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# 1 Introduction

We wish to calculate the gyromagnetic ratio of a particle of arbitrary spin in a loosely bound state system. We can write down an effective Lagrangian which will capture all the necessary effects. Our job is then to calculate the coefficients of this effective particle. By considering the constraints which exist on the electromagnetic current in a general relativistic theory, we can obtain a simple form which holds for arbitrary spin, and then use this to fix the coefficients of the relativistic theory.

While for a particle of general spin we do not have an exact Lagrangian, we do have one for a spin-1 theory. We can use this Lagrangian to perform the same calculation as above, but in an exact theory. We derive the nonrelativistic potential first from the equations of motion, and then from diagrammatic calculations, and obtain a result that agrees with the more general calculation.

## 1.1 Description of problem

We consider a loosely bound system two particles of arbitrary spin, and wish to calculate the correction to the gyromagnetic ratio of the particles. So we consider the case of the two particles interacting with both each other, via a Coulomb potential, and a very weak external magnetic field.

### 1.1.1 Experimental Motivation

Talk about

### 1.1.2 Approach

This system allows for several simplifications. First, because the system is loosely bound all energy scales are nonrelativistic. Second, the magnetic field we consider is both very weak and constant. So we can ignore all corrections which involve derivatives of the magnetic field or which are quadratic in its strength.

The approach we'll take is to first consider the electric potential as an external field acting on a single particle. Next we can consider the bound system as a whole sitting in an external magnetic field, and take into account recoil effects.

## 1.2 Theoretical Background

(Here we talk about existing theoretical work in this area)

### 1.2.1 Spin-1/2

(Here talk about the approach that works for spin 1/2, and why it breaks down in the general case.)

### 1.2.2 BMT equation

(Here discuss the BMT equation, which holds for general spin and can be used to derive the g-factor corrections)

### 1.2.3 Khriplovich general spin work

(Discuss the work Khriplovich did with general spinors.)

## 2 NRQED

We can construct an effective, nonrelativistic Lagrangian for a charged particle interacting with an electromagnetic field. — (quick outline) Background

Construction - Define the interaction – physical situation and the fields involved - enumerate building blocks – ( $\bar{S}$  text here) -Define order of terms -Symmetry of terms -Enumerating terms – Start by discussing how spin/position operators combine. – More indices on spin operator - $i$  higher order position space operator. – Make clear that there are exactly 4 orders of terms ( $v^2, v^4, B, Bv^2$ ) – First construct all order unity objects – Construct position space terms for each order, going through number of indices – (then contracting with the order unity objects) – Now we have all terms in the Lagrangian Calculation

### 2.1 The NRQED Lagrangian

We want to construct an effective Lagrangian in the nonrelativistic limit. Our goal is to calculate the leading order corrections to the  $g$ -factor, which are corrections of order  $\alpha^2$ . To this end, we need terms in the effective nonrelativistic Lagrangian which are equivalent corrections.

#### 2.1.1 position space operators

	Order	P	T	$\dagger$
$eE_i$	$m^2v^3$	-	+	+
$eB_i$	$m^2v^2$	+	-	+
$D_i$	$mv$	-	+	-
$D_0$	$mv$	+	-	-
$S_i$	1	+	-	+
$i$	1	+	-	-

There are several symmetries that terms in the Lagrangian must preserve: spatial reflection, time reversal, gauge invariance, and Hermiticity. Gauge invariance is taken care of by only considering gauge-invariant operators.

All of the possible spin-space structures preserve spatial parity. If the term in the Lagrangian is to obey such a symmetry, then any allowed collection of position-space operators must be themselves invariant under reflection.

Each term must also be invariant under time reversal. The spin operator  $S_i$  flips sign under time reversal. Thus,  $\bar{S}_{ij}$  is even, and  $\bar{S}_{ijk}$  odd, under the same transformation.  $E_i$  and  $D_i$  are even, while  $B_i$  is odd. Any set of operators with definite behavior under time reversal, though, can be made invariant by including an extra factor of  $i$ . So even if some term  $ABC$  is odd,  $iABC$  will be even.

Finally, any sequence of operators can be made Hermitian by simply adding the Hermitian conjugate; if we wish to include a term  $ABC$  in the Lagrangian, we add  $C^\dagger B^\dagger A^\dagger$ . Since all the operators under consideration are either Hermitian or anti-Hermitian, this means that exactly one of  $ABC \pm CBA$  will be allowed. The spin space operators are all Hermitian of themselves, and commute with all other operators, so only the position space operators are nontrivial.

Thus for any possible set of operators, first it is determined if such a set is allowed by considerations of parity. Then, whether an additional factor of  $i$  is needed is determined by examining the properties under time reversal. Finally, the allowed Hermitian terms are enumerated.

Of the “building blocks” for the general Lagrangian, those odd in parity are also odd in  $v$ . And those even in parity are of an order even in  $v$ . So only terms of an order even in  $v$  can possibly be allowed under parity, and at the same time every such term will have the correct symmetry properties under spatial reflection.

First consider terms of leading order  $mv^2$ . What collections of position space operators exist?  $E$  is already too high of an order, leaving only that with two powers of  $D$ :

$$\text{Sets of } \mathcal{O}(mv^2) \text{ combinations} = \left\{ \frac{1}{m} D_i D_j \right\} \quad (2.1)$$

This is in addition to the single permitted term containing  $e\Phi$  which is not, by itself, gauge-invariant.

Next consider terms of order  $mv^4$ . We could have up to a single power of  $E$ , or only the long derivative operators:

$$\text{Set of } \mathcal{O}(mv^2) \text{ combinations} = \left\{ \frac{1}{m^3} D_i D_j D_k D_\ell, \frac{1}{m^2} e E_i D_j \right\} \quad (2.2)$$

The leading order term in  $B$  is just  $\frac{e}{m} B_i$ .

The order  $\frac{e}{m} B_i v^2$  terms are drawn from:

$$\text{Set of } \mathcal{O}\left(\frac{e}{m} B v^2\right) \text{ combinations} = \left\{ \frac{e}{m^3} D_i D_j B_k \right\} \quad (2.3)$$

### 2.1.2 Contraction of terms

In this nonrelativistic theory the Lagrangian need not be Lorentz invariant, but must still be Galilean invariant. All the operators we consider transform as 3-vectors or higher order 3-tensors. To form allowed terms, all indices must be somehow contracted.

Above, all relevant sets of position space operators are considered for each order of term. The greatest number of indices free was four. These must be contracted with order unity tensors, which as well as the spin operators include  $\delta_{ij}$  and the completely antisymmetric tensor  $\epsilon_{ijk}$ .

### 2.1.3 Order of terms

We consider constant, infinitesimal external magnetic fields, so we need only consider terms linear in  $\mathbf{B}$ .

The velocity of the particles in our bound state system will be  $v \sim \alpha$ .

The electric field we consider is the Coulomb field, so  $e\Phi \sim mZ\alpha^2 \sim mv^2$ , and  $eE \sim m^2 v^3$ .

Each derivative of the electric field will add an additional factor of  $mv$ , so the operator  $\mathbf{D}$  can be taken to be of this order.

We need to keep terms up to order  $mv^4$  and  $\frac{B}{m}v^2$  in order to calculate the  $g$ -factor to the necessary precision. We include  $mv^4$  terms so we can be sure that there are no effects entering from second-order perturbation theory.

### 2.1.4 Constraints on the form of the Lagrangian

The Lagrangian is constrained to obey several symmetries. It must be invariant under the symmetries of parity and time reversal. It must also be invariant under Galilean transformations. The Lagrangian must also be Hermitian, and gauge invariant.

What are the gauge invariant building blocks we can use to construct this Lagrangian? We have the external fields  $\mathbf{E}$  and  $\mathbf{B}$ , the spin operators  $\mathbf{S}$ , and the long derivative  $\mathbf{D} = \partial - ie\mathbf{A}$ . The fields should always be accompanied by the charge  $e$  of the particle.

When considering the case of higher spin particles, we might consider terms quadratic and above in spin operators. For a particle of spin  $s$ , there must be  $(2s + 1)^2$  independent hermitian operators. We can span this set of operators by considering products of up to  $2s$  spin matrices which are symmetric and traceless in every vector index. For example, for spin-1 we have quadratic, in addition to  $I$  and  $S_i$ , five independent structures of the form  $S_i S_j + S_j S_i + \delta_{ij} \mathbf{S}$ .

We also have the scalar  $D_0$ , however, we need only include a single such term because we insist on having only one power of the time derivative.

To consider possible terms, we need to know how each of the above behave under the discrete transformations and Hermitian conjugate. The signs under these transformations are listed in the table below. (Also included is the imaginary number  $i$ .)

	Order	P	T	†
$eE_i$	$m^2v^3$	-	+	+
$eB_i$	$m^2v^2$	+	-	+
$D_i$	$mv$	-	+	-
$D_0$	$mv$	+	-	-
$S_i$	1	+	-	+
$i$	1	+	-	-

### 2.1.5 Allowed terms

Our strategy in cataloguing terms will be to first list all the combinations of  $E$ ,  $B$  and  $D$  which might be allowed at a particular order, to consider the various ways of contracting these vectors, and finally to eliminate terms which do not obey the proper symmetries. We can always make a particular combination Hermitian, and get the proper behavior under time reversal by adding a factor of  $i$ , but parity will kill several terms. Note that of the structures we can contract with, all are even under parity.

We can also insist that the Lagrangian have the expected form in the absence of external fields, which eliminates terms like  $\bar{S}_{ij}D_iD_j$ . The leading order terms should be of order  $mv^2$  or  $\frac{eB}{m}$ . Combinations of the correct order are:

- The single  $D_0$  term. To have the correct transformation properties this should be  $iD_0$ .
- The kinetic  $\mathbf{D}^2$  term, which must be simply  $\frac{\mathbf{D}^2}{2m}$
- A term with a single power of  $B_i$ . The only way to contract this is with the spin matrix, so the term will have the form  $\frac{e}{m}\mathbf{S} \cdot \mathbf{B}$

All these terms are Hermitian in themselves.

So, the allowed terms at this order are:

$$iD_0, \frac{\mathbf{D}^2}{m}, \frac{e}{m}\mathbf{S} \cdot \mathbf{B}$$

The first two terms have their coefficients fixed, while we wish to honestly calculate the factor before the last.

$$\mathcal{L}_{NRQED} = \Psi^\dagger \left\{ iD_0 + \frac{\mathbf{D}^2}{2m} + c_F \frac{e}{m} \mathbf{S} \cdot \mathbf{B} \right\} \Psi \quad (2.4)$$

Are there any terms of order  $mv^3$  or  $\frac{B}{m}v$  allowed? Possible combinations are:

- Three powers of  $D$ :  $D_iD_jD_k$ . However, this is odd under parity, and so not allowed.
- A term with both the derivative and magnetic field:  $D_iB_j$ . Again, this is odd under parity and so forbidden.
- A single power of  $E$ . Again, odd under parity.

So, all such terms are forbidden by consideration of parity.

Next we consider terms of order  $mv^4$ .

- Four powers of  $D$ : fixed by the kinetic term to be  $\frac{\mathbf{D}^2}{8m^3}$
- One power of  $E_i$  and one of  $D_j$ . This combination is even under parity and odd under Hermitian conjugate. There are three ways of contracting these two fields.
  - With the delta function. The allowed Hermitian term is then  $\delta_{ij}(D_iE_j - E_jD_i)$ .
  - With the combination  $i\epsilon_{ijk}S_k$ . The allowed term is  $i\epsilon_{ijk}(D_iE_jS_k + S_kE_jD_i)$ .

– With the quadratic spin structure  $Q_{ij}$ :  $Q_{ij}(D_i E_j - E_j D_i)$ .

In the Lagrangian we'll write these as:

$$\mathcal{L}_{mv^4} = \Psi^\dagger \left\{ \frac{\mathbf{D}^4}{8m^2} + c_D \frac{e(\mathbf{D} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{D})}{8m^2} + c_Q \frac{eQ_{ij}(D_i E_j - E_i D_j)}{8m^2} + c_S \frac{ie\mathbf{S} \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D})}{8m^2} \right\} \Psi \quad (2.5)$$

Terms of order  $\frac{B}{m}v^2$ . The only allowed combination is  $D_i D_j B_k$ . We can contract two indices with each other and the third with a spin matrix in three different ways:

- $(\mathbf{S} \cdot \mathbf{B})\mathbf{D}^2 + \mathbf{D}^2(\mathbf{S} \cdot \mathbf{B})$
- $S_i D_j B_i D_j$
- $S_i (D_i B_j D_j + D_j B_j D_i)$

We can also contract all indices with a cubic spin structure:

- $\bar{S}_{ijk}(D_i D_j B_k + B_k D_j D_i)$
- $\bar{S}_{ijk} D_i B_j D_k$

In the Lagrangian we'll write these as:

$$\begin{aligned} \mathcal{L}_{Bv^2} = \Psi^\dagger \left\{ c_{W1} \frac{e\mathbf{D}^2 \mathbf{S} \cdot \mathbf{B} + \mathbf{S} \cdot \mathbf{B} \mathbf{D}^2}{8m^3} - c_{W2} \frac{eD_i(\mathbf{S} \cdot \mathbf{B})D_i}{4m^3} + c_{p'p} \frac{e[(\mathbf{S} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{D}) + (\mathbf{B} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{D})]}{8m^3} \right. \\ \left. + c_{T1} \frac{e\bar{S}_{ijk}(D_i D_j B_k + B_k D_j D_i)}{8m^3} + c_{T2} \frac{e\bar{S}_{ijk} D_i B_j D_k}{8m^3} \right\} \Psi \end{aligned} \quad (2.6)$$

### 2.1.6 Full Lagrangian

The full Lagrangian we consider is then:

$$\begin{aligned} \mathcal{L}_{NRQED} = \Psi^\dagger \left\{ iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^2} + c_F \frac{e}{m} \mathbf{S} \cdot \mathbf{B} + c_D \frac{e(\mathbf{D} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{D})}{8m^2} + c_Q \frac{eQ_{ij}(D_i E_j - E_i D_j)}{8m^2} \right. \\ + c_S \frac{ie\mathbf{S} \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D})}{8m^2} + c_{W1} \frac{e\mathbf{D}^2 \mathbf{S} \cdot \mathbf{B} + \mathbf{S} \cdot \mathbf{B} \mathbf{D}^2}{8m^3} - c_{W2} \frac{eD_i(\mathbf{S} \cdot \mathbf{B})D_i}{4m^3} \\ \left. + c_{p'p} \frac{e[(\mathbf{S} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{D}) + (\mathbf{B} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{D})]}{8m^3} + c_{T1} \frac{e\bar{S}_{ijk}(D_i D_j B_k + B_k D_j D_i)}{8m^3} + c_{T2} \frac{e\bar{S}_{ijk} D_i B_j D_k}{8m^3} \right\} \Psi \end{aligned} \quad (2.7)$$

One of the features of this Lagrangian is that every coefficient is fixed by the one-photon interaction. Although some terms might represent two-photon interactions, they are terms like  $\mathbf{S} \cdot \mathbf{A} \times \mathbf{E}$ , whose coefficient is fixed by the gauge-invariant term  $\mathbf{S} \cdot \mathbf{D} \times \mathbf{E}$ . This in turn means that we can calculate the corrections to the  $g$ -factor by considering only one-photon interactions.

## 2.2 Scattering off external field in NRQED

We can write down those terms in the NRQED Lagrangian which have one power of the external field. This set of terms will not, by themselves, be gauge invariant. If, for example, a coefficient exists before a term with both one and two powers of  $A$ , we'll want to make sure we get the same result in both calculations. We'll add a superscript to the coefficient to keep track of this: so below we write  $c_S^1$ .

In writing the expansion of terms like  $\mathbf{D}^4$  it is convenient to use anticommutators.

$$\begin{aligned}\mathcal{L}_A = & \Psi^\dagger(-eA_0 - ie\frac{\{\nabla_i, A_i\}}{2m} - ie\frac{\{\nabla^2, \{\nabla_i, A_i\}\}}{8m^3} + c_F e\frac{\mathbf{S} \cdot \mathbf{B}}{2m} + c_D \frac{e(\nabla \cdot \mathbf{E} - \mathbf{E} \cdot \nabla)}{8m^2} + c_Q \frac{eQ_{ij}(\nabla_i E_j - E_i \nabla_j)}{8m^2} \\ & + c_S^1 \frac{ie\mathbf{S} \cdot (\nabla \times \mathbf{E} - \mathbf{E} \times \nabla)}{8m^2} + c_{W_1} \frac{e[\nabla^2(\mathbf{S} \cdot \mathbf{B}) + (\mathbf{S} \cdot \mathbf{B})\nabla^2]}{8m^3} - c_{W_2} \frac{e\nabla^i(\mathbf{S} \cdot \mathbf{B})\nabla^i}{4m^3} + c_{p'p} \frac{e[(\mathbf{S} \cdot \nabla)(\mathbf{B} \cdot \nabla) + (\mathbf{B} \cdot \nabla)(\mathbf{S} \cdot \nabla)]}{8m^3}\end{aligned}$$

We want to calculate from this a particular process: scattering off an external field, with incoming momentum  $\mathbf{p}$ , outgoing  $\mathbf{p}'$ , and  $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ . There is one diagram associated with each term above, but the total amplitude is just going to be the sum of all these one-photon vertices. These of course can just be read off directly from the Lagrangian: we replace the fields  $\Psi$  with the spinors  $\phi$ , and any operator  $\nabla$  acting will become  $i\mathbf{p}$  if it acts on the right,  $i\mathbf{p}'$  if it is to the left.

We can simplify some expressions involving  $\nabla$  and  $\mathbf{E}$ : Because  $Q_{ij}$  is symmetric:

$$Q_{ij}(\nabla_i E_j - E_i \nabla_j) = Q_{ij}[\nabla_i, E_j] = Q_{ij}(\partial_i E_j)$$

And because  $E_i = -\partial_i \Phi$

$$\nabla \times \mathbf{E} - \mathbf{E} \times \nabla = -2\mathbf{E} \times \nabla$$

And also use that

$$\nabla \cdot \mathbf{E} - \mathbf{E} \cdot \nabla = (\partial_i E_i)$$

Now we can write down the scattering amplitude for scattering off the external field, before we apply any assumptions about the particular process.

$$\begin{aligned}iM = & ie\phi_S^\dagger \left( -A_0 + \frac{\mathbf{A} \cdot (\mathbf{p} + \mathbf{p}')}{2m} - \frac{\mathbf{A} \cdot (\mathbf{p} + \mathbf{p}')\mathbf{p}^2 + \mathbf{p}'^2 \mathbf{A} \cdot (\mathbf{p} + \mathbf{p}')}{8m^3} \right. \\ & + c_F \frac{\mathbf{S} \cdot \mathbf{B}}{2m} + c_D \frac{(\partial_i E_i)}{8m^2} + c_Q \frac{Q_{ij}(\partial_i E_j)}{8m^2} + c_S^1 \frac{\mathbf{E} \times \mathbf{p}}{4m^2} \\ & \left. - c_{W_1} \frac{(\mathbf{S} \cdot \mathbf{B})(\mathbf{p}^2 + \mathbf{p}'^2)}{8m^3} + c_{W_2} \frac{(\mathbf{S} \cdot \mathbf{B})(\mathbf{p} \cdot \mathbf{p}')}{4m^3} - c_{p'p} \frac{(\mathbf{S} \cdot \mathbf{p}')(\mathbf{B} \cdot \mathbf{p}) + (\mathbf{B} \cdot \mathbf{p}')(\mathbf{S} \cdot \mathbf{p})}{8m^3} \right) \phi_S\end{aligned}$$

The above can be simplified somewhat. We choose our gauge such that  $\nabla_i A_i = 0$ . If we specify elastic scattering then kinematics dictate that  $\mathbf{p}'^2 = \mathbf{p}^2$ . Finally, if we consider  $\mathbf{B}$  constant, the  $c_W$  terms become indistinguishable, since  $[\nabla_i, B_j] = 0$ . (It is only this last assumption that costs us any information.) Then the scattering amplitude, as calculated from  $\mathcal{L}_{NRQED}$ , is:

$$\begin{aligned}iM = & ie\phi_S^\dagger \left( -A_0 + \frac{\mathbf{A} \cdot \mathbf{p}}{m} - \frac{(\mathbf{A} \cdot \mathbf{p})\mathbf{p}^2}{2m^3} + c_F \frac{\mathbf{S} \cdot \mathbf{B}}{2m} + c_D \frac{(\partial_i E_i)}{8m^2} + c_Q \frac{Q_{ij}(\partial_i E_j)}{8m^2} \right. \\ & \left. + c_S^1 \frac{\mathbf{E} \times \mathbf{p}}{4m^2} - (c_{W_1} - c_{W_2}) \frac{(\mathbf{S} \cdot \mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p} \frac{(\mathbf{S} \cdot \mathbf{p})(\mathbf{B} \cdot \mathbf{p})}{4m^3} \right) \phi_S\end{aligned}\tag{2.8}$$

### 3 General Spin Formalism

Our ultimate goal is to calculate corrections to the  $g$ -factor of a loosely bound charged particle of arbitrary spin. Our strategy is to obtain an effective Lagrangian in the nonrelativistic limit.

We first consider features of a general-spin formalism in both the relativistic and nonrelativistic cases, and the connection between the wave functions of the free particles. Then we consider how constraints of the relativistic theory let us calculate scattering off an external field. Comparing this result to that done with an effective NRQED Lagrangian, we can obtain the coefficients of that Lagrangian for particles of general spin.

#### 3.1 Spinors for general-spin charged particles

##### 3.1.1 Relativistic bispinors

First we need to work out a formalism that will apply to the general spin case. We want to represent the spin state of the particles by an object that looks like a generalization of the Dirac bispinor.

It is easiest to start with the Dirac basis, where the upper and lower components of the bispinor are objects of opposite helicity, each transforming as an object of spin  $1/2$ .

To that end define an object

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad (3.1)$$

that we wish to have the appropriate properties. Each component should transform as a particle of spin  $s$ , but with opposite helicity. Under reflection the upper and lower components transform into each other.

Representations of the proper Lorentz group are spinors which are separately symmetric in dotted and undotted indices. If  $\xi$  is an object with  $p$  undotted and  $q$  dotted indices

$$\xi = \{\xi^{\alpha_1 \dots \alpha_p}_{\dot{\beta}_1 \dots \dot{\beta}_q}\} \quad (3.2)$$

Then this can be a representation of a particle of spin  $s = (p + q)/2$ .

We have some free choice in how to partition the dotted/undotted indices, and we cannot choose exactly the same scheme for all spin as long as both types of indices are present. However, we can make separately consistent choices for integral and half-integral spin. For integral spin we can say  $p = q = s$ , while for the half-integral case we'll choose  $p = s + \frac{1}{2}$ ,  $q = s - \frac{1}{2}$ .

We want the  $\xi$  and  $\eta$  to transform as objects of opposite helicity. Under reflection they will transform into each other. So

$$\eta = \{\eta^{\beta_1 \dots \beta_q}_{\alpha_1 \dots \alpha_p}\} \quad (3.3)$$

In the rest frame of the particle, they will have clearly defined and identical properties under rotation. The rest frame spinors are equivalent to rank  $2s$  nonrelativistic spinors. So the bispinor in the rest frame looks like

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_0 \\ \xi_0 \end{pmatrix} \quad (3.4)$$

where

$$\xi_0 = \{(\xi_0)_{\alpha_1 \dots \alpha_p \beta_1 \dots \beta_q}\} \quad (3.5)$$

and all indices are symmetric.

We can obtain the spinors in an arbitrary frame by boosting from the rest frame. The upper and lower components we have defined to have opposite helicity, and so will act in opposite ways under boost:

$$\xi = \exp\left(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi_0, \quad \eta = \exp\left(-\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi_0 \quad (3.6)$$



What form should the operator  $\Sigma$  have? Under an infinitesimal boost by a rapidity  $\phi$ , a spinor with a single undotted index is transformed as

$$\xi_\alpha \rightarrow \xi'_\alpha = \left( \delta_{\alpha\beta} + \frac{\phi \cdot \sigma_{\alpha\beta}}{2} \right) \xi_\beta$$

while one with a dotted index will transform as

$$\xi_{\dot{\alpha}} \rightarrow \xi'_{\dot{\alpha}} = \left( \delta_{\dot{\alpha}\dot{\beta}} - \frac{\phi \cdot \sigma_{\dot{\alpha}\dot{\beta}}}{2} \right) \xi_{\dot{\beta}}$$

The infinitesimal transformation of a higher spin object with the first  $p$  indices undotted and the last  $q$  dotted would then be

$$\xi \rightarrow \xi' = \left( 1 + \sum_{a=0}^p \frac{\sigma_a \cdot \phi}{2} - \sum_{a=p+1}^{p+q} \frac{\sigma_a \cdot \phi}{2} \right) \xi$$

where  $a$  denotes which spinor index of  $\xi$  is operated on.

If we define

$$\Sigma = \sum_{a=0}^p \sigma_a - \sum_{a=p+1}^{p+q} \sigma_a \quad (3.7)$$

Then the infinitesimal transformations would be

$$\xi \rightarrow \xi' = \left( 1 + \frac{\Sigma \cdot \phi}{2} \right) \xi$$

$$\eta \rightarrow \eta' = \left( 1 - \frac{\Sigma \cdot \phi}{2} \right) \eta$$

So the exact transformation should be

$$\xi \rightarrow \xi' = \exp \left( \frac{\Sigma \cdot \phi}{2} \right) \xi$$

$$\eta \rightarrow \eta' = \exp \left( -\frac{\Sigma \cdot \phi}{2} \right) \eta$$

Therefore, the bispinor of some particle boosted by  $\phi$  from rest will be

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \exp \left( \frac{\Sigma \cdot \phi}{2} \right) \xi_0 \\ \exp \left( -\frac{\Sigma \cdot \phi}{2} \right) \xi_0 \end{pmatrix} \quad (3.8)$$

In dealing with the relativistic theory, we'll want a basis that separates the particle and antiparticle parts of the wave function. If we want the upper component to be the particle, then in the rest frame the lower component will vanish, and for low momentum will be small compared to the upper component. The unitary transformation which accomplishes this is

$$\Psi' = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

$$\phi = \frac{1}{\sqrt{2}}(\xi + \eta)$$

$$\chi = \frac{1}{\sqrt{2}}(\eta - \xi)$$

Which is equivalent to

$$\Psi' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Psi$$

Then,

$$\phi = \cosh\left(\frac{\Sigma \cdot \phi}{2}\right) \xi_0 \quad (3.9)$$

$$\chi = \sinh\left(\frac{\Sigma \cdot \phi}{2}\right) \xi_0 \quad (3.10)$$

### 3.1.2 Spinors for nonrelativistic theory

We also need to discuss the nonrelativistic, single-particle theory. This is much simpler: the spin state of particles in this theory is represented by symmetric spinors with  $2s + 1$  undotted indices. The only operators we need to consider acting on this space are spin matrices and products of spin matrices.

## 3.2 Electromagnetic Interaction

Knowing how the wave functions themselves behave, we want to see what that tells us about possible electromagnetic interaction. Interaction with a single electromagnetic photon should take the form

$$M = A_\mu j^\mu$$

where  $j^\mu$  is the electromagnetic current.

The electromagnetic current must be built out of the particle's momenta and bilinears of the charged particle fields in such a way that they have the correct Lorentz properties. We must also demand current conservation: the equation  $q_\mu j^\mu = 0$  must hold. Above we already have shown that, in the case of general spin, there exist only two such bilinears, a scalar and a tensor.

There will be two permissible terms in the current. We could consider a scalar bilinear coupled with a single power of external momenta. In order to fulfill the current conservation requirement, it should be

$$\frac{p^\mu + p'^\mu}{2m} \bar{\Psi}^\dagger \Psi$$

This will obey current conservation because  $q = p' - p$ , and  $(p + p') \cdot (p' - p) = p^2 - p'^2 = 0$

We can also consider a tensor term contracted with a power of momenta. To fulfill current conservation, we can demand that the tensor bilinear be antisymmetric, and contract it with  $q$ :

$$\frac{q_\nu}{2m} \bar{\Psi}^\dagger \Sigma^{\mu\nu} \Psi$$

We don't need to worry about higher order tensor bilinears: they will necessitate too many powers of the external momenta.

So the most general current would look like

$$j^\mu = F_e \frac{p^\mu + p'^\mu}{2m} \bar{\Psi}^\dagger \Psi + F_m \frac{q_\nu}{2m} \bar{\Psi}^\dagger \Sigma^{\mu\nu} \Psi \quad (3.11)$$

In general the form factors might have quite complicated dependence on  $q$ , but these corrections will be too small compared to the type of result we're interested in. At leading order  $F_e$  will just be the electric charge of the particle in question, and  $F_m$  will, as we'll see after connecting this result to the nonrelativistic limit, be related to the particle's  $g$ -factor. So to the order we need, we can write the current as

$$j^\mu = e \frac{p^\mu + p'^\mu}{2m} \bar{\Psi}^\dagger \Psi + eg \frac{q_\nu}{2m} \bar{\Psi}^\dagger \Sigma^{\mu\nu} \Psi \quad (3.12)$$

This captures the essence of the interaction between a charged particle of general-spin and a single photon.

### 3.2.1 Connection between the spinors of the two theories

Knowing something of how the relativistic theory behaves, we can find the connection between the relativistic and nonrelativistic spinors. In the rest frame, there are two independent bispinors which represent particle and antiparticle states:

$$\Psi = \begin{pmatrix} \xi_0 \\ 0 \end{pmatrix}$$

or

$$\Psi = \begin{pmatrix} 0 \\ \xi_0 \end{pmatrix}$$

However, when we consider a particle with zero momentum it is not the case that the upper component of the bispinor can be directly associated with the Schrodinger like wave-function of the particle — for instance, it would not be correctly normalized, for there is some mixing with the lower component.

We can obtain a relation between  $\xi_0$  and the Schrodinger amplitude  $\phi_s$  by considering the current density at zero momentum transfer. For  $\phi_s$  it will be  $j_0 = \phi_s^\dagger \phi$ . For the relativistic theory we have, as calculated above:

$$j^0 = F_e \frac{p^0 + p'^0}{2m} \bar{\Psi}^\dagger \Psi + F_m \frac{q_\nu}{2m} \bar{\Psi}^\dagger T^{0\nu} \Psi$$

At  $q = 0$  the expression simplifies

$$j^0(q = 0) = F_e \frac{p_0}{m} \bar{\Psi}^\dagger \Psi$$

$$= F_e \frac{p_0}{m} (\phi^\dagger \phi - \chi^\dagger \chi)$$

$\phi$  and  $\chi$  are both related to the rest frame spinor  $\xi_0$ . So we can write instead

$$j^0 = F_e \frac{p_0}{m} \xi_0^\dagger \left\{ \cosh^2\left(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) - \sinh^2\left(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \right\} \xi_0 = F_e \frac{p_0}{m} \xi_0^\dagger \xi_0$$

where the last equality follows from the hyperbolic trig identity.

If we demand that the two current densities be equal to each other, we find

$$\frac{p_0}{m} \xi_0^\dagger \xi_0 = \phi_s^\dagger \phi_s$$

Approximating

$$\left(1 + \frac{\mathbf{p}^2}{2m}\right) \xi_0^\dagger \xi_0 = \phi_s^\dagger \phi_s$$

This will hold to the necessary order if we identify

$$\xi_0 = \left(1 - \frac{\mathbf{p}^2}{4m}\right) \phi_s$$

To write the relativistic bispinors in terms of  $\phi_s$  we will also need approximations to  $\cosh(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2})$  and  $\sinh(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2})$ . We only need the rapidity to the leading order:  $\phi \approx \mathbf{v} \approx \frac{\mathbf{p}}{m}$ .

$$\cosh\left(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \approx 1 + \frac{1}{2} \left(\frac{\mathbf{\Sigma} \cdot \mathbf{p}}{2m}\right)^2$$

$$\sinh\left(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \approx \frac{\mathbf{\Sigma} \cdot \mathbf{p}}{2m}$$

The the two bispinor components are

$$\begin{aligned}\phi &\approx \left(1 + \left[\frac{1}{2} \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{2m}\right]^2\right) \xi_0 \\ &\approx \left(1 + \frac{(\boldsymbol{\Sigma} \cdot \mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m}\right) \phi_S\end{aligned}\tag{3.13}$$

$$\begin{aligned}\chi &\approx \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{2m} \xi_0 \\ &\approx \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{2m} \phi_S\end{aligned}\tag{3.14}$$

### 3.3 Bilinears in terms of nonrelativistic theory

The next step is to express the relativistic bilinears, built out of the bispinors  $\Psi$ , in terms of the Schrodinger like wave functions.

We have above written the bispinors in terms of  $\phi_s$ , so we can use those identities to express the bilinears in the same manner.

#### 3.3.1 Scalar bilinear

$$\begin{aligned}\bar{\Psi}^\dagger(p')\Psi(p) &= \phi^\dagger\phi - \chi^\dagger\chi \\ &= \phi_s^\dagger \left[1 + \frac{(\boldsymbol{\Sigma} \cdot \mathbf{p}')^2}{8m^2} - \frac{\mathbf{p}'^2}{4m^2}\right] \left[1 + \frac{(\boldsymbol{\Sigma} \cdot \mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m^2}\right] \phi_s - \phi_s^\dagger \left[\frac{(\boldsymbol{\Sigma} \cdot \mathbf{p}')(\boldsymbol{\Sigma} \cdot \mathbf{p})}{4m^2}\right] \phi_s \\ &= \phi_s^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2} \{(\boldsymbol{\Sigma} \cdot \mathbf{p}')^2 + (\boldsymbol{\Sigma} \cdot \mathbf{p})^2 - 2(\boldsymbol{\Sigma} \cdot \mathbf{p}')(\boldsymbol{\Sigma} \cdot \mathbf{p})\}\right) \phi_s \\ &= \phi_s^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2} \{[\boldsymbol{\Sigma} \cdot \mathbf{p}, \boldsymbol{\Sigma} \cdot \mathbf{q}] + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2\}\right) \phi_s \\ &= \phi_s^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2} \{[4i\epsilon_{ijk}p_iq_jS_k + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2]\}\right) \phi_s\end{aligned}$$

#### 3.3.2 Tensor $ij$ component

In calculating the nonrelativistic limit of the antisymmetric tensor bilinear, we will treat the  $0i$  and the  $ij$  components seperately. First let us consider  $\bar{\Psi}\Sigma_{ij}\Psi$ .

$$\begin{aligned}\bar{\Psi}\Sigma_{ij}\Psi &= \bar{\Psi}(-2\epsilon_{ijk}S_k)\Psi \\ &= -2i\epsilon_{ijk}(\phi^\dagger S_k\phi - \chi^\dagger S_k\chi) \\ &= -2i\epsilon_{ijk}\left(\phi_s^\dagger \left[1 + \frac{(\boldsymbol{\Sigma} \cdot \mathbf{p}')^2}{8m^2} - \frac{\mathbf{p}'^2}{4m^2}\right] S_k \left[1 + \frac{(\boldsymbol{\Sigma} \cdot \mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m^2}\right] \phi_s - \phi_s^\dagger \frac{(\boldsymbol{\Sigma} \cdot \mathbf{p}')S_k(\boldsymbol{\Sigma} \cdot \mathbf{p})}{4m^2} \phi_s\right) \\ &= -2i\epsilon_{ijk}\phi_s^\dagger \left\{S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{1}{8m^2} [(\boldsymbol{\Sigma} \cdot \mathbf{p}')^2 S_k + S_k(\boldsymbol{\Sigma} \cdot \mathbf{p})^2 - 2(\boldsymbol{\Sigma} \cdot \mathbf{p}')S_k(\boldsymbol{\Sigma} \cdot \mathbf{p})]\right\} \phi_s\end{aligned}$$

We want to write the terms in square brackets explicitly in terms of  $\mathbf{p}$  and  $\mathbf{q}$ .

$$\begin{aligned}(\boldsymbol{\Sigma} \cdot \mathbf{p}')^2 S_k + S_k(\boldsymbol{\Sigma} \cdot \mathbf{p})^2 - 2(\boldsymbol{\Sigma} \cdot \mathbf{p}')S_k(\boldsymbol{\Sigma} \cdot \mathbf{p}) &= (\boldsymbol{\Sigma} \cdot \mathbf{p})^2 S_k + S_k(\boldsymbol{\Sigma} \cdot \mathbf{p}) - 2(\boldsymbol{\Sigma} \cdot \mathbf{p})S_k(\boldsymbol{\Sigma} \cdot \mathbf{p}) \\ &\quad + \{(\boldsymbol{\Sigma} \cdot \mathbf{p})(\boldsymbol{\Sigma} \cdot \mathbf{q}) + (\boldsymbol{\Sigma} \cdot \mathbf{q})(\boldsymbol{\Sigma} \cdot \mathbf{p})\}S_k \\ &\quad - 2(\boldsymbol{\Sigma} \cdot \mathbf{q})S_k(\boldsymbol{\Sigma} \cdot \mathbf{p}) + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2 S_k\end{aligned}$$

We can express many of these terms as commutators

$$\begin{aligned}
&= \boldsymbol{\Sigma} \cdot \mathbf{p} [\boldsymbol{\Sigma} \cdot \mathbf{p}, S_k] + [S_k, \boldsymbol{\Sigma} \cdot \mathbf{p}] \boldsymbol{\Sigma} \cdot \mathbf{p} + 2\boldsymbol{\Sigma} \cdot \mathbf{q} [\boldsymbol{\Sigma} \cdot \mathbf{p}, S_k] - [\boldsymbol{\Sigma} \cdot \mathbf{q}, S_k] \boldsymbol{\Sigma} \cdot \mathbf{p} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2 S_k \\
&= i\epsilon_{ijk} p_j \{(\boldsymbol{\Sigma} \cdot \mathbf{p}) \Sigma_i - \Sigma_i (\boldsymbol{\Sigma} \cdot \mathbf{p})\} + 2i\epsilon_{ijk} \{(\boldsymbol{\Sigma} \cdot \mathbf{q}) \Sigma_i p_j - \Sigma_i (\boldsymbol{\Sigma} \cdot \mathbf{p}) q_j\} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2 S_k \\
&= i\epsilon_{ijk} p_j [(\boldsymbol{\Sigma} \cdot \mathbf{p}), \Sigma_i] + 2i\epsilon_{ijk} \{(\boldsymbol{\Sigma} \cdot \mathbf{q}) \Sigma_i p_j - \Sigma_i (\boldsymbol{\Sigma} \cdot \mathbf{p}) q_j\} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2 S_k \\
&= 4(\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p}) p_k) + 2i\epsilon_{ijk} \{(\boldsymbol{\Sigma} \cdot \mathbf{q}) \Sigma_i p_j - \Sigma_i (\boldsymbol{\Sigma} \cdot \mathbf{p}) q_j\} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2 S_k
\end{aligned}$$

Thus the whole bilinear is

$$-2i\epsilon_{ijk} \phi_S^\dagger \left\{ S_k \left( 1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{1}{8m^2} \left[ 4(\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p}) p_k) + 2i\epsilon_{lmk} \{(\boldsymbol{\Sigma} \cdot \mathbf{q}) \Sigma_\ell p_m - \Sigma_\ell (\boldsymbol{\Sigma} \cdot \mathbf{p}) q_m\} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2 S_k \right] \right\} \phi_S \quad (3.15)$$

### 3.3.3 Tensor $\Sigma_{0i}$ component

We calculate  $\bar{\Psi} \Sigma_{0i} \Psi$ .

$$\begin{aligned}
\bar{\Psi} \Sigma_{0i} \Psi &= \bar{\Psi} \begin{pmatrix} 0 & \Sigma_i \\ \Sigma_i & 0 \end{pmatrix} \Psi \\
&= \phi_S^\dagger \Sigma_i \chi - \chi^\dagger \Sigma_i \phi
\end{aligned}$$

We'll only need  $\phi$  and  $\chi$  to first order here.

$$= \phi_S^\dagger \left( \frac{\Sigma_i \Sigma_j p_j - \Sigma_j \Sigma_i p'_j}{2m} \right) \phi_S$$

Using  $p' = p + q$  the terms involving only  $p$  can be simplified using the commutator of  $\Sigma$  matrices.

$$= \phi_S^\dagger \left( \frac{4i\epsilon_{ijk} p_j S_k - \Sigma_j \Sigma_i q_j}{2m} \right) \phi$$

## 3.4 Current in terms of nonrelativistic wave functions

We derived the four-current (3.11) above; in nonrelativistic notation it is:

$$j_0 = F_e \frac{p_0 + p'_0}{2m} \bar{\Psi}^\dagger \Psi - F_m \frac{q_j}{2m} \bar{\Psi}^\dagger \Sigma^{0j} \Psi \quad (3.16)$$

$$j_i = F_e \frac{p_i + p'_i}{2m} \bar{\Psi}^\dagger \Psi - F_m \frac{q_j}{2m} \bar{\Psi}^\dagger \Sigma^{ij} \Psi + F_m \frac{q_0}{2m} \bar{\Psi}^\dagger \Sigma^{i0} \Psi \quad (3.17)$$

We have expressions for the bilinears in terms of the nonrelativistic wave functions  $\phi_S$ , so it is fairly straight forward to apply them here. The calculation of  $j_0$  is straightforward:

$$\begin{aligned}
F_e \frac{p_0 + p'_0}{2m} \bar{\Psi}^\dagger \Psi &= F_e \left( 1 + \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) \phi_S^\dagger \left( 1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2} \{4i\epsilon_{ijk} p_i q_j S_k + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2\} \right) \phi_S \\
&\approx F_e \phi_S^\dagger \left( 1 + \frac{1}{8m^2} \{4i\mathbf{S} \cdot \mathbf{p} \times \mathbf{q} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2\} \right) \phi_S \\
F_m \frac{q_j}{2m} \bar{\Psi}^\dagger \Sigma^{0j} \Psi &= F_m \frac{q_j}{2m} \phi_S^\dagger \left( \frac{4i\epsilon_{ijk} p_j S_k - \Sigma_j \Sigma_i q_j}{2m} \right) \phi_S \\
&= F_m \phi_S^\dagger \left( \frac{4i\mathbf{S} \cdot \mathbf{q} \times \mathbf{p} - (\boldsymbol{\Sigma} \cdot \mathbf{q})^2}{4m^2} \right) \phi_S
\end{aligned}$$

It turns out that both terms here have the same form, so combining them we get

$$j_0 = \phi_S^\dagger \left( F_e + \frac{F_e + 2F_m}{8m^2} \{4i\mathbf{S} \cdot \mathbf{p} \times \mathbf{q} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2\} \right) \phi_S \quad (3.18)$$

To calculate  $j_i$  we want to first simplify things by considering the constraints of our particular problem. The term with  $\Sigma_i j$  can be simplified by dropping terms with more than one power of  $q$ ; these will turn into derivatives of the magnetic field, and our problem concerns only a constant field. Further, we need only calculate elastic scattering, and so  $q_0 = 0$ . With those simplifications

$$\begin{aligned} \bar{\Psi} \Sigma_{ij} \Psi &\approx -2i\epsilon_{ijk} \phi_S^\dagger \left\{ S_k \left( 1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p}) p_k}{2m^2} \right\} \phi_S \\ F_e \frac{p_i + p'_i}{2m} \bar{\Psi}^\dagger \Psi &= F_e \frac{p_i + p'_i}{2m} \phi_S^\dagger \left( 1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2} \{4i\epsilon_{\ell j k} p_\ell q_j S_k + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2\} \right) \phi_S \\ &\approx F_e \frac{p_i + p'_i}{2m} \phi_S^\dagger \left( 1 + \frac{1}{8m^2} \{4i\epsilon_{\ell j k} p_\ell q_j S_k\} \right) \phi_S \\ F_m \frac{q_j}{2m} \bar{\Psi}^\dagger \Sigma^{ij} \Psi &= -F_m \frac{i\epsilon_{ijk} q_j}{m} \phi_S^\dagger \left\{ S_k \left( 1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p}) p_k}{2m^2} \right\} \phi_S \end{aligned}$$

So the full spatial part of the current is

$$j_i = \phi_S^\dagger \left\{ F_e \frac{p_i + p'_i}{2m} \left( 1 + \frac{i\epsilon_{\ell j k} p_\ell q_j S_k}{2m^2} \right) + F_m \frac{i\epsilon_{ijk} q_j}{m} \left( S_k \left( 1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p}) p_k}{2m^2} \right) \right\} \phi_S \quad (3.19)$$

### 3.5 Scattering off external field

To compare to the NRQED Lagrangian, we want to calculate scattering off an external field for an arbitrary spin particle. We already have the current, so the scattering is just

$$M = e j_\mu A^\mu = e j_0 A_0 - \mathbf{e} \mathbf{j} \cdot \mathbf{A}$$

Above we have expressions for both  $j_0$  (3.18) and  $\mathbf{j}$  (3.19). So we can write down the parts of the amplitude directly:

$$\begin{aligned} e j_0 A_0 &= e A_0 \phi_S^\dagger \left( F_e + \frac{F_e + 2F_m}{8m^2} \{4i\mathbf{S} \cdot \mathbf{p} \times \mathbf{q} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2\} \right) \phi_S \\ \mathbf{e} \mathbf{j} \cdot \mathbf{A} &= A_i \phi_S^\dagger \left\{ F_e \frac{p_i + p'_i}{2m} \left( 1 + \frac{i\epsilon_{\ell j k} p_\ell q_j S_k}{2m^2} \right) + F_m \frac{i\epsilon_{ijk} q_j}{m} \left( S_k \left( 1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p}) p_k}{2m^2} \right) \right\} \phi_S \end{aligned} \quad (3.20)$$

As much as possible we want to express the result in terms of gauge invariant quantities  $\mathbf{B}$  and  $\mathbf{E}$ . We write the relations between these fields and  $A_\mu$  in position space and the equivalent equation in momentum space.

$$\begin{aligned} \mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \rightarrow i\mathbf{q} \times \mathbf{A} \\ \mathbf{E} &= -\boldsymbol{\nabla} A_0 \rightarrow -i\mathbf{q} A_0 \end{aligned}$$

There is one term above that can only be put into gauge-invariant form by considering the kinematic constraints of elastic scattering. If the scattering is elastic, we have  $\mathbf{q} \cdot (\mathbf{p}' + \mathbf{p}) = \mathbf{p}'^2 - \mathbf{p}^2 = 0$ . We can use this identity on the term  $q_j(p'_i + p_i)A_i$  as follows:

$$\begin{aligned} \epsilon_{ijk} B_k &= \partial_i A_j - \partial_j A_i = i(q_i A_j - q_j A_i) \\ (p_i + p'_i) \epsilon B_k &= i(p_i + p'_i)(q_i A_j - q_j A_i) \\ &= -i(p_i + p'_i) q_j A_i \end{aligned}$$

So we have the identity

$$i(p_i + p'_i)q_j A_i = -\epsilon_{ijk} B_k (p_i + p'_i) \quad (3.21)$$

Now we can write each term involving  $q$  in terms of position space quantities.

$$\begin{aligned} i\mathbf{S} \cdot \mathbf{p} \times \mathbf{q} A_0 &= -\mathbf{S} \cdot \mathbf{p} \times \mathbf{E} \\ (\boldsymbol{\Sigma} \cdot \mathbf{q})^2 A_0 &= \Sigma_i \Sigma_j q_i q_j A_0 \\ &= \Sigma_i \Sigma_j \partial_i E_j \\ i\epsilon_{ijk} A_i q_j &= -i(\mathbf{q} \times \mathbf{A})_k \\ &= -B_k \\ A_i(p_i + p'_i) i\epsilon_{\ell j k} p_\ell q_j S_k &= \epsilon_{\ell j k} p_\ell S_k i(p_i + p'_i) q_j A_i \\ &= -\epsilon_{\ell j k} p_\ell S_k \{\epsilon_{ijm} B_m (p_i + p'_i)\} \\ &= -(\delta_{\ell i} \delta_{km} - \delta_{\ell m} \delta_{ik}) p_\ell S_k \{\epsilon_{ijm} B_m (p_i + p'_i)\} \\ &= 2\{(\mathbf{B} \cdot \mathbf{p})(\mathbf{S} \cdot \mathbf{p}) - (\mathbf{B} \cdot \mathbf{S})\mathbf{p}^2\} \end{aligned}$$

Using these

$$\begin{aligned} e j_0 A_0 &= e \phi_S^\dagger \left\{ A_0 + \frac{1 - 2F_2}{8m^2} (4\mathbf{S} \cdot \mathbf{E} \times \mathbf{p} + \Sigma_i \Sigma_j \partial_i E_j) \right\} \\ e \mathbf{j} \cdot \mathbf{A} &= e \phi_S^\dagger \left\{ \frac{\mathbf{p} \cdot \mathbf{A}}{m} + \frac{(\mathbf{B} \cdot \mathbf{p})(\mathbf{S} \cdot \mathbf{p}) - (\mathbf{B} \cdot \mathbf{S})\mathbf{p}^2}{m^2} - F_m \left( \frac{\mathbf{S} \cdot \mathbf{B}}{m} \left\{ 1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right\} + \frac{(\mathbf{B} \cdot \mathbf{p})(\mathbf{S} \cdot \mathbf{p}) - (\mathbf{B} \cdot \mathbf{S})\mathbf{p}^2}{2m^2} \right) \right\} \phi \\ &= e \phi_S^\dagger \left\{ \frac{\mathbf{p} \cdot \mathbf{A}}{m} + [1 - 2F_m] \frac{(\mathbf{B} \cdot \mathbf{p})(\mathbf{S} \cdot \mathbf{p})}{m^2} - \mathbf{S} \cdot \mathbf{B} \frac{\mathbf{p}^2}{m^2} - \frac{F_m}{m} \mathbf{S} \cdot \mathbf{B} \right\} \end{aligned}$$

From this we can see that our  $F_m$  is actually  $g/2$ , so in such terms

$$\begin{aligned} e j_0 A_0 &= e \phi_S^\dagger \left\{ A_0 - \frac{g-1}{2m^2} \left( \mathbf{S} \cdot \mathbf{E} \times \mathbf{p} + \frac{1}{4} \Sigma_i \Sigma_j \partial_i E_j \right) \right\} \\ e \mathbf{j} \cdot \mathbf{A} &= e \phi_S^\dagger \left\{ \frac{\mathbf{p} \cdot \mathbf{A}}{m} - [g-1] \frac{(\mathbf{B} \cdot \mathbf{p})(\mathbf{S} \cdot \mathbf{p})}{m^2} - \mathbf{S} \cdot \mathbf{B} \frac{\mathbf{p}^2}{m^2} - \frac{g}{2m} \mathbf{S} \cdot \mathbf{B} \right\} \end{aligned}$$

So the entire scattering process is

$$e \phi_S^\dagger \left\{ A_0 - \frac{g-1}{2m^2} \left( \mathbf{S} \cdot \mathbf{E} \times \mathbf{p} + \frac{1}{4} \Sigma_i \Sigma_j \partial_i E_j \right) \frac{\mathbf{p} \cdot \mathbf{A}}{m} - [g-1] \frac{(\mathbf{B} \cdot \mathbf{p})(\mathbf{S} \cdot \mathbf{p})}{m^2} - \mathbf{S} \cdot \mathbf{B} \frac{\mathbf{p}^2}{m^2} - \frac{g}{2m} \mathbf{S} \cdot \mathbf{B} \right\} \phi_S \quad (3.22)$$

### 3.6 Comparison with relativistic result

Having calculated the same process in both the relativistic theory and in our NRQED effective theory, we can compare the two amplitudes and fix the coefficients.

The NRQED amplitude (2.8) is

$$\begin{aligned} iM &= ie \phi^\dagger \left( -A_0 + \frac{\mathbf{A} \cdot \mathbf{p}}{m} - \frac{(\mathbf{A} \cdot \mathbf{p})\mathbf{p}^2}{2m^3} + c_F \frac{\mathbf{S} \cdot \mathbf{B}}{2m} + c_D \frac{(\partial_i E_i)}{8m^2} + c_Q \frac{Q_{ij}(\partial_i E_j)}{8m^2} \right. \\ &\quad \left. + c_S^1 \frac{\mathbf{E} \times \mathbf{p}}{4m^2} - (c_{W1} - c_{W2}) \frac{(\mathbf{S} \cdot \mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p} \frac{(\mathbf{S} \cdot \mathbf{p})(\mathbf{B} \cdot \mathbf{p})}{4m^3} \right) \phi \end{aligned}$$

While the relativistic amplitude was

$$iM_{REL} = -ie\phi^\dagger \left( A_0 - \frac{\mathbf{p} \cdot \mathbf{A}}{m} + \frac{\mathbf{p} \cdot \mathbf{A} \mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3} \{ \nabla \cdot \mathbf{E} - \mathbf{S} \cdot \mathbf{p} \times \mathbf{E} - S_i S_j \nabla_i E_j \} - g \frac{1}{2m} \mathbf{S} \cdot \mathbf{B} + \mathbf{S} \cdot \mathbf{B} \frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3} (\mathbf{S} \cdot \mathbf{p})(\mathbf{B} \cdot \mathbf{p}) \right) \phi$$

We should rewrite term  $\nabla \cdot \mathbf{E} - S_i S_j \nabla_i E_j$  using the quadropole moment tensor  $Q_{ij} = \frac{1}{2}(S_i S_j + S_j S_i - \frac{2}{3} \mathbf{S}^2)$ .

Remember that  $\nabla_i E_j$  is actually symmetric under exchange of  $i$  and  $j$ . Then we can write

$$\begin{aligned} S_i S_j \nabla_i E_j &= \frac{1}{2}(S_i S_j + S_j S_i) = (Q_{ij} + \frac{1}{3} \mathbf{S}^2 \delta_{ij}) \nabla_i E_j \\ &= Q_{ij} \nabla_i E_j + \frac{2}{3} \nabla \cdot \mathbf{E} \end{aligned}$$

Written using this identity, the relativistic amplitude is

$$iM_{REL} = -ie\phi^\dagger \left( A_0 - \frac{\mathbf{p} \cdot \mathbf{A}}{m} + \frac{\mathbf{p} \cdot \mathbf{A} \mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3} \left\{ \frac{1}{3} \nabla \cdot \mathbf{E} - \mathbf{S} \cdot \mathbf{p} \times \mathbf{E} - Q_{ij} \nabla_i E_j \right\} - g \frac{1}{2m} \mathbf{S} \cdot \mathbf{B} + \mathbf{S} \cdot \mathbf{B} \frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3} (\mathbf{S} \cdot \mathbf{p})(\mathbf{B} \cdot \mathbf{p}) \right) \phi$$

Comparing the two, the coefficients are:

$$\begin{aligned} c_F &= g \\ c_D &= \frac{4(g-1)}{3} \\ c_Q &= -4(g-1) \\ c_S^1 &= 2(g-1) \\ (c_{W_1} - c_{W_2}) &= 2 \\ c_{p'p} &= (g-2) \end{aligned}$$



### 3.7 Exact equations of motion for spin-1

$$\mathcal{L} = -\frac{1}{2}(D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) + m^2 W^{\mu\dagger} W_\mu - i\lambda e W^{\mu\dagger} W^\nu F_{\mu\nu} \quad (3.23)$$

where as usual  $D$  is the long derivative  $D^\mu = \partial^\mu + ieA^\mu$ .

Obtain the equations of motion from the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial [\partial_\mu W^{\dagger\alpha}]} = 0$$

Or equivalently,

$$\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} - D_\mu \frac{\partial \mathcal{L}}{\partial [D_\mu W^{\dagger\alpha}]} = 0$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} &= \frac{\partial}{\partial W^{\dagger\alpha}} \left( m^2 W^{\mu\dagger} W_\mu - i\lambda e W^{\mu\dagger} W^\nu F_{\mu\nu} \right) \\ &= m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial [D_\gamma W^{\dagger\alpha}]} &= -\frac{1}{2} \frac{\partial}{\partial [D_\gamma W^{\dagger\alpha}]} (D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) \\ &= -\frac{1}{2} (g^{\mu\gamma} g_\alpha^\nu - g^{\nu\gamma} g_\alpha^\mu) (D_\mu W_\nu - D_\nu W_\mu) \\ &= D_\alpha W^\gamma - D^\gamma W_\alpha \end{aligned}$$

So the complete equation from Euler-Lagrange is

$$m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} + D_\mu (D^\mu W_\alpha - D_\alpha W^\mu) = 0 \quad (3.24)$$

This is a set of four coupled second order equations for the field  $W$ . We rewrite as a set of first order equations by introducing a field  $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$ . So (3.24) becomes

$$m^2 W_\alpha - ie\lambda W^\mu F_{\alpha\mu} + D^\mu W_{\mu\alpha} = 0 \quad (3.25)$$

$W^{\mu\nu}$  is antisymmetric and so has six degrees of freedom, corresponding to six independent fields. Together with  $W^\mu$  this represents a total of ten fields. However, upon examination only some of these fields are dynamic. The fields  $W^{0i}$  and  $W^i$  appear in the equations with time derivatives, while the fields  $W^{ij}$  and  $W^0$  never do. So it is only necessary to consider the former six fields. So that these six fields all have the same dimension, we will define  $\frac{W^{i0}}{m} = i\eta^i$ .

We will now eliminate the extraneous fields and solve for  $iD_0 W^i$ ,  $iD_0 \eta^i$ .

$$m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} + D_\mu (D^\mu W_\alpha - D_\alpha W^\mu) = 0 \quad (3.26)$$

Define  $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$ . Then

$$m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} + D^\mu W_{\mu\alpha} = 0 \quad (3.27)$$

To get the exact Hamiltonian of a bispinor, we can eliminate nondynamic fields.

First consider (3.27) with  $\alpha = 0$ .

$$m^2 W_0 - ie\lambda W^\nu F_{0\nu} + D^\mu W_{\mu 0} \quad (3.28)$$

$$m^2 W_0 - ie\lambda W^j F_{0j} + D^j W_{j0} \quad (3.29)$$

Solve this for  $W_0$

$$W_0 = \frac{1}{m^2} (ie\lambda W^j F_{0j} - D^j W_{j0}) \quad (3.30)$$

Now, consider (3.27) with  $\alpha = i$

$$m^2 W_i - ie\lambda W^\mu F_{i\mu} + D^\mu W_{\mu i} = 0 \quad (3.31)$$

$$m^2 W_i - ie\lambda W^0 F_{i0} + D^0 W_{0i} - ie\lambda W^j F_{ij} + D^j W_{ji} = 0 \quad (3.32)$$

Using (3.30) we can replace  $W_0$

$$m^2 W_i - \frac{ie\lambda}{m^2} (ie\lambda W^\nu F_{0\nu} - D^j W_{j0}) F_{i0} + D^0 W_{0i} - ie\lambda W^j F_{ij} + D^j W_{ji} = 0 \quad (3.33)$$

Using  $W_{ji} = D_j W_i - D_i W_j$

$$m^2 W_i - \frac{ie\lambda}{m^2} (ie\lambda W^\nu F_{0\nu} - D^j W_{j0}) F_{i0} + D^0 W_{0i} - ie\lambda W^j F_{ij} + D^j (D_j W_i - D_i W_j) = 0 \quad (3.34)$$

Solve this for  $D^0 W_{0i}$ :

$$D^0 W_{0i} = -m^2 W_i + \frac{ie\lambda}{m^2} (ie\lambda W^\nu F_{0\nu} - D^j W_{j0}) F_{i0} + ie\lambda W^j F_{ij} - D^j (D_j W_i - D_i W_j) \quad (3.35)$$

To get a similar equation for  $W_i$ , consider

$$W_{i0} = D_i W_0 - D_0 W_i \quad (3.36)$$

Then

$$D^0 W_i = D_i W_0 - W_{i0} \quad (3.37)$$

$$D^0 W_i = D_i \frac{1}{m^2} (ie\lambda W^j F_{0j} - D^j W_{j0}) - W_{i0} \quad (3.38)$$

We now have equations that tell us the time evolution of a total of six fields:  $W_i$  and  $W_{i0}$ . We want to treat these as the components of some sort of bispinor. To that end, first define  $\eta_i = -i/m W_{i0}$  so that we have a pair of fields with the same mass dimension and hermiticity. (Since  $W_{i0} = D_i W_0 - D_0 W_i$  would pick up another minus sign under complex conjugation compared to  $W_i$ .) Going the other way  $W_{i0} = im\eta_i$ .

Since eventually we want a nonrelativistic expression, we should write the spatial components of four vectors as three vectors. To that end also write the components of the tensor  $F_{\mu\nu}$  in terms of three vectors.

$$F_{0i} = E^i, \quad F_{ij} = -\epsilon_{ijk} B^k$$

Regular spatial vectors are “naturally raised” while  $D_i$  is “naturally lowered”. Then we can rewrite (3.35).

$$-imD_0\eta_i = -m^2 W_i - \frac{ie\lambda}{m^2} (ie\lambda W^j E^j - D^j W_{j0}) E^i + ie\lambda W^j \epsilon_{ijk} B^k - D^j (D_j W_i - D_i W_j) \quad (3.39)$$

$$iD_0\eta^i = mW^i + \frac{ie\lambda}{m^3} (ie\lambda W^j E^j - D^j W_{j0}) E^i - \frac{ie\lambda}{m} W^j \epsilon_{ijk} B^k + \frac{1}{m} D_j (D_j W^i - D_i W^j) \quad (3.40)$$

And likewise (3.38) becomes

$$D^0 W_i = D_i \frac{1}{m^2} (-ie\lambda W^j E^j - imD^j \eta_j) - im\eta_i \quad (3.41)$$

$$iD^0 W^i = -iD_i \frac{1}{m^2} (-ie\lambda W^j E^j - imD^j \eta_j) + m\eta^i \quad (3.42)$$

$$iD^0 W^i = -\frac{1}{m^2} D_i e\lambda \mathbf{W} \cdot \mathbf{E} - \frac{1}{m} D_i \mathbf{D} \cdot \boldsymbol{\eta} + m\eta^i \quad (3.43)$$

### 3.7.1 Spin identities

To obtain some Schrodinger like equation for a bispinor, we need to express the time derivative of the bispinor in terms of other operators which can be interpreted as the Hamiltonian:  $i\partial_0\Psi = \hat{H}\Psi$ . We have expressions for the time derivatives of  $W_i$  and  $\eta_i$ , but some mixing of the indices is involved. To write the Hamiltonian in a block form we can introduce spin matrices whose action will mix the components of the fields.

The spin matrix for a spin one particle can be represented as:

$$(S^k)_{ij} = -i\epsilon_{ijk} \quad (3.44)$$

which leads to the following identities:

$$(\mathbf{a} \times \mathbf{v})_i = -i(\mathbf{S} \cdot \mathbf{a})_{ij}v_j \quad (3.45)$$

$$\{\mathbf{a} \times (\mathbf{a} \times \mathbf{v})\}_i = -(\{\mathbf{S} \cdot \mathbf{a}\}^2)_{ij}v_j \quad (3.46)$$

$$a_i(\mathbf{b} \cdot \mathbf{v}) = \{\mathbf{a} \cdot \mathbf{b} \delta_{ij} - (S^k S^\ell)_{ij} a^\ell b^k\}v_j \quad (3.47)$$

Using these identities

$$iD^0\mathbf{W} = -\frac{e\lambda}{m^2}\mathbf{D}(\mathbf{W} \cdot \mathbf{E}) + \frac{1}{m}\mathbf{D}(\mathbf{D} \cdot \boldsymbol{\eta}) + m\boldsymbol{\eta} \quad (3.48)$$

$$iD^0\mathbf{W} = -\frac{1}{m^2}\{\mathbf{D} \cdot \mathbf{E} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D})\}\mathbf{W} + \frac{1}{m}\{\mathbf{D}^2 - (\mathbf{S} \cdot \mathbf{D})^2\}\boldsymbol{\eta} + m\boldsymbol{\eta} \quad (3.49)$$

and for the other equation

$$iD_0\eta^i = mW^i - \frac{ie\lambda}{m^3}(ie\lambda W^j E^j - D^j W_{j0})E^i + \frac{ie\lambda}{m}W^j \epsilon_{ijk}B^k - \frac{1}{m}D_j(D_j W^i - D_i W^j) \quad (3.50)$$

$$iD_0\eta^i = mW^i - \frac{ie\lambda}{m^3}(ie\lambda W^j E^j - D^j W_{j0})E^i + \frac{ie\lambda}{m}W^j \epsilon_{ijk}B^k - \frac{1}{m}D_j(D_j W^i - D_i W^j) \quad (3.51)$$

$$iD_0\boldsymbol{\eta} = m\mathbf{W} + \frac{e^2\lambda^2}{m^3}\mathbf{E}(\mathbf{W} \cdot \mathbf{E}) - \frac{1}{m^2}\mathbf{E}(\mathbf{D} \cdot \boldsymbol{\eta}) + \frac{ie\lambda}{m}(\mathbf{W} \times \mathbf{B}) + \frac{1}{m}\mathbf{D} \times (\mathbf{D} \times \mathbf{W}) \quad (3.52)$$

$$iD_0\boldsymbol{\eta} = m\mathbf{W} + \frac{e^2\lambda^2}{m^3}\{\mathbf{E}^2 - (\mathbf{S} \cdot \mathbf{E})^2\}\mathbf{W} - \frac{1}{m^2}\{\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D})\}\boldsymbol{\eta} + \frac{e\lambda}{m}(\mathbf{S} \cdot \mathbf{B})\mathbf{W} - \frac{1}{m}(\mathbf{S} \cdot \mathbf{D})^2\mathbf{W} \quad (3.53)$$

$$iD_0\boldsymbol{\eta} = \left(m + \frac{e^2\lambda^2}{m^3}\{\mathbf{E}^2 - (\mathbf{S} \cdot \mathbf{E})^2\} - \frac{e\lambda}{m}(\mathbf{S} \cdot \mathbf{B}) - \frac{1}{m}(\mathbf{S} \cdot \mathbf{D})^2\right)\mathbf{W} + \frac{1}{m^2}\{\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D})\}\boldsymbol{\eta} \quad (3.54)$$

Now that the equations are in the correct form, we can write them as follows:

$$iD_0 \begin{pmatrix} W \\ \eta \end{pmatrix} = \begin{pmatrix} \lambda \frac{e}{m^2} [\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D}) + i\mathbf{S} \cdot \mathbf{E} \times \mathbf{D}] & m - \frac{1}{m}(\mathbf{S} \cdot \mathbf{D})^2 - \lambda \frac{e}{m}\mathbf{S} \cdot \mathbf{B} + \lambda^2 \frac{e^2}{m^3} [\mathbf{E}^2 - (\mathbf{S} \cdot \mathbf{E})^2] \\ m - \frac{1}{m}[\mathbf{D}^2 - (\mathbf{S} \cdot \mathbf{D})^2 + e\mathbf{S} \cdot \mathbf{B}] & -\lambda \frac{e}{m^2} [\mathbf{D} \cdot \mathbf{E} - (\mathbf{S} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{E}) + i\mathbf{S} \cdot \mathbf{D} \times \mathbf{E}] \end{pmatrix} \begin{pmatrix} W \\ \eta \end{pmatrix} \quad (3.55)$$

### 3.7.2 Current

We can also derive the conserved current from the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}(D \times W)^\dagger \cdot (D \times W) + m_w^2 W^\dagger W - \lambda i e W^{\dagger\mu} W^\nu F_{\mu\nu}$$

Where

$$D^\mu = \partial^\mu - i e A^\mu, \quad D \times W = D^\mu W^\nu - D^\nu W^\mu$$

We want the conserved current corresponding to the transformation  $W_i \rightarrow e^{i\alpha} W_i$ , which in infinitesimal form is:

$$W_\mu \rightarrow W_\mu + i\alpha W_\mu, \quad W_\mu^\dagger \rightarrow W_\mu^\dagger - i\alpha W_\mu^\dagger$$

The 4-current density will be:

$$j^\sigma = -i \frac{\partial \mathcal{L}}{\partial W_{\mu,\sigma}} W_\mu + i \frac{\partial \mathcal{L}}{\partial W_{\mu,\sigma}^\dagger} W_\mu^\dagger$$

Only one term contains derivatives of the field:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial W_{\alpha,\sigma}} &= \frac{\partial}{\partial W_{\alpha,\sigma}} \left\{ -\frac{1}{2} (D_\mu W_\nu - D_\nu W_\mu)^\dagger (D^\mu W^\nu - D^\nu W^\mu) \right\} \\ &= -\frac{1}{2} (D_\mu W_\nu - D_\nu W_\mu)^\dagger (g_{\sigma\mu} g_{\alpha\nu} - g_{\sigma\nu} g_{\alpha\mu}) \\ &= -(D_\alpha W_\sigma - D_\sigma W_\alpha)^\dagger \end{aligned}$$

Likewise:

$$\frac{\partial \mathcal{L}}{\partial W_{\alpha,\sigma}^\dagger} = -(D_\alpha W_\sigma - D_\sigma W_\alpha)$$

If we define  $W_{\mu\nu} = D^\mu W^\nu - D^\nu W^\mu$  then the 4-current and charge density are:

$$\begin{aligned} j_\sigma &= i W_{\sigma\mu}^\dagger W^\mu - i W_{\sigma\mu} W^{\dagger\mu} \\ j_0 &= i W_{0\mu}^\dagger W^\mu - i W_{0\mu} W^{\dagger\mu} \\ &= i W_{0i}^\dagger W^i - i W_{0i} W^{\dagger i} \end{aligned}$$

Where the last equality follows from the antisymmetry of  $W_{\mu\nu}$ .

Now, we defined the fields  $\eta_i = -i \frac{W_{i0}}{m}$ . In terms of these fields,  $j_0 = m(\eta_i^\dagger W^i + \eta_i W^{\dagger i})$ .

We can do the same to find the vector part of the current.

$$\begin{aligned} j_i &= i W_{i\mu}^\dagger W^\mu - i W_{i\mu} W^{\dagger\mu} \\ &= i W_j^\dagger W_{ij} + i W_{i0}^\dagger W_0 + c.c. \end{aligned}$$

We have  $W_{ij} = D_i W_j - D_j W_i$ . Using the identities developed in the appendix, we can obtain

$$D_j W_i = D_i W_j - D_k (S_i S_k)_{ja} W_a$$

Then

$$W_{ij} = D_k (S_i S_k)_{ja} W_a$$

In the absence of an electric field  $E$ ,  $W_0 = \frac{i}{m} D_j \eta_j$ , with  $W_{i0} = im\eta_i$ .

$$W_{i0}^\dagger W_0 = -\eta_i^\dagger D_j \eta_j$$

Again we introduce spin matrices to get the equation in the desired form, and obtain

$$W_{i0}^\dagger W_0 = -\eta_j^\dagger D_k (\delta_{ik} - S_k S_i) \eta_j$$

This leads to

$$j_i = iW_j^\dagger D_k (S_i S_k W)_j - i\eta_j^\dagger D_k ([\delta_{ik} - S_k S_i] \eta)_j + c.c.$$

Writing this in terms of the bispinor  $\begin{pmatrix} \eta \\ W \end{pmatrix}$ , the expression for the current is

$$j_i = \frac{i}{2} \begin{pmatrix} \eta^\dagger & W^\dagger \end{pmatrix} \left[ (\{S_i, S_j\} - \delta_{ij}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - ([S_i, S_j] + \delta_{ij}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] D_j \begin{pmatrix} \eta \\ W \end{pmatrix} + c.c. \quad (3.56)$$

### 3.7.3 Hermiticity of Hamiltonian

It's clear from inspection that the above Hamiltonian is not Hermitian in the standard sense. (Of the operators in use, the only one which is not self adjoint is  $D_i^\dagger = -D_i$ .) Noticing that

$$[\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D}) + i\mathbf{S} \cdot \mathbf{E} \times \mathbf{D}]^\dagger = -[\mathbf{D} \cdot \mathbf{E} - (\mathbf{S} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{E}) + i\mathbf{S} \cdot \mathbf{D} \times \mathbf{E}]$$

we can see it has the general form

$$H = \begin{pmatrix} A & B \\ C & A^\dagger \end{pmatrix}$$

where the off diagonal blocks are Hermitian in the normal sense:  $B^\dagger = B$  and  $C^\dagger = C$ .

At this point we should consider that an operator is defined as Hermitian with respect to a particular inner product. In quantum mechanics this inner product is normally defined as:

$$\langle \Psi, \phi \rangle = \int d^3x \Psi^\dagger \phi$$

This definition, however, does not produce sensible results for the states in question. We need the inner product of a state with itself to be conserved; in other words, it should play the role of a conserved charge. From the considerations above we already have one such quantity: the conserved charge  $\int d^3x j_0$ , where

$$j_0 = m[\eta^\dagger W + W^\dagger \eta] = m \begin{pmatrix} \eta^\dagger & W^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ W \end{pmatrix}$$

A more general definition of the inner product includes some weight  $M$ :

$$\langle \Psi, \phi \rangle = \int d^3x \Psi^\dagger M \phi$$

In normal quantum mechanics  $M$  would be the identity matrix, but here, as implied by the charge density, we want  $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Such a definition will lead to the inner product  $\langle \Psi, \Psi \rangle$  being conserved.

An operator  $H$  is hermitian with respect to this inner product if

$$\langle H\Psi, \phi \rangle = \langle \Psi, H\phi \rangle \rightarrow \int d^3x \Psi^\dagger H^\dagger M \phi = \int d^3x \Psi^\dagger M H \phi$$

For this equality to hold, it is sufficient for  $H^\dagger M = M H$ . With  $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  this condition reduces to

$$\begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$$

Our Hamiltonian fulfills exactly this requirement, and so is Hermitian with respect to this particular inner product.

### 3.8 Non-relativistic Hamiltonian

Now we will consider the non-relativistic limit of the above Hamiltonian. To work in this regime constrains the order of both the momentum and the electromagnetic field strength.

$$\begin{aligned} D &\sim mv \\ \Phi &\sim mv^2 \\ E &\sim m^2 v^3 \\ B &\sim m^2 v^2 \end{aligned}$$

We can write the Hamiltonian matrix in terms of the basis of 2x2 Hermitian matrices:  $(\mathbf{I}, \rho_i)$ :

$$\begin{aligned} H &= a_0 \mathbf{I} + a_i \rho_i \\ a_0 &= \frac{1}{2}(H_{11} + H_{22}) = e\Phi + \lambda \frac{e}{2m^2} [\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D}) + i\mathbf{S} \cdot \mathbf{E} \times \mathbf{D} - \mathbf{D} \cdot \mathbf{E} + (\mathbf{S} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{E}) - i\mathbf{S} \cdot \mathbf{D} \times \mathbf{E}] \\ a_3 &= \frac{1}{2}(H_{11} - H_{22}) = \lambda s \frac{e}{2m^2} [\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D}) + i\mathbf{S} \cdot \mathbf{E} \times \mathbf{D} + \mathbf{D} \cdot \mathbf{E} - (\mathbf{S} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{E}) + i\mathbf{S} \cdot \mathbf{D} \times \mathbf{E}] \\ ia_2 &= \frac{1}{2}(H_{21} - H_{12}) = - \left[ \frac{\mathbf{D}^2}{2m} - \frac{1}{m}(\mathbf{S} \cdot \mathbf{D})^2 - \frac{\lambda - 1}{2} \frac{e}{m} \mathbf{S} \cdot \mathbf{B} + \frac{e^2}{2m^3} (\mathbf{E}^2 - (\mathbf{S} \cdot \mathbf{E})^2) \right] \\ a_1 &= \frac{1}{2}(H_{12} - H_{21}) = \left[ m - \frac{\mathbf{D}^2}{2m} - \frac{1 + \lambda}{2} \frac{e}{m} \mathbf{S} \cdot \mathbf{B} + \frac{e^2}{2m^3} (\mathbf{E}^2 - (\mathbf{S} \cdot \mathbf{E})^2) \right] \end{aligned}$$

We can see that to leading order, the Hamiltonian is

$$H = m\rho_1 = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$$

Since we wish to separate positive and negative energy states, this poses a problem. So we first switch to a basis where at least the rest energies are separate. Then, remaining off-diagonal elements can be treated as perturbations.

An appropriate transformation which meets our requirements is a "rotation" in the space spanned by  $\rho_i$  matrices, about the  $\rho_1 + \rho_3$  axis. This has the explicit form:

$$U = \frac{1}{\sqrt{2}}(\rho_1 + \rho_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (3.57)$$

Any transformation U can be described by it's action on the basis of 4x4 matrices. This takes:

$$\begin{aligned} \mathbf{I} &\rightarrow \mathbf{I} \\ \rho_1 &\rightarrow \rho_3 \\ \rho_3 &\rightarrow \rho_1 \\ \rho_2 &\rightarrow -\rho_2 \\ \begin{pmatrix} \eta \\ W \end{pmatrix} &\rightarrow \begin{pmatrix} \Psi_u \\ \Psi_\ell \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta + W \\ \eta - W \end{pmatrix} \end{aligned}$$

(This transformation will transform the current to

$$j_0 = m(\Psi_u^\dagger \Psi_u - \Psi_\ell^\dagger \Psi_\ell)$$

and the weight M, used in the inner product, to

$$M \rightarrow M' = U^\dagger M U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The transformed Hamiltonian will of course be Hermitian with respect to the transformed inner product.)  
Our equation is now of the following form:

$$i\partial_0 \begin{pmatrix} \Psi_u \\ \Psi_\ell \end{pmatrix} = H' \begin{pmatrix} \Psi_u \\ \Psi_\ell \end{pmatrix}$$

We can see that the Hamiltonian still contains off-diagonal elements, so this represents a pair of coupled equations for the upper and lower components of  $\Psi$ . But the off-diagonal terms are small, so we can consider the case where  $\Psi_\ell$  is small compared to  $\Psi_u$ . Solving for  $\Psi_\ell$  in terms of  $\Psi_u$ :

$$\begin{aligned} E\Psi_\ell &= H'_{21}\Psi_u + H'_{22}\Psi_\ell \\ \Psi_\ell &= (E - H'_{22})^{-1}H'_{21}\Psi_u \end{aligned}$$

This gives the exact formula:

$$E\Psi_u = (H'_{11} + H'_{12}[E - H'_{22}]^{-1}H'_{21})\Psi_u$$

However, we only need corrections to the magnetic moment of order  $v^2$ . With  $\frac{e}{m}\mathbf{S} \cdot \mathbf{B} \sim mv^2$ , this means we only need the Hamiltonian to at most order  $mv^4$ . Examining the leading order terms of the matrix  $H'$ , the diagonal elements are order  $m$  while the off-diagonal elements are order  $mv^2$ . To leading order the term  $[E - H'_{22}]^{-1} = \frac{1}{2m}$ . So we'll need  $H'_{11}$  to  $\mathcal{O}(v^4)$ , and  $H'_{12}$ ,  $H'_{21}$ , and  $[E - H'_{22}]^{-1}$  each to only leading order.

$$E\Psi_u = \left( H'_{11} + \frac{1}{2m}H'_{12}H'_{21} + \mathcal{O}(mv^6) \right) \Psi_u$$

The needed terms of  $H$  are, using  $\lambda = g - 1$

$$\begin{aligned} H'_{11} = a_0 + a_3 &= m + e\Phi - \frac{D^2}{2m} - \frac{g}{2}\frac{e}{m}\mathbf{S} \cdot \mathbf{B} \\ &\quad + (g-1)\frac{e}{2m^2}[\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D}) + i\mathbf{S} \cdot \mathbf{E} \times \mathbf{D} - \mathbf{D} \cdot \mathbf{E} + (\mathbf{S} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{E}) - i\mathbf{S} \cdot \mathbf{D} \times \mathbf{E}] + \mathcal{O}(mv^4) \\ H'_{12} = a_1 - ia_2 &= \frac{D^2}{2m} - \frac{1}{m}(\mathbf{S} \cdot \mathbf{D})^2 - \frac{g-2}{2}\frac{e}{m}\mathbf{S} \cdot \mathbf{B} + \mathcal{O}(mv^4) \\ H'_{21} = a_1 + ia_2 &= -\frac{D^2}{2m} + \frac{1}{m}(\mathbf{S} \cdot \mathbf{D})^2 + \frac{g-2}{2}\frac{e}{m}\mathbf{S} \cdot \mathbf{B} + \mathcal{O}(mv^4) \end{aligned}$$

The product  $H'_{12}H'_{21}$  is calculated in the appendix. To first order in the magnetic field strength it is:

$$\begin{aligned} \frac{1}{2m}H'_{12}H'_{21} &= -\frac{1}{2m^3} \left( \frac{\mathbf{D}^2}{2} - (\mathbf{S} \cdot \mathbf{D})^2 - \frac{g-2}{2}e\mathbf{S} \cdot \mathbf{B} \right)^2 \\ &= -\frac{1}{2m^3} \left( \frac{\pi^4}{4} - e\mathbf{p}^2\mathbf{S} \cdot \mathbf{B} - \frac{g-2}{2}e(\mathbf{S} \cdot \mathbf{p})(\mathbf{B} \cdot \mathbf{p}) \right) \end{aligned}$$

So finally, replacing all  $\mathbf{D}$  with  $\boldsymbol{\pi} \equiv \mathbf{p} - e\mathbf{A}$ , we have a direct expression for  $\Psi_u$ :

$$\begin{aligned} E\Psi_u &= \left\{ m + e\Phi + \frac{\pi^2}{2m} - \frac{g}{2}\frac{e}{m}\mathbf{S} \cdot \mathbf{B} - \frac{\pi^4}{8m^3} + \frac{e\mathbf{p}^2(\mathbf{S} \cdot \mathbf{B})}{2m^3} + (g-2)\frac{e}{4m^3}(\mathbf{S} \cdot \mathbf{p})(\mathbf{B} \cdot \mathbf{p}) \right. \\ &\quad \left. + (g-1)\frac{ie}{2m^2}[\mathbf{E} \cdot \boldsymbol{\pi} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \boldsymbol{\pi}) + i\mathbf{S} \cdot \mathbf{E} \times \boldsymbol{\pi} - \boldsymbol{\pi} \cdot \mathbf{E} + (\mathbf{S} \cdot \boldsymbol{\pi})(\mathbf{S} \cdot \mathbf{E}) - i\mathbf{S} \cdot \boldsymbol{\pi} \times \mathbf{E}] \right\} \Psi_u \end{aligned}$$

The complicated expression in square brackets can be cleaned up a bit:

$$\begin{aligned}
\mathbf{E} \cdot \boldsymbol{\pi} - \boldsymbol{\pi} \cdot \mathbf{E} &= [E_i, \pi_i] \\
&= [E_i, -i\partial_i] \\
&= i(\partial_i E_i) \\
(\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \boldsymbol{\pi}) - (\mathbf{S} \cdot \boldsymbol{\pi})(\mathbf{S} \cdot \mathbf{E}) &= S_i S_j E_i \pi_j - S_i S_j \pi_i E_j \\
&= (S_i S_j)(E_i \pi_j - E_j \pi_i - [\pi_i, E_j]) \\
&= [S_i, S_j](E_i \pi_j) - (S_i S_j)(-i\nabla_i E_j) \\
&= (i\epsilon_{ijk} S_k) E_j \pi_i - (S_i S_j)(-i\nabla_i E_j) \\
&= i\mathbf{S} \cdot \mathbf{E} \times \boldsymbol{\pi} + iS_i S_j \nabla_i E_j \\
(\mathbf{E} \times \boldsymbol{\pi} - \boldsymbol{\pi} \times \mathbf{E})_k &= \epsilon_{ijk}(E_i \pi_j - \pi_i E_j) \\
&= \epsilon_{ijk}(E_i \pi_j + \pi_j E_i) \\
&= \epsilon_{ijk}(2E_i \pi_j + [\pi_i, E_j]) \\
&= 2\epsilon_{ijk} E_i \pi_j \\
&= 2(\mathbf{E} \times \boldsymbol{\pi})_k
\end{aligned}$$

Using these identities and collecting terms, and then writing everything in terms of  $g$ ,  $g - 2$

$$\begin{aligned}
E\Psi_u &= \left\{ m + e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} - \frac{e}{m} \mathbf{S} \cdot \mathbf{B} \left( \frac{g}{2} - \frac{p^2}{2m^2} \right) + (g - 2) \frac{e}{4m^3} (\mathbf{S} \cdot \mathbf{p})(\mathbf{B} \cdot \mathbf{p}) \right. \\
&\quad \left. - (g - 1) \frac{e}{2m^2} [\nabla \cdot \mathbf{E} - S_i S_j \nabla_i E_j + \mathbf{S} \cdot \mathbf{E} \times \boldsymbol{\pi}] \right\} \Psi_u \\
&= \left\{ m + e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} - \frac{g}{2} \frac{e}{m} \mathbf{S} \cdot \mathbf{B} \left( 1 - \frac{p^2}{2m^2} \right) - \frac{g - 2}{2} \frac{e}{m} \frac{p^2}{2m^2} \mathbf{S} \cdot \mathbf{B} + (g - 2) \frac{e}{4m^3} (\mathbf{S} \cdot \mathbf{p})(\mathbf{B} \cdot \mathbf{p}) \right. \\
&\quad \left. - \left( \frac{g}{2} + \frac{g - 2}{2} \right) \frac{e}{2m^2} [\nabla \cdot \mathbf{E} - S_i S_j \nabla_i E_j + \mathbf{S} \cdot \mathbf{E} \times \boldsymbol{\pi}] \right\} \Psi_u
\end{aligned}$$

We have a Hamiltonian for the upper component of the bispinor, as desired. But is it truly Schrodinger-like? In the general case we would need to perform the Fouldy-Wouthyusen transformation, to a representation where all the physics up to the desired order is contained in the single spinor equation. But we can show that, at the order we are working at, the Hamiltonian above is correct.



## 4 Comparison with Silenko

We can compare to a result in Silenko (arXiv:hep-th/0401183v1). Silenko determines a Hamiltonian which is equivalent to that used above. Dropping terms manifestly beyond  $\mathcal{O}(mv^4)$ :

$$H = \rho_3 \epsilon' + e\Phi + \frac{e}{4m} \left[ \left\{ \left( \frac{g-2}{2} + \frac{m}{\epsilon' + m} \right) \frac{1}{\epsilon'}, (\mathbf{S} \cdot \boldsymbol{\pi} \times \mathbf{E} - \mathbf{S} \cdot \mathbf{E} \times \boldsymbol{\pi}) \right\}_+ - \rho_3 \left\{ \left( g - 2 + \frac{2m}{\epsilon'} \right), \mathbf{S} \cdot \mathbf{B} \right\}_+ \right. \\ \left. + \rho_3 \left\{ \frac{g-2}{2\epsilon'(\epsilon' + m)}, \{\mathbf{S} \cdot \boldsymbol{\pi}, \boldsymbol{\pi} \cdot \mathbf{B}\}_+ \right\}_+ \right] + \frac{e(g-1)}{4m^2} \{\mathbf{S} \cdot \boldsymbol{\nabla}, \mathbf{S} \cdot \mathbf{E}\}_+ - \frac{e(g-1)}{2m^2} \boldsymbol{\nabla} \cdot \mathbf{E}$$

We'll take each term and expand to  $\mathcal{O}(mv^4)$ . First the operator  $\epsilon'$ :

$$\begin{aligned} \epsilon' &= \sqrt{m^2 + \pi^2} \\ &= m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^2} + \mathcal{O}(mv^6) \end{aligned}$$

Using this, we get

$$H = \rho_3 \epsilon' + e\Phi + \frac{e}{4m} \left[ \left\{ \frac{g-1}{2m}, (\mathbf{S} \cdot \boldsymbol{\pi} \times \mathbf{E} - \mathbf{S} \cdot \mathbf{E} \times \boldsymbol{\pi}) \right\}_+ - \rho_3 \left\{ g - \frac{\pi^2}{m^2}, \mathbf{S} \cdot \mathbf{B} \right\}_+ \right. \\ \left. + \rho_3 \left\{ \frac{g-2}{4m^2}, \{\mathbf{S} \cdot \boldsymbol{\pi}, \boldsymbol{\pi} \cdot \mathbf{B}\}_+ \right\}_+ \right] + \frac{e(g-1)}{4m^2} \{\mathbf{S} \cdot \boldsymbol{\nabla}, \mathbf{S} \cdot \mathbf{E}\}_+ - \frac{e(g-1)}{2m^2} \boldsymbol{\nabla} \cdot \mathbf{E}$$

Now we'll expand the anticommutators.

We have already shown that  $\boldsymbol{\pi} \times \mathbf{E} = -\mathbf{E} \times \boldsymbol{\pi}$ , and that such a term is  $\mathcal{O}(mv^4)$

$$\left\{ \frac{g-1}{2m}, (\mathbf{S} \cdot \boldsymbol{\pi} \times \mathbf{E} - \mathbf{S} \cdot \mathbf{E} \times \boldsymbol{\pi}) \right\}_+ = -\frac{2(g-1)}{m} \mathbf{S} \cdot \mathbf{E} \times \boldsymbol{\pi}$$

Because we specialize to constant magnetic fields,  $\boldsymbol{\pi}$  commutes with  $\mathbf{B}$ , so

$$\left\{ g - \frac{\pi^2}{m^2}, \mathbf{S} \cdot \mathbf{B} \right\}_+ = (2g - \frac{2\pi^2}{m^2}) \mathbf{S} \cdot \mathbf{B}$$

and

$$\left\{ \frac{g-2}{4m^2}, \{\mathbf{S} \cdot \boldsymbol{\pi}, \boldsymbol{\pi} \cdot \mathbf{B}\}_+ \right\}_+ = \frac{g-2}{m^2} (\mathbf{S} \cdot \boldsymbol{\pi})(\boldsymbol{\pi} \cdot \mathbf{B})$$

Using the identity that  $[\nabla_i, E_j] = 0$ ,  $\boldsymbol{\nabla} \times \mathbf{E} = 0$ :

$$\begin{aligned} \{\mathbf{S} \cdot \boldsymbol{\nabla}, \mathbf{S} \cdot \mathbf{E}\}_+ &= S_i S_j \nabla_i E_j + S_j S_i E_j \nabla_i \\ &= (S_i S_j + S_j S_i) \nabla_i E_j \\ &= (2S_i S_j + [S_j, S_i]) \nabla_i E_j \\ &= (2S_i S_j + iS_k \epsilon_{ijk}) \nabla_i E_j \\ &= 2S_i S_j \nabla_i E_j \end{aligned}$$

So

$$\begin{aligned} H &= \rho_3 \left( m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^2} \right) + e\Phi - (g-1) \frac{e}{2m^2} [\mathbf{S} \cdot \mathbf{E} \times \boldsymbol{\pi} - S_i S_j \nabla_i E_j + \boldsymbol{\nabla} \cdot \mathbf{E}] \\ &\quad + \rho_3 (g-2) \frac{2}{4m^2} (\mathbf{S} \cdot \boldsymbol{\pi})(\boldsymbol{\pi} \cdot \mathbf{B}) - \rho_3 \frac{e}{m} \left( \frac{g}{2} - \frac{\pi^2}{2m^2} \right) \mathbf{S} \cdot \mathbf{B} \\ &= \rho_3 \left( m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^2} \right) + e\Phi - \left( \frac{g-2}{2} + \frac{g}{2} \right) \frac{e}{2m^2} [\mathbf{S} \cdot \mathbf{E} \times \boldsymbol{\pi} - S_i S_j \nabla_i E_j + \boldsymbol{\nabla} \cdot \mathbf{E}] \\ &\quad + \rho_3 (g-2) \frac{2}{4m^2} (\mathbf{S} \cdot \boldsymbol{\pi})(\boldsymbol{\pi} \cdot \mathbf{B}) - \rho_3 \frac{e}{m} \left[ \frac{g}{2} \left( 1 - \frac{\pi^2}{2m^2} \right) + \frac{g-2}{2} \frac{\pi^2}{2m^2} \right] \mathbf{S} \cdot \mathbf{B} \end{aligned}$$

## Magnetic Moment

Now we'll keep only those terms which contribute to the magnetic moment, using  $\boldsymbol{\pi} = \mathbf{p} - e\mathbf{A}$

$$\begin{aligned} H_{S \cdot B} &= \left( \frac{g-2}{2} + \frac{g}{2} \right) \frac{e^2}{2m^2} \mathbf{S} \cdot \mathbf{E} \times \mathbf{A} + (g-2) \frac{e}{4m^2} (\mathbf{S} \cdot \mathbf{p})(\mathbf{p} \cdot \mathbf{B}) - \frac{e}{m} \left[ \frac{g}{2} \left( 1 - \frac{p^2}{2m^2} \right) + \frac{g-2}{2} \frac{p^2}{2m^2} \right] \mathbf{S} \cdot \mathbf{B} \\ &= -\frac{e}{2m} \left\{ g \left( 1 - \frac{p^2}{2m^2} \right) \mathbf{S} \cdot \mathbf{B} + (g-2) \frac{p^2}{2m^2} \mathbf{S} \cdot \mathbf{B} - (g-2) \frac{(\mathbf{S} \cdot \mathbf{p})(\mathbf{p} \cdot \mathbf{B})}{2m^2} - \frac{e}{m} \left( \frac{g-2}{2} + \frac{g}{2} \right) \mathbf{S} \cdot \mathbf{E} \times \mathbf{A} \right\} \end{aligned}$$

This exactly matches with what was found before.

## 5 Identities

Simplify  $\mathbf{W} \times \mathbf{B}$ :

$$\begin{aligned} (\mathbf{W} \times \mathbf{B})_i &= \epsilon_{ijk} W_j B_k \\ &= i(S_k)_{ij} W_j B_k \\ &= i(\mathbf{S} \cdot \mathbf{B})_{ij} W_j \\ &= i([\mathbf{S} \cdot \mathbf{B}]\mathbf{W})_i \end{aligned}$$

Simplify  $\mathbf{D} \times (\mathbf{D} \times \mathbf{W})$

$$\begin{aligned} (\mathbf{D} \times [\mathbf{D} \times \mathbf{W}])_i &= \epsilon_{ijk} D_j (\mathbf{D} \times \mathbf{W})_k \\ &= \epsilon_{ijk} \epsilon_{klm} D_j D_\ell W_m \\ &= -(S_j)_{ki} (S_\ell)_{mk} D_j D_\ell W_m \\ &= -(\mathbf{S} \cdot \mathbf{D})_{ki} (\mathbf{S} \cdot \mathbf{D})_{mk} W_m \\ &= -([\mathbf{S} \cdot \mathbf{D}]^2)_{im} W_m \\ &= -([\mathbf{S} \cdot \mathbf{D}]^2 \mathbf{W})_i \end{aligned}$$

Our representation defines the spin matrices as follows:

$$(S_k)_{ij} = -i\epsilon_{ijk}$$

They have the commutator

$$[S_i, S_j] = i\epsilon_{ijk} S_k$$

The product of two such spin matrices is given by:

$$\begin{aligned} (S_k S_\ell)_{ij} &= (S_k)_{ia} (S_\ell)_{aj} \\ &= -\epsilon_{iak} \epsilon_{ajl} \\ &= (\delta_{kl} \delta_{ij} - \delta_{kj} \delta_{li}) \end{aligned}$$

This implies that:

$$\begin{aligned} (S_i S_j A_j B_i \mathbf{v})_l &= (S_i)_{lm} (S_j)_{mn} A_j B_i v_n \\ &= (S_i S_j)_{ln} A_j B_i v_n \\ &= (\delta_{ij} \delta_{ln} - \delta_{in} \delta_{lj}) A_j B_i v_n \\ &= (\mathbf{A} \cdot \mathbf{B}) v_l - A_l (\mathbf{B} \cdot \mathbf{v}) \end{aligned}$$

Or

$$\mathbf{A}(\mathbf{B} \cdot \mathbf{v}) = (\mathbf{A} \cdot \mathbf{B} - S_i S_j A_j B_i) \mathbf{v}$$

Now we can use this to establish some identities:

$$\begin{aligned} E^i \mathbf{D} \cdot \boldsymbol{\eta} &= ([\mathbf{E} \cdot \mathbf{D} - S_j S_k E_k D_j] \boldsymbol{\eta})^i \\ &= ([\mathbf{E} \cdot \mathbf{D} - (S_k S_j - [S_k, S_j]) E_k D_j] \boldsymbol{\eta})^i \\ &= ([\mathbf{E} \cdot \mathbf{D} - (S_k S_j - i \epsilon_{kjl} S_l) E_k D_j] \boldsymbol{\eta})^i \\ &= ([\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D}) + i S_l (\mathbf{E} \times \mathbf{D})_l] \boldsymbol{\eta})^i \\ &= ([\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D}) + i \mathbf{S} \cdot (\mathbf{E} \times \mathbf{D})] \boldsymbol{\eta})^i \end{aligned}$$

Since  $\mathbf{E}$  and  $\mathbf{W}$  commute:

$$\begin{aligned} E^i \mathbf{W} \cdot \mathbf{E} &= ([\mathbf{E}^2 - S_j S_k E_k E_j] \mathbf{W})^i \\ &= ([\mathbf{E}^2 - (\mathbf{S} \cdot \mathbf{E})^2] \mathbf{W})^i \end{aligned}$$

And

$$\begin{aligned} D^i \mathbf{D} \cdot \boldsymbol{\eta} &= ([\mathbf{D}^2 - S_j S_k D_k D_j] \boldsymbol{\eta})^i \\ &= ([\mathbf{D}^2 - (S_k S_j + [S_j, S_k]) D_k D_j] \boldsymbol{\eta})^i \\ &= ([\mathbf{D}^2 - (\mathbf{S} \cdot \mathbf{D})^2 + i \epsilon_{jkl} S_l D_k D_j] \boldsymbol{\eta})^i \\ &= ([\mathbf{D}^2 - (\mathbf{S} \cdot \mathbf{D})^2 + i \mathbf{S} \cdot (\mathbf{D} \times \mathbf{D})] \boldsymbol{\eta})^i \\ &= ([\mathbf{D}^2 - (\mathbf{S} \cdot \mathbf{D})^2 + e \mathbf{S} \cdot \mathbf{B}] \boldsymbol{\eta})^i \end{aligned}$$

Similarly:

$$\begin{aligned} D^i \mathbf{E} \cdot \mathbf{W} &= ([\mathbf{D} \cdot \mathbf{E} - S_j S_k D_k E_j] \boldsymbol{\eta})^i \\ &= ([\mathbf{D} \cdot \mathbf{E} - (S_k S_j + [S_j, S_k]) D_k E_j] \boldsymbol{\eta})^i \\ &= ([\mathbf{D} \cdot \mathbf{E} - (S_k S_j + i \epsilon_{jkl} S_l) D_k E_j] \boldsymbol{\eta})^i \\ &= ([\mathbf{D} \cdot \mathbf{E} - (\mathbf{S} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{E}) + i \mathbf{S} \cdot (\mathbf{D} \times \mathbf{E})] \boldsymbol{\eta})^i \end{aligned}$$

### Product of $H_{12}H_{21}$

We need to calculate  $\left( \frac{\pi^2}{2} - (\mathbf{S} \cdot \boldsymbol{\pi})^2 + (g-2) \frac{2}{m} \mathbf{S} \cdot \mathbf{B} \right)^2$  to first order in magnetic field strength. As a first step of simplification

$$\left( \frac{\pi^2}{2} - (\mathbf{S} \cdot \boldsymbol{\pi})^2 + \frac{g-2}{2} \frac{e}{m} \mathbf{S} \cdot \mathbf{B} \right)^2 = \left( \frac{\pi^2}{2} - (\mathbf{S} \cdot \boldsymbol{\pi})^2 \right)^2 + \frac{g-2}{2} \frac{e}{m} \left\{ \frac{p^2}{2} - (\mathbf{S} \cdot \mathbf{p})^2, \mathbf{S} \cdot \mathbf{B} \right\}$$

### First term

To simplify the first term, consider one element of this matrix operator:

$$\begin{aligned}
\left\{ \left( \frac{\pi^2}{2} - (\mathbf{S} \cdot \boldsymbol{\pi})^2 \right)^2 \right\}_{ac} &= \left( \frac{\pi^2}{2} - S_i S_j \pi_i \pi_j \right)_{ab} \left( \frac{\pi^2}{2} - S_l S_m \pi_l \pi_m \right)_{bc} \\
&= \left( \frac{\pi^2}{2} \delta_{ab} - [S_i S_j]_{ab} \pi_i \pi_j \right) \left( \frac{\pi^2}{2} \delta_{bc} - [S_l S_m]_{bc} \pi_l \pi_m \right) \\
&= \left( \frac{\pi^2}{2} \delta_{ab} - [\delta_{ab} \delta_{ij} - \delta_{aj} \delta_{bi}] \pi_i \pi_j \right) \left( \frac{\pi^2}{2} \delta_{bc} - [\delta_{bc} \delta_{lm} - \delta_{bm} \delta_{cl}] \pi_l \pi_m \right) \\
&= \left( -\frac{\pi^2}{2} \delta_{ab} + \pi_b \pi_a \right) \left( -\frac{\pi^2}{2} \delta_{bc} + \pi_c \pi_b \right) \\
&= \frac{\pi^4}{4} \delta_{ac} - \pi_c \pi_a \frac{\pi^2}{2} - \frac{\pi^2}{2} \pi_c \pi_a + \pi_b \pi_a \pi_c \pi_b
\end{aligned}$$

It's very useful to have the following identity:

$$\begin{aligned}
e(\mathbf{S} \cdot \mathbf{B})_{ab} &= e(S_i)_{ab} B_i \\
&= -ie \epsilon_{iab} B_i \\
&= -ie \epsilon_{iab} (\epsilon_{ijk} \partial_j A_k) \\
&= -ie \epsilon_{iab} \epsilon_{ijk} \frac{1}{2} (\partial_j A_k \partial_k A_j) \\
&= -\epsilon_{iab} \epsilon_{ijk} \frac{1}{2} [\pi_j, \pi_k] \\
&= -\epsilon_{iab} \epsilon_{ijk} \pi_j \pi_k \\
&= -(\delta_{aj} \delta_{bk} - \delta_{ak} \delta_{bj}) \pi_j \pi_k \\
&= \pi_b \pi_a - \pi_a \pi_b
\end{aligned}$$

Therefore,

$$e(\mathbf{S} \cdot \mathbf{B})_{ab} = [\pi_b, \pi_a]$$

Using this, and the fact that  $\pi$  commutes with  $\mathbf{S}$  and  $\mathbf{B}$ :

$$\begin{aligned}
\pi_b \pi_a \pi_c \pi_b &= \pi_b \pi_c \pi_a \pi_b - \pi_b (e\mathbf{S} \cdot \mathbf{B})_{ac} \pi_b \\
&= \pi_b \pi_c \pi_a \pi_b - \pi^2 (e\mathbf{S} \cdot \mathbf{B})_{ac}
\end{aligned}$$

Also,

$$\begin{aligned}
\pi_b \pi_c \pi_a \pi_b &= \pi_b \pi_c \pi_b \pi_a + \pi_b \pi_c (e\mathbf{S} \cdot \mathbf{B})_{ba} \\
&= \pi_b \pi_b \pi_c \pi_a + \pi_b \pi_a (e\mathbf{S} \cdot \mathbf{B})_{bc} + \pi_b \pi_c (e\mathbf{S} \cdot \mathbf{B})_{ba}
\end{aligned}$$

So now:

$$\begin{aligned}
\pi_b \pi_a \pi_c \pi_b - \pi_c \pi_a \frac{\pi^2}{2} - \frac{\pi^2}{4} \pi_c \pi_a &= \pi^2 \pi_c \pi_a + \pi_b \pi_a (e\mathbf{S} \cdot \mathbf{B})_{bc} + \pi_b \pi_c (e\mathbf{S} \cdot \mathbf{B})_{ba} \\
&\quad - \pi^2 (e\mathbf{S} \cdot \mathbf{B})_{ac} - \pi_c \pi_a \frac{\pi^2}{2} - \frac{\pi^2}{4} \pi_c \pi_a \\
&= \frac{1}{2} [\pi^2, \pi_c \pi_a] + \pi_b \pi_a (e\mathbf{S} \cdot \mathbf{B})_{bc} + \pi_b \pi_c (e\mathbf{S} \cdot \mathbf{B})_{ba} - \pi^2 (e\mathbf{S} \cdot \mathbf{B})_{ac}
\end{aligned}$$

Now evaluate the commutator:

$$\begin{aligned} [\pi_b, \pi_c \pi_a] &= [\pi_b, \pi_c] \pi_a - \pi_c [\pi_a, \pi_b] \\ &= (e\mathbf{S} \cdot \mathbf{B})_{cb} \pi_a - \pi_c (e\mathbf{S} \cdot \mathbf{B})_{ba} \end{aligned}$$

$$\begin{aligned} [\pi^2, \pi_c \pi_a] &= [\pi_b \pi_b, \pi_c \pi_a] \\ &= \pi_b [\pi_b, \pi_c \pi_a] + [\pi_b, \pi_c \pi_a] \pi_b \\ &= (e\mathbf{S} \cdot \mathbf{B})_{cb} (\pi_b \pi_a + \pi_a \pi_b) - (e\mathbf{S} \cdot \mathbf{B})_{ba} (\pi_b \pi_c + \pi_c \pi_b) \end{aligned}$$

This gives the result:

$$\begin{aligned} \pi_b \pi_a \pi_c \pi_b - \pi_c \pi_a \frac{\pi^2}{2} - \frac{\pi^2}{4} \pi_c \pi_a &= \frac{1}{2} [(e\mathbf{S} \cdot \mathbf{B})_{cb} (\pi_b \pi_a + \pi_a \pi_b) - (e\mathbf{S} \cdot \mathbf{B})_{ba} (\pi_b \pi_c + \pi_c \pi_b)] \\ &\quad + \pi_b \pi_a (e\mathbf{S} \cdot \mathbf{B})_{bc} + \pi_b \pi_c (e\mathbf{S} \cdot \mathbf{B})_{ba} - \pi^2 (e\mathbf{S} \cdot \mathbf{B})_{ac} \\ &= \frac{1}{2} [(e\mathbf{S} \cdot \mathbf{B})_{cb} (\pi_a \pi_b - \pi_b \pi_a) + (e\mathbf{S} \cdot \mathbf{B})_{ba} (\pi_b \pi_c - \pi_c \pi_b)] - \pi^2 (e\mathbf{S} \cdot \mathbf{B})_{ac} \\ &= \frac{1}{2} [(e\mathbf{S} \cdot \mathbf{B})_{cb} (e\mathbf{S} \cdot \mathbf{B})_{ba} + (e\mathbf{S} \cdot \mathbf{B})_{ba} (e\mathbf{S} \cdot \mathbf{B})_{cb}] - \pi^2 (e\mathbf{S} \cdot \mathbf{B})_{ac} \\ &= (e\mathbf{S} \cdot \mathbf{B})_{ab} (e\mathbf{S} \cdot \mathbf{B})_{bc} - \pi^2 (e\mathbf{S} \cdot \mathbf{B})_{ac} \\ &= [(e\mathbf{S} \cdot \mathbf{B})^2]_{ac} - \pi^2 (e\mathbf{S} \cdot \mathbf{B})_{ac} \end{aligned}$$

Since we can throw away terms of order  $B^2$ , the final result tells us that, to first order in  $\mathbf{B}$ :

$$\begin{aligned} \left( \frac{\pi^2}{2} - (\mathbf{S} \cdot \boldsymbol{\pi})^2 \right)^2 &= \frac{\pi^4}{4} - \pi^2 (e\mathbf{S} \cdot \mathbf{B}) \\ &= \frac{\pi^4}{4} - e\mathbf{p}^2 \mathbf{S} \cdot \mathbf{B} \end{aligned}$$

## Second term

To simplify the second term, we need

$$\begin{aligned} \left\{ \frac{\mathbf{p}^2}{2} - (\mathbf{S} \cdot \mathbf{p})^2, \mathbf{S} \cdot \mathbf{B} \right\} &= \mathbf{p}^2 \mathbf{S} \cdot \mathbf{B} - [(\mathbf{S} \cdot \mathbf{p})^2 \mathbf{S} \cdot \mathbf{B} + \mathbf{S} \cdot \mathbf{B} (\mathbf{S} \cdot \mathbf{p})^2] \\ &= \mathbf{p}^2 \mathbf{S} \cdot \mathbf{B} - (S_i S_j S_k + S_k S_j S_i) p_i p_j B_k \end{aligned}$$

To simplify that triple product of spin matrices, we can use their explicit form:

$$\begin{aligned} (S_i S_j S_k)_{ab} &= i \epsilon_{aci} \epsilon_{cdj} \epsilon_{dbk} \\ &= i (\delta_{id} \delta_{aj} - \delta_{ij} \delta_{ad}) \epsilon_{dbk} \\ &= i (\delta_{aj} \epsilon_{ibk} - \delta_{ij} \epsilon_{abk}) \\ (S_i S_j S_k + S_k S_j S_i)_{ab} &= i (\delta_{aj} \epsilon_{ibk} + \delta_{aj} \epsilon_{kbi} - \delta_{ij} \epsilon_{abk} - \delta_{kj} \epsilon_{abi}) \\ &= -i (\delta_{ij} \epsilon_{abk} + \delta_{kj} \epsilon_{abi}) \\ &= \delta_{ij} (S_k)_a b + \delta_{kj} (S_i)_{ab} \end{aligned}$$

Now

$$\begin{aligned} \mathbf{p}^2 \mathbf{S} \cdot \mathbf{B} - (S_i S_j S_k + S_k S_j S_i) p_i p_j B_k &= \mathbf{p}^2 \mathbf{S} \cdot \mathbf{B} - (\delta_{ij} S_k + \delta_{kj} S_i) p_i p_j B_k \\ &= -(\mathbf{S} \cdot \mathbf{p})(\mathbf{B} \cdot \mathbf{p}) \end{aligned}$$

## Result

At last, the final result is that, to the order we care about

$$\left( \frac{\pi^2}{2} - (\mathbf{S} \cdot \boldsymbol{\pi})^2 + (g-2) \frac{e}{m} \mathbf{S} \cdot \mathbf{B} \right)^2 = \frac{\pi^4}{4} - e \mathbf{p}^2 \mathbf{S} \cdot \mathbf{B} - \frac{g-2}{2} \frac{e}{m} (\mathbf{S} \cdot \mathbf{p})(\mathbf{B} \cdot \mathbf{p})$$