Contents

Chapter 1

Introduction

We wish to calculate the gyromagnetic ratio of a particle of arbitrary spin in a loosely bound state system. We can write down an effective Lagrangian which will capture all the necessary effects. Our job is then to calculate the coefficients of this effective particle. By considering the constraints which exist on the electromagnetic current in a general relativistic theory, we can obtain a simple form which holds for arbitrary spin, and then use this to fix the coefficients of the relativistic theory.

While for a particle of general spin we do not have an exact Lagrangian, we do have one for a spin-1 theory. We can use this Lagrangian to perform the same calculation as above, but in an exact theory. We derive the nonrelativistic potential first from the equations of motion, and then from diagrammatic calculations, and obtain a result that agrees with the more general calculation.

1.0.1 Description of problem

We consider a loosely bound system two particles of arbitrary spin, and wish to calculate the correction to the gyromagnetic ratio of the particles. So we consider the case of the two particles interacting with both each other, via a Coulomb potential, and a very weak external magnetic field.

Experimental Motivation

Talk about

Approach

This system allows for several simplifications. First, because the system is loosely bound all energy scales are nonrelativistic. Second, the magnetic field we consider is both very weak and constant. So we can ignore all corrections which involve derivatives of the magnetic field or which are quadratic in its strength.

The approach we'll take is to first consider the electric potential as an external field acting on a single particle. Next we can consider the bound system as a whole sitting in an external magnetic field, and take into account recoil effects.

1.0.2 Theoretical Background

(Here we talk about existing theoretical work in this area)

Spin-1/2

(Here talk about the approach that works for spin 1/2, and why it breaks down in the general case.)

BMT equation

(Here discuss the BMT equation, which holds for general spin and can be used to derive the g-factor corrections)

Khriplovich general spin work

(Discuss the work Khriplovich did with general spinors.)

Chapter 2

NRQED

We can construct an effective, nonrelativistic Lagrangian for a charged particle interacting with an electromagnetic field. —— (quick outline) Background

Construction - Define the interaction – physical situation and the fields involved - enumerate building blocks – ($\bar{S}$ text here) -Define order of terms -Symmetry of terms -Enumerating terms – Start by discussing how spin/position operators combine. – More indices on spin operator -¿ higher order position space operator. – Make clear that there are exactly 4 orders of terms (v^2, v^4, B, Bv^2) – First construct all order unity objects – Construct position space terms for each order, going through number of indice s – (then contracting with the order unity objects) – Now we have all terms in the Lagrangian Calculation – section on diagrams from the Lagrangian –

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2.0.3 Effective Field Theories

2.0.4 The NRQED Lagrangian

We want to construct an effective Lagrangian in the nonrelativistic limit. Our goal is to calculate the leading order corrections to the g-factor, which are corrections of order α^2. To this end, we need terms in the effective nonrelativistic Lagrangian which are equivalent corrections.

	Order	P	T	†
eE_i	m^2v^3	-	+	+
eB_i	m^2v^2	+	-	+
D_i	mv	-	+	-
D_0	mv^2	+	-	-
S_i	1	+	-	+
i	1	+	-	-

Energy Scales

The effective Lagrangian contains an infinite number of terms. But these terms may be organised by the order of their contribution to scattering. Then, up to a particular order there are only a finite number of terms — and for the first couple of orders this finite number is quite small and manageable.

Consider again the particular physical situation of interest: a loosely bound state system of two charged particles, each with arbitrary spin, placed in an infinitesimal magnetic field. It is crucial that the system is loosely bound, so that the entire system is nonrelativistic. For now it is assumed that recoil effects are ignored.

Then what are the energy scales of this particular problem? In addition to the mass of the particle, there are two: one is related to the external magnetic field, an energy scale of $e/m\,|\mathbf{B}|$. The other is related to the kinetic energy or momentum of the particle. There is an electric field present as well, representing the interaction between the bound state particles. But by the quantum virial theorem the Coulomb potential is proportional to the kinetic energy, and so is not an independent energy scale. In general kinetic energy and momentum might be considered seprate energy scales, but in the nonrelativistic regime are just mv and $(1/2)mv^2$.

Both of these energy scales are nonrelativistic — that is, small compared to the mass of the particle. So the effective Lagrangian represents an expansion in the small quantities e/m^2B and $p/m = v$. However, the calculation of the gyromagnetic ratio involves only linear terms in the magnetic field. (The g-factor is defined by the linear response to an infinitesimal magnetic field.) So no terms quadratic in B are needed.

The goal is to calculate corrections to the g-factor of up to $\mathcal{O}(\alpha^2)$. In the bound state system $v \sim Z\alpha$, so in the Lagrangian should definitely be included terms of up to order $(e/m)Bv^2$. It will also be necessary to consider terms of order mv^4, for they *might* effect the calculation of the g-factor through perturbation theory.

What is the exact order of each term? The magnetic field defines its own scale, but the scale of operators D, A_0, and E come from the particular bound state.

The term involving the Coulomb potential, $eA_0 = Ze^2/r^2$ is of order mv^2. The first derivative $\partial_i\Phi = E_i$ is of order mv^3. The operator $\boldsymbol{\nabla}$ may then be said to be of order mv.

position space operators

There are several symmetries that terms in the Lagrangian must preserve: spatial reflection, time reversal, gauge invariance, and Hermiticity. Gauge invariance is taken care of by only considering gauge-invariant operators.

All of the possible spin-space structures preserve spatial parity. If the term in the Lagrangian is to obey such a symmetry, then any allowed collection of position-space operators must be themselves invariant under reflection.

Each term must also be invariant under time reversal. The spin operator S_i flips sign under time reversal. Thus, $\bar{S}_{ij}$ is even, and $\bar{S}_{ijk}$ odd, under the same transformation. E_i and D_i are even, while B_i is odd. Any set of operators with definite behavior under time reversal, though, can be made invariant by including an extra factor of i. So even if some term ABC is odd, $iABC$ will be even.

Finally, any sequence of operators can be made Hermitian by simply adding the Hermitian conjugate; if we wish to include a term ABC in the Lagrangian, we add $C^\dagger B^\dagger A^\dagger$. Since all the operator s under consideration are either Hermitian or anti-Hermitian, this means that exactly one of $ABC \pm CBA$ will be allowed. The spin space operators are all Hermitian of themselves, and commute with all other operators, so only the position space operators are nontrivial.

Thus for any possible set of operators, first it is determined if such a set is allowed by considerations of parity. Then, whether an additional factor of i is needed is determined by examining the properties under time reversal. Finally, the allowed Hermitian terms are enumerated.

Of the "building blocks" for the general Lagrangian, those odd in parity are also odd in v. And those even in parity are of an order even in v. So only terms of an order even in v can possibly be allowed under parity, and at the same time every such term will have the correct symmetry properties under spatial reflection.

First consider terms of leading order mv^2. What collections of position space operators exist? E is already too high of an order, leaving only that with two powers of D:

$$\text{Sets of } \mathcal{O}(mv^2) \text{ combinations} = \{\frac{1}{m}D_iD_j\} \tag{2.0.1}$$

This is in addition to the single permitted term containing $e\Phi$ which is not, by itself, gauge-invariant.

Next consider terms of order mv^4. We could have up to a single power of E, or only the long derivative operators:

$$\text{Set of } \mathcal{O}(mv^2) \text{ combinations} = \{\frac{1}{m^3}D_iD_jD_kD_\ell,\ \frac{1}{m^2}eE_iD_j\} \tag{2.0.2}$$

The leading order term in B is just $\frac{e}{m}B_i$.

The order $\frac{e}{m}B_i v^2$ terms are drawn from:

$$\text{Set of } \mathcal{O}(\frac{e}{m}Bv^2) \text{ combinations} = \{\frac{e}{m^3}D_i D_j B_k\} \tag{2.0.3}$$

Contraction of terms

In this nonrelativistic theory the Lagrangian need not be Lorentz invariant, but must still be Galilean invariant. All the operators we consider transform as 3-vectors or higher order 3-tensors. To form allowed terms, all indices must be somehow contracted.

Above, all relevant sets of position space operators are considered for each order of term. The greatest number of indices free was four. These must be contracted with order unity structures, which as well as the spin operators with up to four indices (S_i, $\bar{S}_{ij}$, $\bar{S}_{ijk}$,$\bar{S}_{ijk\ell}$) include δ_{ij} and the completely antisymmetric tensor ϵ_{ijk}.

- The only structure with one index is just S_i.
- With two indices, there are the simple structures $\bar{S}_{ij}$ and δ_{ij}, and also $S_i\epsilon_{ijk}$.
- With three indices, there are ϵ_{ijk} and $\bar{S}_{ijk}$ as well as $\delta_{ij}S_k$ or $\bar{S}_{ij}\epsilon_{jk\ell}$.
- The only position space operator with four indices is just $D_iD_jD_kD_\ell$. While there are a fairly large number of order unity structures with four indices, any with spin operators are forbidden because of the kinetic nature of the term. So only $\delta_{ij}\delta_{k\ell}$ need be considered.

Above are categorized the possible sets of position space operators for each order, and the ways of contracting them by number of indices. The next step is to write all Hermitian combinations we can form from these operators. This will form a complete catalogue of allowed terms in the Lagrangian.

Some terms which on the surface appear distinct might in fact be identical. One identity of this nature is

$$D_iD_j\epsilon_{ijk} = -ieB_k \tag{2.0.4}$$

Which means that any term where two long derivative terms are contracted with the antisymmetric tensor will be equivalent to some already catalogued term involving B.

Similarly,

$$[D_i, D_j] = -ie(\partial_i A_j - \partial_j A_i) = -ieF_{ij} = -ie\epsilon_{ijk}B_k \tag{2.0.5}$$

so terms which appear distinct because of the ordering of two long derivative operators are really the same, up to some difference that is absorbed into a term involving B.

Finally, while in the general case $[D_i, B_j] \neq 0$, it is always true that $[D_i, B_j] = 0$.

So many apparently distinct terms may be rewritten and absorbed as the combination of several already accounted for terms. As an example, it is not necessary to write a term such as $D_iD_jD_iD_j$ in the Lagrangian. It can be rewritten as

$$\begin{aligned}
D_iD_jD_iD_j &= D_iD_iD_jD_j + D_i[D_i, D_j]D_j \\
&= \mathbf{D}^4 - ieD_i\epsilon_{ijk}B_kD_j \\
&= \mathbf{D}^4 - ie\epsilon_{ijk}(D_iD_jB_k + [B_k, D_j]) \\
&= \mathbf{D}^4 - e^2B^2 + ie\epsilon_{ijk}D_i(\partial_j B_k)
\end{aligned}$$

The term D_iD_j has two indices. There are three ways to contract it: with δ_{ij}, $\bar{S}_{ij}$, or $S_k\epsilon_{ijk}$. Since kinetic terms with spin are not considered, that only leaves one possible term: $\mathbf{D}^2$. This term is Hermitian in and of itself. It has mass dimension two, so the term as it appears in the Lagrangian will be:

$$D_iD_j \to \frac{\mathbf{D^2}}{2m} \tag{2.0.6}$$

Terms from the set E_iD_j also have two indices. It can be contracted with all three of δ_{ij}, $\bar{S}_{ij}$, or $iS_k\epsilon_{ijk}$. E_i, δ_{ij} and $\bar{S}_{ij}$ are Hermitian, while D_j and $iS_k\epsilon_{ijk}$ are anti-Hermitian. It is convenient to use that $[D_j, E_i] = \boldsymbol{\nabla}_j E_i$.

The Hermitian combination with δ_{ij} is

$$(D_iE_j - E_jD_i)\delta_{ij} \rightarrow \frac{\boldsymbol{\nabla}\cdot\mathbf{E}}{4m^2} = \frac{\boldsymbol{\nabla}\cdot\mathbf{E}}{4m^2} \tag{2.0.7}$$

The Hermitian combination with $\bar{S}_{ij}$ is

$$(D_iE_j - E_jD_i)\bar{S}_{ij} \rightarrow \frac{\bar{S}_{ij}\partial_i E_j}{4m^2} \tag{2.0.8}$$

The Hermitian combination with $iS_k\epsilon_{ijk}$ is

$$(D_iE_j + E_jD_i) \rightarrow iS_k\epsilon_{ijk} \rightarrow \frac{i\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E}+\mathbf{E}\times\mathbf{D})}{4m^2} \tag{2.0.9}$$

A single power of the magnetic field, B_i may only be contracted with S_i. So the term is

$$\frac{e}{m}\mathbf{S}\cdot\mathbf{B} \tag{2.0.10}$$

The second order terms involving the magnetic field are drawn from the set $D_iD_jB_k$. There are four order-unity structures which have three indices, but only two of those need be considered. This is because $\mathbf{D}\times\mathbf{D}\sim\mathbf{B}$ and $\mathbf{D}\times\mathbf{B}=0$. If is not assumed that $[D_i, B_j] = 0$ then the allowed Hermitian combinations will be with $D_iD_jB_k + B_kD_jD_i$ and $D_iB_jD_k$.

While $[D_i, D_j] \neq 0$, it does produce another term proportional to B and thus, smaller than considered. So for instance, $D_iD_jB_i$ need not be considered separate from $D_jD_iB_i$.

$$\begin{aligned}
(D_iD_jB_k + B_kD_jD_i)\delta_{ij}S_k &\rightarrow \frac{e}{m}\frac{\mathbf{D}^2(\mathbf{S}\cdot\mathbf{B}) + (\mathbf{S}\cdot\mathbf{B})\mathbf{D}^2}{4m^2} \\
(D_iD_jB_k + B_kD_jD_i)\delta_{jk}S_i &\rightarrow \frac{e}{m}\frac{(\mathbf{S}\cdot\mathbf{D})(\mathbf{D}\cdot\mathbf{B}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})}{4m^2} \\
D_iB_jD_k\delta_{ij}S_k &\rightarrow \frac{e}{m}\frac{(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{D}\cdot\mathbf{B})(\mathbf{S}\cdot\mathbf{B})}{4m^2}
\end{aligned}$$

$$\begin{aligned}
(D_iD_jB_k + B_kD_jD_i)\bar{S}_{ijk} &\rightarrow \frac{e}{m}\frac{\bar{S}_{ijk}(D_iD_jB_k + B_kD_jD_i)}{4m^2} \\
D_iB_jD_k\bar{S}_{ijk} &\rightarrow \frac{e}{m}\frac{\bar{S}_{ijk}D_iB_jD_k}{4m^2}
\end{aligned}$$

Finally, there is only one way to contract $D_iD_jD_kD_\ell$, which is as $\mathbf{D}^4$. Again, variations such as $D_i\mathbf{D}^2D_i$ need not be considered because they just reproduce already considered terms with B.

$$D_iD_jD_kD_\ell\delta_{ij}\delta_{k\ell} \rightarrow \frac{\mathbf{D}^4}{8m^3}$$

$$\mathcal{L}_{NRQED} = \Psi^\dagger\left\{iD_0 + \frac{\mathbf{D}^2}{2m} + c_F\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\right\}\Psi \tag{2.0.11}$$

$$\mathcal{L}_{mv^4} = \Psi^\dagger\left\{\frac{\mathbf{D}^4}{8m^2} + c_D\frac{e(\mathbf{D}\cdot\mathbf{E}-\mathbf{E}\cdot\mathbf{D})}{8m^2} + c_Q\frac{eQ_{ij}(D_iE_j - E_iD_j)}{8m^2} + c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D}}{8m^2}\right\}\Psi \tag{2.0.12}$$

$$\mathcal{L}_{Bv^2} = \Psi^\dagger \Bigg\{ c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B}+\mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3} + c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D})+(\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3}$$
$$+ c_{T_1}\frac{e\bar{S}_{ijk}(D_iD_jB_k+B_kD_jD_i)}{8m^3} + c_{T_2}\frac{e\bar{S}_{ijk}D_iB_jD_k}{8m^3}\Bigg\}\Psi \tag{2.0.13}$$

Full Lagrangian

The full Lagrangian we consider is then:

$$\mathcal{L}_{NRQED} = \Psi^\dagger \Bigg\{ iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^2} + c_F\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + c_D\frac{e(\mathbf{D}\cdot\mathbf{E}-\mathbf{E}\cdot\mathbf{D})}{8m^2} + c_Q\frac{eQ_{ij}(D_iE_j-E_iD_j)}{8m^2}$$
$$+ c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D}}{8m^2} + c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B}+\mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3}$$
$$+ c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D})+(\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3} + c_{T_1}\frac{e\bar{S}_{ijk}(D_iD_jB_k+B_kD_jD_i)}{8m^3} + c_{T_2}\frac{e\bar{S}_{ijk}D_iB_jD_k}{8m^3}\Bigg\}\Psi \tag{2.0.14}$$

One of the features of this Lagrangian is that every coefficient is fixed by the one-photon interaction. Although some terms might represent two-photon interactions, they are terms like $\mathbf{S}\cdot\mathbf{A}\times\mathbf{E}$, whose coefficient is fixed by the gauge-invariant term $\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}$. This in turn means that we can calculate the corrections to the g-factor by considering only one-photon interactions.

Order of terms

We consider constant, infinitesimal external magnetic fields, so we need only consider terms linear in $\mathbf{B}$.

The velocity of the particles in our bound state system will be $v \sim \alpha$.

The electric field we consider is the Coulomb field, so $e\Phi \sim mZ\alpha^2 \sim mv^2$, and $eE \sim m^2v^3$.

Each derivative of the electric field will add an additional factor of mv, so the operator $\mathbf{D}$ can be taken to be of this order.

We need to keep terms up to order mv^4 and $\frac{B}{m}v^2$ in order to calculate the g-factor to the necessary precision. We include mv^4 terms so we can be sure that there are no effects entering from second-order perturbation theory.

Constraints on the form of the Lagrangian

The Lagrangian is constrained to obey several symmetries. It must be invariant under the symmetries of parity and time reversal. It must also be invariant under Galilean transformations. The Lagrangian must also be Hermitian, and gauge invariant.

What are the gauge invariant building blocks we can use to construct this Lagrangian? We have the external fields $\mathbf{E}$ and $\mathbf{B}$, the spin operators $\mathbf{S}$, and the long derivative $\mathbf{D} = \partial - ie\mathbf{A}$. The fields should always be accompanied by the charge e of the particle.

When considering the case of higher spin particles, we might consider terms quadratic and above in spin operators. For a particle of spin s, there must be $(2s+1)^2$ independent hermitian operators. We can span this set of operators by considering products of up to $2s$ spin matrices which are symmetric and traceless in every vector index. For example, for spin-1 we have quadratic, in addition to I and S_i, five independent structures of the form $S_iS_j + S_jS_i + \delta_{ij}\mathbf{S}$.

We also have the scalar D_0, however, we need only include a single such term because we insist on having only one power of the time derivative.

To consider possible terms, we need to know how each of the above behave under the discrete transformations and Hermitian conjugate. The signs under these transformations are listed in the table below. (Also included is the imaginary number i.)

	Order	P	T	†
eE_i	m^2v^3	-	+	+
eB_i	m^2v^2	+	-	+
D_i	mv	-	+	-
D_0	mv	+	-	-
S_i	1	+	-	+
i	1	+	-	-

What about terms such as $D_iD_j\epsilon_{ijk}$? First write them using $\mathbf{D} = \boldsymbol{\nabla} - ie\mathbf{A}$:

$$(\mathbf{D} \times \mathbf{D}) = D_iD_j\epsilon_{ijk} = (\boldsymbol{\nabla}_i - ieA_i)(\boldsymbol{\nabla}_j - ieA_j)\epsilon_{ijk} \tag{2.0.15}$$

Or in full as

$$= (\boldsymbol{\nabla}_i\boldsymbol{\nabla}_j - ie\boldsymbol{\nabla}_iA_j - ie\boldsymbol{\nabla}_jA_i - e^2A_iA_j)\epsilon_{ijk} \tag{2.0.16}$$

The antisymmetric tensor will kill the first and last terms, leaving

$$= -ie(\boldsymbol{\nabla}_iA_j + \boldsymbol{\nabla}_jA_i)\epsilon_{ijk} \tag{2.0.17}$$

We can use put this in the form of a commutator by switching indices in the second term: $\boldsymbol{\nabla}_jA_i\epsilon_{ijk} = -\boldsymbol{\nabla}_iA_j\epsilon_{ijk}$

$$= -ie[\boldsymbol{\nabla}_i, A_j]\epsilon_{ijk} \tag{2.0.18}$$

and of course the commutator between $\boldsymbol{\nabla}$ and any field will just be the derivative of that field.

$$= -ie\partial_iA_j\epsilon_{ijk} \tag{2.0.19}$$

Finally, from Maxwell's equations this is just $-ieB_k$, so

$$(\mathbf{D} \times \mathbf{D})_k = -ieB_k \tag{2.0.20}$$

Going the other way, the commutator $[D_i, B_i]$ can be shown to vanish:

$$[D_i, B_i] = D_i(\epsilon_{ijk}D_jD_k) - (\epsilon_{ijk}D_jD_k)D_i = \epsilon_{ijk}(D_iD_jD_k - D_jD_kD_i) \tag{2.0.21}$$

Taking the second term and taking an even permutation of the indices: $i \to k \to j \to i$ we show that

$$[D_i, B_i] = \epsilon_{ijk}(D_iD_jD_k - D_iD_jD_k) = 0 \tag{2.0.22}$$

In formulating NRQED for general spin particles, we need to consider all the possible operators might show up in the Lagrangian. The state-space of a spin-s particle is the direct product of it's spin-state and all the other state information. We want to write down all possible operators which act only on the spin space.

For a particular representation, we can always write a bilinear as the spin operators and other operators acting between two spinors:

$$\Psi^\dagger\mathcal{O}_S\mathcal{O}_X\Psi \tag{2.0.23}$$

The two types of operators will always commute, since they act on orthogonal spaces, so it doesn't matter what order they're written in. All such bilinears must be Galilean invariant, but individual operators might not be. The non-spin operators we consider, such as $\mathbf{D}$, $\mathbf{B}$ or contractions with the tensor ϵ_{ijk} are already all written as 3-vectors or (in combination) as higher rank tensors. Therefore, it will be most convenient to write spin-operators in the same way, so that writing Galilean-invariant combinations of the two types of operators is done just by contracting indices.

The spinors Ψ are written with $2s+1$ independent components. The spin operators will be matrices acting on these components, and so for a spin-s particle will be $(2s+1) \times (2s+1)$ matrices. The combined operator $\mathcal{O}_S\mathcal{O}_X$ must be Hermitian, but without loss of generality we can require any $\mathcal{O}_S$, $\mathcal{O}_X$ to be Hermitian separately. So there is the additional constraint that the matrices be Hermitian, and this means a total of $(2s+1)^2$ degrees of freedom.

For spin-0 there is only one component to the spinor, so the only possible matrix is equivalent to the identity.

For spin-1/2 we have, in addition to the identity matrix, the spin matrices $s_i = \frac{1}{2}\sigma_i$. This is four independent matrices, and since the space has $(2s+1)^2 = 4$ degrees of freedom, exactly spans the space of all spin-operators. If we try to construct terms which are bilinear in spin matrices, they just reduce through the identity $\sigma_i\sigma_j = \delta_{ij} + \epsilon_{ijk}\sigma_k$, which we can already construct through combinations of the four operators we already have. Since those four operators form a basis for the space, independent bilinears were forbidden even without an explicit form for the equation.

What about spin-1? We need 9 independent operators to span the space. All the operators that exist in spin-1/2 will work here as well, though the spin matrices will have a different representation. That leaves 5 operators to construct. It is natural to try to construct these from bilinear combinations of spin matrices. Naively S_iS_j would itself be 9 independent structures, but clearly some of these are expressible in terms of the lower order operators.

Regardless of their representation, the spin operators always fulfill certain identities based on their Lie group. Namely

$$S_iS_j\delta_{ij} \sim I,\ [S_i, S_j] = \epsilon_{ijk}S_k \tag{2.0.24}$$

and it is these identities which allow certain combinations of S_iS_j to be related to lower order operators.

If instead of general spin bilinears we consider only combinations which are

- Symmetric in i, j
- Traceless

then such a structure will be independent of the set of operators $\{I, S_1, S_2, S_3\}$. This is a set of 4 constraints, so from the original 9 degrees of freedom possessed by combinations of S_iS_j are left only 5. Together with the 4 lower order operators this is exactly enough to span the space.

We can explicitly write this symmetric, traceless structure as

$$S_iS_j + S_jS_i + \delta_{ij}S^2 \tag{2.0.25}$$

Finally consider the general spin case, proceeding inductively using the same rough idea as for the case of spin-1. We construct all the possible operators from the set of all "lower order" operators which were used for lower spin particles, and by introducing new operators which are of higher degree in the spin matrices.

So suppose that for a spin-$s-1$ particle we have a set of operators written as $\bar{S}^0, \bar{S}^1, \ldots \bar{S}^{(s-1)}$, where a structure $\bar{S}^n$ carries n Galilean indices and is symmetric and traceless between any pair of indices, that is:

$$\bar{S}^n_{..i..j..} = \bar{S}^n_{..j..i..},\ \delta_{ij}\bar{S}^n_{..i..j..} = 0 \tag{2.0.26}$$

(From above, $\bar{S}^0 = I$, $\bar{S}^1_i = S_i$, and $\bar{S}^2 = S_iS_j + S_jS_i + \delta_{ij}S^2$.)

The objects $\bar{S}^n$ are built as follows: start with all combinations involving the product of exactly n spin matrices. (There are 3^n such structures.) Form them into combinations which are symmetric in all indices. Each index has three possible values, so we can label each structure by how many indices are equal to 1 and 2. If a is the number of indices equal to 1, and b the number of indices equal to 2, then for a given a there are $n+1-a$ possible choices for b. The total number of symmetric structures is then

$$\Sigma^n_{a=0}(n-a+1) = \frac{1}{2}(n+1)(n+2) \tag{2.0.27}$$

We have the additional constraint that the $\bar{S}^n$ be traceless in all indices. This is an additional constrain for each pair of indices, and there are $n(n-1)/2$ pairs of indices. The total degrees of freedom left are

$$\frac{1}{2}(n+1)(n+2)-\frac{1}{2}n(n-1)=\frac{1}{2}\left(n^2+3n+2-n^2+n\right)=2n+1 \tag{2.0.28}$$

In combination with the lower order spin operators, this is exactly the number of independent operators we need to span the space.

2.0.5 Scattering off external field in NRQED

We can write down those terms in the NRQED Lagrangian which have one power of the external field. This set of terms will not, by themselves, be gauge invariant. If, for example, a coefficient exists before a term with both one and two powers of A, we'll want to make sure we get the same result in both calculations. We'll add a superscript to the coefficient to keep track of this: so below we write c_S^1.

In writing the expansion of terms like $\mathbf{D}^4$ it is convenient to use anticommutators.

$$\begin{aligned}\mathcal{L}_A \quad = \quad & \Psi^\dagger(-eA_0-ie\frac{\{\nabla_i,A_i\}}{2m}-ie\frac{\{\boldsymbol{\nabla}^2,\{\nabla_i,A_i\}\}}{8m^3}+c_Fe\frac{\mathbf{S}\cdot\mathbf{B}}{2m}+c_D\frac{e(\boldsymbol{\nabla}\cdot\mathbf{E}-\mathbf{E}\cdot\boldsymbol{\nabla})}{8m^2}+c_Q\frac{eQ_{ij}(\nabla_iE_j-E_i\nabla_j)}{8m^2}\\ & +c_S^1\frac{ie\mathbf{S}\cdot(\boldsymbol{\nabla}\times\mathbf{E}-\mathbf{E}\times\boldsymbol{\nabla})}{8m^2}+c_{W_1}\frac{e[\boldsymbol{\nabla}^2(\mathbf{S}\cdot\mathbf{B})+(\mathbf{S}\cdot\mathbf{B})\boldsymbol{\nabla}^2]}{8m^3}-c_{W_2}\frac{e\nabla^i(\mathbf{S}\cdot\mathbf{B})\nabla^i}{4m^3}+c_{p'p}\frac{e[(\mathbf{S}\cdot\boldsymbol{\nabla})(\mathbf{B}\cdot\boldsymbol{\nabla})+(\mathbf{B}\cdot\boldsymbol{\nabla})(\mathbf{S}\cdot\boldsymbol{\nabla}}{8m^3}\end{aligned}$$

We want to calculate from this a particular process: scattering off an external field, with incoming momentum $\mathbf{p}$, outgoing $\mathbf{p}'$, and $\mathbf{q}=\mathbf{p}'-\mathbf{p}$. There is one diagram associated with each term above, but the total amplitude is just going to be the sum of all these one-photon vertices. These of course can just be read off directly from the Lagrangian: we replace the fields Ψ with the spinors ϕ, and any operator $\boldsymbol{\nabla}$ acting will become $i\mathbf{p}$ if it acts on the right, $i\mathbf{p}'$ if it is to the left.

We can simplify some expressions involving $\boldsymbol{\nabla}$ and $\mathbf{E}$: Because Q_{ij} is symmetric:

$$Q_{ij}(\nabla_iE_j-E_i\nabla_j)=Q_{ij}[\nabla_i,E_j]=Q_{ij}(\partial_iE_j)$$

And because $E_i=-\partial_i\Phi$

$$\boldsymbol{\nabla}\times\mathbf{E}-\mathbf{E}\times\boldsymbol{\nabla}=-2\mathbf{E}\times\boldsymbol{\nabla}$$

And also use that

$$\boldsymbol{\nabla}\cdot\mathbf{E}-\mathbf{E}\cdot\boldsymbol{\nabla}=(\partial_iE_i)$$

Now we can write down the scattering amplitude for scattering off the external field, before we apply any assumptions about the particular process.

$$\begin{aligned}iM \quad = \quad & ie\phi_S^\dagger\Bigg(-A_0+\frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')}{2m}-\frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')\mathbf{p}^2+\mathbf{p}'^2\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')}{8m^3}\\ & +c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m}+c_D\frac{(\partial_iE_i)}{8m^2}+c_Q\frac{Q_{ij}(\partial_iE_j)}{8m^2}+c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2}\\ & -c_{W_1}\frac{(\mathbf{S}\cdot\mathbf{B})(\mathbf{p}^2+\mathbf{p}'^2)}{8m^3}+c_{W_2}\frac{(\mathbf{S}\cdot\mathbf{B})(\mathbf{p}\cdot\mathbf{p}')}{4m^3}-c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p}')(\mathbf{B}\cdot\mathbf{p})+(\mathbf{B}\cdot\mathbf{p}')(\mathbf{S}\cdot\mathbf{p})}{8m^3}\Bigg)\phi_S\end{aligned}$$

The above can be simplified somewhat. We choose our gauge such that $\nabla_iA_i=0$. If we specify elastic scattering then kinematics dictate that $\mathbf{p}'^2=\mathbf{p}^2$. Finally, if we consider $\mathbf{B}$ constant, the c_W terms become indistinguishable, since $[\nabla_i,B_j]=0$. (It is only this last assumption that costs us any information.) Then the scattering amplitude, as calculated from $\mathcal{L}_{NRQED}$, is:

$$\begin{aligned}iM =& ie\phi_S^\dagger\Bigg(-A_0+\frac{\mathbf{A}\cdot\mathbf{p}}{m}-\frac{(\mathbf{A}\cdot\mathbf{p})\mathbf{p}^2}{2m^3}+c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m}+c_D\frac{(\partial_iE_i)}{8m^2}+c_Q\frac{Q_{ij}(\partial_iE_j)}{8m^2}\\ &+c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2}-(c_{W_1}-c_{W_2})\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{4m^3}-c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3}\Bigg)\phi_S\end{aligned} \tag{2.0.29}$$

Chapter 3

General Spin Formalism

Our ultimate goal is to calculate corrections to the g-factor of a loosely bound charged particle of arbitrary spin. Our strategy is to obtain an effective Lagrangian in the nonrelativistic limit.

We first consider features of a general-spin formalism in both the relativistic and nonrelativistic cases, and the connection between the wave functions of the free particles. Then we consider how constraints of the relativistic theory let us calculate scattering off an external field. Comparing this result to that done with an effective NRQED Lagrangian, we can obtain the coefficients of that Lagrangian for particles of general spin.

3.0.6 Spinors for general-spin charged particles

Relativistic bispinors

First we need to work out a formalism that will apply to the general spin case. We want to represent the spin state of the particles by an object that looks like a generalization of the Dirac bispinor.

It is easiest to start with the Dirac basis, where the upper and lower components of the bispinor are objects of opposite helicity, each transforming as an object of spin 1/2.

To that end define an object

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{3.0.1}$$

that we wish to have the appropriate properties. Each component should transform as a particle of spin s, but with opposite helicity. Under reflection the upper and lower components transform into each other.

Representations of the proper Lorentz group are spinors which are separately symmetric in dotted and undotted indices. If ξ is an object with p undotted and q dotted indices

$$\xi = \{\xi^{\alpha_1 \ldots \alpha_p}_{\dot{\beta}_1 \ldots \dot{\beta}_q}\} \tag{3.0.2}$$

Then this can be a representation of a particle of spin $s = (p+q)/2$.

We have some free choice in how to partition the dotted/undotted indices, and we cannot choose exactly the same scheme for all spin as long as both types of indices are present. However, we can make separately consistent choices for integral and half-integral spin. For integral spin we can say $p = q = s$, while for the half-integral case we'll choose $p = s + \frac{1}{2}$, $q = s - \frac{1}{2}$.

We want the ξ and η to transform as objects of opposite helicity. Under reflection they will transform into each other. So

$$\eta = \{\eta^{\beta_1 \ldots \beta_q}_{\dot{\alpha}_1 \ldots \dot{\alpha}_p}\} \tag{3.0.3}$$

In the rest frame of the particle, they will have clearly defined and identical properties under rotation. The rest frame spinors are equivalent to rank $2s$ nonrelativistic spinors. So the bispinor in the rest frame looks like

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_0 \\ \xi_0 \end{pmatrix} \tag{3.0.4}$$

where

$$\xi_0 = \{(\xi_0)_{\alpha_1 \dots \alpha_p \beta_1 \dots \beta_q}\} \tag{3.0.5}$$

and all indices are symmetric.

We can obtain the spinors in an arbitrary frame by boosting from the rest frame. The upper and lower components we have defined to have opposite helicity, and so will act in opposite ways under boost:

$$\xi = \exp\left(\frac{\boldsymbol{\Sigma} \cdot \phi}{2}\right)\xi_0, \qquad \eta = \exp\left(-\frac{\boldsymbol{\Sigma} \cdot \phi}{2}\right)\xi_0 \tag{3.0.6}$$

What form should the operator $\boldsymbol{\Sigma}$ have? Under an infinitesimal boost by a rapidity ϕ, a spinor with a single undotted index is transformed as

$$\xi_\alpha \to \xi'_\alpha = \left(\delta_{\alpha\beta} + \frac{\boldsymbol{\phi} \cdot \boldsymbol{\sigma}_{\alpha\beta}}{2}\right)\xi_\beta$$

while one with a dotted index will transform as

$$\xi_{\dot\alpha} \to \xi'_{\dot\alpha} = \left(\delta_{\dot\alpha\dot\beta} - \frac{\boldsymbol{\phi} \cdot \boldsymbol{\sigma}_{\dot\alpha\dot\beta}}{2}\right)\xi_{\dot\beta}$$

The infinitesimal transformation of a higher spin object with the first p indices undotted and the last q dotted would then be

$$\xi \to \xi' = \left(1 + \sum_{a=0}^{p} \frac{\boldsymbol{\sigma}_a \cdot \boldsymbol{\phi}}{2} - \sum_{a=p+1}^{p+q} \frac{\boldsymbol{\sigma}_a \cdot \boldsymbol{\phi}}{2}\right)\xi$$

where a denotes which spinor index of ξ is operated on.

If we define

$$\boldsymbol{\Sigma} = \sum_{a=0}^{p} \boldsymbol{\sigma}_a - \sum_{a=p+1}^{p+q} \boldsymbol{\sigma}_a \tag{3.0.7}$$

Then the infinitesimal transformations would be

$$\xi \to \xi' = \left(1 + \frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right)\xi$$

$$\eta \to \eta' = \left(1 - \frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right)\eta$$

So the exact transformation should be

$$\xi \to \xi' = \exp\left(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right)\xi$$

$$\eta \to \eta' = \exp\left(-\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right)\eta$$

Therefore, the bispinor of some particle boosted by $\boldsymbol{\phi}$ from rest will be

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \exp\left(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right)\xi_0 \\ \exp\left(\frac{-\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right)\xi_0 \end{pmatrix} \tag{3.0.8}$$

In dealing with the relativistic theory, we'll want a basis that separates the particle and antiparticle parts of the wave function. If we want the upper component to be the particle, then in the rest frame the lower component will vanish, and for low momentum will be small compared to the upper component. The unitary transformation which accomplishes this is

$$\Psi' = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

$$\phi = \frac{1}{\sqrt{2}}(\xi + \eta)$$

$$\chi = \frac{1}{\sqrt{2}}(\eta - \xi)$$

Which is equivalent to

$$\Psi' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Psi$$

Then,

$$\phi = \cosh\left(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi_0 \tag{3.0.9}$$

$$\chi = \sinh\left(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi_0 \tag{3.0.10}$$

Spinors for nonrelativistic theory

We also need to discuss the nonrelativistic, single-particle theory. This is much simpler: the spin state of particles in this theory is represented by symmetric spinors with $2s + 1$ undotted indices. The only operators we need to consider acting on this space are spin matrices and products of spin matrices.

3.0.7 Electromagnetic Interaction

Knowing how the wave functions themselves behave, we want to see what that tells us about possible electromagnetic interaction. Interaction with a single electromagnetic photon should take the form

$$M = A_\mu j^\mu$$

where j^μ is the electromagnetic current.

The electromagnetic current must be built out of the particle's momenta and bilinears of the charged particle fields in such a way that they have the correct Lorentz properties. We must also demand current conservation: the equation $q_\mu j^\mu = 0$ must hold. Above we already have shown that, in the case of general spin, there exist only two such bilinears, a scalar and a tensor.

There will be two permissible terms in the current. We could consider a scalar bilinear coupled with a single power of external momenta. In order to fulfill the current conservation requirement, it should be

$$\frac{p^\mu + p'^\mu}{2m} \bar{\Psi}^\dagger \Psi$$

This will obey current conservation because $q = p' - p$, and $(p + p') \cdot (p' - p) = p^2 - p'^2 = 0$

We can also consider a tensor term contracted with a power of momenta. To fulfill current conservation, we can demand that the tensor bilinear be antisymmetric, and contract it with q:

$$\frac{q_\nu}{2m} \bar{\Psi}^\dagger \Sigma^{\mu\nu} \Psi$$

We don't need to worry about higher order tensor bilinears: they will necessitate too many powers of the external momenta.

So the most general current would look like

$$j^\mu = F_e \frac{p^\mu + p'^\mu}{2m} \bar{\Psi}^\dagger \Psi + F_m \frac{q_\nu}{2m} \bar{\Psi}^\dagger \Sigma^{\mu\nu} \Psi \tag{3.0.11}$$

In general the form factors might have quite complicated dependence on q, but these corrections will be too small compared to the type of result we're interested in. At leading order F_e will just be the electric charge of the particle in question, and F_m will, as we'll see after connecting this result to the nonrelativistic limit, be related to the particle's g-factor. So to the order we need, we can write the current as

$$j^\mu = e \frac{p^\mu + p'^\mu}{2m} \bar{\Psi}^\dagger \Psi + eg \frac{q_\nu}{2m} \bar{\Psi}^\dagger \Sigma^{\mu\nu} \Psi \tag{3.0.12}$$

This captures the essence of the interaction between a charged particle of general-spin and a single photon.

Connection between the spinors of the two theories

Knowing something of how the relativistic theory behaves, we can find the connection between the relativistic and nonrelativistic spinors. In the rest frame, there are two independent bispinors which represent particle and antiparticle states:

$$\Psi = \begin{pmatrix} \xi_0 \\ 0 \end{pmatrix}$$

or

$$\Psi = \begin{pmatrix} 0 \\ \xi_0 \end{pmatrix}$$

However, when we consider a particle with zero momentum it is not the case that the upper component of the bispinor can be directly associated with the Schrodinger like wave-function of the particle — for instance, it would not be correctly normalized, for there is some mixing with the lower component.

We can obtain a relation between ξ_0 and the Schrodinger amplitude ϕ_s by considering the current density at zero momentum transfer. For ϕ_s it will be $j_0 = \phi_s^\dagger \phi$. For the relativistic theory we have, as calculated above:

$$j^0 = F_e \frac{p^0 + p'^0}{2m} \bar{\Psi}^\dagger \Psi + F_m \frac{q_\nu}{2m} \bar{\Psi}^\dagger T^{0\nu} \Psi$$

At $q = 0$ the expression simplifies

$$j^0(q = 0) = F_e \frac{p_0}{m} \bar{\Psi}^\dagger \Psi$$

$$= F_e \frac{p_0}{m} (\phi^\dagger \phi - \chi^\dagger \chi)$$

ϕ and χ are both related to the rest frame spinor ξ_0. So we can write instead

$$j^0 = F_e \frac{p_0}{m} \xi_0^\dagger \left\{ \cosh^2(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}) - \sinh^2(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}) \right\} \xi_0 = F_e \frac{p_0}{m} \xi_0^\dagger \xi_0$$

where the last equality follows from the hyperbolic trig identity.

If we demand that the two current densities be equal to each other, we find

$$\frac{p_0}{m} \xi_0^\dagger \xi_0 = \phi_s^\dagger \phi_s$$

Approximating

$$\left(1 + \frac{\mathbf{p}^2}{2m}\right) \xi_0^\dagger \xi_0 = \phi_s^\dagger \phi_s$$

This will hold to the necessary order if we identify

$$\xi_0 = \left(1 - \frac{\mathbf{p}^2}{4m}\right)\phi_s$$

To write the relativistic bispinors in terms of ϕ_s we will also need approximations to $\cosh(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2})$ and $\sinh(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2})$. We only need the rapidity to the leading order: $\boldsymbol{\phi} \approx \mathbf{v} \approx \frac{\mathbf{p}}{m}$.

$$\cosh(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2}) \approx 1 + \frac{1}{2}\left(\frac{\boldsymbol{\Sigma}\cdot\mathbf{p}}{2m}\right)^2$$
$$\sinh(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2}) \approx \frac{\boldsymbol{\Sigma}\cdot\mathbf{p}}{2m}$$

The the two bispinor components are

$$\begin{aligned}\phi &\approx \left(1 + \left[\frac{1}{2}\frac{\boldsymbol{\Sigma}\cdot\mathbf{p}}{2m}\right]^2\right)\xi_0 \\ &\approx \left(1 + \frac{(\boldsymbol{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m}\right)\phi_S \qquad (3.0.13)\\ \chi &\approx \frac{\boldsymbol{\Sigma}\cdot\mathbf{p}}{2m}\xi_0 \\ &\approx \frac{\boldsymbol{\Sigma}\cdot\mathbf{p}}{2m}\phi_S \qquad (3.0.14)\end{aligned}$$

3.0.8 Bilinears in terms of nonrelativistic theory

The next step is to express the relativistic bilinears, built out of the bispinors Ψ, in terms of the Schrodinger like wave functions.

We have above written the bispinors in terms of ϕ_s, so we can use those identities to express the bilinears in the same manner.

Scalar bilinear

$$\begin{aligned}\bar{\Psi}^\dagger(p')\Psi(p) &= \phi^\dagger\phi - \chi^\dagger\chi \\ &= \phi_s^\dagger\left[1 + \frac{(\boldsymbol{\Sigma}\cdot\mathbf{p}')^2}{8m^2} - \frac{\mathbf{p}'^2}{4m^2}\right]\left[1 + \frac{(\boldsymbol{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m^2}\right]\phi_s - \phi_s^\dagger\left[\frac{(\boldsymbol{\Sigma}\cdot\mathbf{p}')(\boldsymbol{\Sigma}\cdot\mathbf{p})}{4m^2}\right]\phi_s \\ &= \phi_s^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{(\boldsymbol{\Sigma}\cdot\mathbf{p}')^2 + (\boldsymbol{\Sigma}\cdot\mathbf{p})^2 - 2(\boldsymbol{\Sigma}\cdot\mathbf{p}')(\boldsymbol{\Sigma}\cdot\mathbf{p})\right\}\right)\phi_s \\ &= \phi_s^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{[\boldsymbol{\Sigma}\cdot\mathbf{p}, \boldsymbol{\Sigma}\cdot\mathbf{q}] + (\boldsymbol{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_s \\ &= \phi_s^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{[4i\epsilon_{ijk}p_i q_j S_k + (\boldsymbol{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_s\end{aligned}$$

Tensor ij component

In calculating the nonrelativistic limit of the antisymmetric tensor bilinear, we will treat the $0i$ and the ij components seperately. First let us consider $\bar{\Psi}\Sigma_{ij}\Psi$.

$$
\begin{aligned}
\bar{\Psi}\Sigma_{ij}\Psi &= \bar{\Psi}(-2\epsilon_{ijk}S_k)\Psi \\
&= -2i\epsilon_{ijk}(\phi^\dagger S_k\phi - \chi^\dagger S_k\chi) \\
&= -2i\epsilon_{ijk}\Big(\phi_S^\dagger\left[1+\frac{(\mathbf{\Sigma}\cdot\mathbf{p}')^2}{8m^2}-\frac{\mathbf{p}'^2}{4m^2}\right]S_k\left[1+\frac{(\mathbf{\Sigma}\cdot\mathbf{p})^2}{8m^2}-\frac{\mathbf{p}^2}{4m^2}\right]\phi_S - \phi_S^\dagger\frac{(\mathbf{\Sigma}\cdot\mathbf{p}')S_k(\mathbf{\Sigma}\cdot\mathbf{p})}{4m^2}\phi_S\Big) \\
&= -2i\epsilon_{ijk}\phi_S^\dagger\left\{S_k\left(1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right)+\frac{1}{8m^2}\Big[(\mathbf{\Sigma}\cdot\mathbf{p}')^2S_k+S_k(\mathbf{\Sigma}\cdot\mathbf{p})^2-2(\mathbf{\Sigma}\cdot\mathbf{p}')S_k(\mathbf{\Sigma}\cdot\mathbf{p})\Big]\right\}\phi_S
\end{aligned}
$$

We want to write the terms in square brackets explicitly in terms of $\mathbf{p}$ and $\mathbf{q}$.

$$
\begin{aligned}
(\mathbf{\Sigma}\cdot\mathbf{p}')^2S_k+S_k(\mathbf{\Sigma}\cdot\mathbf{p})^2-2(\mathbf{\Sigma}\cdot\mathbf{p}')S_k(\mathbf{\Sigma}\cdot\mathbf{p}) &= (\mathbf{\Sigma}\cdot\mathbf{p})^2S_k+S_k(\mathbf{\Sigma}\cdot\mathbf{p})-2(\mathbf{\Sigma}\cdot\mathbf{p})S_k(\mathbf{\Sigma}\cdot\mathbf{p}) \\
&\quad +\{(\mathbf{\Sigma}\cdot\mathbf{p})(\mathbf{\Sigma}\cdot\mathbf{q})+(\mathbf{\Sigma}\cdot\mathbf{q})(\mathbf{\Sigma}\cdot\mathbf{p})\}S_k \\
&\quad -2(\mathbf{\Sigma}\cdot\mathbf{q})S_k(\mathbf{\Sigma}\cdot\mathbf{p})+(\mathbf{\Sigma}\cdot\mathbf{q})^2S_k
\end{aligned}
$$

We can express many of these terms as commutators

$$= \mathbf{\Sigma}\cdot\mathbf{p}[\mathbf{\Sigma}\cdot\mathbf{p},S_k]+[S_k,\mathbf{\Sigma}\cdot\mathbf{p}]\mathbf{\Sigma}\cdot\mathbf{p}+2\mathbf{\Sigma}\cdot\mathbf{q}[\mathbf{\Sigma}\cdot\mathbf{p},S_k]-[\mathbf{\Sigma}\cdot\mathbf{q},S_k]\mathbf{\Sigma}\cdot\mathbf{p}+(\mathbf{\Sigma}\cdot\mathbf{q})^2S_k$$

$$= i\epsilon_{ijk}p_j\{(\mathbf{\Sigma}\cdot\mathbf{p})\Sigma_i-\Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})\}+2i\epsilon_{ijk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_ip_j-\Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})q_j)\}+(\mathbf{\Sigma}\cdot\mathbf{q})^2S_k$$

$$= i\epsilon_{ijk}p_j[(\mathbf{\Sigma}\cdot\mathbf{p}),\Sigma_i]+2i\epsilon_{ijk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_ip_j-\Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})q_j)\}+(\mathbf{\Sigma}\cdot\mathbf{q})^2S_k$$

$$= 4(\mathbf{p}^2S_k-(\mathbf{S}\cdot\mathbf{p})p_k)+2i\epsilon_{ijk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_ip_j-\Sigma_i(\mathbf{\Sigma}\cdot\mathbf{p})q_j)\}+(\mathbf{\Sigma}\cdot\mathbf{q})^2S_k$$

Thus the whole bilinear is

$$-2i\epsilon_{ijk}\phi_S^\dagger\left\{S_k\left(1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right)+\frac{1}{8m^2}\Big[4(\mathbf{p}^2S_k-(\mathbf{S}\cdot\mathbf{p})p_k)+2i\epsilon_{\ell mk}\{(\mathbf{\Sigma}\cdot\mathbf{q})\Sigma_\ell p_m-\Sigma_\ell(\mathbf{\Sigma}\cdot\mathbf{p})q_m)\}+(\mathbf{\Sigma}\cdot\mathbf{q})^2S_k\Big]\right\}\phi_S \tag{3.0.15}$$

Tensor Σ_{0i} component

We calculate $\bar{\Psi}\Sigma_{0i}\Psi$.

$$\bar{\Psi}\Sigma_{0i}\Psi = \bar{\Psi}\begin{pmatrix}0 & \Sigma_i\\ \Sigma_i & 0\end{pmatrix}\Psi$$

$$= \phi^\dagger\Sigma_i\chi-\chi^\dagger\Sigma_i\phi$$

We'll only need ϕ and χ to first order here.

$$= \phi_S^\dagger\left(\frac{\Sigma_i\Sigma_jp_j-\Sigma_j\Sigma_ip'j}{2m}\right)\phi_S$$

Using $p'=p+q$ the terms involving only p can be simplified using the commutator of Σ matrices.

$$= \phi_S^\dagger\left(\frac{4i\epsilon_{ijk}p_jS_k-\Sigma_j\Sigma_iq_j}{2m}\right)\phi$$

3.0.9 Current in terms of nonrelativistic wave functions

We derived the four-current (3.0.11) above; in nonrelativistic notation it is:

$$j_0 = F_e \frac{p_0 + p_0'}{2m} \bar{\Psi}^\dagger \Psi - F_m \frac{q_j}{2m} \bar{\Psi}^\dagger \Sigma^{0j} \Psi \tag{3.0.16}$$

$$j_i = F_e \frac{p_i + p_i'}{2m} \bar{\Psi}^\dagger \Psi - F_m \frac{q_j}{2m} \bar{\Psi}^\dagger \Sigma^{ij} \Psi + F_m \frac{q_0}{2m} \bar{\Psi}^\dagger \Sigma^{i0} \Psi \tag{3.0.17}$$

We have expressions for the bilinears in terms of the nonrelativistic wave functions ϕ_S, so it is fairly straight forward to apply them here. The calculation of j_0 is straightforward:

$$\begin{aligned}
F_e \frac{p_0 + p_0'}{2m} \bar{\Psi}^\dagger \Psi &= F_e \left(1 + \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) \phi_S^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2} \left\{4i\epsilon_{ijk} p_i q_j S_k + (\mathbf{\Sigma} \cdot \mathbf{q})^2\right\}\right) \phi_S \\
&\approx F_e \phi_S^\dagger \left(1 + \frac{1}{8m^2} \left\{4i \mathbf{S} \cdot \mathbf{p} \times \mathbf{q} + (\mathbf{\Sigma} \cdot \mathbf{q})^2\right\}\right) \phi_S \\
F_m \frac{q_j}{2m} \bar{\Psi}^\dagger \Sigma^{0j} \Psi &= F_m \frac{q_i}{2m} \phi_S^\dagger \left(\frac{4i\epsilon_{ijk} p_j S_k - \Sigma_j \Sigma_i q_j}{2m}\right) \phi_S \\
&= F_m \phi_S^\dagger \left(\frac{4i \mathbf{S} \cdot \mathbf{q} \times \mathbf{p} - (\mathbf{\Sigma} \cdot \mathbf{q})^2}{4m^2}\right) \phi_S
\end{aligned}$$

It turns out that both terms here have the same form, so combining them we get

$$j_0 = \phi_S^\dagger \left(F_e + \frac{F_e + 2F_m}{8m^2} \left\{4i \mathbf{S} \cdot \mathbf{p} \times \mathbf{q} + (\mathbf{\Sigma} \cdot \mathbf{q})^2\right\}\right) \phi_S \tag{3.0.18}$$

To calculate j_i we want to first simplify things by considering the constraints of our particular problem. The term with $\Sigma_i j$ can be simplified by dropping terms with more than one power of q; these will turn into derivatives of the magnetic field, and our problem concerns only a constant field. Further, we need only calculate elastic scattering, and so $q_0 = 0$. With those simplifications

$$\bar{\Psi} \Sigma_{ij} \Psi \approx -2i\epsilon_{ijk} \phi_S^\dagger \left\{ S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p}) p_k}{2m^2} \right\} \phi_S$$

$$\begin{aligned}
F_e \frac{p_i + p_i'}{2m} \bar{\Psi}^\dagger \Psi &= F_e \frac{p_i + p_i'}{2m} \phi_S^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2} \left\{4i\epsilon_{\ell jk} p_\ell q_j S_k + (\mathbf{\Sigma} \cdot \mathbf{q})^2\right\}\right) \phi_S \\
&\approx F_e \frac{p_i + p_i'}{2m} \phi_S^\dagger \left(1 + \frac{1}{8m^2} \left\{4i\epsilon_{\ell jk} p_\ell q_j S_k\right\}\right) \phi_S \\
F_m \frac{q_j}{2m} \bar{\Psi}^\dagger \Sigma^{ij} \Psi &= -F_m \frac{i\epsilon_{ijk} q_j}{m} \phi_S^\dagger \left\{ S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p}) p_k}{2m^2} \right\} \phi_S
\end{aligned}$$

So the full spatial part of the current is

$$j_i = \phi_S^\dagger \left\{ F_e \frac{p_i + p_i'}{2m} \left(1 + \frac{i\epsilon_{\ell jk} p_\ell q_j S_k}{2m^2}\right) + F_m \frac{i\epsilon_{ijk} q_j}{m} \left(S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p}) p_k}{2m^2} \right) \right\} \phi_S \tag{3.0.19}$$

3.0.10 Scattering off external field

To compare to the NRQED Lagrangian, we want to calculate scattering off an external field for an arbitrary spin particle. We already have the current, so the scattering is just

$$M = ej_\mu A^\mu = ej_0 A_0 - e\mathbf{j}\cdot\mathbf{A}$$

Above we have expressions for both j_0 (3.0.18) and $\mathbf{j}$ (3.0.19). So we can write down the parts of the amplitude directly:

$$\begin{aligned} ej_0A_0 =& eA_0\phi_S^\dagger\left(F_e + \frac{F_e+2F_m}{8m^2}\left\{4i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q} + (\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_S \\ e\mathbf{j}\cdot\mathbf{A} =& A_i\phi_S^\dagger\left\{F_e\frac{p_i+p_i'}{2m}\left(1+\frac{i\epsilon_{\ell jk}p_\ell q_j S_k}{2m^2}\right) + F_m\frac{i\epsilon_{ijk}q_j}{m}\left(S_k\left(1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right)+\frac{\mathbf{p}^2S_k-(\mathbf{S}\cdot\mathbf{p})p_k}{2m^2}\right)\right\}\phi_S \end{aligned} \tag{3.0.20}$$

As much as possible we want to express the result in terms of gauge invariant quantities $\mathbf{B}$ and $\mathbf{E}$. We write the relations between these fields and A_μ in position space and the equivalent equation in momentum space.

$$\begin{aligned} \mathbf{B} = \mathbf{\nabla}\times\mathbf{A} \to i\mathbf{q}\times\mathbf{A} \\ \mathbf{E} = -\mathbf{\nabla}A_0 \to -i\mathbf{q}A_0 \end{aligned}$$

There is one term above that can only be put into gauge-invariant form by considering the kinematic constraints of elastic scattering. If the scattering is elastic, we have $\mathbf{q}\cdot(\mathbf{p}'+\mathbf{p}) = \mathbf{p}'^2 - \mathbf{p}^2 = 0$. We can use this identity on the term $q_j(p_i'+p_i)A_i$ as follows:

$$\begin{aligned} \epsilon_{ijk}B_k &= \partial_i A_j - \partial_j A_i = i(q_iA_j - q_jA_i) \\ (p_i+p_i')\epsilon B_k &= i(p_i+p_i')(q_iA_j - q_jA_i) \\ &= -i(p_i+p_i')q_jA_i \end{aligned}$$

So we have the identity

$$i(p_i+p_i')q_jA_i = -\epsilon_{ijk}B_k(p_i+p_i') \tag{3.0.21}$$

Now we can write each term involving q in terms of position space quantities.

$$\begin{aligned} i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q}A_0 &= -\mathbf{S}\cdot\mathbf{p}\times\mathbf{E} \\ (\mathbf{\Sigma}\cdot\mathbf{q})^2A_0 &= \Sigma_i\Sigma_j q_iq_jA_0 \\ &= \Sigma_i\Sigma_j\partial_iE_j \\ i\epsilon_{ijk}A_iq_j &= -i(\mathbf{q}\times\mathbf{A})_k \\ &= = -B_k \\ A_i(p_i+p_i')i\epsilon_{\ell jk}p_\ell q_jS_k &= \epsilon_{\ell jk}p_\ell S_k i(p_i+p_i')q_jA_i \\ &= -\epsilon_{\ell jk}p_\ell S_k\{\epsilon_{ijm}B_m(p_i+p_i')\} \\ &= -(\delta_{\ell i}\delta_{km} - \delta\ell m\delta_{ik})p_\ell S_k\{\epsilon_{ijm}B_m(p_i+p_i')\} \\ &= 2\{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p}) - (\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2\} \end{aligned}$$

Using these

$$ej_0A_0 = e\phi_S^\dagger\left\{A_0 + \frac{1-2F_2}{8m^2}\left(4\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \Sigma_i\Sigma_j\partial_iE_j\right)\right\}$$

$$e\mathbf{j}\cdot\mathbf{A} = e\phi_S^\dagger\left\{\frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})-(\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2}{m^2} - F_m\left(\frac{\mathbf{S}\cdot\mathbf{B}}{m}\left\{1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right\} + \frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})-(\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2}{2m^2}\right)\right\}\phi$$

$$= e\phi_S^\dagger \left\{ \frac{\mathbf{p}\cdot\mathbf{A}}{m} + [1-2F_m]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{F_m}{m}\mathbf{S}\cdot\mathbf{B} \right\}$$

From this we can see that our F_m is actually $g/2$, so in such terms

$$ej_0 A_0 = e\phi_S^\dagger \left\{ A_0 - \frac{g-1}{2m^2}\left(\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \frac{1}{4}\Sigma_i\Sigma_j\partial_i E_j \right) \right\}$$

$$e\mathbf{j}\cdot\mathbf{A} = e\phi_S^\dagger \left\{ \frac{\mathbf{p}\cdot\mathbf{A}}{m} - [g-1]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{g}{2m}\mathbf{S}\cdot\mathbf{B} \right\}$$

So the entire scattering process is

$$e\phi_S^\dagger \left\{ A_0 - \frac{g-1}{2m^2}\left(\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \frac{1}{4}\Sigma_i\Sigma_j\partial_i E_j \right) \frac{\mathbf{p}\cdot\mathbf{A}}{m} - [g-1]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{g}{2m}\mathbf{S}\cdot\mathbf{B} \right\} \phi_S \tag{3.0.22}$$

3.0.11 Comparison with relativistic result

Having calculated the same process in both the relativistic theory and in our NRQED effective theory, we can compare the two amplitudes and fix the coefficients.

The NRQED amplitude (2.0.29) is

$$\begin{aligned} iM &= ie\phi^\dagger \Bigg(-A_0 + \frac{\mathbf{A}\cdot\mathbf{p}}{m} - \frac{(\mathbf{A}\cdot\mathbf{p})\mathbf{p}^2}{2m^3} + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2} + c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2} \\ &\quad + c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2} - (c_{W_1} - c_{W_2})\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3} \Bigg)\phi \end{aligned}$$

While the relativistic amplitude was

$$iM_{REL} = -ie\phi^\dagger \Big(A_0 - \frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{\mathbf{p}\cdot\mathbf{A}\mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3}\{\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - S_iS_j\boldsymbol{\nabla}_i E_j\} - g\frac{1}{2m}\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \Big)\phi$$

We should rewrite term $\boldsymbol{\nabla}\cdot\mathbf{E} - S_iS_j\boldsymbol{\nabla}_i E_j$ using the quadropole moment tensor $Q_{ij} = \frac{1}{2}(S_iS_j + S_jS_i - \frac{2}{3}\mathbf{S}^2)$.

Remember that $\nabla_i E_j$ is actually symmetric under exchange of i and j. Then we can write

$$\begin{aligned} S_iS_j\nabla_i E_j &= \frac{1}{2}(S_iS_j + S_jS_i) = (Q_{ij} + \frac{1}{3}\mathbf{S}^2\delta_{ij})\nabla_i E_j \\ &= Q_ij\nabla_i E_j + \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{E} \end{aligned}$$

Written using this identity, the relativistic amplitude is

$$iM_{REL} = -ie\phi^\dagger \Big(A_0 - \frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{\mathbf{p}\cdot\mathbf{A}\mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3}\{\frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - Q_{ij}\boldsymbol{\nabla}_i E_j\} - g\frac{1}{2m}\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \Big)\phi$$

Comparing the two, the coefficients are:

$$\begin{aligned} c_F &= g \\ c_D &= \frac{4(g-1)}{3} \\ c_Q &= -4(g-1) \\ c_S^1 &= 2(g-1) \\ (c_{W_1} - c_{W_2}) &= 2 \\ c_{p'p} &= (g-2) \end{aligned}$$

3.0.12 Exact equations of motion for spin-1

$$\mathcal{L} = -\frac{1}{2}(D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) + m^2 W^{\mu\dagger} W_\mu - i\lambda e W^{\mu\dagger} W^\nu F_{\mu\nu} \tag{3.0.23}$$

where as usual D is the long derivative $D^\mu = \partial^\mu + ieA^\mu$.

Obtain the equations of motion from the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial[\partial_\mu W^{\dagger\alpha}]} = 0$$

Or equivalently,

$$\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} - D_\mu \frac{\partial \mathcal{L}}{\partial[D_\mu W^{\dagger\alpha}]} = 0$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} &= \frac{\partial}{\partial W^{\dagger\alpha}} \left(m^2 W^{\mu\dagger} W_\mu - i\lambda e W^{\mu\dagger} W^\nu F_{\mu\nu} \right) \\
&= m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial[D_\gamma W^{\dagger\alpha}]} &= -\frac{1}{2} \frac{\partial}{\partial[D_\gamma W^{\dagger\alpha}]} (D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) \\
&= -\frac{1}{2} (g^{\mu\gamma} g^\nu_\alpha - g^{\nu\gamma} g^\mu_\alpha)(D_\mu W_\nu - D_\nu W_\mu) \\
&= D_\alpha W^\gamma - D^\gamma W_\alpha
\end{aligned}$$

So the complete equation from Euler-Lagrange is

$$m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} + D_\mu (D^\mu W_\alpha - D_\alpha W^\mu) = 0 \tag{3.0.24}$$

This is a set of four coupled second order equations for the field W. We rewrite as a set of first order equations by introducing a field $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$. So (3.0.24) becomes

$$m^2 W_\alpha - ie\lambda W^\mu F_{\alpha\mu} + D^\mu W_{\mu\alpha} = 0 \tag{3.0.25}$$

$W^{\mu\nu}$ is antisymmetric and so has six degrees of freedom, corresponding to six independent fields. Together with W^μ this represents a total of ten fields. However, upon examination only some of these fields are dynamic. The fields W^{0i} and W^i appear in the equations with time derivatives, while the fields W^{ij} and W^0 never do. So it is only necessary to consider the former six fields. So that these six fields all have the same dimension, we will define $\frac{W^{i0}}{m} = i\eta^i$.

We will now eliminate the extraneous fields and solve for $iD_0 W^i$, $iD_0 \eta^i$.

$$m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} + D_\mu (D^\mu W_\alpha - D_\alpha W^\mu) = 0 \tag{3.0.26}$$

Define $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$. Then

$$m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} + D^\mu W_{\mu\alpha} = 0 \tag{3.0.27}$$

To get the exact Hamiltonian of a bispinor, we can eliminate nondynamic fields.

First consider (3.0.27) with $\alpha = 0$.

$$m^2 W_0 - ie\lambda W^\nu F_{0\nu} + D^\mu W_{\mu 0} \tag{3.0.28}$$

$$m^2 W_0 - ie\lambda W^j F_{0j} + D^j W_{j0} \tag{3.0.29}$$

Solve this for W_0

$$W_0 = \frac{1}{m^2}\left(ie\lambda W^j F_{0j} - D^j W_{j0}\right) \tag{3.0.30}$$

Now, consider (3.0.27) with $\alpha = i$

$$m^2 W_i - ie\lambda W^\mu F_{i\mu} + D^\mu W_{\mu i} = 0 \tag{3.0.31}$$

$$m^2 W_i - ie\lambda W^0 F_{i0} + D^0 W_{0i} - ie\lambda W^j F_{ij} + D^j W_{ji} = 0 \tag{3.0.32}$$

Using (3.0.30) we can replace W_0

$$m^2 W_i - \frac{ie\lambda}{m^2}\left(ie\lambda W^\nu F_{0\nu} - D^j W_{j0}\right) F_{i0} + D^0 W_{0i} - ie\lambda W^j F_{ij} + D^j W_{ji} = 0 \tag{3.0.33}$$

Using $W_{ji} = D_j W_i - D_i W_j$

$$m^2 W_i - \frac{ie\lambda}{m^2}\left(ie\lambda W^\nu F_{0\nu} - D^j W_{j0}\right) F_{i0} + D^0 W_{0i} - ie\lambda W^j F_{ij} + D^j (D_j W_i - D_i W_j) = 0 \tag{3.0.34}$$

Solve this for $D^0 W_{0i}$:

$$D^0 W_{0i} = -m^2 W_i + \frac{ie\lambda}{m^2}\left(ie\lambda W^\nu F_{0\nu} - D^j W_{j0}\right) F_{i0} + ie\lambda W^j F_{ij} - D^j (D_j W_i - D_i W_j) \tag{3.0.35}$$

To get a similar equation for W_i, consider

$$W_{i0} = D_i W_0 - D_0 W_i \tag{3.0.36}$$

Then

$$D^0 W_i = D_i W_0 - W_{i0} \tag{3.0.37}$$

$$D^0 W_i = D_i \frac{1}{m^2}\left(ie\lambda W^j F_{0j} - D^j W_{j0}\right) - W_{i0} \tag{3.0.38}$$

We now have equations that tell us the time evolution of a total of six fields: W_i and W_{i0}. We want to treat these as the components of some sort of bispinor. To that end, first define $\eta_i = -i/m W_{i0}$ so that we have a pair of fields with the same mass dimension and hermiticity. (Since $W_{i0} = D_i W_0 - D_0 W_i$ would pick up another minus sign under complex conjugation compared to W_i.) Going the other way $W_{i0} = im\eta_i$.

Since eventually we want a nonrelativistic expression, we should write the spatial components of four vectors as three vectors. To that end also write the components of the tensor $F_{\mu\nu}$ in terms of three vectors.

$$F_{0i} = E^i, \; F_{ij} = -\epsilon_{ijk} B^k$$

Regular spatial vectors are "naturally raised" while D_i is "naturally lowered". Then we can rewrite (3.0.35).

$$-imD_0\eta_i = -m^2 W_i - \frac{ie\lambda}{m^2}\left(ie\lambda W^j E^j - D^j W_{j0}\right) E^i + ie\lambda W^j \epsilon_{ijk} B^k - D^j (D_j W_i - D_i W_j) \tag{3.0.39}$$

$$iD_0\eta^i = mW^i + \frac{ie\lambda}{m^3}\left(ie\lambda W^j E^j - D^j W_{j0}\right) E^i - \frac{ie\lambda}{m} W^j \epsilon_{ijk} B^k + \frac{1}{m} D_j (D_j W^i - D_i W^j) \tag{3.0.40}$$

And likewise (3.0.38) becomes

$$D^0 W_i = D_i \frac{1}{m^2}\left(-ie\lambda W^j E^j - imD^j \eta_j\right) - im\eta_i \tag{3.0.41}$$

$$iD^0 W^i = -iD_i \frac{1}{m^2}\left(-ie\lambda W^j E^j - imD_j \eta^j\right) + m\eta^i \tag{3.0.42}$$

$$iD^0 W^i = -\frac{1}{m^2} D_i e\lambda \mathbf{W}\cdot\mathbf{E} - \frac{1}{m} D_i \mathbf{D}\cdot\eta + m\eta^i \tag{3.0.43}$$

Spin identities

To obtain some Schrodinger like equation for a bispinor, we need to express the time derivative of the bispinor in terms of other operators which can be interpreted as the Hamiltonian: $i\partial_0\Psi = \hat{H}\Psi$. We have expressions for the time derivatives of W_i and η_i, but some mixing of the indices is involved. To write the Hamiltonian in a block form we can introduce spin matrices whose action will mix the components of the fields.

The spin matrix for a spin one particle can represented as:

$$(S^k)_{ij} = -i\epsilon_{ijk} \tag{3.0.44}$$

which leads to the following identities:

$$(\mathbf{a}\times\mathbf{v})_i = -i(\mathbf{S}\cdot\mathbf{a})_{ij}v_j \tag{3.0.45}$$

$$\{\mathbf{a}\times(\mathbf{a}\times\mathbf{v})\}_i = -(\{\mathbf{S}\cdot\mathbf{a}\}^2)_{ij}v_j \tag{3.0.46}$$

$$a_i(\mathbf{b}\cdot\mathbf{v}) = \{\mathbf{a}\cdot\mathbf{b}\,\delta_{ij} - (S^kS^\ell)_{ij}a^\ell b^k\}v_j \tag{3.0.47}$$

Using these identities

$$iD^0\mathbf{W} = -\frac{e\lambda}{m^2}\mathbf{D}(\mathbf{W}\cdot\mathbf{E}) + \frac{1}{m}\mathbf{D}(\mathbf{D}\cdot\boldsymbol{\eta}) + m\boldsymbol{\eta} \tag{3.0.48}$$

$$iD^0\mathbf{W} = -\frac{1}{m^2}\{\mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\}\mathbf{W} + \frac{1}{m}\{\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2\}\boldsymbol{\eta} + m\boldsymbol{\eta} \tag{3.0.49}$$

and for the other equation

$$iD_0\eta^i = mW^i - \frac{ie\lambda}{m^3}\left(ie\lambda W^jE^j - D^jW_{j0}\right)E^i + \frac{ie\lambda}{m}W^j\epsilon_{ijk}B^k - \frac{1}{m}D_j(D_jW^i - D_iW^j) \tag{3.0.50}$$

$$iD_0\eta^i = mW^i - \frac{ie\lambda}{m^3}\left(ie\lambda W^jE^j - D^jW_{j0}\right)E^i + \frac{ie\lambda}{m}W^j\epsilon_{ijk}B^k - \frac{1}{m}D_j(D_jW^i - D_iW^j) \tag{3.0.51}$$

$$iD_0\boldsymbol{\eta} = m\mathbf{W} + \frac{e^2\lambda^2}{m^3}\mathbf{E}(\mathbf{W}\cdot\mathbf{E}) - \frac{1}{m^2}\mathbf{E}(\mathbf{D}\cdot\boldsymbol{\eta}) + \frac{ie\lambda}{m}(\mathbf{W}\times\mathbf{B}) + \frac{1}{m}\mathbf{D}\times(\mathbf{D}\times\mathbf{W}) \tag{3.0.52}$$

$$iD_0\boldsymbol{\eta} = m\mathbf{W} + \frac{e^2\lambda^2}{m^3}\{\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\}\mathbf{W} - \frac{1}{m^2}\{\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\}\boldsymbol{\eta} + \frac{e\lambda}{m}(\mathbf{S}\cdot\mathbf{B})\mathbf{W} - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2\mathbf{W} \tag{3.0.53}$$

$$iD_0\boldsymbol{\eta} = \left(m + \frac{e^2\lambda^2}{m^3}\{\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\} - \frac{e\lambda}{m}(\mathbf{S}\cdot\mathbf{B}) - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2\right)\mathbf{W} + \frac{1}{m^2}\{\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\}\boldsymbol{\eta} \tag{3.0.54}$$

Now that the equations are in the correct form, we can write them as follows:

$$iD_0\begin{pmatrix}W\\ \eta\end{pmatrix} = \begin{pmatrix}\lambda\frac{e}{m^2}\left[\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot\mathbf{E}\times\mathbf{D}\right] & m - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2 - \lambda\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + \lambda^2\frac{e^2}{m^3}\left[\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\right]\\ m - \frac{1}{m}\left[\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2 + e\mathbf{S}\cdot\mathbf{B}\right] & -\lambda\frac{e}{m^2}\left[\mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) + i\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}\right]\end{pmatrix} \tag{3.0.55}$$

Current

We can also derive the conserved current from the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}(D \times W)^\dagger \cdot (D \times W) + m_w^2 W^\dagger W - \lambda ie W^{\dagger\mu} W^\nu F_{\mu\nu}$$

Where

$$D^\mu = \partial^\mu - ieA^\mu, \qquad D \times W = D^\mu W^\nu - D^\nu W^\mu$$

We want the conserved current corresponding to the transformation $W_i \to e^{i\alpha} W_i$, which in infinitesimal form is:

$$W_\mu \to W_\mu + i\alpha W_\mu,\; W_\mu^\dagger \to W_\mu^\dagger - i\alpha W_\mu^\dagger$$

The 4-current density will be:

$$j^\sigma = -i\frac{\partial \mathcal{L}}{\partial W_{\mu,\sigma}} W_\mu + i\frac{\partial \mathcal{L}}{\partial W^\dagger_{\mu,\sigma}} W_\mu^\dagger$$

Only one term contains derivatives of the field:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial W_{\alpha,\sigma}} &= \frac{\partial}{\partial W_{\alpha,\sigma}}\left\{-\frac{1}{2}(D_\mu W_\nu - D_\nu W_\mu)^\dagger (D^\mu W^\nu - D^\nu W^\mu)\right\} \\ &= -\frac{1}{2}(D_\mu W_\nu - D_\nu W_\mu)^\dagger (g_{\sigma\mu} g_{\alpha\nu} - g_{\sigma\nu} g_{\alpha\mu}) \\ &= -(D_\alpha W_\sigma - D_\sigma W_\alpha)^\dagger\end{aligned}$$

Likewise:

$$\frac{\partial \mathcal{L}}{\partial W^\dagger_{\alpha,\sigma}} = -(D_\alpha W_\sigma - D_\sigma W_\alpha)$$

If we define $W_{\mu\nu} = D^\mu W^\nu - D^\nu W^\mu$ then the 4-current and charge density are:

$$\begin{aligned} j_\sigma &= iW^\dagger_{\sigma\mu} W^\mu - iW_{\sigma\mu} W^{\dagger\mu} \\ j_0 &= iW^\dagger_{0\mu} W^\mu - iW_{0\mu} W^{\dagger\mu} \\ &= iW^\dagger_{0i} W^i - iW_{0i} W^{\dagger i}\end{aligned}$$

Where the last equality follows from the antisymmetry of $W_{\mu\nu}$.

Now, we defined the fields $\eta_i = -i\frac{W_{i0}}{m}$. In terms of these fields, $j_0 = m(\eta_i^\dagger W^i + \eta_i W^{\dagger i})$.

We can do the same to find the vector part of the current.

$$\begin{aligned} j_i &= iW^\dagger_{i\mu} W^\mu - iW_{i\mu} W^{\dagger\mu} \\ &= iW_j^\dagger W_{ij} + iW^\dagger_{i0} W_0 + c.c.\end{aligned}$$

We have $W_{ij} = D_i W_j - D_j W_i$. Using the identities developed in the appendix, we can obtain

$$D_j W_i = D_i W_j - D_k (S_i S_k)_{ja} W_a$$

Then

$$W_{ij} = D_k (S_i S_k)_{ja} W_a$$

In the absence of an electric field E, $W_0 = \frac{i}{m} D_j \eta_j$, with $W_{i0} = im\eta_i$.

$$W_{i0}^\dagger W_0 = -\eta_i^\dagger D_j \eta_j$$

Again we introduce spin matrices to get the equation in the desired form, and obtain

$$W_{i0}^\dagger W_0 = -\eta_j^\dagger D_k (\delta_{ik} - S_k S_i) \eta_j$$

This leads to

$$j_i = iW_j^\dagger D_k (S_i S_k W)_j - i\eta_j^\dagger D_k ([\delta_{ik} - S_k S_i]\eta)_j + c.c.$$

Writing this in terms of the bispinor $\begin{pmatrix} \eta \\ W \end{pmatrix}$, the expression for the current is

$$j_i = \frac{i}{2} \begin{pmatrix} \eta^\dagger & W^\dagger \end{pmatrix} \left[(\{S_i, S_j\} - \delta_{ij}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - ([S_i, S_j] + \delta_{ij}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] D_j \begin{pmatrix} \eta \\ W \end{pmatrix} + c.c. \tag{3.0.56}$$

Hermiticity of Hamiltonian

It's clear from inspection that the above Hamiltonian is not Hermitian in the standard sense. (Of the operators in use, the only one which is not self adjoint is $D_i^\dagger = -D_i$.) Noticing that

$$[\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D}) + i\mathbf{S} \cdot \mathbf{E} \times \mathbf{D}]^\dagger = -[\mathbf{D} \cdot \mathbf{E} - (\mathbf{S} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{E}) + i\mathbf{S} \cdot \mathbf{D} \times \mathbf{E}]$$

we can see it has the general form

$$H = \begin{pmatrix} A & B \\ C & A^\dagger \end{pmatrix}$$

where the off diagonal blocks are Hermitian in the normal sense: $B^\dagger = B$ and $C^\dagger = C$.

At this point we should consider that an operator is defined as Hermitian with respect to a particular inner product. In quantum mechanics this inner product is normally defined as:

$$\langle \Psi, \phi \rangle = \int d^3x \Psi^\dagger \phi$$

This definition, however, does not produce sensible results for the states in question. We need the inner product of a state with itself to be conserved; in other words, it should play the role of a conserved charge. From the considerations above we already have one such quantity: the conserved charge $\int d^3x j_0$, where

$$j_0 = m[\eta^\dagger W + W^\dagger \eta] = m \begin{pmatrix} \eta^\dagger & W^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ W \end{pmatrix}$$

A more general definition of the inner product includes some weight M:

$$\langle \Psi, \phi \rangle = \int d^3x \Psi^\dagger M \phi$$

In normal quantum mechanics M would be the identity matrix, but here, as implied by the charge density, we want $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Such a definition will lead to the inner product $< \Psi, \Psi >$ being conserved.

An operator H is hermitian with respect to this inner product if

$$\langle H\Psi, \phi \rangle = \langle \Psi, H\phi \rangle \rightarrow \int d^3x \Psi^\dagger H^\dagger M \phi = \int d^3x \Psi^\dagger M H \phi$$

For this equality to hold, it is sufficient for $H^\dagger M = MH$. With $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ this condition reduces to

$$\begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$$

Our Hamiltonian fulfills exactly this requirement, and so is Hermitian with respect to this particular inner product.

3.0.13 Non-relativistic Hamiltonian

Now we will consider the non-relativistic limit of the above Hamiltonian. To work in this regime constrains the order of both the momentum and the electromagnetic field strength.

$$
\begin{aligned}
D &\sim mv \\
\Phi &\sim mv^2 \\
E &\sim m^2v^3 \\
B &\sim m^2v^2
\end{aligned}
$$

We can write the Hamiltonian matrix in terms of the basis of 2x2 Hermitian matrices: $(\mathbf{I}, \rho_i)$:

$$
\begin{aligned}
H &= a_0\mathbf{I} + a_i\rho_i \\
a_0 &= \frac{1}{2}(H_{11}+H_{22}) = e\Phi + \lambda\frac{e}{2m^2}\left[\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot\mathbf{E}\times\mathbf{D} - \mathbf{D}\cdot\mathbf{E} + (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) - i\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}\right] \\
a_3 &= \frac{1}{2}(H_{11}-H_{22}) = \lambda s\frac{e}{2m^2}\left[\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot\mathbf{E}\times\mathbf{D} + \mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) + i\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}\right] \\
ia_2 &= \frac{1}{2}(H_{21}-H_{12}) = -\left[\frac{\mathbf{D}^2}{2m} - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2 - \frac{\lambda-1}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + \frac{e^2}{2m^3}(\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2)\right] \\
a_1 &= \frac{1}{2}(H_{12}-H_{21}) = \left[m - \frac{\mathbf{D}^2}{2m} - \frac{1+\lambda}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + \frac{e^2}{2m^3}(\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2)\right]
\end{aligned}
$$

We can see that to leading order, the Hamiltonian is

$$
H = m\rho_1 = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}
$$

Since we wish to separate positive and negative energy states, this poses a problem. So we first switch to a basis where at least the rest energies are separate. Then, remaining off-diagonal elements can be treated as perturbations.

An appropriate transformation which meets our requirements is a "rotation" in the space spanned by ρ_i matrices, about the $\rho_1 + \rho_3$ axis. This has the explicit form:

$$
U = \frac{1}{\sqrt{2}}(\rho_1 + \rho_3) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3.0.57}
$$

Any transformation U can be described by it's action on the basis of 4x4 matrices. This takes:

$$
\begin{aligned}
\mathbf{I} &\to \mathbf{I} \\
\rho_1 &\to \rho_3 \\
\rho_3 &\to \rho_1 \\
\rho_2 &\to -\rho_2 \\
\begin{pmatrix} \eta \\ W \end{pmatrix} &\to \begin{pmatrix} \Psi_u \\ \Psi_\ell \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \eta + W \\ \eta - W \end{pmatrix}
\end{aligned}
$$

(This transformation will transform the current to

$$
j_0 = m(\Psi_u^\dagger\Psi_u - \Psi_\ell^\dagger\Psi_\ell)
$$

and the weight M, used in the inner product, to

$$
M \to M' = U^\dagger M U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

The transformed Hamiltonian will of course be Hermitian with respect to the transformed inner product.)

Our equation is now of the following form:

$$i\partial_0 \begin{pmatrix} \Psi_u \\ \Psi_\ell \end{pmatrix} = H' \begin{pmatrix} \Psi_u \\ \Psi_\ell \end{pmatrix}$$

We can see that the Hamiltonian still contains off-diagonal elements, so this represents a pair of coupled equations for the upper and lower components of $\mathbf{\Psi}$. But the off-diagonal terms are small, so we can consider the case where Ψ_ℓ is small compared to Ψ_u. Solving for Ψ_ℓ in terms of Ψ_u:

$$\begin{aligned} E\Psi_\ell &= H'_{21}\Psi_u + H'_{22}\Psi_\ell \\ \Psi_\ell &= (E - H'_{22})^{-1} H'_{21}\Psi_u \end{aligned}$$

This gives the exact formula:

$$E\Psi_u = \left(H'_{11} + H'_{12}[E - H'_{22}]^{-1} H'_{21}\right)\Psi_u$$

However, we only need corrections to the magnetic moment of order v^2. With $\frac{e}{m}\mathbf{S}\cdot\mathbf{B} \sim mv^2$, this means we only need the Hamiltonian to at most order mv^4. Examining the leading order terms of the matrix H', the diagonal elements are order m while the off-diagonal elements are order mv^2. To leading order the term $[E - H'_{22}]^{-1} = \frac{1}{2m}$. So we'll need H'_{11} to $\mathcal{O}(v^4)$, and H'_{12}, H'_{21}, and $[E - H'_{22}]^{-1}$ each to only leading order.

$$E\Psi_u = \left(H'_{11} + \frac{1}{2m}H'_{12}H'_{21} + \mathcal{O}(mv^6)\right)\Psi_u$$

The needed terms of H are, using $\lambda = g - 1$

$$\begin{aligned} H'_{11} = a_0 + a_3 &= m + e\Phi - \frac{D^2}{2m} - \frac{g}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} \\ &\quad + (g-1)\frac{e}{2m^2}\left[\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot\mathbf{E}\times\mathbf{D} - \mathbf{D}\cdot\mathbf{E} + (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) - i\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}\right] + \mathcal{O}(mv \\ H'_{12} = a_1 - ia_2 &= \frac{D^2}{2m} - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2 - \frac{g-2}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + \mathcal{O}(mv^4) \\ H'_{21} = a_1 + ia_2 &= -\frac{D^2}{2m} + \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2 + \frac{g-2}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + \mathcal{O}(mv^4) \end{aligned}$$

The product $H'_{12}H'_{21}$ is calculated in the appendix. To first order in the magnetic field strength it is:

$$\begin{aligned} \frac{1}{2m}H'_{12}H'_{21} &= -\frac{1}{2m^3}\left(\frac{\mathbf{D}^2}{2} - (\mathbf{S}\cdot\mathbf{D})^2 - \frac{g-2}{2}e\mathbf{S}\cdot\mathbf{B}\right)^2 \\ &= -\frac{1}{2m^3}\left(\frac{\boldsymbol{\pi}^4}{4} - e\mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - \frac{g-2}{2}e(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\right) \end{aligned}$$

So finally,replacing all $\mathbf{D}$ with $\boldsymbol{\pi} \equiv \mathbf{p} - e\mathbf{A}$, we have a direct expression for Ψ_u:

$$\begin{aligned} E\Psi_u = &\left\{ m + e\Phi + \frac{\pi^2}{2m} - \frac{g}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} - \frac{\pi^4}{8m^3} + \frac{e\mathbf{p}^2(\mathbf{S}\cdot\mathbf{B})}{2m^3} + (g-2)\frac{e}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \right. \\ &\left. + (g-1)\frac{ie}{2m^2}\left[\mathbf{E}\cdot\boldsymbol{\pi} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\boldsymbol{\pi}) + i\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi} - \boldsymbol{\pi}\cdot\mathbf{E} + (\mathbf{S}\cdot\boldsymbol{\pi})(\mathbf{S}\cdot\mathbf{E}) - i\mathbf{S}\cdot\boldsymbol{\pi}\times\mathbf{E}\right] \right\}\Psi_u \end{aligned}$$

The complicated expression in square brackets can be cleaned up a bit:

$$\begin{aligned}
\mathbf{E}\cdot\boldsymbol{\pi}-\boldsymbol{\pi}\cdot\mathbf{E} &= [E_i,\pi_i] \\
&= [E_i,-i\partial_i] \\
&= i(\partial_i E_i) \\
(\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\boldsymbol{\pi})-(\mathbf{S}\cdot\boldsymbol{\pi})(\mathbf{S}\cdot\mathbf{E}) &= S_iS_jE_i\pi_j - S_iS_j\pi_iE_j \\
&= (S_iS_j)(E_i\pi_j - E_j\pi_i - [\pi_i,E_j]) \\
&= [S_i,S_j](E_i\pi_j) - (S_iS_j)(-i\nabla_iE_j) \\
&= (i\epsilon_{ijk}S_k)E_j\pi_i - (S_iS_j)(-i\nabla_iE_j) \\
&= i\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi} + iS_iS_j\nabla_iE_j) \\
(\mathbf{E}\times\boldsymbol{\pi}-\boldsymbol{\pi}\times\mathbf{E})_k &= \epsilon_{ijk}(E_i\pi_j - \pi_iE_j) \\
&= \epsilon_{ijk}(E_i\pi_j + \pi_jE_i) \\
&= \epsilon_{ijk}(2E_i\pi_j + [\pi_i,E_j]) \\
&= 2\epsilon_{ijk}E_i\pi_j \\
&= 2(\mathbf{E}\times\boldsymbol{\pi})_k
\end{aligned}$$

Using these identities and collecting terms, and then writing everything in terms of g, $g-2$

$$\begin{aligned}
E\Psi_u &= \Bigg\{m+e\Phi+\frac{\pi^2}{2m}-\frac{\pi^4}{8m^3}-\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\left(\frac{g}{2}-\frac{p^2}{2m^2}\right)+(g-2)\frac{e}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \\
&\quad -(g-1)\frac{e}{2m^2}\left[\nabla\cdot\mathbf{E}-S_iS_j\nabla_iE_j+\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi}\right]\Bigg\}\Psi_u \\
&= \Bigg\{m+e\Phi+\frac{\pi^2}{2m}-\frac{\pi^4}{8m^3}-\frac{g}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\left(1-\frac{p^2}{2m^2}\right)-\frac{g-2}{2}\frac{e}{m}\frac{p^2}{2m^2}\mathbf{S}\cdot\mathbf{B}+(g-2)\frac{e}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \\
&\quad -\left(\frac{g}{2}+\frac{g-2}{2}\right)\frac{e}{2m^2}\left[\nabla\cdot\mathbf{E}-S_iS_j\nabla_iE_j+\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi}\right]\Bigg\}\Psi_u
\end{aligned}$$

We have a Hamiltonian for the upper component of the bispinor, as desired. But is it truly Schrodinger-like? In the general case we would need to perform the Fouldy-Wouthyusen transformation, to a representation where all the physics up to the desired order is contained in the single spinor equation. But we can show that, at the order we are working at, the Hamiltonian above is correct.

3.1 Comparison with Silenko

We can compare to a result in Silenko (arXiv:hep-th/0401183v1). Silenko determines a Hamiltonian which is equivalent to that used above. Dropping terms manifestly beyond $\mathcal{O}(mv^4)$:

$$\begin{aligned} H &= \rho_3\epsilon' + e\Phi + \frac{e}{4m}\left[\left\{\left(\frac{g-2}{2} + \frac{m}{\epsilon'+m}\right)\frac{1}{\epsilon'}, (\mathbf{S}\cdot\boldsymbol{\pi}\times\boldsymbol{E} - \mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi})\right\}_+ - \rho_3\left\{\left(g-2+\frac{2m}{\epsilon'}\right), \mathbf{S}\cdot\mathbf{B}\right\}_+ \right. \\ &\left. +\rho_3\left\{\frac{g-2}{2\epsilon'(\epsilon'+m)}, \{\mathbf{S}\cdot\boldsymbol{\pi}, \boldsymbol{\pi}\cdot\mathbf{B}\}_+\right\}_+\right] + \frac{e(g-1)}{4m^2}\{\mathbf{S}\cdot\boldsymbol{\nabla}, \mathbf{S}\cdot\mathbf{E}\}_+ - \frac{e(g-1)}{2m^2}\nabla\cdot\mathbf{E} \end{aligned}$$

We'll take each term and expand to $\mathcal{O}(mv^4)$. First the operator ϵ':

$$\begin{aligned} \epsilon' &= \sqrt{m^2+\pi^2} \\ &= m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^2} + \mathcal{O}(mv^6) \end{aligned}$$

Using this, we get

$$\begin{aligned} H &= \rho_3\epsilon' + e\Phi + \frac{e}{4m}\left[\left\{\frac{g-1}{2m}, (\mathbf{S}\cdot\boldsymbol{\pi}\times\boldsymbol{E} - \mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi})\right\}_+ - \rho_3\left\{g - \frac{\pi^2}{m^2}, \mathbf{S}\cdot\mathbf{B}\right\}_+ \right. \\ &\left. +\rho_3\left\{\frac{g-2}{4m^2}, \{\mathbf{S}\cdot\boldsymbol{\pi}, \boldsymbol{\pi}\cdot\mathbf{B}\}_+\right\}_+\right] + \frac{e(g-1)}{4m^2}\{\mathbf{S}\cdot\boldsymbol{\nabla}, \mathbf{S}\cdot\mathbf{E}\}_+ - \frac{e(g-1)}{2m^2}\nabla\cdot\mathbf{E} \end{aligned}$$

Now we'll expand the anticommutators.
We have already shown that $\boldsymbol{\pi}\times\mathbf{E} = -\mathbf{E}\times\boldsymbol{\pi}$, and that such a term is $\mathcal{O}(mv^4)$

$$\left\{\frac{g-1}{2m}, (\mathbf{S}\cdot\boldsymbol{\pi}\times\boldsymbol{E} - \mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi})\right\}_+ = -\frac{2(g-1)}{m}\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi}$$

Because we specialize to constant magnetic fields, $\boldsymbol{\pi}$ commutes with $\mathbf{B}$, so

$$\left\{g - \frac{\pi^2}{m^2}, \mathbf{S}\cdot\mathbf{B}\right\}_+ = (2g - \frac{2\pi^2}{m^2})\mathbf{S}\cdot\mathbf{B}$$

and

$$\left\{\frac{g-2}{4m^2}, \{\mathbf{S}\cdot\boldsymbol{\pi}, \boldsymbol{\pi}\cdot\mathbf{B}\}_+\right\}_+ = \frac{g-2}{m^2}(\mathbf{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\mathbf{B})$$

Using the identity that $[\nabla_i, E_j] = 0$, $\nabla\times\mathbf{E} = 0$:

$$\begin{aligned} \{\mathbf{S}\cdot\boldsymbol{\nabla}, \mathbf{S}\cdot\mathbf{E}\}_+ &= S_iS_j\nabla_iE_j + S_jS_iE_j\nabla_i \\ &= (S_iS_j + S_jS_i)\nabla_iE_j \\ &= (2S_iS_j + [S_j, S_i])\nabla_iE_j \\ &= (2S_iS_j + iS_k\epsilon_{ijk})\nabla_iE_j \\ &= 2S_iS_j\nabla_iE_j \end{aligned}$$

So

$$\begin{aligned} H &= \rho_3(m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^2}) + e\Phi - (g-1)\frac{e}{2m^2}\left[\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi} - S_iS_j\nabla_iE_j + \nabla\cdot\mathbf{E}\right] \\ &+\rho_3(g-2)\frac{2}{4m^2}(\mathbf{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\mathbf{B}) - \rho_3\frac{e}{m}(\frac{g}{2} - \frac{\pi^2}{2m^2})\mathbf{S}\cdot\mathbf{B} \\ &= \rho_3\left(m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^2}\right) + e\Phi - \left(\frac{g-2}{2} + \frac{g}{2}\right)\frac{e}{2m^2}\left[\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi} - S_iS_j\nabla_iE_j + \nabla\cdot\mathbf{E}\right] \\ &+\rho_3(g-2)\frac{2}{4m^2}(\mathbf{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\mathbf{B}) - \rho_3\frac{e}{m}\left[\frac{g}{2}\left(1 - \frac{\pi^2}{2m^2}\right) + \frac{g-2}{2}\frac{\pi^2}{2m^2})\right]\mathbf{S}\cdot\mathbf{B} \end{aligned}$$

Magnetic Moment

Now we'll keep only those terms which contribute to the magnetic moment, using $\boldsymbol{\pi} = \mathbf{p} - e\mathbf{A}$

$$\begin{aligned}
H_{S\cdot B} &= \left(\frac{g-2}{2} + \frac{g}{2}\right)\frac{e^2}{2m^2}\mathbf{S}\cdot\mathbf{E}\times\mathbf{A} + (g-2)\frac{e}{4m^2}(\mathbf{S}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{B}) - \frac{e}{m}\left[\frac{g}{2}\left(1-\frac{p^2}{2m^2}\right) + \frac{g-2}{2}\frac{p^2}{2m^2}\right]\mathbf{S}\cdot\mathbf{B} \\
&= -\frac{e}{2m}\left\{g\left(1-\frac{p^2}{2m^2}\right)\mathbf{S}\cdot\mathbf{B} + (g-2)\frac{p^2}{2m^2}\mathbf{S}\cdot\mathbf{B} - (g-2)\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{B})}{2m^2} - \frac{e}{m}\left(\frac{g-2}{2}+\frac{g}{2}\right)\mathbf{S}\cdot\mathbf{E}\times\mathbf{A}\right\}
\end{aligned}$$

This exactly matches with what was found before.

3.2 Identities

Simplify $\mathbf{W}\times\mathbf{B}$:

$$\begin{aligned}
(\mathbf{W}\times\mathbf{B})_i &= \epsilon_{ijk}W_jB_k \\
&= i(S_k)_{ij}W_jB_k \\
&= i(\mathbf{S}\cdot\mathbf{B})_{ij}W_j \\
&= i([\mathbf{S}\cdot\mathbf{B}]\mathbf{W})_i
\end{aligned}$$

Simplify $\mathbf{D}\times(\mathbf{D}\times\mathbf{W})$

$$\begin{aligned}
(\mathbf{D}\times[\mathbf{D}\times\mathbf{W}])_i &= \epsilon_{ijk}D_j(\mathbf{D}\times\mathbf{W})_k \\
&= \epsilon_{ijk}\epsilon_{k\ell m}D_jD_\ell W_m \\
&= -(S_j)_{ki}(S_\ell)_{mk}D_jD_\ell W_m \\
&= -(\mathbf{S}\cdot\mathbf{D})_{ki}(\mathbf{S}\cdot\mathbf{D})_{mk}W_m \\
&= -\left([\mathbf{S}\cdot\mathbf{D}]^2\right)_{im}W_m \\
&= -\left([\mathbf{S}\cdot\mathbf{D}]^2\mathbf{W}\right)_i
\end{aligned}$$

Our representation defines the spin matrices as follows:

$$(S_k)_{ij} = -i\epsilon_{ijk}$$

They have the commutator

$$[S_i, S_j] = i\epsilon_{ijk}S_k$$

The product of two such spin matrices is given by:

$$\begin{aligned}
(S_kS_\ell)_{ij} &= (S_k)_{ia}(S_\ell)_{aj} \\
&= -\epsilon_{iak}\epsilon_{aj\ell} \\
&= (\delta_{k\ell}\delta_{ij} - \delta_{kj}\delta_{\ell i})
\end{aligned}$$

This implies that:

$$\begin{aligned}
(S_iS_jA_jB_i\mathbf{v})_l &= (S_i)_{lm}(S_j)_{mn}A_jB_iv_n \\
&= (S_iS_j)_{ln}A_jB_iv_n \\
&= (\delta_{ij}\delta_{ln} - \delta_{in}\delta_{lj})A_jB_iv_n \\
&= (\mathbf{A}\cdot\mathbf{B})v_l - A_l(\mathbf{B}\cdot\mathbf{v})
\end{aligned}$$

Or

$$\mathbf{A}(\mathbf{B}\cdot\mathbf{v}) = (\mathbf{A}\cdot\mathbf{B} - S_i S_j A_j B_i)\mathbf{v}$$

Now we can use this to establish some identities:

$$\begin{aligned}
E^i\mathbf{D}\cdot\eta &= ([\mathbf{E}\cdot\mathbf{D} - S_j S_k E_k D_j]\,\eta)^i \\
&= ([\mathbf{E}\cdot\mathbf{D} - (S_k S_j - [S_k, S_j])E_k D_j]\,\eta)^i \\
&= ([\mathbf{E}\cdot\mathbf{D} - (S_k S_j - i\epsilon_{kjl}S_l)E_k D_j]\,\eta)^i \\
&= ([\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + iS_l(\mathbf{E}\times\mathbf{D})_l]\,\eta)^i \\
&= ([\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot(\mathbf{E}\times\mathbf{D})]\,\eta)^i
\end{aligned}$$

Since E and W commute:

$$\begin{aligned}
E^i\mathbf{W}\cdot\mathbf{E} &= \left(\left[\mathbf{E}^2 - S_j S_k E_k E_j\right]\mathbf{W}\right)^i \\
&= \left(\left[\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\right]\mathbf{W}\right)^i
\end{aligned}$$

And

$$\begin{aligned}
D^i\mathbf{D}\cdot\eta &= \left(\left[\mathbf{D}^2 - S_j S_k D_k D_j\right]\eta\right)^i \\
&= \left(\left[\mathbf{D}^2 - (S_k S_j + [S_j, S_k])D_k D_j\right]\eta\right)^i \\
&= \left(\left[\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2 + i\epsilon_{jkl}S_l D_k D_j)\right]\eta\right)^i \\
&= \left(\left[\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2 + i\mathbf{S}\cdot(\mathbf{D}\times\mathbf{D})\right]\eta\right)^i \\
&= \left(\left[\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2 + e\mathbf{S}\cdot\mathbf{B})\right]\eta\right)^i
\end{aligned}$$

Similarly:

$$\begin{aligned}
D^i\mathbf{E}\cdot\mathbf{W} &= ([\mathbf{D}\cdot\mathbf{E} - S_j S_k D_k E_j]\,\eta)^i \\
&= ([\mathbf{D}\cdot\mathbf{E} - (S_k S_j + [S_j, S_k])D_k E_j]\,\eta)^i \\
&= ([\mathbf{D}\cdot\mathbf{E} - (S_k S_j + i\epsilon_{jkl}S_l)D_k E_j]\,\eta)^i \\
&= ([\mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) + i\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E})]\,\eta)^i
\end{aligned}$$

Product of $H_{12}H_{21}$

We need to calculate $\left(\frac{\boldsymbol{\pi}^2}{2} - (\mathbf{S}\cdot\boldsymbol{\pi})^2 + (g-2)\frac{2}{m}\mathbf{S}\cdot\mathbf{B}\right)^2$ to first order in magnetic field strength. As a first step of simplification

$$\left(\frac{\boldsymbol{\pi}^2}{2} - (\mathbf{S}\cdot\boldsymbol{\pi})^2 + \frac{g-2}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\right)^2 = \left(\frac{\boldsymbol{\pi}^2}{2} - (\mathbf{S}\cdot\boldsymbol{\pi})^2\right)^2 + \frac{g-2}{2}\frac{e}{m}\left\{\frac{p^2}{2} - (\mathbf{S}\cdot\mathbf{p})^2, \mathbf{S}\cdot\mathbf{B}\right\}$$

First term

To simplify the first term, consider one element of this matrix operator:

$$
\begin{aligned}
\left\{\left(\frac{\boldsymbol{\pi}^2}{2}-(\mathbf{S}\cdot\boldsymbol{\pi})^2\right)^2\right\}_{ac} &= \left(\frac{\boldsymbol{\pi}^2}{2}-S_iS_j\pi_i\pi_j\right)_{ab}\left(\frac{\boldsymbol{\pi}^2}{2}-S_lS_m\pi_l\pi_m\right)_{bc} \\
&= \left(\frac{\boldsymbol{\pi}^2}{2}\delta_{ab}-[S_iS_j]_{ab}\pi_i\pi_j\right)\left(\frac{\boldsymbol{\pi}^2}{2}\delta_{bc}-[S_lS_m]_{bc}\pi_l\pi_m\right) \\
&= \left(\frac{\boldsymbol{\pi}^2}{2}\delta_{ab}-[\delta_{ab}\delta_{ij}-\delta_{aj}\delta_{bi}]\pi_i\pi_j\right)\left(\frac{\boldsymbol{\pi}^2}{2}\delta_{bc}-[\delta_{bc}\delta_{lm}-\delta_{bm}\delta_{cl}]\pi_l\pi_m\right) \\
&= \left(-\frac{\boldsymbol{\pi}^2}{2}\delta_{ab}+\pi_b\pi_a\right)\left(-\frac{\boldsymbol{\pi}^2}{2}\delta_{bc}+\pi_c\pi_b\right) \\
&= \frac{\boldsymbol{\pi}^4}{4}\delta_{ac}-\pi_c\pi_a\frac{\boldsymbol{\pi}^2}{2}-\frac{\boldsymbol{\pi}^2}{2}\pi_c\pi_a+\pi_b\pi_a\pi_c\pi_b
\end{aligned}
$$

It's very useful to have the following identity:

$$
\begin{aligned}
e(\mathbf{S}\cdot\mathbf{B})_{ab} &= e(S_i)_{ab}B_i \\
&= -ie\epsilon_{iab}B_i \\
&= -ie\epsilon_{iab}(\epsilon_{ijk}\partial_jA_k) \\
&= -ie\epsilon_{iab}\epsilon_{ijk}\frac{1}{2}(\partial_jA_k\partial_kA_j) \\
&= -\epsilon_{iab}\epsilon_{ijk}\frac{1}{2}[\pi_j,\pi_k] \\
&= -\epsilon_{iab}\epsilon_{ijk}\pi_j\pi_k \\
&= -(\delta_{aj}\delta_{bk}-\delta_{ak}\delta_{bj})\pi_j\pi_k \\
&= \pi_b\pi_a-\pi_a\pi_b
\end{aligned}
$$

Therefore,

$$e(\mathbf{S}\cdot\mathbf{B})_{ab}=[\pi_b,\pi_a]$$

Using this, and the fact that π commutes with S and B:

$$
\begin{aligned}
\pi_b\pi_a\pi_c\pi_b &= \pi_b\pi_c\pi_a\pi_b-\pi_b(e\mathbf{S}\cdot\mathbf{B})_{ac}\pi_b \\
&= \pi_b\pi_c\pi_a\pi_b-\boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac}
\end{aligned}
$$

Also,

$$
\begin{aligned}
\pi_b\pi_c\pi_a\pi_b &= \pi_b\pi_c\pi_b\pi_a+\pi_b\pi_c(e\mathbf{S}\cdot\mathbf{B})_{ba} \\
&= \pi_b\pi_b\pi_c\pi_a+\pi_b\pi_a(e\mathbf{S}\cdot\mathbf{B})_{bc}+\pi_b\pi_c(e\mathbf{S}\cdot\mathbf{B})_{ba}
\end{aligned}
$$

So now:

$$
\begin{aligned}
\pi_b\pi_a\pi_c\pi_b-\pi_c\pi_a\frac{\boldsymbol{\pi}^2}{2}-\frac{\boldsymbol{\pi}^2}{4}\pi_c\pi_a &= \boldsymbol{\pi}^2\pi_c\pi_a+\pi_b\pi_a(e\mathbf{S}\cdot\mathbf{B})_{bc}+\pi_b\pi_c(e\mathbf{S}\cdot\mathbf{B})_{ba} \\
&\quad -\boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac}-\pi_c\pi_a\frac{\boldsymbol{\pi}^2}{2}-\frac{\boldsymbol{\pi}^2}{4}\pi_c\pi_a \\
&= \frac{1}{2}[\boldsymbol{\pi}^2,\pi_c\pi_a]+\pi_b\pi_a(e\mathbf{S}\cdot\mathbf{B})_{bc}+\pi_b\pi_c(e\mathbf{S}\cdot\mathbf{B})_{ba}-\boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac}
\end{aligned}
$$

Now evaluate the commutator:

$$\begin{aligned}
[\pi_b, \pi_c\pi_a] &= [\pi_b, \pi_c]\pi_a - \pi_c[\pi_a, \pi_b] \\
&= (e\mathbf{S}\cdot\mathbf{B})_{cb}\pi_a - \pi_c(e\mathbf{S}\cdot\mathbf{B})_{ba}
\end{aligned}$$

$$\begin{aligned}
[\boldsymbol{\pi}^2, \pi_c\pi_a] &= [\pi_b\pi_b, \pi_c\pi_a] \\
&= \pi_b[\pi_b, \pi_c\pi_a] + [\pi_b, \pi_c\pi_a]\pi_b \\
&= (e\mathbf{S}\cdot\mathbf{B})_{cb}(\pi_b\pi_a + \pi_a\pi_b) - (e\mathbf{S}\cdot\mathbf{B})_{ba}(\pi_b\pi_c + \pi_c\pi_b)
\end{aligned}$$

This gives the result:

$$\begin{aligned}
\pi_b\pi_a\pi_c\pi_b - \pi_c\pi_a\frac{\boldsymbol{\pi}^2}{2} - \frac{\boldsymbol{\pi}^2}{4}\pi_c\pi_a &= \frac{1}{2}\left[(e\mathbf{S}\cdot\mathbf{B})_{cb}(\pi_b\pi_a + \pi_a\pi_b) - (e\mathbf{S}\cdot\mathbf{B})_{ba}(\pi_b\pi_c + \pi_c\pi_b)\right] \\
&\quad + \pi_b\pi_a(e\mathbf{S}\cdot\mathbf{B})_{bc} + \pi_b\pi_c(e\mathbf{S}\cdot\mathbf{B})_{ba} - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac} \\
&= \frac{1}{2}[(e\mathbf{S}\cdot\mathbf{B})_{cb}(\pi_a\pi_b - \pi_b\pi_a) + (e\mathbf{S}\cdot\mathbf{B})_{ba}(\pi_b\pi_c - \pi_c\pi_b)] - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac} \\
&= \frac{1}{2}[(e\mathbf{S}\cdot\mathbf{B})_{cb}(e\mathbf{S}\cdot\mathbf{B})_{ba} + (e\mathbf{S}\cdot\mathbf{B})_{ba}(e\mathbf{S}\cdot\mathbf{B})_{cb}] - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac} \\
&= (e\mathbf{S}\cdot\mathbf{B})_{ab}(e\mathbf{S}\cdot\mathbf{B})_{bc}) - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac} \\
&= \left[(e\mathbf{S}\cdot\mathbf{B})^2\right]_{ac} - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac}
\end{aligned}$$

Since we can throw away terms of order B^2, the final result tells us that, to first order in B:

$$\begin{aligned}
\left(\frac{\boldsymbol{\pi}^2}{2} - (\mathbf{S}\cdot\boldsymbol{\pi})^2\right)^2 &= \frac{\boldsymbol{\pi}^4}{4} - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B}) \\
&= \frac{\boldsymbol{\pi}^4}{4} - e\mathbf{p}^2\mathbf{S}\cdot\mathbf{B}
\end{aligned}$$

Second term

To simplify the second term, we need

$$\begin{aligned}
\left\{\frac{\mathbf{p}^2}{2} - (\mathbf{S}\cdot\mathbf{p})^2, \mathbf{S}\cdot\mathbf{B}\right\} &= \mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - [(\mathbf{S}\cdot\mathbf{p})^2\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}(\mathbf{S}\cdot\mathbf{p})^2] \\
&= \mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - (S_iS_jS_k + S_kS_jS_i)p_ip_jB_k
\end{aligned}$$

To simplify that triple product of spin matrices, we can use their explicit form:

$$\begin{aligned}
(S_iS_jS_k)_{ab} &= i\epsilon_{aci}\epsilon_{cdj}\epsilon_{dbk} \\
&= i(\delta_{id}\delta_{aj} - \delta_{ij}\delta_{ad})\epsilon_{dbk} \\
&= i(\delta_{aj}\epsilon_{ibk} - \delta_{ij}\epsilon_{abk}) \\
(S_iS_jS_k + S_kS_jS_i)_{ab} &= i(\delta_aj\epsilon_{ibk} + \delta_{aj}\epsilon_{kbi} - \delta_{ij}\epsilon_{abk} - \delta kj\epsilon_{abi}) \\
&= -i(\delta_{ij}\epsilon_{abk} + \delta_{kj}\epsilon_{abi} \\
&= \delta_{ij}(S_k)_ab + \delta_{kj}(S_i)_{ab}
\end{aligned}$$

Now

$$\begin{aligned}
\mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - (S_iS_jS_k + S_kS_jS_i)p_ip_jB_k &= \mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - (\delta_{ij}S_k + \delta_{kj}S_i)p_ip_jB_k \\
&= -(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})
\end{aligned}$$

Result

At last, the final result is that, to the order we care about

$$\left(\frac{\boldsymbol{\pi}^2}{2} - (\mathbf{S}\cdot\boldsymbol{\pi})^2 + (g-2)\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\right)^2 = \frac{\boldsymbol{\pi}^4}{4} - e\mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - \frac{g-2}{2}\frac{e}{m}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})$$