Universal Binding and Recoil Corrections to Bound State g-Factors

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- Corrections to g-factor of bound particles are well established for spin-1/2
- Measurements in hydrogen-like carbon $^{12}C^{5+}$ or oxygen $^{16}O^{7+}$ need a precise theoretical bound g-factor, and involve systems with nuclear spin 0
- We find leading order recoil and binding corrections for particles of arbitrary spin

Definition of g

- Gyromagnetic ratio: ratio of **magnetic dipole moment** to **angular momentum**
- Here, concerned only with the gyromagnetic ratio associated with spin
- Then g-factor related to magnetic moment by

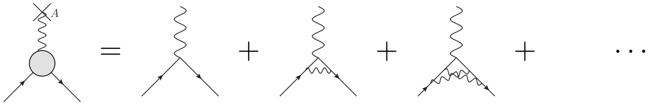
$$\boldsymbol{\mu} = g\frac{e}{2m}\mathbf{S}$$

- Energy difference between spin-flipped states

$$\Delta E = g\frac{e}{m}\mathbf{S}\cdot\mathbf{B}.$$

Free g-factor

- Free g-factor well known for electron
- Leading order $g = 2$, modified by radiative corrections



- Theoretical value agrees well with the measured value, (Hanneke, Fogwell, Gabrielse, 2008):

$$g_e = 2.002\,319\,304\,362\,2(15), \qquad \delta = 7.4 \times 10^{-13}.$$

Bound g-factor

For a particle in a bound state, there are corrections besides radiative.

- Recoil corrections that occur when separating the internal degrees of freedom from the external motion of the whole system. (Order m/M)
- Relativistic or binding corrections. (Because the velocity of a hydrogenic bound system is $v \sim Z\alpha$, corrections of this nature will be an expansion over $(Z\alpha)^2$)
- Additional effect such as due to the finite size of the nucleus — not considered here.

Bound g-factor

- Binding corrections for the electron with $g = 2$ first calculated by Breit (1928)
- Calculating $\langle e\gamma \cdot A \rangle$ gives

$$g_b = \frac{2}{3}\left(1 + 2\sqrt{1-(Z\alpha)^2}\right) = 2\left(1 - \frac{(Z\alpha)^2}{3} - \frac{(Z\alpha)^4}{12} + \cdots\right)$$

- For systems with nuclear spin one-half, the situation is well understood. For instance up to order $\alpha(Z\alpha)^2$:

$$\begin{aligned} g_b = g_e \Big\{ 1 - \frac{1}{3}(Z\alpha)^2 \left[1 - \frac{3}{2}\frac{m}{M} + \frac{3}{2}(1+Z)\frac{m^2}{M^2}\right] \\ + \frac{1}{4\pi}\alpha(Z\alpha)^2 \left[1 - \frac{5}{3}\frac{m}{M} + \frac{6+Z}{3}\frac{m^2}{M^2}\right] \Big\}. \end{aligned}$$

- In general, corrections as a series in $\alpha^n(Z\alpha)^k$ and (m/M).
- But, a result for arbitrary spin is needed

Experiment

- Hydrogen like ion is placed in a weak magnetic field B
- Spin flip frequency ω_L, corresponding to transitions between Zeeman levels, is measured

$$\omega_L = g_b \frac{e}{2m_e} B.$$

- The cyclotron frequency ω_C is

$$\omega_C = (Z-1)\frac{eB}{M}.$$

- Ratio can determine g_b

$$\frac{\omega_L}{\omega_C} = \frac{f_L}{f_C} = g_b \frac{e}{2(Z-1)} \frac{M}{m_e}.$$

- Or, determine electron mass

$$m_e = \frac{g_b}{2(Z-1)} \frac{\omega_C}{\omega_L} M$$

Measurements

- Most sensitive experiments in carbon $^{12}C^{5+}$ or oxygen $^{16}O^{7+}$ (Haffner, Werth, Verdu, 2003)
- Carbon

$$\frac{f_L}{f_C} = 4376.210\,498\,9(23), \qquad \delta = 5.2 \times 10^{-10}.$$

- Oxygen

$$\frac{f_L}{f_C} = 4164.376\,183\,7(32), \qquad \delta = 7.6 \times 10^{-10}.$$

- Precise enough to be best source of electron mass ratio
  - Already factor of 5 improvement over previous result
  - Experiments with smaller errors will further improve measurement
- But, requires theoretical value of g_b with enough precision

Statement of problem

- Consider loosely bound hydrogen-like systems, in a weak, constant magnetic field
- Calculate leading binding and recoil corrections to g_b
- Binding corrections will be of order $(Z\alpha)^2$ relative to zero-order value

Approach

- *Goal*: calculate leading binding and recoil corrections to the bound gyromagnetic ratio
- Possible if an effective nonrelativistic Lagrangian was known to order $1/m^3$
- *Strategy*:
    - Write down the most general form of such a Lagrangian, and fix the coefficients by comparing to relativistic theory
    - Then use this to calculate the interaction potential
    - From the interaction potential can be found g_b.

Approach

Work through this approach in several contexts:

- First, consider well known case — a spin one-half particle like the electron.

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Work through this approach in several contexts:

- First, consider well known case — a spin one-half particle like the electron.
- Next move to a spin one particle, such as the W boson.
- Finally derive an effective Lagrangian valid for arbitrary spin, and compare the result to the specific cases.

How to construct an effective Lagrangian

- Finite number of fields and operators to construct terms in the Lagrangian
- Combinations of these terms are restricted by symmetries and other constraints of the theory (Hermiticity, gauge invariance, etc.)
- Still an infinite number of combinations
- But, at a given level of precision, only a finite number of relevant terms
- Write down a Lagrangian with all such terms:
- Result: a Lagrangian with several coefficients, capturing the details of the high energy/small scale physics
- Fix these coefficients (to some level of precision) by demanding consistency with the higher energy theory.

Constructing the effective NRQED Lagrangian for spin one-half

- Constraints: Invariance under Galilean transforms as well as parity and time reversal, Hermiticity, and gauge invariance
- Gauge invariance will be fulfilled automatically if we use only gauge invariant building blocks with which to construct the Lagrangian
- In addition to the fermion field ψ, several other building blocks:

$$\mathbf{S},\ \mathbf{E},\ \mathbf{B},\ \text{and}\ \mathbf{D} = \boldsymbol{\nabla} - ie\mathbf{A}$$

- For our purposes, only the interaction of a single charged particle with an electromagnetic field is needed. So all terms in the Lagrangian will involve two fermion fields, with various powers of the other building blocks.

Restrictions on relevance of terms

- To calculate the leading ($v^2 \sim (Z\alpha)^2$) binding corrections to g_b, only leading order corrections in the Lagrangian need be considered
- The original nonrelativistic Lagrangian is

$$\mathcal{L} = D_0 + \frac{\mathbf{D}^2}{2m} - g\frac{e}{2m}\mathbf{S}\cdot\mathbf{B} \tag{1}$$

- For a hydrogen like system, it contains terms of $\mathcal{O}(mv^2)$ and $\mathcal{O}(B/m)$.
- So the corrections needed will be $\mathcal{O}(mv^4)$ and $\mathcal{O}(v^2B/m)$.
- Terms quadratic in the magnetic field may be neglected

NRQED Lagrangian

All the allowed terms are

$$\begin{aligned}\mathcal{L}_{NRQED} =&\psi^\dagger \Bigg\{ iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^2} + c_F \frac{e}{m}\mathbf{S}\cdot\mathbf{B} + c_D \frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2} \\ &+ c_S \frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D})}{8m^2} + c_{W1} \frac{e\mathbf{D}^2(\mathbf{S}\cdot\mathbf{B}) + (\mathbf{S}\cdot\mathbf{B})\mathbf{D}^2}{8m^3} \\ &- c_{W2} \frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3} + c_{p'p} \frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3} \Bigg\}\psi\end{aligned}$$

NRQED Lagrangian

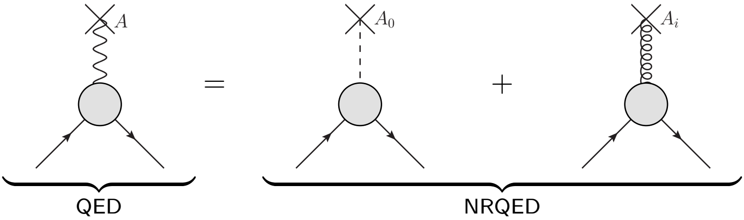
All the allowed terms are

$$
\begin{aligned}
\mathcal{L}_{NRQED} =& \psi^\dagger \Bigg\{ iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^2} + c_F \frac{e}{m} \mathbf{S} \cdot \mathbf{B} + c_D \frac{e(\mathbf{D} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{D})}{8m^2} \\
& + c_S \frac{ie\mathbf{S} \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D})}{8m^2} + (c_{W1} - c_{W2}) c_{W2} \frac{eD_i(\mathbf{S} \cdot \mathbf{B}) D_i}{4m^3} \\
& + c_{p'p} \frac{e[(\mathbf{S} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{D}) + (\mathbf{B} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{D})]}{8m^3} \Bigg\} \psi
\end{aligned}
$$

(Considering only constant magnetic field)

Scattering comparison

- Fix coefficients by comparison of physical processes
- Only need coefficients in front of one-photon terms
- (Terms like $\mathbf{E} \times \mathbf{A}$ contribute, but are part of gauge invariant $\mathbf{E} \times \mathbf{D}$)
- Elastic scattering off external field fixes all desired terms
- Also use stronger constraint that magnetic field is constant



Scattering in QED — form of vertex

- QED one-photon scattering
- Form of vertex captured by just two form factors
- $iM = -ie\bar{u}\Gamma^{\mu}A_{\mu}u$, Γ^{μ} defined as:

$$\Gamma^{\mu} = F_1(q^2)\gamma^{\mu} + F_2(q^2)\frac{i\sigma^{\mu\nu}q_{\nu}}{2m}$$

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- Form of vertex captured by just two form factors
- $iM = -ie\bar{u}\Gamma^{\mu}A_{\mu}u$, Γ^{μ} defined as:

$$\Gamma^{\mu} = F_1(q^2)\frac{p^{\mu} + p'^{\mu}}{2m} + i[F_1(q^2) + F_2(q^2)]\frac{\sigma^{\mu\nu}q_{\nu}}{2m}.$$

- Relation between u and ϕ?
- Demand that current densities match at 0 momentum transfer $(q = 0)$

$$e\frac{p^0}{m}\bar{u}u = e\phi^\dagger\phi, \qquad \text{where } u = \begin{pmatrix}\eta\\ \chi\end{pmatrix} = \begin{pmatrix}\eta\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\eta\end{pmatrix}$$

Discarding terms of $\mathcal{O}(\mathbf{p}^4/m^3)$

$$\eta \approx \left(1 - \frac{\mathbf{p}^2}{8m^2}\right)\phi$$

Scattering in QED — amplitudes in terms of ϕ

- $\bar{u}\Gamma^{\mu}u$ in terms of ϕ

$$
\begin{aligned}
(p+p')^{\mu}\bar{u}u &= (p+p')^{\mu}\phi^{\dagger}\left(1-\frac{\mathbf{p}^{2}+2\mathbf{p}\cdot\mathbf{p}'+\mathbf{p}'}{8m^{2}}-\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{4m^{2}}\right)\phi \\
\bar{u}\frac{i}{2m}q_{j}\sigma^{ij}u &= \frac{i\epsilon_{ijk}q_{j}}{2m}\phi^{\dagger}\left(\sigma_{k}-\frac{\boldsymbol{\sigma}\cdot\mathbf{p}p_{k}}{2m^{2}}\right)\phi \\
\bar{u}\frac{i}{2m}q_{j}\sigma^{0j}u &= -\phi^{\dagger}\left(\frac{\mathbf{q}^{2}}{4m^{2}}-\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}}{2m^{2}}\right)\phi
\end{aligned}
$$

Scattering in QED — amplitudes in terms of ϕ

Amplitudes:

$$eA_0\bar{u}\Gamma^0 u = \phi^\dagger\left(F_1 eA_0 + [F_1 + 2F_2]\left[\frac{e\boldsymbol{\sigma}\cdot\mathbf{E}\times\mathbf{p}}{4m^2} - \frac{e\boldsymbol{\nabla}\cdot\mathbf{E}}{8m^2}\right]\right)\phi,$$

$$\begin{aligned} eA_i\bar{u}\Gamma^i u = \phi^\dagger\Bigg\{ & -F_1\frac{e\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')}{2m}\left(1-\frac{\mathbf{p}^2}{2m^2}\right) - [F_1+F_2]\frac{e\boldsymbol{\sigma}\cdot\mathbf{B}}{2m} \\ & + F_1\frac{e\boldsymbol{\sigma}\cdot\mathbf{B}\mathbf{p}^2}{4m^3} + F_2\frac{(\boldsymbol{\sigma}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^2}\Bigg\}\phi. \end{aligned}$$

Scattering in NRQED

- One photon scattering read directly from Lagrangian
- For instance, $\mathbf{D}^2 \to e(\mathbf{p}+\mathbf{p}')\cdot\mathbf{A}$

Nonrelativistic amplitude

$$iM = ie\phi^\dagger\Bigg(-A_0 + \frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')}{2m} - \frac{\mathbf{A}\cdot(\mathbf{p}+\mathbf{p}')\mathbf{p}^2}{4m^3} + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2}$$
$$+ c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2} - (c_{W_1} - c_{W_2})\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3}\Bigg)\phi.$$

Coefficients

Comparing the two, coefficients are:

Coefficients for spin one-half

$$\begin{aligned} c_F &= g \\ c_D &= (g-1) \end{aligned} \qquad \begin{aligned} c_S &= 2(g-1) \\ c_{W_1} - c_{W_2} &= 2 \\ c_{p'p} &= g-2 \end{aligned}$$

Spin one

- Same type of calculation can be done for a spin one particle
- Standard model contains W^+, W^-
- Known relativistic Lagrangian, so no obstacles in calculating scattering
- NRQED Lagrangian for spin one will contain new terms

NRQED Lagrangian for spin one

- Same building blocks as before
- All terms from spin one-half exist here, too
- New terms arise because quadratic spin terms are allowed
- Quadrupole moment $Q_{ij} = S_i S_j + S_j S_i - \frac{2}{3}\delta_{ij}\mathbf{S}^2$

One new term

$$c_Q \frac{Q_{ij}(D_i E_j - E_j D_i)}{8m^3}$$

NRQED Lagrangian

General form for spin one NRQED Lagrangian is:

$$\mathcal{L}_{NRQED} = \psi^\dagger \Big\{ i(\partial_0 + ieA_0) + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^3} + c_F e \frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D \frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2}$$

$$+ c_Q \frac{eQ_{ij}(D_i E_j - E_i D_j)}{8m^2} + c_S \frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D})}{8m^2}$$

$$+ (c_{W_1} - c_{W_2}) \frac{e(\mathbf{S}\cdot\mathbf{B})\mathbf{D}^2}{4m^3} + c_{p'p} \frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3} \Big\} \psi$$

Relativistic scattering — diagrams

Relativistic Lagrangian

$$\mathcal{L} = -\frac{1}{2}(D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) + m^2 W^{\mu\dagger} W_\mu - i[g-1]eW^{\mu\dagger} W^\nu F_{\mu\nu}$$

From the above Lagrangian, derive Feynman rules and current density

Relativistic scattering — diagrams

Relativistic Lagrangian

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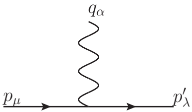
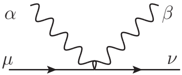
Diagrams are



$$= -ie\Big[g^{\mu\lambda}(p+p')^\alpha - g^{\lambda\alpha}(p' + [g-1]q)^\mu - g^{\alpha\mu}(p - [g-1]q)^\lambda\Big]$$



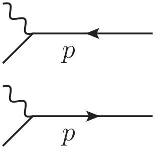
$$= -ie^2(2g^{\mu\nu}g^{\alpha\beta} - g^{\mu\beta}g^{\nu\alpha} - g^{\nu\beta}g^{\mu\alpha})$$

Relativistic scattering — diagrams

Relativistic Lagrangian

$$\mathcal{L} = -\frac{1}{2}(D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) + m^2 W^{\mu\dagger} W_\mu - i[g-1]eW^{\mu\dagger} W^\nu F_{\mu\nu}$$

External charged particle legs



$$= w_\mu(p)$$

$$= w^*_\mu(p)$$

$$p \cdot w(p) = 0$$

Amplitudes in terms of ϕ

Again, $j_0(q=0)$ must match.

$$\phi^\dagger \phi = 2p_0 \mathbf{w}^\dagger \cdot \mathbf{w} - 2\frac{(\mathbf{w}^\dagger \cdot \mathbf{p})(\mathbf{p} \cdot \mathbf{w})}{p_0}$$

Mixing between components can be written as the action of spin operators

$$\mathbf{w} = \frac{1}{\sqrt{2m}}\left(1 + \frac{\mathbf{p}^2}{4m^2} - \frac{(\mathbf{S} \cdot \mathbf{p})^2}{2m^2}\right)\phi$$

QED amplitude

- Amplitude from single vertex

$$iM = iew_\mu(p)w^*_\nu(p')\left[g^{\mu\nu}(p+p')\cdot A + g(q^\nu A^\mu - q^\mu A^\nu)\right]$$

- Split into two parts
- First has no g dependence

$$M_q = iew_\mu(p)w^*_\nu(p')g^{\mu\nu}(p+p')\cdot A$$

- Second proportional to g

$$M_g = iegw_\mu(p)w^*_\nu(p')(q^\nu A^\mu - q^\mu A^\nu)$$

QED amplitude

- Work out the nonrelativistic approximation as before
- Result:

QED amplitude

$$iM_{REL} = -ie\phi^\dagger\Big(A_0 - \frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{\mathbf{p}\cdot\mathbf{A}\mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3}\{\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - S_iS_j\boldsymbol{\nabla}_iE_j\}$$
$$- g\frac{1}{2m}\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\Big)\phi$$

NRQED Amplitude

- Again, read off one photon scattering from the NRQED Lagrangian

NRQED Amplitude

$$
\begin{aligned}
iM &= ie\phi^\dagger\Bigg(-A_0 + \frac{\mathbf{A}\cdot\mathbf{p}}{m} - \frac{(\mathbf{A}\cdot\mathbf{p})\mathbf{p}^2}{2m^3} + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2} + c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2} \\
&\quad + c_S\frac{\mathbf{E}\times\mathbf{p}}{4m^2} - (c_{W_1} - c_{W_2})\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3}\Bigg)\phi
\end{aligned}
$$

Coefficients

- Comparing the two amplitudes, the coefficients are fixed

Spin one coefficients

$$
\begin{aligned}
c_F &= g \\
c_D &= \frac{4(g-1)}{3} \\
c_Q &= -4(g-1)
\end{aligned}
\qquad
\begin{aligned}
c_S &= 2(g-1) \\
c_{W_1} - c_{W_2} &= 2 \\
c_{p'p} &= g-2
\end{aligned}
$$

Coefficients

- Comparing the two amplitudes, the coefficients are fixed

Spin one coefficients

$$\begin{aligned} c_F &= g \\ c_D &= \frac{4(g-1)}{3} \\ c_Q &= -4(g-1) \end{aligned} \qquad \begin{aligned} c_S &= 2(g-1) \\ c_{W_1} - c_{W_2} &= 2 \\ c_{p'p} &= g-2 \end{aligned}$$

- For contrast, the spin half coefficients were

Spin half coefficients

$$\begin{aligned} c_F &= g \\ c_D &= (g-1) \end{aligned} \qquad \begin{aligned} c_S &= 2(g-1) \\ c_{W_1} - c_{W_2} &= 2 \\ c_{p'p} &= g-2 \end{aligned}$$

Arbitrary spin

Now we move on to a formalism for particles of arbitrary spin

- First develop NRQED Lagrangian
- Then an effective relativistic theory

NRQED Lagrangian

- For the case of general spin, more complicated spin polynomials might arise – involving products of three or more spin matrices.

$$\mathcal{L} = \psi^\dagger \Big\{ iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^2} + c_F \frac{e}{m}\mathbf{S}\cdot\mathbf{B} + c_D \frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2} + c_Q \frac{eQ_{ij}(D_i E_j - E_i D_j)}{8m^2}$$
$$+ c_S \frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D})}{8m^2} + c_{W1}\frac{e\mathbf{D}^2\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2}\frac{eD_i(\mathbf{S}\cdot\mathbf{B})D_i}{4m^3}$$
$$+ c_{p'p}\frac{e[(\mathbf{S}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{D})]}{8m^3} + c_{T_1}\frac{e\bar{S}_{ijk}(D_i D_j B_k + B_k D_j D_i)}{8m^3}$$
$$+ c_{T_2}\frac{e\bar{S}_{ijk}D_i B_j D_k}{8m^3}\Big\}\psi.$$

- Terms with one or two powers of the external field
- These come from both one- and two- photon interactions
- But, coefficients are fixed by one-photon interaction

Relativistic theory for arbitrary spin

- Follow the approach of Khriplovich and Pomeransky (1997)
- Define bispinors Ψ by boosting from rest frame spinors

$$\Psi = \begin{pmatrix} \cosh \frac{\boldsymbol{\Sigma}\cdot\phi}{2}\xi_0 \\ \sinh \frac{\boldsymbol{\Sigma}\cdot\phi}{2}\xi_0 \end{pmatrix}$$

- Behavior of spinors in rest frame defined by the spin of the particle
  - 2s indices, split between dotted and undotted types.
  - Two natural operators, $\mathbf{S}$ and $\boldsymbol{\Sigma}$

Features of relativistic theory

Properties of Σ, related to spin

$$\Sigma^2 = 4s + \Delta, \qquad \Delta = 0 \text{ for integer spin}, 1 \text{ otherwise}$$

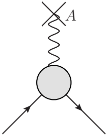
$$\left\langle \Sigma_i \Sigma_j + \Sigma_j \Sigma_i - \frac{2}{3}\delta_{ij}\mathbf{\Sigma}^2 \right\rangle = \lambda \left\langle S_i S_j + S_j S_i - \frac{2}{3}\delta_{ij}\mathbf{S}^2 \right\rangle$$

$$\lambda = -\frac{4}{2s-1} \text{ for integer spin}, -\frac{4}{2s} \text{ otherwise}$$

- λ and $\mathbf{\Sigma}^2$ are spin dependant constants that will show up in NRQED Lagrangian
- For spin one-half, $\Sigma = \sigma$.

One-photon process

One-photon interaction constrained by general considerations of electromagnetic current



- The current $j_\mu = e\bar{\Psi}\Gamma_\mu\Psi$ must
  - transform like a Lorentz 4-vector
  - be gauge invariant
- The most general such expression has

$$\Gamma_\mu = F_e \frac{(p+p')_\mu}{2m} - F_m \frac{\Sigma_{\mu\nu} q^\nu}{2m}$$

  - $\Sigma_{\mu\nu}$ is analogous to $\sigma_{\mu\nu}$ in the spin one-half case
  - The parameters have some dependence on q, but at leading order they are $F_e = 1$ and $F_m = g/2$
  - Other terms can be written, but will not contribute to the NRQED coefficients

Nonrelativistic expansion

To connect back to the NRQED Lagrangian, expand the relativistic expression

- The wave functions are approximated

$$\Psi \approx \begin{pmatrix} \left[1 + \frac{(\boldsymbol{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m^2}\right]\phi \\ \frac{\boldsymbol{\Sigma}\cdot\mathbf{p}}{2m}\phi \end{pmatrix}$$

- Example: expand the zero component of the current as

$$\bar{\Psi}\Gamma_0\Psi \approx \phi^\dagger \left(1 - [g-1]\frac{(\boldsymbol{\Sigma}\cdot\mathbf{q})^2}{8m^2} + [g-1]\frac{i\mathbf{s}\cdot(\mathbf{q}\times\mathbf{p})}{4m^2}\right)\phi$$

- This then gives rise to terms in the potential

$$eA_0\bar{\Psi}\Gamma_0\Psi \to eA_0 - e[g-1]\left(\frac{\boldsymbol{\nabla}\cdot\mathbf{E}}{8m^2}\frac{\boldsymbol{\Sigma}^2}{3} + \lambda\frac{Q_{ij}\boldsymbol{\nabla}_i E_j}{4m^2}\right) - e[g-1]\frac{\mathbf{s}\cdot\mathbf{E}\times\mathbf{p}}{2m^2}$$

NRQED coefficients

- Scattering amplitudes compared like before

Coefficients for arbitrary spin

$$\begin{aligned} c_F &= g \\ c_D &= (g-1)\frac{\mathbf{\Sigma}^2}{3} \\ c_Q &= -2\lambda(g-1) \end{aligned} \qquad \begin{aligned} c_S &= 2(g-1) \\ c_{W_1} - c_{W_2} &= 2 \\ c_{p'p} &= g-2 \end{aligned}$$

- Coefficients before spin trilinears c_T do not appear
- $\mathbf{\Sigma^2}$ and λ are spin dependent constants
- Coefficients for spins $1/2$ and 1 agree with previous results
- Spin dependence associated with derivatives of the electric field

Connection to BMT equation

- Universality of some coefficients can be understood through the BMT equation
- Gives time evolution of spin four-vector a_μ
- Neglecting derivatives of the electromagnetic field, the equation has no dependence on spin magnitude:

$$\frac{da^\mu}{d\tau} = g\frac{e}{2m}F^{\mu\nu}a_\nu - (g-2)\frac{e}{2m}u^\nu F^{\mu\lambda}u_\mu a_\lambda.$$

- Time evolution of spin must also be given by $[H, \mathbf{S}]$
- So to agree with the BMT equation, all relevant coefficients in H must be universal

Interaction

Now we're ready to consider the bound system

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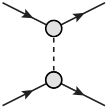
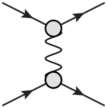
- Idea is to calculate g_b with a regular quantum mechanical Hamiltonian
- The potential between the two bound particles is found by considering scattering
- That scattering is calculated from the NRQED Lagrangian already developed

Interaction

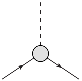
In the absence of an external field, the effective interaction potential can be found from the scattering amplitudes



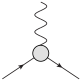
Coulomb exchange



Transverse exchange



$$= \quad ie\left(1 + \tfrac{1}{8m^2}\left[c_D\mathbf{q}^2 + c_Q Q_{ij} q_i q_j - 2ic_S \mathbf{S}\cdot\mathbf{p}\times\mathbf{q}\right]\right)$$



$$= \quad i\,\tfrac{e}{2m}\left(\mathbf{p} + \mathbf{p}' + c_F\, i\mathbf{S}\times\mathbf{q}\right)_i .$$

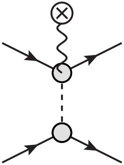
Potential in absence of external field

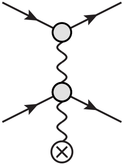
- The scattering is related to the quantum mechanical potential via $M = (\phi_1^\dagger \phi_2^\dagger \phi_2 \phi_1) U(\mathbf{p_1}, \mathbf{p_2}, \mathbf{q})$
- The potential calculated in this way is

$$
\begin{aligned}
\overline{U}(\mathbf{p_1}, \mathbf{p_2}, \mathbf{r}) \quad = \quad & e_1 e_2 \Bigg[\frac{1}{4\pi r} - \frac{1}{8m_2^2} \left(d_D \delta(\mathbf{r}) - 3 d_Q \frac{Q_{2\,ij} r_i r_j}{4\pi r^5} - d_S \frac{\mathbf{r} \cdot \mathbf{S_2} \times \mathbf{p_2}}{2\pi r^3} \right) \\
& - \frac{1}{8m_1^2} \left(c_D \delta(\mathbf{r}) - 3 c_Q \frac{Q_{1\,ij} r_i r_j}{4\pi r^5} + c_S \frac{\mathbf{r} \cdot \mathbf{S_1} \times \mathbf{p_1}}{2\pi r^3} \right) \\
& - \frac{1}{m_1 m_2} \left(\frac{\mathbf{p_1} \cdot \mathbf{p_2}}{8\pi r} + \frac{(\mathbf{p_1} \cdot \mathbf{r})(\mathbf{p_2} \cdot \mathbf{r})}{8\pi r^3} \right) \\
& + \frac{1}{2m_1 m_2} \frac{\mathbf{r} \cdot (d_F \mathbf{S_2} \times \mathbf{p_1} - c_F \mathbf{S_1} \times \mathbf{p_2})}{4\pi r^3} \\
& - \frac{c_F d_F}{4m_1 m_2} \left(\frac{2}{3} \mathbf{S_1} \cdot \mathbf{S_2} \delta(\mathbf{r}) - \frac{1}{4\pi r^3} \left\{ \mathbf{S_1} \cdot \mathbf{S_2} - 3 \frac{(\mathbf{S_1} \cdot \mathbf{r})(\mathbf{S_2} \cdot \mathbf{r})}{r^2} \right\} \right) \Bigg].
\end{aligned}
$$

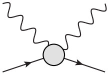
Diagrams with magnetic field

Now also include the magnetic field, with diagrams like:

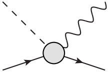




Where the new vertices are



$$= -i\frac{e^2\delta_{ij}}{m}.$$



$$= c_S\frac{e^2}{4m^2}\epsilon_{ijk}q_jS_k.$$

Interaction potential with magnetic field

Result in momentum space:

$$\begin{aligned} U_2(\mathbf{p_1},\mathbf{p_2},\mathbf{q}) \quad = \quad & e_1 e_2 \Big\{ -ic_s \frac{e_1}{4m_1^2} \frac{\mathbf{S_1}\cdot\mathbf{A}_1\times\mathbf{q}}{\mathbf{q}^2} + id_S \frac{e_2}{4m_2^2} \frac{\mathbf{S_2}\cdot\mathbf{A}_2\times\mathbf{q}}{\mathbf{q}^2} \\ & + \frac{e_1}{m_1 m_2}\Big(\frac{\mathbf{p}_2\cdot\mathbf{A}_1}{\mathbf{q}^2} - id_F \frac{\mathbf{S}_2\cdot\mathbf{q}\times\mathbf{A}_1}{2\mathbf{q}^2} - \frac{(\mathbf{p}_2\cdot\mathbf{q})(\mathbf{A}_1\cdot\mathbf{q})}{\mathbf{q}^4} \Big) \\ & + \frac{e_2}{m_1 m_2}\Big(\frac{\mathbf{p}_1\cdot\mathbf{A}_2}{\mathbf{q}^2} + ic_F \frac{\mathbf{S}_1\cdot\mathbf{q}\times\mathbf{A}_2}{2\mathbf{q}^2} - \frac{(\mathbf{p}_1\cdot\mathbf{q})(\mathbf{A}_2\cdot\mathbf{q})}{\mathbf{q}^4} \Big) \Big\} \end{aligned}$$

Relevant part of Hamiltonian

Full Hamiltonian is $H = H_1 + H_2 + H_{\text{int}}$ — each particle's free Hamiltonian added to the interaction

- Extract relevant parts for particle 1
- Only terms linear in the magnetic field and spin (S_1) contribute to g_b
- *Note*: no spin dependent coefficients enter this expression

$$
\begin{aligned}
H^{(1)}_{\text{spin}} = &- g_1 \frac{e_1}{2m_1} \mathbf{S}_1 \cdot \mathbf{B} \left(1 - \frac{\mathbf{p}_1^2}{2m_1^2}\right) \\
&- (g_1 - 2) \frac{e_1}{2m_1^2} \mathbf{S}_1 \cdot \mathbf{B} \frac{\mathbf{p}_1^2}{2m_1^2} + (g_1 - 2) \frac{e_1}{2m_1^2} \frac{(\mathbf{p}_1 \cdot \mathbf{B})(\mathbf{S}_1 \cdot \mathbf{p}_1)}{2m_1^2} \\
&- e_1 e_2 (g_1 - 1) \frac{2\mathbf{S}_1 \cdot \mathbf{r} \times [\mathbf{p}_1 - e_1 \mathbf{A}_1]}{16\pi m_1^2 r^3} - e_1 e_2 g_1 \frac{2\mathbf{S}_1 \cdot \mathbf{r} \times [\mathbf{p}_2 - e_2 \mathbf{A}_2]}{16\pi m_1 m_2 r^3}.
\end{aligned}
$$

Separation of CoM motion in presence of an external field

- Need to separate center of mass motion from internal motion
- Normal procedure is to define internal/CoM position and momentum as

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \qquad \mathbf{R} = \mu_1 \mathbf{r}_1 + \mu_2 \mathbf{r}_2,$$

$$\mathbf{p} = \mu_2 \mathbf{p}_1 - \mu_1 \mathbf{p}_2, \qquad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$$

- External field $\mathbf{A}(\mathbf{r}) = \mathbf{B} \times \mathbf{r}/2$ spoils this separation
- Solution: additional transformation needed

Transformation

Find transformation by demanding the center of mass motion match that of a single particle

No field

$$H = \frac{\mathbf{p}^2}{2m}$$

$\mathbf{p}$ is conserved

With field

$$H = \frac{(\mathbf{p} - e\mathbf{A})^2}{2m}$$

$\mathbf{p} + \mathbf{A}$ is conserved

- In this case of constant magnetic field, can define $\mathbf{A}_R = \mu_1 \mathbf{A}_1 + \mu_2 \mathbf{A}_2$, potential at $\mathbf{R}$.
- If the center of mass motion is separated, $\mathbf{P} + e\mathbf{A}_R$ should be conserved
- Instead $\mathbf{P} + e(\mathbf{A}_1 + \mathbf{A}_2)$ is the conserved quantity

Transformation

Desired quantity is conserved if

$$\mathbf{P} \to U^{-1}\mathbf{P}U = \mathbf{P} - (e_1\mu_2 - e_2\mu_1)\mathbf{A}_r$$

Transformation realised by

$$U = e^{-i(e_1\mu_2 - e_2\mu_1)\mathbf{A}_R\cdot\mathbf{r}}.$$

Other effects of U:

$$\begin{aligned}
\mathbf{p}_1 &\to \mathbf{p}_1 + (e_1\mu_2 - e_2\mu_1)\mathbf{A}_1 \\
\mathbf{p}_2 &\to \mathbf{p}_2 - (e_1\mu_2 - e_2\mu_1)\mathbf{A}_2 \\
\mathbf{p} &\to \mathbf{p} + (e_1\mu_2 - e_2\mu_1)\mathbf{A}_R
\end{aligned}$$

Transformed Hamiltonian

To transform the original Hamiltonian, including perturbations, apply these substitutions.

$$\begin{aligned}
\mathbf{p}_1 - e_1\mathbf{A}(\mathbf{r}_1) &\rightarrow \mu_1[\mathbf{P} - (e_1+e_2)\mathbf{A}(\mathbf{R})] + \left[\mathbf{p} - [e_1 - (e_1+e_2)\mu_1^2]\mathbf{A}(\mathbf{r})\right], \\
\mathbf{p}_2 - e_2\mathbf{A}(\mathbf{r}_1) &\rightarrow \mu_2[\mathbf{P} - (e_1+e_2)\mathbf{A}(\mathbf{R})] + \left[\mathbf{p} - [e_2 - (e_1+e_2)\mu_2^2]\mathbf{A}(\mathbf{r})\right]
\end{aligned}$$

Resultant Hamiltonian

$$\begin{aligned}
H'^{(1)}_{\text{spin}} =& - g_1 \frac{e_1}{2m_1}\mathbf{S}_1 \cdot \mathbf{B}\left(1 - \frac{\mathbf{p}^2}{2m_1^2}\right) \\
&- (g_1 - 2)\frac{e_1}{2m_1^2}\mathbf{S}_1 \cdot \mathbf{B}\frac{\mathbf{p}^2}{2m_1^2} + (g_1-2)\frac{e_1}{2m_1^2}\frac{(\mathbf{p}\cdot\mathbf{B})(\mathbf{S}_1\cdot\mathbf{p})}{2m_1^2} \\
&- e_1 e_2 (g_1 - 1)\frac{2\mathbf{S}_1 \cdot \mathbf{r} \times [\mathbf{p} - (e_1 - [e_1+e_2]\mu_1^2)\mathbf{A}_r]}{16\pi m_1^2 r^3} \\
&- e_1 e_2 g_1 \frac{2\mathbf{S}_1 \cdot \mathbf{r} \times [\mathbf{p} - (e_2 - [e_1+e_2]\mu_2^2)\mathbf{A}_r]}{16\pi m_1 m_2 r^3}.
\end{aligned}$$

Calculation of g_b

- Calculate g_b for S-states of hydrogen like systems
- Calculate matrix elements like

$$\left\langle n \left| \frac{1}{r} \right| n \right\rangle = -\frac{m_r e_1 e_2}{4\pi n^2} = \frac{m_r Z\alpha}{n^2}, \qquad \left\langle n \left| \mathbf{p}^2 \right| n \right\rangle = \frac{m_r^2 e_1^2 e_2^2}{16\pi^2 n^2} = \frac{m_r^2 (Z\alpha)^2}{n^2}.$$

- g_2^{bound} found by exchange of indices

Bound g with recoil and binding corrections

$$\begin{aligned} g_1^{\text{bound}} = & g_1 \left\{ \left(1 - \frac{\mu_2^2 (Z\alpha)^2}{2n^2}\right) + \frac{\mu_2^2 Z^2 \alpha [\alpha + (Z\alpha - \alpha)\mu_1^2]}{6n^2} \right. \\ & \left. - \frac{\mu_1^2 Z^2 \alpha [Z\alpha + (Z\alpha - \alpha)\mu_2^2]}{3n^2} \right\} \\ & + (g_1 - 2) \left\{ \frac{\mu_2^2 (Z\alpha)^2}{3n^2} + \frac{\mu_2^2 Z^2 \alpha [\alpha + (Z\alpha - \alpha)\mu_1^2)\mu_1^2]}{6n^2} \right\} \end{aligned}$$

Results

- NRQED Lagrangian for arbitrary spin up to $\mathcal{O}(1/m^3)$ was developed
- Agrees with known NRQED Lagrangians for spin one and spin one-half
- Quantum mechanical Hamiltonian obtained for two particles of arbitrary spin in a loosely bound system
- Leading order binding and recoil corrections for g_b calculated
- Such corrections were shown to be universal (no dependence on spin magnitude)
- Physical reason for universality related to BMT equation