Contents

Chapter 1

Nonrelativistic Quantum Electrodynamics

1.1 Effective Field Theories

The most generally powerful approach to nonrelativistic bond state theories is the use of the techniques of effective field theory. In this chapter we develop the theory of such an approach.

1.1.1 Brief background

First, a brief background of the development of effective field theories.

The first workable relativistic theory of quantum mechanics came from Dirac. By realising that a relativistic equation for fermions necessitated a four component spinor, he was able to write down what is now known as Dirac's equation. This led to the prediction of the electron's antiparticle the positron.

The relativistic theory necessitated the introduction of an infinite degrees of freedom. Thus, wave functions had to be promoted to fields, and the first quantum field theory arose. It was highly successful at predicting leading order quantities, but when attempting to use perturbation theory within this context, a number of infinities arose.

To resolve this matter, QED and renormalization were developed in the late 1940s. The

key was in changing the parametrization from of the theory from "bare" values, to measured values at some particular energy scale. So to completely define the theory one needed to write down not only the Lagrangian but also the renormalization point. The coupling constants of the theory would depend upon this. With this approach QED was enormously successful, producing very accurate predictions to such quantities as the anomalous magnetic moment.

There was much frustrating work to formulate a theory of the other interactions along the same lines. During this work (such as that by Gell-man and Low) with these types of theories it became clear that there was structure to how the coupling constants behaved as the renormalization point changed.

While working through the Ising model, Wilson, who had a background working with QFTs, realised there were applications of these renormalisation ideas to critical phenomena. Later, working with fixed-source meson theory, had an insight that led to effective field theory. Working with a hierarchy of momentum "slices" and a process of eliminating energy scales, he found that although an infinite number of interactions was generated, the effects at each iteration were bounded – only a finite number of terms were necessary for any particular level of precision. A successful theory with arbitrarily many constants still worked, as long as there was a clear hierarchy of terms.

An effective field theory was not renormalizable in the way QED was, but as Wilson found the proliferation of terms could be contained such that only a finite number mattered for a particular calculation. The constraint is that an effective field theory has application only well below some energy scale. But, the very feature that it is formulated at a particular scale makes it in many ways the natural theory to use at that scale.

1.1.2 A Worked example of Renormalization

Let us review how the process of renormalization works when a theory *is* renormalizable.

The QED Lagrangian defined at some cut-off is:

$$\mathcal{L}_0 = \bar{\Psi}\left(i\partial\cdot\gamma - e_0 A\cdot\gamma - m_0\right)\Psi - \frac{1}{2}(F^{\mu\nu})^2 \tag{1.1.1}$$

and additionally the cut-off regulator Λ_0.

In calculating a process without a cut-off, all intermediate states must be summed over. When kinematics do not strictly dictate the intermediate momenta of some particles, this means integrating over an infinite range. This is exactly the case with an internal loop (as it described in the language of Feynman diagrams), and is what caused the infinities that plauged initial attempts to formulate relativistic quantum mechanics.

By introducing a cut-off, the integrals only take place over a bounded domain of momenta, and are thus finite. The seeming cost is that physical quantities should not depend upon this arbitrary cut-off Λ_0. Still, the parameters can be fixed by comparing with experiment. This theory will then produce results correct up to some terms of $\mathcal{O}(1/\Lambda_0^2)$.

There are two parameters: the bare mass m_0 and the bare charge e_0. These parameters are determined from experiment. Two processes can be calculated (such as electron-electron scattering and the electron scattering off some external field), compared to the experimental measurements, and thus the parameters fixed.

Handed this theory with this high cut-off Λ_0, it is possible to reformulate it in terms of a new, lower cut-off Λ. The new theory will hold valid for processes where the external momenta are much less than Λ.

By introducing this lower cut-off, high energy virtual processes are eliminated from the theory. Rather than being explicitly included, their effects will implicitly be included by corrections to the parameters of the theory.

These corrections can be calculated from the old theory. Before loop-integrals over momenta ran from 0 to the old cut-ff Λ_0. In the new theory, they will run from 0 to Λ. Clearly the difference between the two calculations will be an integral from Λ to Λ_0. Importantly, because Λ is taken to be greater than the energy of any process considered, so will the loop momentum in that sector of the integral.

First consider the calculation of the electron vertex. Call the original value T, which will have contributions from several diagrams. On such contribution is from the diagram (INSERT). Call this contribution $T^{(a)}$.

In the original $\mathcal{L}_0$ theory the contribution would be

$$T(a)(k>0) = -e_0^3 \int_0^{\Lambda_0} \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\left\{\bar{u}(p')\gamma^\mu \frac{1}{(p'-k)\cdot\gamma - m_0} A_{\text{ext}}(p'-p)\cdot\gamma \frac{1}{(p-k)\cdot\gamma - m_0}\gamma_\mu u(p)\right\} \tag{1.1.2}$$

When the momenta are cut-off at the lower point Λ, the part of the old integral missing from the new calculation will be

$$T(a)(k>\Lambda) = -e_0^3 \int_\Lambda^{\Lambda_0} \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\left\{\bar{u}(p')\gamma^\mu \frac{1}{(p'-k)\cdot\gamma - m_0} A_{\text{ext}}(p'-p)\cdot\gamma \frac{1}{(p-k)\cdot\gamma - m_0}\gamma_\mu u(p)\right\} \tag{1.1.3}$$

Remember that Λ is chosen to be a great deal greater than p, p' or m_0, and this then holds for k over the entire range of the integral. Then, if corrections of the type p/Λ are discarded, the integral can be greatly simplified. The approximation used is:

$$\frac{1}{(p'-k)\cdot\gamma - m_0} \approx -\frac{1}{k\cdot\gamma} \tag{1.1.4}$$

Of course for any four vector a, it holds that $(a\cdot\gamma)^2 = a^2$, so

$$-\frac{1}{k\cdot\gamma} = -\frac{k\cdot\gamma}{k^2} \tag{1.1.5}$$

Then

$$\begin{aligned} T^{(a)}(k>\Lambda) &\approx -e_0^3 \int_\Lambda^{\Lambda_0} \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\left\{\bar{u}(p')\gamma^\mu \frac{k\cdot\gamma}{k^2} A_{\text{ext}}(p'-p)\cdot\gamma \frac{k\cdot\gamma}{k^2}\gamma_\mu u(p)\right\} \\ &\approx -e_0^3 \bar{u}(p') A_{\text{ext}}(p'-p)\cdot\gamma u(p) \int_\Lambda^{\Lambda_0} \frac{d^4k}{(2\pi)^4}\frac{1}{k^4} \end{aligned}$$

There are other one electron scattering diagrams. When the above analysis is applied to them all, the total difference is found to be of the form:

$$T^(k>\Lambda) = -ie_0 c_0 (\Lambda/\Lambda_0)\bar{u}(p') A_{\text{ext}}(p'-p)\cdot\gamma u(p) \tag{1.1.6}$$

This is the piece of the electron vertex structure that we are missing if we calculate using the original Lagrangian with the lower cut-off. The correct results will be obtained if we

incorporate into the Lagrangian a new term:

$$\delta\mathcal{L} = -e_0 c_0(\Lambda/\Lambda_0)\bar{u}(p')A_{\text{ext}}(p'-p)\cdot\gamma u(p) \tag{1.1.7}$$

What about the nature of this constant c_0? Consider the above calculation of $T^{(a)}(k > \Lambda)$. It had, like the total correction, had a structure of some constant terms times $\bar{u}(p')A_{\text{ext}}(p'-p)\cdot\gamma u(p)$. The structure of the constant term came from the integral over a function of only k. Since there were no scales involved in this integration, other than the limits of integration, the result must be some function of Λ and Λ_0. And because the integral is dimensionless, it must actually be a function of their ratio Λ/Λ_0. The same logic goes through when the other terms are computed. The final result is that

$$c_0 = -\frac{\alpha_0}{6\pi}\log(\Lambda/\Lambda_0) \tag{1.1.8}$$

That the result of these integrals involves only the limits is contingent upon the approximation made earlier, that the scales p, p' and m_0 are all small compared to Λ. If corrections of that order are important, then there will be additional terms with structures including these momenta and mass. This can be accomplished by, instead of completely neglecting these terms, doing Taylor expansion in terms of p/k, m_0/k and so forth. We'll return to this later.

Going back to the correction $\delta\mathcal{L}$, note that it has almost the same form as a term in the original $\mathcal{L}_0$, the difference being an explicit dependence on the cut-offs Λ, Λ_0. (Although of course, e_0 itself depended on comparing measurements to calculations in the $\mathcal{L}_0$ theory, so it really *was* dependent on Λ_0.) Rather than interpret it as new interaction, then, it can be seen as change in the strength of e_0.

$$-\bar{\Psi}e_0 A\cdot\gamma\Psi \to -\bar{\Psi}e_0[1 + c_0(\Lambda/\Lambda_0)]A\cdot\gamma\Psi \tag{1.1.9}$$

As long as all other scales are considered to small to Λ to enter the calculations, it isn't possible for truly new terms to enter, only corrections to the already existing terms. That means that all that can happen is an adjustment of the existing coupling constants e_0 and

m_0.

There are indeed corrections to m_0, coming from the electron self-energy. An additional correction term is required of the form

$$\delta\mathcal{L} = -m_0\tilde{c}_0(\Lambda/\Lambda_0)\bar{\Psi}\Psi \tag{1.1.10}$$

The Lagrangian valid with the new cut-off Λ can be written in terms of the old as:

$$\mathcal{L}_\Lambda = \bar{\Psi}\left(i\partial\cdot\gamma - e_\Lambda A\cdot\gamma - m_\Lambda\right)\Psi - \frac{1}{2}(F^{\mu\nu})^2 \tag{1.1.11}$$

where the constants are

$$\begin{aligned} e_\Lambda &= e_0(1+c_0) \\ m_\Lambda &= m_0(1+\tilde{c}_0) \end{aligned}$$

Above only two processes were considered, which produced corrections to known interactions. In principle corrections will also arise from other processes, such as electron-electron scattering.

But, consider the form of such cross-sections. They must have a spinor structure something like $\bar{u}u\bar{u}u$ or $\bar{u}\gamma^\mu u\bar{u}\gamma_\mu u$, with an accompanying factor. Certainly it involves four fermion fields. The key point is that the scattering amplitude with four external lines must be dimensionless. Since u has mass dimension $1/2$, d_0 must have dimension -2.

So to write a dimensionless factor d_0 analogous to c_0 above, there must be an additional factor of mass dimension -2. The relevent scale is Λ, so this factor must be $1/\Lambda^2$. (Again, we follow the earlier logic that no other mass scales can enter, being negligible to the highly virtual loop momentum of the correction terms.) So here the correction would need to look something like

$$d_0(\Lambda/\Lambda_0)\frac{1}{\Lambda^2}\bar{u}u\bar{u}u \tag{1.1.12}$$

The actual value of the spinors structure only involves the low energy momenta of the

theory, so it must be suppressed by the larger factor $1/\Lambda^2$. Therefore, these four fermion terms enter at a smaller order than the terms already discussed.

Nonrenormalizable cut-off theories

In the above, corrections small compared to the scale Λ were ignored, terms of order $\mathcal{O}(p/\Lambda)$, $\mathcal{O}(m/\Lambda)$ and so forth. If such terms are important, they may be calculated in the same general manner as outlined for the $\log \Lambda/\Lambda_0$ corrections. However, the logic above that prevented new terms from being introduced now fails, so new types of interaction are to be expected.

There are two key points where terms of this nature were discarded. The first was in the highly virtual loop integrals, where all scales smaller than k were neglected. The second was in considering the types of processes which might introduce corrections to the Lagrangian, where (for example) terms involving four fermions were subjected to dimensional analysis and found to be suppressed by $1/\Lambda^2$.

Going back to the loop integrals, now instead of simply discarding all terms involving p, p', or m_0, a Taylor expansion in p/Λ m_0/Λ (and so on) can be performed. As an example, instead of approximating $1/\{(p-k)\cdot\gamma) - m_0\}$ as $-1/k\cdot\gamma$

$$\begin{aligned}\frac{1}{(p-k)\cdot\gamma) - m_0} &\approx -\frac{1}{k\cdot\gamma(1 - p\cdot\gamma/k\cdot\gamma + m_0/k\cdot\gamma)} \\ &\approx -\frac{k\cdot\gamma}{k^2} + \frac{p\cdot\gamma}{k^2} - \frac{m_0}{k^2}\end{aligned}$$

Systematically using such expansions, the high energy part of the loop calculations unaccounted for by the new cut-off theory can be found. The general form is

$$T(k>\Lambda) = -iec_0\bar{u}(A_{\text{ext}}\cdot\gamma)u - \frac{ie_0m_0c_1}{\Lambda^2}\bar{u}(A^\mu_{\text{ext}}\sigma_{\mu\nu}(p-p')^\nu)u - \frac{ie_0m_0c_2}{\Lambda^2}(p-p')^2\bar{u}(A_{\text{ext}}\cdot\gamma)u \tag{1.1.13}$$

The form of these corrections can in some cases be found from explicit calculation. In such cases, the calculation will also fix the coefficients in the Lagrangian. However, often it

is either difficult or impossible to explicitly "integrate out" high momentum contributions to the theory. One can already see above how much more tedious the integrations over intermediate momenta will become when polynomial terms in p and so forth are included. There will be a proliferation of new terms to account for in the loop integrals, and if pushed to the next order of correction an unpleasant combinatorical explosion will occur. The hope is to use NRQED to *simplify* the calculation.

And there also exist theories where explicit calculation is simply impossible, and the use of an effective Lagrangian is not just a convenience but a necessity. In low energy QCD it is impossible to work with a perturbative expansion of Feynman diagrams. Because of the strength of the coupling, such series do not converge.

In either case, there is an alternate method. For any particular calculation and given level of precision there will be a finite number of terms in the Lagrangian that contribute. The existence of terms is limited by two factors:

- First, there are direct constraints on the form of the Lagrangian: current conservation, Lorentz invariance, chiral symmetry and so on, and these will apply to each term separately
- Second, only terms of up to a particular order are kept

It is this second point that ensures that the number of terms is finite. Typically each building block that might be used to construct a term comes with at least one power of mass dimension. And the greater the mass dimension of the term, the stronger the suppression by $1/\Lambda$. Building blocks of order unity do exist, such as spin space operators, but there will be a finite basis for such.

Once the form of all possible terms is catalogued, how then to fix the coefficients before each? It is important that, no matter what, physical predictions obtained from either the effective theory or the high energy theory must coincide. So if the same physical process is calculated using both theories, then demanding equality of the two results will determine the coefficients of the low energy theory in terms of the original.

Still, that in principle the coefficients can be obtained by integrating out high momentum loops can still tell us something of their behavior. When all other energy scales were ignored,

the constant c_0 could depend only on Λ/Λ_0. When they are included, it may additionally depend on m_0/Λ. It will never depend upon the momentum p, because such terms are instead included as new interactions with separate coefficients.

So c_0 will be the same coefficient calculated earlier, but with additional corrections of order $\mathcal{O}(m_0^2/\Lambda^2)$. The Lagrangian must also be augmented by the new interactions, so there is an additional correction of the form

$$\delta\mathcal{L} = \frac{e_0 m_0 c_1}{\Lambda^2}\bar{\Psi}F^{\mu\nu}\sigma_{\mu\nu}\Psi + \frac{e_0 m_0 c_2}{\Lambda^2}\bar{\Psi}i\partial_\mu F^{\mu\nu}\sigma_{\mu\nu}\partial_\nu\Psi \tag{1.1.14}$$

(In writing down the terms in the Lagrangian, momentum become derivatives of the fields, so for example $q = p' - p$ becomes a derivative of A_{ext}.)

In addition to these new higher order corrections to the already calculated quantities, there will be contributions to the Lagrangian of new processes. For instance, electron-electron scattering enters at the $\mathcal{O}(p^2/m^2)$. But of course it's not the case that all processes now enter — a process with 6 external legs would be suppressed by $1/\Lambda^4$ and not enter at the currently considered order.

These four fermion terms come from the contributions of other process than simple scattering off an external field. Rather, they come from integrating out the high momentum modes of processes such as electron-electron scattering. The loop diagram corrections to such processes involve in the high-energy theory, like the other loops mentioned, integrals over momentum higher than the cut-off. These intermediate states are highly virtual.

The uncertainty principle says that such high energy virtual states are allowed only if they exist for a correspondingly short amount of time. The result is that in the low energy theory they may be treated as effectively instantaneous interactions, appearing as local contact terms in the Lagrangian. This is how new multiple particle interactions arise in an effective theory – the high energy process becomes a new local interaction.

For electron-electron scattering at the order discussed, such terms would be the likes of

$$\delta\mathcal{L}_{4-\text{fermion}} = d_1\frac{e_0^2}{\Lambda^2}(\Psi^\dagger\Psi)^2 + d_2\frac{e_0^2}{\Lambda^2}(\Psi^\dagger\gamma\Psi)\cdot(\Psi^\dagger\gamma\Psi) \tag{1.1.15}$$

These coefficients would be fixed by calculating a process like electron-electron scattering in both theories. However, one would first have to fix the constants c_i. For in the new theory with a low cut-off there will be contributions to the scattering not only from contact terms, but also from tree level diagrams involving two 2-fermion vertices.

1.2 Nonrelativistic Quantum Electrodynamics

Above we explored how, by changing the cut-off in QED, new nonrenormalizable terms appeared, and a new effective theory emerged. It was suitable only for calculations below the new cut-off, which was chosen to be well above the scale of any other momenta or energy in the theory. However, one particularly fruitful use of this technique is the formulation of a theory of nonrelativistic quantum electrodynamics (NRQED).

NRQED is most useful when working with bound state systems. There are many important corrections to bound state energy levels that come from high energy physics. The more precisely one measures these energy levels, the more information is gained about the high energy theory. But only if the predictions of that theory have been worked out with the necessary precision.

While in principle the full QED theory can be used to accomplish this, it is unweildy and unsuited for the task. Most typically the systems studied are loosely bound and non-relativistic. While calculating non-relativistic scattering in QED might not be too bad, the situation is different for a bound state. What spoils everything is the existence of new energy scales.

One such scale is the inverse of the Bohr radius. If μ is the reduced mass of the system and Ze the charge of the center, then

$$p \sim Z\mu\alpha = \frac{1}{r_{\text{Bohr}}} \tag{1.2.1}$$

is the typical momentum scale of the bound system.

In a QED scattering calculation, the order of a term may be addressed as the number of loops in a diagram. Thus we talk about tree level diagrams, one-loop diagrams, two-loop diagrams and so on, with the understanding that each loop carries with it a suppressing

factor of α. In this bound state system the typical momentum scale may enter in a way that exactly cancels the loop factor. So to calculate contributions of order α an infinite number of diagrams must in fact be summed.

There are techniques of doing this, but clearly it becomes trickier to easily sort out what diagrams contribute at a particular order. The matter is made worse by the existence of a third distinct scale, the kinetic energy of each particle:

$$E = \frac{(Z\mu\alpha)^2}{m_i} \tag{1.2.2}$$

further complicating the process of calculating the contributions of a given order.

Well, QED is a high energy theory, while calculations should be easiest in a low energy theory formulated for this nonrelativistic regime. What is desired is a theory that lives at the appropriate scale, and thus avoids the business of summing infinite numbers of diagrams, but never-the-less incorporates all the effects of high energy physics.

Of course, an effective field theory has exactly these characteristics. The procedure is as follows. First write down the most general Lagrangian that obeys the symmetries of the theory. Of course here, in going from the high energy theory to the effective nonrelativistic, Lorentz symmetry is no longer required.

Once the form of the Lagrangian is fixed, the same physical process may be calculated in QED and NRQED. There is no problem in performing the QED calculation in this step because we don't have to choose a bound-state calculation – the scattering of free particles will suffice. The idea here is to find the simplest calculation that will fix the coefficients.

It is these coefficients that then contain all the information about the high energy theory. Bound state calculations may then be performed using the NRQED Lagrangian and diagrams, with the desired result: high energy physics is included, but we have a workable theory in the nonrelativistic regime. Instead of figuring out how to find a correct perturbation series in α, the theory uses expansions in terms of the other two nonrelativistic (and thus small) scales; basically expanding in terms of the velocity v as well as α.

Before we examine how the process works explicitly for NRQED, there is an alternate approach that should be examined first.

1.2.1 The Foldy-Wouthuyesen-Tani Transformation

As sketched in the previous section, to work with bound state problems it is simplest to have a nonrelativistic Lagrangian to work with. After all, in regular quantum mechanics it isn't so hard to calculate the levels of hydrogen. It is using a relativistic theory to crack a nonrelativistic nut that causes problems.

If the goal is to derive a nonrelativistic Lagrangian or Hamiltonian for the system, one approach is to start from the Dirac equation and find a way to express it nonrelativistically.

One such approach is the Foldy-Wouthuyesen-Tani transformation. Consider the Dirac Lagrangian:

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi} \left(i\partial \cdot \gamma - eA \cdot \gamma - m \right) \Psi \tag{1.2.3}$$

If the system is nonrelativistic, then one of the important consequences is that electrons and positrons behave pretty much as independent particles. A representation can be chosen such that the upper and lower components of the bispinors Ψ are roughly equivalent to the electron and positron spinors. There will be some mixing, and it is the FWT transformation that finds a form for the bispinors in such a way that all the operators above become diagonal. Then, the upper and lower components are completely separated, and one can easily treat them as separate fields.

Technically this is impossible when an external field A_{ext} is present. However, a basis where the mixing between "particle" and "anti-particle" is arbitrarily small may be found perturbatively. The result will be an expansion in terms of $|\mathbf{p}|/m$, that relates the original upper and lower components to the desired "Shrodinger-like" spinors.

The second order result will be, for a Dirac bispinor u with upper and lower components ϕ and ξ:

$$u = \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} (1 - \frac{p^2}{8m^2})\phi_s \\ \frac{\sigma \cdot p}{2m}(1 - \frac{3p^2}{8m^2})\phi_s \end{pmatrix}$$

An important feature here is that the lower component is suppressed compared to the upper, by a factor of $\sigma \cdot p/2m$. But the overall normalisation must stay the same. This can actually

be used to find the correct relations between relativistic and nonrelativistic wave functions without formally performing the transformation.

Once this is accomplished, the Lagrangian may be rewritten directly in terms of, say, the electron spinor. The relativistic bispinor has been replaced by a two-component nonrelativistic spinor. Additionally, all the other terms can be rewritten in terms of Galilean three vectors and scalars instead of Lorentz 4-vectors. The result is a manifestly nonrelativistic expression.

If an external electric field is considered acting on a single charged particle of mass m and charge e is considered, then to the second order

$$H = \frac{\mathbf{p}^2}{2m} + e\Phi - \frac{\mathbf{p}^4}{8m^3} - \frac{1}{4m^2}\boldsymbol{\sigma}\cdot\mathbf{E}\times\mathbf{p} - \frac{1}{8m^2}\boldsymbol{\nabla}\cdot\mathbf{E} \tag{1.2.4}$$

In writing the Hamiltonian, of course derivatives become momentum operators instead. Clearly, the higher order terms have relativistic origins — $\mathbf{p}^4/8m^3$, for instance, is the first relativistic correction to the kinetic energy $\mathbf{p}^2/2m$.

From this starting point ordinary Rayleigh-Schrodinger perturbation theory may be used. Because the theory is explicitly nonrelativistic, it will avoid all the scale problems with QED.

However, this isn't as powerful as an effective field theory. The idea here was to discard high energy processes rather than to incorporate them. For instance, the process

$$e^-e^+ \to \gamma \to e^-e^+ \tag{1.2.5}$$

involves a fundamentally relativistic intermediate photon that this formulation hasn't incorporated, and so relativistic corrections to the photon propagator will matter. If instead the techniques of effective field theory are employed, no process with such relativistic internal momenta need be considered. Instead, the effects of this process will be incorporated as four fermion contact terms.

As the uncertainty principle tells us, the higher the energy of the intermediate state, the shorter the length of the time that the state can exist. So a process with a relativistic internal

state as above should happen instantaneously, acting just like a local contact interaction between four fermions.

When higher-order perturbation theory is attempted with the above Hamiltonian, it also will fail. The terms diverge, producing infinities. This of course has the same root cause.

So, if high precision is important and all high energy processes must be accounted for, this method isn't accurate enough. It does, of course, predict the leading order coefficients of NRQED.

1.2.2 An NRQED Calculation

Let us now consider the effective field theory approach to finding, from the relativistic theory, a nonrelativistic Lagrangian suitable for the bound-state problems already mentioned. As an example, consider a hydrogen-like muonium system with an electron and a muon. Formally the idea is to introduce into QED a cut-off at the nonrelativistic energy, λ. This cut-off is somewhere about the energy of the electron mass m_e. The higher energy states are removed, leaving an explicitly nonrelativistic Lagrangian.

As normal for an effective field theory, this Lagrangian will have an inifinte number of terms, but they can be arranged in a hierarchy that allows any particular calculation to be performed with only a finite number. This hierarchy can be treated as an expansion over the large mass scale of the system, $1/m$. In the example calculation, terms up to order $1/m^3$ will be kept.

First a Lagrangian is written that contains all the possible terms that might contribute to the process considered. Lorentz symmetry is no longer required, but the following constraints on the forms of terms exist:

- Galilean invariance
- Invariance under spatial reflection
- Invariance under time reversal
- Gauge invariance

- Hermiticity
- Locality

Once all the terms are catalogued, their coefficients must be determined. The idea is to consider some particular physical process. For the two formulations to be consistent, they must produce the same predictions for any such process. So something like a scattering amplitude can be calculated in both theories, and the result compared.

1.2.3 Comparison of QED and NRQED

The simplest process that will give the required information can be used. So any assumption that still distinguishes between needed terms in the NRQED Lagrangian may be used. In comparing scattering diagrams, it is often most useful to perform the calculation at threshold. Likewise, the frame of reference which most readily simplifies the calculation should be used. Any physical assumption must be applied to each calculation in the same way, of course – it must be the same physical process!

However, given the assumption of gauge invariance, different gauges may be chosen for each calculation. Since the physical measurement will be gauge invariant, there is nothing inconsistent about this. And indeed the most convenient gauges for a relativistic and nonrelativistic calculation often differ.

Remember that the goal is to use NRQED to calculate bound state energies. To this end, it is necessary that the theory has poles in the complex plain and can thus be analytically continued to include off-shell bound states. To ensure this, it is necessary to demand that all external particles be on mass-shell when fixing the coefficients. For the same reason, it is necessary to perform all the intermediate calculations with a finite photon mass. **Why? I don't really follow this point very well.**

So that the processes we compare have the same meaning, it is necessary for the S-matrix to have the same normalisation in both theories. Because of this, the normal relativistic normalisation of the QED spinor cannot be used. It is conventional to set $\bar{u}u = 2m$, but this will make it hard to find a sensible relation between the spinors of QED and NRQED. Instead, the normalisation $u^\dagger u = 1$ will be used.

Given that all these conditions are met, the process of comparing physical results will fix, without ambiguity, the coefficients of NRQED.

1.2.4 Construction of NRQED Lagrangian

In order to perform any comparison with QED, it is first necessary to catalogue all the terms that could arise in the Lagrangian. Let us work first with only the two-fermion terms.

What are the building blocks at our disposal? Each term will have two electron fields, possibly the external electromagnetic field, and derivatives of either type of field. There can also be mixing between the spin-space of the fermion fields.

The first constraint to apply is gauge-invariance. If only gauge invariant combinations are allowed, then instead of any mixture of derivatives and fields, only the long derivate $D = \partial + ieA$ is allowed, along with E, B and their derivatives.

The operators that mix spin for the spin-1/2 electron are just the Pauli spin matrices σ_i. Together with the identity they form a basis for the space of Hermitian spin-operators, so no quadratic terms in spin can appear. Those can be directly reduced with the identity $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}S_k$.

Many of these "blocks" are written as three-vectors. For Galilean invariance to hold, there can be no dangling indices — all terms must be contracted, either with each other or the anti-symmetric tensor ϵ_{ijk}.

Lets consider now the symmetries of each type of term.

- The electric field is Hermitian, odd under parity and even under time reversal.
- The magnetic field is Hermitian, even under parity and odd under time reversal
- The long derivative is anti-Hermitian, odd under parity, and even under time reversal
- Spin is Hermitian, even under parity, odd under time reversal.

Given a collection of these blocks, then as long as the set as a whole isn't anti-Hermitian we can always simply add the Hermitian conjugate to form a Hermitian term. This does reduce the number of independent terms. Likewise, to make a term invariant under time-reversal a factor of i can be appended. However, there is no way to fix a term that is odd

under parity, so that is the one hard constraint. There can be no term $\mathbf{B} \cdot \mathbf{D}$, for instance, because it is not invariant under parity.

So far these constraints still allow for an infinite number of terms. But the greater the number of fields, or derivatives of fields, that exist in a term, the more strongly it will be suppressed. Each additional power of energy will carry a factor of $1/m$ to ensure that the total dimension of the term is allowed.

To figure out how far down the hierarchy of terms we need to go for a particular calculation, we also need to establish the individual order of magnitude of each term and "building block". This will allow us to eliminate all but a finite number of terms from consideration. The example case is an electron in an atomic system, so A_0 is the Coulomb potential, and $\mathbf{B}$ comes from interaction with the nucleus, in this case a muon.

.

$$\begin{aligned} \partial &\sim mv \\ \partial_t &\sim mv^2 \\ eA_0 &\sim mv^2 \\ e\mathbf{A} &\sim mv^3 \\ e\mathbf{E} &\sim m^2v^3 \\ e\mathbf{B} &\sim m^2v^3 \end{aligned}$$

The magnitudes of the fields can be readily derived from their atomic origin. The spin operator is order unity (when we have $\hbar = 1$.)

For the Coulomb potential,

$$e\Phi \sim \frac{Z\alpha}{r} \tag{1.2.6}$$

For an atom, $1/r \sim mv = m(Z\alpha)$. So the order of $e\Phi$ is mv^2.

The electric field is the first derivative of Φ, so

$$eE_i \sim \frac{Z\alpha}{r^3} r_i \sim m^2(Z\alpha)^3 = m^2v^3 \tag{1.2.7}$$

The magnetic field $B \sim$

To be consistent with the Schrodinger equation, $\partial_t \sim mv^2$.

When acting on the fermion wave functions, the spatial derivative will bring down a factor of p. Upon the electromagnetic fields, it gives an overall $1/r$. In both these cases, ∂ can be considered to be of order mv

Knowing both the order and symmetries of each building block, the NRQED Lagrangian can be explicitly constructed.

It is the suppressing powers of $1/m$ that keep the number of terms to be considered finite. A term with six derivatives like D^6 need not be considered, because it must appear in $\mathcal{L}_{NRQED}$ as $\mathbf{D}^6/32m^5$. So the first step is to consider what combination of such terms *are* allowed, if terms of higher order than mv^4 are discarded. The only allowed spin structure is σ.

The leading order terms appear at mv^2. Since E and B are already too high order, the only available terms are A_0, ∂_0, and D_i. Without spin, A_0 and ∂_0 appear by themselves. A single power of D_i has nothing to contract with, and two powers may only be contracted with each other. So the spinless terms are

$$eA_0,\ \partial_0,\ \frac{\mathbf{D}^2}{2m} \tag{1.2.8}$$

A single power of D contracted with σ is disallowed by parity, and something like $(\boldsymbol{\sigma}\cdot\mathbf{D}\,)^2$ is redundant, since quadratic terms in σ reduce. Only linear combinations of σ need be considered. None of the other ingredients have Lorentz indices, so no spin terms appear at this order.

The next order of terms are mv^3. This could be a single power of E, a single power of B, or three powers of D. However, since E and D are both odd under parity, those terms are not allowed. The only one left is a single power of B. Since the index needs to be contracted with something, it must be the order unity spin matrix. Thus, the only term allowed at this order is $\boldsymbol{\sigma}\cdot\mathbf{B}$:

$$\frac{e}{m}\boldsymbol{\sigma}\cdot\mathbf{B} \tag{1.2.9}$$

At order mv^4, there could be four powers of D, or a single power of D accompanied by either E or B. However, something like $\mathbf{B}\cdot\mathbf{D}$ is forbidden by parity. With four powers of D, the only way to contract them is as D^4. With E and D there are two ways to contract them: either with δ_{ij} or with $\sigma_k\epsilon_{ijk}$. In any case the resulting term must be Hermitian and invariant under time reversal, so the two terms are

$$\frac{\mathbf{E}\cdot\mathbf{D}-\mathbf{D}\cdot\mathbf{E}}{4m^2},\ \frac{i\epsilon_{ijk}(E_iD_j-D_iE_j)}{4m^2} \tag{1.2.10}$$

Finally at order mv^5 there are more terms involving B. Parity again terms involving only combinations of E and D, but one power of B and two powers of D is allowed. Because there are an odd number of indices, and contracting ϵ_{ijk} with two powers of D is redundant (producing an $\mathcal{O}(mv^6)$ term B^2), all the terms will involve σ_i.

The allowed combinations are

$$\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B}+\mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3},\ \frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3},\ \frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D})+(\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3} \tag{1.2.11}$$

Of course in the special case where derivatives of B vanish (i.e. B commutes with D) then the first two terms become indistinguishable. That is not the case here.

Now we have the Lagrangian, including terms up to order $1/m^3$.

What will the overall structure of the Lagrangian be? There will be two fermion terms, four-fermion terms, and photon terms. The photon terms are taken as just that of QED at leading order: $\frac{1}{2}(F^{mu\nu})^2$.

Now examine the two-fermion part in more detail. It will include kinetic terms that are not renormalized and thus need no coefficients. The leading order terms will go as $1/m$, and there will then be corrections coming from additional terms.

Call the coefficients in this two-fermion Lagrangian c_i. We can expect that the leading order coefficients of these two-fermion terms should replicate exactly the results of doing a Foldy-Wouthuysen-Tani transformation. So for convenience they can be written such that they are, if they exist at this order, equal to 1. So such coefficients will have the form

$$c_i = 1+\mathcal{O}(\alpha) = 1+c_i^{(1)}\alpha+c_i^{(2)}\alpha^2+\ldots \tag{1.2.12}$$

If we consider terms up to $1/m^3$, then the two-fermion part of the Lagrangian is:

$$\begin{aligned}\mathcal{L}_{NRQED} =&\Psi^\dagger\Bigg\{iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^2} + c_F\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + c_D\frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2} \\ &+ c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}}{8m^2} + c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3} \\ &+ c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3}\Bigg\}\Psi\end{aligned} \tag{1.2.13}$$

In addition to the two-fermion terms, at the order $1/m^2$ four fermion contact terms are allowed. Since the QED Lagrangian has no exact four-fermion contact terms, these are terms that arise from the removal of four-fermion diagrams involving high momenta loops. In contrast with the two-fermion Lagrangian, label the coefficients of such terms d_i.

These coefficients will have a more complex structure than c_i. Any process involving two distinct fermion fields will have a richer set of energy scales and parameters to draw from.

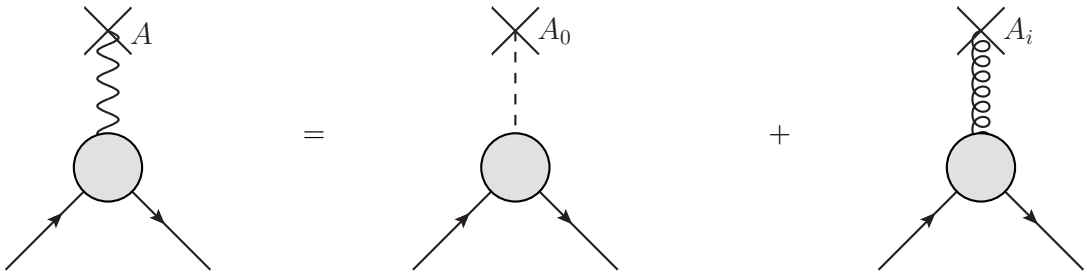
If the additional fermion has mass M and spinor χ, then the contact Lagrangian is:

$$\begin{aligned}\mathcal{L}_{\text{contact}} =&d_1\frac{1}{mM}(\Psi^\dagger\boldsymbol{\sigma}\Psi)\cdot(\chi^\dagger\boldsymbol{\sigma}\chi) + d_2\frac{1}{mM}(\Psi^\dagger\Psi)(\chi^\dagger\chi) \\ &+ d_3\frac{1}{mM}(\Psi^\dagger\boldsymbol{\sigma}\chi)\cdot(\chi^\dagger\boldsymbol{\sigma}\Psi) + d_4\frac{1}{mM}(\Psi^\dagger\chi)(\chi^\dagger\Psi)\end{aligned} \tag{1.2.14}$$

The terms with coefficients d_3 and d_4 only enter if the additional fermion is actually the anti-particle of the original.

1.2.5 Determination of coefficients

To determine the coefficients of the two-fermion piece of the NRQED Lagrangian, it will suffice to calculate the scattering of the electron off an external field. Every coefficient that needs to be fixed accompanies at least one term with a single power of the photon field. First the scattering amplitude will be calculated in QED, then compared to the NRQED terms. Schematically the equivalence can be written as:



A general one-photon vertex in QED may be expressed in terms of form factors $F_1(q^2)$ and $F_2(q^2)$. These encode all the information about radiative corrections, so the QED calculation here can conveniently be expressed with such factors.

The amplitude of scattering off a static vector potential is

$$e\bar{u}(p')\left[-\boldsymbol{\gamma}\cdot\mathbf{A}(\mathbf{q})F_1+\frac{i}{2m}\sigma^{ij}A^iq^jF_2\right]u(p) \tag{1.2.15}$$

or in terms of Pauli spinors instead, up to $1/m^3$

$$\begin{aligned}=&F_1\Psi^\dagger\left[-\frac{e}{2m}(\mathbf{p}'+\mathbf{p})\cdot\mathbf{A}-\frac{ie}{2m}\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A}+\frac{ie}{8m^3}(\mathbf{p}'^2+\mathbf{p}^2)\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A}\right]\Psi\\&+F_2\Psi^\dagger\left[-\frac{ie}{2m}\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A}+\frac{ie}{16m^3}(\mathbf{p}'^2+\mathbf{p}^2)\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A}+\frac{ie}{8m^3}(\boldsymbol{\sigma}\cdot\mathbf{p}')(\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A})(\boldsymbol{\sigma}\cdot\mathbf{p})\right]\Psi\end{aligned} \tag{1.2.16}$$

For scattering off a static potential A_0

$$e\bar{u}(p')\left[\gamma^0A^0F_1-\frac{i}{2m}\sigma^{0j}A^0q^jF_2\right]u(\mathbf{p}) \tag{1.2.17}$$

Again expressing in terms of Pauli spinors

$$\begin{aligned}=&F_1\Psi^\dagger\left[eA^0-\frac{e}{8m^2}\mathbf{q}^2+\frac{ie}{4m^2}\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p})A^0\right]\Psi\\&+F_2\Psi^\dagger\left[-\frac{e}{8m^2}\mathbf{q}^2+\frac{ie}{4m^2}\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p})A^0\right]\Psi\end{aligned} \tag{1.2.18}$$

So the electron-external field scattering, as calculated in QED, has been expressed in terms of mostly nonrelativistic quantities. There are still the form factors to expand, which

depend on $q^2 = (q-0)^2 - \mathbf{q}^2$. But q_0 is of lower order than $\mathbf{q}$, so the leading order corrections will involve only $\mathbf{q}^2$. They are

$$\begin{aligned} F_1(q^2) &= 1 - \frac{\alpha}{3\pi}\left[\frac{\mathbf{q}^2}{m^2}\left(\ln\frac{m}{\lambda}\right)\right] \\ F_2(q^3) &= a_e - \frac{\alpha}{\pi}\frac{\mathbf{q}^2}{12m^2} \end{aligned}$$

They contain an explicit dependence on the finite photon mass λ.

The first part of the calculation of scattering from the NRQED Lagrangian is straightforward — only tree level vertices enter the calculation, so it can be read directly off the Lagrangian. However, this would fix the coefficients as having a direct dependence on the photon mass, rather than, as one might expect, the value of the cut-off. For instance, the coefficient of the Darwin term would be found to be

$$c_D^{QED} = 1 + \frac{\alpha}{\pi}\frac{8}{3}\left[\ln\left(\frac{m}{\lambda}\right) - \frac{3}{8}\right] \tag{1.2.19}$$

But of course, in calculating from NRQED at this level of precision perturbation theory terms will also enter. This introduces a further renormalization of the coefficients, in a similar manner to the production of counterterms in QED. Without further correction these perturbations would spoil the agreement of QED and NRQED. The solution is to introduce an additional term in the NRQED Lagrangian that simply subtracts off the unwanted term.

To again use the Darwin term as an example, it was found that in the absence of perturbation theory the value for c_D labelled c_D^{QED} would cause QED and NRQED to predict the same result. However, when perturbations are taken into account there will be an *additional* contribution, of the same form of the Darwin term, with some coefficient we can call c_D^{NRQED}. So the actual value of c_D should be adjusted in order to bring the two theories back into agreement:

$$c_D = c_D^{QED} - c_D^{NRQED} \tag{1.2.20}$$

The particular value found for c_D^{NRQED} is

$$c_D^{NRQED} = -\frac{\alpha}{\pi}\frac{8}{3}\left[\ln\left(\frac{\lambda}{2\Lambda}\right) + \frac{5}{6}\right] \tag{1.2.21}$$

Which means

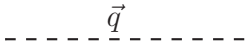
$$c_D = 1 + \frac{\alpha}{\pi}\frac{8}{3}\left[\ln\left(\frac{m}{\lambda}\right) - \frac{3}{8} + \frac{5}{6}\right] \tag{1.2.22}$$

With the result that the final correction to this coefficient depends upon the cut-off, just as with the regular renormalization theory. The same idea goes through to each other coefficient.

1.2.6 Vertices

To formally establish rules for the various vertices written above, it is necessary to strip off the external fields, and write derivatives of such fields in terms of their momentum. The coulomb potential A_0 becomes a Coulomb photon, so (for example) $\boldsymbol{\nabla} \cdot \mathbf{E}$ becomes the Darwin vertex, consisting of two fermion legs and a Coulomb photon, and a vertex with a factor of $-(e/8m^2)\mathbf{q}^2$.

Coulomb photon propagator

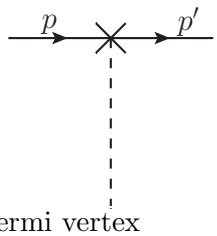


$= \quad \frac{1}{\mathbf{q}^2+\lambda^2}$

Transverse photon propagator

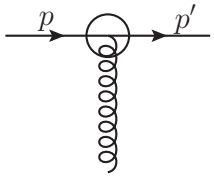
$= \quad \frac{\delta^{ij}-\frac{q^i q^j}{\mathbf{q}^2-\lambda^2}}{(q^0)^2-\mathbf{q}^2-\lambda^2+ie}$

Darwin vertex



$= \quad \frac{-e}{8m^2}\left|\mathbf{p}' - \mathbf{p}\right|^2$

Fermi vertex



$= \quad \frac{ie}{2m}(\mathbf{p}' - \mathbf{p}) \times \boldsymbol{\sigma}$

Chapter 2

Spin one-half

2.1 Structure of spin one-half theory

The goal is to obtain a nonrelativistic theory from the relativistic. To that end it'll help to have a clear understanding of the structure of the two theories.

2.1.1 Relativistic framework for spin one-half

In the relativistic theory, the electrons are part of a fermion field that also includes the positron anti-particle. Since both are spin-1/2, each has two spin orientations. There are, then, a total of four degrees of freedom.

The relativistic Lagrangian for the fermion fields is

$$\mathcal{L} \quad = \quad \bar{\Psi}(i\partial \cdot \gamma - eA \cdot \gamma - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

The fermion fields are Ψ, the photon field is A. m is the particle's mass, and e the electron's charge. The γ matrices mix the different components.

It will be convenient to work in a representation which already suggests the nonrelativistic behavior. At low momenta, it should be expected that the free electron and positron fields act approximately as independent fields. This is exactly the case for the Dirac representation. In the rest frame, a free particle can be said to be definitively an electron or positron, and in the Dirac representation these correspond to the upper and lower parts of

the bispinor.

In this representation, the gamma matrices are written

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1.1}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{2.1.2}$$

The gamma matrices by themselves do not form a complete basis for this space. To find such a basis products of the matrices can be considered. It will make sense to consider combinations such that bilinears are Hermitian.

There is the identity — such bilinears transform as a scalar.

Symmetric combinations are not considered because $\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$. The antisymmetric combinations are explicitly

$$\begin{aligned} [\gamma^0, \gamma^i] &= 2\gamma^0\gamma^i = \begin{pmatrix} 0 & 2\sigma_i \\ 2\sigma_i & 0 \end{pmatrix} \\ [\gamma^i, \gamma^j] &= \begin{pmatrix} -2i\epsilon_{ijk}\sigma_k & 0 \\ 0 & -2i\epsilon_{ijk}\sigma_k \end{pmatrix} \end{aligned}$$

Using these a tensor like structure arises:

$$\sigma^{\mu\nu} = i\frac{1}{2}[\gamma^\mu, \gamma^\nu] \tag{2.1.3}$$

The specific form in the Dirac representation is

$$\sigma^{ij} = \frac{i}{2}[\gamma^i, \gamma^j] = \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \tag{2.1.4}$$

$$\sigma^{0i} = i\gamma^0\gamma^i = \begin{pmatrix} 0 & i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \tag{2.1.5}$$

The product of each gamma matrix in turn gives a pseudo-scalar:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.1.6}$$

The product of γ^5 and γ^μ gives a pseudo-vector

$$\gamma^5\gamma^\mu \tag{2.1.7}$$

$$u = \begin{pmatrix} \eta \\ \chi \end{pmatrix} \tag{2.1.8}$$

2.1.2 Nonrelativistic framework for spin one-half

$$\begin{aligned} \eta &= \left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w \\ \chi &= \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\left(1 - \frac{3\mathbf{p}^2}{8m^2}\right) w \end{aligned}$$

2.2 Foldy-Wouthyusen approach

The goal is to derive a nonrelativistic Hamiltonian or Lagrangian starting from relativistic theory. (Having obtained one, we can easily obtain the other, of course.) One method is to take the relativistic equations of motion and use them to obtain a Schrodinger like equation.

The starting point is the relativistic equations of motion, which can come from the Lagrangian of the relativistic theory. Those equations can then be written in terms of the noncovariant quantities that appear in the nonrelativistic theory. In doing so, the energy of the particle will now explicitly appear.

The relativistic theory will contain not only the particle of interest (the electron) but also its anti-particle (the positron.) The nonrelativistic theory should contain only the electron.

Before obtaining an expression for the nonrelativistic Hamiltonian it will be necessary to somehow disentangle the two fields. This is impossible in the general case, but as long as the energy and momenta in question are nonrelativistic, can be accomplished to any desired order.

Formally this is accomplished by the Foldy-Wouthyusen transformation, the result of which is that all operators are diagonal, the coupling between the particle and anti-particle suppressed to which ever order is desired. However, practically the same result can be obtained by examining the normalisation of the two theory's particles. By demanding that the Schrodinger like wave functions are appropriately normalized, the relationship between relativistic and nonrelativistic spinors can be established.

The result of this procedure will be an equation for the energy of the electron, accurate at some order in the nonrelativistic expansion. However, it will not perfectly replicate the predictions of the high energy theory. Unlike the process of NRQED, it does not truly incorporate the high energy sector of the theory.

2.2.1 Equations of motion

To reestablish the problem considered, the system to be examined is an electron placed in a loosely bound system with another charged particle, subject to an infinitesimal and constant magnetic field. There will be, because of the bound system, an electric field acting on the electron as well as the external magnetic field. When recoil effects are ignored, the electric field can just be taken as given.

The corrections to the gyromagnetic ratio of the electron are to be established. There are two small scales that appear in the problem, the velocity v of the electron and the infinitesimally small magnetic field eB. The precision desired requires terms of up to order mv^4 and $(e/m)Bv^2$.

The starting point will be the relativistic Lagrangian of Dirac. However, remember that the technique to be used simply ignores behavior introduced by the high energy sector of the theory, even if it might effect the low energy behavior. One such effect is corrections to the free gyromagnetic ratio of the electron, which first arise when one loop diagrams are considered. Without such corrections the g-factor will be exactly $g = 2$.

Knowing that there will actually be bound-state corrections proportional to $g-2$, it is necessary to somehow include this anomalous term. The way to do so is to introduce a new local interaction into the Lagrangian, coming from the high energy radiative corrections which dress the electron vertex. The Lagrangian to be used is, then

$$\mathcal{L} \quad = \quad \bar{\Psi}(\not{D} - m)\Psi + \frac{1}{2}\mu' \bar{\Psi}\sigma^{\mu\nu}F_{\mu\nu}\Psi$$

μ' is the correction to the classical moment $\mu_0 = \frac{e}{2m}$, and is equal to $(g-2)/2\mu_0$.

From this Lagrangian the equations of motion of the particle may be obtained from the Euler-Lagrange method. The Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial \bar{\Psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi})} = 0 \tag{2.2.1}$$

In the Lagrangian above, we can consider that all differential operators act only on the right field Ψ. (This freedom of choice comes from being able to rewrite the Lagrangian through integration by parts, without changing its physical meaning.) So the second term in the Euler-Lagrange equation can be ignored, and after differentiating with respect to $\bar{\Psi}$ the following equation is obtained:

$$(\not{p} - m + \frac{1}{2}\mu' \sigma^{\mu\nu}F_{\mu\nu})\Psi = 0 \tag{2.2.2}$$

Writing explicitly in terms of the γ matrices, this is

$$\left((p_\mu - eA_\mu)\gamma^\mu - m + i\mu' \frac{1}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}\right)\Psi = 0 \tag{2.2.3}$$

This equation of motion is invariant under Lorentz transformations. It is written in terms of the Dirac bispinor Ψ, the four momentum p_μ, external fields A_μ and $F_{\mu\nu}$, and the gamma matrices. To apply in to a nonrelativistic problem, the very first step will be to rewrite it in terms of the sorts of quantities that appear in that domain: three-vectors and scalars.

The scalars that appear will be p_0 and $A_0 = \Phi$. The external fields $\mathbf{E}$ and $\mathbf{B}$ will appear explicitly, while the vector field $\mathbf{A}$ will appear in the gauge-invariant operator $\boldsymbol{\pi} = \mathbf{p} - e\mathbf{A}$.

The gamma matrices can be written in terms of the Pauli spin matrices $\boldsymbol{\sigma}$. Finally, the bispinor will be written in terms of its upper and lower components.

Of the terms that appear in (2.2.3), all except the last are trivial to write in this manner. To deal with that last term, the antisymmetric tensor $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ needs to be written explicitly.

Using the antisymmetry of $\sigma^{\mu\nu} \equiv$, and that we deal with time-independent fields:

$$\begin{aligned}
[\gamma^0, \gamma^i] &= \begin{pmatrix} 0 & 2\sigma_i \\ 2\sigma_i & 0 \end{pmatrix} \\
[\gamma^i, \gamma^j] &= \begin{pmatrix} -2i\epsilon_{ijk}\sigma_k & 0 \\ 0 & -2i\epsilon_{ijk}\sigma_k \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
F_{\mu\nu}\sigma^{\mu\nu} &= F_i\sigma^{ij} - F_{0i}\sigma^{0i} - F_{i0}\sigma^{i0} + F_{00}\sigma^{00} \\
&= F_{ij}\sigma^{ij} - 2F_{0i}\sigma^{0i} \\
&= 2\partial_i A_j \sigma^{ij} - 2\partial_i \Phi \sigma^{0i} \\
&= -2i \begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{B} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\mathbf{B} \end{pmatrix} - 2 \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{E} \\ \boldsymbol{\sigma}\cdot\mathbf{E} & 0 \end{pmatrix}
\end{aligned}$$

The bispinor Ψ is written in terms of upper and lower components

$$\Psi = \begin{pmatrix} \eta \\ \chi \end{pmatrix} \tag{2.2.4}$$

With these considerations, the (2.2.3) can be rewritten acting explicitly on the bispinor.

$$\left\{ \begin{pmatrix} p_0 - e\Phi - m & 0 \\ 0 & -p_0 + e\Phi - m \end{pmatrix} + \begin{pmatrix} 0 & -\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \\ \boldsymbol{\sigma}\cdot\boldsymbol{\pi} & 0 \end{pmatrix} + \mu' \left[\begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{B} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\mathbf{B} \end{pmatrix} - i \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{E} \\ \boldsymbol{\sigma}\cdot\mathbf{E} & 0 \end{pmatrix} \right] \right\} \begin{pmatrix} \eta \\ \chi \end{pmatrix} = 0 \tag{2.2.5}$$

This gives rise to exact coupled equations for η and χ. So far this is in principle the same as the relativistic equation, only the form in which it is written is non covariant.

2.2.2 NR limit

The particle under consideration is a nonrelativistic electron. Roughly, the expectation is that η corresponds to the electron field and χ to that of the positron. The off-diagonal terms in the equation above represent some sort of mixing between the electron and positron: the electron wave function still has some small positron component, that decreases as momentum is decreased. The off diagonal component that does *not* vanish at 0 momentum is proportional to μ', the term introduced to account for a high-energy process.

The upshot is that although the equation above is really a set of coupled equations for η and χ, χ will be small compared to η — the very leading order diagonal term will indicate that $\chi \sim \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\eta$.

Because the off diagonal terms are small, the set of coupled equations may be solved perturbatively. The particular quantity of interest is the nonrelativistic energy of the particle $\epsilon = p_0 - m$. For a free particle, this would be

$$\epsilon = p_0 - m \approx \frac{\mathbf{p}^2}{2m} + \frac{\mathbf{p}^4}{8m^3} + \mathcal{O}\left(\frac{\mathbf{p}^6}{m^5}\right) \tag{2.2.6}$$

In order to perform this perturbative analysis the order of various terms needs to be established. It's evident that at leading order $\epsilon \sim mv^2$. From earlier analysis, $\Phi \sim mv^2$, $\pi \sim mv$, and $\mathbf{E} \sim m^2v^3$.

First, find an expression for χ in terms of η. The second of the set of equations represented by (2.2.5) is

$$(-p_0 + e\Phi - m)\chi + \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\eta + \mu'(\boldsymbol{\sigma}\cdot\mathbf{B}\,\chi - i\boldsymbol{\sigma}\cdot\mathbf{E}\,\eta) \tag{2.2.7}$$

Writing $p_0 = \epsilon + m$, and grouping terms, the result is that

$$\left(\epsilon + 2m - \mu'\boldsymbol{\sigma}\cdot\mathbf{B}\right)\chi = (\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - i\boldsymbol{\sigma}\cdot\mathbf{E}\,)\,\eta \tag{2.2.8}$$

It is necessary now to approximate χ in terms of η. Because ϵ and $|B|$ are smaller than m,

and only second order terms are needed for the final result:

$$\chi \approx \frac{1}{2m}\left(1 - \frac{\epsilon - e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B}}{2m}\right)(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - i\mu'\boldsymbol{\sigma}\cdot\mathbf{E})\phi \tag{2.2.9}$$

With this expression χ may be eliminated from the first of the set of equations (at least at the necessary order). The resulting equation will only involve η, and so may be used to solve for $\epsilon\eta$.

The original equation is

$$(p_0 - e\Phi - m)\eta - \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\chi + \mu'(\boldsymbol{\sigma}\cdot\mathbf{B}\,\eta - i\boldsymbol{\sigma}\cdot\mathbf{E}\,\chi) \tag{2.2.10}$$

So again using $p_0 - m = \epsilon$

$$\epsilon\eta = (e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B})\eta + (\boldsymbol{\sigma}\cdot\mathbf{E} + \boldsymbol{\sigma}\cdot\boldsymbol{\pi})\chi \tag{2.2.11}$$

Now the expression for χ in terms of η may be used.

$$\epsilon\eta \approx (e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B})\eta + (\boldsymbol{\sigma}\cdot\mathbf{E} + \boldsymbol{\sigma}\cdot\boldsymbol{\pi})\frac{1}{2m}\left(1 - \frac{\epsilon - e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B}}{2m}\right)(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - i\mu'\boldsymbol{\sigma}\cdot\mathbf{E})\eta \tag{2.2.12}$$

Writing the $1/m$ and $1/m^2$ terms separately:

$$\begin{aligned}\epsilon\eta \approx \Bigg\{ & e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} + i\mu'\boldsymbol{\sigma}\cdot\mathbf{E})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - i\mu'\boldsymbol{\sigma}\cdot\mathbf{E})}{2m} \\ & + \frac{1}{4m^2}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} + i\mu'\boldsymbol{\sigma}\cdot\mathbf{E})(\mu'\boldsymbol{\sigma}\cdot\mathbf{B} - [\epsilon - e\Phi])(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - i\mu'\boldsymbol{\sigma}\cdot\mathbf{E})\Bigg\}\eta\end{aligned} \tag{2.2.13}$$

Several of the terms are of too high order to consider. A term with both E and B, for instance, will be of higher order than $(e/m)\,|B|\,v^2$. Likewise, a term of $E\Phi$ or $E\epsilon$ is also too small. Dropping all such:

$$\epsilon\eta \approx \left\{e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2 - i\mu'[\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,,\boldsymbol{\sigma}\cdot\mathbf{E}]}{2m} + \frac{1}{4m^2}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,(\mu'\boldsymbol{\sigma}\cdot\mathbf{B} - [\epsilon - e\Phi])\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right\}\eta \tag{2.2.14}$$

This is an expression for the energy ϵ of the particle, in terms of operators. This will yield the nonrelativistic Hamiltonian. There is still some manipulation required, though,

because the right hand side also contains ϵ. But since the leading order terms don't, it may be perturbatively solved for. (The above expression could be simplified somewhat, using the properties of σ matrices for instance, but for now it is more convenient to write it compactly.)

To that end, the Hamiltonian can be split into leading order and second order terms. The leading order will be of mv^2 and $(e/m)B$, while the next order will be suppressed by an additional factor of v^2. Because the magnetic field is infinitesimally small no B^2 terms are needed. Since the leading order term in H is $\mathcal{O}(mv^2)$, this suggests we split it into two parts: $H = H_0 + H_1 + \mathcal{O}(mv^6, (e/m)Bv^4)$, where H_1 consists of only second order terms.

$$\begin{aligned} \hat{H}_0 &= e\Phi - \mu' \boldsymbol{\sigma}\cdot\mathbf{B} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{2m} \\ \hat{H}_1 &= -\frac{i\mu'}{2m}[\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,, \boldsymbol{\sigma}\cdot\mathbf{E}] + \frac{1}{4m^2}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,(\mu'\boldsymbol{\sigma}\cdot\mathbf{B} - [\epsilon - e\Phi])\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \end{aligned}$$

H_1 contains ϵ, along with other terms of total order mv^2. So to eliminate ϵ from H_1 it'll only be necessary to find it to leading order.

$$\begin{aligned} \epsilon\eta &= \left(\hat{H}_0 + \mathcal{O}(mv^4)\right)\eta \\ &\approx \left(e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{2m}\right)\eta \end{aligned}$$

The operators on the right hand side, operating on η, produce ϵ. The combination actually needed is $\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\epsilon\boldsymbol{\sigma}\cdot\boldsymbol{\pi}$. To that end, start with $\boldsymbol{\sigma}\cdot\boldsymbol{\pi}^{\,2}\epsilon$ and use commutation relations.

$$\begin{aligned} (\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2(\epsilon - e\Phi)\eta &= (\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\left(\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{2m} - \mu'\boldsymbol{\sigma}\cdot\mathbf{B}\right)\eta \\ \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,(\epsilon - e\Phi)\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\eta &= \left(\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^4}{2m} - \mu'(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\boldsymbol{\sigma}\cdot\mathbf{B} - \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,[e\Phi, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}]\right)\eta \end{aligned}$$

With this ϵ is eliminated, leaving:

$$\hat{H}_1 = -\frac{i\mu'}{2m}[\boldsymbol{\sigma}\boldsymbol{\pi}\,, \boldsymbol{\sigma}\mathbf{E}] + \frac{1}{4m^2}\left(\mu'\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\boldsymbol{\sigma}\cdot\mathbf{B}\,\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^4}{2m} + \mu'(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\boldsymbol{\sigma}\cdot\mathbf{B} + \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,[e\Phi, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}]\right) \tag{2.2.15}$$

Some terms couple A and B; they can be dropped. Some simplification of the structures

involving σ matrices can be done. To start with, simplify $(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2$.

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2 = \sigma_i\sigma_j\pi_i\pi_j = \pi^2 - i\epsilon_{ijk}\pi_i\pi_j\sigma_k \tag{2.2.16}$$

Since $\mathbf{p}\times\mathbf{p} = \mathbf{A}\times\mathbf{A} = 0$, from $\boldsymbol{\pi}\times\boldsymbol{\pi}$ only the cross terms survive:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2 = \pi^2 - ie\epsilon_{ijk}(p_iA_j - A_ip_j) = \pi^2 - e\boldsymbol{\sigma}\cdot\mathbf{B} \tag{2.2.17}$$

Looking at the terms $\mu'\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\boldsymbol{\sigma}\cdot\mathbf{B}\,\boldsymbol{\sigma}\cdot\boldsymbol{\pi} + \mu'(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\boldsymbol{\sigma}\cdot\mathbf{B}$, they contain an anticommutator involving B and p. Because the magnetic field is assumed to be constant, p and B commute, so:

$$\begin{aligned}\{\boldsymbol{\sigma}\cdot\mathbf{B}\,,\boldsymbol{\sigma}\cdot\mathbf{p}\,\} &= B_ip_j\{\sigma_i,\sigma_j\}\\ &= 2\mathbf{B}\cdot\mathbf{p}\end{aligned}$$

The commutator of Φ and a derivative operator should give the electric field E:

$$[\Phi,\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,] = [\Phi,p_i]\sigma_i = -iE_i\sigma_i = -i\boldsymbol{\sigma}\cdot\mathbf{E} \tag{2.2.18}$$

Using these identities, the Hamiltonian can be expressed as:

$$\hat{H}_0 = e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{\pi^2}{2m} - \frac{e}{2m}\boldsymbol{\sigma}\cdot\mathbf{B} \tag{2.2.19}$$

$$\hat{H}_1 = -\frac{\pi^4}{8m^3} + e\frac{p^2}{4m^3}\boldsymbol{\sigma}\cdot\mathbf{B} - i\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\boldsymbol{\sigma}\cdot\mathbf{E}}{4m^2} + \mu'\left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}\,\mathbf{B}\cdot\mathbf{p}}{2m^2} - \frac{i[\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,,\boldsymbol{\sigma}\cdot\mathbf{E}\,]}{2m}\right) \tag{2.2.20}$$

There are still some simplifications that can be made to terms quadratic in σ, but it'll be more convenient for now to keep H written as is.

2.2.3 FW Transform

To find a complete description of a single nonrelativistic particle in normal quantum mechanics, we must work in a basis where the lower component χ is truly negligible, at least at the desired order. While there exists a formal technique for finding this Fouldy-Wouthyusen

transformation, for our purposes we can simply demand that the wave function after transformation $\phi_S = (1+\Delta)\eta$ obeys the normalization $\langle \phi_S, \phi_S \rangle = 1$.

To find the necessary transformation, we can use our expression for χ in terms of η to find the normalization

$$\begin{aligned} \int d^3x(\eta^\dagger\eta + \chi^\dagger\chi) &= \int d^3x \left[\eta^\dagger\phi + \left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\pi}}{2m}\phi\right)^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\pi}}{2m}\eta\right)\right] \\ &= \int d^3x\eta^\dagger \left[1 + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{4m^2}\right]\eta \end{aligned}$$

Since we know that $\langle \Psi, \Psi \rangle = 1$ this shows that if $\eta = \left(1 - \frac{(\boldsymbol{\sigma}\boldsymbol{\pi})^2}{8m^2}\right)\phi_S$, the Schrodinger wave functions are properly normalized.

We now need to find the form of $\hat{H}$ after this transformation. For now work with the general form:

$$\epsilon\eta = (\hat{H}_0 + \hat{H}_1)\eta \tag{2.2.21}$$

After changing to the Schrodinger like wave functions, this becomes:

$$\epsilon\left(1 - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{8m^2}\right)\phi_S = (\hat{H}_0 + \hat{H}_1)\left(1 - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{8m^2}\right)\phi_S \tag{2.2.22}$$

To the order needed the inverse of $1 + \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,^2/8m^2$ is just $1 - \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,^2/8m^2$, so eliminating that on the left hand side gives:

$$\epsilon\phi_S = (1 + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{8m^2})(\hat{H}_0 + \hat{H}_1)(1 - \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{8m^2})\phi_S \tag{2.2.23}$$

Since H_1 is already second order, these further corrections don't involve it directly. Expressing the result as a commutator:

$$\epsilon\phi_S = \left(\hat{H}_0 + \frac{1}{8m^2}[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \hat{H}_0] + \hat{H}_1\right)\phi_S \tag{2.2.24}$$

So under the FW transformation, the leading order term is unchanged, and the second order term is:

$$\hat{H}_1 \to \hat{H}_1' = \hat{H}_1 + \frac{1}{8m^2}[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \hat{H}_0] \tag{2.2.25}$$

The final step is to simplify the commutator $[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \hat{H}_0]$.

$$[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \hat{H}_0] = [(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, e\Phi - \mu'\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2}{2m}] \tag{2.2.26}$$

Obviously $(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2$ commutes with itself, so that term vanishes. Since $\boldsymbol{\sigma}\cdot\mathbf{B}$ is constant, that commutator will also disappear. Writing $\boldsymbol{\sigma}\cdot\boldsymbol{\pi}$ as shown earlier:

$$[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \mu'\boldsymbol{\sigma}\cdot\mathbf{B}\,] = [\pi^2 - e\boldsymbol{\sigma}\cdot\mathbf{B}\,, \mu'\boldsymbol{\sigma}\cdot\mathbf{B}\,] = 0$$

The non-trivial part is the commutation of the derivative operators with the electric potential Φ.

$$\begin{aligned}
[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, \hat{H}_0] &= [(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2, e\Phi] \\
&= e(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,[\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,, \Phi] + [\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,, \Phi]\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,) \\
&= ie(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\boldsymbol{\sigma}\cdot\mathbf{E} + \boldsymbol{\sigma}\cdot\mathbf{E}\,\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,)
\end{aligned}$$

So, writing down the new H_1':

$$\begin{aligned}
\hat{H}_1' &= \hat{H}_1 + \frac{ie}{8m^2}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\boldsymbol{\sigma}\cdot\mathbf{E} + \boldsymbol{\sigma}\cdot\mathbf{E}\,\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,) \\
&= -\frac{\pi^4}{8m^3} + e\frac{p^2}{4m^3}\boldsymbol{\sigma}\cdot\mathbf{B} - \frac{ie}{8m^2}[\boldsymbol{\sigma}\cdot\mathbf{E}\,, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,] + \mu'\left(\frac{(\boldsymbol{\sigma}\cdot\mathbf{p}\,)(B\cdot p)}{2m^2} - \frac{i}{2m}[\boldsymbol{\sigma}\cdot\mathbf{E}\,, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,]\right)
\end{aligned}$$

NR Hamiltonian

From the relativistic equations of motion, a nonrelativistic Hamiltonian was defined in terms of Schrodinger-like wave functions that conserve probability. The final result may now be written down. For comparison with other work, rather than writing in terms of μ', it will be written in terms of the gyromagnetic ratio g. Because terms proportional to $g-2$ may enter seperately, it is convenient to express all g dependent terms as linear combinations of g and $g-2$.

So the entire Hamiltonian, using $\mu' = \frac{g-2}{2}\mu_0 = \frac{g-2}{2}\frac{e}{2m}$, is:

$$\begin{aligned} H &= e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} - (1+\frac{g-2}{2})\frac{e}{2m}\boldsymbol{\sigma}\cdot\mathbf{B} \\ &\quad + e\frac{p^2}{4m^3}\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{ie}{8m^2}(1+(g-2))[\boldsymbol{\sigma}\cdot\mathbf{E}\,,\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,] + (g-2)\frac{e}{2m}\frac{(\boldsymbol{\sigma}\cdot\mathbf{p}\,)(\mathbf{B}\cdot\mathbf{p})}{4m^2} \\ &= e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} \\ &\quad - \frac{e}{2m}\left\{\frac{g}{2}\boldsymbol{\sigma}\cdot\mathbf{B} - \frac{p^2}{2m^2}\boldsymbol{\sigma}\cdot\mathbf{B} - (g-2)\frac{(\boldsymbol{\sigma}\cdot\mathbf{p}\,)(\mathbf{B}\cdot\mathbf{p})}{4m^2} + (g-1)\boldsymbol{\sigma}\cdot(\mathbf{E}\times\boldsymbol{\pi})\right\} \\ &= e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} \\ &\quad - \frac{e}{2m}\left\{\frac{g}{2}\left(1-\frac{p^2}{2m^2}\right)\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{g-2}{2}\frac{p^2}{2m^2}\boldsymbol{\sigma}\cdot\mathbf{B} - \frac{g-2}{2}\frac{(\boldsymbol{\sigma}\cdot\mathbf{p}\,)(\mathbf{B}\cdot\mathbf{p})}{2m^2} + \left(\frac{g}{2}+\frac{g-2}{2}\right)\boldsymbol{\sigma}\cdot(\mathbf{E}\times\boldsymbol{\pi})\right\} \end{aligned}$$

2.3 Method of NRQED

Instead of starting from the relativistic equations of motion and rewriting them in the form of a Schrodinger like equation, the method of NRQED can be employed. The nonrelativistic Lagrangian can be written in the most general way, but ignoring all possible terms that will have too small a contribution to calculations. The coefficients before each term will be unknown, but there is a straightforward method of fixing them.

Within the realm of NRQEDs validity, it should produce the same predictions as NRQED. The same physical process can be calculated in both frameworks, and then the results compared. The calculation in QED will then fix the coefficients in NRQED. In comparing the two calculations it will be necessary to write the QED results nonrelativisitcally, in the same manner as was done for the approach above.

For spin one-half, up to the order desired, the form of the NRQED Lagrangian involving

two fermion fields has been derived in the previous chapter. It is

$$\begin{aligned}\mathcal{L}_{NRQED} =&\Psi^\dagger\Bigg\{iD_0+\frac{\mathbf{D}^2}{2m}+\frac{\mathbf{D}^4}{8m^2}+c_F\frac{e}{m}\mathbf{S}\cdot\mathbf{B}+c_D\frac{e(\mathbf{D}\cdot\mathbf{E}-\mathbf{E}\cdot\mathbf{D})}{8m^2}\\&+c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D}}{8m^2}+c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B}+\mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3}-c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3}\\&+c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D})+(\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3}\Bigg\}\Psi\end{aligned} \tag{2.3.1}$$

Each term contains two fermion fields, and zero, one or two powers of the photon field. Terms with a different number of photon fields, of course, correspond to different physical processes. But because gauge invariance is a necessary feature of the theory, the coefficients of many terms involving the photon field are constrained to be the same as that of terms with a smaller number of such fields. For instance, the gauge invariant term $\mathbf{D}^2/(2m)$ is:

$$\frac{\mathbf{D}^2}{2m}=\frac{\boldsymbol{\nabla}^2-ie(\boldsymbol{\nabla}\cdot\mathbf{A}+e\mathbf{A}\cdot\boldsymbol{\nabla})-e^2\mathbf{A}^2}{2m} \tag{2.3.2}$$

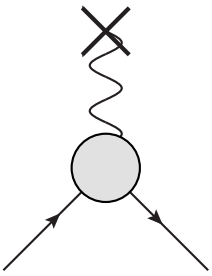
The first term above, containing $\boldsymbol{\nabla}^2$, is a purely kinetic term. The other two represent interactions with one or two photon fields. To fix these coefficients, two physical processes could be calculated and compared with the QED results, but in the end they must have the same coefficient as guaranteed by gauge invariance.

Obviously not every term goes this same way. The term with $\mathbf{S}\cdot\mathbf{B}$, for instance, is in and of itself gauge-invariant, and so there is no option but to calculate it from a process involving a single photon. For the necessary terms appearing in the above Lagrangian, it would suffice to consider just the one photon processes. However, since in principle some of the interesting coefficients could be calculated from two photon diagrams, this also will be done.

To fix these terms, some physical process must be chosen. For the one photon terms, scattering off an external field will be calculated. For the two photon terms, Compton scattering will be used.

2.3.1 Calculation of electron scattering off external field in QED

To fix all the terms in the NRQED Lagrangian which have a single power of the photon field, it suffices to calculate the scattering of an electron off an external field, as represented (in QED) by the diagram



The leading order contributions (in QED) to this interaction come from the fundamental electron vertex. There are radiative corrections to this process, starting at the one loop order. In principle, such calculations (up to some desired order in α) should be included in the QED calculation.

However, it turns out that the actual *form* of the interaction is highly constrained by the symmetries of the theory. No matter the source of contributions to the vertex, their effects can be incorporated into two coefficients or form factors. The NRQED coefficients can then be written in terms of these form factors, which can later be calculated to whatever precision is necessary.

The first symmetry that constrains the interaction is that it must be invariant under Lorentz transformations. Since every term involves the photon field A_μ and has external fermion legs, then the interaction must be proportional to the general form:

$$A_\mu \bar{u}(p')\Gamma^\mu(p',p)u(p) \tag{2.3.3}$$

and $\bar{u}(p')\Gamma^\mu(p',p)u(p)$ must transform as a Lorentz vector. If it did not, the whole would not be Lorentz invariant. To leading order $\Gamma^\mu = \gamma^\mu$, since the fundamental vertex is just that. The corrections, whatever the exact details of the processes which produce them, can only depend on the momenta p and p', in addition to constants m and e, and such structures as may act upon the spinors.

A basis for such structures is known, with well defined Lorentz transformations. That

gives scalar, vector, tensor, pseudo-vector, and pseudoscalar terms. From the momenta in the problem can be constructed scalar and vector quantities. What symmetries control the allowed terms?

In addition to proper Lorentz invariance, it is necessary that the interaction as a whole preserve parity. Since A is a vector, the term $\bar{u}\Gamma^\mu u$ must also behave as a vector under parity transformations. And there is no Lorentz invariant combination of external momenta that can be written that is not even under parity. Because of this, there can be no contribution from the pseudovector and pseudoscalar bilinears.

The vector bilinear $\bar{u}\gamma^\mu u$ already has the correct transformation properties. From the scalar bilinear $\bar{u}u$, a vector can be constructed by adjoining a single power of the momentum. (As, for example, $p^\mu \bar{u}u$.) And after contracting one index of the tensor with a momentum vector, it will also behave as a vector.

The momenta terms available are p and p'. However, the terms constructed from them are not independent, because each must separately obey current conservation. So whatever terms go into Γ^μ, $q_\mu \bar{u}\Gamma^\mu u = 0$. The unique scalar term will be

$$\frac{p^\mu + p'^\mu}{2m}\bar{u}u \tag{2.3.4}$$

which vanishes because $(p' + p) \cdot (p' - p) = p'^2 - p^2 = m^2 - m^2 = 0$.

The unique tensor term allowed will be

$$\frac{q_\nu}{2m}\bar{u}\sigma^{\mu\nu}u \tag{2.3.5}$$

which conserves current because $\sigma^{\mu\nu}$ is antisymmetric. So from these considerations, the general form of the vertex will have three terms, each with a momentum dependant coefficient:

$$\bar{u}\Gamma^\mu u = c_1(p,p')\frac{p^\mu + p'^\mu}{2m}\bar{u}u + c_2(p,p')\bar{u}\gamma^\mu u + c_3(p,p')\frac{q_\mu}{2m}\bar{u}\sigma^{\mu\nu}u \tag{2.3.6}$$

However, there is one more consideration. From the Dirac equation can be derived the

Gordon identity, which relates these three terms:

$$\bar{u}\gamma^{\mu}u = \bar{u}\left(\frac{p^{\mu}+p'^{\mu}}{2m} + \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\right)u \tag{2.3.7}$$

With this, any one of the terms in (2.3.6) can be rewritten as some combination of the other two, and its coefficient effectively absorbed into the other two. To write down the most general form of the vertex, one need only choose two of the three terms. Which two to choose might depend on the nature of the calculation; at any rate, there are always three paths to go down.

In the calculation of scattering off an external field an agnostic approach will be taken at first. The scattering amplitude is some combination of the three bilinears. The goal is to rewrite the scattering amplitude in nonrelativistic language. So, each bilinear in turn will be rewritten in this manner.

The vertex can be written in each of three ways. The form factors will be defined with respect to the particular combination of γ^{μ} and $\sigma^{\mu\nu}$ terms.

$$\Gamma^{\mu} = \gamma^{\mu}F_1(q^2) + i\frac{\sigma^{\mu\nu}q_{\nu}}{2m}F_2(q^2) \tag{2.3.8}$$

Using the Gordon identity to write this in terms of the scalar and vector bilinears, the result is

$$\Gamma^{\mu} = \gamma^{\mu}[F_1(q^2) + F_2(q^2)] - \frac{p^{\mu}+p'^{\mu}}{2m}F_2(q^2) \tag{2.3.9}$$

And writing in terms of the scalar and tensor:

$$\Gamma^{\mu} = \frac{p^{\mu}+p'^{\mu}}{2m}F_1(q^2) + i\frac{\sigma^{\mu\nu}q_{\nu}}{2m}[F_1(q^2) + F_2(q^2)] \tag{2.3.10}$$

The momentum dependence can be uniquely written in terms of q^2. The factors must be Lorentz invariant, so only such combinations of momenta can be considered. From the momenta p and p', only three such quantities can be constructed. But since the external fermions are on mass-shell, $p^2 = p'^2 = m^2$. That leaves only $p \cdot p' = m^2 + p \cdot q$, and this can

be related to q^2 by

$$q \cdot (2p+q) = (p'-p)\cdot(p'+p) = p'^2 - p^2 = 0 \rightarrow q^2 = -\frac{1}{2}p\cdot q \tag{2.3.11}$$

Given that there exist these three ways to write the vertex, there are three ways to perform the calculation. The scattering amplitude will have the form

$$iM = A_\mu \bar{u}(p')\Gamma^\mu(p',p)u(p) \tag{2.3.12}$$

and no matter which way Γ^μ is written, a nonrelativistic expansion in terms of ϕ_S will be needed. For now an agnostic approach will be taken, and the expressions for each of the three possible bilinears found.

Nonrelativistic expressions for the bilinears

To compare with the NRQED scattering amplitude, everything needs to be written with consistent language. We start with the relativistic bilinears, each of which behaves like a four vector, and appears in the amplitude dotted with the external field A. Any dot products $a \cdot b$ should instead be written as $a_0 b_0 - \mathbf{a}\cdot\mathbf{b}$. To this end, it might be necessary to treat the spatial and time-like components of the four-vector bilinears separately.

The bispinors u will be first written in terms of upper and lower components η, χ, and then in terms of the nonrelativistic wave spinors w. The relationship between the two sets are

$$\begin{aligned} \eta(p) &= \left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w(p) \\ \chi(p) &= \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\left(1 - \frac{3\mathbf{p}^2}{8m^2}\right) w(p) \end{aligned}$$

In writing in terms of the upper and lower components, the explicit expressions of the γ matrices, as well as $\sigma^{\mu\nu}$ will be needed.

Of course the above expressions for the spinors in terms of w is approximate. In the NRQED Lagrangian, the terms we wish to fix involve A_0 with up to two powers of mo-

mentum (such as $\boldsymbol{\nabla}\cdot\mathbf{E}$), and A_i with up to three (as in $\boldsymbol{\sigma}\cdot\mathbf{B}\,\mathbf{p}^2$). Because only the case of a constant magnetic field is needed, in any term which will explicitly contain B higher derivatives may be ignored. Throwing away unneeded terms, whatever is left can be used to calculate the scattering amplitude and compare with NRQED.

This same general procedure will be followed for each of the bilinears.

Scalar bilinear: Start with the first bilinear, a scalar coupled with a momentum four-vector. Rewrite it in terms of η and χ.

$$(p+p')^\mu \bar{u}u = (p+p')^\mu \left(\eta^\dagger\eta - \chi^\dagger\chi\right) \tag{2.3.13}$$

Now express in terms of w

$$= (p+p')^\mu \left\{ w^\dagger \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right)\left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w - w^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{2m}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\right) w \right\} \tag{2.3.14}$$

And finally drop terms beyond the order that we need.

$$= (p+p')^\mu w^\dagger \left(1 - \frac{\mathbf{p}^2+\mathbf{p}'^2}{8m^2} - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2}\right) w \tag{2.3.15}$$

Simplifying $\boldsymbol{\sigma}\cdot\mathbf{p}'\,\boldsymbol{\sigma}\cdot\mathbf{p} = \mathbf{p}\cdot\mathbf{p}' + i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}$.

$$(p+p')^\mu \bar{u}u = (p+p')^\mu w^\dagger \left(1 - \frac{\mathbf{p}^2+2\mathbf{p}\cdot\mathbf{p}'+\mathbf{p}'}{8m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w \tag{2.3.16}$$

The spatial part is straightforward, but the time-like part is

$$(p+p')^0 \bar{u}u = (p+p')_0 w^\dagger \left(1 - \frac{\mathbf{p}^2+2\mathbf{p}\cdot\mathbf{p}'+\mathbf{p}'^2}{8m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w \tag{2.3.17}$$

Approximating the relativistic energies gives $p_0 = m + \mathbf{p}^2/(2m)$. So

$$p_0 + p_0' \approx 2m + \frac{\mathbf{p}^2+\mathbf{p}'^2}{2m} = 2m\left(1 + \frac{\mathbf{p}^2+\mathbf{p}'^2}{4m^2}\right) \tag{2.3.18}$$

Then using this correction to the leading order term

$$(p+p')^0\bar{u}u = 2mw^\dagger\left(1+\frac{\mathbf{p}^2-2\mathbf{p}\cdot\mathbf{p}'+\mathbf{p}'}{8m^2}-\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right)w \tag{2.3.19}$$

$$(p+p')^0\bar{u}u \approx 2mw^\dagger\left(1+\frac{\mathbf{q}^2}{8m^2}-\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right)w \tag{2.3.20}$$

Vector bilinear: For the term $\bar{u}\gamma^\mu u$ it'll be necessary to treat the spatial/time-like indices separately, since they have different spinor structure.

The time-like part is:

$$\bar{u}\gamma^0 u = u^\dagger u \tag{2.3.21}$$

which in terms of η and χ is just

$$= \eta^\dagger\eta + \chi^\dagger\chi \tag{2.3.22}$$

Then rewritten with w

$$= w^\dagger\left(1-\frac{\mathbf{p}'^2}{8m^2}\right)\left(1-\frac{\mathbf{p}^2}{8m^2}\right)w + w^\dagger\left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{2m}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\right)w \tag{2.3.23}$$

Which is, at the order needed

$$= w^\dagger\left(1-\frac{\mathbf{p}^2+\mathbf{p}'^2}{8m^2}+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2}\right)w \tag{2.3.24}$$

Simplifying the last term, using $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$, gives

$$\bar{u}\gamma^0 u = w^\dagger\left(1-\frac{\mathbf{q}^2}{8m^2}+\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right)w \tag{2.3.25}$$

The spatial part is

$$\bar{u}\gamma^i u = u^\dagger\gamma^0\gamma^i u \tag{2.3.26}$$

writing the matrices explicitly

$$= \bar{u}^\dagger\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}u \tag{2.3.27}$$

which in terms of spinors is

$$= \eta^\dagger \sigma_i \chi + \chi^\dagger \sigma_i \eta \tag{2.3.28}$$

Replacing the spinors with w gives

$$= w^\dagger \left\{ \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right) \frac{\sigma_i \boldsymbol{\sigma}\cdot\mathbf{p}}{2m} \left(1 - \frac{3\mathbf{p}^2}{8m^2}\right) + \left(1 - \frac{3\mathbf{p}'^2}{8m^2}\right) \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\,\sigma_i}{2m} \left(1 - \frac{\mathbf{p}^2}{8m^2}\right) \right\} w \tag{2.3.29}$$

Using $\mathbf{p}'^2 = \mathbf{p}^2$ gives

$$= w^\dagger \left\{ \frac{\sigma_i \boldsymbol{\sigma}\cdot\mathbf{p} + \boldsymbol{\sigma}\cdot\mathbf{p}'\,\sigma_i}{2m} \left(1 - \frac{\mathbf{p}^2}{2m^2}\right) \right\} w \tag{2.3.30}$$

Then $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$

$$= w^\dagger \left\{ \frac{p_i + \epsilon_{ijk} p_j \sigma_k + p'_i + \epsilon_{jik} p'_j \sigma_k}{2m} \left(1 - \frac{\mathbf{p}^2}{2m^2}\right) \right\} w \tag{2.3.31}$$

So finally

$$\bar{u}\gamma^i u = w^\dagger \left\{ \frac{p_i + p'_i - \epsilon_{ijk} q_j \sigma_k}{2m} \left(1 - \frac{\mathbf{p}^2}{2m^2}\right) \right\} w \tag{2.3.32}$$

Tensor term: The tensor term is subject to an additional simplification. Because the process under considering is elastic scattering off an external static field, terms involving q_0 can be dropped. Under this approximation $q_\nu \sigma^{\mu\nu} \approx q_j \sigma^{\mu j}$.

Dealing first with the case where $\mu = i$, write the tensor structure explicitly as a matrix:

$$\bar{u} \frac{i}{2m} q_j \sigma^{ij} u = \frac{i\epsilon_{ijk} q_j}{2m} \bar{u} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} u \tag{2.3.33}$$

Write in terms of spinors

$$= \frac{i\epsilon_{ijk} q_j}{2m} \left(\eta^\dagger \sigma_k \eta - \chi^\dagger \sigma_k \chi \right) \tag{2.3.34}$$

And then in terms of w

$$= \frac{i\epsilon_{ijk} q_j}{2m} w^\dagger \left\{ \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right) \sigma_k \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right) - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}' \sigma_k \boldsymbol{\sigma}\cdot\mathbf{p}}{4m^2} w \right\} \tag{2.3.35}$$

There now appears a term with a triple product of σ matrices. That can be simplified with

the following expression:

$$\sigma_a\sigma_b\sigma_c = \sigma_a(\delta_{bc} + i\epsilon_{bcd}\sigma_d) = \sigma_a\delta_{bc} - \sigma_b\delta_{ca} + \sigma_c\delta_{ab} + i\epsilon_{abc} \tag{2.3.36}$$

Using that identity,

$$\bar{u}\frac{i}{2m}q_j\sigma^{ij}u = \frac{i\epsilon_{ijk}q_j}{2m}w^\dagger\left\{\sigma_k\left(1 - \frac{\mathbf{p}'^2 + \mathbf{p}^2}{8m^2}\right) - \frac{\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{p}')p_k - \sigma_k\mathbf{p}\cdot\mathbf{p}' + i\epsilon_{akc}q_ap_c}{4m^2}\right\}w \tag{2.3.37}$$

This can be further simplified by combining the like terms $\sigma_k(\mathbf{p}'^2+\mathbf{p}^2)+2\sigma_k\mathbf{p}\cdot\mathbf{p}' = \sigma_k\mathbf{q}^2$.

$$\bar{u}\frac{i}{2m}q_j\sigma^{ij}u = \frac{i\epsilon_{ijk}q_j}{2m}w^\dagger\left\{\sigma_k\left(1 - \frac{\mathbf{q}^2}{8m^2}\right) - \frac{\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{p}')p_k + i\epsilon_{akc}q_ap_c}{4m^2}\right\}w \tag{2.3.38}$$

Now it is necessary to consider exactly what derivatives of the field A_i are to be kept. The assumption is that B is constant and so $\partial_i B_j = 0$. Contracted with A_i above, $\epsilon_{ijk}A_iq_j \sim B_k$. So besides the leading factor, no terms with q are needed. Applying this simplification,

$$\bar{u}\frac{i}{2m}q_j\sigma^{ij}u = \frac{i\epsilon_{ijk}q_j}{2m}w^\dagger\left\{\sigma_k - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}p_k}{2m^2}\right\}w \tag{2.3.39}$$

The case with $\mu = 0$ goes as follows.

$$\bar{u}\frac{i}{2m}q_j\sigma^{0j}u = -\frac{q_j}{2m}\bar{u}\gamma^0\gamma^ju = -\frac{q_j}{2m}u^\dagger\gamma^ju \tag{2.3.40}$$

Explicitly in matrix form

$$= -\frac{q_j}{2m}u^\dagger\begin{pmatrix}0 & \sigma_j\\ \sigma_j & 0\end{pmatrix}u \tag{2.3.41}$$

Written in terms of spinors, and Galilean three-vector $q^j = -q_j$.

$$= \frac{q^j}{2m}\left(\eta^\dagger\sigma_j\chi - \chi^\dagger\sigma_j\eta\right) \tag{2.3.42}$$

And then in terms of w:

$$= \frac{q^j}{2m} w^\dagger \left\{ \left(1 - \frac{\mathbf{p}'^2}{8m^2}\right) \frac{\sigma_j \boldsymbol{\sigma}\cdot\mathbf{p}}{2m} \left(1 - \frac{3\mathbf{p}^2}{8m^2}\right) - \left(1 - \frac{3\mathbf{p}'^2}{8m^2}\right) \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_j}{2m} \left(1 - \frac{\mathbf{p}^2}{8m^2}\right) \right\} w \tag{2.3.43}$$

This bilinear is contracted with A_0, and so actually, only terms involving two additional powers of momenta need be kept.

$$= \frac{q^j}{2m} w^\dagger \left\{ \frac{\sigma_j \boldsymbol{\sigma}\cdot\mathbf{p}}{2m} - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'\sigma_j}{2m} \right\} w \tag{2.3.44}$$

or simplifying using the commutator of σ matrices

$$= \frac{q^j}{2m} w^\dagger \left\{ \frac{2i\epsilon_{jik} p_i \sigma_k - q_i - i\epsilon_{ijk} q_j \sigma_k}{2m} \right\} w \tag{2.3.45}$$

One term above vanishes because of symmetry

$$\bar{u} \frac{i}{2m} q_j \sigma^{0j} u = -w^\dagger \left\{ \frac{\mathbf{q}^2}{4m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{2m^2} \right\} w \tag{2.3.46}$$

Full vertex Now that the three bilinears have been calculated, the complete scattering amplitude can be written down. Each of the three forms should prove equivalent. To simplify comparison, the coupling to A_0 and A_i can be treated separately.

The first has $\Gamma^\mu = \gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)$. Using (2.3.25) and (2.3.46)

$$A_0 \bar{u}\Gamma^0 u = A_0 F_1 w^\dagger \left(1 - \frac{\mathbf{q}^2}{8m^2} + \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w - A_0 F_2 w^\dagger \left\{ \frac{\mathbf{q}^2}{4m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{2m^2} \right\} w \tag{2.3.47}$$

The second is $\Gamma^\mu = \gamma^\mu [F_1(q^2) + F_2(q^2)] - \frac{p^\mu + p'^\mu}{2m} F_2(q^2)$. Using (2.3.25) and (2.3.20).

$$A_0 \bar{u}\Gamma^0 u = A_0 (F_1 + F_2) w^\dagger \left(1 - \frac{\mathbf{q}^2}{8m^2} + \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w - A_0 F_2 w^\dagger \left(1 + \frac{\mathbf{q}^2}{8m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w \tag{2.3.48}$$

The third combination is $\Gamma^\mu = \frac{p^\mu + p'^\mu}{2m} F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m}[F_1(q^2) + F_2(q^2)]$. Using (2.3.20) and

(2.3.46)

$$A_0\bar{u}\Gamma^0 u = F_1 w^\dagger \left(1 + \frac{\mathbf{q}^2}{8m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w - [F_1+F_2] w^\dagger \left\{\frac{\mathbf{q}^2}{4m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{2m^2}\right\} w \quad (2.3.49)$$

Taking any of these three results and collecting like terms gives the result:

$$A_0 w^\dagger \left(F_1 + [F_1+2F_2]\left[\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2} - \frac{\mathbf{q}^2}{8m^2}\right]\right) w \quad (2.3.50)$$

So the calculations are at least consistant with the Gordon identity.

Now turning to the coupling with $\mathbf{A}$. The first has $\Gamma^\mu = \gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)$. Using (2.3.32) and (2.3.39)

$$\begin{aligned} A_i\bar{u}\gamma^i u =& F_1 w^\dagger \left\{\frac{p_i + p'_i - \epsilon_{ijk}q_j\sigma_k}{2m}\left(1 - \frac{\mathbf{p}^2}{2m^2}\right)\right\} w \\ &+ F_2\frac{i\epsilon_{ijk}q_j}{2m} w^\dagger \left\{\sigma_k\left(1 - \frac{\mathbf{q}^2}{8m^2}\right) - \frac{\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{p}')p_k + i\epsilon_{akc}q_a p_c}{4m^2}\right\} w \end{aligned} \quad (2.3.51)$$

The second is $\Gamma^\mu = \gamma^\mu[F_1(q^2)+F_2(q^2)] - \frac{p^\mu+p'^\mu}{2m}F_2(q^2)$. Using (2.3.32) and (2.3.16):

$$\begin{aligned} A_i\bar{u}\gamma^i u =& A_i F_1 w^\dagger \left\{\frac{p_i + p'_i - \epsilon_{ijk}q_j\sigma_k}{2m}\left(1 - \frac{\mathbf{p}^2}{2m^2}\right)\right\} w \\ &+ F_2\frac{(p+p')^i}{2m} w^\dagger \left(1 - \frac{\mathbf{p}^2 + 2\mathbf{p}\cdot\mathbf{p}' + \mathbf{p}'}{8m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w \end{aligned} \quad (2.3.52)$$

The third combination is $\Gamma^\mu = \frac{p^\mu+p'^\mu}{2m}F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m}[F_1(q^2)+F_2(q^2)]$. Using (2.3.16) and (2.3.39)

$$\begin{aligned} A_i\bar{u}\gamma^i u =& A_i F_1 \frac{p^i + p'^i}{2m} w^\dagger \left(1 - \frac{\mathbf{p}^2 + 2\mathbf{p}\cdot\mathbf{p}' + \mathbf{p}'}{8m^2} - \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^2}\right) w \\ &+ A_i(F_1+F_2)\frac{i\epsilon_{ijk}q_j}{2m} w^\dagger \left\{\sigma_k - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}p_k}{2m^2}\right\} w \end{aligned} \quad (2.3.53)$$

2.3.2 Calculation of Compton scattering in QED

The relevant terms in the NRQED Lagrangian can all be fixed by the previous calculation of scattering off an external field, because even though there are terms involving two powers of the photon field, the requirement of gauge invariance means they share a coefficient with

one photon terms. However, for reasons of self consistency it would be good to check that the coefficients really do work out the same if calculated independently.

The easiest process to calculate involving two photons will be Compton scattering.

2.4 The two-photon vertex of NRQED

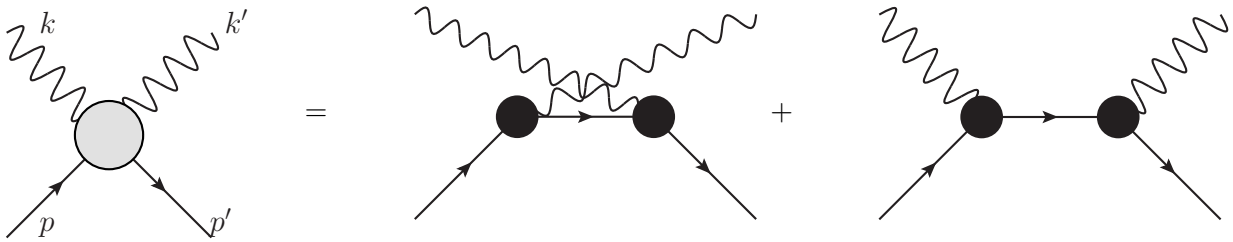
In the NRQED Lagrangian, in addition to the terms involving the fermions interaction with a single photon, there are terms which represent the interaction of a fermion with two photons. At the order needed, all such terms are fixed by gauge invariance. There are terms, such as those involving $\mathbf{E}^2$, that would are by themselves gauge invariant, but these occur at too high an order. (The order of such a term would be $E^2/m^3 \sim mv^6$.)

So though the coefficients of concern are all fixed by considering just the one-photon interactions, they could also be fixed from considering two-photon interactions. Since it *is* possible, it makes sense to do so, as a check of consistency. In this section, the coefficients of two-photon terms in the NRQED Lagrangian will be fixed from QED calculations.

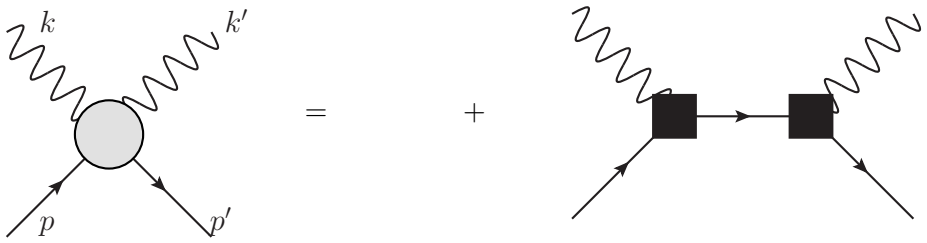
As before, this will involve calculating some physical process in both QED and NRQED, and comparing the result. The simplest two photon process to consider is Compton scattering. By calculating Compton scattering in each theory, the coefficients desired will be obtained.

This is not quite as straightforward as in the case of the one-photon scattering, for the following reason: while the one-photon scattering is a local interaction in both QED and NRQED, Compton scattering will involve some mix of local and non-local diagrams. In QED, there are of course no local interactions between a fermion and two photons. The situation is most readily stated diagrammatically.

In QED, the leading order diagrams contributing to Compton scattering are:



While in NRQED, the following diagrams contribute to the scattering:



In each set of diagrams, the vertices represent the *total* electron vertex. For QED this is determined, as before, by the form factors, and for NRQED it is determined by the calculations of the previous section.

Since the two amplitudes must be equal, in principle the process is this: First calculate the scattering amplitude in QED. Then, calculate the contribution to the scattering amplitude coming from the tree diagrams I and II above. Whatever discrepancy remains must be the value of the local two-photon vertex III.

The process of subtracting the one set of diagrams from the other could be slightly complicated, but luckily it turns out there is a simpler path. By considering the physical origin of the local terms in NRQED, it will be possible to split the QED diagrams into local and non-local parts, where the latter can be shown to be equal to the non-local diagrams in NRQED. Then, comparing the two scattering processes becomes much easier.

Z diagrams

The high energy theory (QED) doesn't contain any two-photon vertices, while the low energy theory (NRQED) does. This is a general feature of effective field theories, that new types of local interactions arise. The high energy theory can have intermediate states that are highly virtual, while the low energy theory doesn't. Instead, as according to the uncertainty principle, intermediate states with extremely high energy can be considered to occur almost instantaneously, giving rise in the effective theory to local interactions.

How does the local two-photon interaction arise in NRQED? Of course there are an infinite number of contributions, but we'll consider just the leading order contributions. These will come from the tree level two photon diagrams as shown above. Compare the

tree-level diagrams in the two theories: in addition to the vertices being different, so are the propagators. The propagator in QED represents some admixture of the electron and positron field, while in NRQED it is only the electron. In both QED and NRQED, a process is calculated as the sum of a series of diagrams, representing an expansion in perturbation theory. However, there is a difference btween the two in the nature of perturbation theory employed.

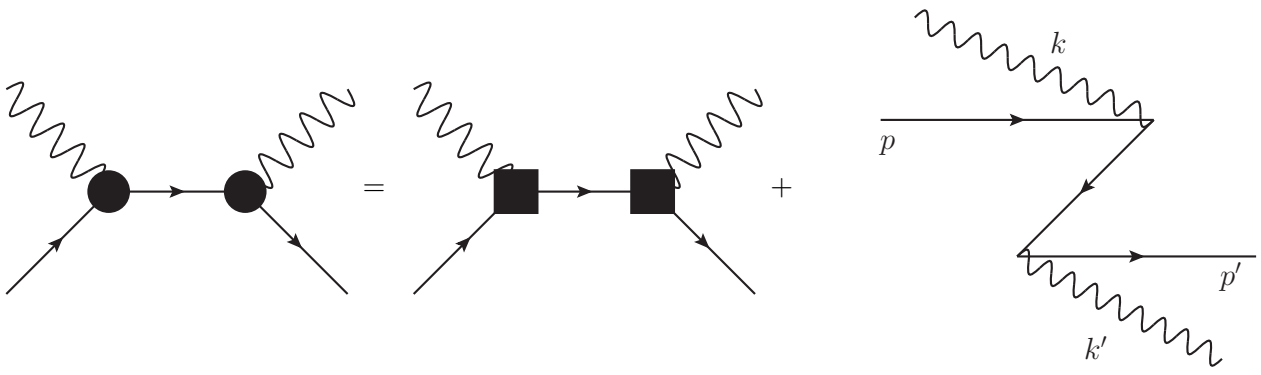
In QED, at each vertex both energy and momentum is conserved. But intermediate particles may be off mass-shell; that is it is no longer the case that for a particle of four-momentum p and mass m that $p^2 = m^2$.

In NRQED, the old Rayleigh-Schrodinger perturbation theory is used. All intermediate particles are on mass-shell. But at the vertices (when represented diagrammatically), although momenta is conserved, energy is not.

$$\Delta = \Sigma_{\text{int}} \frac{\langle \text{out} | V | \text{int} \rangle \langle \text{int} | V | \text{in} \rangle}{E_{\text{in}} - E_{\text{int}}} \tag{2.4.1}$$

The trick, then is to take the relativistic tree-level diagrams of QED and rewrite them in the language of Rayleigh-Schrodinger before trying to compare them to NRQED. In NRQED, only intermediate states involving electrons can be considered, but in QED intermediate states identified with positrons will appear as well. It is *these* processes, involving a large violation of energy conservation, that will appear as contact terms in NRQED.

There are two diagrams in QED, and both can be dealt with in the same general way. First consider the uncrossed diagram:



There are two tree level processes that can be considered in the old time-ordered perturbation theory. The first corresponds to an incoming electron, which first absorbs a photon and then emits one. The second, more complicated process, involves the creation of intermediate positron. While a free electron travels along, an incoming photon decays into an electron and positron. Then, the positron annihilates the incoming electron and emits the outgoing photon. Because of the shape of this diagram, it is called a "Z diagram."

In the Z-diagrams, the electrons and the photons are external, so the sum over the intermediate states is specifically the intermediate states of the positrons. Likewise, the other diagrams are written as a sum over intermediate electron states. But because of the rules of Rayleigh-Schrodinger perturbation theory, all these states are on mass shell. And since the momenta here is fixed, the sum over intermediate states is a sum over spin states.

So the original QED diagram should somehow split into two terms, one involving a sum over electron states and the other a sum over anti-particles.

Call the intermediate momentum $\mathbf{q}$. The initial energy will be q_0, the intermediate energy will be that of the on mass shell particle, $E_q = \sqrt{\mathbf{q}^2 + m^2}$.

$= \quad i\frac{\not{q}-m}{q^2-m^2} = i\frac{1}{\sqrt{\mathbf{q}^2+m^2}}\left(\frac{\Sigma\bar{u}(\mathbf{q})u(\mathbf{q})}{q_0-\sqrt{\mathbf{q}^2+m^2}} + \frac{\Sigma\bar{v}(-\mathbf{q})v(-\mathbf{q})}{q_0+\sqrt{\mathbf{q}^2+m^2}}\right)$

This identity can be technically reproduced as follows. First write the denominator of the propagator as

$$i\frac{\not{q}-m}{q^2-m^2} = i\frac{\not{q}-m}{q_0^2-(\mathbf{q}^2+m^2)} \tag{2.4.2}$$

This could be factored into

$$q_0^2-(\mathbf{q}^2+m^2) = (q_0+\sqrt{\mathbf{q}^2+m^2})(q_0-\sqrt{\mathbf{q}^2+m^2}) \tag{2.4.3}$$

So it implies poles at $q_0 = \pm\sqrt{\mathbf{q}^2+m^2}$. There is one unique way of factoring the original propagator into the two poles:

$$\frac{1}{2\sqrt{\mathbf{q}^2+m^2}}\left(\frac{\gamma^0\sqrt{\mathbf{q}^2+m^2}-\gamma\cdot\mathbf{q}+m}{q_0-\sqrt{\mathbf{q}^2+m^2}} - \frac{\gamma^0\sqrt{\mathbf{q}^2+m^2}+\gamma\cdot\mathbf{q}-m}{q_0+\sqrt{\mathbf{q}^2+m^2}}\right) \tag{2.4.4}$$

The two numerators can be exactly equated to sums over polarisation states of electron and

positron spinors:

$$\begin{aligned} \Sigma u(\mathbf{p})\bar{u}(\mathbf{p}) &= \gamma \cdot p + m \\ \Sigma v(\mathbf{p})\bar{v}(\mathbf{p}) &= \gamma \cdot p - m \end{aligned}$$

These relations hold for particles which are on mass-shell. That is exactly the case here. But then, there is an assumption that the quantity p_0 above is the on mass-shell energy, $\sqrt{\mathbf{p}^2 + m^2}$.

So, noting that particles with momentum $\pm\mathbf{q}$ have the same energy $\sqrt{\mathbf{q}^2 + m^2}$ and that q_0 is the off-mass shell energy from the relativistic diagram, the propagator can be rewritten:

$$\frac{1}{2\sqrt{\mathbf{q}^2 + m^2}} \left(\frac{\Sigma u(\mathbf{q})\bar{u}(\mathbf{q})}{q_0 - \sqrt{\mathbf{q}^2 + m^2}} - \frac{\Sigma v(-\mathbf{q})\bar{v}(-\mathbf{q})}{q_0 + \sqrt{\mathbf{q}^2 + m^2}} \right) \tag{2.4.5}$$

The numerators have now been put into exactly the forms expected for the regular- and Z-diagrams of old perturbation theory. The denominators also correspond to the expected form of $E_{\text{in}} - E_{\text{int}}$.

First consider the regular diagram. The initial energy is q_0, since in relativistic theory the total energy at the vertex is conserved. The intermediate energy is the on-mass shell energy of the electron: $\sqrt{\mathbf{q}^2 + m^2}$. Thus the denominator of $q_0 - \sqrt{\mathbf{q}^2 + m^2}$ is that expected.

Now consider the Z-diagram. The initial energy is still q_0. The intermediate energy is more complicated: there are two photons, two electrons, and a positron present. The total combined energy is:

$$E_{\text{int}} = p_0 + p_0' + k_0 + k_0' + \sqrt{\mathbf{q}^2 + m^2} = 2q_0 + \sqrt{\mathbf{q}^2 + m^2} \tag{2.4.6}$$

Then the difference $E_{\text{in}} - E_{\text{int}} = -q_0 - \sqrt{\mathbf{q}^2 + m^2}$. This correspons to the denominator found above, with the overall negative factor providing the relative minus sign.

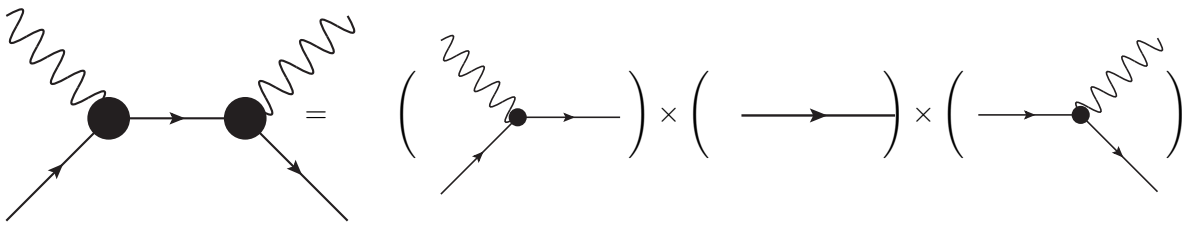
So rewriting the propagator into these two separate terms can be understood as how the fully relativistic process appears in old perturbation theory. In a relativistic theory, inter-

mediate states involving positrons must be accounted for. For higher order QED diagrams more complicated processes will appear, but at the tree level only those discussed above are involved. No approximations are involved — the identity for the propagator is exact. The form is more convenient for discussing nonrelativistic energies, but is still equivalent to the relativistic diagrams.

Relation between NRQED and old perturbation theory

Now it is necessary to show how the diagrams in old perturbation theory relate to those of NRQED.

The normal diagrams can be easily interpreted as the product of vertices and the relativistic Rayleigh-Schrodinger propagator:



Where Recall that the total NRQED one-photon vertex was derived by comparing to the QED vertex, so necessarily

$$\bar{u}\Gamma^0 u = \phi^\dagger V^0 \phi,\ \bar{u}\Gamma^i u = \phi^\dagger V^i \phi \tag{2.4.7}$$

The propagator $1/(q_0 - \sqrt{m^2 + \mathbf{q}^2})$ is equal to the total propagator in NRQED (including relativistic corrections). So it really is the case that the two sets of diagrams are equivalent. Whether the vertices are written in terms of NRQED or QED, the ultimate expression will be the same.

It then follows that the contributions to the local contact term in NRQED come (at tree level) exactly from the Z-diagrams. To calculate the contact terms to the order (in the nonrelativistic expansion) needed, the Z-diagrams need to be approximated just as the one-photon vertex diagrams were in the previous section. Then the NRQED coefficients can be obtained by comparing the two calculations.

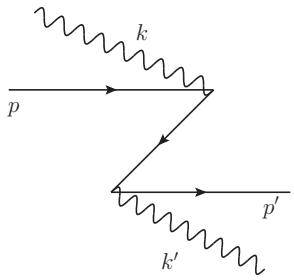
At nonrelativistic energies, it would be expected that the sum over intermediate states now does not resemble a propagator. Because only terms involving up to one power of momentum need be kept, the square root term becomes simply m. q_0, the total incoming energy, will involve both the electron and photon energy. Depending on whether the crossed or uncrossed diagram is considered, it will be either $q_0 = p_0 + k_0$ or $q_0 = p_0 - k_0'$. In either case, it will be m at the leading order, with a first order correction due to the photon energy.

$$\frac{\Sigma\bar{v}(-\mathbf{q})v(-\mathbf{q})}{q_0 + \sqrt{\mathbf{q}^2 + m^2}} \approx \frac{\Sigma\bar{v}(-\mathbf{q})v(-\mathbf{q})}{2m + (q_0 - m)} \tag{2.4.8}$$

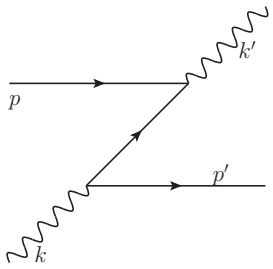
Since $q_0 - m << m$, this can then be written as

$$\approx \left(1 - \frac{q_0 - m}{2m}\right) \frac{\Sigma\bar{v}(-\mathbf{q})v(-\mathbf{q})}{2m} \tag{2.4.9}$$

In this approximation, the two Z diagrams become



$$= \quad \Sigma_{\text{spin}} \left(1 - \tfrac{k_0}{2m}\right) \tfrac{1}{2m} \bar{u}(\mathbf{p}')\gamma^\mu \epsilon_\mu^*(k')v_s(-\mathbf{p} - \mathbf{k})\bar{v}_s(-\mathbf{p} - \mathbf{k})\Gamma^\nu \epsilon_\nu(k)u(\mathbf{p})$$



$$= \quad \Sigma_{\text{spin}} \left(1 + \tfrac{k_0'}{2m}\right) \tfrac{1}{2m} \bar{u}(\mathbf{p}')\gamma^\mu \epsilon_\mu(k)v(\mathbf{k}' - \mathbf{p})_s\bar{v}_s(\mathbf{k}' - \mathbf{p})\Gamma^\nu \epsilon_\nu^*(k')u(\mathbf{p})$$

While the sums over intermediate states could also be expanded, it'll be easiest to calculate these in the above form. The vertices will be the sum of particle-antiparticle bilinears, which can be calculated separately.

2.4.1 Nonrelativistic expressions for Z diagrams

Looking at the equations for the Z diagrams, they are both the product of two types of terms to calculate:

$$\bar{u}(\mathbf{p}')\Gamma^{\mu}(q)v(\boldsymbol{\ell}) \text{ and } \bar{u}(\boldsymbol{\ell})\Gamma^{\mu}(q)v(\mathbf{p}) \tag{2.4.10}$$

Here ℓ is the intermediate momentum of the positron, and q is the momentum of the photon going into the vertex (either k or $-k'$). The form of Γ is

$$\Gamma^{\mu}(q) = F_1\gamma^{\mu} + F_2\frac{q_\nu\sigma^{\mu\nu}}{2m} \tag{2.4.11}$$

To compare the Z diagrams to the contact terms of NRQED, first express the bilinears in the vertices of (2.4.10) in terms of the nonrelativistic quantities. This can be done for each of the two terms in Γ^{μ} separately. The bispinors u will be replaced by the spinor ϕ, as before. Now, the bispinor v will be replaced by a spinor for a positron, which shall be called χ. In doing the expansion only terms up to $\mathcal{O}(1/m)$ need be kept.

To calculate the vector like bilinears, treat the spatial and time-like components separately. First $\mu = 0$:

$$\begin{aligned}
\bar{u}(p')\gamma^0 v(\ell) &= \begin{pmatrix} \varphi^\dagger & \frac{\boldsymbol{\sigma}\mathbf{p}'}{2m}\varphi^\dagger \end{pmatrix} \begin{pmatrix} \frac{\boldsymbol{\sigma}\mathbf{q}}{2m}\chi \\ \chi \end{pmatrix} \\
&= \phi^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\ell + \boldsymbol{\sigma}\cdot\mathbf{p}'}{2m} \right) \chi \\
\bar{v}\ell\gamma^0 u(p) &= \begin{pmatrix} \frac{\boldsymbol{\sigma}\boldsymbol{\ell}}{2m}\chi^\dagger & \chi^\dagger \end{pmatrix} \begin{pmatrix} \varphi \\ \frac{\boldsymbol{\sigma}\mathbf{p}}{2m}\varphi \end{pmatrix} \\
&= \chi^\dagger \left(\frac{\boldsymbol{\sigma}\cdot\ell + \boldsymbol{\sigma}\cdot\mathbf{p}}{2m} \right) \phi
\end{aligned}$$

Then $\mu = i$:

$$\begin{aligned}
\bar{u}(p')\gamma^i v(\ell) &= \begin{pmatrix} \varphi^\dagger & \frac{\boldsymbol{\sigma}\mathbf{p}'}{2m}\varphi^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} \frac{\boldsymbol{\sigma}\boldsymbol{\ell}}{2m}\chi \\ \chi \end{pmatrix} \\
&= \phi^\dagger \sigma_i \chi \\
\bar{v}(\ell)\gamma^i u(p) &= \begin{pmatrix} \frac{\boldsymbol{\sigma}\boldsymbol{\ell}}{2m}\chi^\dagger & \chi^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \frac{\boldsymbol{\sigma}\mathbf{p}}{2m}\varphi \end{pmatrix} \\
&= \chi^\dagger \sigma_i \phi
\end{aligned}$$

In the tensor terms a factor of momentum q_ν appears. The spatial part, q_j is "naturally raised" so $q_j = -(\mathbf{q})_j$. As before the two types of indices should be treated separately. For $\mu = 0$:

$$\begin{aligned}
\frac{iq_\nu}{2m}\bar{u}(p')\sigma^{0\nu}v(\ell) &= \frac{iq_j}{2m}\bar{u}(p')\sigma^{0j}v(\ell) \\
&= -\frac{q_j}{2m}\begin{pmatrix} \varphi^\dagger & \frac{\boldsymbol{\sigma}\mathbf{p}'}{2m}\varphi^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} \frac{\boldsymbol{\sigma}\boldsymbol{\ell}}{2m}\chi \\ \chi \end{pmatrix} \\
&= \phi^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{q}}{2m}\chi \\
\frac{iq_\nu}{2m}\bar{v}(\ell)\sigma^{0\nu}u(p) &= \frac{iq_j}{2m}\bar{v}(\ell)\sigma^{0j}u(p) \\
&= -\frac{q_j}{2m}\bar{v}\ell \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \frac{\boldsymbol{\sigma}\mathbf{p}}{2m}\varphi \end{pmatrix} \\
&= -\chi^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{q}}{2m}\phi
\end{aligned}$$

Then for $\mu = i$

$$\frac{iq_\nu}{2m}\bar{u}(p')\sigma^{i\nu}v(\ell) = \frac{iq_0}{2m}\bar{u}(p')\sigma^{i0}v(\ell) + \frac{iq_j}{2m}\bar{u}(p')\sigma^{ij}v(\ell) \tag{2.4.12}$$

$$= -\frac{iq_0}{2m}\begin{pmatrix} \varphi^\dagger & \frac{\boldsymbol{\sigma}\mathbf{p}'}{2m}\varphi^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} \frac{\boldsymbol{\sigma}\boldsymbol{\ell}}{2m}\chi \\ \chi \end{pmatrix} + \frac{iq_j}{2m}\epsilon_{ijk}\begin{pmatrix} \varphi^\dagger & \frac{\boldsymbol{\sigma}\mathbf{p}'}{2m}\varphi^\dagger \end{pmatrix} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \begin{pmatrix} \frac{\boldsymbol{\sigma}\boldsymbol{\ell}}{2m}\chi \\ \chi \end{pmatrix}$$

The second term will be of order $\mathcal{O}(1/m^2)$ and so can be neglected. So

$$\frac{iq_\nu}{2m}\bar{u}(p')\sigma^{i\nu}v(\ell) = \phi^\dagger \frac{q_0\sigma_i}{2m}\chi \tag{2.4.13}$$

The complementary term is

$$\frac{iq_\nu}{2m}\bar{v}(\ell)\sigma^{i\nu}u(p) = \frac{iq_0}{2m}\bar{v}(\ell)\sigma^{i0}v(\ell) + \frac{iq_j}{2m}\bar{u}(p')\sigma^{ij}u(p) \tag{2.4.14}$$

Again, the second term with σ^{ij} has the same general structure and will be of order $1/m^2$. So

$$\begin{aligned}\frac{iq_\nu}{2m}\bar{v}(\ell)\sigma^{i\nu}u(p) &= \frac{iq_0}{2m}\begin{pmatrix}\frac{\boldsymbol{\sigma}\boldsymbol{\ell}}{2m}\chi^\dagger & \chi^\dagger\end{pmatrix}\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix}\varphi\\ \frac{\boldsymbol{\sigma}\mathbf{p}}{2m}\varphi\end{pmatrix}\\ &= -\chi^\dagger\frac{q_0\sigma^i}{2m}\phi\end{aligned}$$

Now the total vertices $\bar{u}\Gamma^\mu v$ can be expressed nonrelativistically:

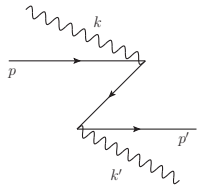
$$\bar{u}(p')\Gamma^0 v(\ell) = \phi^\dagger(p')\left(F_1\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\ell}+\boldsymbol{\sigma}\cdot\mathbf{p}'}{2m} + F_2\frac{\boldsymbol{\sigma}\cdot\mathbf{q}}{2m}\right)\chi(\ell) \tag{2.4.15}$$

$$\bar{v}(\ell)\Gamma^0 v(p) = \chi^\dagger(p')\left(F_1\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\ell}+\boldsymbol{\sigma}\cdot\mathbf{p}}{2m} + F_2\frac{\boldsymbol{\sigma}\cdot\mathbf{q}}{2m}\right)\phi(\ell) \tag{2.4.16}$$

$$\bar{u}(p')\Gamma^i v(\ell) = \phi^\dagger\left(F_1 + F_2\frac{q_0}{2m}\right)\sigma^i\chi \tag{2.4.17}$$

$$\bar{v}(\ell)\Gamma^i v(p) = \chi^\dagger\left(F_1 - F_2\frac{q_0}{2m}\right)\sigma^i\phi \tag{2.4.18}$$

Returning to the Z diagrams, the structures Γ^μ appear contracted with the photon polarization. Because a physical process is being calculated, the result should not depend on the gauge chosen. So it will be easiest to choose the gauge where the photons are transverse and $\epsilon_0 = 0$. Then $\Gamma\cdot\epsilon = -\boldsymbol{\Gamma}\cdot\boldsymbol{\epsilon}$.



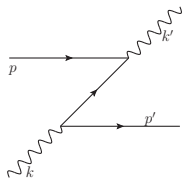
$$= \Sigma_{\text{spin}} \left(1 - \frac{k_0}{2m}\right) \frac{1}{2m} \left[\phi^\dagger \left(F_1 + F_2 \frac{-k_0'}{2m}\right) \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^*(k')\chi\right] \times \left[\epsilon_j(k)\chi^\dagger \left(F_1 - F_2 \frac{k_0}{2m}\right) \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}(k)\phi\right] \tag{2.4.19}$$

The sum over intermediate spin spates just becomes the identity: $\Sigma_{\text{spin}}\chi^\dagger\chi = 1$.

$$= \frac{1}{2m}\left(1 - \frac{k_0}{2m}\right)\phi^\dagger \left[\left(F_1 - F_2\frac{k_0'}{2m}\right)\left(F_1 - F_2\frac{k_0}{2m}\right)\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(k')\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}(k)\right]\phi \tag{2.4.20}$$

Since only terms up to order $1/m$ are needed, this can be simplified. Also, in this approximation $k_0' \approx k_0$, as the total conservation of energy implies $k_0' - k_0 = p_0 - p_0' \sim \mathbf{p}^2/m^2$. So for convenience everything will be written in terms of k_0.

$$= \frac{1}{2m}\phi^\dagger \left[\left(F_1^2 - F_1[F_1 + 2F_2]\frac{k_0}{2m}\right)\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(k')\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}(k)\right]\phi \tag{2.4.21}$$



$$= \Sigma_{\text{spin}} \left(1 + \tfrac{k_0'}{2m}\right) \tfrac{1}{2m} \left[\phi^\dagger \left(F_1 + F_2 \tfrac{k_0}{2m}\right) \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^(k)\chi\right] \left[\epsilon_j(k)\chi^\dagger \left(F_1 + F_2 \tfrac{k_0'}{2m}\right) \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^*(k')\phi\right]$$

After summing over spin states

$$= \frac{1}{2m}\left(1 + \frac{k_0'}{2m}\right)\phi^\dagger \left[\left(F_1 + F_2\frac{k_0}{2m}\right)\left(F_1 + F_2\frac{k_0'}{2m}\right)\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^(k)\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(k')\right]\phi \tag{2.4.22}$$

And then applying the same simplifications as before

$$= \frac{1}{2m}\phi^\dagger \left[\left(F_1^2 + F_1[F_1 + 2F_2]\frac{k_0}{2m}\right)\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^(k)\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(k')\right]\phi \tag{2.4.23}$$

The local interaction comes from the sum of the two diagrams. Adding them together,

$$
\begin{aligned}
\text{(Sum of Z diagrams)} &= \frac{1}{2m}\phi^\dagger\Bigg[\left(F_1^2 + F_1[F_1+2F_2]\frac{k_0}{2m}\right)\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^(k)\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(k') \\
&\quad + \left(F_1^2 - F_1[F_1+2F_2]\frac{k_0}{2m}\right)\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(k')\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}(k)\Bigg]\phi \\
&= \frac{F_1}{2m}\phi^\dagger\Bigg[F_1\{\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon},\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*\} + (F_1+2F_2)\frac{k_0}{2m}[\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon},\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*]\Bigg]\phi \\
&= \frac{F_1}{m}\phi^\dagger\Bigg[F_1\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}^* + (F_1+2F_2)\frac{k_0}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\times\boldsymbol{\epsilon}^*\Bigg]\phi
\end{aligned}
$$