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Chapter 1

Introduction

We wish to calculate the gyromagnetic ratio of a particle of arbitrary spin in a loosely bound state system. We can write down an effective Lagrangian which will capture all the necessary effects. Our job is then to calculate the coefficients of this effective particle. By considering the constraints which exist on the electromagnetic current in a general relativistic theory, we can obtain a simple form which holds for arbitrary spin, and then use this to fix the coefficients of the relativistic theory.

While for a particle of general spin we do not have an exact Lagrangian, we do have one for a spin-1 theory. We can use this Lagrangian to perform the same calculation as above, but in an exact theory. We derive the nonrelativistic potential first from the equations of motion, and then from diagrammatic calculations, and obtain a result that agrees with the more general calculation.

1.0.1 Description of problem

We consider a loosely bound system two particles of arbitrary spin, and wish to calculate the correction to the gyromagnetic ratio of the particles. So we consider the case of the two particles interacting with both each other, via a Coulomb potential, and a very weak external magnetic field.

Experimental Motivation

Talk about

Approach

This system allows for several simplifications. First, because the system is loosely bound all energy scales are nonrelativistic. Second, the magnetic field we consider is both very weak and constant. So we can ignore all corrections which involve derivatives of the magnetic field or which are quadratic in its strength.

The approach we'll take is to first consider the electric potential as an external field acting on a single particle. Next we can consider the bound system as a whole sitting in an external magnetic field, and take into account recoil effects.

1.0.2 Theoretical Background

(Here we talk about existing theoretical work in this area)

Spin-1/2

(Here talk about the approach that works for spin 1/2, and why it breaks down in the general case.)

BMT equation

(Here discuss the BMT equation, which holds for general spin and can be used to derive the g-factor corrections)

Khriplovich general spin work

(Discuss the work Khriplovich did with general spinors.)

Chapter 2

NRQED

We can construct an effective, nonrelativistic Lagrangian for a charged particle interacting with an electromagnetic field.

Chapter 3

Nonrelativistic Quantum Electrodynamics

3.1 Effective Field Theories

The most generally powerful approach to nonrelativistic bond state theories is the use of the techniques of effective field theory. In this chapter we develop the theory of such an approach.

3.1.1 Brief background

First, a brief background of the development of effective field theories.

The first workable relativistic theory of quantum mechanics came from Dirac. By realising that a relativistic equation for fermions necessitated a four component spinor, he was able to write down what is now known as Dirac's equation. This led to the prediction of the electron's antiparticle the positron.

The relativistic theory necessitated the introduction of an infinite degrees of freedom. Thus, wave functions had to be promoted to fields, and the first quantum field theory arose. It was highly successful at predicting leading order quantities, but when attempting to use perturbation theory within this context, a number of infinities arose.

To resolve this matter, QED and renormalization were developed in the late 1940s. The key was in changing the parametrization from of the theory from “bare” values, to measured values at some particular energy scale. So to completely define the theory one needed to write down not only the Lagrangian but also the renormalization point. The coupling constants of the theory would depend upon this. With this approach QED was enormously successful, producing very accurate predictions to such quantities as the anomolous magnetic moment.

There was much frustrating work to formulate a theory of the other interactions along the same lines. During this work (such as that by Gell-man and Low) with these types of theories it became clear that there was structure to how the coupling constants behaved as the renormalization point changed.

While working through the Ising model, Wilson, who had a background working with QFTs, realised there were applications of these renormalisation ideas to critical phenomena. Later, working with fixed-source meson theory, had an insight that led to effective field theory. Working with a hierarchy of momentum “slices” and a process of eliminating energy scales, he found that although an infinite number of interactions was generated, the effects at each iteration were bounded – only a finite number of terms were necessary for any particular level of precision. A successful theory with arbitrarily many constants still worked, as long as there was a clear hierarchy of terms.

An effective field theory was not renormalizable in the way QED was, but as Wilson found the proliferation of terms could be contained such that only a finite number mattered for a particular calculation. The constraint is that an effective field theory has application only well below some energy scale. But, the very feature that it is formulated at a particular scale makes it in many ways the natural theory to use at that scale.

3.1.2 A Worked example of Renormalization

Let us review how the process of renormalization works when a theory *is* renormalizable.

The QED Lagrangian defined at some cut-off is:

$$\mathcal{L}_0 = \bar{\Psi}\left(i\partial\cdot\gamma - e_0 A\cdot\gamma - m_0\right)\Psi - \frac{1}{2}(F^{\mu\nu})^2 \tag{3.1.1}$$

and additionally the cut-off regulator Λ_0.

In calculating a process without a cut-off, all intermediate states must be summed over. When kinematics do not strictly dictate the intermediate momenta of some particles, this means integrating over an infinite range. This is exactly the case with an internal loop (as it described in the language of Feynman diagrams), and is what caused the infinities that plauged initial attempts to formulate relativistic quantum mechanics.

By introducing a cut-off, the integrals only take place over a bounded domain of momenta, and are thus finite. The seeming cost is that physical quantities should not depend upon this arbitrary cut-off Λ_0. Still, the parameters can be fixed by comparing with experiment. This theory will then produce results correct up to some terms of $\mathcal{O}(1/\Lambda_0^2)$.

There are two parameters: the bare mass m_0 and the bare charge e_0. These parameters are determined from experiment. Two processes can be calculated (such as electron-electron scattering and the electron scattering off some external field), compared to the experimental measurements, and thus the parameters fixed.

Handed this theory with this high cut-off Λ_0, it is possible to reformulate it in terms of a new, lower cut-off Λ. The new theory will hold valid for processes where the external momenta are much less than Λ.

By introducing this lower cut-off, high energy virtual processes are eliminated from the theory. Rather than being explicitly included, their effects will implicitly be included by corrections to the parameters of the theory.

These corrections can be calculated from the old theory. Before loop-integrals over momenta ran from 0 to the old cut-ff Λ_0. In the new theory, they will run from 0 to Λ. Clearly the difference between the two calculations will be an integral from Λ to Λ_0. Importantly, because Λ is taken to be greater than the energy of any process considered, so will the loop momentum in that sector of the integral.

First consider the calculation of the electron vertex. Call the original value T, which will have contributions from several diagrams. On such contribution is from the diagram (INSERT). Call this contribution $T^{(a)}$.

In the original $\mathcal{L}_0$ theory the contribution would be

$$T(a)(k>0) = -e_0^3 \int_0^{\Lambda_0} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \left\{ \bar{u}(p')\gamma^\mu \frac{1}{(p'-k)\cdot\gamma - m_0} A_{\text{ext}}(p'-p)\cdot\gamma \frac{1}{(p-k)\cdot\gamma - m_0} \gamma_\mu u(p) \right\} \tag{3.1.2}$$

When the momenta are cut-off at the lower point Λ, the part of the old integral missing from the new calculation will be

$$T(a)(k>\Lambda) = -e_0^3 \int_\Lambda^{\Lambda_0} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \left\{ \bar{u}(p')\gamma^\mu \frac{1}{(p'-k)\cdot\gamma - m_0} A_{\text{ext}}(p'-p)\cdot\gamma \frac{1}{(p-k)\cdot\gamma - m_0} \gamma_\mu u(p) \right\} \tag{3.1.3}$$

Remember that Λ is chosen to be a great deal greater than p, p' or m_0, and this then holds for k over the entire range of the integral. Then, if corrections of the type p/Λ are discarded, the integral can be greatly simplified. The approximation used is:

$$\frac{1}{(p'-k)\cdot\gamma - m_0} \approx -\frac{1}{k\cdot\gamma} \tag{3.1.4}$$

Of course for any four vector a, it holds that $(a\cdot\gamma)^2 = a^2$, so

$$-\frac{1}{k\cdot\gamma} = -\frac{k\cdot\gamma}{k^2} \tag{3.1.5}$$

Then

$$\begin{aligned} T^{(a)}(k>\Lambda) &\approx -e_0^3 \int_\Lambda^{\Lambda_0} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \left\{ \bar{u}(p')\gamma^\mu \frac{k\cdot\gamma}{k^2} A_{\text{ext}}(p'-p)\cdot\gamma \frac{k\cdot\gamma}{k^2} \gamma_\mu u(p) \right\} \\ &\approx -e_0^3 \bar{u}(p') A_{\text{ext}}(p'-p)\cdot\gamma u(p) \int_\Lambda^{\Lambda_0} \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} \end{aligned}$$

There are other one electron scattering diagrams. When the above analysis is applied to them all, the total difference is found to be of the form:

$$T^(k > \Lambda) = -ie_0 c_0(\Lambda/\Lambda_0)\bar{u}(p')A_{\text{ext}}(p' - p) \cdot \gamma u(p) \tag{3.1.6}$$

This is the piece of the electron vertex structure that we are missing if we calculate using the original Lagrangian with the lower cut-off. The correct results will be obtained if we incorporate into the Lagrangian a new term:

$$\delta\mathcal{L} = -e_0 c_0(\Lambda/\Lambda_0)\bar{u}(p')A_{\text{ext}}(p' - p) \cdot \gamma u(p) \tag{3.1.7}$$

What about the nature of this constant c_0? Consider the above calculation of $T^{(a)}(k > \Lambda)$. It had, like the total correction, had a structure of some constant terms times $\bar{u}(p')A_{\text{ext}}(p' - p) \cdot \gamma u(p)$. The structure of the constant term came from the integral over a function of only k. Since there were no scales involved in this integration, other than the limits of integration, the result must be some function of Λ and Λ_0. And because the integral is dimensionless, it must actually be a function of their ratio Λ/Λ_0. The same logic goes through when the other terms are computed. The final result is that

$$c_0 = -\frac{\alpha_0}{6\pi}\log(\Lambda/\Lambda_0) \tag{3.1.8}$$

That the result of these integrals involves only the limits is contingent upon the approximation made earlier, that the scales p, p' and m_0 are all small compared to Λ. If corrections of that order are important, then there will be additional terms with structures including these momenta and mass. This can be accomplished by, instead of completely neglecting these terms, doing Taylor expansion in terms of p/k, m_0/k and so forth. We'll return to this later.

Going back to the correction $\delta\mathcal{L}$, note that it has almost the same form as a term in the original $\mathcal{L}_0$, the difference being an explicit dependence on the cut-offs Λ, Λ_0. (Although of course, e_0 itself depended on comparing measurements to calculations in the $\mathcal{L}_0$ theory, so it really *was* dependent on Λ_0.) Rather than interpret it as new interaction, then, it can be seen as change in the strength of e_0.

$$-\bar{\Psi}e_0 A\cdot\gamma\Psi \to -\bar{\Psi}e_0[1+c_0(\Lambda/\Lambda_0)]A\cdot\gamma\Psi \tag{3.1.9}$$

As long as all other scales are considered to small to Λ to enter the calculations, it isn't possible for truly new terms to enter, only corrections to the already existing terms. That means that all that can happen is an adjustment of the existing coupling constants e_0 and m_0.

There are indeed corrections to m_0, coming from the electron self-energy. An additional correction term is required of the form

$$\delta\mathcal{L} = -m_0\tilde{c}_0(\Lambda/\Lambda_0)\bar{\Psi}\Psi \tag{3.1.10}$$

The Lagrangian valid with the new cut-off Λ can be written in terms of the old as:

$$\mathcal{L}_\Lambda = \bar{\Psi}\left(i\partial\cdot\gamma - e_\Lambda A\cdot\gamma - m_\Lambda\right)\Psi - \frac{1}{2}(F^{\mu\nu})^2 \tag{3.1.11}$$

where the constants are

$$\begin{aligned} e_\Lambda &= e_0(1+c_0) \\ m_\Lambda &= m_0(1+\tilde{c}_0) \end{aligned}$$

Above only two processes were considered, which produced corrections to known interactions. In principle corrections will also arise from other processes, such as electron-electron scattering.

But, consider the form of such cross-sections. They must have a spinor structure something like $\bar{u}u\bar{u}u$ or $\bar{u}\gamma^\mu u\bar{u}\gamma_\mu u$, with an accompanying factor. Certainly it involves four fermion fields. The key point is that the scattering amplitude with four external lines must be dimensionless. Since u has mass dimension $1/2$, d_0 must have dimension -2.

So to write a dimensionless factor d_0 analogous to c_0 above, there must be an additional factor of mass dimension -2. The relevent scale is Λ, so this factor must be $1/\Lambda^2$. (Again, we follow the earlier logic that no other mass scales can enter, being negligible to the highly virtual loop momentum of the correction terms.) So here the correction would need to look something like

$$d_0(\Lambda/\Lambda_0)\frac{1}{\Lambda^2}\bar{u}u\bar{u}u \tag{3.1.12}$$

The actual value of the spinors structure only involves the low energy momenta of the theory, so it must be suppressed by the larger factor $1/\Lambda^2$. Therefore, these four fermion terms enter at a smaller order than the terms already discussed.

Nonrenormalizable cut-off theories

In the above, corrections small compared to the scale Λ were ignored, terms of order $\mathcal{O}(p/\Lambda)$, $\mathcal{O}(m/\Lambda)$ and so forth. If such terms are important, they may be calculated in the same general manner as outlined for the $\log \Lambda/\Lambda_0$ corrections. However, the logic above that prevented new terms from being introduced now fails, so new types of interaction are to be expected.

There are two key points where terms of this nature were discarded. The first was in the highly virtual loop integrals, where all scales smaller than k were neglected. The second was in considering the types of processes which might introduce corrections to the Lagrangian, where (for example) terms involving four fermions were subjected to dimensional analysis and found to be suppressed by $1/\Lambda^2$.

Going back to the loop integrals, now instead of simply discarding all terms involving p, p', or m_0, a Taylor expansion in p/Λ m_0/Λ (and so on) can be performed. As an example, instead of approximating $1/\{(p-k)\cdot\gamma) - m_0\}$ as $-1/k\cdot\gamma$

$$
\begin{aligned}
\frac{1}{(p-k)\cdot\gamma)-m_0} &\approx -\frac{1}{k\cdot\gamma(1-p\cdot\gamma/k\cdot\gamma+m_0/k\cdot\gamma)} \\
&\approx -\frac{k\cdot\gamma}{k^2}+\frac{p\cdot\gamma}{k^2}-\frac{m_0}{k^2}
\end{aligned}
$$

Systematically using such expansions, the high energy part of the loop calculations unaccounted for by the new cut-off theory can be found. The general form is

$$
T(k>\Lambda) = -iec_0\bar{u}(A_{\text{ext}}\cdot\gamma)u - \frac{ie_0m_0c_1}{\Lambda^2}\bar{u}(A^{\mu}_{\text{ext}}\sigma_{\mu\nu}(p-p')^{\nu})u - \frac{ie_0m_0c_2}{\Lambda^2}(p-p')^2\bar{u}(A_{\text{ext}}\cdot\gamma)u \tag{3.1.13}
$$

The form of these corrections can in some cases be found from explicit calculation. In such cases, the calculation will also fix the coefficients in the Lagrangian. However, often it is either difficult or impossible to explicitly "integrate out" high momentum contributions to the theory. One can already see above how much more tedious the integrations over intermediate momenta will become when polynomial terms in p and so forth are included. There will be a proliferation of new terms to account for in the loop integrals, and if pushed to the next order of correction an unpleasant combinatorical explosion will occur. The hope is to use NRQED to *simplify* the calculation.

And there also exist theories where explicit calculation is simply impossible, and the use of an effective Lagrangian is not just a convenience but a necessity. In low energy QCD it is impossible to work with a perturbative expansion of Feynman diagrams. Because of the strength of the coupling, such series do not converge.

In either case, there is an alternate method. For any particular calculation and given level of precision there will be a finite number of terms in the Lagrangian that contribute. The existence of terms is limited by two factors:

- First, there are direct constraints on the form of the Lagrangian: current conservation, Lorentz invariance, chiral symmetry and so on, and these will apply to each term separately

- Second, only terms of up to a particular order are kept

It is this second point that ensures that the number of terms is finite. Typically each building block that might be used to construct a term comes with at least one power of mass dimension. And the greater the mass dimension of the term, the stronger the suppression by $1/\Lambda$. Building blocks of order unity do exist, such as spin space operators, but there will be a finite basis for such.

Once the form of all possible terms is catalogued, how then to fix the coefficients before each? It is important that, no matter what, physical predictions obtained from either the effective theory or the high energy theory must coincide. So if the same physical process is calculated using both theories, then demanding equality of the two results will determine the coefficients of the low energy theory in terms of the original.

Still, that in principle the coefficients can be obtained by integrating out high momentum loops can still tell us something of their behavior. When all other energy scales were ignored, the constant c_0 could depend only on Λ/Λ_0. When they are included, it may additionally depend on m_0/Λ. It will never depend upon the momentum p, because such terms are instead included as new interactions with separate coefficients.

So c_0 will be the same coefficient calculated earlier, but with additional corrections of order $\mathcal{O}(m_0^2/\Lambda^2)$. The Lagrangian must also be augmented by the new interactions, so there is an additional correction of the form

$$\delta\mathcal{L} = \frac{e_0 m_0 c_1}{\Lambda^2}\bar{\Psi}F^{\mu\nu}\sigma_{\mu\nu}\Psi + \frac{e_0 m_0 c_2}{\Lambda^2}\bar{\Psi}i\partial_\mu F^{\mu\nu}\sigma_{\mu\nu}\partial_\nu\Psi \tag{3.1.14}$$

(In writing down the terms in the Lagrangian, momentum become derivatives of the fields, so for example $q = p' - p$ becomes a derivative of A_{ext}.)

In addition to these new higher order corrections to the already calculated quantities, there will be contributions to the Lagrangian of new processes. For instance, electron-electron scattering enters at the $\mathcal{O}(p^2/m^2)$. But of course it's not the case that all processes now enter — a process with 6 external legs would be suppressed by $1/\Lambda^4$ and not enter at the currently considered order.

These four fermion terms come from the contributions of other process than simple scattering off an external field. Rather, they come from integrating out the high momentum modes of processes such as electron-electron scattering. The loop diagram corrections to such processes involve in the high-energy theory, like the other loops mentioned, integrals over momentum higher than the cut-off. These intermediate states are highly virtual.

The uncertainty principle says that such high energy virtual states are allowed only if they exist for a correspondingly short amount of time. The result is that in the low energy theory they may be treated as effectively instantaneous interactions, appearing as local contact terms in the Lagrangian. This is how new multiple particle interactions arise in an effective theory – the high energy process becomes a new local interaction.

For electron-electron scattering at the order discussed, such terms would be the likes of

$$\delta\mathcal{L}_{4-\text{fermion}} = d_1 \frac{e_0^2}{\Lambda^2}(\Psi^\dagger\Psi)^2 + d_2 \frac{e_0^2}{\Lambda^2}(\Psi^\dagger\gamma\Psi)\cdot(\Psi^\dagger\gamma\Psi) \tag{3.1.15}$$

These coefficients would be fixed by calculating a process like electron-electron scattering in both theories. However, one would first have to fix the constants c_i. For in the new theory with a low cut-off there will be contributions to the scattering not only from contact terms, but also from tree level diagrams involving two 2-fermion vertices.

3.2 Nonrelativistic Quantum Electrodynamics

Above we explored how, by changing the cut-off in QED, new nonrenormalizable terms appeared, and a new effective theory emerged. It was suitable only for calculations below the new cut-off, which was chosen to be well above the scale of any other momenta or energy in the theory. However, one particularly fruitful use of this technique is the formulation of a theory of nonrelativistic quantum electrodynamics (NRQED).

NRQED is most useful when working with bound state systems. There are many important corrections to bound state energy levels that come from high energy physics. The more precisely one measures these energy levels, the more information is gained about the high energy theory. But only if the predictions of that theory have been worked out with the necessary precision.

While in principle the full QED theory can be used to accomplish this, it is unweildy and unsuited for the task. Most typically the systems studied are loosely bound and nonrelativistic. While calculating non-relativistic scattering in QED might not be too bad, the situation is different for a bound state. What spoils everything is the existence of new energy scales.

One such scale is the inverse of the Bohr radius. If μ is the reduced mass of the system and Ze the charge of the center, then

$$p \sim Z\mu\alpha = \frac{1}{r_{\text{Bohr}}} \tag{3.2.1}$$

is the typical momentum scale of the bound system.

In a QED scattering calculation, the order of a term may be addressed as the number of loops in a diagram. Thus we talk about tree level diagrams, one-loop diagrams, two-loop diagrams and so on, with the understanding that each loop carries with it a suppressing factor of α. In this bound state system the typical momentum scale may enter in a way that exactly cancels the loop factor. So to calculate contributions of order α an infinite number of diagrams must in fact be summed.

There are techniques of doing this, but clearly it becomes trickier to easily sort out what diagrams contribute at a particular order. The matter is made worse by the existence of a third distinct scale, the kinetic energy of each particle:

$$E = \frac{(Z\mu\alpha)^2}{m_i} \tag{3.2.2}$$

further complicating the process of calculating the contributions of a given order.

Well, QED is a high energy theory, while calculations should be easiest in a low energy theory formulated for this nonrelativistic regime. What is desired is a theory that lives at the appropriate scale, and thus avoids the business of summing infinite numbers of diagrams, but never-the-less incorporates all the effects of high energy physics.

Of course, an effective field theory has exactly these characteristics. The procedure is as follows. First write down the most general Lagrangian that obeys the symmetries of the theory. Of course here, in going from the high energy theory to the effective nonrelativistic, Lorentz symmetry is no longer required.

Once the form of the Lagrangian is fixed, the same physical process may be calculated in QED and NRQED. There is no problem in performing the QED calculation in this step because we don't have to choose a bound-state calculation – the scattering of free particles will suffice. The idea here is to find the simplest calculation that will fix the coefficients.

It is these coefficients that then contain all the information about the high energy theory. Bound state calculations may then be performed using the NRQED Lagrangian and diagrams, with the desired result: high energy physics is included, but we have a workable theory in the nonrelativistic regime. Instead of figuring out how to find a correct perturbation series in α, the theory uses expansions in terms of the other two nonrelativistic (and thus small) scales; basically expanding in terms of the velocity v as well as α.

Before we examine how the process works explicitly for NRQED, there is an alternate approach that should be examined first.

3.2.1 The Foldy-Wouthuyesen-Tani Transformation

As sketched in the previous section, to work with bound state problems it is simplest to have a nonrelativistic Lagrangian to work with. After all, in regular quantum mechanics it isn't so hard to calculate the levels of hydrogen. It is using a relativistic theory to crack a nonrelativistic nut that causes problems.

If the goal is to derive a nonrelativistic Lagrangian or Hamiltonian for the system, one approach is to start from the Dirac equation and find a way to express it nonrelativistically.

One such approach is the Foldy-Wouthuyesen-Tani transformation. Consider the Dirac

Lagrangian:

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi}\left(i\partial \cdot \gamma - eA \cdot \gamma - m\right)\Psi \tag{3.2.3}$$

If the system is nonrelativistic, then one of the important consequences is that electrons and positrons behave pretty much as independent particles. A representation can be chosen such that the upper and lower components of the bispinors Ψ are roughly equivalent to the electron and positron spinors. There will be some mixing, and it is the FWT transformation that finds a form for the bispinors in such a way that all the operators above become diagonal. Then, the upper and lower components are completely separated, and one can easily treat them as separate fields.

Technically this is impossible when an external field A_{ext} is present. However, a basis where the mixing between "particle" and "anti-particle" is arbitrarily small may be found perturbatively. The result will be an expansion in terms of $|\mathbf{p}|/m$, that relates the original upper and lower components to the desired "Shrodinger-like" spinors.

The second order result will be, for a Dirac bispinor u with upper and lower components ϕ and ξ:

$$u = \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} (1 - \frac{p^2}{8m^2})\phi_s \\ \frac{\sigma \cdot p}{2m}(1 - \frac{3p^2}{8m^2})\phi_s \end{pmatrix}$$

An important feature here is that the lower component is suppressed compared to the upper, by a factor of $\sigma \cdot p/2m$. But the overall normalisation must stay the same. This can actually be used to find the correct relations between relativistic and nonrelativistic wave functions without formally performing the transformation.

Once this is accomplished, the Lagrangian may be rewritten directly in terms of, say, the electron spinor. The relativistic bispinor has been replaced by a two-component nonrelativistic spinor. Additionally, all the other terms can be rewritten in terms of Galilean three vectors and scalars instead of Lorentz 4-vectors. The result is a manifestly nonrelativistic expression.

If an external electric field is considered acting on a single charged particle of mass m and charge e is considered, then to the second order

$$H = \frac{\mathbf{p}^2}{2m} + e\Phi - \frac{\mathbf{p}^4}{8m^3} - \frac{1}{4m^2}\boldsymbol{\sigma} \cdot \mathbf{E} \times \mathbf{p} - \frac{1}{8m^2}\boldsymbol{\nabla} \cdot \mathbf{E} \tag{3.2.4}$$

In writing the Hamiltonian, of course derivatives become momentum operators instead. Clearly, the higher order terms have relativistic origins — $\mathbf{p}^4/8m^3$, for instance, is the first relativistic correction to the kinetic energy $\mathbf{p}^2/2m$.

From this starting point ordinary Rayleigh-Schrodinger perturbation theory may be used. Because the theory is explicitly nonrelativistic, it will avoid all the scale problems with QED.

However, this isn't as powerful as an effective field theory. The idea here was to discard high energy processes rather than to incorporate them. For instance, the process

$$e^- e^+ \to \gamma \to e^- e^+ \tag{3.2.5}$$

involves a fundamentally relativistic intermediate photon that this formulation hasn't incorporated, and so relativistic corrections to the photon propagator will matter. If instead the techniques of effective field theory are employed, no process with such relativistic internal momenta need be considered. Instead, the effects of this process will be incorporated as four fermion contact terms.

As the uncertainty principle tells us, the higher the energy of the intermediate state, the shorter the length of the time that the state can exist. So a process with a relativistic internal state as above should happen instantaneously, acting just like a local contact interaction between four fermions.

When higher-order perturbation theory is attempted with the above Hamiltonian, it also will fail. The terms diverge, producing infinities. This of course has the same root cause.

So, if high precision is important and all high energy processes must be accounted for, this method isn't accurate enough. It does, of course, predict the leading order coefficients of NRQED.

3.2.2 An NRQED Calculation

Let us now consider the effective field theory approach to finding, from the relativistic theory, a nonrelativistic Lagrangian suitable for the bound-state problems already mentioned. As an example, consider a hydrogen-like muonium system with an electron and a muon. Formally the idea is to introduce into QED a cut-off at the nonrelativistic energy, λ. This cut-off is somewhere about the energy of the electron mass m_e. The higher energy states are removed, leaving an explicitly nonrelativistic Lagrangian.

As normal for an effective field theory, this Lagrangian will have an inifinte number of terms, but they can be arranged in a hierarchy that allows any particular calculation to be performed with only a finite number. This hierarchy can be treated as an expansion over the large mass scale of the system, $1/m$. In the example calculation, terms up to order $1/m^3$ will be kept.

First a Lagrangian is written that contains all the possible terms that might contribute to the process considered. Lorentz symmetry is no longer required, but the following constraints on the forms of terms exist:

- Galilean invariance
- Invariance under spatial reflection
- Invariance under time reversal
- Gauge invariance
- Hermiticity
- Locality

Once all the terms are catalogued, their coefficients must be determined. The idea is to consider some particular physical process. For the two formulations to be consistent, they must produce the same predictions for any such process. So something like a scattering amplitude can be calculated in both theories, and the result compared.

3.2.3 Comparison of QED and NRQED

The simplest process that will give the required information can be used. So any assumption that still distinguishes between needed terms in the NRQED Lagrangian may be used. In comparing scattering diagrams, it is often most useful to perform the calculation at threshold. Likewise, the frame of reference which most readily simplifies the calculation should be used. Any physical assumption must be applied to each calculation in the same way, of course – it must be the same physical process!

However, given the assumption of gauge invariance, different gauges may be chosen for each calculation. Since the physical measurement will be gauge invariant, there is nothing inconsistent about this. And indeed the most convenient gauges for a relativistic and nonrelativistic calculation often differ.

Remember that the goal is to use NRQED to calculate bound state energies. To this end, it is necessary that the theory has poles in the complex plain and can thus be analytically continued to include off-shell bound states. To ensure this, it is necessary to demand that all external particles be on mass-shell when fixing the coefficients. For the same reason, it is necessary to perform all the intermediate calculations with a finite photon mass. **Why? I don't really follow this point very well.**

So that the processes we compare have the same meaning, it is necessary for the S-matrix to have the same normalisation in both theories. Because of this, the normal relativistic normalisation of the QED spinor cannot be used. It is conventional to set $\bar{u}u = 2m$, but this will make it hard to find a sensible relation between the spinors of QED and NRQED. Instead, the normalisation $u^\dagger u = 1$ will be used.

Given that all these conditions are met, the process of comparing physical results will fix, without ambiguity, the coefficients of NRQED.

3.2.4 Construction of NRQED Lagrangian

In order to perform any comparison with QED, it is first necessary to catalogue all the terms that could arise in the Lagrangian. Let us work first with only the two-fermion terms.

What are the building blocks at our disposal? Each term will have two electron fields, possibly the external electromagnetic field, and derivatives of either type of field. There can also be mixing between the spin-space of the fermion fields.

The first constraint to apply is gauge-invariance. If only gauge invariant combinations are allowed, then instead of any mixture of derivatives and fields, only the long derivate $D = \partial + ieA$ is allowed, along with E, B and their derivatives.

The operators that mix spin for the spin-1/2 electron are just the Pauli spin matrices σ_i. Together with the identity they form a basis for the space of Hermitian spin-operators, so no quadratic terms in spin can appear. Those can be directly reduced with the identity $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk} S_k$.

Many of these "blocks" are written as three-vectors. For Galilean invariance to hold, there can be no dangling indices — all terms must be contracted, either with each other or the anti-symmetric tensor ϵ_{ijk}.

Lets consider now the symmetries of each type of term.

- The electric field is Hermitian, odd under parity and even under time reversal.
- The magnetic field is Hermitian, even under parity and odd under time reversal
- The long derivative is anti-Hermitian, odd under parity, and even under time reversal
- Spin is Hermitian, even under parity, odd under time reversal.

Given a collection of these blocks, then as long as the set as a whole isn't anti-Hermitian we can always simply add the Hermitian conjugate to form a Hermitian term. This does reduce the number of independent terms. Likewise, to make a term invariant under time-reversal a factor of i can be appended. However, there is no way to fix a term that is odd under parity, so that is the one hard constraint. There can be no term $\mathbf{B} \cdot \mathbf{D}$, for instance, because it is not invariant under parity.

To figure out how far down the hierarchy of terms we need to go for a particular calculation, we also need to establish the individual order of magnitude of each term. This will allow us to eliminate all but a finite number of terms from consideration. The example case is an electron in an atomic system, so A_0 is the Coulomb potential, and $\mathbf{B}$ comes from the spin of the muon.

$$\begin{aligned}
\partial &\sim mv \\
\partial_t &\sim mv^2 \\
eA_0 &\sim mv^2 \\
e\mathbf{A} &\sim mv^3 \\
e\mathbf{E} &\sim mv^3 \\
e\mathbf{B} &\sim mv^3
\end{aligned}$$

The magnitudes of the fields can be readily derived from their atomic origin. That the derivative operators have order of $p = mv$ when acting on the charged field is sensible, and the derivative of A_0 also gains a factor of mv. The spin operator is order unity (when we have $\hbar = 1$.)

Suppose we build the Lagrangian, including terms up to order $1/m^3$.

What will the overall structure of the Lagrangian be? There will be two fermion terms, four-fermion terms, and photon terms. The photon terms are taken as just that of QED at leading order: $\frac{1}{2}(F^{mu\nu})^2$.

Now examine the two-fermion part in more detail. It will include kinetic terms that are not renormalized and thus need no coefficients. The leading order terms will go as $1/m$, and there will then be corrections coming from additional terms.

Call the coefficients in this two-fermion Lagrangian c_i. We can expect that the leading order coefficients of these two-fermion terms should replicate exactly the results of doing a Foldy-Wouthuysen-Tani transformation. So for convenience they can be written such that they are, if they exist at this order, equal to 1. So such coefficients will have the form

$$c_i = 1 + \mathcal{O}(\alpha) = 1 + c_i^{(1)}\alpha + c_i^{(2)}\alpha^2 + \dots \tag{3.2.6}$$

If we consider terms up to $1/m^3$, then the two-fermion part of the Lagrangian is:

$$\begin{aligned}\mathcal{L}_{NRQED} =& \Psi^\dagger \Bigg\{ iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^2} + c_F\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + c_D\frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2} \\ &+ c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D})}{8m^2} + c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3} \\ &+ c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3}\Bigg\}\Psi\end{aligned} \tag{3.2.7}$$

In addition to the two-fermion terms, at the order $1/m^2$ four fermion contact terms are allowed. Since the QED Lagrangian has no exact four-fermion contact terms, these are terms that arise from the removal of four-fermion diagrams involving high momenta loops. In contrast with the two-fermion Lagrangian, label the coefficients of such terms d_i.

These coefficients will have a more complex structure than c_i. Any process involving two distinct fermion fields will have a richer set of energy scales and parameters to draw from.

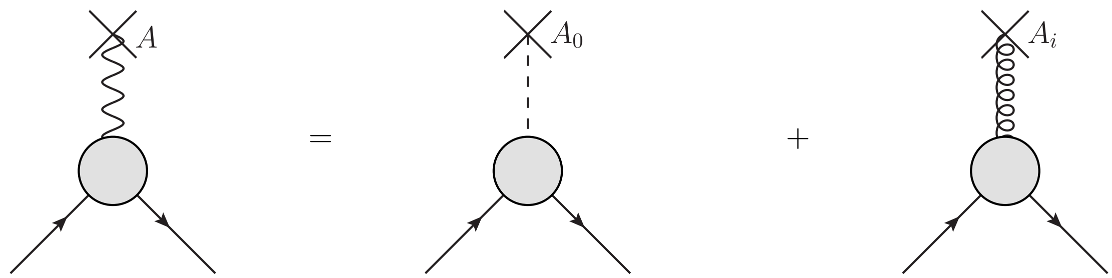
If the additional fermion has mass M and spinor χ, then the contact Lagrangian is:

$$\begin{aligned}\mathcal{L}_{\text{contact}} =& d_1\frac{1}{mM}(\Psi^\dagger\boldsymbol{\sigma}\Psi)\cdot(\chi^\dagger\boldsymbol{\sigma}\chi) + d_2\frac{1}{mM}(\Psi^\dagger\Psi)(\chi^\dagger\chi) \\ &+ d_3\frac{1}{mM}(\Psi^\dagger\boldsymbol{\sigma}\chi)\cdot(\chi^\dagger\boldsymbol{\sigma}\Psi) + d_4\frac{1}{mM}(\Psi^\dagger\chi)(\chi^\dagger\Psi)\end{aligned} \tag{3.2.8}$$

The terms with coefficients d_3 and d_4 only enter if the additional fermion is actually the anti-particle of the original.

3.2.5 Determination of coefficients

To determine the coefficients of the two-fermion piece of the NRQED Lagrangian, it will suffice to calculate the scattering of the electron off an external field. Every coefficient that needs to be fixed accompanies at least one term with a single power of the photon field. First the scattering amplitude will be calculated in QED, then compared to the NRQED terms. Schematically the equivalence can be written as:



A general one-photon vertex in QED may be expressed in terms of form factors $F_1(q^2)$ and $F_2(q^2)$. These encode all the information about radiative corrections, so the QED calculation here can conveniently be expressed with such factors.

The amplitude of scattering off a static vector potential is

$$e\bar{u}(p')\left[-\boldsymbol{\gamma}\cdot\mathbf{A}(\mathbf{q})F_1+\frac{i}{2m}\sigma^{ij}A^iq^jF_2\right]u(p) \tag{3.2.9}$$

or in terms of Pauli spinors instead, up to $1/m^3$

$$\begin{aligned}=&F_1\Psi^\dagger\left[-\frac{e}{2m}(\mathbf{p}'+\mathbf{p})\cdot\mathbf{A}-\frac{ie}{2m}\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A}+\frac{ie}{8m^3}(\mathbf{p}'^2+\mathbf{p}^2)\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A}\right]\Psi\\&+F_2\Psi^\dagger\left[-\frac{ie}{2m}\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A}+\frac{ie}{16m^3}(\mathbf{p}'^2+\mathbf{p}^2)\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A}+\frac{ie}{8m^3}\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A}\boldsymbol{\sigma}\cdot\mathbf{p}\right]\Psi\end{aligned} \tag{3.2.10}$$

For scattering off a static potential A_0

$$e\bar{u}(p')\left[\gamma^0 A^0 F_1 - \frac{i}{2m}\sigma^{0j}A^0 q^j F_2\right]u(\mathbf{p}) \tag{3.2.11}$$

Again expressing in terms of Pauli spinors

$$\begin{aligned}=&F_1\Psi^\dagger\left[eA^0 - \frac{e}{8m^2}\mathbf{q}^2 + \frac{ie}{4m^2}\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p}A^0\right]\Psi \\ &+F_2\Psi^\dagger\left[-\frac{e}{8m^2} + \frac{ie}{4m^2}\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p}A^0\right]\Psi\end{aligned} \tag{3.2.12}$$

So the electron-external field scattering, as calculated in QED, has been expressed in terms of mostly nonrelativistic quantities. There are still the form factors to expand, which depend on $q^2 = (q-0)^2 - \mathbf{q}^2$. But q_0 is of lower order than $\mathbf{q}$, so the leading order corrections will involve only $\mathbf{q}^2$. They are

$$\begin{aligned}F_1(q^2) &= 1 - \frac{\alpha}{3\pi}\left[\frac{\mathbf{q}^2}{m^2}\left(\ln\left(\frac{m}{\lambda}\right)\right)\right] \\ F_2(q^3) &= a_e - \frac{\alpha}{\pi}\frac{\mathbf{q}^2}{12m^2}\end{aligned}$$

They contain an explicit dependence on the finite photon mass λ.

The first part of the calculation of scattering from the NRQED Lagrangian is straightforward — only tree level vertices enter the calculation, so it can be read directly off the Lagrangian. However, this would fix the coefficients as having a direct dependence on the photon mass, rather than, as one might expect, the value of the cut-off. For instance, the coefficient of the Darwin term would be found to be

$$c_D^{QED} = 1 + \frac{\alpha}{\pi}\frac{8}{3}\left[\ln\left(\frac{m}{\lambda}\right) - \frac{3}{8}\right] \tag{3.2.13}$$

But of course, in calculating from NRQED at this level of precision perturbation theory terms will also enter. This introduces a further renormalization of the coefficients, in a similar manner to the production of counterterms in QED. Without further correction these perturbations would spoil the agreement of QED and NRQED. The solution is to introduce an additional term in the NRQED Lagrangian that simply subtracts off the unwanted term.

To again use the Darwin term as an example, it was found that in the absence of perturbation theory the value for c_D labelled c_D^{QED} would cause QED and NRQED to predict the same result. However, when perturbations are taken into account there will be an *additional* contribution, of the same form of the Darwin term, with some coefficient we can call c_D^{NRQED}. So the actual value of c_D should be adjusted in order to bring the two theories back into agreement:

$$c_D = c_D^{QED} - c_D^{NRQED} \tag{3.2.14}$$

The particular value found for c_D^{NRQED} is

$$c_D^{NRQED} = -\frac{\alpha}{\pi}\frac{8}{3}\left[\ln\left(\frac{\lambda}{2\Lambda}\right) + \frac{5}{6}\right] \tag{3.2.15}$$

Which means

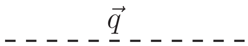
$$c_D = 1 + \frac{\alpha}{\pi}\frac{8}{3}\left[\ln\left(\frac{m}{\lambda}\right) - \frac{3}{8} + \frac{5}{6}\right] \tag{3.2.16}$$

With the result that the final correction to this coefficient depends upon the cut-off, just as with the regular renormalization theory. The same idea goes through to each other coefficient.

3.2.6 Vertices

To formally establish rules for the various vertices written above, it is necessary to strip off the external fields, and write derivatives of such fields in terms of their momentum. The coulomb potential A_0 becomes a Coulomb photon, so (for example) $\boldsymbol{\nabla}\cdot\mathbf{E}$ becomes the Darwin vertex, consisting of two fermion legs and a Coulomb photon, and a vertex with a factor of $-(e/8m^2)\mathbf{q}^2$.

Coulomb photon propagator

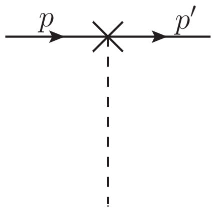


$$= \frac{1}{\mathbf{q}^2+\lambda^2}$$

Transverse photon propagator

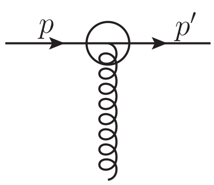
$$= \frac{\delta^{ij} - \frac{q^i q^j}{\mathbf{q}^2-\lambda^2}}{(q^0)^2 - \mathbf{q}^2 - \lambda^2 + ie}$$

Darwin vertex



$$= \frac{-e}{8m^2}\left|\mathbf{p'} - \mathbf{p}\right|^2$$

Fermi vertex



$$= \frac{ie}{2m}(\mathbf{p'} - \mathbf{p}) \times \boldsymbol{\sigma}$$

3.3 Constructing the NRQED Lagrangian

We want to construct an effective Lagrangian in the nonrelativistic limit. Our goal is to calculate the leading order corrections to the g-factor, which are corrections of order α^2. To this end, we need terms in the effective nonrelativistic Lagrangian which are equivalent corrections.

3.3.1 Constraints on the form of the Lagrangian

The Lagrangian is constrained to obey several symmetries. It must be invariant under the symmetries of parity and time reversal. It must also be invariant under Galilean transformations. The Lagrangian must also be Hermitian, and gauge invariant.

What are the gauge invariant building blocks we can use to construct this Lagrangian? We have the fields $\mathbf{E}$ and $\mathbf{B}$, the spin operators $\mathbf{S}$, and the long derivative $\mathbf{D} = \partial - ie\mathbf{A}$. The fields should always be accompanied by the charge e of the particle.

When considering the case of higher spin particles, we might consider terms quadratic and above in spin operators. For a particle of spin s, there must be $(2s+1)^2$ independent hermitian operators. We can span this set of operators by considering products of up to $2s$ spin matrices which are symmetric and traceless in every vector index. For example, for spin-1 we have quadratic, in addition to I and S_i, five independent structures of the form $S_iS_j + S_jS_i + \delta_{ij}\mathbf{S}$. This idea is pursued further in a section below.

We also have the scalar D_0, however, we need only include a single such term because we insist on having only one power of the time derivative.

Energy Scales

The effective Lagrangian contains an infinite number of terms. But these terms may be organised by the order of their contribution to scattering. Then, up to a particular order there are only a finite number of terms — and for the first couple of orders this finite number is quite small and manageable.

Consider again the particular physical situation of interest: a loosely bound state system of two charged particles, each with arbitrary spin, placed in an infinitesimal magnetic field. It is crucial that the system is loosely bound, so that the entire system is nonrelativistic. For now it is assumed that recoil effects are ignored.

Then what are the energy scales of this particular problem? In addition to the mass of the particle, there are two: one is related to the external magnetic field, an energy scale of $e/m\,|\mathbf{B}|$. The other is related to the kinetic energy or momentum of the particle. There is an electric field present as well, representing the interaction between the bound state particles. But by the quantum virial theorem the Coulomb potential is proportional to the kinetic energy, and so is not an independent energy scale. In general kinetic energy and momentum might be considered seprate energy scales, but in the nonrelativistic regime are just mv and $(1/2)mv^2$.

Both of these energy scales are nonrelativistic — that is, small compared to the mass of the particle. So the effective Lagrangian represents an expansion in the small quantities e/m^2B and $p/m = v$. However, the calculation of the gyromagnetic ratio involves only linear terms in the magnetic field. (The g-factor is defined by the linear response to an infinitesimal magnetic field.) So no terms quadratic in B are needed.

The goal is to calculate corrections to the g-factor of up to $\mathcal{O}(\alpha^2)$. In the bound state system $v \sim Z\alpha$, so in the Lagrangian should definitely be included terms of up to order $(e/m)Bv^2$. It will also be necessary to consider terms of order mv^4, for they *might* effect the calculation of the g-factor through perturbation theory.

What is the exact order of each term? The magnetic field defines its own scale, but the scale of operators D, A_0, and E come from the particular bound state.

The term involving the Coulomb potential, $eA_0 = Ze^2/r^2$ is of order mv^2. The first derivative $\partial_i \Phi = E_i$ is of order mv^3. The operator $\boldsymbol{\nabla}$ may then be said to be of order mv.

Discrete Constraints

All the building blocks have been chosen so that their transformation under rotation is clear. If a term has an index, it transforms like a 3-vector. (Considering this nonrelativistic theory, there are no upper or lower indices.) The Lagrangian must also be symmetric under the discrete symmetries of refleciton and time reversal, and terms must be Hermitian.

To construct terms that obey these symmetries, it is necessary to know how the building blocks themselves transform under spatial reflections and time reversal, as well as under complex conjugation.

- Under spatial reflections, $\mathbf{E}$ and $\mathbf{D}$ are vectors, and so are odd under P. While $\mathbf{B}$ and $\mathbf{S}$ are pseudovectors, even under P.

- E and D are even under time reversal, while spin and the magnetic field are odd. Also, under time reversal imaginary constant i is also odd.

- Under complex conjugation E, B and S are Hermitian, while D_i, D_0 and (obviously) i change sign.

For reference, all these properties are tabulated below:

	Order	P	T	†
eE_i	m^2v^3	-	+	+
eB_i	m^2v^2	+	-	+
D_i	mv	-	+	-
D_0	mv^2	+	-	-
S_i	1	+	-	+
i	1	+	-	-

3.3.2 Properties of D_i

Before going any further, note that it would be possible to write the entire Lagrangian without explicitly writing B. Instead we could write antisymmetric combinations of D_i.

Consider a term containing $D_i D_j \epsilon_{ijk}$. First write them using $\mathbf{D} = \boldsymbol{\nabla} - ie\mathbf{A}$:

$$(\mathbf{D} \times \mathbf{D}) = D_i D_j \epsilon_{ijk} = (\boldsymbol{\nabla}_i - ieA_i)(\boldsymbol{\nabla}_j - ieA_j)\epsilon_{ijk} \tag{3.3.1}$$

Or in full as

$$= (\boldsymbol{\nabla}_i \boldsymbol{\nabla}_j - ie\boldsymbol{\nabla}_i A_j - ie\boldsymbol{\nabla}_j A_i - e^2 A_i A_j)\epsilon_{ijk} \tag{3.3.2}$$

The antisymmetric tensor will kill the first and last terms, leaving

$$= -ie(\boldsymbol{\nabla}_i A_j + \boldsymbol{\nabla}_j A_i)\epsilon_{ijk} \tag{3.3.3}$$

We can use put this in the form of a commutator by switching indices in the second term: $\boldsymbol{\nabla}_j A_i \epsilon_{ijk} = -\boldsymbol{\nabla}_i A_j \epsilon_{ijk}$

$$= -ie[\boldsymbol{\nabla}_i, A_j]\epsilon_{ijk} \tag{3.3.4}$$

and of course the commutator between $\boldsymbol{\nabla}$ and any field will just be the derivative of that field.

$$= -ie\partial_i A_j \epsilon_{ijk} \tag{3.3.5}$$

Finally, from Maxwell's equations this is just $-ieB_k$, so

$$(\mathbf{D}\times\mathbf{D})_k = -ieB_k \tag{3.3.6}$$

Going the other way, the commutator $[D_i, B_i]$ can be shown to vanish:

$$[D_i, B_i] = D_i(\epsilon_{ijk}D_jD_k) - (\epsilon_{ijk}D_jD_k)D_i = \epsilon_{ijk}(D_iD_jD_k - D_jD_kD_i) \tag{3.3.7}$$

Taking the second term and taking an even permutation of the indices: $i \to k \to j \to i$ we show that

$$[D_i, B_i] = \epsilon_{ijk}(D_iD_jD_k - D_iD_jD_k) = 0 \tag{3.3.8}$$

3.3.3 Properties of spin operators

In formulating NRQED for general spin particles, we need to consider all the possible operators might show up in the Lagrangian. The state-space of a spin-s particle is the direct product of it's spin-state and all the other state information. Because the spaces are orthogonal, we can treat separately operators in the two spaces. The operators and fields which exist in position space are the same for a particle of any spin, but unsurpsingly the operators allowed in spin space do depend upon the spin of the particle. As the spin of the particle is increased, and thus its spin degrees of freedom rise, there are more ways to mix these components, and thus a greater number of spin operators to consider.

For a particular representation, we can always write a bilinear as the spin operators and other operators acting between two spinors:

$$\Psi^\dagger \mathcal{O}_S \mathcal{O}_X \Psi \tag{3.3.9}$$

The two types of operators will always commute, since they act on orthogonal spaces, so it doesn't matter what order they're written in. All such bilinears must be Galilean invariant, but individual operators might not be. The non-spin operators we consider, such as **D**, **B** or contractions with the tensor ϵ_{ijk} are already all written as 3-vectors or (in combination) as higher rank tensors. Therefore, it will be most convenient to write spin-operators in the same way, so that writing Galilean-invariant combinations of the two types of operators is done just by contracting indices.

Even though the number of spin opeartors does depend upon the spin of the particle, it is still possible to proceed in such a way that the same notation may be used no matter the spin. There are a few requirements:

- We write all high spin operators in terms of combinations of S_i, since these have universal properties regardless of the representation they are written in.
- If an operator exists and is non-zero in the representation of spin-s, it also exists in spin-$s+1$
- All operators introduced to account for the additional degrees of freedom in higher spin representations vanish when written in a lower spin theory. (As an example of the last point, the operator $S_iS_j + S_jS_i - \delta_{ij}S^2$ is needed to account for the degrees of freedom in a spin-1 theory, but vanishes in spin-1/2.
-)

If these requirements are met a consistent spin-agnostic notation can be adopted. Now we attempt to construct operators that meet these conditions.

The spinors Ψ are written with $2s+1$ independent components. The spin operators will be matrices acting on these components, and so for a spin-s particle will be $(2s+1)\times(2s+1)$ matrices. The combined operator $\mathcal{O}_S\mathcal{O}_X$ must be Hermitian, but without loss of generality we can require any $\mathcal{O}_S$, $\mathcal{O}_X$ to be Hermitian separately. So there is the additional constraint that the matrices be Hermitian, and this means a total of $(2s+1)^2$ degrees of freedom.

For spin-0 there is only one component to the spinor, so the only possible matrix is equivalent to the identity.

For spin-1/2 we have, in addition to the identity matrix, the spin matrices $s_i = \frac{1}{2}\sigma_i$. This is four independent matrices, and since the space has $(2s+1)^2 = 4$ degrees of freedom, exactly spans the space of all spin-operators. If we try to construct terms which are bilinear in spin matrices, they just reduce through the identity $\sigma_i\sigma_j = \delta_{ij} + \epsilon_{ijk}\sigma_k$, which we can already construct through combinations of the four operators we already have. Since those four operators form a basis for the space, independent bilinears were forbidden even without an explicit form for the equation.

What about spin-1? We need 9 independent operators to span the space. All the operators that exist in spin-1/2 will work here as well, though the spin matrices will have a different representation. That leaves 5 operators to construct. It is natural to try to construct these from bilinear combinations of spin matrices. Naively S_iS_j would itself be 9 independent structures, but clearly some of these are expressible in terms of the lower order operators. (By the order of a spin operator we mean its greatest degree in S_i)

Regardless of their representation, the spin operators always fulfill certain identities based on their Lie group. Namely

$$S_iS_j\delta_{ij} \sim I,\ [S_i, S_j] = \epsilon_{ijk}S_k \tag{3.3.10}$$

and it is these identities which allow certain combinations of S_iS_j to be related to lower order operators.

If instead of general spin bilinears we consider only combinations which are

- Symmetric in i, j
- Traceless

then such a structure will be independent of the set of operators $\{I, S_1, S_2, S_3\}$. Because it is symmetric no combination may be related using the commutator, and because it is traceless there is no combination that reduces due to the other identity.

This conditions form a set of 4 constraints, so from the original 9 degrees of freedom possessed by combinations of S_iS_j are left only 5. Together with the 4 lower order operators this is exactly enough to span the space.

We can explicitly write this symmetric, traceless structure as

$$S_i S_j + S_j S_i - \delta_{ij} S^2 \tag{3.3.11}$$

Having explored how the procedure works for spin-1, move on to consider the general spin case. The idea is to proceed inductively using the same rough attack as for the case of spin-1. In addition to all the "lower order" operators which were used for lower spin representations introduce new operators which are of higher degree in the spin matrices and guaranteed to be independent of the lower spin operators.

So suppose that for a spin-$s-1$ particle we have a set of operators written as $\bar{S}^0, \bar{S}^1, \ldots \bar{S}^{(s-1)}$, where a structure $\bar{S}^n$ carries n Galilean indices and is symmetric and traceless between any pair of indices, that is:

$$\bar{S}^n_{..i..j..} = \bar{S}^n_{..j..i..},\ \delta_{ij}\bar{S}^n_{..i..j..} = 0 \tag{3.3.12}$$

(From above, $\bar{S}^0 = I$, $\bar{S}^1_i = S_i$, and $\bar{S}^2 = S_i S_j + S_j S_i + \delta_{ij} S^2$.)

The objects $\bar{S}^n$ are built as follows: start with all combinations involving the product of exactly n spin matrices. (There are 3^n such structures.) Form them into combinations which are symmetric in all indices. Each index has three possible values, so we can label each structure by how many indices are equal to 1 and 2. If a is the number of indices equal to 1, and b the number of indices equal to 2, then for a given a there are $n + 1 - a$ possible choices for b. The total number of symmetric structures is then

$$\Sigma^n_{a=0}(n - a + 1) = \frac{1}{2}(n + 1)(n + 2) \tag{3.3.13}$$

We want to apply the additional constraint that the $\bar{S}^n$ be traceless in all indices. This will involve subtracting all the lower order structures which result when the trace of the completely symmetric combinations is taken.

It introduces an additional constraint on $\bar{S}^n$ for each pair of indices, and there are $n(n - 1)/2$ distinct pairs of indices. The total degrees of freedom left are

$$\frac{1}{2}(n+1)(n+2) - \frac{1}{2}n(n-1) = \frac{1}{2}\left(n^2 + 3n + 2 - n^2 + n\right) = 2n + 1 \tag{3.3.14}$$

In combination with the lower order spin operators, this is exactly the number of independent operators we need to span the space. Combined with the lower order operators this is a complete basis, so we know we haven't missed any terms. Because they are constructed to be independent from all the lower order operators, it must necessarily be true that they will vanish in lower spin representations.

Using this notation we can write down terms in the Lagrangian that are valid for particles of any spin. By writing all spin operators in terms of S_i they are representation agnostic, and by construction they will vanish for low spin particles where they do not "fit".

3.3.4 Composition of position space operators

There are several symmetries that terms in the Lagrangian must preserve: spatial reflection, time reversal, gauge invariance, and Hermiticity. Gauge invariance is taken care of by only considering gauge-invariant operators.

All of the possible spin-space structures preserve spatial parity. If the term in the Lagrangian is to obey such a symmetry, then any allowed collection of position-space operators must be themselves invariant under reflection.

Each term must also be invariant under time reversal. The spin operator S_i flips sign under time reversal. Thus, $\bar{S}_{ij}$ is even, and $\bar{S}_{ijk}$ odd, under the same transformation. E_i and D_i are even, while B_i is odd. Any set of operators with definite behavior under time reversal, though, can be made invariant by including an extra factor of i. So even if some term ABC is odd, $iABC$ will be even.

Finally, any sequence of operators can be made Hermitian by simply adding the Hermitian conjugate; if we wish to include a term ABC in the Lagrangian, we add $C^\dagger B^\dagger A^\dagger$. Since all the operator s under consideration are either Hermitian or anti-Hermitian, this means that exactly one of $ABC \pm CBA$ will be allowed. The spin space operators are all Hermitian of themselves, and commute with all other operators, so only the position space operators are nontrivial.

Thus for any possible set of operators, first it is determined if such a set is allowed by considerations of parity. Then, whether an additional factor of i is needed is determined by examining the properties under time reversal. Finally, the allowed Hermitian terms are enumerated.

Of the "building blocks" for the general Lagrangian, those odd in parity are also odd in v. And those even in parity are of an order even in v. So only terms of an order even in v can possibly be allowed under parity, and at the same time every such term will have the correct symmetry properties under spatial reflection.

First consider terms of leading order mv^2. What collections of position space operators exist? E is already too high of an order, leaving only that with two powers of D:

$$\text{Sets of } \mathcal{O}(mv^2) \text{ combinations} = \{\frac{1}{m}D_iD_j\} \tag{3.3.15}$$

This is in addition to the single permitted term containing $e\Phi$ which is not, by itself, gauge-invariant.

Next consider terms of order mv^4. We could have up to a single power of E, or only the long derivative operators:

$$\text{Set of } \mathcal{O}(mv^2) \text{ combinations} = \{\frac{1}{m^3}D_iD_jD_kD_\ell,\, \frac{1}{m^2}eE_iD_j\} \tag{3.3.16}$$

The leading order term in B is just $\frac{e}{m}B_i$.

The order $\frac{e}{m}B_iv^2$ terms are drawn from:

$$\text{Set of } \mathcal{O}(\frac{e}{m}Bv^2) \text{ combinations} = \{\frac{e}{m^3}D_iD_jB_k\} \tag{3.3.17}$$

Contraction of terms

In this nonrelativistic theory the Lagrangian need not be Lorentz invariant, but must still be Galilean invariant. All the operators we consider transform as 3-vectors or higher order 3-tensors. To form allowed terms, all indices must be somehow contracted.

Above, all relevant sets of position space operators are considered for each order of term. The greatest number of indices free was four. These must be contracted with order unity structures, which as well as the spin operators with up to four indices (S_i, $\bar{S}_{ij}$, $\bar{S}_{ijk}$,$\bar{S}_{ijk\ell}$) include δ_{ij} and the completely antisymmetric tensor ϵ_{ijk}.

- The only structure with one index is just S_i.
- With two indices, there are the simple structures $\bar{S}_{ij}$ and δ_{ij}, and also $S_i\epsilon_{ijk}$.
- With three indices, there are ϵ_{ijk} and $\bar{S}_{ijk}$ as well as $\delta_{ij}S_k$ or $\bar{S}_{ij}\epsilon_{jk\ell}$.
- The only position space operator with four indices is just $D_iD_jD_kD_\ell$. While there are a fairly large number of order unity structures with four indices, any with spin operators are forbidden because of the kinetic nature of the term. So only $\delta_{ij}\delta_{k\ell}$ need be considered.

Above are categorized the possible sets of position space operators for each order, and the ways of contracting them by number of indices. The next step is to write all Hermitian combinations we can form from these operators. This will form a complete catalogue of allowed terms in the Lagrangian.

Some terms which on the surface appear distinct might in fact be identical. One identity of this nature is

$$D_iD_j\epsilon_{ijk} = -ieB_k \tag{3.3.18}$$

Which means that any term where two long derivative terms are contracted with the anti-symmetric tensor will be equivalent to some already catalogued term involving B.

Similarly,

$$[D_i, D_j] = -ie(\partial_iA_j - \partial_jA_i) = -ieF_{ij} = -ie\epsilon_{ijk}B_k \tag{3.3.19}$$

so terms which appear distinct because of the ordering of two long derivative operators are really the same, up to some difference that is absorbed into a term involving B.

Finally, while in the general case $[D_i, B_j] \neq 0$, it is always true that $[D_i, B_j] = 0$.

So many apparently distinct terms may be rewritten and absorbed as the combination of several already accounted for terms. As an example, it is not necessary to write a term such as $D_iD_jD_iD_j$ in the Lagrangian. It can be rewritten as

$$\begin{aligned} D_iD_jD_iD_j &= D_iD_iD_jD_j + D_i[D_i, D_j]D_j \\ &= \mathbf{D}^4 - ieD_i\epsilon_{ijk}B_kD_j \\ &= \mathbf{D}^4 - ie\epsilon_{ijk}(D_iD_jB_k + [B_k, D_j]) \\ &= \mathbf{D}^4 - e^2B^2 + ie\epsilon_{ijk}D_i(\partial_jB_k) \end{aligned}$$

The term D_iD_j has two indices. There are three ways to contract it: with δ_{ij}, $\bar{S}_{ij}$, or $S_k\epsilon_{ijk}$. Since kinetic terms with spin are not considered, that only leaves one possible term: $\mathbf{D}^2$. This term is Hermitian in and of itself. It has mass dimension two, so the term as it appears in the Lagrangian will be:

$$D_iD_j \rightarrow \frac{\mathbf{D^2}}{2m} \tag{3.3.20}$$

Terms from the set E_iD_j also have two indices. It can be contracted with all three of δ_{ij}, $\bar{S}_{ij}$, or $iS_k\epsilon_{ijk}$. E_i, δ_{ij} and $\bar{S}_{ij}$ are Hermitian, while D_j and $iS_k\epsilon_{ijk}$ are anti-Hermitian. It is convenient to use that $[D_j, E_i] = \boldsymbol{\nabla}_jE_i$.

The Hermitian combination with δ_{ij} is

$$(D_iE_j - E_jD_i)\delta_{ij} \rightarrow \frac{\boldsymbol{\nabla}\cdot\mathbf{E}}{4m^2} = \frac{\boldsymbol{\nabla}\cdot\mathbf{E}}{4m^2} \tag{3.3.21}$$

The Hermitian combination with $\bar{S}_{ij}$ is

$$(D_iE_j - E_jD_i)\bar{S}_{ij} \rightarrow \frac{\bar{S}_{ij}\partial_iE_j}{4m^2} \tag{3.3.22}$$

The Hermitian combination with $iS_k\epsilon_{ijk}$ is

$$(D_iE_j + E_jD_i) \to iS_k\epsilon_{ijk} \to \frac{i\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E}+\mathbf{E}\times\mathbf{D})}{4m^2} \tag{3.3.23}$$

A single power of the magnetic field, B_i may only be contracted with S_i. So the term is

$$\frac{e}{m}\mathbf{S}\cdot\mathbf{B} \tag{3.3.24}$$

The second order terms involving the magnetic field are drawn from the set $D_iD_jB_k$. There are four order-unity structures which have three indices, but only two of those need be considered. This is because $\mathbf{D}\times\mathbf{D} \sim \mathbf{B}$ and $\mathbf{D}\times\mathbf{B} = 0$. If is not assumed that $[D_i, B_j] = 0$ then the allowed Hermitian combinations will be with $D_iD_jB_k + B_kD_jD_i$ and $D_iB_jD_k$.

While $[D_i, D_j] \neq 0$, it does produce another term proportional to B and thus, smaller than considered. So for instance, $D_iD_jB_i$ need not be considered separate from $D_jD_iB_i$.

$$\begin{aligned}
(D_iD_jB_k + B_kD_jD_i)\delta_{ij}S_k &\to \frac{e}{m}\frac{\mathbf{D}^2(\mathbf{S}\cdot\mathbf{B})+(\mathbf{S}\cdot\mathbf{B})\mathbf{D}^2}{4m^2}\\
(D_iD_jB_k + B_kD_jD_i)\delta_{jk}S_i &\to \frac{e}{m}\frac{(\mathbf{S}\cdot\mathbf{D})(\mathbf{D}\cdot\mathbf{B})+(\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})}{4m^2}\\
D_iB_jD_k\delta_{ij}S_k &\to \frac{e}{m}\frac{(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D})+(\mathbf{D}\cdot\mathbf{B})(\mathbf{S}\cdot\mathbf{B})}{4m^2}
\end{aligned}$$

$$\begin{aligned}
(D_iD_jB_k + B_kD_jD_i)\bar{S}_{ijk} &\to \frac{e}{m}\frac{\bar{S}_{ijk}(D_iD_jB_k + B_kD_jD_i)}{4m^2}\\
D_iB_jD_k\bar{S}_{ijk} &\to \frac{e}{m}\frac{\bar{S}_{ijk}D_iB_jD_k}{4m^2}
\end{aligned}$$

Finally, there is only one way to contract $D_iD_jD_kD_\ell$, which is as $\mathbf{D}^4$. Again, variations such as $D_i\mathbf{D}^2D_i$ need not be considered because they just reproduce already considered terms with B.

$$D_i D_j D_k D_\ell \delta_{ij}\delta_{k\ell} \quad \rightarrow \quad \frac{\mathbf{D}^4}{8m^3}$$

Having organised all the operators into all allowed combinations, we can write down the different orders of the Lagrangian.

The leading order terms in the Lagrangian, of order mv^2 and $(e/m)B$, are:

$$\mathcal{L}_{mv^2} = \Psi^\dagger \left\{ iD_0 + \frac{\mathbf{D}^2}{2m} + c_F \frac{e}{m}\mathbf{S}\cdot\mathbf{B} \right\} \Psi \tag{3.3.25}$$

The part of the Lagrangian of order mv^4 is:

$$\mathcal{L}_{mv^4} = \Psi^\dagger \left\{ \frac{\mathbf{D}^4}{8m^2} + c_D \frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2} + c_Q \frac{eQ_{ij}(D_iE_j - E_iD_j)}{8m^2} + c_S \frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}}{8m^2} \right\} \Psi \tag{3.3.26}$$

And the terms which are corrections to the magnetic field terms:

$$\begin{aligned} \mathcal{L}_{Bv^2} =& \Psi^\dagger \Bigg\{ c_{W1} \frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2} \frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3} + c_{p'p} \frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3} \\ & + c_{T_1} \frac{e\bar{S}_{ijk}(D_iD_jB_k + B_kD_jD_i)}{8m^3} + c_{T_2} \frac{e\bar{S}_{ijk}D_iB_jD_k}{8m^3} \Bigg\} \Psi \end{aligned} \tag{3.3.27}$$

Full Lagrangian

Combining all of the above, the full Lagrangian we consider is then:

$$
\begin{aligned}
\mathcal{L}_{NRQED} =&\Psi^\dagger\Bigg\{iD_0+\frac{\mathbf{D}^2}{2m}+\frac{\mathbf{D}^4}{8m^2}+c_F\frac{e}{m}\mathbf{S}\cdot\mathbf{B}+c_D\frac{e(\mathbf{D}\cdot\mathbf{E}-\mathbf{E}\cdot\mathbf{D})}{8m^2}+c_Q\frac{eQ_{ij}(D_iE_j-E_iD_j)}{8m^2}\\
&+c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D}}{8m^2}+c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B}+\mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3}-c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3}\\
&+c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D})+(\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3}+c_{T_1}\frac{e\bar{S}_{ijk}(D_iD_jB_k+B_kD_jD_i)}{8m^3}+c_{T_2}\frac{e\bar{S}_{ijk}D_iB_jD_k}{8m^3}\Bigg\}\Psi
\end{aligned}
\tag{3.3.28}
$$

One of the features of this Lagrangian is that every coefficient is fixed by the one-photon interaction. Although some terms might represent two-photon interactions, they are terms like $\mathbf{S}\cdot\mathbf{A}\times\mathbf{E}$, whose coefficient is fixed by the gauge-invariant term $\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}$. This in turn means that we can calculate the corrections to the g-factor by considering only one-photon interactions.

3.4 Scattering off external field in NRQED

We now have established the general form of the Lagrangian for NRQED of arbitrary spin. In order to fix the coefficients, it will be necessary to compare the predictions of NRQED with some other source, such as an exact relativistic theory.

3.4.1 One-photon Lagrangian

The simplest (and most important for our purposes) set of coefficients to fix are those involved in the process of the charged particle scattering off an external field. We can write down those terms in the NRQED Lagrangian which have one power of the external field. This set of terms will not, by themselves, be gauge invariant. However, we had previously taken care to group terms in a gauge invariant way. So first we must untangle from these gauge invariant terms (involving one or more powers of $\mathbf{D}$), those which involve a single field.

First take the kinetic terms. $\mathbf{D} = \boldsymbol{\nabla} - ie\mathbf{A}$, so

$$\mathbf{D}^2 = (\nabla_i - ieA_i)(\nabla_i - ieA_i) \tag{3.4.1}$$

It is a mixture of terms with two, one, or no powers of the external field A. If just the terms with one power of $\mathbf{A}$ are included, what remains is

$$-ie(A_i\nabla_i + A_i\nabla_i = -ie\{A_i, \nabla_i\} \tag{3.4.2}$$

So from $\mathbf{D}^2/2m$ we can write $-ie\{\nabla_i, A_i\}/2m$ in $\mathcal{L}_A$.

The second kinetic term is $\mathbf{D}^4$:

$$\mathbf{D}^4 = (\nabla_i - ieA_i)(\nabla_i - ieA_i)(\nabla_j - ieA_j)(\nabla_j - ieA_j) \tag{3.4.3}$$

or

$$\mathbf{D}^4 = (\boldsymbol{\nabla}^2 - ie\{A_i, \nabla_i\} - e^2\mathbf{A}^2)(\boldsymbol{\nabla}^2 - ie\{A_j, \nabla_j\} - e^2\mathbf{A}^2) \tag{3.4.4}$$

Keeping again only the terms with a single power of A, what remains may be expressed as the double anti-commutator

$$-ie\{\boldsymbol{\nabla}^2, \{\nabla_i, A_i\}\} \tag{3.4.5}$$

There are then several terms involving one or more powers of D combined with either E or B. Since only terms with a single power of the field are of interest, from all such $\mathbf{D}$ only the part involving $\boldsymbol{\nabla}$ is kept.

So finally, writing all such terms from the original Lagrangian involved in scattering off an external field leaves:

$$\begin{aligned}\mathcal{L}_A \;=\;& \Psi^\dagger\Big(-eA_0 - ie\frac{\{\nabla_i, A_i\}}{2m} - ie\frac{\{\boldsymbol{\nabla}^2, \{\nabla_i, A_i\}\}}{8m^3} + c_F e\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{e(\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{E}\cdot\boldsymbol{\nabla})}{8m^2} + c_Q\frac{eQ_{ij}(\nabla_i E_j - E_i\nabla_j)}{8m^2} \\ & + c_S^1\frac{ie\mathbf{S}\cdot(\boldsymbol{\nabla}\times\mathbf{E} - \mathbf{E}\times\boldsymbol{\nabla})}{8m^2} + c_{W_1}\frac{e[\boldsymbol{\nabla}^2(\mathbf{S}\cdot\mathbf{B}) + (\mathbf{S}\cdot\mathbf{B})\boldsymbol{\nabla}^2]}{8m^3} - c_{W_2}\frac{e\nabla^i(\mathbf{S}\cdot\mathbf{B})\nabla^i}{4m^3} + c_{p'p}\frac{e[(\mathbf{S}\cdot\boldsymbol{\nabla})(\mathbf{B}\cdot\boldsymbol{\nabla}) + \cdots}{8m^3 \cdots}\end{aligned}$$

3.4.2 Calculation

Now we have the Lagrangian that will account for all interactions with a single photon. We want to calculate from this a particular process: scattering off an external field, with incoming momentum $\mathbf{p}$, outgoing $\mathbf{p}'$, and $\mathbf{q} = \mathbf{p}' - \mathbf{p}$. There is one diagram associated with each term above, but the total amplitude is just going to be the sum of all these one-photon vertices. These of course can just be read off directly from the Lagrangian.

It is necessary to switch to the language of momentum space. The recipe is this: replace the fields Ψ with the spinors ϕ, and any operator $\boldsymbol{\nabla}$ acting will become $i\mathbf{p}$ if it acts on the right, $i\mathbf{p}'$ if it is to the left. Go through term by term.

The terms originating from D^2 are:

$$-ie\frac{\{\nabla_i, A_i\}}{2m} \tag{3.4.6}$$

which in position space become

$$e\frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')}{2m} \tag{3.4.7}$$

While the terms arising from D^4

$$-ie\frac{\{\boldsymbol{\nabla}^2, \{\nabla_i, A_i\}\}}{8m^3} \tag{3.4.8}$$

become

$$-e\frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')(\mathbf{p}'^2+\mathbf{p}^2)}{8m^3} \tag{3.4.9}$$

We can simplify some expressions involving $\boldsymbol{\nabla}$ and $\mathbf{E}$: Because Q_{ij} is symmetric:

$$Q_{ij}(\nabla_i E_j - E_i \nabla_j) = Q_{ij}[\nabla_i, E_j] = Q_{ij}(\partial_i E_j)$$

And because $E_i = -\partial_i \Phi$

$$\boldsymbol{\nabla}\times\mathbf{E} - \mathbf{E}\times\boldsymbol{\nabla} = -2\mathbf{E}\times\boldsymbol{\nabla}$$

And also use that

$$\boldsymbol{\nabla}\cdot\mathbf{E}-\mathbf{E}\cdot\boldsymbol{\nabla}=(\partial_i E_i)$$

Now we can write down the scattering amplitude for scattering off the external field, before we apply any assumptions about the particular process.

$$\begin{aligned}
iM \;=\;& ie\phi_S^\dagger\Bigg(-A_0+\frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')}{2m}-\frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')\mathbf{p}^2+\mathbf{p}'^2\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')}{8m^3}\\
&+c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m}+c_D\frac{(\partial_i E_i)}{8m^2}+c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2}+c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2}\\
&-c_{W_1}\frac{(\mathbf{S}\cdot\mathbf{B})(\mathbf{p}^2+\mathbf{p}'^2)}{8m^3}+c_{W_2}\frac{(\mathbf{S}\cdot\mathbf{B})(\mathbf{p}\cdot\mathbf{p}')}{4m^3}-c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p}')(\mathbf{B}\cdot\mathbf{p})+(\mathbf{B}\cdot\mathbf{p}')(\mathbf{S}\cdot\mathbf{p})}{8m^3}\Bigg)\phi_S
\end{aligned}$$

The above can be simplified somewhat. We choose our gauge such that $\nabla_i A_i=0$. If we specify elastic scattering then kinematics dictate that $\mathbf{p}'^2=\mathbf{p}^2$. Finally, if we consider $\mathbf{B}$ constant, the c_W terms become indistinguishable, since $[\nabla_i,B_j]=0$. (It is only this last assumption that costs us any information.) Then the scattering amplitude, as calculated from $\mathcal{L}_{NRQED}$, is:

$$\begin{aligned}
iM =& ie\phi_S^\dagger\Bigg(-A_0+\frac{\mathbf{A}\cdot\mathbf{p}}{m}-\frac{(\mathbf{A}\cdot\mathbf{p})\mathbf{p}^2}{2m^3}+c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m}+c_D\frac{(\partial_i E_i)}{8m^2}+c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2}\\
&+c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2}-(c_{W_1}-c_{W_2})\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{4m^3}-c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3}\Bigg)\phi_S
\end{aligned}\tag{3.4.10}$$

This may be compared with a relativistic calculation to fix the coefficients. (Although of course it does not fix c_{W_1} and c_{W_2} separately.)

3.4.3 Two photon scattering in NRQED

The calculation above suffices to fix all coefficients in the NRQED Lagrangian needed to calculate corrections to the g-factor. While there is a contribution that comes from a term with two electromagnetic fields, the coefficient of that term is still fixed by just considering the one-photon interaction, due to the necessity of the overall gauge-invariance of the term.

Still, we *can* calculate the coefficient separately from an exact theory, so it would be good to be able to compare the results to make sure no surprises occur. To that end it will be necessary to calculate a two-photon process, Compton scattering.

So, we want to calculate the Compton scattering in the nonrelativistic theory. We choose the gauge such that the photon polarisations obey $a_0 = 0$. In general we should consider both terms arising from two-photon vertices, and those from tree level diagrams of two one-photon vertices. However, because of the approach we take in calculating the process from the relativistic Lagrangian, we only need the former terms. That is, we know how to separate contact terms (which arise from some combination of Z-diagrams and two-photon vertices in the relativistic theory) from the rest.

Further, ultimately we only care about a few of these terms, ignoring terms which have both $\mathbf{B}$ and $\mathbf{A}$.

Note that both $(\mathbf{A}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{A}) = 0$ and $Q_{ij}(A_iE_j - E_iA_j) = 0$ by symmetry.
The remaining terms we're interested in are:

$$\mathcal{L}_{A^2} = \Psi^\dagger\left(-\frac{e^2\mathbf{A}^2}{2m} - e^2\frac{\{\boldsymbol{\nabla}^2, \mathbf{A}^2\}}{8m^3} - e^2\frac{\{\nabla_i, A_i\}\{\nabla_j, A_j\}}{8m^3} + c_S^2\frac{e^2\mathbf{S}\cdot(\mathbf{A}\times\mathbf{E} - \mathbf{E}\times\mathbf{A})}{8m^2}\right)\Psi$$

The process we consider has an incoming photon with momentum k and polarisation a, and an outgoing photon with momentum $-k'$ and polarisation a'. The charged particle has incoming momentum p and outgoing $p' = p + k + k'$.

As in the single-photon calculation, it is simply a matter of reading terms off the Lagrangian. If we want the scattering amplitude, we replace Ψ with ϕ, and replace $\mathbf{A}$ with photon polarisations a and a'. In the gauge chosen, $\mathbf{E}(k) = -\partial_0\mathbf{a} = ik_0\mathbf{a}(k)$.

Contracted with the photon of momentum k we get $\mathbf{E} \to ik_0\mathbf{a}$, while with the photon of momentum k' we have $\mathbf{E} \to ik_0'\mathbf{a}'$. Both processes must be considered in calculating the scattering, so:

$$\mathbf{A}\times\mathbf{E} = -\mathbf{A}\times(\partial_0\mathbf{A}) \to -i(k_0'\mathbf{a}\times\mathbf{a}' + k_0\mathbf{a}'\times\mathbf{a}) = i(k_0' - k_0)\mathbf{a}\times\mathbf{a}'$$

And

$$\mathbf{E} \times \mathbf{A} = -(\partial_0 \mathbf{A}) \times \mathbf{A} \to -i(k_0 \mathbf{a} \times \mathbf{a}' + k_0' \mathbf{a}' \times \mathbf{a}) = -i(k_0' - k_0)\mathbf{a} \times \mathbf{a}'$$

So from the term in the Lagrangian

$$c_S \Psi^\dagger \frac{e^2 \mathbf{S} \cdot (\mathbf{A} \times \mathbf{E} - \mathbf{E} \times \mathbf{A})}{8m^2})\Psi$$

we get in the scattering amplitude

$$-ic_S \phi^\dagger \Big(\frac{e^2}{4m^2} i(k_0 - k_0')\mathbf{a} \times \mathbf{a}' \Big) \phi = c_S \frac{e^2}{4m^2} \phi^\dagger \Big((k_0 - k_0')\mathbf{a} \times \mathbf{a}' \Big) \phi$$

This is the only part of the amplitude we wanted to compare for reasons of consistency.

Chapter 4

Spin one-half

4.0.4 Conventions

Spinors:

$$u = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \tag{4.0.1}$$

$$\begin{aligned} \eta &= \left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w \\ \xi &= \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2m} \left(1 - \frac{3\mathbf{p}^2}{8m^2}\right) w \end{aligned}$$

structures (in the useful represenation)

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \tag{4.0.2}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{4.0.3}$$

$$\sigma^{\mu\nu} = i\frac{1}{2}[\gamma^\mu, \gamma^\nu] \tag{4.0.4}$$

$$[\gamma^0, \gamma^i] = 2\gamma^0\gamma^i = 2\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{4.0.5}$$

$$\gamma^i\gamma^j = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_i\sigma_j & 0 \\ 0 & -\sigma_i\sigma_j \end{pmatrix} \tag{4.0.6}$$

$$[\gamma^i, \gamma^j] = \begin{pmatrix} [\sigma_j, \sigma_i] & 0 \\ 0 & [\sigma_j, \sigma_i] \end{pmatrix} = i\epsilon_{jik}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} = -i\epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \tag{4.0.7}$$

4.0.5 QED calculations

First form:

$$(p+p')^\mu \bar{u}u = (p+p')^\mu \left(\eta^\dagger\eta - \xi^\dagger\xi\right) \tag{4.0.8}$$

$$= (p+p')^\mu \left\{ w^\dagger \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right)\left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w - w^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{2m}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\right) w \right\} \tag{4.0.9}$$

$$= (p+p')^\mu w^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{8m^2} - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2}\right) w \tag{4.0.10}$$

Second form: For the term $\bar{u}\gamma^\mu u$ it'll be necessary to treat the spatial/time-like indices separately.

time-like

$$\bar{u}\gamma^0 u = u^\dagger u \tag{4.0.11}$$

$$= \eta^\dagger\eta + \xi^\dagger\xi \tag{4.0.12}$$

$$= w^\dagger \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right)\left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w + w^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{2m}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\right) w \tag{4.0.13}$$

$$= w^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{8m^2} + \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2}\right) w \tag{4.0.14}$$

spatial

$$\bar{u}\gamma^i u = u^\dagger\gamma^0\gamma^i u \tag{4.0.15}$$

$$= \bar{u}^\dagger \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} u \tag{4.0.16}$$

$$= \eta^\dagger\sigma_i\xi + \xi^\dagger\sigma_i\eta \tag{4.0.17}$$

$$= w^\dagger \left\{ \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right)\sigma_k\left(1 - \frac{3\mathbf{p}^2}{8m^2}\right) - \left(1 - \frac{3\mathbf{p}'^2}{8m^2}\right)\sigma_k\left(1 - \frac{\mathbf{p}^2}{8m^2}\right) - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_k\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2}\right\} \tag{4.0.18}$$

Third type (tensor)

$$\bar{u}\frac{i}{2m}q_j\sigma^{ij}u = \frac{i\epsilon_{ijk}q_j}{2m}\bar{u}\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} u \tag{4.0.19}$$

$$\frac{i\epsilon_{ijk}q_j}{2m}\left(\eta^\dagger\sigma_k\eta - \xi^\dagger\sigma_k\xi\right) \tag{4.0.20}$$

$$\frac{i\epsilon_{ijk}q_j}{2m}w^\dagger\left\{\left(1 - \frac{\mathbf{p}'^2}{8m^2}\right)\sigma_k\left(1 - \frac{\mathbf{p}'^2}{8m^2}\right) - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_k\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2}w\right\} \tag{4.0.21}$$

Need triple sigma identity

$$\sigma_a\sigma_b\sigma_c = \sigma_a(\delta_{bc} + i\epsilon_{bcd}\sigma_d) = \sigma_a\delta_{bc} - \sigma_b\delta_{ca} + \sigma_c\delta_{ab} + i\epsilon_{abc} \tag{4.0.22}$$

Then using above

$$\bar{u}\frac{i}{2m}q_j\sigma^{ij}u = \frac{i\epsilon_{ijk}q_j}{2m}w^\dagger\left\{\sigma_k\left(1-\frac{\mathbf{p}'^2+\mathbf{p}^2}{8m^2}\right)-\frac{\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{p}')p_k-\sigma_k\mathbf{p}\cdot\mathbf{p}'+i\epsilon_{akc}q_ap_c}{4m^2}\right\}w \tag{4.0.23}$$

The 'time-like' part of the tensor term

$$\bar{u}\frac{i}{2m}q_j\sigma^{0j}u = -\frac{q_j}{2m}\bar{u}\gamma^0\gamma^j u \tag{4.0.24}$$

$$= -\frac{q_j}{2m}u^\dagger\gamma^j u \tag{4.0.25}$$

$$= -\frac{q_j}{2m}\left(\eta^\dagger\sigma_j\chi-\chi^\dagger\sigma_j\eta\right) \tag{4.0.26}$$

$$= -\frac{q_j}{2m}w^\dagger\left\{\left(1-\frac{\mathbf{p}'^2}{8m^2}\right)\frac{\sigma_j\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\left(1-\frac{3\mathbf{p}^2}{8m^2}\right)-\left(1-\frac{3\mathbf{p}'^2}{8m^2}\right)\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_j}{2m}\left(1-\frac{\mathbf{p}^2}{8m^2}\right)\right\}w \tag{4.0.27}$$

Dropping terms quadratic in q, all p' can be written just as p.

$$\approx -\frac{q_j}{2m}w^\dagger\left\{\frac{\sigma_j\boldsymbol{\sigma}\cdot\mathbf{p}-\boldsymbol{\sigma}\cdot\mathbf{p}\sigma_j}{2m}\left(1-\frac{\mathbf{p}^2}{2m^2}\right)\right\}w \tag{4.0.28}$$

$$= \frac{q_j}{2m}w^\dagger\left\{\frac{i\epsilon_{ijk}\sigma_kp_i}{2m}\left(1-\frac{\mathbf{p}^2}{2m^2}\right)\right\}w \tag{4.0.29}$$

$$= w^\dagger\left\{\frac{i\epsilon_{ijk}p_iq_j\sigma_k}{4m^2}\left(1-\frac{\mathbf{p}^2}{2m^2}\right)\right\}w \tag{4.0.30}$$

4.1 Fouldy-Wouthyusen approach

Chapter 5

General Spin Formalism

Our ultimate goal is to calculate corrections to the g-factor of a loosely bound charged particle of arbitrary spin. Our strategy is to obtain an effective Lagrangian in the nonrelativistic limit.

We first consider features of a general-spin formalism in both the relativistic and nonrelativistic cases, and the connection between the wave functions of the free particles. Then we consider how constraints of the relativistic theory let us calculate scattering off an external field. Comparing this result to that done with an effective NRQED Lagrangian, we can obtain the coefficients of that Lagrangian for particles of general spin.

5.0.1 Spinors for general-spin charged particles

Relativistic bispinors

First we need to work out a formalism that will apply to the general spin case. We want to represent the spin state of the particles by an object that looks like a generalization of the Dirac bispinor.

It is easiest to start with the Dirac basis, where the upper and lower components of the bispinor are objects of opposite helicity, each transforming as an object of spin 1/2.

To that end define an object

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{5.0.1}$$

that we wish to have the appropriate properties. Each component should transform as a particle of spin s, but with opposite helicity. Under reflection the upper and lower components transform into each other.

Representations of the proper Lorentz group are spinors which are separately symmetric in dotted and undotted indices. If ξ is an object with p undotted and q dotted indices

$$\xi = \{\xi^{\alpha_1 \ldots \alpha_p}_{\dot{\beta}_1 \ldots \dot{\beta}_q}\} \tag{5.0.2}$$

Then this can be a representation of a particle of spin $s = (p+q)/2$.

We have some free choice in how to partition the dotted/undotted indices, and we cannot choose exactly the same scheme for all spin as long as both types of indices are present. However, we can make separately consistent choices for integral and half-integral spin. For integral spin we can say $p = q = s$, while for the half-integral case we'll choose $p = s + \frac{1}{2}$, $q = s - \frac{1}{2}$.

We want the ξ and η to transform as objects of opposite helicity. Under reflection they will transform into each other. So

$$\eta = \{\eta^{\beta_1 \ldots \beta_q}_{\dot{\alpha}_1 \ldots \dot{\alpha}_p}\} \tag{5.0.3}$$

In the rest frame of the particle, they will have clearly defined and identical properties under rotation. The rest frame spinors are equivalent to rank $2s$ nonrelativistic spinors. So the bispinor in the rest frame looks like

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_0 \\ \xi_0 \end{pmatrix} \tag{5.0.4}$$

where

$$\xi_0 = \{(\xi_0)_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q}\} \tag{5.0.5}$$

and all indices are symmetric.

We can obtain the spinors in an arbitrary frame by boosting from the rest frame. The upper and lower components we have defined to have opposite helicity, and so will act in opposite ways under boost:

$$\xi = \exp\left(\frac{\mathbf{\Sigma} \cdot \phi}{2}\right)\xi_0, \qquad \eta = \exp\left(-\frac{\mathbf{\Sigma} \cdot \phi}{2}\right)\xi_0 \tag{5.0.6}$$

What form should the operator $\mathbf{\Sigma}$ have? Under an infinitesimal boost by a rapidity ϕ, a spinor with a single undotted index is transformed as

$$\xi_\alpha \to \xi'_\alpha = \left(\delta_{\alpha\beta} + \frac{\boldsymbol{\phi} \cdot \boldsymbol{\sigma}_{\alpha\beta}}{2}\right)\xi_\beta$$

while one with a dotted index will transform as

$$\xi_{\dot\alpha} \to \xi'_{\dot\alpha} = \left(\delta_{\dot\alpha\dot\beta} - \frac{\boldsymbol{\phi} \cdot \boldsymbol{\sigma}_{\dot\alpha\dot\beta}}{2}\right)\xi_{\dot\beta}$$

The infinitesimal transformation of a higher spin object with the first p indices undotted and the last q dotted would then be

$$\xi \to \xi' = \left(1 + \sum_{a=0}^{p} \frac{\boldsymbol{\sigma}_a \cdot \boldsymbol{\phi}}{2} - \sum_{a=p+1}^{p+q} \frac{\boldsymbol{\sigma}_a \cdot \boldsymbol{\phi}}{2}\right)\xi$$

where a denotes which spinor index of ξ is operated on.

If we define

$$\mathbf{\Sigma} = \sum_{a=0}^{p} \boldsymbol{\sigma}_a - \sum_{a=p+1}^{p+q} \boldsymbol{\sigma}_a \tag{5.0.7}$$

Then the infinitesimal transformations would be

$$\xi \to \xi' = \left(1 + \frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi$$

$$\eta \to \eta' = \left(1 - \frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \eta$$

So the exact transformation should be

$$\xi \to \xi' = \exp\left(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi$$

$$\eta \to \eta' = \exp\left(-\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \eta$$

Therefore, the bispinor of some particle boosted by $\boldsymbol{\phi}$ from rest will be

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \exp\left(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2}\right) \xi_0 \\ \exp\left(\frac{-\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2}\right) \xi_0 \end{pmatrix} \tag{5.0.8}$$

In dealing with the relativistic theory, we'll want a basis that separates the particle and antiparticle parts of the wave function. If we want the upper component to be the particle, then in the rest frame the lower component will vanish, and for low momentum will be small compared to the upper component. The unitary transformation which accomplishes this is

$$\Psi' = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

$$\phi = \frac{1}{\sqrt{2}}(\xi + \eta)$$

$$\chi = \frac{1}{\sqrt{2}}(\eta - \xi)$$

Which is equivalent to

$$\Psi' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Psi$$

Then,

$$\phi = \cosh\left(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi_0 \tag{5.0.9}$$

$$\chi = \sinh\left(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi_0 \tag{5.0.10}$$

Spinors for nonrelativistic theory

We also need to discuss the nonrelativistic, single-particle theory. This is much simpler: the spin state of particles in this theory is represented by symmetric spinors with $2s+1$ undotted indices. The only operators we need to consider acting on this space are spin matrices and products of spin matrices.

5.0.2 Electromagnetic Interaction

Knowing how the wave functions themselves behave, we want to see what that tells us about possible electromagnetic interaction. Interaction with a single electromagnetic photon should take the form

$$M = A_\mu j^\mu$$

where j^μ is the electromagnetic current.

The electromagnetic current must be built out of the particle's momenta and bilinears of the charged particle fields in such a way that they have the correct Lorentz properties. We must also demand current conservation: the equation $q_\mu j^\mu = 0$ must hold. Above we already have shown that, in the case of general spin, there exist only two such bilinears, a scalar and a tensor.

There will be two permissible terms in the current. We could consider a scalar bilinear coupled with a single power of external momenta. In order to fulfill the current conservation requirement, it should be

$$\frac{p^\mu + p'^\mu}{2m}\bar{\Psi}^\dagger\Psi$$

This will obey current conservation because $q = p' - p$, and $(p + p') \cdot (p' - p) = p^2 - p'^2 = 0$

We can also consider a tensor term contracted with a power of momenta. To fulfill current conservation, we can demand that the tensor bilinear be antisymmetric, and contract it with q:

$$\frac{q_\nu}{2m}\bar{\Psi}^\dagger\Sigma^{\mu\nu}\Psi$$

We don't need to worry about higher order tensor bilinears: they will necessitate too many powers of the external momenta.

So the most general current would look like

$$j^\mu = F_e\frac{p^\mu + p'^\mu}{2m}\bar{\Psi}^\dagger\Psi + F_m\frac{q_\nu}{2m}\bar{\Psi}^\dagger\Sigma^{\mu\nu}\Psi \tag{5.0.11}$$

In general the form factors might have quite complicated dependence on q, but these corrections will be too small compared to the type of result we're interested in. At leading order F_e will just be the electric charge of the particle in question, and F_m will, as we'll see after connecting this result to the nonrelativistic limit, be related to the particle's g-factor. So to the order we need, we can write the current as

$$j^\mu = e\frac{p^\mu + p'^\mu}{2m}\bar{\Psi}^\dagger\Psi + eg\frac{q_\nu}{2m}\bar{\Psi}^\dagger\Sigma^{\mu\nu}\Psi \tag{5.0.12}$$

This captures the essence of the interaction between a charged particle of general-spin and a single photon.

Connection between the spinors of the two theories

Knowing something of how the relativistic theory behaves, we can find the connection between the relativistic and nonrelativistic spinors. In the rest frame, there are two independent bispinors which represent particle and antiparticle states:

$$\Psi = \begin{pmatrix} \xi_0 \\ 0 \end{pmatrix}$$

or

$$\Psi = \begin{pmatrix} 0 \\ \xi_0 \end{pmatrix}$$

However, when we consider a particle with zero momentum it is not the case that the upper component of the bispinor can be directly associated with the Schrodinger like wave-function of the particle — for instance, it would not be correctly normalized, for there is some mixing with the lower component.

We can obtain a relation between ξ_0 and the Schrodinger amplitude ϕ_s by considering the current density at zero momentum transfer. For ϕ_s it will be $j_0 = \phi_s^\dagger \phi$. For the relativistic theory we have, as calculated above:

$$j^0 = F_e \frac{p^0 + p'^0}{2m} \bar{\Psi}^\dagger \Psi + F_m \frac{q_\nu}{2m} \bar{\Psi}^\dagger T^{0\nu} \Psi$$

At $q = 0$ the expression simplifies

$$j^0(q=0) = F_e \frac{p_0}{m} \bar{\Psi}^\dagger \Psi$$

$$= F_e \frac{p_0}{m} (\phi^\dagger \phi - \chi^\dagger \chi)$$

ϕ and χ are both related to the rest frame spinor ξ_0. So we can write instead

$$j^0 = F_e \frac{p_0}{m} \xi_0^\dagger \left\{ \cosh^2(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}) - \sinh^2(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}) \right\} \xi_0 = F_e \frac{p_0}{m} \xi_0^\dagger \xi_0$$

where the last equality follows from the hyperbolic trig identity.

If we demand that the two current densities be equal to each other, we find

$$\frac{p_0}{m}\xi_0^\dagger \xi_0 = \phi_s^\dagger \phi_s$$

Approximating

$$\left(1+\frac{\mathbf{p}^2}{2m}\right)\xi_0^\dagger \xi_0 = \phi_s^\dagger \phi_s$$

This will hold to the necessary order if we identify

$$\xi_0 = \left(1-\frac{\mathbf{p}^2}{4m}\right)\phi_s$$

To write the relativistic bispinors in terms of ϕ_s we will also need approximations to $\cosh(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2})$ and $\sinh(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2})$. We only need the rapidity to the leading order: $\boldsymbol{\phi} \approx \mathbf{v} \approx \frac{\mathbf{p}}{m}$.

$$\cosh(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}) \approx 1+\frac{1}{2}\left(\frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\right)^2$$

$$\sinh(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}) \approx \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}$$

The the two bispinor components are

$$\begin{aligned}\phi &\approx \left(1+\left[\frac{1}{2}\frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\right]^2\right)\xi_0 \\ &\approx \left(1+\frac{(\mathbf{\Sigma}\cdot\mathbf{p})^2}{8m^2}-\frac{\mathbf{p}^2}{4m}\right)\phi_S \qquad (5.0.13)\end{aligned}$$

$$\begin{aligned}\chi &\approx \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\xi_0 \\ &\approx \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\phi_S \qquad (5.0.14)\end{aligned}$$

5.0.3 Bilinears in terms of nonrelativistic theory

The next step is to express the relativistic bilinears, built out of the bispinors Ψ, in terms of the Schrodinger like wave functions.

We have above written the bispinors in terms of ϕ_s, so we can use those identities to express the bilinears in the same manner.

Scalar bilinear

$$\begin{aligned}
\bar{\Psi}^{\dagger}(p')\Psi(p) &= \phi^{\dagger}\phi - \chi^{\dagger}\chi \\
&= \phi_s^{\dagger}\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p}')^2}{8m^2} - \frac{\mathbf{p}'^2}{4m^2}\right]\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m^2}\right]\phi_s - \phi_s^{\dagger}\left[\frac{(\mathbf{\Sigma}\cdot\mathbf{p}')(\mathbf{\Sigma}\cdot\mathbf{p})}{4m^2}\right]\phi_s \\
&= \phi_s^{\dagger}\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{(\mathbf{\Sigma}\cdot\mathbf{p}')^2 + (\mathbf{\Sigma}\cdot\mathbf{p})^2 - 2(\mathbf{\Sigma}\cdot\mathbf{p}')(\mathbf{\Sigma}\cdot\mathbf{p})\right\}\right)\phi_s \\
&= \phi_s^{\dagger}\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{[\mathbf{\Sigma}\cdot\mathbf{p}, \mathbf{\Sigma}\cdot\mathbf{q}] + (\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_s \\
&= \phi_s^{\dagger}\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{[4i\epsilon_{ijk}p_i q_j S_k + (\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_s
\end{aligned}$$

Tensor ij component

In calculating the nonrelativistic limit of the antisymmetric tensor bilinear, we will treat the $0i$ and the ij components seperately. First let us consider $\bar{\Psi}\Sigma_{ij}\Psi$.

$$\begin{aligned}
\bar{\Psi}\Sigma_{ij}\Psi &= \bar{\Psi}(-2\epsilon_{ijk}S_k)\Psi \\
&= -2i\epsilon_{ijk}(\phi^{\dagger}S_k\phi - \chi^{\dagger}S_k\chi) \\
&= -2i\epsilon_{ijk}\Big(\phi_S^{\dagger}\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p}')^2}{8m^2} - \frac{\mathbf{p}'^2}{4m^2}\right]S_k\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m^2}\right]\phi_S - \phi_S^{\dagger}\frac{(\mathbf{\Sigma}\cdot\mathbf{p}')S_k(\mathbf{\Sigma}\cdot\mathbf{p})}{4m^2}\phi_S\Big) \\
&= -2i\epsilon_{ijk}\phi_S^{\dagger}\left\{S_k\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{1}{8m^2}\Big[(\mathbf{\Sigma}\cdot\mathbf{p}')^2S_k + S_k(\mathbf{\Sigma}\cdot\mathbf{p})^2 - 2(\mathbf{\Sigma}\cdot\mathbf{p}')S_k(\mathbf{\Sigma}\cdot\mathbf{p})\Big]\right\}\phi_S
\end{aligned}$$

We want to write the terms in square brackets explicitly in terms of $\mathbf{p}$ and $\mathbf{q}$.

$$\begin{aligned}
(\mathbf{\Sigma}\cdot\mathbf{p}')^2S_k + S_k(\mathbf{\Sigma}\cdot\mathbf{p})^2 - 2(\mathbf{\Sigma}\cdot\mathbf{p}')S_k(\mathbf{\Sigma}\cdot\mathbf{p}) &= (\mathbf{\Sigma}\cdot\mathbf{p})^2S_k + S_k(\mathbf{\Sigma}\cdot\mathbf{p}) - 2(\mathbf{\Sigma}\cdot\mathbf{p})S_k(\mathbf{\Sigma}\cdot\mathbf{p}) \\
&\quad + \{(\mathbf{\Sigma}\cdot\mathbf{p})(\mathbf{\Sigma}\cdot\mathbf{q}) + (\mathbf{\Sigma}\cdot\mathbf{q})(\mathbf{\Sigma}\cdot\mathbf{p})\}S_k \\
&\quad - 2(\mathbf{\Sigma}\cdot\mathbf{q})S_k(\mathbf{\Sigma}\cdot\mathbf{p}) + (\mathbf{\Sigma}\cdot\mathbf{q})^2S_k
\end{aligned}$$

We can express many of these terms as commutators

$$= \mathbf{\Sigma}\cdot\mathbf{p}[\mathbf{\Sigma}\cdot\mathbf{p}, S_k] + [S_k, \mathbf{\Sigma}\cdot\mathbf{p}]\mathbf{\Sigma}\cdot\mathbf{p} + 2\mathbf{\Sigma}\cdot\mathbf{q}[\mathbf{\Sigma}\cdot\mathbf{p}, S_k] - [\mathbf{\Sigma}\cdot\mathbf{q}, S_k]\mathbf{\Sigma}\cdot\mathbf{p} + (\mathbf{\Sigma}\cdot\mathbf{q})^2 S_k$$

$$= i\epsilon_{ijk}p_j\{(\mathbf{\Sigma}\cdot\mathbf{p})\Sigma_i - \Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})\} + 2i\epsilon_{ijk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_i p_j - \Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})q_j)\} + (\mathbf{\Sigma}\cdot\mathbf{q})^2 S_k$$

$$= i\epsilon_{ijk}p_j[(\mathbf{\Sigma}\cdot\mathbf{p}), \Sigma_i] + 2i\epsilon_{ijk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_i p_j - \Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})q_j)\} + (\mathbf{\Sigma}\cdot\mathbf{q})^2 S_k$$

$$= 4(\mathbf{p}^2 S_k - (\mathbf{S}\cdot\mathbf{p})p_k) + 2i\epsilon_{ijk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_i p_j - \Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})q_j)\} + (\mathbf{\Sigma}\cdot\mathbf{q})^2 S_k$$

Thus the whole bilinear is

$$-2i\epsilon_{ijk}\phi_S^\dagger\left\{S_k\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{1}{8m^2}\Big[4(\mathbf{p}^2 S_k - (\mathbf{S}\cdot\mathbf{p})p_k) + 2i\epsilon_{\ell mk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_\ell p_m - \Sigma_\ell(\mathbf{\Sigma}\cdot\mathbf{p})q_m)\} + (\mathbf{\Sigma}\cdot\mathbf{q}\right. \tag{5.0.15}$$

Tensor Σ_{0i} component

We calculate $\bar{\Psi}\Sigma_{0i}\Psi$.

$$\bar{\Psi}\Sigma_{0i}\Psi = \bar{\Psi}\begin{pmatrix} 0 & \Sigma_i \\ \Sigma_i & 0 \end{pmatrix}\Psi$$

$$= \phi^\dagger\Sigma_i\chi - \chi^\dagger\Sigma_i\phi$$

We'll only need ϕ and χ to first order here.

$$= \phi_S^\dagger\left(\frac{\Sigma_i\Sigma_j p_j - \Sigma_j\Sigma_i p'j}{2m}\right)\phi_S$$

Using $p' = p + q$ the terms involving only p can be simplified using the commutator of Σ matrices.

$$= \phi_S^\dagger \left(\frac{4i\epsilon_{ijk} p_j S_k - \Sigma_j \Sigma_i q_j}{2m} \right) \phi$$

5.0.4 Current in terms of nonrelativistic wave functions

We derived the four-current (5.0.11) above; in nonrelativistic notation it is:

$$j_0 = F_e \frac{p_0 + p_0'}{2m} \bar{\Psi}^\dagger \Psi - F_m \frac{q_j}{2m} \bar{\Psi}^\dagger \Sigma^{0j} \Psi \tag{5.0.16}$$

$$j_i = F_e \frac{p_i + p_i'}{2m} \bar{\Psi}^\dagger \Psi - F_m \frac{q_j}{2m} \bar{\Psi}^\dagger \Sigma^{ij} \Psi + F_m \frac{q_0}{2m} \bar{\Psi}^\dagger \Sigma^{i0} \Psi \tag{5.0.17}$$

We have expressions for the bilinears in terms of the nonrelativistic wave functions ϕ_S, so it is fairly straight forward to apply them here. The calculation of j_0 is straightforward:

$$\begin{aligned}
F_e \frac{p_0 + p_0'}{2m} \bar{\Psi}^\dagger \Psi &= F_e \left(1 + \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) \phi_S^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2} \left\{4i\epsilon_{ijk} p_i q_j S_k + (\mathbf{\Sigma} \cdot \mathbf{q})^2\right\}\right) \phi_S \\
&\approx F_e \phi_S^\dagger \left(1 + \frac{1}{8m^2} \left\{4i\mathbf{S} \cdot \mathbf{p} \times \mathbf{q} + (\mathbf{\Sigma} \cdot \mathbf{q})^2\right\}\right) \phi_S \\
F_m \frac{q_j}{2m} \bar{\Psi}^\dagger \Sigma^{0j} \Psi &= F_m \frac{q_i}{2m} \phi_S^\dagger \left(\frac{4i\epsilon_{ijk} p_j S_k - \Sigma_j \Sigma_i q_j}{2m} \right) \phi_S \\
&= F_m \phi_S^\dagger \left(\frac{4i\mathbf{S} \cdot \mathbf{q} \times \mathbf{p} - (\mathbf{\Sigma} \cdot \mathbf{q})^2}{4m^2} \right) \phi_S
\end{aligned}$$

It turns out that both terms here have the same form, so combining them we get

$$j_0 = \phi_S^\dagger \left(F_e + \frac{F_e + 2F_m}{8m^2} \left\{4i\mathbf{S} \cdot \mathbf{p} \times \mathbf{q} + (\mathbf{\Sigma} \cdot \mathbf{q})^2\right\}\right) \phi_S \tag{5.0.18}$$

To calculate j_i we want to first simplify things by considering the constraints of our particular problem. The term with $\Sigma_i j$ can be simplified by dropping terms with more than one power of q; these will turn into derivatives of the magnetic field, and our problem concerns only a constant field. Further, we need only calculate elastic scattering, and so $q_0 = 0$. With those simplifications

$$\bar{\Psi}\Sigma_{ij}\Psi \approx -2i\epsilon_{ijk}\phi_S^\dagger \left\{ S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S}\cdot\mathbf{p})p_k}{2m^2} \right\} \phi_S$$

$$\begin{aligned} F_e \frac{p_i + p_i'}{2m}\bar{\Psi}^\dagger\Psi &= F_e \frac{p_i + p_i'}{2m}\phi_S^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{4i\epsilon_{\ell jk}p_\ell q_j S_k + (\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_S \\ &\approx F_e \frac{p_i + p_i'}{2m}\phi_S^\dagger \left(1 + \frac{1}{8m^2}\left\{4i\epsilon_{\ell jk}p_\ell q_j S_k\right\}\right)\phi_S \\ F_m \frac{q_j}{2m}\bar{\Psi}^\dagger\Sigma^{ij}\Psi &= -F_m \frac{i\epsilon_{ijk}q_j}{m}\phi_S^\dagger \left\{ S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S}\cdot\mathbf{p})p_k}{2m^2} \right\} \phi_S \end{aligned}$$

So the full spatial part of the current is

$$j_i = \phi_S^\dagger \left\{ F_e \frac{p_i + p_i'}{2m}\left(1 + \frac{i\epsilon_{\ell jk}p_\ell q_j S_k}{2m^2}\right) + F_m \frac{i\epsilon_{ijk}q_j}{m}\left(S_k\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S}\cdot\mathbf{p})p_k}{2m^2}\right)\right\}\phi_S \tag{5.0.19}$$

5.0.5 Scattering off external field

To compare to the NRQED Lagrangian, we want to calculate scattering off an external field for an arbitrary spin particle. We already have the current, so the scattering is just

$$M = e j_\mu A^\mu = e j_0 A_0 - e\mathbf{j}\cdot\mathbf{A}$$

Above we have expressions for both j_0 (5.0.18) and $\mathbf{j}$ (5.0.19). So we can write down the parts of the amplitude directly:

$$\begin{aligned} ej_0 A_0 =& eA_0 \phi_S^\dagger \left(F_e + \frac{F_e + 2F_m}{8m^2} \left\{ 4i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q} + (\mathbf{\Sigma}\cdot\mathbf{q})^2 \right\} \right) \phi_S \\ e\mathbf{j}\cdot\mathbf{A} =& A_i \phi_S^\dagger \left\{ F_e \frac{p_i + p_i'}{2m} \left(1 + \frac{i\epsilon_{\ell jk} p_\ell q_j S_k}{2m^2} \right) + F_m \frac{i\epsilon_{ijk} q_j}{m} \left(S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S}\cdot\mathbf{p}) p_k}{2m^2} \right) \right\} \phi_S \end{aligned} \tag{5.0.20}$$

As much as possible we want to express the result in terms of gauge invariant quantities $\mathbf{B}$ and $\mathbf{E}$. We write the relations between these fields and A_μ in position space and the equivalent equation in momentum space.

$$\mathbf{B} = \mathbf{\nabla} \times \mathbf{A} \to i\mathbf{q} \times \mathbf{A}$$

$$\mathbf{E} = -\mathbf{\nabla} A_0 \to -i\mathbf{q} A_0$$

There is one term above that can only be put into gauge-invariant form by considering the kinematic constraints of elastic scattering. If the scattering is elastic, we have $\mathbf{q}\cdot(\mathbf{p}' + \mathbf{p}) = \mathbf{p}'^2 - \mathbf{p}^2 = 0$. We can use this identity on the term $q_j(p_i' + p_i)A_i$ as follows:

$$\begin{aligned} \epsilon_{ijk} B_k &= \partial_i A_j - \partial_j A_i = i(q_i A_j - q_j A_i) \\ (p_i + p_i')\epsilon B_k &= i(p_i + p_i')(q_i A_j - q_j A_i) \\ &= -i(p_i + p_i') q_j A_i \end{aligned}$$

So we have the identity

$$i(p_i + p_i') q_j A_i = -\epsilon_{ijk} B_k (p_i + p_i') \tag{5.0.21}$$

Now we can write each term involving q in terms of position space quantities.

$$\begin{aligned}
i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q}A_0 &= -\mathbf{S}\cdot\mathbf{p}\times\mathbf{E} \\
(\mathbf{\Sigma}\cdot\mathbf{q})^2A_0 &= \Sigma_i\Sigma_j q_i q_j A_0 \\
&= \Sigma_i\Sigma_j\partial_i E_j \\
i\epsilon_{ijk}A_i q_j &= -i(\mathbf{q}\times\mathbf{A})_k \\
&= = -B_k \\
A_i(p_i+p_i')i\epsilon_{\ell jk}p_\ell q_j S_k &= \epsilon_{\ell jk}p_\ell S_k i(p_i+p_i')q_j A_i \\
&= -\epsilon_{\ell jk}p_\ell S_k\{\epsilon_{ijm}B_m(p_i+p_i')\} \\
&= -(\delta_{\ell i}\delta_{km}-\delta\ell m\delta_{ik})p_\ell S_k\{\epsilon_{ijm}B_m(p_i+p_i')\} \\
&= 2\{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})-(\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2\}
\end{aligned}$$

Using these

$$ej_0A_0 = e\phi_S^\dagger\left\{A_0+\frac{1-2F_2}{8m^2}\left(4\mathbf{S}\cdot\mathbf{E}\times\mathbf{p}+\Sigma_i\Sigma_j\partial_iE_j\right)\right\}$$

$$e\mathbf{j}\cdot\mathbf{A} = e\phi_S^\dagger\left\{\frac{\mathbf{p}\cdot\mathbf{A}}{m}+\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})-(\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2}{m^2}-F_m\left(\frac{\mathbf{S}\cdot\mathbf{B}}{m}\left\{1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right\}+\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})-(\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2}{2m^2}\right.\right.$$

$$= e\phi_S^\dagger\left\{\frac{\mathbf{p}\cdot\mathbf{A}}{m}+[1-2F_m]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2}-\mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2}-\frac{F_m}{m}\mathbf{S}\cdot\mathbf{B}\right\}$$

From this we can see that our F_m is actually $g/2$, so in such terms

$$ej_0A_0 = e\phi_S^\dagger\left\{A_0-\frac{g-1}{2m^2}\left(\mathbf{S}\cdot\mathbf{E}\times\mathbf{p}+\frac{1}{4}\Sigma_i\Sigma_j\partial_iE_j\right)\right\}$$

$$e\mathbf{j}\cdot\mathbf{A} = e\phi_S^\dagger\left\{\frac{\mathbf{p}\cdot\mathbf{A}}{m}-[g-1]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2}-\mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2}-\frac{g}{2m}\mathbf{S}\cdot\mathbf{B}\right\}$$

So the entire scattering process is

$$e\phi_S^\dagger \left\{ A_0 - \frac{g-1}{2m^2}\left(\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \frac{1}{4}\Sigma_i\Sigma_j\partial_i E_j\right)\frac{\mathbf{p}\cdot\mathbf{A}}{m} - [g-1]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{g}{2m}\mathbf{S}\cdot\mathbf{B}\right\}\phi_S \tag{5.0.22}$$

5.0.6 Comparison with relativistic result

Having calculated the same process in both the relativistic theory and in our NRQED effective theory, we can compare the two amplitudes and fix the coefficients.

The NRQED amplitude (3.4.10) is

$$\begin{aligned} iM &= ie\phi^\dagger\Bigg(-A_0 + \frac{\mathbf{A}\cdot\mathbf{p}}{m} - \frac{(\mathbf{A}\cdot\mathbf{p})\mathbf{p}^2}{2m^3} + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2} + c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2} \\ &\quad + c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2} - (c_{W_1} - c_{W_2})\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3}\Bigg)\phi \end{aligned}$$

While the relativistic amplitude was

$$iM_{REL} = -ie\phi^\dagger\Big(A_0 - \frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{\mathbf{p}\cdot\mathbf{A}\mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3}\{\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - S_iS_j\boldsymbol{\nabla}_iE_j\} - g\frac{1}{2m}\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3}(\mathbf{S}\cdot\mathbf{p})$$

We should rewrite term $\boldsymbol{\nabla}\cdot\mathbf{E} - S_iS_j\boldsymbol{\nabla}_iE_j$ using the quadropole moment tensor $Q_{ij} = \frac{1}{2}(S_iS_j + S_jS_i - \frac{2}{3}\mathbf{S}^2)$.

Remember that ∇_iE_j is actually symmetric under exchange of i and j. Then we can write

$$\begin{aligned} S_iS_j\nabla_iE_j &= \frac{1}{2}(S_iS_j + S_jS_i) = (Q_{ij} + \frac{1}{3}\mathbf{S}^2\delta_{ij})\nabla_iE_j \\ &= Q_ij\nabla_iE_j + \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{E} \end{aligned}$$

Written using this identity, the relativistic amplitude is

$$iM_{REL} = -ie\phi^\dagger\Big(A_0 - \frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{\mathbf{p}\cdot\mathbf{A}\mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3}\{\frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - Q_{ij}\boldsymbol{\nabla}_iE_j\} - g\frac{1}{2m}\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3}(\mathbf{S}\cdot\mathbf{p}$$

Comparing the two, the coefficients are:

$$
\begin{aligned}
c_F &= g \\
c_D &= \frac{4(g-1)}{3} \\
c_Q &= -4(g-1) \\
c_S^1 &= 2(g-1) \\
(c_{W_1} - c_{W_2}) &= 2 \\
c_{p'p} &= (g-2)
\end{aligned}
$$

Chapter 6

Exact equations of motion for spin-1

$$\mathcal{L} = -\frac{1}{2}(D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) + m^2 W^{\mu\dagger} W_\mu - i\lambda e W^{\mu\dagger} W^\nu F_{\mu\nu} \tag{6.0.1}$$

where as usual D is the long derivative $D^\mu = \partial^\mu + ieA^\mu$.

Obtain the equations of motion from the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial[\partial_\mu W^{\dagger\alpha}]} = 0$$

Or equivalently,

$$\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} - D_\mu \frac{\partial \mathcal{L}}{\partial[D_\mu W^{\dagger\alpha}]} = 0$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} &= \frac{\partial}{\partial W^{\dagger\alpha}} \left(m^2 W^{\mu\dagger} W_\mu - i\lambda e W^{\mu\dagger} W^\nu F_{\mu\nu} \right) \\ &= m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} \end{aligned}$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial [D_\gamma W^{\dagger\alpha}]} &= -\frac{1}{2}\frac{\partial}{\partial [D_\gamma W^{\dagger\alpha}]}(D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) \\
&= -\frac{1}{2}(g^{\mu\gamma} g^\nu_\alpha - g^{\nu\gamma} g^\mu_\alpha)(D_\mu W_\nu - D_\nu W_\mu) \\
&= D_\alpha W^\gamma - D^\gamma W_\alpha
\end{aligned}
$$

So the complete equation from Euler-Lagrange is

$$
m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} + D_\mu(D^\mu W_\alpha - D_\alpha W^\mu) = 0 \tag{6.0.2}
$$

This is a set of four coupled second order equations for the field W. We rewrite as a set of first order equations by introducing a field $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$. So (6.0.2) becomes

$$
m^2 W_\alpha - ie\lambda W^\mu F_{\alpha\mu} + D^\mu W_{\mu\alpha} = 0 \tag{6.0.3}
$$

$W^{\mu\nu}$ is antisymmetric and so has six degrees of freedom, corresponding to six independent fields. Together with W^μ this represents a total of ten fields. However, upon examination only some of these fields are dynamic. The fields W^{0i} and W^i appear in the equations with time derivatives, while the fields W^{ij} and W^0 never do. So it is only necessary to consider the former six fields. So that these six fields all have the same dimension, we will define $\frac{W^{i0}}{m} = i\eta^i$.

We will now eliminate the extraneous fields and solve for $iD_0 W^i$, $iD_0\eta^i$.

$$
m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} + D_\mu(D^\mu W_\alpha - D_\alpha W^\mu) = 0 \tag{6.0.4}
$$

Define $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$. Then

$$
m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} + D^\mu W_{\mu\alpha} = 0 \tag{6.0.5}
$$

To get the exact Hamiltonian of a bispinor, we can eliminate nondynamic fields.

First consider (6.0.5) with $\alpha = 0$.

$$m^2 W_0 - ie\lambda W^\nu F_{0\nu} + D^\mu W_{\mu 0} \tag{6.0.6}$$

$$m^2 W_0 - ie\lambda W^j F_{0j} + D^j W_{j0} \tag{6.0.7}$$

Solve this for W_0

$$W_0 = \frac{1}{m^2}\left(ie\lambda W^j F_{0j} - D^j W_{j0}\right) \tag{6.0.8}$$

Now, consider (6.0.5) with $\alpha = i$

$$m^2 W_i - ie\lambda W^\mu F_{i\mu} + D^\mu W_{\mu i} = 0 \tag{6.0.9}$$

$$m^2 W_i - ie\lambda W^0 F_{i0} + D^0 W_{0i} - ie\lambda W^j F_{ij} + D^j W_{ji} = 0 \tag{6.0.10}$$

Using (6.0.8) we can replace W_0

$$m^2 W_i - \frac{ie\lambda}{m^2}\left(ie\lambda W^\nu F_{0\nu} - D^j W_{j0}\right) F_{i0} + D^0 W_{0i} - ie\lambda W^j F_{ij} + D^j W_{ji} = 0 \tag{6.0.11}$$

Using $W_{ji} = D_j W_i - D_i W_j$

$$m^2 W_i - \frac{ie\lambda}{m^2}\left(ie\lambda W^\nu F_{0\nu} - D^j W_{j0}\right) F_{i0} + D^0 W_{0i} - ie\lambda W^j F_{ij} + D^j (D_j W_i - D_i W_j) = 0 \tag{6.0.12}$$

Solve this for $D^0 W_{0i}$:

$$D^0 W_{0i} = -m^2 W_i + \frac{ie\lambda}{m^2}\left(ie\lambda W^\nu F_{0\nu} - D^j W_{j0}\right) F_{i0} + ie\lambda W^j F_{ij} - D^j (D_j W_i - D_i W_j) \tag{6.0.13}$$

To get a similar equation for W_i, consider

$$W_{i0} = D_i W_0 - D_0 W_i \tag{6.0.14}$$

Then

$$D^0 W_i = D_i W_0 - W_{i0} \tag{6.0.15}$$

$$D^0 W_i = D_i \frac{1}{m^2}\left(ie\lambda W^j F_{0j} - D^j W_{j0}\right) - W_{i0} \tag{6.0.16}$$

We now have equations that tell us the time evolution of a total of six fields: W_i and W_{i0}. We want to treat these as the components of some sort of bispinor. To that end, first define $\eta_i = -i/mW_{i0}$ so that we have a pair of fields with the same mass dimension and hermiticity. (Since $W_{i0} = D_i W_0 - D_0 W_i$ would pick up another minus sign under complex conjugation compared to W_i.) Going the other way $W_{i0} = im\eta_i$.

Since eventually we want a nonrelativistic expression, we should write the spatial components of four vectors as three vectors. To that end also write the components of the tensor $F_{\mu\nu}$ in terms of three vectors.

$$F_{0i} = E^i,\ F_{ij} = -\epsilon_{ijk} B^k$$

Regular spatial vectors are "naturally raised" while D_i is "naturally lowered". Then we can rewrite (6.0.13).

$$-imD_0\eta_i = -m^2 W_i - \frac{ie\lambda}{m^2}\left(ie\lambda W^j E^j - D^j W_{j0}\right) E^i + ie\lambda W^j \epsilon_{ijk} B^k - D^j(D_j W_i - D_i W_j) \tag{6.0.17}$$

$$iD_0\eta^i = mW^i + \frac{ie\lambda}{m^3}\left(ie\lambda W^j E^j - D^j W_{j0}\right) E^i - \frac{ie\lambda}{m} W^j \epsilon_{ijk} B^k + \frac{1}{m} D_j(D_j W^i - D_i W^j) \tag{6.0.18}$$

And likewise (6.0.16) becomes

$$D^0 W_i = D_i \frac{1}{m^2}\left(-ie\lambda W^j E^j - imD^j \eta_j\right) - im\eta_i \tag{6.0.19}$$

$$iD^0 W^i = -iD_i \frac{1}{m^2}\left(-ie\lambda W^j E^j - imD_j \eta^j\right) + m\eta^i \tag{6.0.20}$$

$$iD^0 W^i = -\frac{1}{m^2} D_i e\lambda \mathbf{W}\cdot\mathbf{E} - \frac{1}{m} D_i \mathbf{D}\cdot\eta + m\eta^i \tag{6.0.21}$$

Spin identities

To obtain some Schrodinger like equation for a bispinor, we need to express the time derivative of the bispinor in terms of other operators which can be interpreted as the Hamiltonian: $i\partial_0\Psi = \hat{H}\Psi$. We have expressions for the time derivatives of W_i and η_i, but some mixing of the indices is involved. To write the Hamiltonian in a block form we can introduce spin matrices whose action will mix the components of the fields.

The spin matrix for a spin one particle can represented as:

$$(S^k)_{ij} = -i\epsilon_{ijk} \tag{6.0.22}$$

which leads to the following identities:

$$(\mathbf{a}\times\mathbf{v})_i = -i(\mathbf{S}\cdot\mathbf{a})_{ij} v_j \tag{6.0.23}$$

$$\{\mathbf{a}\times(\mathbf{a}\times\mathbf{v})\}_i = -(\{\mathbf{S}\cdot\mathbf{a}\}^2)_{ij} v_j \tag{6.0.24}$$

$$a_i(\mathbf{b}\cdot\mathbf{v}) = \{\mathbf{a}\cdot\mathbf{b}\,\delta_{ij} - (S^k S^\ell)_{ij} a^\ell b^k\} v_j \tag{6.0.25}$$

Using these identities

$$iD^0\mathbf{W} = -\frac{e\lambda}{m^2}\mathbf{D}(\mathbf{W}\cdot\mathbf{E}) + \frac{1}{m}\mathbf{D}(\mathbf{D}\cdot\boldsymbol{\eta}) + m\boldsymbol{\eta} \tag{6.0.26}$$

$$iD^0\mathbf{W} = -\frac{1}{m^2}\{\mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\}\mathbf{W} + \frac{1}{m}\{\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2\}\boldsymbol{\eta} + m\boldsymbol{\eta} \tag{6.0.27}$$

and for the other equation

$$iD_0\eta^i = mW^i - \frac{ie\lambda}{m^3}\left(ie\lambda W^j E^j - D^j W_{j0}\right)E^i + \frac{ie\lambda}{m}W^j\epsilon_{ijk}B^k - \frac{1}{m}D_j(D_jW^i - D_iW^j) \tag{6.0.28}$$

$$iD_0\eta^i = mW^i - \frac{ie\lambda}{m^3}\left(ie\lambda W^j E^j - D^j W_{j0}\right)E^i + \frac{ie\lambda}{m}W^j\epsilon_{ijk}B^k - \frac{1}{m}D_j(D_jW^i - D_iW^j) \tag{6.0.29}$$

$$iD_0\boldsymbol{\eta} = m\mathbf{W} + \frac{e^2\lambda^2}{m^3}\mathbf{E}(\mathbf{W}\cdot\mathbf{E}) - \frac{1}{m^2}\mathbf{E}(\mathbf{D}\cdot\boldsymbol{\eta}) + \frac{ie\lambda}{m}(\mathbf{W}\times\mathbf{B}) + \frac{1}{m}\mathbf{D}\times(\mathbf{D}\times\mathbf{W}) \tag{6.0.30}$$

$$iD_0\boldsymbol{\eta} = m\mathbf{W} + \frac{e^2\lambda^2}{m^3}\{\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\}\mathbf{W} - \frac{1}{m^2}\{\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\}\boldsymbol{\eta} + \frac{e\lambda}{m}(\mathbf{S}\cdot\mathbf{B})\mathbf{W} - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2\mathbf{W} \tag{6.0.31}$$

$$iD_0\boldsymbol{\eta} = \left(m + \frac{e^2\lambda^2}{m^3}\{\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\} - \frac{e\lambda}{m}(\mathbf{S}\cdot\mathbf{B}) - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2\right)\mathbf{W} + \frac{1}{m^2}\{\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\}\boldsymbol{\eta} \tag{6.0.32}$$

Now that the equations are in the correct form, we can write them as follows:

$$iD_0 \begin{pmatrix} W \\ \eta \end{pmatrix} = \begin{pmatrix} \lambda\frac{e}{m^2}\left[\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot\mathbf{E}\times\mathbf{D}\right] & m - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2 - \lambda\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + \lambda^2\frac{e^2}{m^3}\left[\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E}) \right. \\ m - \frac{1}{m}\left[\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2 + e\mathbf{S}\cdot\mathbf{B}\right] & -\lambda\frac{e}{m^2}\left[\mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) + i\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}\right] \end{pmatrix} \tag{6.0.33}$$

Current

We can also derive the conserved current from the Lagrangian:

$$\mathcal{L} \;=\; -\frac{1}{2}(D\times W)^\dagger\cdot(D\times W) + m_w^2 W^\dagger W - \lambda ie W^{\dagger\mu}W^\nu F_{\mu\nu}$$

Where

$$D^\mu = \partial^\mu - ieA^\mu, \qquad D\times W = D^\mu W^\nu - D^\nu W^\mu$$

We want the conserved current corresponding to the transformation $W_i \to e^{i\alpha}W_i$, which in infinitesimal form is:

$$W_\mu \to W_\mu + i\alpha W_\mu,\; W_\mu^\dagger \to W_\mu^\dagger - i\alpha W_\mu^\dagger$$

The 4-current density will be:

$$j^\sigma = -i\frac{\partial\mathcal{L}}{\partial W_{\mu,\sigma}}W_\mu + i\frac{\partial\mathcal{L}}{\partial W^\dagger_{\mu,\sigma}}W_\mu^\dagger$$

Only one term contains derivatives of the field:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial W_{\alpha,\sigma}} &= \frac{\partial}{\partial W_{\alpha,\sigma}} \left\{ -\frac{1}{2}(D_\mu W_\nu - D_\nu W_\mu)^\dagger (D^\mu W^\nu - D^\nu W^\mu) \right\} \\
&= -\frac{1}{2}(D_\mu W_\nu - D_\nu W_\mu)^\dagger (g_{\sigma\mu} g_{\alpha\nu} - g_{\sigma\nu} g_{\alpha\mu}) \\
&= -(D_\alpha W_\sigma - D_\sigma W_\alpha)^\dagger
\end{aligned}
$$

Likewise:

$$
\frac{\partial \mathcal{L}}{\partial W^\dagger_{\alpha,\sigma}} = -(D_\alpha W_\sigma - D_\sigma W_\alpha)
$$

If we define $W_{\mu\nu} = D^\mu W^\nu - D^\nu W^\mu$ then the 4-current and charge density are:

$$
\begin{aligned}
j_\sigma &= i W^\dagger_{\sigma\mu} W^\mu - i W_{\sigma\mu} W^{\dagger\mu} \\
j_0 &= i W^\dagger_{0\mu} W^\mu - i W_{0\mu} W^{\dagger\mu} \\
&= i W^\dagger_{0i} W^i - i W_{0i} W^{\dagger i}
\end{aligned}
$$

Where the last equality follows from the antisymmetry of $W_{\mu\nu}$.

Now, we defined the fields $\eta_i = -i\frac{W_{i0}}{m}$. In terms of these fields, $j_0 = m(\eta_i^\dagger W^i + \eta_i W^{\dagger i})$.

We can do the same to find the vector part of the current.

$$
\begin{aligned}
j_i &= i W^\dagger_{i\mu} W^\mu - i W_{i\mu} W^{\dagger\mu} \\
&= i W^\dagger_j W_{ij} + i W^\dagger_{i0} W_0 + c.c.
\end{aligned}
$$

We have $W_{ij} = D_i W_j - D_j W_i$. Using the identities developed in the appendix, we can obtain

$$
D_j W_i = D_i W_j - D_k (S_i S_k)_{ja} W_a
$$

Then

$$W_{ij} = D_k(S_iS_k)_{ja}W_a$$

In the absence of an electric field E, $W_0 = \frac{i}{m}D_j\eta_j$, with $W_{i0} = im\eta_i$.

$$W_{i0}^\dagger W_0 = -\eta_i^\dagger D_j\eta_j$$

Again we introduce spin matrices to get the equation in the desired form, and obtain

$$W_{i0}^\dagger W_0 = -\eta_j^\dagger D_k(\delta_{ik} - S_kS_i)\eta_j$$

This leads to

$$j_i = iW_j^\dagger D_k(S_iS_kW)_j - i\eta_j^\dagger D_k([\delta_{ik} - S_kS_i]\eta)_j + c.c.$$

Writing this in terms of the bispinor $\begin{pmatrix}\eta \\ W\end{pmatrix}$, the expression for the current is

$$j_i = \frac{i}{2}\begin{pmatrix}\eta^\dagger & W^\dagger\end{pmatrix}\left[(\{S_i,S_j\} - \delta_{ij})\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} - ([S_i,S_j] + \delta_{ij})\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\right] D_j \begin{pmatrix}\eta \\ W\end{pmatrix} + c.c. \tag{6.0.34}$$

Hermiticity of Hamiltonian

It's clear from inspection that the above Hamiltonian is not Hermitian in the standard sense. (Of the operators in use, the only one which is not self adjoint is $D_i^\dagger = -D_i$.) Noticing that

$$[\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot\mathbf{E}\times\mathbf{D}]^\dagger = -[\mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) + i\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}]$$

we can see it has the general form

$$H = \begin{pmatrix} A & B \\ C & A^\dagger \end{pmatrix}$$

where the off diagonal blocks are Hermitian in the normal sense: $B^\dagger = B$ and $C^\dagger = C$.

At this point we should consider that an operator is defined as Hermitian with respect to a particular inner product. In quantum mechanics this inner product is normally defined as:

$$\langle \Psi, \phi \rangle = \int d^3x \Psi^\dagger \phi$$

This definition, however, does not produce sensible results for the states in question. We need the inner product of a state with itself to be conserved; in other words, it should play the role of a conserved charge. From the considerations above we already have one such quantity: the conserved charge $\int d^3x j_0$, where

$$j_0 = m[\eta^\dagger W + W^\dagger \eta] = m \begin{pmatrix} \eta^\dagger & W^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ W \end{pmatrix}$$

A more general definition of the inner product includes some weight M:

$$\langle \Psi, \phi \rangle = \int d^3x \Psi^\dagger M \phi$$

In normal quantum mechanics M would be the identity matrix, but here, as implied by the charge density, we want $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Such a definition will lead to the inner product $< \Psi, \Psi >$ being conserved.

An operator H is hermitian with respect to this inner product if

$$\langle H\Psi, \phi \rangle = \langle \Psi, H\phi \rangle \rightarrow \int d^3x \Psi^\dagger H^\dagger M \phi = \int d^3x \Psi^\dagger M H \phi$$

For this equality to hold, it is sufficient for $H^\dagger M = MH$. With $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ this condition reduces to

$$\begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$$

Our Hamiltonian fulfills exactly this requirement, and so is Hermitian with respect to this particular inner product.

6.0.7 Non-relativistic Hamiltonian

Now we will consider the non-relativistic limit of the above Hamiltonian. To work in this regime constrains the order of both the momentum and the electromagnetic field strength.

$$\begin{aligned} D &\sim mv \\ \Phi &\sim mv^2 \\ E &\sim m^2v^3 \\ B &\sim m^2v^2 \end{aligned}$$

We can write the Hamiltonian matrix in terms of the basis of 2x2 Hermitian matrices: $(\mathbf{I}, \rho_i)$:

$$
\begin{aligned}
H &= a_0\mathbf{I} + a_i\rho_i \\
a_0 &= \frac{1}{2}(H_{11}+H_{22}) = e\Phi + \lambda\frac{e}{2m^2}\left[\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot\mathbf{E}\times\mathbf{D} - \mathbf{D}\cdot\mathbf{E} + (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) - i\mathbf{S}\cdot\mathbf{D}\right. \\
a_3 &= \frac{1}{2}(H_{11}-H_{22}) = \lambda s\frac{e}{2m^2}\left[\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot\mathbf{E}\times\mathbf{D} + \mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) + i\mathbf{S}\cdot\mathbf{D}\times\right. \\
ia_2 &= \frac{1}{2}(H_{21}-H_{12}) = -\left[\frac{\mathbf{D}^2}{2m} - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2 - \frac{\lambda-1}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + \frac{e^2}{2m^3}(\mathbf{E}^2-(\mathbf{S}\cdot\mathbf{E})^2)\right] \\
a_1 &= \frac{1}{2}(H_{12}-H_{21}) = \left[m - \frac{\mathbf{D}^2}{2m} - \frac{1+\lambda}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + \frac{e^2}{2m^3}(\mathbf{E}^2-(\mathbf{S}\cdot\mathbf{E})^2)\right]
\end{aligned}
$$

We can see that to leading order, the Hamiltonian is

$$
H = m\rho_1 = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}
$$

Since we wish to separate positive and negative energy states, this poses a problem. So we first switch to a basis where at least the rest energies are separate. Then, remaining off-diagonal elements can be treated as perturbations.

An appropriate transformation which meets our requirements is a "rotation" in the space spanned by ρ_i matrices, about the $\rho_1+\rho_3$ axis. This has the explicit form:

$$
U = \frac{1}{\sqrt{2}}(\rho_1+\rho_3) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{6.0.35}
$$

Any transformation U can be described by it's action on the basis of 4x4 matrices. This takes:

$$\begin{aligned}
\mathbf{I} &\rightarrow \mathbf{I} \\
\rho_1 &\rightarrow \rho_3 \\
\rho_3 &\rightarrow \rho_1 \\
\rho_2 &\rightarrow -\rho_2 \\
\begin{pmatrix} \eta \\ W \end{pmatrix} &\rightarrow \begin{pmatrix} \Psi_u \\ \Psi_\ell \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta + W \\ \eta - W \end{pmatrix}
\end{aligned}$$

(This transformation will transform the current to

$$j_0 = m(\Psi_u^\dagger \Psi_u - \Psi_\ell^\dagger \Psi_\ell)$$

and the weight M, used in the inner product, to

$$M \rightarrow M' = U^\dagger M U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The transformed Hamiltonian will of course be Hermitian with respect to the transformed inner product.)

Our equation is now of the following form:

$$i\partial_0 \begin{pmatrix} \Psi_u \\ \Psi_\ell \end{pmatrix} = H' \begin{pmatrix} \Psi_u \\ \Psi_\ell \end{pmatrix}$$

We can see that the Hamiltonian still contains off-diagonal elements, so this represents a pair of coupled equations for the upper and lower components of $\mathbf{\Psi}$. But the off-diagonal terms are small, so we can consider the case where Ψ_ℓ is small compared to Ψ_u. Solving for Ψ_ℓ in terms of Ψ_u:

$$
\begin{aligned}
E\Psi_\ell &= H'_{21}\Psi_u + H'_{22}\Psi_\ell \\
\Psi_\ell &= (E - H'_{22})^{-1} H'_{21}\Psi_u
\end{aligned}
$$

This gives the exact formula:

$$
E\Psi_u = \left(H'_{11} + H'_{12}[E - H'_{22}]^{-1} H'_{21}\right)\Psi_u
$$

However, we only need corrections to the magnetic moment of order v^2. With $\frac{e}{m}\mathbf{S}\cdot\mathbf{B} \sim mv^2$, this means we only need the Hamiltonian to at most order mv^4. Examining the leading order terms of the matrix H', the diagonal elements are order m while the off-diagonal elements are order mv^2. To leading order the term $[E - H'_{22}]^{-1} = \frac{1}{2m}$. So we'll need H'_{11} to $\mathcal{O}(v^4)$, and H'_{12}, H'_{21}, and $[E - H'_{22}]^{-1}$ each to only leading order.

$$
E\Psi_u = \left(H'_{11} + \frac{1}{2m} H'_{12} H'_{21} + \mathcal{O}(mv^6)\right)\Psi_u
$$

The needed terms of H are, using $\lambda = g - 1$

$$
\begin{aligned}
H'_{11} = a_0 + a_3 &= m + e\Phi - \frac{D^2}{2m} - \frac{g}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} \\
&\quad + (g-1)\frac{e}{2m^2}\left[\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot\mathbf{E}\times\mathbf{D} - \mathbf{D}\cdot\mathbf{E} + (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) - i\mathbf{S}\cdot\mathbf{D}\times\right. \\
H'_{12} = a_1 - ia_2 &= \frac{D^2}{2m} - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2 - \frac{g-2}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + \mathcal{O}(mv^4) \\
H'_{21} = a_1 + ia_2 &= -\frac{D^2}{2m} + \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2 + \frac{g-2}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + \mathcal{O}(mv^4)
\end{aligned}
$$

The product $H'_{12}H'_{21}$ is calculated in the appendix. To first order in the magnetic field strength it is:

$$\begin{aligned}
\frac{1}{2m}H'_{12}H'_{21} &= -\frac{1}{2m^3}\left(\frac{\mathbf{D}^2}{2} - (\mathbf{S}\cdot\mathbf{D})^2 - \frac{g-2}{2}e\mathbf{S}\cdot\mathbf{B}\right)^2 \\
&= -\frac{1}{2m^3}\left(\frac{\boldsymbol{\pi}^4}{4} - e\mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - \frac{g-2}{2}e(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\right)
\end{aligned}$$

So finally,replacing all $\mathbf{D}$ with $\boldsymbol{\pi} \equiv \mathbf{p} - e\mathbf{A}$, we have a direct expression for Ψ_u:

$$\begin{aligned}
E\Psi_u &= \left\{m + e\Phi + \frac{\pi^2}{2m} - \frac{g}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} - \frac{\pi^4}{8m^3} + \frac{e\mathbf{p}^2(\mathbf{S}\cdot\mathbf{B})}{2m^3} + (g-2)\frac{e}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\right. \\
&\left. +(g-1)\frac{ie}{2m^2}\left[\mathbf{E}\cdot\boldsymbol{\pi} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\boldsymbol{\pi}) + i\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi} - \boldsymbol{\pi}\cdot\mathbf{E} + (\mathbf{S}\cdot\boldsymbol{\pi})(\mathbf{S}\cdot\mathbf{E}) - i\mathbf{S}\cdot\boldsymbol{\pi}\times\mathbf{E}\right]\right\}\Psi_u
\end{aligned}$$

The complicated expression in square brackets can be cleaned up a bit:

$$\begin{aligned}
\mathbf{E}\cdot\boldsymbol{\pi} - \boldsymbol{\pi}\cdot\mathbf{E} &= [E_i, \pi_i] \\
&= [E_i, -i\partial_i] \\
&= i(\partial_i E_i) \\
(\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\boldsymbol{\pi}) - (\mathbf{S}\cdot\boldsymbol{\pi})(\mathbf{S}\cdot\mathbf{E}) &= S_iS_jE_i\pi_j - S_iS_j\pi_iE_j \\
&= (S_iS_j)(E_i\pi_j - E_j\pi_i - [\pi_i, E_j]) \\
&= [S_i, S_j](E_i\pi_j) - (S_iS_j)(-i\nabla_iE_j) \\
&= (i\epsilon_{ijk}S_k)E_j\pi_i - (S_iS_j)(-i\nabla_iE_j) \\
&= i\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi} + iS_iS_j\nabla_iE_j) \\
(\mathbf{E}\times\boldsymbol{\pi} - \boldsymbol{\pi}\times\mathbf{E})_k &= \epsilon_{ijk}(E_i\pi_j - \pi_iE_j) \\
&= \epsilon_{ijk}(E_i\pi_j + \pi_jE_i) \\
&= \epsilon_{ijk}(2E_i\pi_j + [\pi_i, E_j]) \\
&= 2\epsilon_{ijk}E_i\pi_j \\
&= 2(\mathbf{E}\times\boldsymbol{\pi})_k
\end{aligned}$$

Using these identities and collecting terms, and then writing everything in terms of g, $g-2$

$$
\begin{aligned}
E\Psi_u &= \left\{ m + e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} - \frac{e}{m}\mathbf{S}\cdot\mathbf{B}\left(\frac{g}{2} - \frac{p^2}{2m^2}\right) + (g-2)\frac{e}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \right. \\
&\left. -(g-1)\frac{e}{2m^2}\left[\nabla\cdot\mathbf{E} - S_iS_j\nabla_iE_j + \mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi}\right] \right\} \Psi_u \\
&= \left\{ m + e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} - \frac{g}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\left(1 - \frac{p^2}{2m^2}\right) - \frac{g-2}{2}\frac{e}{m}\frac{p^2}{2m^2}\mathbf{S}\cdot\mathbf{B} + (g-2)\frac{e}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \right. \\
&\left. -\left(\frac{g}{2} + \frac{g-2}{2}\right)\frac{e}{2m^2}\left[\nabla\cdot\mathbf{E} - S_iS_j\nabla_iE_j + \mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi}\right] \right\} \Psi_u
\end{aligned}
$$

We have a Hamiltonian for the upper component of the bispinor, as desired. But is it truly Schrodinger-like? In the general case we would need to perform the Fouldy-Wouthyusen transformation, to a representation where all the physics up to the desired order is contained in the single spinor equation. But we can show that, at the order we are working at, the Hamiltonian above is correct.

6.1 Comparison with Silenko

We can compare to a result in Silenko (arXiv:hep-th/0401183v1). Silenko determines a Hamiltonian which is equivalent to that used above. Dropping terms manifestly beyond $\mathcal{O}(mv^4)$:

$$\begin{aligned} H &= \rho_3\epsilon' + e\Phi + \frac{e}{4m}\left[\left\{\left(\frac{g-2}{2} + \frac{m}{\epsilon'+m}\right)\frac{1}{\epsilon'}, (\mathbf{S}\cdot\boldsymbol{\pi}\times\boldsymbol{E} - \mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi})\right\}_+ - \rho_3\left\{\left(g-2+\frac{2m}{\epsilon'}\right), \mathbf{S}\cdot\mathbf{B}\right\}\right. \\ &\left. +\rho_3\left\{\frac{g-2}{2\epsilon'(\epsilon'+m)}, \{\mathbf{S}\cdot\boldsymbol{\pi}, \boldsymbol{\pi}\cdot\mathbf{B}\}_+\right\}_+\right] + \frac{e(g-1)}{4m^2}\{\mathbf{S}\cdot\boldsymbol{\nabla}, \mathbf{S}\cdot\mathbf{E}\}_+ - \frac{e(g-1)}{2m^2}\nabla\cdot\mathbf{E} \end{aligned}$$

We'll take each term and expand to $\mathcal{O}(mv^4)$. First the operator ϵ':

$$\begin{aligned} \epsilon' &= \sqrt{m^2+\pi^2} \\ &= m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^2} + \mathcal{O}(mv^6) \end{aligned}$$

Using this, we get

$$\begin{aligned} H &= \rho_3\epsilon' + e\Phi + \frac{e}{4m}\left[\left\{\frac{g-1}{2m}, (\mathbf{S}\cdot\boldsymbol{\pi}\times\boldsymbol{E} - \mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi})\right\}_+ - \rho_3\left\{g - \frac{\pi^2}{m^2}, \mathbf{S}\cdot\mathbf{B}\right\}_+\right. \\ &\left. +\rho_3\left\{\frac{g-2}{4m^2}, \{\mathbf{S}\cdot\boldsymbol{\pi}, \boldsymbol{\pi}\cdot\mathbf{B}\}_+\right\}_+\right] + \frac{e(g-1)}{4m^2}\{\mathbf{S}\cdot\boldsymbol{\nabla}, \mathbf{S}\cdot\mathbf{E}\}_+ - \frac{e(g-1)}{2m^2}\nabla\cdot\mathbf{E} \end{aligned}$$

Now we'll expand the anticommutators.

We have already shown that $\boldsymbol{\pi}\times\mathbf{E} = -\mathbf{E}\times\boldsymbol{\pi}$, and that such a term is $\mathcal{O}(mv^4)$

$$\left\{\frac{g-1}{2m}, (\mathbf{S}\cdot\boldsymbol{\pi}\times\boldsymbol{E} - \mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi})\right\}_+ = -\frac{2(g-1)}{m}\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi}$$

Because we specialize to constant magnetic fields, $\boldsymbol{\pi}$ commutes with $\mathbf{B}$, so

$$\left\{g - \frac{\pi^2}{m^2}, \mathbf{S}\cdot\mathbf{B}\right\}_+ = (2g - \frac{2\pi^2}{m^2})\mathbf{S}\cdot\mathbf{B}$$

and

$$\left\{\frac{g-2}{4m^2}, \{\mathbf{S}\cdot\boldsymbol{\pi}, \boldsymbol{\pi}\cdot\mathbf{B}\}_+\right\}_+ = \frac{g-2}{m^2}(\mathbf{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\mathbf{B})$$

Using the identity that $[\nabla_i, E_j] = 0$, $\nabla\times\mathbf{E} = 0$:

$$\begin{aligned}
\{\mathbf{S}\cdot\boldsymbol{\nabla}, \mathbf{S}\cdot\mathbf{E}\}_+ &= S_iS_j\nabla_iE_j + S_jS_iE_j\nabla_i \\
&= (S_iS_j + S_jS_i)\nabla_iE_j \\
&= (2S_iS_j + [S_j, S_i])\nabla_iE_j \\
&= (2S_iS_j + iS_k\epsilon_{ijk})\nabla_iE_j \\
&= 2S_iS_j\nabla_iE_j
\end{aligned}$$

So

$$\begin{aligned}
H &= \rho_3(m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^2}) + e\Phi - (g-1)\frac{e}{2m^2}\left[\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi} - S_iS_j\nabla_iE_j + \nabla\cdot\mathbf{E}\right] \\
&\quad +\rho_3(g-2)\frac{2}{4m^2}(\mathbf{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\mathbf{B}) - \rho_3\frac{e}{m}(\frac{g}{2} - \frac{\pi^2}{2m^2})\mathbf{S}\cdot\mathbf{B} \\
&= \rho_3\left(m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^2}\right) + e\Phi - \left(\frac{g-2}{2} + \frac{g}{2}\right)\frac{e}{2m^2}\left[\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi} - S_iS_j\nabla_iE_j + \nabla\cdot\mathbf{E}\right] \\
&\quad +\rho_3(g-2)\frac{2}{4m^2}(\mathbf{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\mathbf{B}) - \rho_3\frac{e}{m}\left[\frac{g}{2}\left(1 - \frac{\pi^2}{2m^2}\right) + \frac{g-2}{2}\frac{\pi^2}{2m^2})\right]\mathbf{S}\cdot\mathbf{B}
\end{aligned}$$

Magnetic Moment

Now we'll keep only those terms which contribute to the magnetic moment, using $\boldsymbol{\pi} = \mathbf{p} - e\mathbf{A}$

$$\begin{aligned}
H_{S\cdot B} &= \left(\frac{g-2}{2} + \frac{g}{2}\right)\frac{e^2}{2m^2}\mathbf{S}\cdot\mathbf{E}\times\mathbf{A} + (g-2)\frac{e}{4m^2}(\mathbf{S}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{B}) - \frac{e}{m}\left[\frac{g}{2}\left(1 - \frac{p^2}{2m^2}\right) + \frac{g-2}{2}\frac{p^2}{2m^2}\right]\mathbf{S}\cdot \\
&= -\frac{e}{2m}\left\{g\left(1 - \frac{p^2}{2m^2}\right)\mathbf{S}\cdot\mathbf{B} + (g-2)\frac{p^2}{2m^2}\mathbf{S}\cdot\mathbf{B} - (g-2)\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{B})}{2m^2} - \frac{e}{m}\left(\frac{g-2}{2} + \frac{g}{2}\right)\mathbf{S}\cdot\mathbf{E}\times\right.
\end{aligned}$$

This exactly matches with what was found before.

6.2 Identities

Simplify $\mathbf{W} \times \mathbf{B}$:

$$\begin{aligned}
(\mathbf{W} \times \mathbf{B})_i &= \epsilon_{ijk} W_j B_k \\
&= i(S_k)_{ij} W_j B_k \\
&= i(\mathbf{S} \cdot \mathbf{B})_{ij} W_j \\
&= i([\mathbf{S} \cdot \mathbf{B}]\mathbf{W})_i
\end{aligned}$$

Simplify $\mathbf{D} \times (\mathbf{D} \times \mathbf{W})$

$$\begin{aligned}
(\mathbf{D} \times [\mathbf{D} \times \mathbf{W}])_i &= \epsilon_{ijk} D_j (\mathbf{D} \times \mathbf{W})_k \\
&= \epsilon_{ijk} \epsilon_{k\ell m} D_j D_\ell W_m \\
&= -(S_j)_{ki} (S_\ell)_{mk} D_j D_\ell W_m \\
&= -(\mathbf{S} \cdot \mathbf{D})_{ki} (\mathbf{S} \cdot \mathbf{D})_{mk} W_m \\
&= -\left([\mathbf{S} \cdot \mathbf{D}]^2\right)_{im} W_m \\
&= -\left([\mathbf{S} \cdot \mathbf{D}]^2 \mathbf{W}\right)_i
\end{aligned}$$

Our representation defines the spin matrices as follows:

$$(S_k)_{ij} = -i\epsilon_{ijk}$$

They have the commutator

$$[S_i, S_j] = i\epsilon_{ijk} S_k$$

The product of two such spin matrices is given by:

$$\begin{aligned}
(S_k S_\ell)_{ij} &= (S_k)_{ia}(S_\ell)_{aj} \\
&= -\epsilon_{iak}\epsilon_{aj\ell} \\
&= (\delta_{k\ell}\delta_{ij} - \delta_{kj}\delta_{\ell i})
\end{aligned}$$

This implies that:

$$\begin{aligned}
(S_i S_j A_j B_i \mathbf{v})_l &= (S_i)_{lm}(S_j)_{mn} A_j B_i v_n \\
&= (S_i S_j)_{ln} A_j B_i v_n \\
&= (\delta_{ij}\delta_{ln} - \delta_{in}\delta_{lj}) A_j B_i v_n \\
&= (\mathbf{A}\cdot\mathbf{B}) v_l - A_l(\mathbf{B}\cdot\mathbf{v})
\end{aligned}$$

Or

$$\mathbf{A}(\mathbf{B}\cdot\mathbf{v}) = (\mathbf{A}\cdot\mathbf{B} - S_i S_j A_j B_i)\mathbf{v}$$

Now we can use this to establish some identities:

$$\begin{aligned}
E^i \mathbf{D}\cdot\eta &= \left([\mathbf{E}\cdot\mathbf{D} - S_j S_k E_k D_j]\,\eta\right)^i \\
&= \left([\mathbf{E}\cdot\mathbf{D} - (S_k S_j - [S_k, S_j]) E_k D_j]\,\eta\right)^i \\
&= \left([\mathbf{E}\cdot\mathbf{D} - (S_k S_j - i\epsilon_{kjl} S_l) E_k D_j]\,\eta\right)^i \\
&= \left([\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + iS_l(\mathbf{E}\times\mathbf{D})_l]\,\eta\right)^i \\
&= \left([\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot(\mathbf{E}\times\mathbf{D})]\,\eta\right)^i
\end{aligned}$$

Since E and W commute:

$$
\begin{aligned}
E^i \mathbf{W}\cdot\mathbf{E} &= \left(\left[\mathbf{E}^2 - S_j S_k E_k E_j\right]\mathbf{W}\right)^i \\
&= \left(\left[\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\right]\mathbf{W}\right)^i
\end{aligned}
$$

And

$$
\begin{aligned}
D^i \mathbf{D}\cdot\eta &= \left(\left[\mathbf{D}^2 - S_j S_k D_k D_j\right]\eta\right)^i \\
&= \left(\left[\mathbf{D}^2 - (S_k S_j + [S_j, S_k]) D_k D_j\right]\eta\right)^i \\
&= \left(\left[\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2 + i\epsilon_{jkl} S_l D_k D_j)\right]\eta\right)^i \\
&= \left(\left[\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2 + i\mathbf{S}\cdot(\mathbf{D}\times\mathbf{D})\right]\eta\right)^i \\
&= \left(\left[\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2 + e\mathbf{S}\cdot\mathbf{B})\right]\eta\right)^i
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
D^i \mathbf{E}\cdot\mathbf{W} &= ([\mathbf{D}\cdot\mathbf{E} - S_j S_k D_k E_j]\,\eta)^i \\
&= ([\mathbf{D}\cdot\mathbf{E} - (S_k S_j + [S_j, S_k]) D_k E_j]\,\eta)^i \\
&= ([\mathbf{D}\cdot\mathbf{E} - (S_k S_j + i\epsilon_{jkl} S_l) D_k E_j]\,\eta)^i \\
&= ([\mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) + i\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E})]\,\eta)^i
\end{aligned}
$$

Product of $H_{12}H_{21}$

We need to calculate $\left(\frac{\boldsymbol{\pi}^2}{2} - (\mathbf{S}\cdot\boldsymbol{\pi})^2 + (g-2)\frac{2}{m}\mathbf{S}\cdot\mathbf{B}\right)^2$ to first order in magnetic field strength. As a first step of simplification

$$\left(\frac{\boldsymbol{\pi}^2}{2} - (\mathbf{S}\cdot\boldsymbol{\pi})^2 + \frac{g-2}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\right)^2 = \left(\frac{\boldsymbol{\pi}^2}{2} - (\mathbf{S}\cdot\boldsymbol{\pi})^2\right)^2 + \frac{g-2}{2}\frac{e}{m}\left\{\frac{p^2}{2} - (\mathbf{S}\cdot\mathbf{p})^2, \mathbf{S}\cdot\mathbf{B}\right\}$$

First term

To simplify the first term, consider one element of this matrix operator:

$$\begin{aligned}
\left\{\left(\frac{\boldsymbol{\pi}^2}{2} - (\mathbf{S}\cdot\boldsymbol{\pi})^2\right)^2\right\}_{ac} &= \left(\frac{\boldsymbol{\pi}^2}{2} - S_iS_j\pi_i\pi_j\right)_{ab}\left(\frac{\boldsymbol{\pi}^2}{2} - S_lS_m\pi_l\pi_m\right)_{bc} \\
&= \left(\frac{\boldsymbol{\pi}^2}{2}\delta_{ab} - [S_iS_j]_{ab}\pi_i\pi_j\right)\left(\frac{\boldsymbol{\pi}^2}{2}\delta_{bc} - [S_lS_m]_{bc}\pi_l\pi_m\right) \\
&= \left(\frac{\boldsymbol{\pi}^2}{2}\delta_{ab} - [\delta_{ab}\delta_{ij} - \delta_{aj}\delta_{bi}]\pi_i\pi_j\right)\left(\frac{\boldsymbol{\pi}^2}{2}\delta_{bc} - [\delta_{bc}\delta_{lm} - \delta_{bm}\delta_{cl}]\pi_l\pi_m\right) \\
&= \left(-\frac{\boldsymbol{\pi}^2}{2}\delta_{ab} + \pi_b\pi_a\right)\left(-\frac{\boldsymbol{\pi}^2}{2}\delta_{bc} + \pi_c\pi_b\right) \\
&= \frac{\boldsymbol{\pi}^4}{4}\delta_{ac} - \pi_c\pi_a\frac{\boldsymbol{\pi}^2}{2} - \frac{\boldsymbol{\pi}^2}{2}\pi_c\pi_a + \pi_b\pi_a\pi_c\pi_b
\end{aligned}$$

It's very useful to have the following identity:

$$
\begin{aligned}
e(\mathbf{S}\cdot\mathbf{B})_{ab} &= e(S_i)_{ab}B_i \\
&= -ie\epsilon_{iab}B_i \\
&= -ie\epsilon_{iab}(\epsilon_{ijk}\partial_j A_k) \\
&= -ie\epsilon_{iab}\epsilon_{ijk}\frac{1}{2}(\partial_j A_k \partial_k A_j) \\
&= -\epsilon_{iab}\epsilon_{ijk}\frac{1}{2}[\pi_j, \pi_k] \\
&= -\epsilon_{iab}\epsilon_{ijk}\pi_j\pi_k \\
&= -(\delta_{aj}\delta_{bk} - \delta_{ak}\delta_{bj})\pi_j\pi_k \\
&= \pi_b\pi_a - \pi_a\pi_b
\end{aligned}
$$

Therefore,

$$e(\mathbf{S}\cdot\mathbf{B})_{ab} = [\pi_b, \pi_a]$$

Using this, and the fact that π commutes with S and B:

$$
\begin{aligned}
\pi_b\pi_a\pi_c\pi_b &= \pi_b\pi_c\pi_a\pi_b - \pi_b(e\mathbf{S}\cdot\mathbf{B})_{ac}\pi_b \\
&= \pi_b\pi_c\pi_a\pi_b - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac}
\end{aligned}
$$

Also,

$$
\begin{aligned}
\pi_b\pi_c\pi_a\pi_b &= \pi_b\pi_c\pi_b\pi_a + \pi_b\pi_c(e\mathbf{S}\cdot\mathbf{B})_{ba} \\
&= \pi_b\pi_b\pi_c\pi_a + \pi_b\pi_a(e\mathbf{S}\cdot\mathbf{B})_{bc} + \pi_b\pi_c(e\mathbf{S}\cdot\mathbf{B})_{ba}
\end{aligned}
$$

So now:

$$\begin{aligned}
\pi_b\pi_a\pi_c\pi_b - \pi_c\pi_a\frac{\boldsymbol{\pi}^2}{2} - \frac{\boldsymbol{\pi}^2}{4}\pi_c\pi_a &= \boldsymbol{\pi}^2\pi_c\pi_a + \pi_b\pi_a(e\mathbf{S}\cdot\mathbf{B})_{bc} + \pi_b\pi_c(e\mathbf{S}\cdot\mathbf{B})_{ba} \\
&\quad -\boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac} - \pi_c\pi_a\frac{\boldsymbol{\pi}^2}{2} - \frac{\boldsymbol{\pi}^2}{4}\pi_c\pi_a \\
&= \frac{1}{2}[\boldsymbol{\pi}^2, \pi_c\pi_a] + \pi_b\pi_a(e\mathbf{S}\cdot\mathbf{B})_{bc} + \pi_b\pi_c(e\mathbf{S}\cdot\mathbf{B})_{ba} - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac}
\end{aligned}$$

Now evaluate the commutator:

$$\begin{aligned}
[\pi_b, \pi_c\pi_a] &= [\pi_b, \pi_c]\pi_a - \pi_c[\pi_a, \pi_b] \\
&= (e\mathbf{S}\cdot\mathbf{B})_{cb}\pi_a - \pi_c(e\mathbf{S}\cdot\mathbf{B})_{ba}
\end{aligned}$$

$$\begin{aligned}
[\boldsymbol{\pi}^2, \pi_c\pi_a] &= [\pi_b\pi_b, \pi_c\pi_a] \\
&= \pi_b[\pi_b, \pi_c\pi_a] + [\pi_b, \pi_c\pi_a]\pi_b \\
&= (e\mathbf{S}\cdot\mathbf{B})_{cb}(\pi_b\pi_a + \pi_a\pi_b) - (e\mathbf{S}\cdot\mathbf{B})_{ba}(\pi_b\pi_c + \pi_c\pi_b)
\end{aligned}$$

This gives the result:

$$
\begin{aligned}
\pi_b\pi_a\pi_c\pi_b - \pi_c\pi_a\frac{\boldsymbol{\pi}^2}{2} - \frac{\boldsymbol{\pi}^2}{4}\pi_c\pi_a &= \frac{1}{2}\left[(e\mathbf{S}\cdot\mathbf{B})_{cb}(\pi_b\pi_a+\pi_a\pi_b) - (e\mathbf{S}\cdot\mathbf{B})_{ba}(\pi_b\pi_c+\pi_c\pi_b)\right] \\
&\quad +\pi_b\pi_a(e\mathbf{S}\cdot\mathbf{B})_{bc} + \pi_b\pi_c(e\mathbf{S}\cdot\mathbf{B})_{ba} - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac} \\
&= \frac{1}{2}[(e\mathbf{S}\cdot\mathbf{B})_{cb}(\pi_a\pi_b-\pi_b\pi_a) + (e\mathbf{S}\cdot\mathbf{B})_{ba}(\pi_b\pi_c-\pi_c\pi_b)] - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac} \\
&= \frac{1}{2}[(e\mathbf{S}\cdot\mathbf{B})_{cb}(e\mathbf{S}\cdot\mathbf{B})_{ba} + (e\mathbf{S}\cdot\mathbf{B})_{ba}(e\mathbf{S}\cdot\mathbf{B})_{cb}] - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac} \\
&= (e\mathbf{S}\cdot\mathbf{B})_{ab}(e\mathbf{S}\cdot\mathbf{B})_{bc}) - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac} \\
&= \left[(e\mathbf{S}\cdot\mathbf{B})^2\right]_{ac} - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac}
\end{aligned}
$$

Since we can throw away terms of order B^2, the final result tells us that, to first order in B:

$$
\begin{aligned}
\left(\frac{\boldsymbol{\pi}^2}{2} - (\mathbf{S}\cdot\boldsymbol{\pi})^2\right)^2 &= \frac{\boldsymbol{\pi}^4}{4} - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B}) \\
&= \frac{\boldsymbol{\pi}^4}{4} - e\mathbf{p}^2\mathbf{S}\cdot\mathbf{B}
\end{aligned}
$$

Second term

To simplify the second term, we need

$$
\begin{aligned}
\left\{\frac{\mathbf{p}^2}{2} - (\mathbf{S}\cdot\mathbf{p})^2, \mathbf{S}\cdot\mathbf{B}\right\} &= \mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - [(\mathbf{S}\cdot\mathbf{p})^2\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}(\mathbf{S}\cdot\mathbf{p})^2] \\
&= \mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - (S_iS_jS_k + S_kS_jS_i)p_ip_jB_k
\end{aligned}
$$

To simplify that triple product of spin matrices, we can use their explicit form:

$$\begin{aligned}
(S_i S_j S_k)_{ab} &= i\epsilon_{aci}\epsilon_{cdj}\epsilon_{dbk} \\
&= i(\delta_{id}\delta_{aj} - \delta_{ij}\delta_{ad})\epsilon_{dbk} \\
&= i(\delta_{aj}\epsilon_{ibk} - \delta_{ij}\epsilon_{abk}) \\
(S_i S_j S_k + S_k S_j S_i)_{ab} &= i(\delta_a j\epsilon_{ibk} + \delta_{aj}\epsilon_{kbi} - \delta_{ij}\epsilon_{abk} - \delta k j\epsilon_{abi}) \\
&= -i(\delta_{ij}\epsilon_{abk} + \delta_{kj}\epsilon_{abi} \\
&= \delta_{ij}(S_k)_a b + \delta_{kj}(S_i)_{ab}
\end{aligned}$$

Now

$$\begin{aligned}
\mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - (S_i S_j S_k + S_k S_j S_i)p_i p_j B_k &= \mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - (\delta_{ij}S_k + \delta_{kj}S_i)p_i p_j B_k \\
&= -(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})
\end{aligned}$$

Result

At last, the final result is that, to the order we care about

$$\left(\frac{\boldsymbol{\pi}^2}{2} - (\mathbf{S}\cdot\boldsymbol{\pi})^2 + (g-2)\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\right)^2 = \frac{\boldsymbol{\pi}^4}{4} - e\mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - \frac{g-2}{2}\frac{e}{m}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})$$