Infinite Graphs and Randomness

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***Abstract*—This survey explores foundational concepts in both finite and infinite random graphs, emphasizing probabilistic constructions, key structural and spectral properties, and applications to modeling large-scale dynamic networks such as the World Wide Web. We begin with the classical Erdős–Rényi model $\mathcal{G}(n, p)$, detailing threshold phenomena, adjacency axioms like n-existential closure, and examples of global graph properties. Next, we introduce quasi-randomness and its equivalent combinatorial and spectral characterizations, illustrated by explicit algebraic constructions of Paley graphs. Finally, we transition to countable infinite graphs, focusing on the Rado (infinite random) graph: its back-and-forth construction, uniqueness, universality, and connections to model theory and dynamic graph processes.**

***Index Terms*—random graphs, Erdős–Rényi, quasi-randomness, Paley graphs, Rado graph, infinite graphs, existential closure, back-and-forth**

I. Introduction

Graphs model complex systems in mathematics, computer science, physics, and social networks. Formally, a *graph* $G = (V, E)$ consists of a vertex set V and an edge set $E \subseteq \binom{V}{2}$; we focus on simple, undirected graphs without loops or multi-edges. Common definitions:

- **Order and Size:** $|V| = n$ is the *order*, $|E|$ the *size*.
- **Degree:** For vertex v, $\deg(v) = |\{u : \{u, v\} \in E\}|$.
- **Induced Subgraph:** For $U \subseteq V$, the subgraph $G[U]$ has vertex set U and all edges of E with both endpoints in U.
- **Complement:** $\overline{G} = (V, \binom{V}{2} \setminus E)$.
- **Path and Cycle:** A path is a sequence of distinct vertices with consecutive edges; a cycle is a closed path.
- **Diameter:** The maximum distance (length of shortest path) between any two vertices.
- **Connectivity:** A graph is connected if every pair of vertices is joined by a path.

II. Basic Graph Theory Definitions

Before diving into advanced concepts, we establish some additional fundamental definitions:

- **Clique:** A complete subgraph where every pair of vertices is adjacent.
- **Independent Set:** A set of vertices with no edges between them.
- **Automorphism:** A graph isomorphism from a graph to itself.
- **Vertex-Transitivity:** All vertices are equivalent under automorphisms.
- **Edge-Transitivity:** All edges are equivalent under automorphisms.
- **Girth:** Length of the shortest cycle in the graph.
- **Chromatic Number:** Minimum number of colors needed for a proper vertex coloring.

This paper covers:

1) Finite random graphs ($\mathcal{G}(n, p)$), threshold phenomena, and adjacency axioms.
2) Quasi-random graphs: spectral and combinatorial characterizations, Paley graphs.
3) Infinite random graphs: the Rado graph, back-and-forth construction, uniqueness, and universality.
4) Applications to evolving networks and open directions.

III. Finite Random Graphs: The Erdős–Rényi Model

A. Definition and Distribution

Let $V = [n] = \{1, 2, \ldots, n\}$ and fix $p \in [0, 1]$. The Erdős–Rényi model $\mathcal{G}(n, p)$ selects each of the $\binom{n}{2}$ possible edges independently with probability p. For any fixed labelled graph G on n vertices:

$$\Pr[G] = p^{|E(G)|}(1-p)^{\binom{n}{2} - |E(G)|}.$$

B. Threshold Phenomena

A property P has a *threshold* $p_c(n)$ if:

$$\lim_{n\to\infty} \Pr[\mathcal{G}(n, p) \models P] = \begin{cases} 0, & p \ll p_c(n), \\ 1, & p \gg p_c(n). \end{cases}$$

Classic thresholds:

- Connectivity: $p_c(n) = \frac{\log n}{n}$.
- Emergence of a Giant Component: $p_c(n) = \frac{1}{n}$.
- Containment of fixed H: $p_c(n) = n^{-1/m(H)}$, where $m(H) = \max_{H' \subseteq H} e(H')/v(H')$.

Proofs use first/second moment methods and sharp-threshold results (e.g. Friedgut–Kalai).

IV. Adjacency Axioms and Existential Closure

A. n-Existentially Closed Graphs

A graph $G = (V, E)$ is *n-existentially closed* (n-e.c.) if for every disjoint $A, B \subset V$ with $|A \cup B| = n$, there exists $z \in V \setminus (A \cup B)$ with z adjacent to all of A and to none of B. This property captures several important aspects:

- **Local Diversity:** Every possible local connection pattern appears.
- **Extension Property:** Any partial adjacency pattern extends to a vertex.
- **Universality:** n-e.c. graphs contain all small subgraphs.
- **Homogeneity:** High symmetry in local structures.

The n-e.c. property strengthens with increasing n:

- 1-e.c.: Both neighbors and non-neighbors exist for each vertex.
- 2-e.c.: Common neighbors and non-neighbors exist for each vertex pair.
- Higher n: Increasingly complex local patterns are guaranteed.

B. Asymptotic Satisfaction and Construction

For $\mathcal{G}(n,p)$ with constant $p \in (0,1)$, we can prove n-e.c. via:

1) For fixed k, consider all $\binom{n}{k}2^k$ possible (A,B) pairs.
2) Each desired extension occurs with probability $p^{|A|}(1-p)^{|B|}$.
3) Union bound over failure probability $\approx n^k(1-p^{|A|}(1-p)^{|B|})^{n-k}$.
4) This tends to 0 as $n \to \infty$ for fixed k.

V. Cartesian Products of Graphs

For graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, the Cartesian product $G \square H$ has vertex set $V_G \times V_H$. Edges:

$$\begin{aligned}(g_1,h_1) \sim (g_2,h_2) \iff \; & \big(g_1 = g_2 \wedge h_1 \sim_H h_2\big) \\ \vee \; & \big(h_1 = h_2 \wedge g_1 \sim_G g_2\big)\end{aligned}$$

Properties:

- If G is k_G-regular and H is k_H-regular, then $G \square H$ is $(k_G + k_H)$-regular.
- $\mathrm{diam}(G \square H) = \mathrm{diam}(G) + \mathrm{diam}(H)$.
- Useful for constructing high-dimensional grid graphs and product codes.

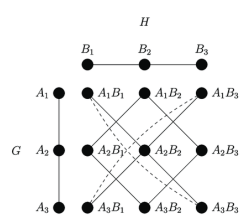

Fig. 1. Illustration of the Cartesian product $G \square H$ of two graphs. Vertices are of the form (g,h) with adjacency inherited from G and H in each coordinate.

VI. Quasi-Random Graphs

A. Motivation and Intuition

Quasi-random graphs capture the essence of random graph properties deterministically. They exhibit behavior statistically similar to $\mathcal{G}(n,p)$ despite being constructed explicitly. Key insights:

- Local patterns appear in expected proportions
- Edge distribution is uniform across vertex subsets
- Eigenvalue spectrum resembles that of random graphs
- Small subgraph frequencies match random expectations

B. Equivalent Characterizations

A sequence (G_n) on n vertices with edge-density p is *quasi-random* if it satisfies these equivalent properties:

1) Spectral Properties:

- Principal eigenvalue: $\lambda_1 = pn + o(n)$
- Spectral gap: $\max_{i\geq 2}|\lambda_i| = o(n)$
- Expander-like mixing properties
- Fast random walk mixing time

2) Combinatorial Properties:

1) **Subgraph Counts:** For each fixed graph H:
$$N_s(H,G_n) = (1+o(1))p^{e_H}n^{v_H}$$
where $N_s(H,G)$ counts subgraphs isomorphic to H
2) **Edge Distribution:** For all $S \subset V(G_n)$:
$$|e(S) - p\binom{|S|}{2}| = o(n^2)$$
3) **Codegree Regularity:** For most vertex pairs (u,v):
$$|\{w : w \sim u \text{ and } w \sim v\} - p^2 n| = o(n)$$
4) **Cut Properties:** For all partitions (A,B) of $V(G_n)$:
$$|e(A,B) - p|A||B|| = o(n^2)$$

C. Construction Methods

Several methods exist for building quasi-random graphs:

- **Paley Graphs:** Based on quadratic residues
- **Cayley Graphs:** From carefully chosen generating sets
- **Ramanujan Graphs:** Optimal spectral expansion
- **Random Cayley Graphs:** Almost surely quasi-random

VII. Example: Paley Graphs

Let $q \equiv 1 \pmod 4$ be a prime power. The Paley graph $P(q)$ on $\mathbb{F}_q$ joins distinct x, y iff $x - y$ is a nonzero square. Properties:

- $P(q)$ is $(q-1)/2$-regular and self-complementary.
- Eigenvalues: $\{(q-1)/2, (-1 \pm \sqrt{q})/2\}$.
- Pseudorandom subgraph counts match $\mathcal{G}(q, 1/2)$ up to lower-order terms.

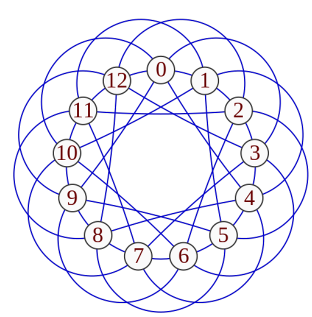

Fig. 2. Paley graph $P(9)$ on $\mathbb{F}_9$.

VIII. Infinite Random Graphs: The Rado Graph

A. Dynamic Web Motivation

Dynamic, user-generated networks grow without a fixed size. The Rado graph models an unbounded evolving network with rich local structure and symmetry.

B. Construction and e.c. Property

Build $G_0 = K_1$, then for $t \geq 0$ add v_{t+1} joining to G_t with probability p. The resulting R_p satisfies the existential-closure property: any finite pattern of adjacency/non-adjacency is realized by some new vertex.

C. Back-and-Forth Construction

The back-and-forth method proves uniqueness of countable homogeneous structures:

1) Setup: Given two countable graphs G, H satisfying the e.c. property:

- Enumerate vertices: $G = \{a_0, a_1, \dots\}$, $H = \{b_0, b_1, \dots\}$
- Maintain partial isomorphism $f_t : A_t \to B_t$
- Alternately extend domain and range

2) Algorithm: At step t:

1) If t even: Select least unused a_i
    - Find b_j matching a_i's adjacencies to A_t
    - Extend f_t by mapping $a_i \mapsto b_j$
2) If t odd: Select least unused b_j
    - Find a_i matching b_j's adjacencies to B_t
    - Extend f_t by defining $f_t^{-1}(b_j) = a_i$
3) The e.c. property ensures required vertices exist

3) Properties: This construction yields:

- A full isomorphism $f = \bigcup_t f_t$
- Proof of uniqueness up to isomorphism
- Demonstration of homogeneity
- Verification of universality

D. Universality

Any countable graph H embeds into R_p via an inductive process: map vertices one by one, using e.c. to ensure correct adjacency.

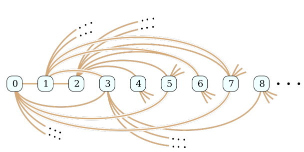

Fig. 3. Back-and-forth schematic for the Rado graph.

IX. Conclusion and Future Directions

Random and quasi-random finite graphs, together with infinite models like the Rado graph, offer complementary insights: probabilistic thresholds, deterministic pseudo-randomness, and logical universality. Applications include network evolution and sparse graph limits. Open questions: limits of growing quasi-random sequences, and model-theoretic properties of dynamic networks.

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