Contents

1   Introduction

We wish to calculate the gyromagnetic ratio of a particle of arbitrary spin in a loosely bound state system. We can write down an effective Lagrangian which will capture all the necessary effects. Our job is then to calculate the coefficients of this effective particle. By considering the constraints which exist on the electromagnetic current in a general relativistic theory, we can obtain a simple form which holds for arbitrary spin, and then use this to fix the coefficients of the relativistic theory.

While for a particle of general spin we do not have an exact Lagrangian, we do have one for a spin-1 theory. We can use this Lagrangian to perform the same calculation as above, but in an exact theory. We derive the nonrelativistic potential first from the equations of motion, and then from diagrammatic calculations, and obtain a result that agrees with the more general calculation.

1.1   Description of problem

We consider a loosely bound system two particles of arbitrary spin, and wish to calculate the correction to the gyromagnetic ratio of the particles. So we consider the case of the two particles interacting with both each other, via a Coulomb potential, and a very weak external magnetic field.

1.1.1   Experimental Motivation

Talk about

1.1.2   Approach

This system allows for several simplifications. First, because the system is loosely bound all energy scales are nonrelativistic. Second, the magnetic field we consider is both very weak and constant. So we can ignore all corrections which involve derivatives of the magnetic field or which are quadratic in its strength.

The approach we'll take is to first consider the electric potential as an external field acting on a single particle. Next we can consider the bound system as a whole sitting in an external magnetic field, and take into account recoil effects.

1.2   Theoretical Background

(Here we talk about existing theoretical work in this area)

1.2.1   Spin-$1/2$

(Here talk about the approach that works for spin $1/2$, and why it breaks down in the general case.)

1.2.2   BMT equation

(Here discuss the BMT equation, which holds for general spin and can be used to derive the g-factor corrections)

1.2.3   Khriplovich general spin work

(Discuss the work Khriplovich did with general spinors.)

2 NRQED

We can construct an effective, nonrelativistic Lagrangian for a charged particle interacting with an electromagnetic field.

2.1 The NRQED Lagrangian

We want to construct an effective Lagrangian in the nonrelativistic limit. Our goal is to calculate the leading order corrections to the g-factor, which are corrections of order α^2. To this end, we need terms in the effective nonrelativistic Lagrangian which are equivalent corrections.

2.1.1 Order of terms

We consider constant, infinitesimal external magnetic fields, so we need only consider terms linear in $\mathbf{B}$.

The velocity of the particles in our bound state system will be $v \sim \alpha$.

The electric field we consider is the Coulomb field, so $e\Phi \sim mZ\alpha^2 \sim mv^2$, and $eE \sim m^2v^3$.

Each derivative of the electric field will add an additional factor of mv, so the operator $\mathbf{D}$ can be taken to be of this order.

We need to keep terms up to order mv^4 and $\frac{B}{m}v^2$ in order to calculate the g-factor to the necessary precision. We include mv^4 terms so we can be sure that there are no effects entering from second-order perturbation theory.

2.1.2 Constraints on the form of the Lagrangian

The Lagrangian is constrained to obey several symmetries. It must be invariant under the symmetries of parity and time reversal. It must also be invariant under Galilean transformations. The Lagrangian must also be Hermitian, and gauge invariant.

What are the gauge invariant building blocks we can use to construct this Lagrangian? We have the external fields $\mathbf{E}$ and $\mathbf{B}$, the spin operators $\mathbf{S}$, and the long derivative $\mathbf{D} = \partial - ie\mathbf{A}$. The fields should always be accompanied by the charge e of the particle.

When considering the case of higher spin particles, we might consider terms quadratic and above in spin operators. For a particle of spin s, there must be $(2s + 1)^2$ independent hermitian operators. We can span this set of operators by considering products of up to $2s$ spin matrices which are symmetric and traceless in every vector index. For example, for spin-1 we have quadratic, in addition to I and S_i, five independent structures of the form $S_iS_j + S_jS_i + \delta_{ij}\mathbf{S}$.

We also have the scalar D_0, however, we need only include a single such term because we insist on having only one power of the time derivative.

To consider possible terms, we need to know how each of the above behave under the discrete transformations and Hermitian conjugate. The signs under these transformations are listed in the table below. (Also included is the imaginary number i.)

	Order	P	T	†
eE_i	m^2v^3	-	+	+
eB_i	m^2v^2	+	-	+
D_i	mv	-	+	-
D_0	mv	+	-	-
S_i	1	+	-	+
i	1	+	-	-

2.1.3 Allowed terms

Our strategy in cataloguing terms will be to first list all the combinations of E, B and D which might be allowed at a particular order, to consider the various ways of contracting these vectors, and finally to eliminate terms which do not obey the proper symmetries. We can always make a particular combination

3

Hermitian, and get the proper behavior under time reversal by adding a factor of i, but parity will kill several terms. Note that of the structures we can contract with, all are even under parity.

We can also insist that the Lagrangian have the expected form in the absence of external fields, which eliminates terms like $\bar{S}_{ij}D_iD_j$. The leading order terms should be of order mv^2 or $\frac{eB}{m}$. Combinations of the correct order are:

- The single D_0 term. To have the correct transformation properties this should be iD_0.

- The kinetic $\mathbf{D}^2$ term, which must be simply $\frac{\mathbf{D}^2}{2m}$

- A term with a single power of B_i. The only way to contract this is with the spin matrix, so the term will have the form $\frac{e}{m}\mathbf{S}\cdot\mathbf{B}$

All these terms are Hermitian in themselves.

So, the allowed terms at this order are:

$$iD_0, \frac{\mathbf{D}^2}{m}, \frac{e}{m}\mathbf{S}\cdot\mathbf{B}$$

The first two terms have their coefficients fixed, while we wish to honestly calculate the factor before the last.

$$\mathcal{L}_{NRQED} = \Psi^\dagger\left\{iD_0 + \frac{\mathbf{D}^2}{2m} + c_F\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\right\}\Psi \tag{2.1}$$

Are there any terms of order mv^3 or $\frac{B}{m}v$ allowed? Possible combinations are:

- Three powers of D: $D_iD_jD_k$. However, this is odd under parity, and so not allowed.

- A term with both the derivative and magnetic field: D_iB_j. Again, this is odd under parity and so forbidden.

- A single power of E. Again, odd under parity.

So, all such terms are foribdden by consideration of parity.

Next we consider terms of order mv^4.

- Four powers of D: fixed by the kinetic term to be $\frac{\mathbf{D}^2}{8m^3}$

- One power of E_i and one of D_j. This combination is even under parity and odd under Hermitian conjugate. There are three ways of contracting these two fields.

  - With the delta function. The allowed Hermitian term is then $\delta_{ij}(D_iE_j - E_jD_i)$.
  - With the combination $i\epsilon_{ijk}S_k$. The allowed term is $i\epsilon_{ijk}(D_iE_jS_k + S_kE_jD_i)$.
  - With the quadratic spin structure Q_{ij}: $Q_{ij}(D_iE_j - E_jD_i)$.

In the Lagrangian we'll write these as:

$$\mathcal{L}_{mv^4} = \Psi^\dagger\left\{\frac{\mathbf{D}^4}{8m^2} + c_D\frac{e(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2} + c_Q\frac{eQ_{ij}(D_iE_j - E_iD_j)}{8m^2} + c_S\frac{ie\mathbf{S}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D})}{8m^2}\right\}\Psi \tag{2.2}$$

Terms of order $\frac{B}{m}v^2$. The only allowed combination is $D_iD_jB_k$. We can contract two indices with each other and the third with a spin matrix in three different ways:

- $(\mathbf{S}\cdot\mathbf{B})\mathbf{D}^2 + \mathbf{D}^2(\mathbf{S}\cdot\mathbf{B})$

- $S_iD_jB_iD_j$

4

- $S_i(D_i B_j D_j + D_j B_j D_i)$

We can also contract all indices with a cubic spin structure:

- $\bar{S}_{ijk}(D_i D_j B_k + B_k D_j D_i)$

- $\bar{S}_{ijk} D_i B_j D_k$

In the Lagrangian we'll write these as:

$$\mathcal{L}_{Bv^2} = \Psi^\dagger \left\{ c_{W1} \frac{e\mathbf{D}^2 \mathbf{S} \cdot \mathbf{B} + \mathbf{S} \cdot \mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2} \frac{eD_i(\mathbf{S} \cdot \mathbf{B})D_i}{4m^3} + c_{p'p} \frac{e[(\mathbf{S} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{D}) + (\mathbf{B} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{D})]}{8m^3} \right.$$
$$\left. + c_{T_1} \frac{e\bar{S}_{ijk}(D_i D_j B_k + B_k D_j D_i)}{8m^3} + c_{T_2} \frac{e\bar{S}_{ijk} D_i B_j D_k}{8m^3} \right\} \Psi \tag{2.3}$$

2.1.4   Full Lagrangian

The full Lagrangian we consider is then:

$$\mathcal{L}_{NRQED} = \Psi^\dagger \left\{ iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^2} + c_F \frac{e}{m} \mathbf{S} \cdot \mathbf{B} + c_D \frac{e(\mathbf{D} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{D})}{8m^2} + c_Q \frac{eQ_{ij}(D_i E_j - E_i D_j)}{8m^2} \right.$$
$$+ c_S \frac{ie\mathbf{S} \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D})}{8m^2} + c_{W1} \frac{e\mathbf{D}^2 \mathbf{S} \cdot \mathbf{B} + \mathbf{S} \cdot \mathbf{B}\mathbf{D}^2}{8m^3} - c_{W2} \frac{eD_i(\mathbf{S} \cdot \mathbf{B})D_i}{4m^3}$$
$$\left. + c_{p'p} \frac{e[(\mathbf{S} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{D}) + (\mathbf{B} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{D})]}{8m^3} + c_{T_1} \frac{e\bar{S}_{ijk}(D_i D_j B_k + B_k D_j D_i)}{8m^3} + c_{T_2} \frac{e\bar{S}_{ijk} D_i B_j D_k}{8m^3} \right\} \Psi \tag{2.4}$$

One of the features of this Lagrangian is that every coefficient is fixed by the one-photon interaction. Although some terms might represent two-photon interactions, they are terms like $\mathbf{S} \cdot \mathbf{A} \times \mathbf{E}$, whose coefficient is fixed by the gauge-invariant term $\mathbf{S} \cdot \mathbf{D} \times \mathbf{E}$. This in turn means that we can calculate the corrections to the g-factor by considering only one-photon interactions.

2.2   Scattering off external field in NRQED

We can write down those terms in the NRQED Lagrangian which have one power of the external field. This set of terms will not, by themselves, be gauge invariant. If, for example, a coefficient exists before a term with both one and two powers of A, we'll want to make sure we get the same result in both calculations. We'll add a superscript to the coefficient to keep track of this: so below we write c_S^1.

In writing the expansion of terms like $\mathbf{D}^4$ it is convenient to use anticommutators.

$$\mathcal{L}_A = \Psi^\dagger(-eA_0 - ie\frac{\{\nabla_i, A_i\}}{2m} - ie\frac{\{\nabla^2, \{\nabla_i, A_i\}\}}{8m^3} + c_F e \frac{\mathbf{S} \cdot \mathbf{B}}{2m} + c_D \frac{e(\nabla \cdot \mathbf{E} - \mathbf{E} \cdot \nabla)}{8m^2} + c_Q \frac{eQ_{ij}(\nabla_i E_j - E_i \nabla_j)}{8m^2}$$
$$+ c_S^1 \frac{ie\mathbf{S} \cdot (\nabla \times \mathbf{E} - \mathbf{E} \times \nabla)}{8m^2} + c_{W_1} \frac{e[\nabla^2(\mathbf{S} \cdot \mathbf{B}) + (\mathbf{S} \cdot \mathbf{B})\nabla^2]}{8m^3} - c_{W_2} \frac{e\nabla^i(\mathbf{S} \cdot \mathbf{B})\nabla^i}{4m^3} + c_{p'p} \frac{e[(\mathbf{S} \cdot \nabla)(\mathbf{B} \cdot \nabla) + (\mathbf{B} \cdot \nabla)(\mathbf{S} \cdot \nabla)]}{8m^3}$$

We want to calculate from this a particular process: scattering off an external field, with incoming momentum $\mathbf{p}$, outgoing $\mathbf{p'}$, and $\mathbf{q} = \mathbf{p'} - \mathbf{p}$. There is one diagram associated with each term above, but the total amplitude is just going to be the sum of all these one-photon vertices. These of course can just be read off directly from the Lagrangian: we replace the fields Ψ with the spinors ϕ, and any operator ∇ acting will become $i\mathbf{p}$ if it acts on the right, $i\mathbf{p'}$ if it is to the left.

We can simplify some expressions involving ∇ and $\mathbf{E}$: Because Q_{ij} is symmetric:

$$Q_{ij}(\nabla_i E_j - E_i \nabla_j) = Q_{ij}[\nabla_i, E_j] = Q_{ij}(\partial_i E_j)$$

And because $E_i = -\partial_i \Phi$

$$\mathbf{\nabla} \times \mathbf{E} - \mathbf{E} \times \mathbf{\nabla} = -2\mathbf{E} \times \mathbf{\nabla}$$

And also use that

$$\mathbf{\nabla} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{\nabla} = (\partial_i E_i)$$

Now we can write down the scattering amplitude for scattering off the external field, before we apply any assumptions about the particular process.

$$
\begin{aligned}
iM \;=\;\; & ie\phi_S^\dagger \Bigg(-A_0 + \frac{\mathbf{A} \cdot (\mathbf{p} + \mathbf{p}')}{2m} - \frac{\mathbf{A} \cdot (\mathbf{p} + \mathbf{p}')\mathbf{p}^2 + \mathbf{p}'^2 \mathbf{A} \cdot (\mathbf{p} + \mathbf{p}')}{8m^3} \\
& + c_F \frac{\mathbf{S} \cdot \mathbf{B}}{2m} + c_D \frac{(\partial_i E_i)}{8m^2} + c_Q \frac{Q_{ij}(\partial_i E_j)}{8m^2} + c_S^1 \frac{\mathbf{E} \times \mathbf{p}}{4m^2} \\
& - c_{W_1} \frac{(\mathbf{S} \cdot \mathbf{B})(\mathbf{p}^2 + \mathbf{p}'^2)}{8m^3} + c_{W_2} \frac{(\mathbf{S} \cdot \mathbf{B})(\mathbf{p} \cdot \mathbf{p}')}{4m^3} - c_{p'p} \frac{(\mathbf{S} \cdot \mathbf{p}')(\mathbf{B} \cdot \mathbf{p}) + (\mathbf{B} \cdot \mathbf{p}')(\mathbf{S} \cdot \mathbf{p})}{8m^3} \Bigg) \phi_S
\end{aligned}
$$

The above can be simplified somewhat. We choose our gauge such that $\nabla_i A_i = 0$. If we specify elastic scattering then kinematics dictate that $\mathbf{p}'^2 = \mathbf{p}^2$. Finally, if we consider $\mathbf{B}$ constant, the c_W terms become indistinguishable, since $[\nabla_i, B_j] = 0$. (It is only this last assumption that costs us any information.) Then the scattering amplitude, as calculated from $\mathcal{L}_{NRQED}$, is:

$$
\begin{aligned}
iM =\;& ie\phi_S^\dagger \Bigg(-A_0 + \frac{\mathbf{A} \cdot \mathbf{p}}{m} - \frac{(\mathbf{A} \cdot \mathbf{p})\mathbf{p}^2}{2m^3} + c_F \frac{\mathbf{S} \cdot \mathbf{B}}{2m} + c_D \frac{(\partial_i E_i)}{8m^2} + c_Q \frac{Q_{ij}(\partial_i E_j)}{8m^2} \\
& + c_S^1 \frac{\mathbf{E} \times \mathbf{p}}{4m^2} - (c_{W_1} - c_{W_2})\frac{(\mathbf{S} \cdot \mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p}\frac{(\mathbf{S} \cdot \mathbf{p})(\mathbf{B} \cdot \mathbf{p})}{4m^3} \Bigg) \phi_S
\end{aligned}
\tag{2.5}
$$

3 General Spin Formalism

Our ultimate goal is to calculate corrections to the g-factor of a loosely bound charged particle of arbitrary spin. Our strategy is to obtain an effective Lagrangian in the nonrelativistic limit.

We first consider features of a general-spin formalism in both the relativistic and nonrelativistic cases, and the connection between the wave functions of the free particles. Then we consider how constraints of the relativistic theory let us calculate scattering off an external field. Comparing this result to that done with an effective NRQED Lagrangian, we can obtain the coefficients of that Lagrangian for particles of general spin.

3.1 Spinors for general-spin charged particles

3.1.1 Relativistic bispinors

First we need to work out a formalism that will apply to the general spin case. We want to represent the spin state of the particles by an object that looks like a generalization of the Dirac bispinor.

It is easiest to start with the Dirac basis, where the upper and lower components of the bispinor are objects of opposite helicity, each transforming as an object of spin $1/2$.

To that end define an object

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{3.1}$$

that we wish to have the appropriate properties. Each component should transform as a particle of spin s, but with opposite helicity. Under reflection the upper and lower components transform into each other.

Representations of the proper Lorentz group are spinors which are separately symmetric in dotted and undotted indices. If ξ is an object with p undotted and q dotted indices

$$\xi = \{\xi^{\alpha_1 \ldots \alpha_p}_{\dot{\beta}_1 \ldots \dot{\beta}_q}\} \tag{3.2}$$

Then this can be a representation of a particle of spin $s = (p + q)/2$.

We have some free choice in how to partition the dotted/undotted indices, and we cannot choose exactly the same scheme for all spin as long as both types of indices are present. However, we can make separately consistent choices for integral and half-integral spin. For integral spin we can say $p = q = s$, while for the half-integral case we'll choose $p = s + \frac{1}{2}$, $q = s - \frac{1}{2}$.

We want the ξ and η to transform as objects of opposite helicity. Under reflection they will transform into each other. So

$$\eta = \{\eta^{\beta_1 \ldots \beta_q}_{\dot{\alpha}_1 \ldots \dot{\alpha}_p}\} \tag{3.3}$$

In the rest frame of the particle, they will have clearly defined and identical properties under rotation. The rest frame spinors are equivalent to rank $2s$ nonrelativistic spinors. So the bispinor in the rest frame looks like

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_0 \\ \xi_0 \end{pmatrix} \tag{3.4}$$

where

$$\xi_0 = \{(\xi_0)_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q}\} \tag{3.5}$$

and all indices are symmetric.

We can obtain the spinors in an arbitrary frame by boosting from the rest frame. The upper and lower components we have defined to have opposite helicity, and so will act in opposite ways under boost:

$$\xi = \exp\left(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}\right)\xi_0, \qquad \eta = \exp\left(-\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}\right)\xi_0 \tag{3.6}$$

What form should the operator $\boldsymbol{\Sigma}$ have? Under an infinitesimal boost by a rapidity ϕ, a spinor with a single undotted index is transformed as

$$\xi_\alpha \to \xi'_\alpha = \left(\delta_{\alpha\beta} + \frac{\boldsymbol{\phi} \cdot \boldsymbol{\sigma}_{\alpha\beta}}{2}\right) \xi_\beta$$

while one with a dotted index will transform as

$$\xi_{\dot\alpha} \to \xi'_{\dot\alpha} = \left(\delta_{\dot\alpha\dot\beta} - \frac{\boldsymbol{\phi} \cdot \boldsymbol{\sigma}_{\dot\alpha\dot\beta}}{2}\right) \xi_{\dot\beta}$$

The infinitesimal transformation of a higher spin object with the first p indices undotted and the last q dotted would then be

$$\xi \to \xi' = \left(1 + \sum_{a=0}^{p} \frac{\boldsymbol{\sigma}_a \cdot \boldsymbol{\phi}}{2} - \sum_{a=p+1}^{p+q} \frac{\boldsymbol{\sigma}_a \cdot \boldsymbol{\phi}}{2}\right) \xi$$

where a denotes which spinor index of ξ is operated on.

If we define

$$\boldsymbol{\Sigma} = \sum_{a=0}^{p} \boldsymbol{\sigma}_a - \sum_{a=p+1}^{p+q} \boldsymbol{\sigma}_a \tag{3.7}$$

Then the infinitesimal transformations would be

$$\xi \to \xi' = \left(1 + \frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi$$

$$\eta \to \eta' = \left(1 - \frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \eta$$

So the exact transformation should be

$$\xi \to \xi' = \exp\left(\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \xi$$

$$\eta \to \eta' = \exp\left(-\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{\phi}}{2}\right) \eta$$

Therefore, the bispinor of some particle boosted by ϕ from rest will be

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \exp\left(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2}\right) \xi_0 \\ \exp\left(\frac{-\boldsymbol{\Sigma}\cdot\boldsymbol{\phi}}{2}\right) \xi_0 \end{pmatrix} \tag{3.8}$$

In dealing with the relativistic theory, we'll want a basis that separates the particle and antiparticle parts of the wave function. If we want the upper component to be the particle, then in the rest frame the lower component will vanish, and for low momentum will be small compared to the upper component. The unitary transformation which accomplishes this is

$$\Psi' = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

$$\phi = \frac{1}{\sqrt{2}} (\xi + \eta)$$

$$\chi = \frac{1}{\sqrt{2}} (\eta - \xi)$$

Which is equivalent to

$$\Psi' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Psi$$

Then,

$$\phi = \cosh\left(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi_0 \tag{3.9}$$

$$\chi = \sinh\left(\frac{\mathbf{\Sigma}\cdot\boldsymbol{\phi}}{2}\right)\xi_0 \tag{3.10}$$

3.1.2 Spinors for nonrelativistic theory

We also need to discuss the nonrelativistic, single-particle theory. This is much simpler: the spin state of particles in this theory is represented by symmetric spinors with $2s+1$ undotted indices. The only operators we need to consider acting on this space are spin matrices and products of spin matrices.

3.2 Electromagnetic Interaction

Knowing how the wave functions themselves behave, we want to see what that tells us about possible electromagnetic interaction. Interaction with a single electromagnetic photon should take the form

$$M = A_\mu j^\mu$$

where j^μ is the electromagnetic current.

The electromagnetic current must be built out of the particle's momenta and bilinears of the charged particle fields in such a way that they have the correct Lorentz properties. We must also demand current conservation: the equation $q_\mu j^\mu = 0$ must hold. Above we already have shown that, in the case of general spin, there exist only two such bilinears, a scalar and a tensor.

There will be two permissible terms in the current. We could consider a scalar bilinear coupled with a single power of external momenta. In order to fulfill the current conservation requirement, it should be

$$\frac{p^\mu + p'^\mu}{2m}\bar{\Psi}^\dagger\Psi$$

This will obey current conservation because $q = p' - p$, and $(p+p')\cdot(p'-p) = p^2 - p'^2 = 0$

We can also consider a tensor term contracted with a power of momenta. To fulfill current conservation, we can demand that the tensor bilinear be antisymmetric, and contract it with q:

$$\frac{q_\nu}{2m}\bar{\Psi}^\dagger\Sigma^{\mu\nu}\Psi$$

We don't need to worry about higher order tensor bilinears: they will necessitate too many powers of the external momenta.

So the most general current would look like

$$j^\mu = F_e\frac{p^\mu + p'^\mu}{2m}\bar{\Psi}^\dagger\Psi + F_m\frac{q_\nu}{2m}\bar{\Psi}^\dagger\Sigma^{\mu\nu}\Psi \tag{3.11}$$

In general the form factors might have quite complicated dependence on q, but these corrections will be too small compared to the type of result we're interested in. At leading order F_e will just be the electric charge of the particle in question, and F_m will, as we'll see after connecting this result to the nonrelativistic limit, be related to the particle's g-factor. So to the order we need, we can write the current as

$$j^\mu = e\frac{p^\mu + p'^\mu}{2m}\bar{\Psi}^\dagger\Psi + eg\frac{q_\nu}{2m}\bar{\Psi}^\dagger\Sigma^{\mu\nu}\Psi \tag{3.12}$$

This captures the essence of the interaction between a charged particle of general-spin and a single photon.

3.2.1 Connection between the spinors of the two theories

Knowing something of how the relativistic theory behaves, we can find the connection between the relativistic and nonrelativistic spinors. In the rest frame, there are two independent bispinors which represent particle and antiparticle states:

$$\Psi = \begin{pmatrix} \xi_0 \\ 0 \end{pmatrix}$$

or

$$\Psi = \begin{pmatrix} 0 \\ \xi_0 \end{pmatrix}$$

However, when we consider a particle with zero momentum it is not the case that the upper component of the bispinor can be directly associated with the Schrodinger like wave-function of the particle — for instance, it would not be correctly normalized, for there is some mixing with the lower component.

We can obtain a relation between ξ_0 and the Schrodinger amplitude ϕ_s by considering the current density at zero momentum transfer. For ϕ_s it will be $j_0 = \phi_s^\dagger \phi$. For the relativistic theory we have, as calculated above:

$$j^0 = F_e \frac{p^0 + p'^0}{2m} \bar{\Psi}^\dagger \Psi + F_m \frac{q_\nu}{2m} \bar{\Psi}^\dagger T^{0\nu} \Psi$$

At $q = 0$ the expression simplifies

$$j^0(q = 0) = F_e \frac{p_0}{m} \bar{\Psi}^\dagger \Psi$$

$$= F_e \frac{p_0}{m} (\phi^\dagger \phi - \chi^\dagger \chi)$$

ϕ and χ are both related to the rest frame spinor ξ_0. So we can write instead

$$j^0 = F_e \frac{p_0}{m} \xi_0^\dagger \left\{ \cosh^2(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}) - \sinh^2(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}) \right\} \xi_0 = F_e \frac{p_0}{m} \xi_0^\dagger \xi_0$$

where the last equality follows from the hyperbolic trig identity.

If we demand that the two current densities be equal to each other, we find

$$\frac{p_0}{m} \xi_0^\dagger \xi_0 = \phi_s^\dagger \phi_s$$

Approximating

$$\left(1 + \frac{\mathbf{p}^2}{2m} \right) \xi_0^\dagger \xi_0 = \phi_s^\dagger \phi_s$$

This will hold to the necessary order if we identify

$$\xi_0 = \left(1 - \frac{\mathbf{p}^2}{4m} \right) \phi_s$$

To write the relativistic bispinors in terms of ϕ_s we will also need approximations to $\cosh(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2})$ and $\sinh(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2})$. We only need the rapidity to the leading order: $\boldsymbol{\phi} \approx \mathbf{v} \approx \frac{\mathbf{p}}{m}$.

$$\cosh(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}) \approx 1 + \frac{1}{2} \left(\frac{\mathbf{\Sigma} \cdot \mathbf{p}}{2m} \right)^2$$

$$\sinh(\frac{\mathbf{\Sigma} \cdot \boldsymbol{\phi}}{2}) \approx \frac{\mathbf{\Sigma} \cdot \mathbf{p}}{2m}$$

The the two bispinor components are

$$\phi \approx \left(1 + \left[\frac{1}{2}\frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\right]^2\right)\xi_0$$

$$\approx \left(1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m}\right)\phi_S \tag{3.13}$$

$$\chi \approx \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\xi_0$$

$$\approx \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{2m}\phi_S \tag{3.14}$$

3.3 Bilinears in terms of nonrelativistic theory

The next step is to express the relativistic bilinears, built out of the bispinors Ψ, in terms of the Schrodinger like wave functions.

We have above written the bispinors in terms of ϕ_s, so we can use those identities to express the bilinears in the same manner.

3.3.1 Scalar bilinear

$$
\begin{aligned}
\bar{\Psi}^\dagger(p')\Psi(p) &= \phi^\dagger\phi - \chi^\dagger\chi \\
&= \phi_s^\dagger\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p}')^2}{8m^2} - \frac{\mathbf{p}'^2}{4m^2}\right]\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m^2}\right]\phi_s - \phi_s^\dagger\left[\frac{(\mathbf{\Sigma}\cdot\mathbf{p}')(\mathbf{\Sigma}\cdot\mathbf{p})}{4m^2}\right]\phi_s \\
&= \phi_s^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{(\mathbf{\Sigma}\cdot\mathbf{p}')^2 + (\mathbf{\Sigma}\cdot\mathbf{p})^2 - 2(\mathbf{\Sigma}\cdot\mathbf{p}')(\mathbf{\Sigma}\cdot\mathbf{p})\right\}\right)\phi_s \\
&= \phi_s^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{[\mathbf{\Sigma}\cdot\mathbf{p}, \mathbf{\Sigma}\cdot\mathbf{q}] + (\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_s \\
&= \phi_s^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{[4i\epsilon_{ijk}p_iq_jS_k + (\mathbf{\Sigma}\cdot\mathbf{q})^2\right\}\right)\phi_s
\end{aligned}
$$

3.3.2 Tensor ij component

In calculating the nonrelativistic limit of the antisymmetric tensor bilinear, we will treat the $0i$ and the ij components seperately. First let us consider $\bar{\Psi}\Sigma_{ij}\Psi$.

$$
\begin{aligned}
\bar{\Psi}\Sigma_{ij}\Psi &= \bar{\Psi}(-2\epsilon_{ijk}S_k)\Psi \\
&= -2i\epsilon_{ijk}(\phi^\dagger S_k\phi - \chi^\dagger S_k\chi) \\
&= -2i\epsilon_{ijk}\left(\phi_S^\dagger\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p}')^2}{8m^2} - \frac{\mathbf{p}'^2}{4m^2}\right]S_k\left[1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{p})^2}{8m^2} - \frac{\mathbf{p}^2}{4m^2}\right]\phi_S - \phi_S^\dagger\frac{(\mathbf{\Sigma}\cdot\mathbf{p}')S_k(\mathbf{\Sigma}\cdot\mathbf{p})}{4m^2}\phi_S\right) \\
&= -2i\epsilon_{ijk}\phi_S^\dagger\left\{S_k\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{1}{8m^2}\left[(\mathbf{\Sigma}\cdot\mathbf{p}')^2S_k + S_k(\mathbf{\Sigma}\cdot\mathbf{p})^2 - 2(\mathbf{\Sigma}\cdot\mathbf{p}')S_k(\mathbf{\Sigma}\cdot\mathbf{p})\right]\right\}\phi_S
\end{aligned}
$$

We want to write the terms in square brackets explicitly in terms of $\mathbf{p}$ and $\mathbf{q}$.

$$
\begin{aligned}
(\mathbf{\Sigma}\cdot\mathbf{p}')^2S_k + S_k(\mathbf{\Sigma}\cdot\mathbf{p})^2 - 2(\mathbf{\Sigma}\cdot\mathbf{p}')S_k(\mathbf{\Sigma}\cdot\mathbf{p}) &= (\mathbf{\Sigma}\cdot\mathbf{p})^2S_k + S_k(\mathbf{\Sigma}\cdot\mathbf{p}) - 2(\mathbf{\Sigma}\cdot\mathbf{p})S_k(\mathbf{\Sigma}\cdot\mathbf{p}) \\
&\quad + \{(\mathbf{\Sigma}\cdot\mathbf{p})(\mathbf{\Sigma}\cdot\mathbf{q}) + (\mathbf{\Sigma}\cdot\mathbf{q})(\mathbf{\Sigma}\cdot\mathbf{p})\}S_k \\
&\quad - 2(\mathbf{\Sigma}\cdot\mathbf{q})S_k(\mathbf{\Sigma}\cdot\mathbf{p}) + (\mathbf{\Sigma}\cdot\mathbf{q})^2S_k
\end{aligned}
$$

We can express many of these terms as commutators

$$= \boldsymbol{\Sigma} \cdot \mathbf{p}[\boldsymbol{\Sigma} \cdot \mathbf{p}, S_k] + [S_k, \boldsymbol{\Sigma} \cdot \mathbf{p}]\boldsymbol{\Sigma} \cdot \mathbf{p} + 2\boldsymbol{\Sigma} \cdot \mathbf{q}[\boldsymbol{\Sigma} \cdot \mathbf{p}, S_k] - [\boldsymbol{\Sigma} \cdot \mathbf{q}, S_k]\boldsymbol{\Sigma} \cdot \mathbf{p} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2 S_k$$

$$= i\epsilon_{ijk}p_j\{(\boldsymbol{\Sigma} \cdot \mathbf{p})\Sigma_i - \Sigma_i(\boldsymbol{\Sigma} \cdot \mathbf{p})\} + 2i\epsilon_{ijk}\{(\boldsymbol{\Sigma} \cdot \mathbf{q})\Sigma_i p_j - \Sigma_i(\boldsymbol{\Sigma} \cdot \mathbf{p})q_j)\} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2 S_k$$

$$= i\epsilon_{ijk}p_j[(\boldsymbol{\Sigma} \cdot \mathbf{p}), \Sigma_i] + 2i\epsilon_{ijk}\{(\boldsymbol{\Sigma} \cdot \mathbf{q})\Sigma_i p_j - \Sigma_i(\boldsymbol{\Sigma} \cdot \mathbf{p})q_j)\} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2 S_k$$

$$= 4(\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p})p_k) + 2i\epsilon_{ijk}\{(\boldsymbol{\Sigma} \cdot \mathbf{q})\Sigma_i p_j - \Sigma_i(\boldsymbol{\Sigma} \cdot \mathbf{p})q_j)\} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2 S_k$$

Thus the whole bilinear is

$$-2i\epsilon_{ijk}\phi_S^\dagger \left\{ S_k\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right) + \frac{1}{8m^2}\left[4(\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p})p_k) + 2i\epsilon_{\ell mk}\{(\boldsymbol{\Sigma} \cdot \mathbf{q})\Sigma_\ell p_m - \Sigma_\ell(\boldsymbol{\Sigma} \cdot \mathbf{p})q_m)\} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2 S_k\right]\right\}\phi_S \tag{3.15}$$

3.3.3 Tensor Σ_{0i} component

We calculate $\bar{\Psi}\Sigma_{0i}\Psi$.

$$\bar{\Psi}\Sigma_{0i}\Psi = \bar{\Psi}\begin{pmatrix} 0 & \Sigma_i \\ \Sigma_i & 0 \end{pmatrix}\Psi$$

$$= \phi^\dagger \Sigma_i \chi - \chi^\dagger \Sigma_i \phi$$

We'll only need ϕ and χ to first order here.

$$= \phi_S^\dagger \left(\frac{\Sigma_i \Sigma_j p_j - \Sigma_j \Sigma_i p'j}{2m}\right)\phi_S$$

Using $p' = p + q$ the terms involving only p can be simplified using the commutator of Σ matrices.

$$= \phi_S^\dagger \left(\frac{4i\epsilon_{ijk}p_j S_k - \Sigma_j \Sigma_i q_j}{2m}\right)\phi$$

3.4 Current in terms of nonrelativistic wave functions

We derived the four-current (3.11) above; in nonrelativistic notation it is:

$$j_0 = F_e\frac{p_0 + p'_0}{2m}\bar{\Psi}^\dagger\Psi - F_m\frac{q_j}{2m}\bar{\Psi}^\dagger\Sigma^{0j}\Psi \tag{3.16}$$

$$j_i = F_e\frac{p_i + p'_i}{2m}\bar{\Psi}^\dagger\Psi - F_m\frac{q_j}{2m}\bar{\Psi}^\dagger\Sigma^{ij}\Psi + F_m\frac{q_0}{2m}\bar{\Psi}^\dagger\Sigma^{i0}\Psi \tag{3.17}$$

We have expressions for the bilinears in terms of the nonrelativistic wave functions ϕ_S, so it is fairly straight forward to apply them here. The calculation of j_0 is straightforward:

$$F_e\frac{p_0 + p'_0}{2m}\bar{\Psi}^\dagger\Psi = F_e\left(1 + \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right)\phi_S^\dagger\left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2}\left\{4i\epsilon_{ijk}p_i q_j S_k + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2\right\}\right)\phi_S$$

$$\approx F_e\phi_S^\dagger\left(1 + \frac{1}{8m^2}\left\{4i\mathbf{S} \cdot \mathbf{p} \times \mathbf{q} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2\right\}\right)\phi_S$$

$$F_m\frac{q_j}{2m}\bar{\Psi}^\dagger\Sigma^{0j}\Psi = F_m\frac{q_i}{2m}\phi_S^\dagger\left(\frac{4i\epsilon_{ijk}p_j S_k - \Sigma_j \Sigma_i q_j}{2m}\right)\phi_S$$

$$= F_m\phi_S^\dagger\left(\frac{4i\mathbf{S} \cdot \mathbf{q} \times \mathbf{p} - (\boldsymbol{\Sigma} \cdot \mathbf{q})^2}{4m^2}\right)\phi_S$$

It turns out that both terms here have the same form, so combining them we get

$$j_0 = \phi_S^\dagger \left(F_e + \frac{F_e + 2F_m}{8m^2} \left\{ 4i\mathbf{S} \cdot \mathbf{p} \times \mathbf{q} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2 \right\} \right) \phi_S \tag{3.18}$$

To calculate j_i we want to first simplify things by considering the constraints of our particular problem. The term with Σ_{ij} can be simplified by dropping terms with more than one power of q; these will turn into derivatives of the magnetic field, and our problem concerns only a constant field. Further, we need only calculate elastic scattering, and so $q_0 = 0$. With those simplifications

$$\bar{\Psi} \Sigma_{ij} \Psi \approx -2i\epsilon_{ijk}\phi_S^\dagger \left\{ S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p})p_k}{2m^2} \right\} \phi_S$$

$$\begin{aligned} F_e \frac{p_i + p_i'}{2m} \bar{\Psi}^\dagger \Psi &= F_e \frac{p_i + p_i'}{2m} \phi_S^\dagger \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} + \frac{1}{8m^2} \left\{ 4i\epsilon_{\ell jk}p_\ell q_j S_k + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2 \right\} \right) \phi_S \\ &\approx F_e \frac{p_i + p_i'}{2m} \phi_S^\dagger \left(1 + \frac{1}{8m^2} \left\{ 4i\epsilon_{\ell jk}p_\ell q_j S_k \right\} \right) \phi_S \\ F_m \frac{q_j}{2m} \bar{\Psi}^\dagger \Sigma^{ij} \Psi &= -F_m \frac{i\epsilon_{ijk}q_j}{m} \phi_S^\dagger \left\{ S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p})p_k}{2m^2} \right\} \phi_S \end{aligned}$$

So the full spatial part of the current is

$$j_i = \phi_S^\dagger \left\{ F_e \frac{p_i + p_i'}{2m} \left(1 + \frac{i\epsilon_{\ell jk}p_\ell q_j S_k}{2m^2} \right) + F_m \frac{i\epsilon_{ijk}q_j}{m} \left(S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p})p_k}{2m^2} \right) \right\} \phi_S \tag{3.19}$$

3.5 Scattering off external field

To compare to the NRQED Lagrangian, we want to calculate scattering off an external field for an arbitrary spin particle. We already have the current, so the scattering is just

$$M = ej_\mu A^\mu = ej_0 A_0 - e\mathbf{j} \cdot \mathbf{A}$$

Above we have expressions for both j_0 (3.18) and $\mathbf{j}$ (3.19). So we can write down the parts of the amplitude directly:

$$\begin{aligned} ej_0 A_0 &= eA_0 \phi_S^\dagger \left(F_e + \frac{F_e + 2F_m}{8m^2} \left\{ 4i\mathbf{S} \cdot \mathbf{p} \times \mathbf{q} + (\boldsymbol{\Sigma} \cdot \mathbf{q})^2 \right\} \right) \phi_S \\ e\mathbf{j} \cdot \mathbf{A} &= A_i \phi_S^\dagger \left\{ F_e \frac{p_i + p_i'}{2m} \left(1 + \frac{i\epsilon_{\ell jk}p_\ell q_j S_k}{2m^2} \right) + F_m \frac{i\epsilon_{ijk}q_j}{m} \left(S_k \left(1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2} \right) + \frac{\mathbf{p}^2 S_k - (\mathbf{S} \cdot \mathbf{p})p_k}{2m^2} \right) \right\} \phi_S \end{aligned} \tag{3.20}$$

As much as possible we want to express the result in terms of gauge invariant quantities $\mathbf{B}$ and $\mathbf{E}$. We write the relations between these fields and A_μ in position space and the equivalent equation in momentum space.

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \to i\mathbf{q} \times \mathbf{A}$$
$$\mathbf{E} = -\boldsymbol{\nabla} A_0 \to -i\mathbf{q}A_0$$

There is one term above that can only be put into gauge-invariant form by considering the kinematic constraints of elastic scattering. If the scattering is elastic, we have $\mathbf{q} \cdot (\mathbf{p}' + \mathbf{p}) = \mathbf{p}'^2 - \mathbf{p}^2 = 0$. We can use this identity on the term $q_j(p_i' + p_i)A_i$ as follows:

$$\begin{aligned} \epsilon_{ijk}B_k &= \partial_i A_j - \partial_j A_i = i(q_i A_j - q_j A_i) \\ (p_i + p_i')\epsilon B_k &= i(p_i + p_i')(q_i A_j - q_j A_i) \\ &= -i(p_i + p_i')q_j A_i \end{aligned}$$

13

So we have the identity

$$i(p_i + p_i')q_j A_i = -\epsilon_{ijk}B_k(p_i + p_i')$$ (3.21)

Now we can write each term involving q in terms of position space quantities.

$$
\begin{aligned}
i\mathbf{S}\cdot\mathbf{p}\times\mathbf{q}A_0 &= -\mathbf{S}\cdot\mathbf{p}\times\mathbf{E} \\
(\mathbf{\Sigma}\cdot\mathbf{q})^2 A_0 &= \Sigma_i\Sigma_j q_i q_j A_0 \\
&= \Sigma_i\Sigma_j \partial_i E_j \\
i\epsilon_{ijk}A_i q_j &= -i(\mathbf{q}\times\mathbf{A})_k \\
&= -B_k \\
A_i(p_i + p_i')i\epsilon_{\ell jk}p_\ell q_j S_k &= \epsilon_{\ell jk}p_\ell S_k i(p_i + p_i')q_j A_i \\
&= -\epsilon_{\ell jk}p_\ell S_k\{\epsilon_{ijm}B_m(p_i + p_i')\} \\
&= -(\delta_{\ell i}\delta_{km} - \delta\ell m\delta_{ik})p_\ell S_k\{\epsilon_{ijm}B_m(p_i + p_i')\} \\
&= 2\{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p}) - (\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2\}
\end{aligned}
$$

Using these

$$ej_0 A_0 = e\phi_S^\dagger\left\{A_0 + \frac{1-2F_2}{8m^2}\left(4\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \Sigma_i\Sigma_j\partial_i E_j\right)\right\}$$

$$e\mathbf{j}\cdot\mathbf{A} = e\phi_S^\dagger\left\{\frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p}) - (\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2}{m^2} - F_m\left(\frac{\mathbf{S}\cdot\mathbf{B}}{m}\left\{1 - \frac{\mathbf{p}^2 + \mathbf{p}'^2}{4m^2}\right\} + \frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p}) - (\mathbf{B}\cdot\mathbf{S})\mathbf{p}^2}{2m^2}\right)\right\}\phi$$

$$= e\phi_S^\dagger\left\{\frac{\mathbf{p}\cdot\mathbf{A}}{m} + [1-2F_m]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{F_m}{m}\mathbf{S}\cdot\mathbf{B}\right\}$$

From this we can see that our F_m is actually $g/2$, so in such terms

$$ej_0 A_0 = e\phi_S^\dagger\left\{A_0 - \frac{g-1}{2m^2}\left(\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \frac{1}{4}\Sigma_i\Sigma_j\partial_i E_j\right)\right\}$$

$$e\mathbf{j}\cdot\mathbf{A} = e\phi_S^\dagger\left\{\frac{\mathbf{p}\cdot\mathbf{A}}{m} - [g-1]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{g}{2m}\mathbf{S}\cdot\mathbf{B}\right\}$$

So the entire scattering process is

$$e\phi_S^\dagger\left\{A_0 - \frac{g-1}{2m^2}\left(\mathbf{S}\cdot\mathbf{E}\times\mathbf{p} + \frac{1}{4}\Sigma_i\Sigma_j\partial_i E_j\right)\frac{\mathbf{p}\cdot\mathbf{A}}{m} - [g-1]\frac{(\mathbf{B}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p})}{m^2} - \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{m^2} - \frac{g}{2m}\mathbf{S}\cdot\mathbf{B}\right\}\phi_S$$ (3.22)

3.6 Comparison with relativistic result

Having calculated the same process in both the relativistic theory and in our NRQED effective theory, we can compare the two amplitudes and fix the coefficients.

The NRQED amplitude (2.5) is

$$
\begin{aligned}
iM &= ie\phi^\dagger\Bigg(-A_0 + \frac{\mathbf{A}\cdot\mathbf{p}}{m} - \frac{(\mathbf{A}\cdot\mathbf{p})\mathbf{p}^2}{2m^3} + c_F\frac{\mathbf{S}\cdot\mathbf{B}}{2m} + c_D\frac{(\partial_i E_i)}{8m^2} + c_Q\frac{Q_{ij}(\partial_i E_j)}{8m^2} \\
&\quad + c_S^1\frac{\mathbf{E}\times\mathbf{p}}{4m^2} - (c_{W_1} - c_{W_2})\frac{(\mathbf{S}\cdot\mathbf{B})\mathbf{p}^2}{4m^3} - c_{p'p}\frac{(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})}{4m^3}\Bigg)\phi
\end{aligned}
$$

14

While the relativistic amplitude was

$$iM_{REL} = -ie\phi^\dagger \Big(A_0 - \frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{\mathbf{p}\cdot\mathbf{A}\mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3}\{\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - S_iS_j\boldsymbol{\nabla}_iE_j\} - g\frac{1}{2m}\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\Big)\phi$$

We should rewrite term $\boldsymbol{\nabla}\cdot\mathbf{E} - S_iS_j\boldsymbol{\nabla}_iE_j$ using the quadropole moment tensor $Q_{ij} = \frac{1}{2}(S_iS_j + S_jS_i - \frac{2}{3}\mathbf{S}^2)$.

Remember that ∇_iE_j is actually symmetric under exchange of i and j. Then we can write

$$S_iS_j\nabla_iE_j = \frac{1}{2}(S_iS_j + S_jS_i) = (Q_{ij} + \frac{1}{3}\mathbf{S}^2\delta_{ij})\nabla_iE_j$$

$$= Q_{ij}\nabla_iE_j + \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{E}$$

Written using this identity, the relativistic amplitude is

$$iM_{REL} = -ie\phi^\dagger \Big(A_0 - \frac{\mathbf{p}\cdot\mathbf{A}}{m} + \frac{\mathbf{p}\cdot\mathbf{A}\mathbf{p}^2}{2m^3} - \frac{g-1}{2m^3}\{\frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{p}\times\mathbf{E} - Q_{ij}\boldsymbol{\nabla}_iE_j\} - g\frac{1}{2m}\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}\frac{\mathbf{p}^2}{2m^3} + \frac{g-2}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\Big)\phi$$

Comparing the two, the coefficients are:

$$\begin{aligned}
c_F &= g \\
c_D &= \frac{4(g-1)}{3} \\
c_Q &= -4(g-1) \\
c_S^1 &= 2(g-1) \\
(c_{W_1} - c_{W_2}) &= 2 \\
c_{p'p} &= (g-2)
\end{aligned}$$

4 Spin-1 equations of motion

4.1 Exact equations of motion for spin-1

$$\mathcal{L} = -\frac{1}{2}(D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) + m^2 W^{\mu\dagger} W_\mu - i\lambda e W^{\mu\dagger} W^\nu F_{\mu\nu} \tag{4.1}$$

where as usual D is the long derivative $D^\mu = \partial^\mu + ieA^\mu$.

Obtain the equations of motion from the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial [\partial_\mu W^{\dagger\alpha}]} = 0$$

Or equivalently,

$$\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} - D_\mu \frac{\partial \mathcal{L}}{\partial [D_\mu W^{\dagger\alpha}]} = 0$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} &= \frac{\partial}{\partial W^{\dagger\alpha}} \left(m^2 W^{\mu\dagger} W_\mu - i\lambda e W^{\mu\dagger} W^\nu F_{\mu\nu} \right) \\
&= m^2 W_\alpha - i\lambda W^\nu F_{\alpha\nu}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial [D_\gamma W^{\dagger\alpha}]} &= -\frac{1}{2} \frac{\partial}{\partial [D_\gamma W^{\dagger\alpha}]} (D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) \\
&= \frac{1}{2}(g^{\mu\gamma} g^\nu_\alpha - g^{\nu\gamma} g^\mu_\alpha)(D_\mu W_\nu - D_\nu W_\mu) \\
&= D^\gamma W_\alpha - D_\alpha W^\gamma
\end{aligned}$$

So the complete equation from Euler-Lagrange is

$$m^2 W_\alpha - i\lambda W^\nu F_{\alpha\nu} - D_\mu(D^\mu W_\alpha - D_\alpha W^\mu) = 0 \tag{4.2}$$

This is a set of four coupled second order equations for the field W. We rewrite as a set of first order equations by introducing a field $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$. So (4.2) becomes

$$m^2 W_\alpha - i\lambda W^\mu F_{\alpha\mu} - D^\mu W_{\mu\alpha} = 0 \tag{4.3}$$

$W^{\mu\nu}$ is antisymmetric and so has six degrees of freedom, corresponding to six independent fields. Together with W^μ this represents a total of ten fields. However, upon examination only some of these fields are dynamic. The fields W^{0i} and W^i appear in the equations with time derivatives, while the fields W^{ij} and W^0 never do. So it is only necessary to consider the former six fields. So that these six fields all have the same dimension, we will define $\frac{W^{i0}}{m} = i\eta^i$.

We will now eliminate the extraneous fields and solve for $iD_0 W^i$, $iD_0\eta^i$.

$$m^2 W_\alpha - i\lambda W^\nu F_{\alpha\nu} - D_\mu(D^\mu W_\alpha - D_\alpha W^\mu) = 0 \tag{4.4}$$

Define $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$. Then

$$m^2 W_\alpha - i\lambda W^\nu F_{\alpha\nu} - D^\mu W_{\mu\alpha} = 0 \tag{4.5}$$

To get the exact Hamiltonian of a bispinor, we can eliminate nondynamic fields.

First consider (4.5) with $\alpha = 0$.

$$m^2 W_0 - i\lambda W^\nu F_{0\nu} - D^\mu W_{\mu 0} \tag{4.6}$$

$$m^2 W_0 - i\lambda W^\nu F_{0\nu} - D^j W_{j0} \tag{4.7}$$

Solve this for W_0

$$W_0 = \frac{1}{m^2}\left(i\lambda W^j F_{0j} + D^j W_{j0}\right) \tag{4.8}$$

Now, consider (4.5) with $\alpha = i$

$$m^2 W_i - i\lambda W^\mu F_{i\mu} - D^\mu W_{\mu i} = 0 \tag{4.9}$$

$$m^2 W_i - i\lambda W^0 F_{i0} - D^0 W_{0i} - i\lambda W^j F_{ij} - D^j W_{ji} = 0 \tag{4.10}$$

Using (4.8) we can replace W_0

$$m^2 W_i - \frac{i\lambda}{m^2}\left(i\lambda W^\nu F_{0\nu} + D^j W_{j0}\right)F_{i0} - D^0 W_{0i} - i\lambda W^j F_{ij} - D^j W_{ji} = 0 \tag{4.11}$$

Using $W_{ji} = D_j W_i - D_i W_j$

$$m^2 W_i - \frac{i\lambda}{m^2}\left(i\lambda W^\nu F_{0\nu} + D^j W_{j0}\right)F_{i0} - D^0 W_{0i} - i\lambda W^j F_{ij} - D^j(D_j W_i - D_i W_j) = 0 \tag{4.12}$$

Solve this for $D^0 W_{0i}$:

$$D^0 W_{0i} = m^2 W_i - \frac{i\lambda}{m^2}\left(i\lambda W^\nu F_{0\nu} + D^j W_{j0}\right)F_{i0} - i\lambda W^j F_{ij} - D^j(D_j W_i - D_i W_j) \tag{4.13}$$

To get a similar equation for W_i, consider

$$W_{i0} = D_i W_0 - D_0 W_i \tag{4.14}$$

Then

$$D^0 W_i = D_i W_0 - W_{i0} \tag{4.15}$$

$$D^0 W_i = D_i \frac{1}{m^2}\left(i\lambda W^j F_{0j} + D^j W_{j0}\right) - W_{i0} \tag{4.16}$$

We now have equations that tell us the time evolution of a total of six fields: W_i and W_{i0}. We want to treat these as the components of some sort of bispinor. To that end, first define $\eta_i = -i/m W_{i0}$ so that we have a pair of fields with the same mass dimension and hermiticity. (Since $W_{i0} = D_i W_0 - D_0 W_i$ would pick up another minus sign under complex conjugation compared to W_i.) Going the other way $W_{i0} = im\eta_i$.

Since eventually we want a nonrelativistic expression, we should write the spatial components of four vectors as three vectors. To that end also write the components of the tensor $F_{\mu\nu}$ in terms of three vectors.

$$F_{0i} = E^i, \ F_{ij} = -\epsilon_{ijk}B^k$$

Regular spatial vectors are "naturally raised" while D_i is "naturally lowered". Then we can rewrite (4.13).

$$-imD_0\eta_i = m^2 W_i + \frac{i\lambda}{m^2}\left(i\lambda W^j E^j + D^j W_{j0}\right)E^i - i\lambda W^j \epsilon_{ijk}B^k - D^j(D_j W_i - D_i W_j) \tag{4.17}$$

$$iD_0\eta^i = -mW^i + \frac{i\lambda}{m^3}\left(i\lambda W^j E^j + D^j W_{j0}\right)E^i - \frac{i\lambda}{m}W^j \epsilon_{ijk}B^k - \frac{1}{m}D_j(D_j W^i - D_i W^j) \tag{4.18}$$

And likewise (4.16) becomes

$$D^0 W_i = D_i \frac{1}{m^2}\left(-i\lambda W^j E^j + imD^j \eta_j\right) - im\eta_i \tag{4.19}$$

$$iD^0 W^i = -iD_i \frac{1}{m^2}\left(-i\lambda W^j E^j + imD_j \eta^j\right) + m\eta^i \tag{4.20}$$

$$iD^0 W^i = -\frac{1}{m^2}D_i \lambda \mathbf{W} \cdot \mathbf{E} + \frac{1}{m}D_i \mathbf{D} \cdot \eta + m\eta^i \tag{4.21}$$

17

4.1.1 Spin identities

To obtain some Shrodinger like equation for a bispinor, we need to express the time derivative of the bispinor in terms of other operators which can be interpreted as the Hamiltonian: $i\partial_0\Psi = \hat{H}\Psi$. We have expressions for the time derivatives of W_i and η_i, but some mixing of the indices is involved. To write the Hamiltonian in a block form we can introduce spin matrices whose action will mix the components of the fields.

The spin matrix for a spin one particle can represented as:

$$(S^k)_{ij} = -i\epsilon_{ijk} \tag{4.22}$$

which leads to the following identities:

$$(\mathbf{a} \times \mathbf{v})_i = -i(\mathbf{S} \cdot \mathbf{a})_{ij} v_j \tag{4.23}$$

$$\{\mathbf{a} \times (\mathbf{a} \times \mathbf{v})\}_i = -(\{\mathbf{S} \cdot \mathbf{a}\}^2)_{ij} v_j \tag{4.24}$$

$$a_i(\mathbf{b} \cdot \mathbf{v}) = \{\mathbf{a} \cdot \mathbf{b}\, \delta_{ij} - (S^k S^\ell)_{ij} a^\ell b^k\} v_j \tag{4.25}$$

Using these identities

$$iD^0\mathbf{W} = -\frac{\lambda}{m^2}\mathbf{D}(\mathbf{W} \cdot \mathbf{E}) + \frac{1}{m}\mathbf{D}(\mathbf{D} \cdot \boldsymbol{\eta}) + m\boldsymbol{\eta} \tag{4.26}$$

$$iD^0\mathbf{W} = -\frac{1}{m^2}\{\mathbf{D} \cdot \mathbf{E} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D})\}\mathbf{W} + \frac{1}{m}\{\mathbf{D}^2 - (\mathbf{S} \cdot \mathbf{D})^2\}\boldsymbol{\eta} + m\boldsymbol{\eta} \tag{4.27}$$

and for the other equation

$$iD_0\eta^i = -mW^i + \frac{i\lambda}{m^3}\left(i\lambda W^j E^j + D^j W_{j0}\right)E^i - \frac{i\lambda}{m}W^j\epsilon_{ijk}B^k - \frac{1}{m}D_j(D_j W^i - D_i W^j) \tag{4.28}$$

$$iD_0\eta^i = -mW^i + \frac{i\lambda}{m^3}\left(i\lambda W^j E^j + D^j W_{j0}\right)E^i - \frac{i\lambda}{m}W^j\epsilon_{ijk}B^k - \frac{1}{m}D_j(D_j W^i - D_i W^j) \tag{4.29}$$

$$iD_0\boldsymbol{\eta} = -m\mathbf{W} - \frac{\lambda^2}{m^3}\mathbf{E}(\mathbf{W} \cdot \mathbf{E}) - \frac{1}{m^2}\mathbf{E}(\mathbf{D} \cdot \boldsymbol{\eta}) - \frac{i\lambda}{m}(\mathbf{W} \times \mathbf{B}) + \frac{1}{m}\mathbf{D} \times (\mathbf{D} \times \mathbf{W}) \tag{4.30}$$

$$iD_0\boldsymbol{\eta} = -m\mathbf{W} - \frac{\lambda^2}{m^3}\{\mathbf{E}^2 - (\mathbf{S} \cdot \mathbf{E})^2\}\mathbf{W} - \frac{1}{m^2}\{\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D})\}\boldsymbol{\eta} - \frac{\lambda}{m}(\mathbf{S} \cdot \mathbf{B})\mathbf{W} - \frac{1}{m}(\mathbf{S} \cdot \mathbf{D})^2\mathbf{W} \tag{4.31}$$

$$iD_0\boldsymbol{\eta} = \left(-m - \frac{\lambda^2}{m^3}\{\mathbf{E}^2 - (\mathbf{S} \cdot \mathbf{E})^2\} - \frac{\lambda}{m}(\mathbf{S} \cdot \mathbf{B}) - \frac{1}{m}(\mathbf{S} \cdot \mathbf{D})^2\right)\mathbf{W} - \frac{1}{m^2}\{\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D})\}\boldsymbol{\eta} \tag{4.32}$$

Now that the equations are in the correct form, we can write them as follows:

$$iD_0 \begin{pmatrix} W \\ \eta \end{pmatrix}$$

4.2 Derivation of relativistic Hamiltonian

We want to derive a "Schrodinger-like" equation governing the nonrelativistic motion of the vector particle; an equation of the form $i\partial_t \Psi = \hat{H}\Psi$. Since it is a vector particle, Ψ is a spinor with three components. We will first find such an equation for a bispinor in the relativistic theory, having the form:

$$i\partial_t \begin{pmatrix} \Psi_u \\ \Psi_l \end{pmatrix} = \hat{H} \begin{pmatrix} \Psi_u \\ \Psi_l \end{pmatrix}$$

If we have a Lagrangian describing the interaction, such an equation can be obtained from the Euler-Lagrange equations.

We have such a Lagrangian:

$$\mathcal{L} = -\frac{1}{2}(D \times W)^\dagger \cdot (D \times W) + m_w^2 W^\dagger W - ieW^{\dagger\mu} W^\nu F_{\mu\nu}$$

Where

$$D^\mu = \partial^\mu - ieA^\mu, \qquad D \times W = D^\mu W^\nu - D^\nu W^\mu$$

The last term in the Lagrangian, $-ieW^{\dagger\mu} W^\nu F_{\mu\nu}$, is needed to produce the "natural" magnetic moment of g=2. We can introduce an anomalous magnetic moment by changing the coefficient of that term, from 1 to $1 + (g-2)$. For now call this coefficient λ. Now

$$\mathcal{L} = -\frac{1}{2}(D \times W)^\dagger \cdot (D \times W) + m_w^2 W^\dagger W - \lambda ieW^{\dagger\mu} W^\nu F_{\mu\nu}$$

We can find the equations of motion with the Euler-Lagrange method.

$$\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\partial_t W^{\dagger\alpha})} = 0$$

$$\frac{\partial}{\partial W^{\dagger\alpha}}(D \times W)^\dagger_{\mu\nu}(D \times W)^{\mu\nu} = \frac{\partial}{\partial W^{\dagger\alpha}}(D_\mu W_\nu - D_\nu W_\mu)^\dagger(D^\mu W^\nu - D^\nu W^\mu)$$

$$= \frac{\partial}{\partial W^{\dagger\alpha}}2(D_\mu W_\nu)^\dagger(D^\mu W^\nu - D^\nu W^\mu)$$

$$= \frac{\partial}{\partial W^{\dagger\alpha}}2W^\dagger_\mu D_\nu(D^\mu W^\nu - D^\nu W^\mu)$$

$$= 2g^\alpha_\mu D_\nu(D^\mu W^\nu - D^\nu W^\mu)$$

$$= 2D_\nu(D^\alpha W^\nu - D^\nu W^\alpha)$$

$$\frac{\partial}{\partial W^{\dagger\alpha}}m_W^2 W^{\dagger\mu} W_\mu = m_W^2 W_\alpha$$

$$\frac{\partial}{\partial W^{\dagger\alpha}}\lambda ieW^{\dagger\mu} W^\nu F_{\mu\nu}s = \lambda ieg^{\alpha\mu} W^\nu F_{\mu\nu}$$

$$= -\lambda ieW_\nu F^{\nu\mu}$$

To obtain a coupled set of first order equations, we introduce the fields $W^{\mu\nu} = D^\mu W^\nu - D^\nu W^\mu$.

$$D_\mu W^{\mu\nu} + m_W^2 W^\nu + \lambda ieW_\mu F^{\mu\nu} = 0 \tag{4.33}$$

$$W^{\mu\nu} = D^\mu W^\nu - D^\nu W^\mu \tag{4.34}$$

$W^{\mu\nu}$ is antisymmetric and so has six degrees of freedom, corresponding to six independent fields. Together with W^μ this represents a total of ten fields. However, upon examination only some of these fields are dynamic. The fields W^{0i} and W^i appear in the equations with time derivatives, while the fields W^{ij} and

W^0 never do. So it is only necessary to consider the former six fields. So that these six fields all have the same dimension, we will define $\frac{W^{i0}}{m} = i\eta^i$.

We will now eliminate the extraneous fields and solve for iD_0W^i, $iD_0\eta^i$.

Consider (1) with $\nu = 0$:

$$-D^i W^{i0} + m^2 W^0 - \lambda i e W^i F^{i0} = 0$$

or equivalently, using $F^{i0} = -E^i$:

$$\begin{aligned}
W^0 &= \frac{1}{m}\left[D^i \frac{W^{i0}}{m} + \lambda i e \frac{\mathbf{W}}{m} \cdot \mathbf{E}\right] \\
&= \frac{i}{m}\left[\mathbf{D} \cdot \boldsymbol{\eta} + \lambda i e \frac{\mathbf{W}}{m} \cdot \mathbf{E}\right]
\end{aligned}$$

Now consider (1) with $\nu = i$:

$$D^0 W^{0i} - D^j W^{ji} + m^2 W^i + \lambda i e W^0 F^{0i} - \lambda i e W^j F^{ji} = 0$$

Using the definition of W^{ij}, the previously defined η^i, and that $F^{ij} = \epsilon_{ijk}B_k$

$$-iD^0 m\eta^i - D^j(D^j W^i - D^i W^j) + m^2 W^i - \lambda i e W^0 E^i - \lambda i e W^j \epsilon_{jik}B_k = 0$$

This gives

$$iD^0\eta^i = \frac{1}{m}\left(\mathbf{D} \times (\mathbf{D} \times \mathbf{W})\right)^i + mW^i + \lambda \frac{e}{m^2}E^i\left(\mathbf{D} \cdot \boldsymbol{\eta} + \lambda \frac{e}{m}\mathbf{W} \cdot \mathbf{E}\right) + \lambda \frac{ie}{m}(\mathbf{W} \times \mathbf{B})^i \qquad (4.35)$$

Now, look at (2) with $\nu = 0, \mu = i$:

$$\begin{aligned}
W^{i0} &= D^i W^0 - D^0 W^i \\
&= \frac{i}{m}D^i\left[\mathbf{D} \cdot \boldsymbol{\eta} + \lambda i e \frac{\mathbf{W}}{m} \cdot \mathbf{E}\right] - D^0 W^i
\end{aligned}$$

This yields an equation for $iD^0 W^i$:

$$iD^0 W^i = m\eta^i - \frac{1}{m}D^i\left[\mathbf{D} \cdot \boldsymbol{\eta} + \lambda i e \frac{\mathbf{W}}{m} \cdot \mathbf{E}\right] \qquad (4.36)$$

We have two equations for $iD_0 W^i$ and $iD_0\eta^i$. But we need these equations written so that each term should be written as an operator acting on either $\mathbf{W}$ or $\boldsymbol{\eta}$. For this purpose, we introduce into the equations the spin matrices S_i. Then, using the identities shown in the appendix, we can rewrite (4.35) it as:

$$\begin{aligned}
iD^0\boldsymbol{\eta} = &-\frac{1}{m}(\mathbf{S} \cdot \mathbf{D})^2\mathbf{W} + m\mathbf{W} + \lambda \frac{e}{m^2}\left[\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D}) + i\mathbf{S} \cdot \mathbf{E} \times \mathbf{D}\right]\boldsymbol{\eta} \\
&- \lambda \frac{e}{m}\mathbf{S} \cdot \mathbf{B}\mathbf{W} + \lambda^2 \frac{e^2}{m^3}\left[\mathbf{E}^2 - (\mathbf{S} \cdot \mathbf{E})^2\right]\mathbf{W}
\end{aligned} \qquad (4.37)$$

Likewise, write (4.36) as:

$$iD^0\mathbf{W} = m\boldsymbol{\eta} - \frac{1}{m}\left[\mathbf{D}^2 - (\mathbf{S} \cdot \mathbf{D})^2 + e\mathbf{S} \cdot \mathbf{B}\right]\boldsymbol{\eta} - \lambda \frac{e}{m^2}\left[\mathbf{D} \cdot \mathbf{E} - (\mathbf{S} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{E}) + i\mathbf{S} \cdot \mathbf{D} \times \mathbf{E}\right]\mathbf{W} \qquad (4.38)$$

Now, these two equations can be written together in matrix form, from which the Hamiltonian can be easily obtained.

$$iD_0\begin{pmatrix} \eta \\ W \end{pmatrix} = \begin{pmatrix} \lambda \frac{e}{m^2}\left[\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D}) + i\mathbf{S} \cdot \mathbf{E} \times \mathbf{D}\right] & m - \frac{1}{m}(\mathbf{S} \cdot \mathbf{D})^2 - \lambda \frac{e}{m}\mathbf{S} \cdot \mathbf{B} + \lambda^2 \frac{e^2}{m^3}\left[\mathbf{E}^2 - (\mathbf{S} \cdot \mathbf{E})^2\right] \\ m - \frac{1}{m}\left[\mathbf{D}^2 - (\mathbf{S} \cdot \mathbf{D})^2 + e\mathbf{S} \cdot \mathbf{B}\right] & -\lambda \frac{e}{m^2}\left[\mathbf{D} \cdot \mathbf{E} - (\mathbf{S} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{E}) + i\mathbf{S} \cdot \mathbf{D} \times \mathbf{E}\right] \end{pmatrix}\begin{pmatrix} \eta \\ W \end{pmatrix}$$

4.2.1 Current

We can also derive the conserved current from the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}(D \times W)^\dagger \cdot (D \times W) + m_w^2 W^\dagger W - \lambda i e W^{\dagger\mu} W^\nu F_{\mu\nu}$$

Where

$$D^\mu = \partial^\mu - ieA^\mu, \qquad D \times W = D^\mu W^\nu - D^\nu W^\mu$$

We want the conserved current corresponding to the transformation $W_i \to e^{i\alpha} W_i$, which in infinitesimal form is:

$$W_\mu \to W_\mu + i\alpha W_\mu, \ W_\mu^\dagger \to W_\mu^\dagger - i\alpha W_\mu^\dagger$$

The 4-current density will be:

$$j^\sigma = -i\frac{\partial \mathcal{L}}{\partial W_{\mu,\sigma}} W_\mu + i\frac{\partial \mathcal{L}}{\partial W_{\mu,\sigma}^\dagger} W_\mu^\dagger$$

Only one term contains derivatives of the field:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial W_{\alpha,\sigma}} &= \frac{\partial}{\partial W_{\alpha,\sigma}}\left\{ -\frac{1}{2}(D_\mu W_\nu - D_\nu W_\mu)^\dagger (D^\mu W^\nu - D^\nu W^\mu) \right\} \\
&= -\frac{1}{2}(D_\mu W_\nu - D_\nu W_\mu)^\dagger (g_{\sigma\mu} g_{\alpha\nu} - g_{\sigma\nu} g_{\alpha\mu}) \\
&= -(D_\alpha W_\sigma - D_\sigma W_\alpha)^\dagger
\end{aligned}
$$

Likewise:

$$\frac{\partial \mathcal{L}}{\partial W_{\alpha,\sigma}^\dagger} = -(D_\alpha W_\sigma - D_\sigma W_\alpha)$$

If we define $W_{\mu\nu} = D^\mu W^\nu - D^\nu W^\mu$ then the 4-current and charge density are:

$$
\begin{aligned}
j_\sigma &= iW_{\sigma\mu}^\dagger W^\mu - iW_{\sigma\mu} W^{\dagger\mu} \\
j_0 &= iW_{0\mu}^\dagger W^\mu - iW_{0\mu} W^{\dagger\mu} \\
&= iW_{0i}^\dagger W^i - iW_{0i} W^{\dagger i}
\end{aligned}
$$

Where the last equality follows from the antisymmetry of $W_{\mu\nu}$.

Now, we defined the fields $\eta_i = -i\frac{W_{i0}}{m}$. In terms of these fields, $j_0 = m(\eta_i^\dagger W^i + \eta_i W^{\dagger i})$.

We can do the same to find the vector part of the current.

$$
\begin{aligned}
j_i &= iW_{i\mu}^\dagger W^\mu - iW_{i\mu} W^{\dagger\mu} \\
&= iW_j^\dagger W_{ij} + iW_{i0}^\dagger W_0 + c.c.
\end{aligned}
$$

We have $W_{ij} = D_i W_j - D_j W_i$. Using the identities developed in the appendix, we can obtain

$$D_j W_i = D_i W_j - D_k (S_i S_k)_{ja} W_a$$

Then

$$W_{ij} = D_k (S_i S_k)_{ja} W_a$$

In the absence of an electric field E, $W_0 = \frac{i}{m} D_j \eta_j$, with $W_{i0} = im\eta_i$.

$$W_{i0}^\dagger W_0 = -\eta_i^\dagger D_j \eta_j$$

Again we introduce spin matrices to get the equation in the desired form, and obtain

$$W_{i0}^\dagger W_0 = -\eta_j^\dagger D_k (\delta_{ik} - S_k S_i) \eta_j$$

This leads to

$$j_i = iW_j^\dagger D_k (S_i S_k W)_j - i\eta_j^\dagger D_k ([\delta_{ik} - S_k S_i] \eta)_j + c.c.$$

Writing this in terms of the bispinor $\begin{pmatrix} \eta \\ W \end{pmatrix}$, the expression for the current is

$$j_i = \frac{i}{2} \begin{pmatrix} \eta^\dagger & W^\dagger \end{pmatrix} \left[(\{S_i, S_j\} - \delta_{ij}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - ([S_i, S_j] + \delta_{ij}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] D_j \begin{pmatrix} \eta \\ W \end{pmatrix} + c.c. \qquad (4.39)$$

4.2.2 Hermiticity of Hamiltonian

It's clear from inspection that the above Hamiltonian is not Hermitian in the standard sense. (Of the operators in use, the only one which is not self adjoint is $D_i^\dagger = -D_i$.) Noticing that

$$[\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D}) + i\mathbf{S} \cdot \mathbf{E} \times \mathbf{D}]^\dagger = -[\mathbf{D} \cdot \mathbf{E} - (\mathbf{S} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{E}) + i\mathbf{S} \cdot \mathbf{D} \times \mathbf{E}]$$

we can see it has the general form

$$H = \begin{pmatrix} A & B \\ C & A^\dagger \end{pmatrix}$$

where the off diagonal blocks are Hermitian in the normal sense: $B^\dagger = B$ and $C^\dagger = C$.

At this point we should consider that an operator is defined as Hermitian with respect to a particular inner product. In quantum mechanics this inner product is normally defined as:

$$< \Psi, \phi > = \int d^3 x \Psi^\dagger \phi$$

This definition, however, does not produce sensible results for the states in question. We need the inner product of a state with itself to be conserved; in other words, it should play the role of a conserved charge. From the considerations above we already have one such quantity: the conserved charge $\int d^3 x j_0$, where

$$j_0 = m[\eta^\dagger W + W^\dagger \eta] = m \begin{pmatrix} \eta^\dagger & W^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ W \end{pmatrix}$$

A more general definition of the inner product includes some weight M:

$$< \Psi, \phi > = \int d^3 x \Psi^\dagger M \phi$$

In normal quantum mechanics M would be the identity matrix, but here, as implied by the charge density, we want $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Such a definition will lead to the inner product $< \Psi, \Psi >$ being conserved.

An operator H is hermitian with respect to this inner product if

$$< H\Psi, \phi > = < \Psi, H\phi > \rightarrow \int d^3 x \Psi^\dagger H^\dagger M \phi = \int d^3 x \Psi^\dagger M H \phi$$

For this equality to hold, it is sufficient for $H^\dagger M = MH$. With $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ this condition reduces to

$$\begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$$

Our Hamiltonian fulfills exactly this requirement, and so is Hermitian with respect to this particular inner product.

4.3 Non-relativistic Hamiltonian

Now we will consider the non-relativistic limit of the above Hamiltonian. To work in this regime constrains the order of both the momentum and the electromagnetic field strength.

$$
\begin{aligned}
D &\sim mv \\
\Phi &\sim mv^2 \\
E &\sim m^2 v^3 \\
B &\sim m^2 v^2
\end{aligned}
$$

We can write the Hamiltonian matrix in terms of the basis of 2x2 Hermitian matrices: $(\mathbf{I}, \rho_i)$:

$$
\begin{aligned}
H &= a_0 \mathbf{I} + a_i \rho_i \\
a_0 &= \frac{1}{2}(H_{11} + H_{22}) = e\Phi + \lambda \frac{e}{2m^2}\left[\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot\mathbf{E}\times\mathbf{D} - \mathbf{D}\cdot\mathbf{E} + (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) - i\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}\right] \\
a_3 &= \frac{1}{2}(H_{11} - H_{22}) = \lambda s \frac{e}{2m^2}\left[\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot\mathbf{E}\times\mathbf{D} + \mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) + i\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}\right] \\
ia_2 &= \frac{1}{2}(H_{21} - H_{12}) = -\left[\frac{\mathbf{D}^2}{2m} - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2 - \frac{\lambda-1}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + \frac{e^2}{2m^3}(\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2)\right] \\
a_1 &= \frac{1}{2}(H_{12} - H_{21}) = \left[m - \frac{\mathbf{D}^2}{2m} - \frac{1+\lambda}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + \frac{e^2}{2m^3}(\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2)\right]
\end{aligned}
$$

We can see that to leading order, the Hamiltonian is

$$
H = m\rho_1 = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}
$$

Since we wish to separate positive and negative energy states, this poses a problem. So we first switch to a basis where at least the rest energies are separate. Then, remaining off-diagonal elements can be treated as perturbations.

An appropriate transformation which meets our requirements is a "rotation" in the space spanned by ρ_i matrices, about the $\rho_1 + \rho_3$ axis. This has the explicit form:

$$
U = \frac{1}{\sqrt{2}}(\rho_1 + \rho_3) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{4.40}
$$

Any transformation U can be described by it's action on the basis of 4x4 matrices. This takes:

$$
\begin{aligned}
\mathbf{I} &\to \mathbf{I} \\
\rho_1 &\to \rho_3 \\
\rho_3 &\to \rho_1 \\
\rho_2 &\to -\rho_2 \\
\begin{pmatrix} \eta \\ W \end{pmatrix} &\to \begin{pmatrix} \Psi_u \\ \Psi_\ell \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \eta + W \\ \eta - W \end{pmatrix}
\end{aligned}
$$

(This transformation will transform the current to

$$
j_0 = m(\Psi_u^\dagger \Psi_u - \Psi_\ell^\dagger \Psi_\ell)
$$

and the weight M, used in the inner product, to

$$
M \to M' = U^\dagger M U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

The transformed Hamiltonian will of course be Hermitian with respect to the transformed inner product.)

Our equation is now of the following form:

$$i\partial_0 \begin{pmatrix} \Psi_u \\ \Psi_\ell \end{pmatrix} = H' \begin{pmatrix} \Psi_u \\ \Psi_\ell \end{pmatrix}$$

We can see that the Hamiltonian still contains off-diagonal elements, so this represents a pair of coupled equations for the upper and lower components of $\boldsymbol{\Psi}$. But the off-diagonal terms are small, so we can consider the case where Ψ_ℓ is small compared to Ψ_u. Solving for Ψ_ℓ in terms of Ψ_u:

$$E\Psi_\ell = H'_{21}\Psi_u + H'_{22}\Psi_\ell$$
$$\Psi_\ell = (E - H'_{22})^{-1} H'_{21}\Psi_u$$

This gives the exact formula:

$$E\Psi_u = \left(H'_{11} + H'_{12}[E - H'_{22}]^{-1} H'_{21}\right)\Psi_u$$

However, we only need corrections to the magnetic moment of order v^2. With $\frac{e}{m}\mathbf{S}\cdot\mathbf{B} \sim mv^2$, this means we only need the Hamiltonian to at most order mv^4. Examining the leading order terms of the matrix H', the diagonal elements are order m while the off-diagonal elements are order mv^2. To leading order the term $[E - H'_{22}]^{-1} = \frac{1}{2m}$. So we'll need H'_{11} to $\mathcal{O}(v^4)$, and H'_{12}, H'_{21}, and $[E - H'_{22}]^{-1}$ each to only leading order.

$$E\Psi_u = \left(H'_{11} + \frac{1}{2m}H'_{12}H'_{21} + \mathcal{O}(mv^6)\right)\Psi_u$$

The needed terms of H are, using $\lambda = g - 1$

$$H'_{11} = a_0 + a_3 = m + e\Phi - \frac{D^2}{2m} - \frac{g}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B}$$
$$+(g-1)\frac{e}{2m^2}\left[\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D}) + i\mathbf{S}\cdot\mathbf{E}\times\mathbf{D} - \mathbf{D}\cdot\mathbf{E} + (\mathbf{S}\cdot\mathbf{D})(\mathbf{S}\cdot\mathbf{E}) - i\mathbf{S}\cdot\mathbf{D}\times\mathbf{E}\right] + \mathcal{O}(mv$$

$$H'_{12} = a_1 - ia_2 = \frac{D^2}{2m} - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2 - \frac{g-2}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + \mathcal{O}(mv^4)$$
$$H'_{21} = a_1 + ia_2 = -\frac{D^2}{2m} + \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2 + \frac{g-2}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} + \mathcal{O}(mv^4)$$

The product $H'_{12}H'_{21}$ is calculated in the appendix. To first order in the magnetic field strength it is:

$$\frac{1}{2m}H'_{12}H'_{21} = -\frac{1}{2m^3}\left(\frac{\mathbf{D}^2}{2} - (\mathbf{S}\cdot\mathbf{D})^2 - \frac{g-2}{2}e\mathbf{S}\cdot\mathbf{B}\right)^2$$
$$= -\frac{1}{2m^3}\left(\frac{\boldsymbol{\pi}^4}{4} - e\mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - \frac{g-2}{2}e(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\right)$$

So finally, replacing all $\mathbf{D}$ with $\boldsymbol{\pi} \equiv \mathbf{p} - e\mathbf{A}$, we have a direct expression for Ψ_u:

$$E\Psi_u = \left\{m + e\Phi + \frac{\boldsymbol{\pi}^2}{2m} - \frac{g}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B} - \frac{\boldsymbol{\pi}^4}{8m^3} + \frac{e\mathbf{p}^2(\mathbf{S}\cdot\mathbf{B})}{2m^3} + (g-2)\frac{e}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\right.$$
$$\left. +(g-1)\frac{ie}{2m^2}\left[\mathbf{E}\cdot\boldsymbol{\pi} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\boldsymbol{\pi}) + i\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi} - \boldsymbol{\pi}\cdot\mathbf{E} + (\mathbf{S}\cdot\boldsymbol{\pi})(\mathbf{S}\cdot\mathbf{E}) - i\mathbf{S}\cdot\boldsymbol{\pi}\times\mathbf{E}\right]\right\}\Psi_u$$

The complicated expression in square brackets can be cleaned up a bit:

$$
\begin{aligned}
\mathbf{E}\cdot\boldsymbol{\pi} - \boldsymbol{\pi}\cdot\mathbf{E} &= [E_i, \pi_i] \\
&= [E_i, -i\partial_i] \\
&= i(\partial_i E_i) \\
(\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\boldsymbol{\pi}) - (\mathbf{S}\cdot\boldsymbol{\pi})(\mathbf{S}\cdot\mathbf{E}) &= S_i S_j E_i \pi_j - S_i S_j \pi_i E_j \\
&= (S_i S_j)(E_i \pi_j - E_j \pi_i - [\pi_i, E_j]) \\
&= [S_i, S_j](E_i \pi_j) - (S_i S_j)(-i\nabla_i E_j) \\
&= (i\epsilon_{ijk} S_k) E_j \pi_i - (S_i S_j)(-i\nabla_i E_j) \\
&= i\mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi} + iS_i S_j \nabla_i E_j) \\
(\mathbf{E}\times\boldsymbol{\pi} - \boldsymbol{\pi}\times\mathbf{E})_k &= \epsilon_{ijk}(E_i \pi_j - \pi_i E_j) \\
&= \epsilon_{ijk}(E_i \pi_j + \pi_j E_i) \\
&= \epsilon_{ijk}(2E_i \pi_j + [\pi_i, E_j]) \\
&= 2\epsilon_{ijk} E_i \pi_j \\
&= 2(\mathbf{E}\times\boldsymbol{\pi})_k
\end{aligned}
$$

Using these identities and collecting terms, and then writing everything in terms of g, $g-2$

$$
\begin{aligned}
E\Psi_u &= \left\{ m + e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} - \frac{e}{m}\mathbf{S}\cdot\mathbf{B}\left(\frac{g}{2} - \frac{p^2}{2m^2}\right) + (g-2)\frac{e}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \right. \\
&\quad \left. -(g-1)\frac{e}{2m^2}[\nabla\cdot\mathbf{E} - S_i S_j \nabla_i E_j + \mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi}] \right\}\Psi_u \\
&= \left\{ m + e\Phi + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} - \frac{g}{2}\frac{e}{m}\mathbf{S}\cdot\mathbf{B}\left(1 - \frac{p^2}{2m^2}\right) - \frac{g-2}{2}\frac{e}{m}\frac{p^2}{2m^2}\mathbf{S}\cdot\mathbf{B} + (g-2)\frac{e}{4m^3}(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p}) \right. \\
&\quad \left. -\left(\frac{g}{2} + \frac{g-2}{2}\right)\frac{e}{2m^2}[\nabla\cdot\mathbf{E} - S_i S_j \nabla_i E_j + \mathbf{S}\cdot\mathbf{E}\times\boldsymbol{\pi}] \right\}\Psi_u
\end{aligned}
$$

We have a Hamiltonian for the upper component of the bispinor, as desired. But is it truly Schrodinger-like? In the general case we would need to perform the Fouldy-Wouthyusen transformation, to a representation where all the physics up to the desired order is contained in the single spinor equation. But we can show that, at the order we are working at, the Hamiltonian above is correct.

4.4 Normalization

What we want is a single equation for the non-relativistic particle. Because the lower component of the bispinor is small but nonzero, it is not necessarily true that the above equation accurately captures the physics. The FW transformation to a basis where the lower component is truly negligible might be necessary.

However, here it can be shown that such a transformation will have no effect at the desired order. The transformation U would have the form:: $U = e^{iS}$ where S is Hermitian. Because the Hamiltonian is diagonal at leading order, S must be small, and the transformation will affect Ψ_u as

$$
\Psi_u \to \Psi_u' = (1 + \Delta)\Psi_u
$$

where Δ is some small operator. The probability density must be unaffected by this change, so: On the one hand, with $\Psi_\ell = \epsilon\Psi_u$, $\epsilon \sim \mathcal{O}(v^2)$

$$
\begin{aligned}
\int d^3x(\Psi_u^\dagger\Psi_u - \Psi_\ell^\dagger{}_\ell\Psi_\ell) &= \int d^3x(\Psi_u^\dagger\Psi_u - (\epsilon\Psi_u)^\dagger\epsilon\Psi_u) \\
&= \int d^3x\Psi_u^\dagger(1 + \mathcal{O}(v^4))\Psi_u
\end{aligned}
$$

And on the other hand:

$$\int d^3x \Psi'^\dagger_u \Psi'_u \;=\; \int d^3x ([1+\Delta]\Psi_u)^\dagger (1+\Delta)\Psi_u$$

Comparing the two, it can be seen that Δ must be no larger than $\mathcal{O}(v^4)$. Now considering the new equation for Ψ'_u:

$$(E-m)\Psi_u \;=\; \left(H'_{11} + \frac{1}{2m}H'_{12}H'_{21}\right)\Psi_u$$

$$(E-m)(1-\Delta)\Psi'_u \;=\; \left(H'_{11} + \frac{1}{2m}H'_{12}H'_{21}\right)(1-\Delta)\Psi'_u$$

$$(E-m)\Psi'_u \;=\; (1+\Delta)\left(H'_{11} + \frac{1}{2m}H'_{12}H'_{21}\right)(1-\Delta)\Psi'_u$$

Since $H' \sim \mathcal{O}(mv^2)$ and $\Delta \sim \mathcal{O}(v^4)$, to $\mathcal{O}(v^4)$, Ψ'_u obeys exactly the same equation as Ψ_u:

$$(E-m)\Psi'_u \;=\; \left(H'_{11} + \frac{1}{2m}H'_{12}H'_{21}\right)\Psi'_u$$

5   Comparison with Silenko

We can compare to a result in Silenko (arXiv:hep-th/0401183v1). Silenko determines a Hamiltonian which is equivalent to that used above. Dropping terms manifestly beyond $\mathcal{O}(mv^4)$:

$$
\begin{aligned}
H &= \rho_3 \epsilon' + e\Phi + \frac{e}{4m}\left[\left\{\left(\frac{g-2}{2} + \frac{m}{\epsilon'+m}\right)\frac{1}{\epsilon'}, (\mathbf{S}\cdot\boldsymbol{\pi}\times\boldsymbol{E} - \mathbf{S}\cdot\boldsymbol{E}\times\boldsymbol{\pi})\right\}_+ - \rho_3\left\{\left(g - 2 + \frac{2m}{\epsilon'}\right), \mathbf{S}\cdot\mathbf{B}\right\}_+ \right.\\
&\quad \left. + \rho_3\left\{\frac{g-2}{2\epsilon'(\epsilon'+m)}, \{\mathbf{S}\cdot\boldsymbol{\pi}, \boldsymbol{\pi}\cdot\mathbf{B}\}_+\right\}_+\right] + \frac{e(g-1)}{4m^2}\{\mathbf{S}\cdot\boldsymbol{\nabla}, \mathbf{S}\cdot\boldsymbol{E}\}_+ - \frac{e(g-1)}{2m^2}\boldsymbol{\nabla}\cdot\mathbf{E}
\end{aligned}
$$

We'll take each term and expand to $\mathcal{O}(mv^4)$. First the operator ϵ':

$$
\begin{aligned}
\epsilon' &= \sqrt{m^2 + \pi^2}\\
&= m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^2} + \mathcal{O}(mv^6)
\end{aligned}
$$

Using this, we get

$$
\begin{aligned}
H &= \rho_3 \epsilon' + e\Phi + \frac{e}{4m}\left[\left\{\frac{g-1}{2m}, (\mathbf{S}\cdot\boldsymbol{\pi}\times\boldsymbol{E} - \mathbf{S}\cdot\boldsymbol{E}\times\boldsymbol{\pi})\right\}_+ - \rho_3\left\{g - \frac{\pi^2}{m^2}, \mathbf{S}\cdot\mathbf{B}\right\}_+ \right.\\
&\quad \left. + \rho_3\left\{\frac{g-2}{4m^2}, \{\mathbf{S}\cdot\boldsymbol{\pi}, \boldsymbol{\pi}\cdot\mathbf{B}\}_+\right\}_+\right] + \frac{e(g-1)}{4m^2}\{\mathbf{S}\cdot\boldsymbol{\nabla}, \mathbf{S}\cdot\boldsymbol{E}\}_+ - \frac{e(g-1)}{2m^2}\boldsymbol{\nabla}\cdot\mathbf{E}
\end{aligned}
$$

Now we'll expand the anticommutators.

We have already shown that $\boldsymbol{\pi}\times\mathbf{E} = -\mathbf{E}\times\boldsymbol{\pi}$, and that such a term is $\mathcal{O}(mv^4)$

$$
\left\{\frac{g-1}{2m}, (\mathbf{S}\cdot\boldsymbol{\pi}\times\boldsymbol{E} - \mathbf{S}\cdot\boldsymbol{E}\times\boldsymbol{\pi})\right\}_+ = -\frac{2(g-1)}{m}\mathbf{S}\cdot\boldsymbol{E}\times\boldsymbol{\pi}
$$

Because we specialize to constant magnetic fields, $\boldsymbol{\pi}$ commutes with $\mathbf{B}$, so

$$
\left\{g - \frac{\pi^2}{m^2}, \mathbf{S}\cdot\mathbf{B}\right\}_+ = (2g - \frac{2\pi^2}{m^2})\mathbf{S}\cdot\mathbf{B}
$$

and

$$
\left\{\frac{g-2}{4m^2}, \{\mathbf{S}\cdot\boldsymbol{\pi}, \boldsymbol{\pi}\cdot\mathbf{B}\}_+\right\}_+ = \frac{g-2}{m^2}(\mathbf{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\mathbf{B})
$$

Using the identity that $[\nabla_i, E_j] = 0$, $\boldsymbol{\nabla}\times\mathbf{E} = 0$:

$$
\begin{aligned}
\{\mathbf{S}\cdot\boldsymbol{\nabla}, \mathbf{S}\cdot\boldsymbol{E}\}_+ &= S_i S_j \nabla_i E_j + S_j S_i E_j \nabla_i\\
&= (S_i S_j + S_j S_i)\nabla_i E_j\\
&= (2S_i S_j + [S_j, S_i])\nabla_i E_j\\
&= (2S_i S_j + iS_k\epsilon_{ijk})\nabla_i E_j\\
&= 2S_i S_j \nabla_i E_j
\end{aligned}
$$

So

$$
\begin{aligned}
H &= \rho_3(m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^2}) + e\Phi - (g-1)\frac{e}{2m^2}[\mathbf{S}\cdot\boldsymbol{E}\times\boldsymbol{\pi} - S_i S_j \nabla_i E_j + \boldsymbol{\nabla}\cdot\mathbf{E}]\\
&\quad + \rho_3(g-2)\frac{2}{4m^2}(\mathbf{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\mathbf{B}) - \rho_3\frac{e}{m}(\frac{g}{2} - \frac{\pi^2}{2m^2})\mathbf{S}\cdot\mathbf{B}\\
&= \rho_3\left(m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^2}\right) + e\Phi - \left(\frac{g-2}{2} + \frac{g}{2}\right)\frac{e}{2m^2}[\mathbf{S}\cdot\boldsymbol{E}\times\boldsymbol{\pi} - S_i S_j \nabla_i E_j + \boldsymbol{\nabla}\cdot\mathbf{E}]\\
&\quad + \rho_3(g-2)\frac{2}{4m^2}(\mathbf{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\mathbf{B}) - \rho_3\frac{e}{m}\left[\frac{g}{2}\left(1 - \frac{\pi^2}{2m^2}\right) + \frac{g-2}{2}\frac{\pi^2}{2m^2}\right]\mathbf{S}\cdot\mathbf{B}
\end{aligned}
$$

Magnetic Moment

Now we'll keep only those terms which contribute to the magnetic moment, using $\boldsymbol{\pi} = \mathbf{p} - e\mathbf{A}$

$$
\begin{aligned}
H_{S \cdot B} &= \left(\frac{g-2}{2} + \frac{g}{2} \right) \frac{e^2}{2m^2} \mathbf{S} \cdot \mathbf{E} \times \mathbf{A} + (g-2) \frac{e}{4m^2} (\mathbf{S} \cdot \mathbf{p})(\mathbf{p} \cdot \mathbf{B}) - \frac{e}{m} \left[\frac{g}{2} \left(1 - \frac{p^2}{2m^2} \right) + \frac{g-2}{2} \frac{p^2}{2m^2} \right] \mathbf{S} \cdot \mathbf{B} \\
&= -\frac{e}{2m} \left\{ g \left(1 - \frac{p^2}{2m^2} \right) \mathbf{S} \cdot \mathbf{B} + (g-2) \frac{p^2}{2m^2} \mathbf{S} \cdot \mathbf{B} - (g-2) \frac{(\mathbf{S} \cdot \mathbf{p})(\mathbf{p} \cdot \mathbf{B})}{2m^2} - \frac{e}{m} \left(\frac{g-2}{2} + \frac{g}{2} \right) \mathbf{S} \cdot \mathbf{E} \times \mathbf{A} \right\}
\end{aligned}
$$

This exactly matches with what was found before.

6 Identities

Simplify $\mathbf{W} \times \mathbf{B}$:

$$
\begin{aligned}
(\mathbf{W} \times \mathbf{B})_i &= \epsilon_{ijk} W_j B_k \\
&= i(S_k)_{ij} W_j B_k \\
&= i(\mathbf{S} \cdot \mathbf{B})_{ij} W_j \\
&= i([\mathbf{S} \cdot \mathbf{B}]\mathbf{W})_i
\end{aligned}
$$

Simplify $\mathbf{D} \times (\mathbf{D} \times \mathbf{W})$

$$
\begin{aligned}
(\mathbf{D} \times [\mathbf{D} \times \mathbf{W}])_i &= \epsilon_{ijk} D_j (\mathbf{D} \times \mathbf{W})_k \\
&= \epsilon_{ijk} \epsilon_{k\ell m} D_j D_\ell W_m \\
&= -(S_j)_{ki} (S_\ell)_{mk} D_j D_\ell W_m \\
&= -(\mathbf{S} \cdot \mathbf{D})_{ki} (\mathbf{S} \cdot \mathbf{D})_{mk} W_m \\
&= - \left([\mathbf{S} \cdot \mathbf{D}]^2 \right)_{im} W_m \\
&= - \left([\mathbf{S} \cdot \mathbf{D}]^2 \mathbf{W} \right)_i
\end{aligned}
$$

Our representation defines the spin matrices as follows:

$$
(S_k)_{ij} = -i\epsilon_{ijk}
$$

They have the commutator

$$
[S_i, S_j] = i\epsilon_{ijk} S_k
$$

The product of two such spin matrices is given by:

$$
\begin{aligned}
(S_k S_\ell)_{ij} &= (S_k)_{ia} (S_\ell)_{aj} \\
&= -\epsilon_{iak} \epsilon_{aj\ell} \\
&= (\delta_{k\ell}\delta_{ij} - \delta_{kj}\delta_{\ell i})
\end{aligned}
$$

This implies that:

$$
\begin{aligned}
(S_i S_j A_j B_i \mathbf{v})_l &= (S_i)_{lm} (S_j)_{mn} A_j B_i v_n \\
&= (S_i S_j)_{ln} A_j B_i v_n \\
&= (\delta_{ij}\delta_{ln} - \delta_{in}\delta_{lj}) A_j B_i v_n \\
&= (\mathbf{A} \cdot \mathbf{B}) v_l - A_l (\mathbf{B} \cdot \mathbf{v})
\end{aligned}
$$

Or

$$\mathbf{A}(\mathbf{B} \cdot \mathbf{v}) = (\mathbf{A} \cdot \mathbf{B} - S_i S_j A_j B_i)\mathbf{v}$$

Now we can use this to establish some identities:

$$
\begin{aligned}
E^i \mathbf{D} \cdot \eta &= ([\mathbf{E} \cdot \mathbf{D} - S_j S_k E_k D_j]\, \eta)^i \\
&= ([\mathbf{E} \cdot \mathbf{D} - (S_k S_j - [S_k, S_j]) E_k D_j]\, \eta)^i \\
&= ([\mathbf{E} \cdot \mathbf{D} - (S_k S_j - i\epsilon_{kjl} S_l) E_k D_j]\, \eta)^i \\
&= ([\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D}) + i S_l (\mathbf{E} \times \mathbf{D})_l]\, \eta)^i \\
&= ([\mathbf{E} \cdot \mathbf{D} - (\mathbf{S} \cdot \mathbf{E})(\mathbf{S} \cdot \mathbf{D}) + i \mathbf{S} \cdot (\mathbf{E} \times \mathbf{D})]\, \eta)^i
\end{aligned}
$$

Since E and W commute:

$$
\begin{aligned}
E^i \mathbf{W} \cdot \mathbf{E} &= \left([\mathbf{E}^2 - S_j S_k E_k E_j]\, \mathbf{W}\right)^i \\
&= \left([\mathbf{E}^2 - (\mathbf{S} \cdot \mathbf{E})^2]\, \mathbf{W}\right)^i
\end{aligned}
$$

And

$$
\begin{aligned}
D^i \mathbf{D} \cdot \eta &= \left([\mathbf{D}^2 - S_j S_k D_k D_j]\, \eta\right)^i \\
&= \left([\mathbf{D}^2 - (S_k S_j + [S_j, S_k]) D_k D_j]\, \eta\right)^i \\
&= \left([\mathbf{D}^2 - (\mathbf{S} \cdot \mathbf{D})^2 + i\epsilon_{jkl} S_l D_k D_j)]\, \eta\right)^i \\
&= \left([\mathbf{D}^2 - (\mathbf{S} \cdot \mathbf{D})^2 + i\mathbf{S} \cdot (\mathbf{D} \times \mathbf{D})]\, \eta\right)^i \\
&= \left([\mathbf{D}^2 - (\mathbf{S} \cdot \mathbf{D})^2 + e\mathbf{S} \cdot \mathbf{B}]\, \eta\right)^i
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
D^i \mathbf{E} \cdot \mathbf{W} &= ([\mathbf{D} \cdot \mathbf{E} - S_j S_k D_k E_j]\, \eta)^i \\
&= ([\mathbf{D} \cdot \mathbf{E} - (S_k S_j + [S_j, S_k]) D_k E_j]\, \eta)^i \\
&= ([\mathbf{D} \cdot \mathbf{E} - (S_k S_j + i\epsilon_{jkl} S_l) D_k E_j]\, \eta)^i \\
&= ([\mathbf{D} \cdot \mathbf{E} - (\mathbf{S} \cdot \mathbf{D})(\mathbf{S} \cdot \mathbf{E}) + i\mathbf{S} \cdot (\mathbf{D} \times \mathbf{E})]\, \eta)^i
\end{aligned}
$$

Product of $H_{12} H_{21}$

We need to calculate $\left(\frac{\pi^2}{2} - (\mathbf{S} \cdot \boldsymbol{\pi})^2 + (g-2)\frac{2}{m}\mathbf{S} \cdot \mathbf{B}\right)^2$ to first order in magnetic field strength. As a first step of simplification

$$\left(\frac{\pi^2}{2} - (\mathbf{S} \cdot \boldsymbol{\pi})^2 + \frac{g-2}{2}\frac{e}{m}\mathbf{S} \cdot \mathbf{B}\right)^2 = \left(\frac{\pi^2}{2} - (\mathbf{S} \cdot \boldsymbol{\pi})^2\right)^2 + \frac{g-2}{2}\frac{e}{m}\left\{\frac{p^2}{2} - (\mathbf{S} \cdot \mathbf{p})^2, \mathbf{S} \cdot \mathbf{B}\right\}$$

First term

To simplify the first term, consider one element of this matrix operator:

$$\left\{ \left(\frac{\boldsymbol{\pi}^2}{2} - (\mathbf{S} \cdot \boldsymbol{\pi})^2 \right)^2 \right\}_{ac} = \left(\frac{\boldsymbol{\pi}^2}{2} - S_i S_j \pi_i \pi_j \right)_{ab} \left(\frac{\boldsymbol{\pi}^2}{2} - S_l S_m \pi_l \pi_m \right)_{bc}$$

$$= \left(\frac{\boldsymbol{\pi}^2}{2} \delta_{ab} - [S_i S_j]_{ab} \pi_i \pi_j \right) \left(\frac{\boldsymbol{\pi}^2}{2} \delta_{bc} - [S_l S_m]_{bc} \pi_l \pi_m \right)$$

$$= \left(\frac{\boldsymbol{\pi}^2}{2} \delta_{ab} - [\delta_{ab}\delta_{ij} - \delta_{aj}\delta_{bi}]\pi_i \pi_j \right) \left(\frac{\boldsymbol{\pi}^2}{2} \delta_{bc} - [\delta_{bc}\delta_{lm} - \delta_{bm}\delta_{cl}]\pi_l \pi_m \right)$$

$$= \left(-\frac{\boldsymbol{\pi}^2}{2} \delta_{ab} + \pi_b \pi_a \right) \left(-\frac{\boldsymbol{\pi}^2}{2} \delta_{bc} + \pi_c \pi_b \right)$$

$$= \frac{\boldsymbol{\pi}^4}{4} \delta_{ac} - \pi_c \pi_a \frac{\boldsymbol{\pi}^2}{2} - \frac{\boldsymbol{\pi}^2}{2} \pi_c \pi_a + \pi_b \pi_a \pi_c \pi_b$$

It's very useful to have the following identity:

$$e(\mathbf{S} \cdot \mathbf{B})_{ab} = e(S_i)_{ab} B_i$$

$$= -ie\epsilon_{iab} B_i$$

$$= -ie\epsilon_{iab}(\epsilon_{ijk}\partial_j A_k)$$

$$= -ie\epsilon_{iab}\epsilon_{ijk} \frac{1}{2}(\partial_j A_k \partial_k A_j)$$

$$= -\epsilon_{iab}\epsilon_{ijk} \frac{1}{2}[\pi_j, \pi_k]$$

$$= -\epsilon_{iab}\epsilon_{ijk}\pi_j \pi_k$$

$$= -(\delta_{aj}\delta_{bk} - \delta_{ak}\delta_{bj})\pi_j \pi_k$$

$$= \pi_b \pi_a - \pi_a \pi_b$$

Therefore,

$$e(\mathbf{S} \cdot \mathbf{B})_{ab} = [\pi_b, \pi_a]$$

Using this, and the fact that π commutes with S and B:

$$\pi_b \pi_a \pi_c \pi_b = \pi_b \pi_c \pi_a \pi_b - \pi_b(e\mathbf{S} \cdot \mathbf{B})_{ac}\pi_b$$

$$= \pi_b \pi_c \pi_a \pi_b - \boldsymbol{\pi}^2(e\mathbf{S} \cdot \mathbf{B})_{ac}$$

Also,

$$\pi_b \pi_c \pi_a \pi_b = \pi_b \pi_c \pi_b \pi_a + \pi_b \pi_c(e\mathbf{S} \cdot \mathbf{B})_{ba}$$

$$= \pi_b \pi_b \pi_c \pi_a + \pi_b \pi_a(e\mathbf{S} \cdot \mathbf{B})_{bc} + \pi_b \pi_c(e\mathbf{S} \cdot \mathbf{B})_{ba}$$

So now:

$$\pi_b \pi_a \pi_c \pi_b - \pi_c \pi_a \frac{\boldsymbol{\pi}^2}{2} - \frac{\boldsymbol{\pi}^2}{4} \pi_c \pi_a = \boldsymbol{\pi}^2 \pi_c \pi_a + \pi_b \pi_a(e\mathbf{S} \cdot \mathbf{B})_{bc} + \pi_b \pi_c(e\mathbf{S} \cdot \mathbf{B})_{ba}$$

$$-\boldsymbol{\pi}^2(e\mathbf{S} \cdot \mathbf{B})_{ac} - \pi_c \pi_a \frac{\boldsymbol{\pi}^2}{2} - \frac{\boldsymbol{\pi}^2}{4} \pi_c \pi_a$$

$$= \frac{1}{2}[\boldsymbol{\pi}^2, \pi_c \pi_a] + \pi_b \pi_a(e\mathbf{S} \cdot \mathbf{B})_{bc} + \pi_b \pi_c(e\mathbf{S} \cdot \mathbf{B})_{ba} - \boldsymbol{\pi}^2(e\mathbf{S} \cdot \mathbf{B})_{ac}$$

Now evaluate the commutator:

$$[\pi_b, \pi_c\pi_a] = [\pi_b, \pi_c]\pi_a - \pi_c[\pi_a, \pi_b]$$
$$= (e\mathbf{S}\cdot\mathbf{B})_{cb}\pi_a - \pi_c(e\mathbf{S}\cdot\mathbf{B})_{ba}$$

$$[\boldsymbol{\pi}^2, \pi_c\pi_a] = [\pi_b\pi_b, \pi_c\pi_a]$$
$$= \pi_b[\pi_b, \pi_c\pi_a] + [\pi_b, \pi_c\pi_a]\pi_b$$
$$= (e\mathbf{S}\cdot\mathbf{B})_{cb}(\pi_b\pi_a + \pi_a\pi_b) - (e\mathbf{S}\cdot\mathbf{B})_{ba}(\pi_b\pi_c + \pi_c\pi_b)$$

This gives the result:

$$\pi_b\pi_a\pi_c\pi_b - \pi_c\pi_a\frac{\boldsymbol{\pi}^2}{2} - \frac{\boldsymbol{\pi}^2}{4}\pi_c\pi_a = \frac{1}{2}[(e\mathbf{S}\cdot\mathbf{B})_{cb}(\pi_b\pi_a + \pi_a\pi_b) - (e\mathbf{S}\cdot\mathbf{B})_{ba}(\pi_b\pi_c + \pi_c\pi_b)]$$
$$+ \pi_b\pi_a(e\mathbf{S}\cdot\mathbf{B})_{bc} + \pi_b\pi_c(e\mathbf{S}\cdot\mathbf{B})_{ba} - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac}$$
$$= \frac{1}{2}[(e\mathbf{S}\cdot\mathbf{B})_{cb}(\pi_a\pi_b - \pi_b\pi_a) + (e\mathbf{S}\cdot\mathbf{B})_{ba}(\pi_b\pi_c - \pi_c\pi_b)] - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac}$$
$$= \frac{1}{2}[(e\mathbf{S}\cdot\mathbf{B})_{cb}(e\mathbf{S}\cdot\mathbf{B})_{ba} + (e\mathbf{S}\cdot\mathbf{B})_{ba}(e\mathbf{S}\cdot\mathbf{B})_{cb}] - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac}$$
$$= (e\mathbf{S}\cdot\mathbf{B})_{ab}(e\mathbf{S}\cdot\mathbf{B})_{bc}) - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac}$$
$$= [(e\mathbf{S}\cdot\mathbf{B})^2]_{ac} - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})_{ac}$$

Since we can throw away terms of order B^2, the final result tells us that, to first order in B:

$$\left(\frac{\boldsymbol{\pi}^2}{2} - (\mathbf{S}\cdot\boldsymbol{\pi})^2\right)^2 = \frac{\boldsymbol{\pi}^4}{4} - \boldsymbol{\pi}^2(e\mathbf{S}\cdot\mathbf{B})$$
$$= \frac{\boldsymbol{\pi}^4}{4} - e\mathbf{p}^2\mathbf{S}\cdot\mathbf{B}$$

Second term

To simplify the second term, we need

$$\left\{\frac{\mathbf{p}^2}{2} - (\mathbf{S}\cdot\mathbf{p})^2, \mathbf{S}\cdot\mathbf{B}\right\} = \mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - [(\mathbf{S}\cdot\mathbf{p})^2\mathbf{S}\cdot\mathbf{B} + \mathbf{S}\cdot\mathbf{B}(\mathbf{S}\cdot\mathbf{p})^2]$$
$$= \mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - (S_iS_jS_k + S_kS_jS_i)p_ip_jB_k$$

To simplify that triple product of spin matrices, we can use their explicit form:

$$(S_iS_jS_k)_{ab} = i\epsilon_{aci}\epsilon_{cdj}\epsilon_{dbk}$$
$$= i(\delta_{id}\delta_{aj} - \delta_{ij}\delta_{ad})\epsilon_{dbk}$$
$$= i(\delta_{aj}\epsilon_{ibk} - \delta_{ij}\epsilon_{abk})$$
$$(S_iS_jS_k + S_kS_jS_i)_{ab} = i(\delta_{aj}\epsilon_{ibk} + \delta_{aj}\epsilon_{kbi} - \delta_{ij}\epsilon_{abk} - \delta_{kj}\epsilon_{abi})$$
$$= -i(\delta_{ij}\epsilon_{abk} + \delta_{kj}\epsilon_{abi}$$
$$= \delta_{ij}(S_k)_ab + \delta_{kj}(S_i)_{ab}$$

Now

$$\mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - (S_iS_jS_k + S_kS_jS_i)p_ip_jB_k = \mathbf{p}^2\mathbf{S}\cdot\mathbf{B} - (\delta_{ij}S_k + \delta_{kj}S_i)p_ip_jB_k$$
$$= -(\mathbf{S}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})$$

Result

At last, the final result is that, to the order we care about

$$\left(\frac{\boldsymbol{\pi}^2}{2} - (\mathbf{S} \cdot \boldsymbol{\pi})^2 + (g-2)\frac{e}{m}\mathbf{S} \cdot \mathbf{B}\right)^2 = \frac{\boldsymbol{\pi}^4}{4} - e\mathbf{p}^2\mathbf{S} \cdot \mathbf{B} - \frac{g-2}{2}\frac{e}{m}(\mathbf{S} \cdot \mathbf{p})(\mathbf{B} \cdot \mathbf{p})$$