1 Introduction

This is an introductory section.

1.1 A part of the intro

more stuff

2 Genral Spin Formalism

2.1 Formalism

First we need to work out a formalism that will apply to the general spin case. We want to represent the spin state of the particles by an object that looks like a generalization of the Dirac bispinor.

It is easiest to start with the Dirac basis, where the upper and lower components of the bispinor are objects of opposite helicity, each transforming as an object of spin 1/2.

To that end define an object

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

that we wish to have the appropriate properties. Each component should transform as a particle of spin s, but with opposite helicity. Under reflection the upper and lower components transform into each other.

The irreducible representations of the proper Lorentz group are spinors which are seperately symmetric in dotted and undotted indices. The spin of the particle will be half the total number of indices. So if ξ is an object with p undotted and q dotted indices

$$\xi = \{\xi^{\alpha_1 \ldots \alpha_p}_{\dot\beta_1 \ldots \dot\beta_q}\}$$

Then this is a represenation of a particle of spin $s = (p+q)/2$.

We have some free choice in how to partition the dotted/undotted indices, and we cannot choose exactly the same scheme for all spin as long as both types of indices are present. However, we can make separately consistent choices for integral and half-integral spin. For integral spin we can say $p = q = s$, while for the half-integral case we'll choose $p = s + \frac{1}{2}$, $q = s - \frac{1}{2}$.

We want the ξ and η to transform as objects of opposite helicity. Under reflection they will transform into each other. So

$$\eta = \{\eta^{\beta_1 \ldots \beta_q}_{\dot\alpha_1 \ldots \dot\alpha_p}\}$$

In the rest frame of the particle, they will have clearly defined and identical properties under rotation. The rest frame spinors are equivalent to rank $2s$ nonrelativistic spinors. So the bispinor in the rest frame looks like

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_0 \\ \xi_0 \end{pmatrix}$$

where

$$\xi_0 = \{(\xi_0)_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q}\}$$

and all indices are symmetric.

We can obtain the spinors in an arbitrary frame by boosting from the rest frame. The upper and lower components we have defined to have opposite helicity, and so will act in opposite ways under boost:

$$\xi = \exp\left(\frac{\mathbf{\Sigma} \cdot \phi}{2}\right)\xi_0, \qquad \eta = \exp\left(-\frac{\mathbf{\Sigma} \cdot \phi}{2}\right)\xi_0$$

What form should the operator $\mathbf{\Sigma}$ have? Under an infitesimal boost by a rapidity ϕ, a spinor with a single undotted index is transformed as

$$\xi_\alpha \to \xi'_\alpha = \left(\delta_{\alpha\beta} + \frac{\boldsymbol{\phi} \cdot \boldsymbol{\sigma}_{\alpha\beta}}{2}\right)\xi_\beta$$

while one with a dotted index will transform as

$$\xi_{\dot\alpha} \to \xi'_{\dot\alpha} = \left(\delta_{\dot\alpha\dot\beta} - \frac{\boldsymbol{\phi} \cdot \boldsymbol{\sigma}_{\dot\alpha\dot\beta}}{2}\right)\xi_{\dot\beta}$$

The infinitesmal transformation of a higher spin object with the first p indices undotted and the last q dotted would then be

$$\xi \to \xi' = \left(1 + \sum_{a=0}^{p} \frac{\boldsymbol{\sigma}_a \cdot \boldsymbol{\phi}}{2} - \sum_{a=p+1}^{p+q} \frac{\boldsymbol{\sigma}_a \cdot \boldsymbol{\phi}}{2}\right) \xi$$

where a denotes which spinor index of ξ is operated on.

If we define

$$\boldsymbol{\Sigma} = \sum_{a=0}^{p} \boldsymbol{\sigma}_a - \sum_{a=p+1}^{p+q} \boldsymbol{\sigma}_a$$

Then the infinitesmal transformations would be

$$\xi \to \xi' = \left(1 + \frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi$$

$$\eta \to \eta' = \left(1 - \frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \eta$$

So the exact transformation should be

$$\xi \to \xi' = \exp\left(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi$$

$$\eta \to \eta' = \exp\left(-\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \eta$$

Therefore, the bispinor of some particle boosted by ***phi*** will be

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \exp\left(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi_0 \\ \exp\left(\frac{-\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi_0 \end{pmatrix}$$

In dealing with the relativistic theory, we'll want a basis that seperates the particle and antiparticle parts of the wave function. If we want the upper component to be the particle, then in the rest frame the lower component will vanish, and for low momentum will be small compared to the upper component. The unitary transformation which accomplishes this is

$$\Psi' = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

$$\phi = \frac{1}{\sqrt{2}}(\xi + \eta)$$

$$\chi = \frac{1}{\sqrt{2}}(\eta - \xi)$$

Which is equivalent to

$$\Psi' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Psi$$

Then,

$$\phi = \cosh\left(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi_0$$

$$\chi = \sinh\left(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi_0$$

2.2 General bilinear transform

We have the transformation of the spinor under small boosts:

$$\Psi \quad \to \quad \Psi' = \Psi + \frac{\eta_i}{2}\begin{pmatrix} 0 & \Sigma_i \\ \Sigma_i & 0 \end{pmatrix}\Psi$$

$$\bar{\Psi} \quad \to \quad \bar{\Psi}' = \bar{\Psi} - \frac{\eta_i}{2}\bar{\Psi}\begin{pmatrix} 0 & \Sigma_i \\ \Sigma_i & 0 \end{pmatrix}$$

We can also see the transformation of the spinor under parity: simply put, because the upper component is even in $\mathbf{\Sigma} \cdot \mathbf{p}$, whereas the lower component is odd, we obtain

$$\Psi \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Psi$$

$$\bar{\Psi} \to \bar{\Psi}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So

$$\bar{\Psi}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\Psi \to \bar{\Psi}\begin{pmatrix} A & -B \\ -C & D \end{pmatrix}\Psi$$

From these facts we can examine the general behavior of bilinears under Lorentz transformations.

Now we'll examine the behavior of bilinears under boosts. We can write the general structure of the bilinear as

$$\bar{\Psi}T\Psi = \bar{\Psi}\begin{pmatrix} A+D & B+C \\ B-C & A-D \end{pmatrix}\Psi$$

or using a different notation

$$\bar{\Psi}T\Psi = \bar{\Psi}\left[A \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + D \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)]\Psi$$

Under an infitesimal Lorentz boost $\boldsymbol{\eta}$ this will transform into

$$\bar{\Psi}T\Psi \to \bar{\Psi}T\Psi + \frac{\eta_i}{2}\bar{\Psi}\left(\begin{pmatrix} A+D & B+C \\ B-C & A-D \end{pmatrix}\begin{pmatrix} 0 & \Sigma_i \\ \Sigma_i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \Sigma_i \\ \Sigma_i & 0 \end{pmatrix}\begin{pmatrix} A+D & B+C \\ B-C & A-D \end{pmatrix}\right)\Psi$$

We can express this in terms of commutators and anti-commutators

$$\bar{\Psi}T\Psi \to \bar{\Psi}T\Psi + \frac{\eta_i}{2}\bar{\Psi}\left[[B,\Sigma_i] \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + [A,\Sigma_i] \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \{C,\Sigma_i\} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \{D,\Sigma_i\} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)]\Psi$$

We can note here that, using only the matrices $\mathbf{\Sigma}$ and $\boldsymbol{S}$ we can build three structures invariant under rotations: and S^2, Σ^2, and $\Sigma \cdot S$. All three of these structures commute with both S_i and Σ_i, and their value depends only on the particular representation we're working with. So for our purposes here, they can just be treated as pure numbers.

2.2.1 Scalar bilinears

Since the scalar must be invariant to rotation, then by the logic above it's block elements are proportional to the identity.

It must also be unchanged under boosts. We can see that this necessitates that $C = D = 0$, while providing no constraint on A and B. So the general form of a bilinear invariant under boosts is

$$\bar{\Psi}T\Psi = \bar{\Psi}\begin{pmatrix} A & B \\ B & A \end{pmatrix}\Psi$$

where A and B are proportional to the identity.

Under the discrete partiy transformation this will transform into

$$\bar{\Psi}T'\Psi = \bar{\Psi}\begin{pmatrix} A & -B \\ -B & A \end{pmatrix}\Psi$$

This shows that for a true scalar, $B = 0$.

2.2.2 Vector bilinears

To attempt to construct a vector bilinear, we can start by considering the time-like part of it. Since this must be invariant under spatial rotations, then by the same logic as above it must essentially be composed of four blocks proportional to the identity. We also know that under boosts the time-like part is transformed into the spatial and vice versa, so we can use these linked transformations to obtain constraints on the bilinear.

If T^μ is a vector we know that, under an infinitesimal boost, it's transformation will be

$$\begin{aligned} T^0 &\rightarrow T^0 + \eta_i T^i \\ T^i &\rightarrow T^i + \eta_i T^0 \end{aligned}$$

We again write

$$T^\mu = \begin{pmatrix} A^\mu + D^\mu & B^\mu + C^\mu \\ B^\mu - C^\mu & A^\mu - D^\mu \end{pmatrix}$$

Then we see that under boost, T^0 transforms as

$$\bar{\Psi}T^0\Psi \rightarrow \bar{\Psi}T^0\Psi + \eta_i\bar{\Psi}\left[C^0\Sigma_i \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + D^0\Sigma_i \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]\Psi$$

where we've used that fact that all the components of T^0 commute with Σ_i. This tells us that for T^μ to be a 4-vector, the following must be true.

$$\begin{aligned} A^i &= 0 \\ B^i &= 0 \\ C^i &= D^0\Sigma^i \\ D^i &= C^0\Sigma^i \end{aligned}$$

We can now consider how T^i changes under a boost, and discover

$$\begin{aligned} \bar{\Psi}T^i\Psi &\rightarrow \bar{\Psi}T^i\Psi + \frac{\eta_j}{2}\bar{\Psi}\left[\{C^i, \Sigma^j\} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \{D^i, \Sigma^j\} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)]\Psi \\ &= \bar{\Psi}T^i\Psi + \frac{\eta_j}{2}\bar{\Psi}\left[D^0\{\Sigma^i, \Sigma^j\} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + C^0\{\Sigma^i, \Sigma^j\} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)]\Psi \end{aligned}$$

Again considering our demand that T^μ transform like a 4-vector, we get

$$\begin{aligned} A^0 &= 0 \\ B^0 &= 0 \\ C^0\delta^{ij} &= C^0\frac{1}{2}\{\Sigma^i, \Sigma^j\} \\ D^0\delta^{ij} &= D^0\frac{1}{2}\{\Sigma^i, \Sigma^j\} \end{aligned}$$

The last two constraints are met in the spin-1/2 case, but not for higher spins. This tells us that there's no way to, in the higher spin case, construct a vector bilinear using only I, $\mathbf{\Sigma}$, and $\mathbf{S}$.

For spin-1/2, where $\Sigma_i = \sigma_i$, we see that a true vector bilinear (with correct transformation properties under parity) will be proportional to

$$(T^0, \mathbf{T}) = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \right)$$

which, of course, is exactly what we knew already.

2.2.3 Tensor bilinears

Here we'll be a little less ambitious. We can tell from the above considerations that, under boosts, we effectively mix A and B components seperately from the C and D blocks. What's more, we need anti-commutation relationships to deal with the latter transformations. So we'll just consider tensors that look like

$$T^{\mu\nu} = \begin{pmatrix} A^{\mu\nu} & B^{\mu\nu} \\ B^{\mu\nu} & A^{\mu\nu} \end{pmatrix}$$

Furthermore, we'll consider only anti-symmetric tensors for now. Now we can basically procede as in the vector case, knowing how an anti-symmetric tensor should transform:

$$\begin{aligned} T^{0i} &\rightarrow T^{0i} + \eta_j T^{ji} \\ T^{ij} &\rightarrow T^{ij} + \eta_i T^{0j} + \eta_j T^{i0} \end{aligned}$$

Start by considering the components of $T^{0i} = -Ti0$. They must transform as vectors under rotations. We have two vectors available to us, so we can write

$$\begin{aligned} A^{0i} &= \alpha \Sigma^i + \beta S^i \\ B^{0i} &= \gamma \Sigma^i + \delta S^i \end{aligned}$$

Under a boost, we find the relation that

$$\begin{aligned} A^{ji} &= [B^{0i}, \Sigma^j] \\ B^{ji} &= [A^{0i}, \Sigma^j] \end{aligned}$$

And then looking at how T^{ij} transforms, we get the constraint

$$\begin{aligned} \eta_k \left[[A^{0i}, \Sigma^j], \Sigma^k \right] &= \eta_j A^{0i} - \eta_i A^{0j} \\ \eta_k \left[[B^{0i}, \Sigma^j], \Sigma^k \right] &= \eta_j B^{0i} - \eta_i B^{0j} \end{aligned}$$

We're assuming that both A^{0i} and B^{0i} are linear combinations of Σ_i and S_i. So what we need are the relationships

$$\begin{aligned}
[\Sigma^i, \Sigma^j] &= 4i\epsilon_{ijk}S^k \\
[S^i, \Sigma^j] &= i\epsilon_{ijk}\Sigma^k \\
[[\Sigma^i, \Sigma^j], \Sigma^k] &= 4i\epsilon_{ij\ell}[S^\ell, \Sigma^k] \\
&= -4\epsilon_{ij\ell}\epsilon_{\ell km}\Sigma^m \\
&= -4(\delta_{ik}\Sigma^j - \delta_{jk}\Sigma^i) \\
[[S^i, \Sigma^j], \Sigma^k] &= i\epsilon_{ij\ell}[\Sigma^\ell, \Sigma^k] \\
&= -4\epsilon_{ij\ell}\epsilon_{\ell km}S^m \\
&= -4(\delta_{ik}S^j - \delta_{jk}S^i)
\end{aligned}$$

So we can see that, no matter what α and β are, we get the relation

$$[[A^{0i}, \Sigma^j], \Sigma^k] = -4(\delta ikA^{0i} - \delta_{jk}A^{0j}$$

And so necessarily,

$$\eta_k[[A^{0i}, \Sigma^j], \Sigma^k] = 4(\eta_j A^{0i} - \eta_i A^{0j})$$

So any arbitrary combination of $\mathbf{S}$ and $\mathbf{\Sigma}$ will allow us to construct a bilinear that transforms as a tensor.

(In fact, it's not hard to generalise this to any operator expressable as a linear combination of σ_i^A, where the index A represents which spinor index σ operates on.)

2.3 Electromagnetic Interaction

Knowing how the wave functions themselves behave, we want to see what that tells us about possible electromagnetic interaction. Interaction with a single electromagnetic photon should take the form

$$M = A_\mu j^\mu$$

where j^μ is the electromagnetic current.

The electromagnetic current must be built out of the particle's momenta and bilinears of the charged particle fields in such a way that they have the correct Lorentz properties. We must also demand current conservation: the equation $q_\mu j^\mu = 0$ must hold. Above we already have shown that, in the case of general spin, there exist only two such bilinears, a scalar and a tensor.

There will be two permissible terms in the current. We could consider a scalar bilinear coupled with a single power of external momenta. In order to fufill the current conservation requirement, it should be

$$\frac{p^\mu + p'^\mu}{2m}\bar{\Psi}^\dagger\Psi$$

This will obey current conservation because $q = p' - p$, and $(p + p') \cdot (p' - p) = p^2 - p'^2 = 0$

We can also consider a tensor term contracted with a power of momenta. To fufill current conservation, we can demand that the tensor bilinear be antisymmetric, and contract it with q:

$$\frac{q_\nu}{2m}\bar{\Psi}^\dagger T^{\mu\nu}\Psi$$

We don't need to worry about higher order tensor bilinears: they will necessitate too many powers of the external momenta.

So the most general current would look like

$$j^\mu = F_e\frac{p^\mu + p'^\mu}{2m}\bar{\Psi}^\dagger\Psi + F_m\frac{q_\nu}{2m}\bar{\Psi}^\dagger T^{\mu\nu}\Psi$$

In general the form factors might have quite complicated dependence on q, but these corrections will be too small compared to the type of result we're interested in. At leading order F_e will just be the electric charge of the particle in question, and F_m will, as we'll see after connecting this result to the nonrelativistic limit, be related to the particle's g-factor. So to the order we need, we can write the current as

$$j^\mu = e\frac{p^\mu + p'^\mu}{2m}\bar{\Psi}^\dagger\Psi + eg\frac{q_\nu}{2m}\bar{\Psi}^\dagger T^{\mu\nu}\Psi$$

The tensor bilinear $\bar{\Psi}^\dagger T^{\mu\nu}\Psi$ itself has some free parameters. However, it'll turn out that, when computing actual nonrelativistic amplitudes, the two types of anti-symmetric tensors collapse into the same general form.

This captures the essence of the interaction between a charged particle of general-spin and a single photon.

2.4 Connection to nonrelativistic spinors

In the rest frame, there are two independant bispinors which represent particle and antiparticle states:

$$\Psi = \begin{pmatrix} \xi_0 \\ 0 \end{pmatrix}$$

or

$$\Psi = \begin{pmatrix} 0 \\ \xi_0 \end{pmatrix}$$

However, when we consider a particle with zero momentum it is not the case that the upper component of the bispinor can be directly associated with the Shrodinger like wave-function of the particle — for instance, it would not be correctly normalized, for there is some mixing with the lower component.

We can obtain a relation between ξ_0 and the Shrodinger amplitude ϕ_s by considering the current density at zero momentum transfer. For ϕ_s it will be $j_0 = e\phi_s^\dagger\phi$. For the relativistic theory we have, as calculated above:

$$j^0 = e\frac{p^0 + p'^0}{2m}\bar{\Psi}^\dagger\Psi + F_m\frac{q_\nu}{2m}\bar{\Psi}^\dagger T^{0\nu}\Psi$$

At $q = 0$ the particle energies become just m.

$$j^0(q=0) = e\bar{\Psi}^\dagger\Psi$$

$$= e\phi^\dagger\phi - \chi^\dagger\chi$$

ϕ and χ are both related to the rest frame spinor ξ_0. So we can write instead

$$j^0 = e\xi_0^\dagger\left\{\cosh^2(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\varphi}}{2}) - \sinh^2(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\varphi}}{2})\right\}\xi_0 = e\xi_0^\dagger\xi_0$$

where the last line follows from the hyperbolic trig identity.

So we can in fact identify ξ_0 with the nonrelativistic amplitude ϕ_s.

To write the relativistic bispinors in terms of ϕ_s we will also need approximations to $\cosh(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\varphi}}{2})$ nad $\sinh(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\varphi}}{2})$ First we should relate φ to $\mathbf{p}$.

$$\cosh(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\varphi}}{2})$$

2.5 Bilinears in terms of nonrelativistic theory

The next step is to express the relativistic bilinears, built out of the bispinors Ψ, in terms of the Shrodinger like wave functions.

We have above written the b