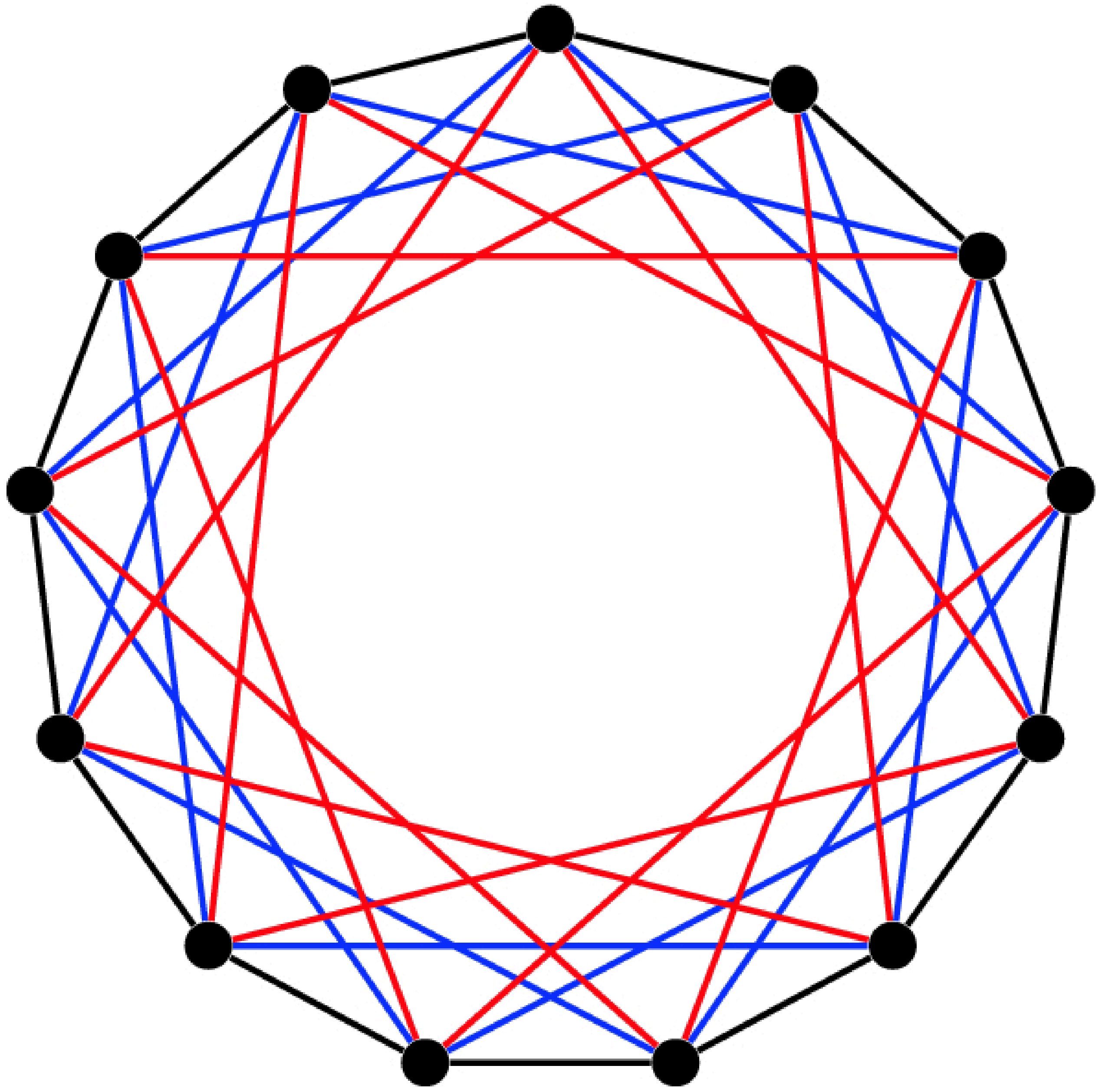
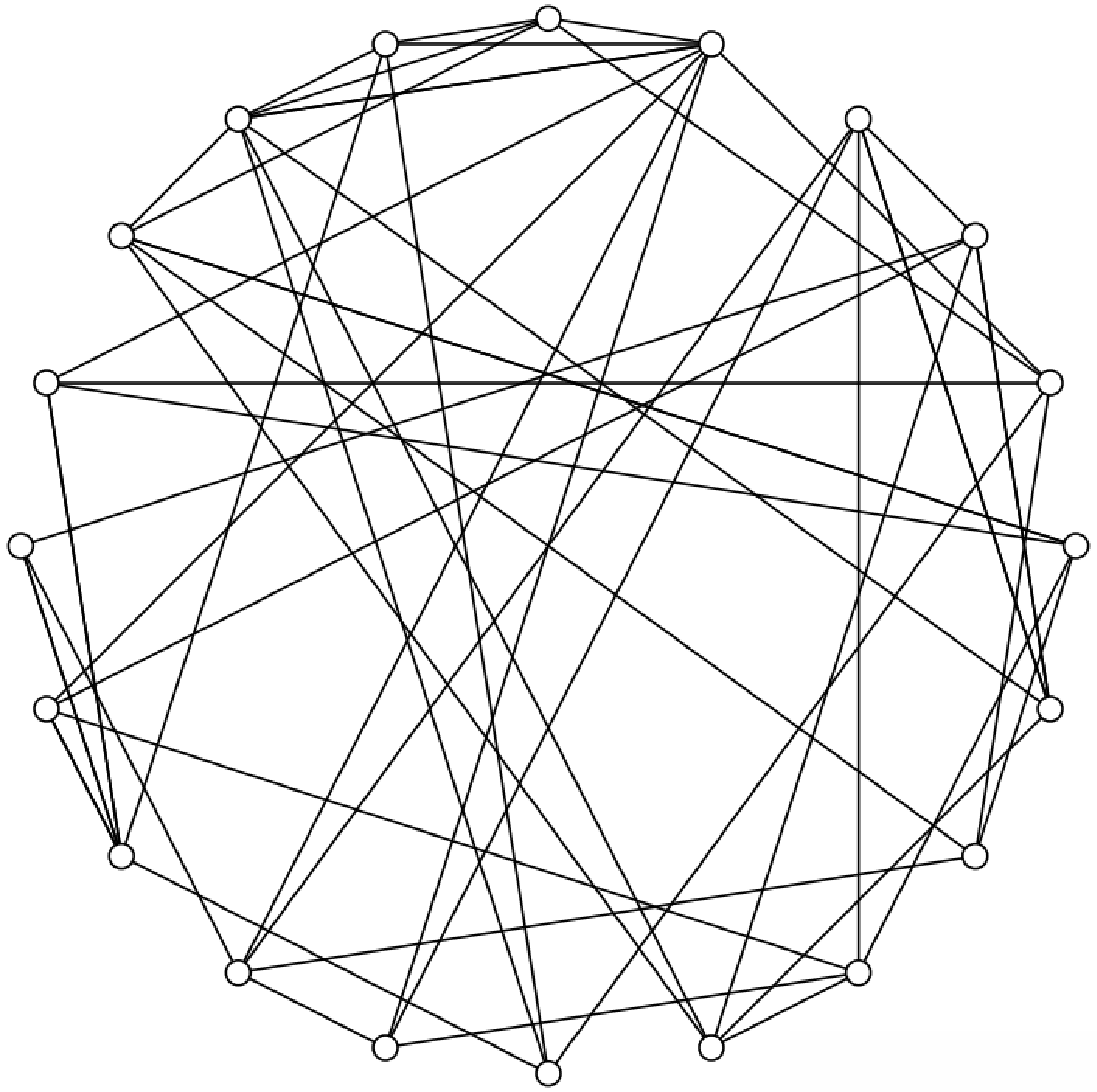
Infinite Graphs and Randomness





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Random Graphs

3.2. What is a Random Graph?

We may define a probability space on graphs of a given order $n \geq 1$ as follows. Fix a vertex set V consisting of n distinct elements, usually taken as $[n] = \{1, 2, \ldots, n\}$, and fix $p \in [0, 1]$. Define the space of *random graphs of order n with edge probability p*, written $G(n, p)$, with sample space equalling the set of all $2^{\binom{n}{2}}$ (labelled) graphs with vertices V, and

$$\mathbb{P}(G) = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}.$$

Graph Property:

A property preserved by isomorphism;
for example, being planar and possessing Hamilton cycles

A.A.S: Asymptotically almost surely

We say that G element of G(n, p) satisfies Property, P asymptotically almost surely (or a. a. s. for short) if when n tends to infinity Probability(G element of G(n, p) satisfies P) tends to 1.

In this case we say any random G element of G(n, p) satisfies P with high probability (w.h.p)

Adjacency Properties:

Let us look into a few properties which are asymptotically satisfied for a random graph.

An adjacency property is a global property of a graph asserting that
for every set S of vertices of some fixed type,
there is a vertex joined to some of the vertices of S in a prescribed way

n-e.c Adjacency Property

A graph is n-existentially closed or n-e.c., if
for all disjoint sets of vertices A and B with |A U B| = n (one of A or B can be empty),
there is a vertex z not in A U B joined to each vertex of A and no vertex of B.

Cartesian Product of Graphs

The Cartesian product of two graphs $G=(V_G,E_G)$ and $H=(V_H,E_H)$ denoted as $G\square H$, is a graph whose vertex set and edge set are defined as follows:

- Vertex Set:
- $V(G\square H)=V_G\times V_H$
- This means each vertex in $G\square H$ is a pair (g,h) where $g\in V_G$ and $h\in V_H$.
- Edge Set:
- Two vertices (g_1, h_1) and (g_2, h_2) in $G\square H$ are adjacent if and only if:
  - $g_1 = g_2$ and $(h_1,h_2)\in E_H$ (i.e., the second coordinate changes according to an edge in H), or
  - $h_1=h_2$ and $(g_1,g_2)\in E_G$ (i.e., the first coordinate changes according to an edge in G).

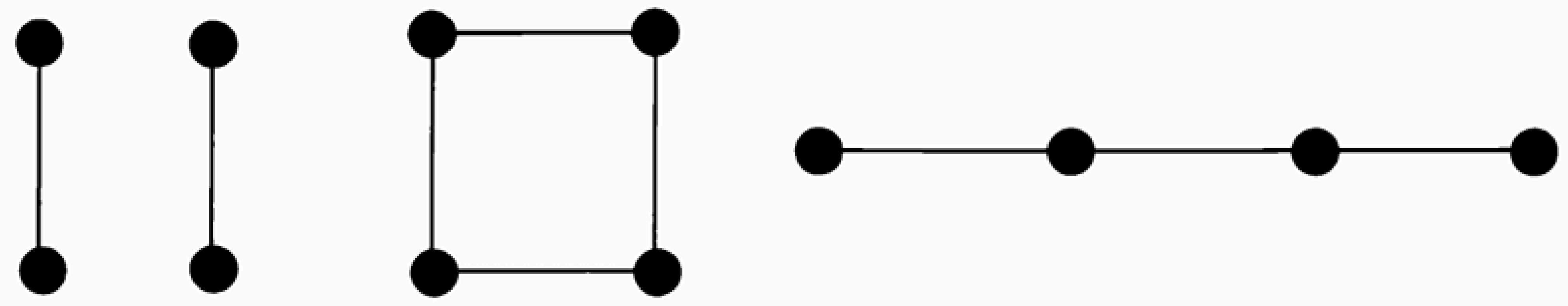


Figure 3.2. The 1-e.c. graphs of order 4.

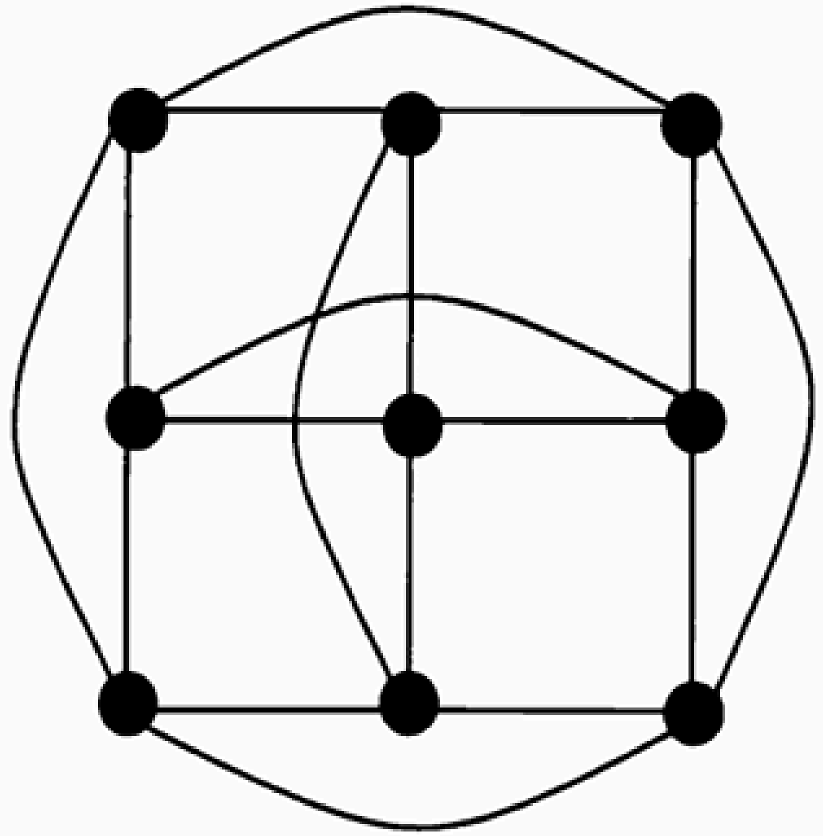
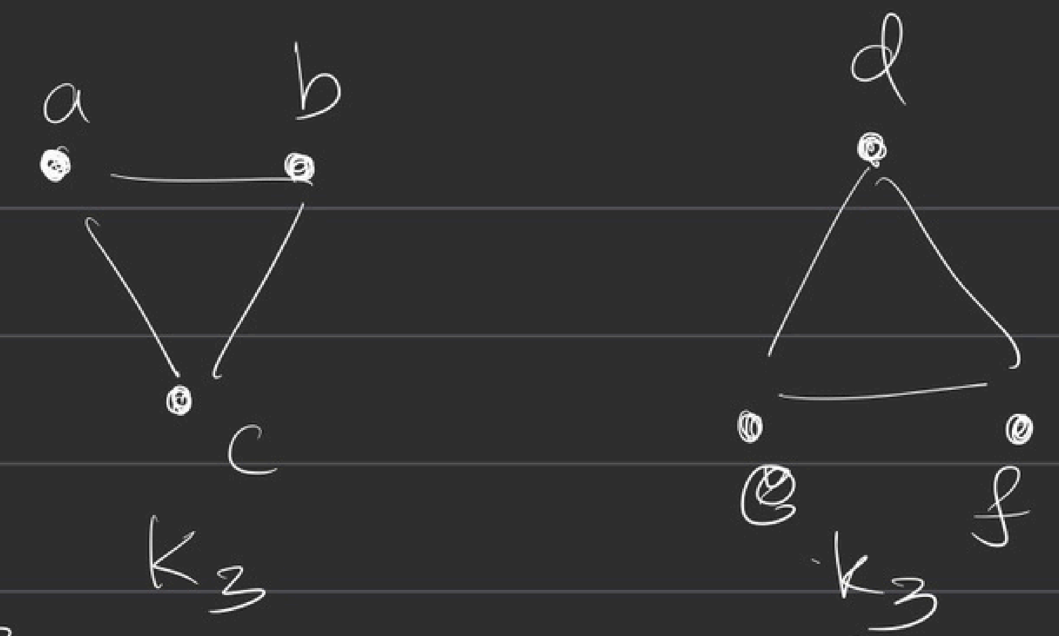


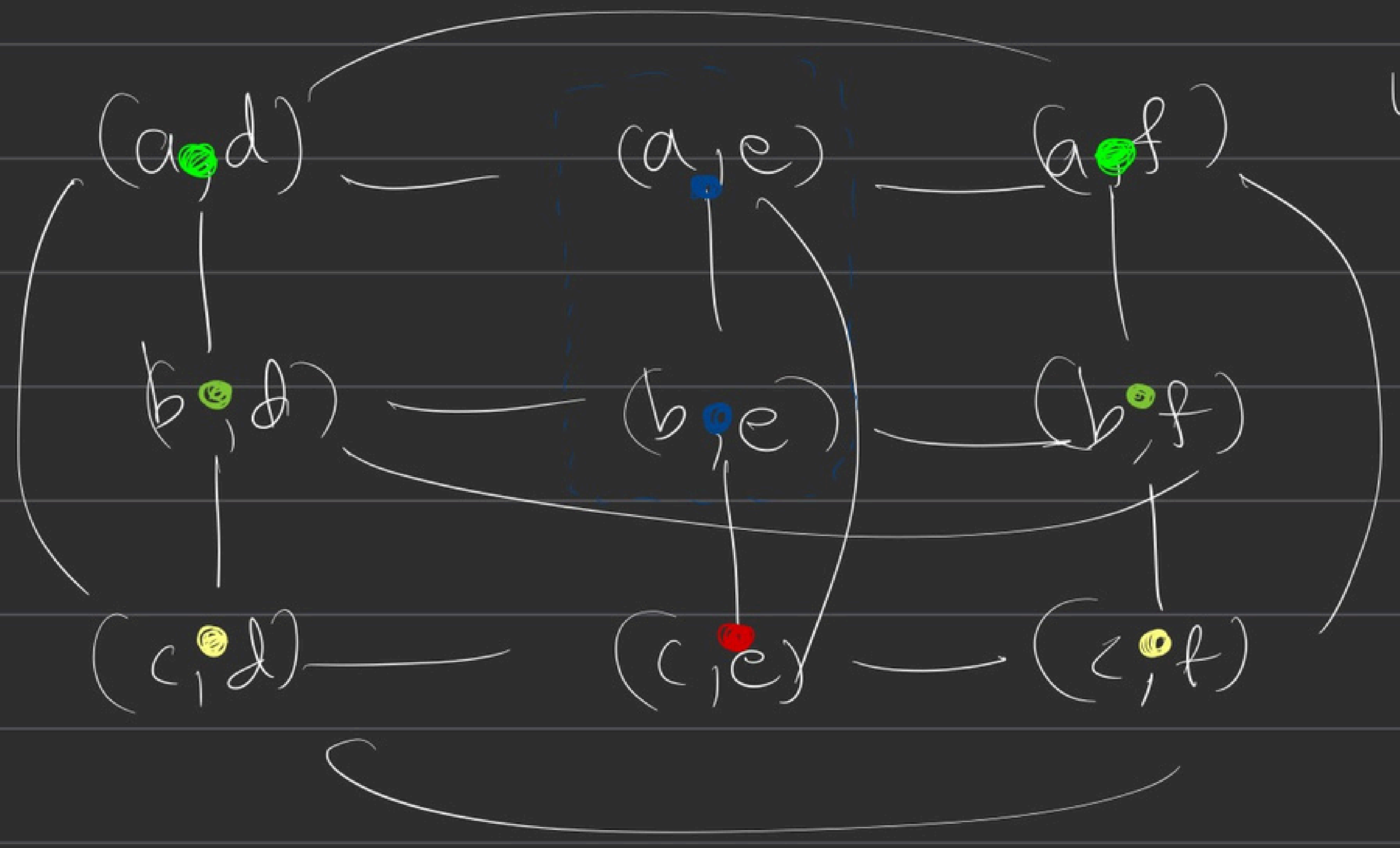
Figure 3.3. The graph $K_3 \Box K_3$.

$$20 \leq m_{ec}(3) \leq 28.$$



Cartesian product $K_3 \square K_3$

$m_{ec}(2) = 9$



we need $2^2 = 4$ vertices for each 2-set

Quasi Randomness

A graph is considered "quasi-random" if it exhibits many of the same properties as a truly random graph of the same size and edge density, even though it might not be generated by a truly random process.

Notations involved-Quasi Randomness

To each graph G of order n we may associate its n x n adjacency matrix A(G).

As A(G) is a real-symmetric matrix, it has n real eigenvalues

which are ordered by absolute value: $|\lambda 1| >= |\lambda 2| \ldots >= |\lambda n|$

Ns (H, G) be the number of labelled subgraphs of G isomorphic to H,

while Nis(H, G) is the number of labelled induced subgraphs of G isomorphic to H

For X C V_G, let e(X) is the number of edges in the subgraph induced by X on G.

For vertices u and v, define s(u, v) to be the set of vertices joined to both u and v or neither u nor v.

Quasi Randomness - Properties

Any graph that is quasi random satisfies a few properties

These properties all hold a.a.s. in G(n, 1/2)

Any deterministic graph that satisfies any of these properties satisfy all of them

(P1) For all $X \subseteq V(G_n)$

$$e(X) = \frac{1}{4}|X|^2 + o(n^2).$$

(P2) $e(G) \geq (1 + o(1))\frac{n^2}{4}$, and $N_S(C_4, G_n) \leq (1 + o(1))\frac{n^4}{16}$.

(P3) $e(G) \geq (1+o(1))\frac{n^2}{4}$, and for any fixed graph H of order $4 \leq t \leq n$,

$$N_{IS}(H, G_n) = (1 + o(1))n^t 2^{-\binom{t}{2}}.$$

Quasi Randomness - Properties

(P4) $\sum_{u,v \in V(G_n)} \left| |N(u) \cap N(v)| - \frac{n}{4} \right| = o(n^3)$.

(P5) $\sum_{u,v \in V(G_n)} \left| s(u,v) - \frac{n}{2} \right| = o(n^3)$.

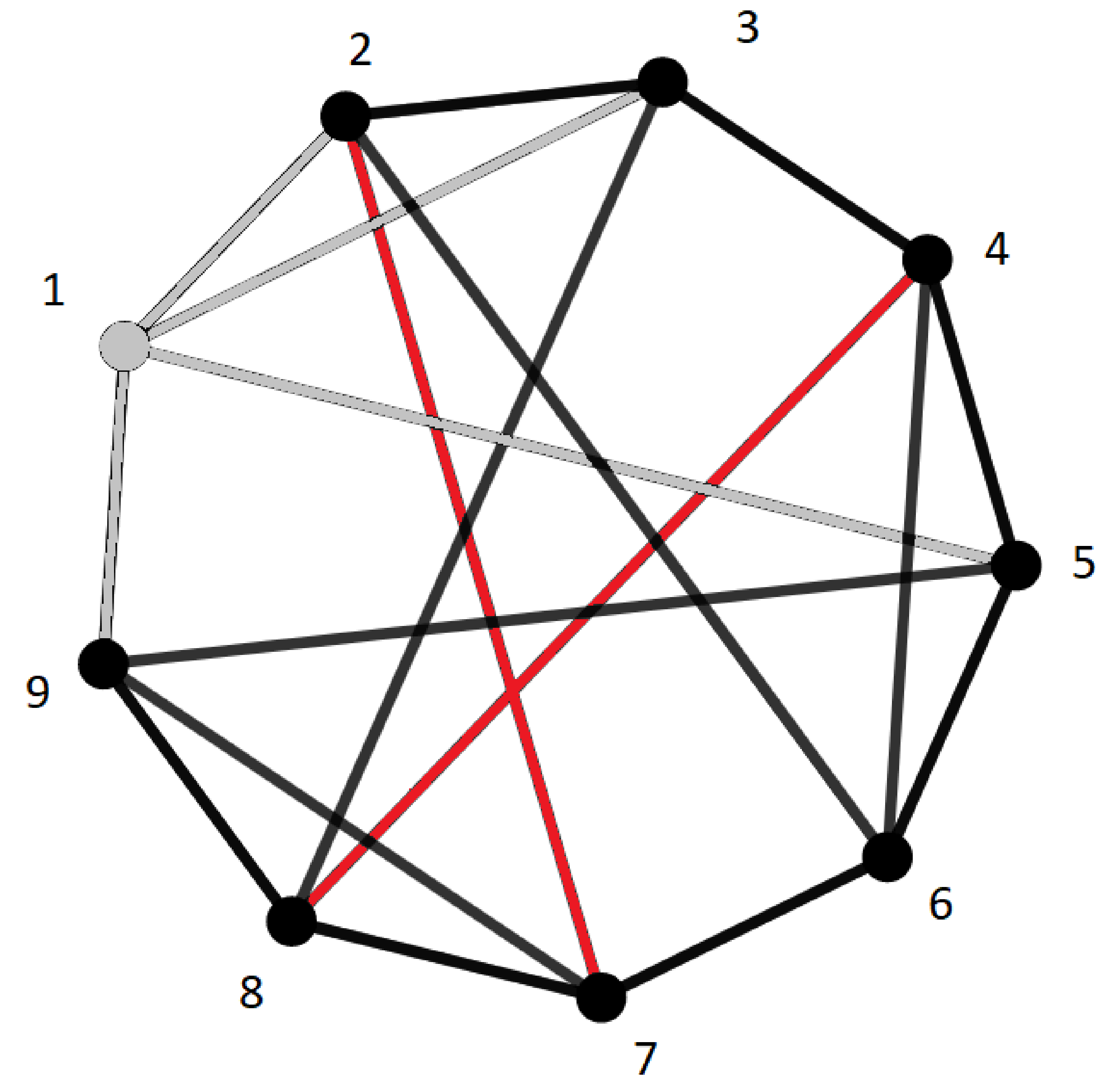
(P6) $e(G) \geq (1 + o(1))\frac{n^2}{4}$ and for $2 \leq i \leq n$

$$\lambda_1 = (1 + o(1))\frac{n}{2}, \ \lambda_n = o(n).$$

THEOREM 3.4 ([70]). *If G_n satisfies any one of the six properties above, then it satisfies all of them.*

Strongly Regular Graphs(SRG)

A k-regular graph G with v vertices, such that each pair of joined vertices has exactly A common neighbors, and each pair of non-joined vertices has exactly p common neighbors is called a strongly regular graph; we say that G is an SRG (v, k, A, p)



Example - Paley graphs

Paley Graphs

Quadratic Residue

A quadratic residue modulo an integer n (typically a prime p) is an integer a such that there exists a non-zero integer x satisfying

x^2≡a(modn).

In other words, a is a quadratic residue modulo n if it is congruent to a perfect square modulo n. If no such x exists, then a is called a quadratic non-residue.

For example consider n = 7. The set of quadratic residues is {1,2,4}, while the set of quadratic non-residues is {3,5,6}

QUADRATIC RESIDUES

$$6^2 = 36 \equiv 1 \ (mod\ 7)$$

$$5^2 = 25 \equiv 4 \ (mod\ 7)$$

$$3^2 = 9 \equiv 2 \ (mod\ 7)$$

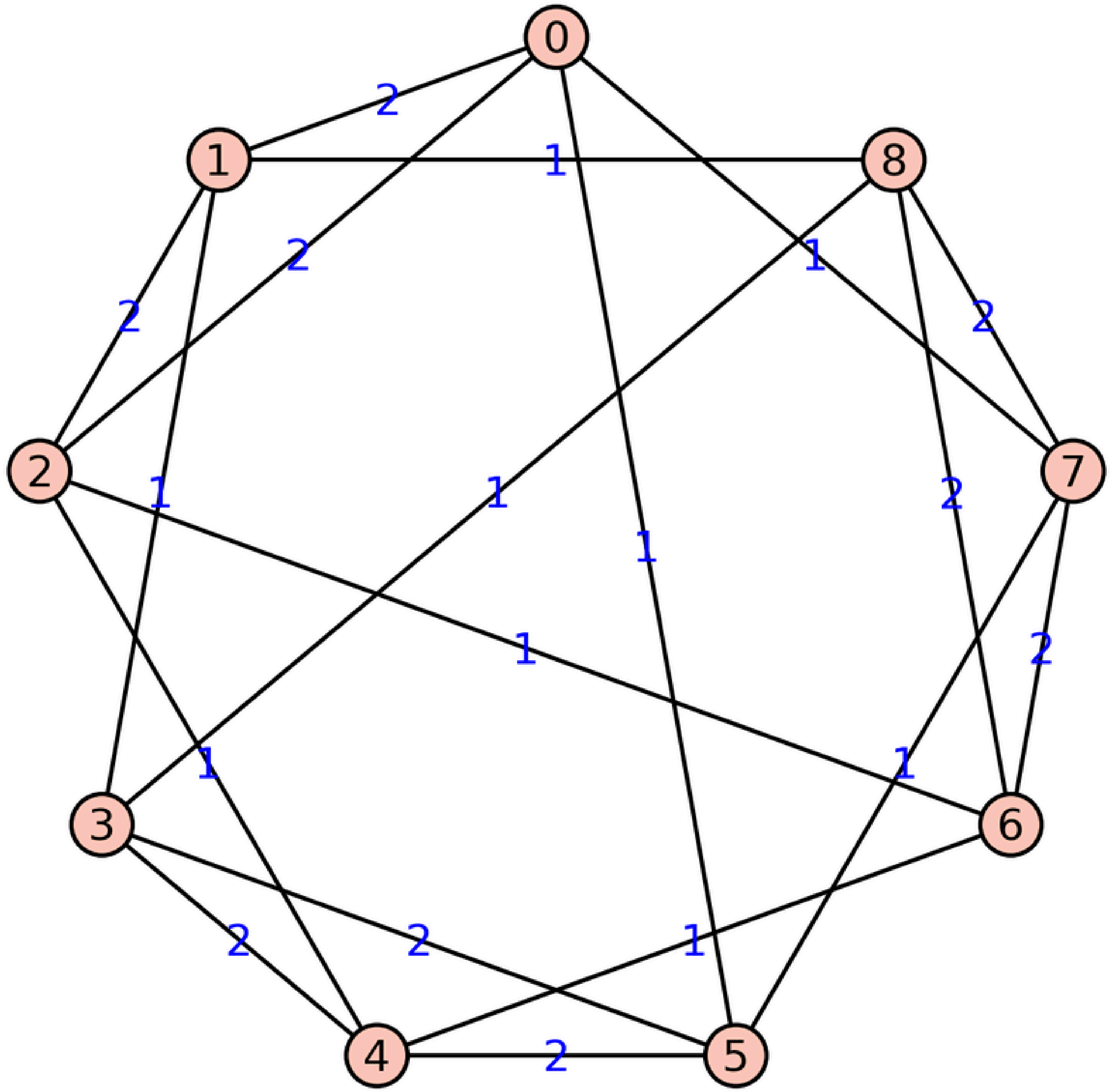
Paley Graphs

Vertex Set:

Choose a positive integer q (often a prime or a prime power) such that
$q \equiv 1 \pmod 4$.
Label the vertices with the numbers
$0,1,2,\ldots,q-1$.

Edge Set:

For any two distinct vertices i and j (considered modulo q), connect i and j with an edge if and only if the difference $(i-j)$ (computed modulo q) is a quadratic residue modulo q.



Paley Graph - Quasi Random

Paley Graphs are Quasi Random because they satisfy P5

$$(\mathrm{P5}) \quad \sum_{u,v \in V(G_n)} \left| s(u,v) - \frac{n}{2} \right| = o(n^3).$$

Proof:

Let us consider an element z of the field for it to be vertex contributing to s(u,v) it has to be satisfy:
either
1)z-u is a quadratic residue and z-v is a quadratic residue
or
2)z-u is not a quadratic residue and z-v is not a quadratic residue

Paley Graph - Quasi Random

Proof:

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either
1)z-u is a quadratic residue and z-v is a quadratic residue
or
2)z-u is not a quadratic residue and z-v is not a quadratic residue
We can include both these conditions in one expression, following term must be a quadratic residue

$$\frac{z-u}{z-v} = a \qquad \frac{z-u}{z-v} = 1 + \frac{v-u}{z-v} = a,$$

Here there exists a 1 to 1 relation between a and z and a is a square
No of squares in a the field is (q-1)/2 and we need to subtract 1 because a can not take value 1

Paley Graph - Quasi Random

Proof:

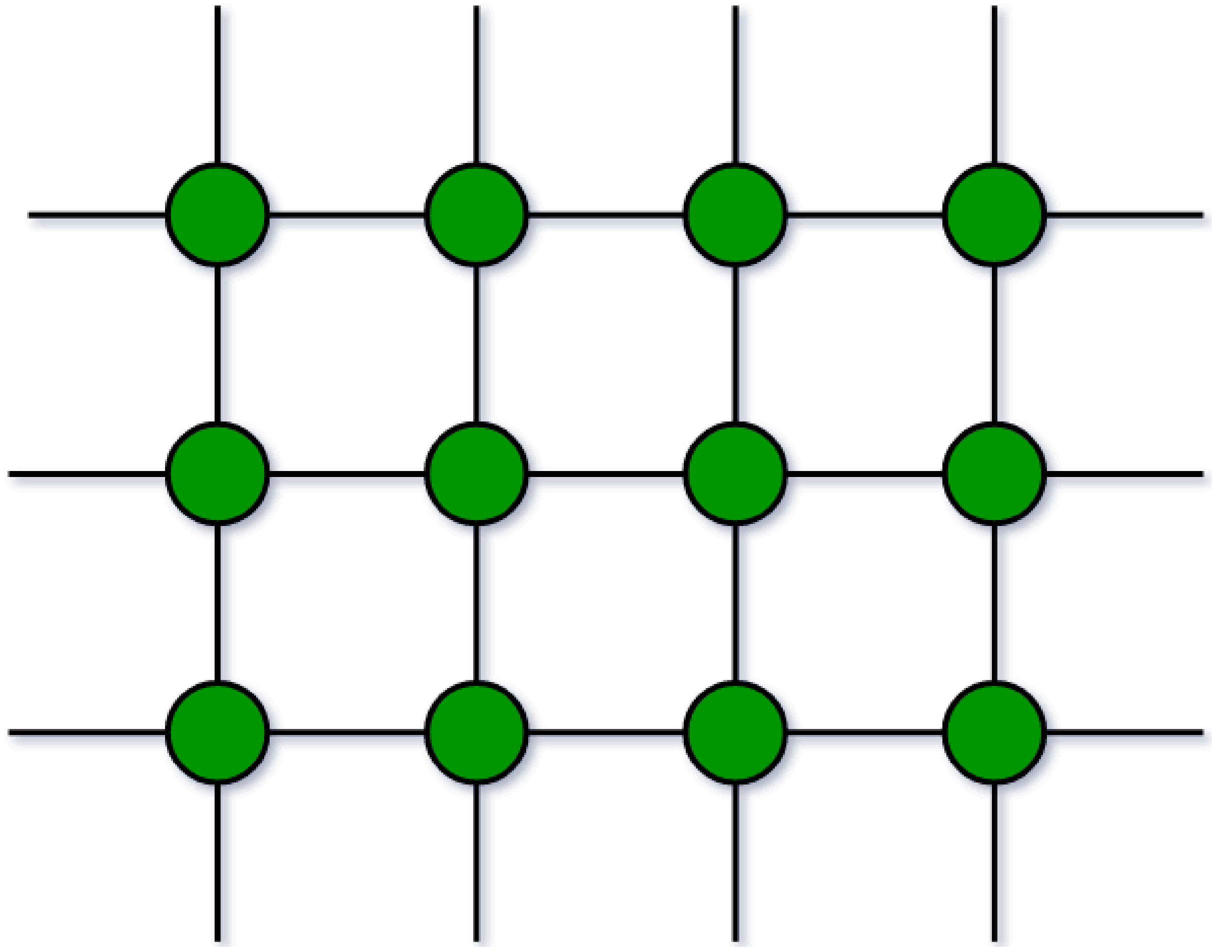
it follows that $s(u,v) = \frac{1}{2}(q-3)$. Hence,

$$\begin{aligned}\sum_{u,v \in V(G_n)} \left| s(u,v) - \frac{q}{2} \right| &= \sum_{u,v \in V(G_n)} \left| \frac{1}{2}(q-3) - \frac{q}{2} \right| \\ &= \binom{q}{2}\frac{3}{2} = \frac{3\left(q^2-q\right)}{4} \\ &= o(q^3). \quad \square\end{aligned}$$

THEOREM 3.7 ([**29, 38**]). *If* $q > n^2 2^{2n-2}$, *then* P_q *is* n-*e.c.*

THE INFINITE WEB

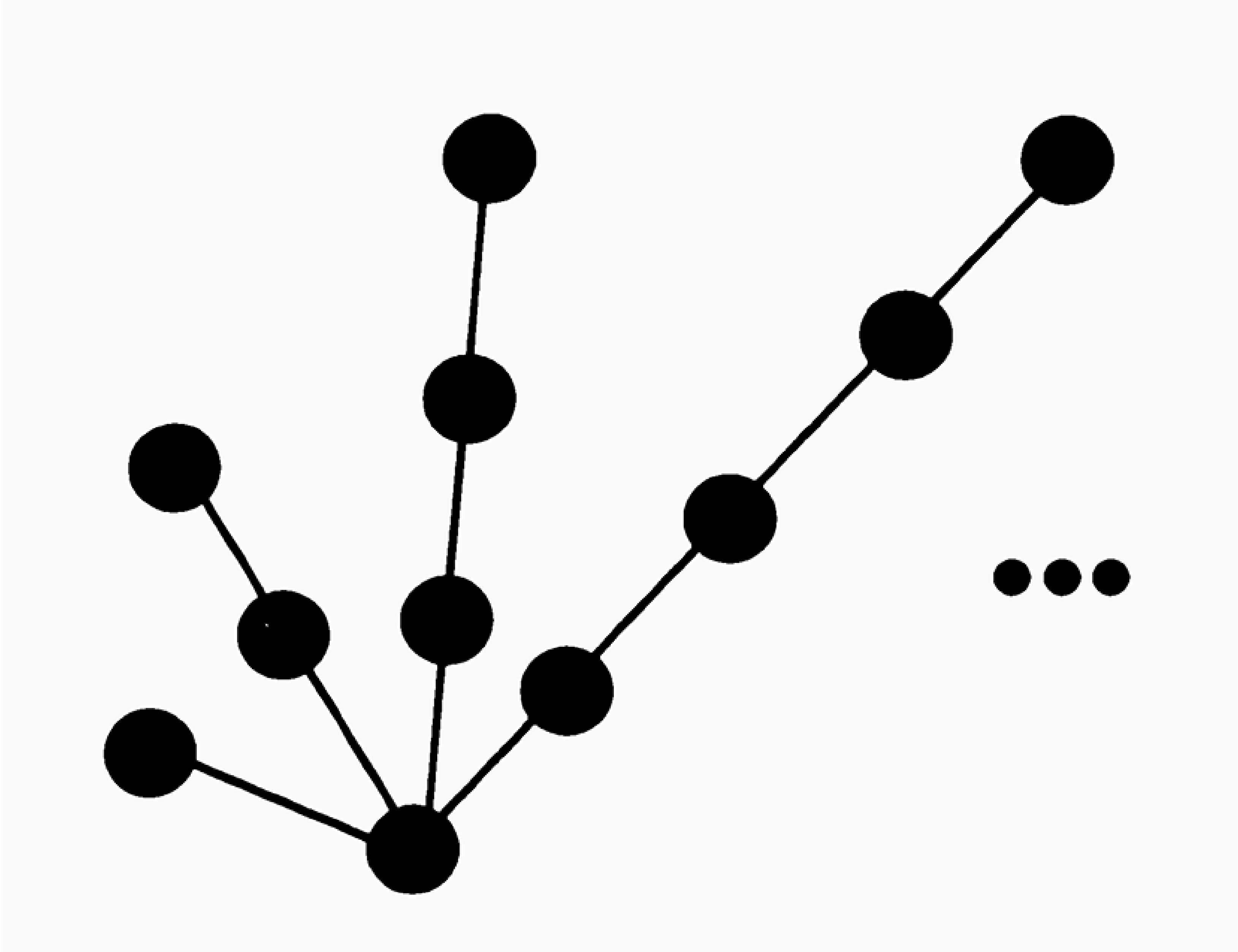
The **static web** consists of a finite number of pages (~54 billion), while the **dynamic web** can generate infinitely many pages based on user input or parameters. Thus, the web graph appears infinite in practice, though bounded by physical constraints.



The view of G as an infinite graph presents an interesting perspective for a mathematician

Considering a graph as infinite gives rise to many interesting properties.

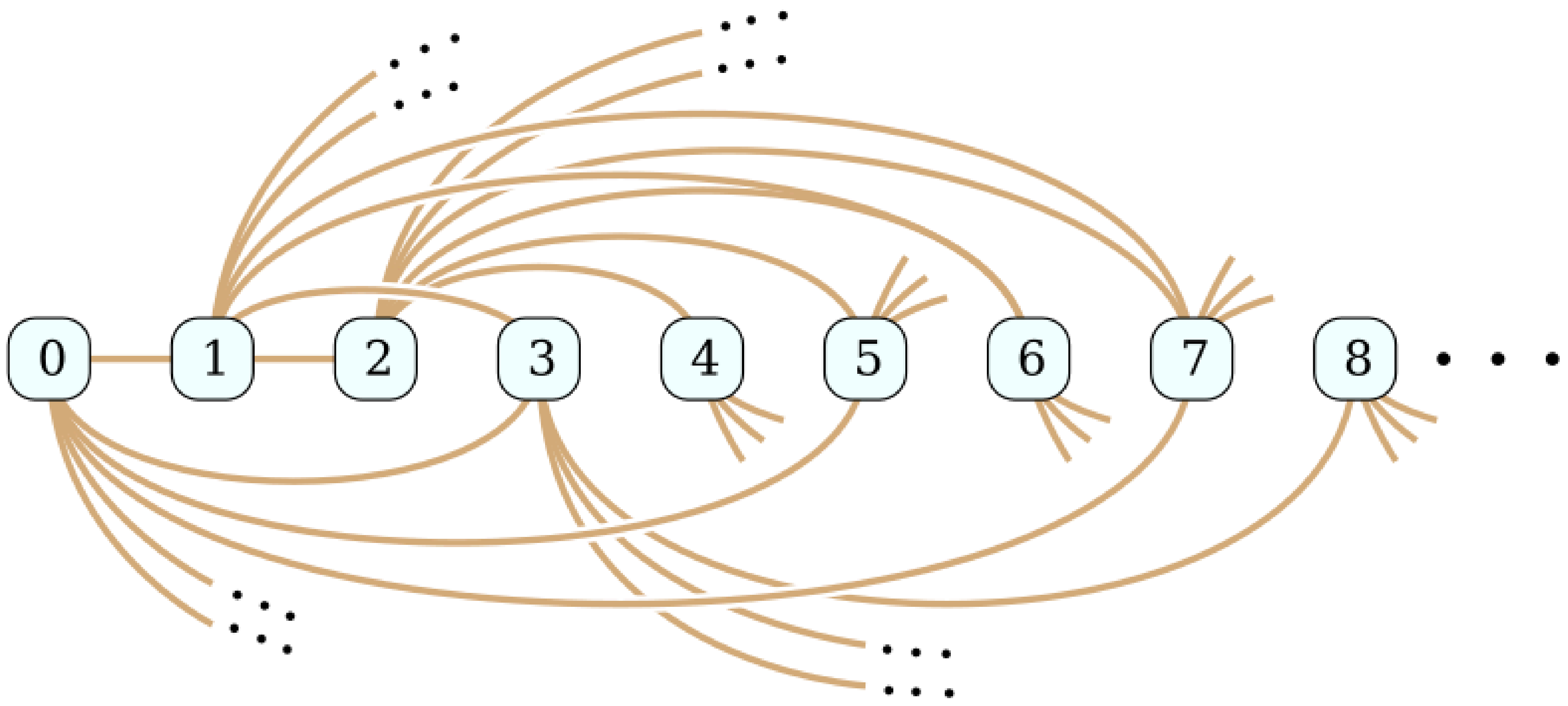
Consider the graph G, shown in the figure. To form G, attach a path of each finite length to a root vertex. The graph G has the property that there exists an infinite number of graphs H such that G and H are mutually embeddable ie - G <= H and H<= G.
This kind of property exists only for infinite graphs, as for finite graphs, observe that the only graph that is mutually embeddable with it, is the graph itself.



Infinite Random Graph

Consider the probability space G(N, p) consisting of graphs with V = N, the set of natural numbers in which every distinct pair of integers is joined independently with probability p.

The space G(N , p) is an infinite analogue of the finite random graph G(n, p)



Infinite Random Graph

THEOREM 6.1. *With probability* 1, *all* $G \in G(\mathbb{N}, p)$ *are isomorphic.*

All graphs in this space are mutually isomorphic, that is, structurally apart from a relabelling of vertices all graphs can considered the same

Proof: To prove this we will consider 2 other theorems and use them. We will also use a very unique and interesting method of proving which is called back-and-forth.

Infinite Graph R*

Start with a single-vertex graph R0, which is K1 (a graph with one vertex and no edges)

- For each non-negative integer t>0, construct the next graph Rt+1 from Rt as follows
- For every subset S of the vertex set V(Rt) (including the empty set), add a new vertex zS.

$$|V(R_{t+1})| = |V(R_t)| + 2^{|V(R_t)|}$$

Age

Let $J = \lim_{t\to\infty} H_t$ be a limit of a chain $C = (H_t : t \in \mathbb{N})$ of graphs, where $H_t \prec H_{t+1}$ for all $t \in \mathbb{N}$. Define ages: $\mathrm{aged}(x) = t$ for $x \in V(J)$ by

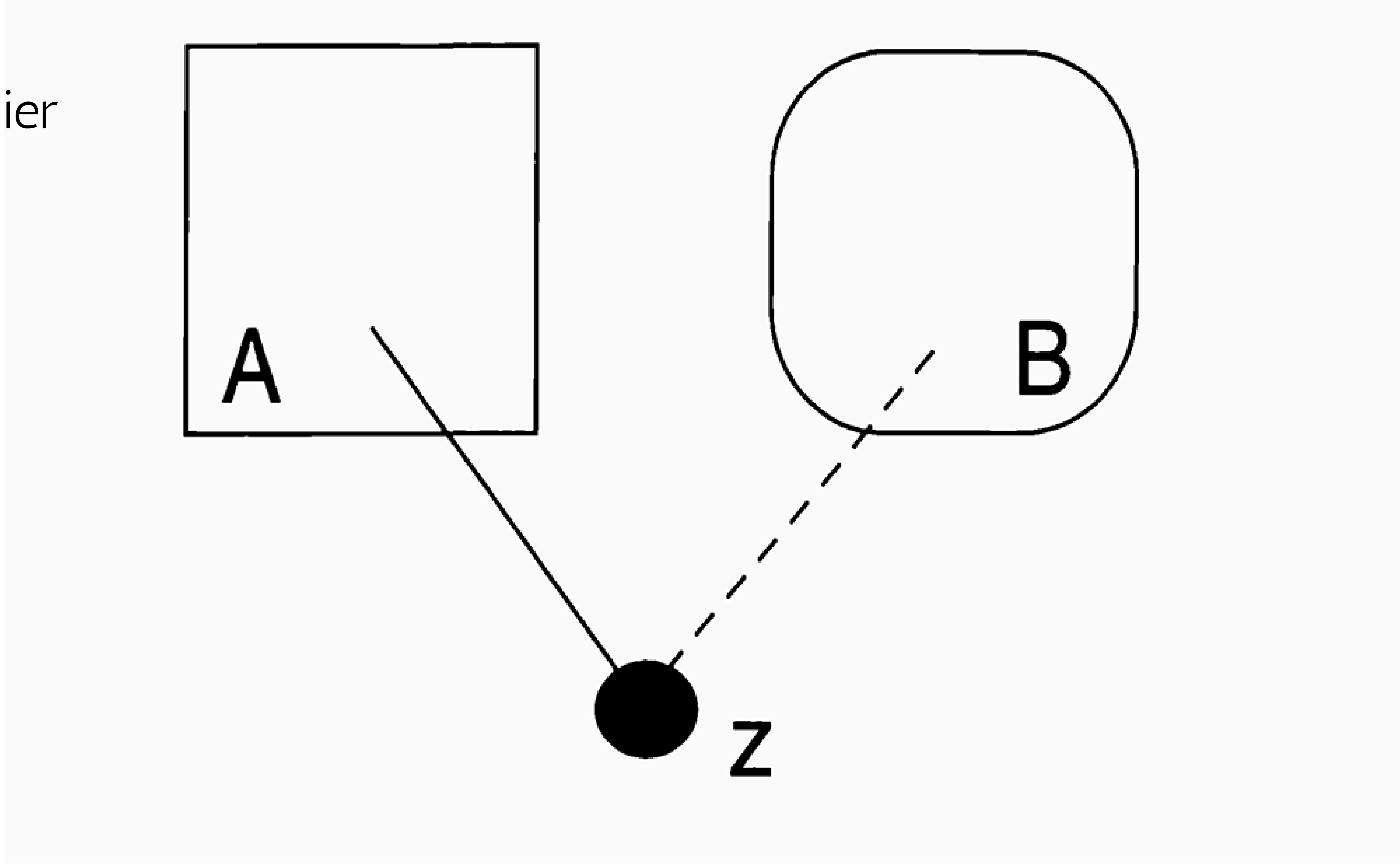
$$\mathrm{aged}(x) = \begin{cases} t & \text{if } x \in V(H_t) \setminus V(H_{t-1}) \text{ where } t > 0 \\ 0 & \text{else} \end{cases}$$

The age of a finite subset, written age(s), is max{age(x) : x E S}.

The e.c property

A graph G is existentially closed or e. c. if for all disjoint finite sets of vertices A and B (one of which may be empty), there is a vertex z joined to all of A and to no vertex of B.

This is very similar to the n. e. c. property that was earlier defined for finite graphs, and is a conjunction of those properties.



THEOREM 6.2. *The graph R^* is e.c.*

Proof: Fix finite disjoint A and B in V(R*). Let to = age(A U B). The vertex z = zA in Rto+1 is correctly joined(satisfies the ec property) to A and B.

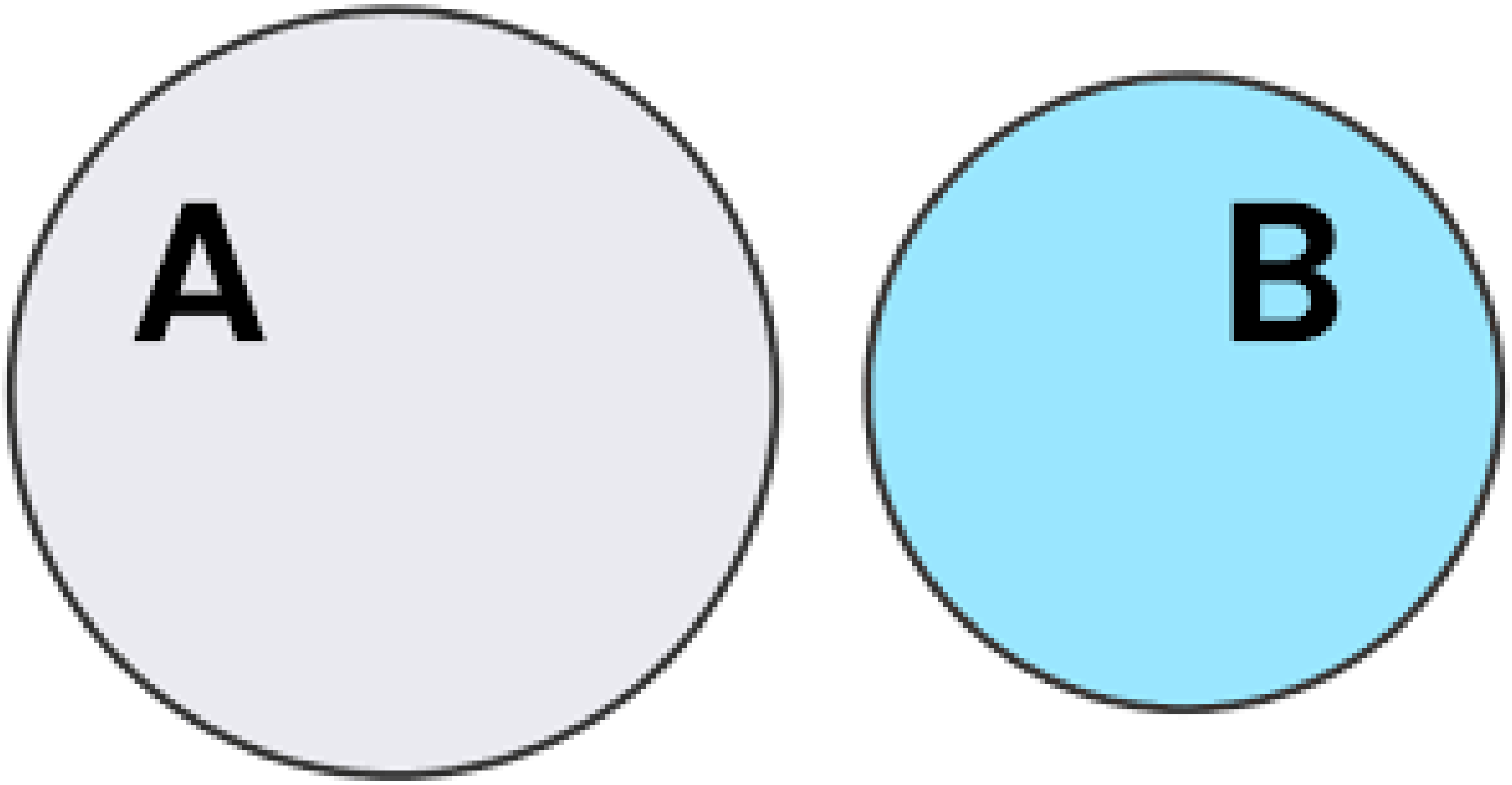
By the definition of age(x), we have that all the vertices that appear in the finite sets A and B are present in Rt0.

We may think of G(N, p) as the limit of an on-line random graph process, similar to how we defined R*. At time 0, let Xp be K1. Assuming a graph Xt at time t is defined and finite, at time t+1, add a new vertex z to form Xt+1 The vertex z is joined to each existing vertex independently with probability p. Then G(N, p) consists of limits limt-> infinity Xt.

Theorem 6.3. *With probability 1, $G \in G(\mathbb{N}, p)$ is e.c.*

Proof: We, consider two disjoint sets A and B and let |A| = i and |B| = j. The probability that an edge occurs is p, and denote q = 1-p as the probability that an edge does not occur.

Now we need to link it to the e .c property, so let us consider a vertex Z correctly joined to these two sets.



Key Idea:

What will be the probability of such a vertex z (not occurring) as the number of edges t tends to infinity?

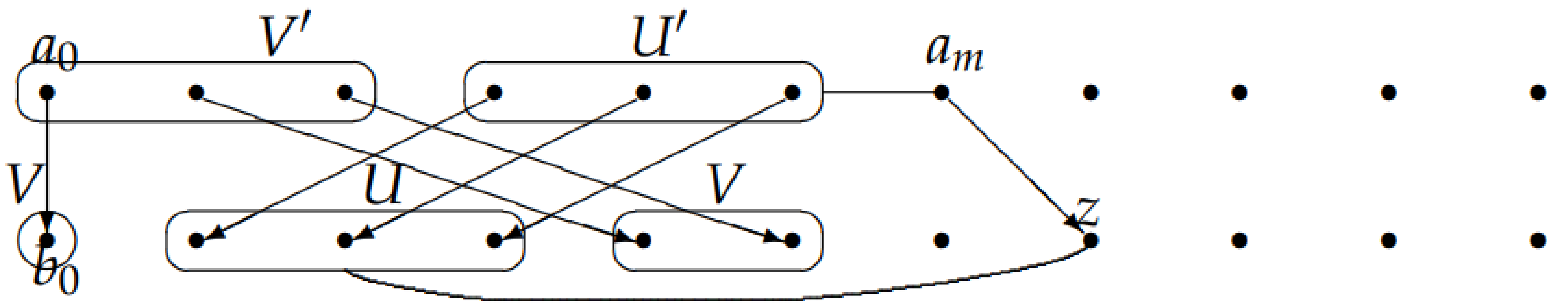
Back and Forth method

THEOREM 6.4. *If G and H are e.c. graphs, then $G \cong H$.*

We use a method known to logicians as "back and forth".
Suppose that $\Gamma 1$ and $\Gamma 2$ are countable graphs satisfying ($\ast$):
enumerate their vertex sets as (a0, a1, . . .) and (b0, b1, . . .). We build an isomorphism φ between them in stages.

Proof:

At stage 0, map a0 to b0. At even-numbered stages, let am the first unmapped ai . Let U` and V` be its neighbours and non-neighbours among the vertices already mapped, and let U and V be their images under φ. Use (∗) in graph Γ2 to find a witness v for U and V. Then map am to z.



At odd-numbered stages, go in the other direction, using (∗) in Γ1 to choose a pre-image of the first unmapped vertex in Γ2. This approach guarantees that every vertex of Γ1 occurs in the domain, and every vertex of Γ2 in the range, of φ

The graph R* is Universal

An existensially closed countable graph is universal in the sense every other countable graph embed in it.

Let G be any countable graph with vertices $V(G) = \{x_i : i \in \mathbb{N}\}$
Define G_t as the subgraph induced by vertices $\{x_i : 0 \leqslant i \leqslant t\}$, so $G = \lim_{t\to\infty} G_t$

Base case: $f_0: G_0 \to R$

Hypothesis: f_t exists

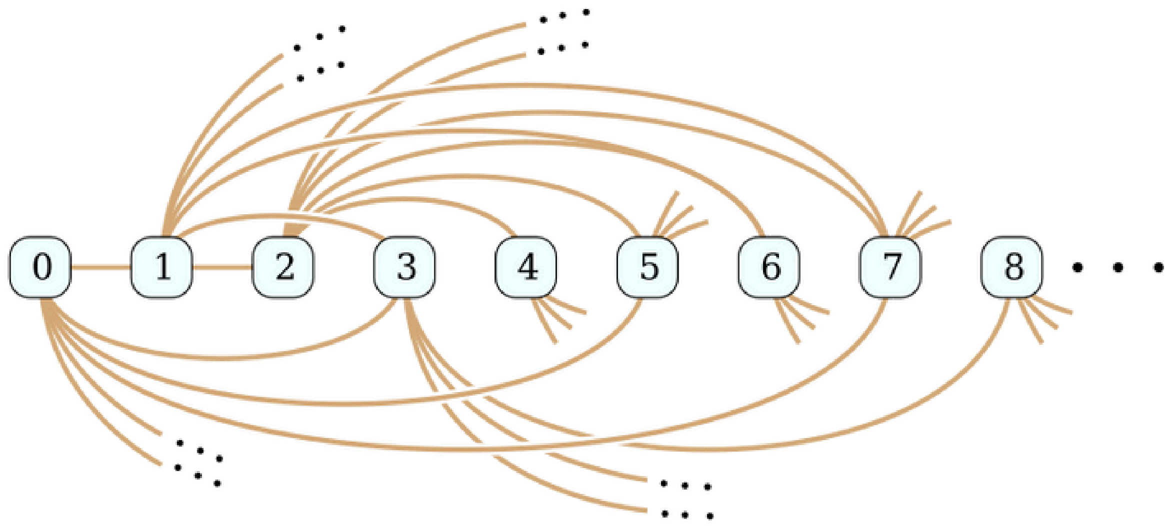
Induction:
Extend f_t to f_{t+1}, vertex x_{t+1} connects to some set S of vertices in G_t
By the e.c. property of R, find some z in R joined only to $f_t(S)$
Define f_{t+1} by extending f_t and mapping x_{t+1} to z

The limit mapping $F = \lim_{t\to\infty} f_t$ gives a complete embedding of G into R1

Countable infinite graph constructions:

Rado Graph

A Rado graph can be constructed by connecting each pair of distinct natural numbers i<j with an edge if the i-th bit in the binary expansion of j is 1.



$$A = \{i_1, \ldots, i_m\} \quad B = \{j_1, \ldots, j_n\}$$

$$i_1 < \cdots < i_m \text{ and } j_1 < \cdots < j_n$$

$$z = 2^{i_1} + \cdots + 2^{i_m} + 2^{j_n+1}.$$

THANK YOU !