &= -F_m \frac{i\epsilon_{ijk} q_j}{m} \phi_S^\dagger \left\{ S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p}) p_k}{2m^2} \right\} \phi_S
\end{aligned}
$$

So the full spatial part of the current is

$$
j_i = \phi_S^\dagger \left\{ F_e \frac{p_i + p_i'}{2m} \left(1 + \frac{i\epsilon_{\ell jk} p_\ell q_j S_k}{2m^2}\right) + F_m \frac{i\epsilon_{ijk} q_j}{m} \left(S_k \left\{1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right\} + \frac{\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p}) p_k}{2m^2} \right) \right\} \phi_S \tag{1.3.22}
$$

Scattering off an external field

To compare to the NRQED Lagrangian, scattering off an external field needs to be calculated for an arbitrary spin particle. The hardest part was calculating the current; with that in hand the amplitude can be found just as

$$M = ej_\mu A^\mu = ej_0 A_0 - e\mathbf{j}\cdot\mathbf{A}$$

With the expressions for both j_0 (1.3.20) and $\mathbf{j}$ (1.3.22) both parts of the amplitude can be written down directly.

$$\begin{aligned}
ej_0A_0 =& eA_0\phi_S^\dagger\left(F_e + \frac{F_e+2F_m}{8m^2}\left\{4i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q} + (\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_S \\
e\mathbf{j}\cdot\mathbf{A} =& A_i\phi_S^\dagger\left\{F_e\frac{p_i+p_i'}{2m}\left(1+\frac{i\epsilon_{\ell jk}p_\ell q_j S_k}{2m^2}\right) + F_m\frac{i\epsilon_{ijk}q_j}{m}\left(S_k\left(1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right)+\frac{\mathbf{p}^2S_k-(\mathbf{S}\cdot\mathbf{p})p_k}{2m^2}\right)\right\}\phi_S
\end{aligned} \tag{1.3.23}$$

To compare to the NRQED result, as much as possible terms should be expressed in gauge invariant quantities $\mathbf{B}$ and $\mathbf{E}$. The relations between these fields and A_μ in position space and the equivalent equation in momentum space are:

$$\mathbf{B} = \mathbf{\nabla}\times\mathbf{A} \to i\mathbf{q}\times\mathbf{A}$$
$$\mathbf{E} = -\mathbf{\nabla}A_0 \to -i\mathbf{q}A_0$$

There is one term above that can only be put into gauge-invariant form by considering the kinematic constraints of elastic scattering. As previously derived in (**??**), the identity is

$$i(p_i+p_i')q_jA_i = -\epsilon_{ijk}B_k(p_i+p_i') \tag{1.3.24}$$

Now each term involving q can be written in terms of E or B.

$$\begin{aligned}
i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q}A_0 &= -\mathbf{S}\cdot\mathbf{p}\times\mathbf{E} \\
(\mathbf{\Sigma}\cdot\mathbf{q})^2 A_0 &= \Sigma_i\Sigma_j q_i q_j A_0 \\
&= \Sigma_i\Sigma_j \partial_i E_j \\
i\epsilon_{ijk}A_i q_j &= -i(\mathbf{q}\times\mathbf{A})_k \\
&= = -B_k \\
A_i(p_i+p_i')i\epsilon_{\ell jk}p_\ell q_j S_k &= \epsilon_{\ell jk}p_\ell S_k i(p_i+p_i')q_j A_i \\
&= -\epsilon_{\ell jk}p_\ell S_k\{\epsilon_{ijm}B_m(p_i+p_i')\} \\
&= -(\delta_{\ell i}\delta_{km}-\delta\ell m\delta_{ik})p_\ell S_k\{\epsilon_{ijm}B_m(p_i+p_i')\} \\
&= 2\{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})-(\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2\}
\end{aligned}$$

Using these

$$ej_0A_0 = e\phi_S^\dagger\left\{A_0+\frac{1-2F_2}{8m^2}\left(4\mathbf{S}\cdot\mathbf{E}\times\mathbf{p}+\Sigma_i\Sigma_j\partial_i E_j\right)\right\}$$

$$\begin{aligned}
e\mathbf{j}\cdot\mathbf{A} =& e\phi_S^\dagger\Bigg\{\frac{\mathbf{p}\cdot\mathbf{A}}{m}+\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})-(\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2}{m^2} \\
&-F_m\left(\frac{\mathbf{S}\cdot\mathbf{B}}{m}\left\{1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right\}+\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})-(\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2}{2m^2}\right)\Bigg\}\phi
\end{aligned}$$

$$= e\phi_S^\dagger\left\{\frac{\mathbf{p}\cdot\mathbf{A}}{m}+[1-2F_m]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2}-\mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2}-\frac{F_m}{m}\mathbf{S}\cdot\mathbf{B}\right\}$$

From looking at the leading order coefficient of $\mathbf{S}\cdot\mathbf{B}$ it can be seen that F_m is actually $g/2$, so in such terms

$$ej_0A_0 = e\phi_S^\dagger\left\{A_0-\frac{g-1}{2m^2}\left(\mathbf{S}\cdot\mathbf{E}\times\mathbf{p}+\frac{1}{4}\Sigma_i\Sigma_j\partial_i E_j\right)\right\}$$

$$e\mathbf{j}\cdot\mathbf{A} = e\phi_S^\dagger\left\{\frac{\mathbf{p}\cdot\mathbf{A}}{m}-[g-1]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2}-\mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2}-\frac{g}{2m}\mathbf{S}\cdot\mathbf{B}\right\}$$

The complete expression for the amplitude is

$$M = e\phi_S^\dagger \Bigg\{ A_0 - \frac{g-1}{2m^2}\left(\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \frac{1}{4}\Sigma_i\Sigma_j\partial_i E_j\right) + \frac{\mathbf{p}\cdot\mathbf{A}}{m}$$
$$-[g-1]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{g}{2m}\mathbf{S}\cdot\mathbf{B}\Bigg\}\phi_S \tag{1.3.25}$$

There is one last vestige of the relativistic notation: the structures Σ_i are not appropriate for comparing to the nonrelativistic theory. In a previous section, identities were developed for scalars and symmetric, traceless tensors built out of Σ. So first, express $\Sigma_i\Sigma_j$ in such terms:

$$\Sigma_i\Sigma_j\partial_i E_j = \frac{1}{2}\partial_i E_j(\Sigma_i\Sigma_j + \Sigma_j\Sigma_i - \frac{2}{3}\delta_{ij}\boldsymbol{\Sigma}^2) + \frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{E}\Sigma^2 \tag{1.3.26}$$

Then using (1.2.16), this may be written

$$\Sigma_i\Sigma_j\partial_i E_j = \frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{E}\Sigma^2 - \frac{\lambda}{2}\partial_i E_j(S_iS_j + S_jS_i - \frac{2}{3}\mathbf{S}^2) = \frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{E}\Sigma^2 - \frac{\lambda}{2}\partial_i E_j Q_{ij} \tag{1.3.27}$$

The spin dependence found in (1.2.14) and (1.2.16) is most easily written separately for integer and half-integer cases.

For integer spin

$$(\Sigma^2)_1 = 4I \qquad\qquad \lambda_1 = -\frac{4}{2I-1}$$

While for half-integer

$$(\Sigma^2)_{\frac{1}{2}} = 4I+1 \qquad\qquad \lambda_{\frac{1}{2}} = -\frac{4}{2I}$$

Where I is the magnitude of the particle's spin.

$$M = e\phi_S^\dagger \Bigg\{ A_0 + \frac{\mathbf{p}\cdot\mathbf{A}}{m} - \frac{g-1}{2m^2}\left(\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \frac{\Sigma^2}{12}\boldsymbol{\nabla}\cdot\mathbf{E} + \frac{\lambda}{8}Q_{ij}\right)$$
$$-[g-1]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{g}{2m}\mathbf{S}\cdot\mathbf{B}\Bigg\}\phi_S \tag{1.3.28}$$

Fixing the nonrelativistic coefficients

Having calculated the same process in both the relativistic theory and in the NRQED effective theory, the two amplitudes can be compared, thus fixing the coefficients of NRQED.

The NRQED amplitude (**??**) is

$$iM = ie\phi^\dagger\Bigg(- A_0 + \frac{\mathbf{A}\cdot\mathbf{p}}{m} - \frac{(\mathbf{A}\cdot\mathbf{p})\mathbf{p}^2}{2m^3} + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2} + c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2}$$
$$+c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2} - (c_{W_1} - c_{W_2})\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3}\Bigg)\phi \tag{1.3.29}$$

While the relativistic amplitude was

$$iM_{REL} = -ie\phi^\dagger\Big(A_0 - \frac{\mathbf{p}\cdot\mathbf{A}}{m} - \frac{g-1}{2m^2}\left\{\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \frac{\Sigma^2}{12}\boldsymbol{\nabla}\cdot\mathbf{E} + \frac{\lambda}{8}Q_{ij}\right\}$$
$$-g\frac{1}{2m}\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\Big)\phi \tag{1.3.30}$$

With both amplitudes, a straightforward comparison gives the coefficients in the NRQED Lagrangian. There are no terms cubic in spin which arise, so

$$c_{T_1} + c_{T_2} = 0 \tag{1.3.31}$$

The non-zero coefficients are:

$$\begin{aligned}
c_F &= g \\
c_D &= \frac{(g-1)}{3}\Sigma^2 \\
c_Q &= \frac{g-1}{2}\lambda \\
c_S^1 &= 2(g-1) \\
(c_{W_1} - c_{W_2}) &= 2 \\
c_{p'p} &= (g-2)
\end{aligned}$$

1.4 Interpretation of universality in light of BMT equation

The time evolution of a particle's spin is given by

$$\dot{\mathbf{S}} = i[H, \mathbf{S}]. \tag{1.4.1}$$

In nonrelativistic quantum mechanics, when a magnetic field is present, the above becomes

$$\dot{\mathbf{S}} = g\frac{e}{2m}\mathbf{S} \times \mathbf{B}. \tag{1.4.2}$$

Let a be a four-vector that is the relativistic generalisation of $\mathbf{S}$. The proper time evolution of this vector is constrained in several ways: it must be linear in the electromagnetic field and the vector a_μ, and must be built out of those quantities as well as the velocity $u = p/m$. With such constraints, only two terms are possible, so the time evolution must have the form

$$\frac{da^\mu}{d\tau} = \alpha F^{\mu\nu} a_\nu + \beta u^\nu F^{\mu\lambda} u_\mu a_\lambda. \tag{1.4.3}$$

Since the nonrelativistic limit is known, the constants α and β can be determined by comparison. In the rest frame, $a = (0, \mathbf{S})$ and $u = (1, 0)$. Then

$$\frac{da^i}{d\tau} = \alpha F^{ij} a_j, \tag{1.4.4}$$

or in terms of $\mathbf{S}$

$$\frac{d\mathbf{S}}{d\tau} = \alpha \mathbf{S} \times \mathbf{B} \tag{1.4.5}$$

Comparing to the nonrelativistic limit (1.4.2), it must be that $\alpha = g\frac{e}{2m}$.

To fix β, consider the time evolution of the velocity. According to the classical motion of a particle in a field,

$$m\frac{du^\mu}{d\tau} = eF^{\mu\nu} u_\nu. \tag{1.4.6}$$

With $a \cdot u = 0$, it follows that

$$u \cdot \frac{da}{d\tau} = -a \cdot \frac{du}{d\tau} = \frac{e}{m} F^{\mu\nu} u_\mu a_\nu. \tag{1.4.7}$$

Dotting u into the evolution equation earlier gives

$$u \cdot \frac{da}{d\tau} = \alpha F^{\mu\nu} a_\nu u_\mu + \beta u^2 F^{\mu\lambda} u_\mu a_\lambda, \tag{1.4.8}$$

and substituting the values of α and $u \cdot \frac{da}{d\tau}$,

$$\frac{e}{m} F^{\mu\nu} u_\mu a_\nu = \left(g\frac{e}{2m} + \beta\right) F^{\mu\nu} u_\mu a_\nu \tag{1.4.9}$$

So

$$\beta = -(g-2)\frac{e}{2m}. \tag{1.4.10}$$

So the original equation for spin evolution is

$$\frac{da^\mu}{d\tau} = g\frac{e}{2m}F^{\mu\nu}a_\nu - (g-2)\frac{e}{2m}u^\nu F^{\mu\lambda}u_\mu a_\lambda. \tag{1.4.11}$$

Written in terms of nonrelativistic quantities, this becomes

$$\frac{d\mathbf{S}}{dt} = \frac{e}{2m}\mathbf{S}\times\left\{\left(g-2+\frac{2}{\gamma}\right)\mathbf{B} \; - \frac{(g-2)\gamma}{1+\gamma}(\mathbf{v}\cdot\mathbf{B})\mathbf{v} + \left(g - \frac{2\gamma}{1+\gamma}\right)\mathbf{E}\times\mathbf{v}\right\}. \tag{1.4.12}$$

Keeping only first order relativistic corrections, and writing $\mathbf{p} - e\mathbf{A}$ instead of $\mathbf{v}$, this becomes

$$\frac{d\mathbf{S}}{dt} = \frac{e}{2m}\mathbf{S}\times\left\{\left(g-\frac{\mathbf{p}^2}{m^2}\right)\mathbf{B} \; - \frac{(g-2)}{2m^2}(\mathbf{p}\cdot\mathbf{B})\mathbf{p} + (g-1)\,\frac{\mathbf{E}\times(\mathbf{p}-e\mathbf{A})}{m}\right\}. \tag{1.4.13}$$

The logic that led up to (1.4.13) depended only on relativistic symmetries and classical limits. These considerations would hold for particles of any spin. So the BMT equation is universal and independent of spin.

Recall again that the time evolution of $\mathbf{S}$ can also be obtained from the commutator $[H,\mathbf{S}]$. If the general NRQED Lagrangian (1.1.8) developed previously is used, then for a constant magnetic field,

$$\frac{d\mathbf{S}}{dt} = \tag{1.4.14}$$

That the BMT equation does not depend on spin then constrains the coefficients appearing in $[H,\mathbf{S}]$ to likewise be universal.

Bibliography