0.1 Exact equations of motion for spin-1

$$\mathcal{L} = -\frac{1}{2}(D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) + m^2 W^{\mu\dagger} W_\mu - i\lambda e W^{\mu\dagger} W^\nu F_{\mu\nu} \tag{0.1}$$

where as usual D is the long derivative $D^\mu = \partial^\mu + ieA^\mu$.

Obtain the equations of motion from the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial[\partial_\mu W^{\dagger\alpha}]} = 0$$

Or equivalently,

$$\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} - D_\mu \frac{\partial \mathcal{L}}{\partial[D_\mu W^{\dagger\alpha}]} = 0$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial W^{\dagger\alpha}} &= \frac{\partial}{\partial W^{\dagger\alpha}} \left(m^2 W^{\mu\dagger} W_\mu - i\lambda e W^{\mu\dagger} W^\nu F_{\mu\nu} \right) \\
&= m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial[D_\gamma W^{\dagger\alpha}]} &= -\frac{1}{2} \frac{\partial}{\partial[D_\gamma W^{\dagger\alpha}]} (D^\mu W^\nu - D^\nu W^\mu)^\dagger (D_\mu W_\nu - D_\nu W_\mu) \\
&= -\frac{1}{2} (g^{\mu\gamma} g^\nu_\alpha - g^{\nu\gamma} g^\mu_\alpha)(D_\mu W_\nu - D_\nu W_\mu) \\
&= D^\gamma W_\alpha - D_\alpha W^\gamma
\end{aligned}$$

So the complete equation from Euler-Lagrange is

$$m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} - D_\mu (D^\mu W_\alpha - D_\alpha W^\mu) = 0 \tag{0.2}$$

This is a set of four coupled second order equations for the field W. We rewrite as a set of first order equations by introducing a field $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$. So (0.2) becomes

$$m^2 W_\alpha - ie\lambda W^\mu F_{\alpha\mu} - D^\mu W_{\mu\alpha} = 0 \tag{0.3}$$

$W^{\mu\nu}$ is antisymmetric and so has six degrees of freedom, corresponding to six independent fields. Together with W^μ this represents a total of ten fields. However, upon examination only some of these fields are dynamic. The fields W^{0i} and W^i appear in the equations with time derivatives, while the fields W^{ij} and W^0 never do. So it is only necessary to consider the former six fields. So that these six fields all have the same dimension, we will define $\frac{W^{i0}}{m} = i\eta^i$.

We will now eliminate the extraneous fields and solve for $iD_0 W^i$, $iD_0 \eta^i$.

$$m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} - D_\mu (D^\mu W_\alpha - D_\alpha W^\mu) = 0 \tag{0.4}$$

Define $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$. Then

$$m^2 W_\alpha - ie\lambda W^\nu F_{\alpha\nu} - D^\mu W_{\mu\alpha} = 0 \tag{0.5}$$

To get the exact Hamiltonian of a bispinor, we can eliminate nondynamic fields.

First consider (0.5) with $\alpha = 0$.

$$m^2 W_0 - ie\lambda W^\nu F_{0\nu} - D^\mu W_{\mu 0} \tag{0.6}$$

$$m^2W_0 - ie\lambda W^j F_{0j} - D^j W_{j0} \tag{0.7}$$

Solve this for W_0

$$W_0 = \frac{1}{m^2}\left(ie\lambda W^j F_{0j} + D^j W_{j0}\right) \tag{0.8}$$

Now, consider (0.5) with $\alpha = i$

$$m^2W_i - ie\lambda W^\mu F_{i\mu} - D^\mu W_{\mu i} = 0 \tag{0.9}$$

$$m^2W_i - ie\lambda W^0 F_{i0} - D^0 W_{0i} - ie\lambda W^j F_{ij} - D^j W_{ji} = 0 \tag{0.10}$$

Using (0.8) we can replace W_0

$$m^2W_i - \frac{ie\lambda}{m^2}\left(ie\lambda W^\nu F_{0\nu} + D^j W_{j0}\right) F_{i0} - D^0 W_{0i} - ie\lambda W^j F_{ij} - D^j W_{ji} = 0 \tag{0.11}$$

Using $W_{ji} = D_j W_i - D_i W_j$

$$m^2W_i - \frac{ie\lambda}{m^2}\left(ie\lambda W^\nu F_{0\nu} + D^j W_{j0}\right) F_{i0} - D^0 W_{0i} - ie\lambda W^j F_{ij} - D^j(D_j W_i - D_i W_j) = 0 \tag{0.12}$$

Solve this for $D^0 W_{0i}$:

$$D^0 W_{0i} = m^2W_i - \frac{ie\lambda}{m^2}\left(ie\lambda W^\nu F_{0\nu} + D^j W_{j0}\right) F_{i0} - ie\lambda W^j F_{ij} - D^j(D_j W_i - D_i W_j) \tag{0.13}$$

To get a similar equation for W_i, consider

$$W_{i0} = D_i W_0 - D_0 W_i \tag{0.14}$$

Then

$$D^0 W_i = D_i W_0 - W_{i0} \tag{0.15}$$

$$D^0 W_i = D_i \frac{1}{m^2}\left(ie\lambda W^j F_{0j} + D^j W_{j0}\right) - W_{i0} \tag{0.16}$$

We now have equations that tell us the time evolution of a total of six fields: W_i and W_{i0}. We want to treat these as the components of some sort of bispinor. To that end, first define $\eta_i = -i/mW_{i0}$ so that we have a pair of fields with the same mass dimension and hermiticity. (Since $W_{i0} = D_i W_0 - D_0 W_i$ would pick up another minus sign under complex conjugation compared to W_i.) Going the other way $W_{i0} = im\eta_i$.

Since eventually we want a nonrelativistic expression, we should write the spatial components of four vectors as three vectors. To that end also write the components of the tensor $F_{\mu\nu}$ in terms of three vectors.

$$F_{0i} = E^i,\ F_{ij} = -\epsilon_{ijk} B^k$$

Regular spatial vectors are "naturally raised" while D_i is "naturally lowered". Then we can rewrite (0.13).

$$-imD_0\eta_i = m^2W_i + \frac{ie\lambda}{m^2}\left(ie\lambda W^j E^j + D^j W_{j0}\right) E^i - ie\lambda W^j \epsilon_{ijk} B^k - D^j(D_j W_i - D_i W_j) \tag{0.17}$$

$$iD_0\eta^i = -mW^i + \frac{ie\lambda}{m^3}\left(ie\lambda W^j E^j + D^j W_{j0}\right) E^i - \frac{ie\lambda}{m} W^j \epsilon_{ijk} B^k - \frac{1}{m} D_j(D_j W^i - D_i W^j) \tag{0.18}$$

And likewise (0.16) becomes

$$D^0 W_i = D_i \frac{1}{m^2}\left(-ie\lambda W^j E^j + imD^j \eta_j\right) - im\eta_i \tag{0.19}$$

$$iD^0 W^i = -iD_i \frac{1}{m^2}\left(-ie\lambda W^j E^j + imD_j \eta^j\right) + m\eta^i \tag{0.20}$$

$$iD^0 W^i = -\frac{1}{m^2} D_i e\lambda \mathbf{W}\cdot\mathbf{E} + \frac{1}{m} D_i \mathbf{D}\cdot\eta + m\eta^i \tag{0.21}$$

0.1.1 Spin identities

To obtain some Shrodinger like equation for a bispinor, we need to express the time derivative of the bispinor in terms of other operators which can be interpreted as the Hamiltonian: $i\partial_0\Psi = \hat{H}\Psi$. We have expressions for the time derivatives of W_i and η_i, but some mixing of the indices is involved. To write the Hamiltonian in a block form we can introduce spin matrices whose action will mix the components of the fields.

The spin matrix for a spin one particle can represented as:

$$(S^k)_{ij} = -i\epsilon_{ijk} \tag{0.22}$$

which leads to the following identities:

$$(\mathbf{a}\times\mathbf{v})_i = -i(\mathbf{S}\cdot\mathbf{a})_{ij}v_j \tag{0.23}$$

$$\{\mathbf{a}\times(\mathbf{a}\times\mathbf{v})\}_i = -(\{\mathbf{S}\cdot\mathbf{a}\}^2)_{ij}v_j \tag{0.24}$$

$$a_i(\mathbf{b}\cdot\mathbf{v}) = \{\mathbf{a}\cdot\mathbf{b}\,\delta_{ij} - (S^kS^\ell)_{ij}a^\ell b^k\}v_j \tag{0.25}$$

Using these identities

$$iD^0\mathbf{W} = -\frac{e\lambda}{m^2}\mathbf{D}(\mathbf{W}\cdot\mathbf{E}) + \frac{1}{m}\mathbf{D}(\mathbf{D}\cdot\boldsymbol{\eta}) + m\boldsymbol{\eta} \tag{0.26}$$

$$iD^0\mathbf{W} = -\frac{1}{m^2}\{\mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\}\mathbf{W} + \frac{1}{m}\{\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2\}\boldsymbol{\eta} + m\boldsymbol{\eta} \tag{0.27}$$

and for the other equation

$$iD_0\eta^i = -mW^i + \frac{ie\lambda}{m^3}\left(ie\lambda W^jE^j + D^jW_{j0}\right)E^i - \frac{ie\lambda}{m}W^j\epsilon_{ijk}B^k - \frac{1}{m}D_j(D_jW^i - D_iW^j) \tag{0.28}$$

$$iD_0\eta^i = -mW^i + \frac{ie\lambda}{m^3}\left(ie\lambda W^jE^j + D^jW_{j0}\right)E^i - \frac{ie\lambda}{m}W^j\epsilon_{ijk}B^k - \frac{1}{m}D_j(D_jW^i - D_iW^j) \tag{0.29}$$

$$iD_0\boldsymbol{\eta} = -m\mathbf{W} - \frac{e^2\lambda^2}{m^3}\mathbf{E}(\mathbf{W}\cdot\mathbf{E}) - \frac{1}{m^2}\mathbf{E}(\mathbf{D}\cdot\boldsymbol{\eta}) - \frac{ie\lambda}{m}(\mathbf{W}\times\mathbf{B}) + \frac{1}{m}\mathbf{D}\times(\mathbf{D}\times\mathbf{W}) \tag{0.30}$$

$$iD_0\boldsymbol{\eta} = -m\mathbf{W} - \frac{e^2\lambda^2}{m^3}\{\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\}\mathbf{W} - \frac{1}{m^2}\{\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\}\boldsymbol{\eta} - \frac{e\lambda}{m}(\mathbf{S}\cdot\mathbf{B})\mathbf{W} - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2\mathbf{W} \tag{0.31}$$

$$iD_0\boldsymbol{\eta} = \left(-m - \frac{e^2\lambda^2}{m^3}\{\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\} - \frac{e\lambda}{m}(\mathbf{S}\cdot\mathbf{B}) - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2\right)\mathbf{W} - \frac{1}{m^2}\{\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\}\boldsymbol{\eta} \tag{0.32}$$

Now that the equations are in the correct form, we can write them as follows:

$$iD_0\begin{pmatrix}W\\ \eta\end{pmatrix} = \begin{pmatrix} -\frac{1}{m^2}\{\mathbf{D}\cdot\mathbf{E} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\} & m + \frac{1}{m}\{\mathbf{D}^2 - (\mathbf{S}\cdot\mathbf{D})^2\} \\ -m - \frac{e^2\lambda^2}{m^3}\{\mathbf{E}^2 - (\mathbf{S}\cdot\mathbf{E})^2\} - \frac{e\lambda}{m}(\mathbf{S}\cdot\mathbf{B}) - \frac{1}{m}(\mathbf{S}\cdot\mathbf{D})^2 & -\frac{1}{m^2}\{\mathbf{E}\cdot\mathbf{D} - (\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\mathbf{D})\} \end{pmatrix} \tag{0.33}$$