6220HW1

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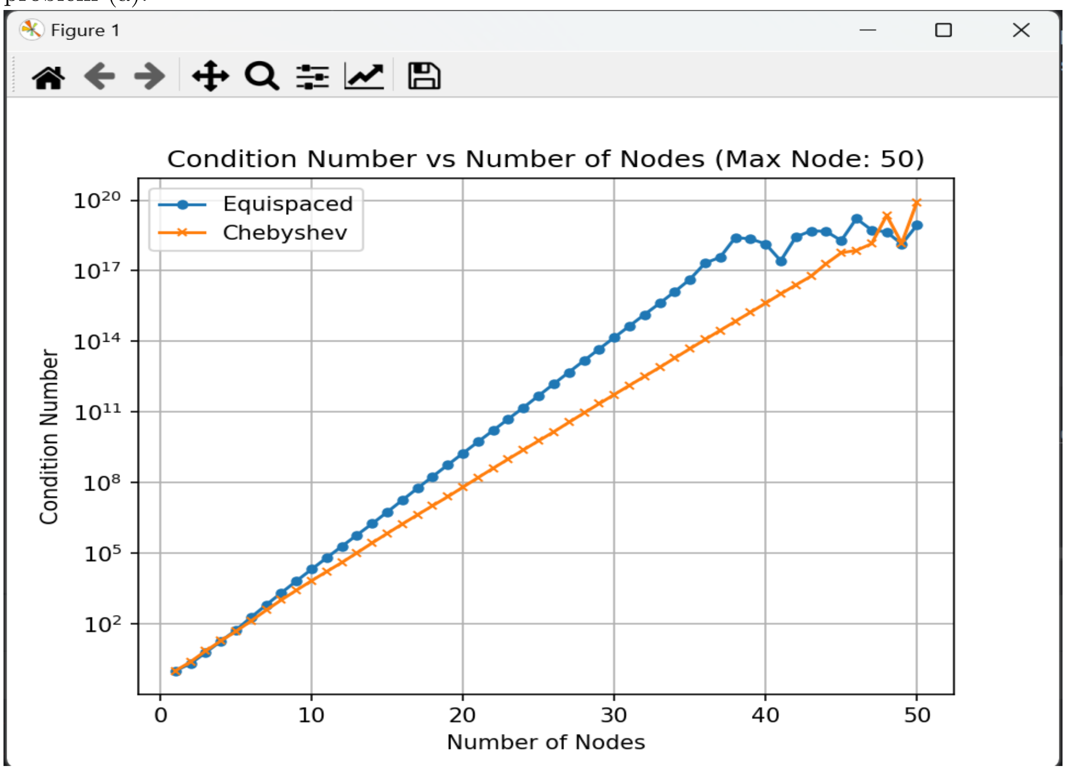
1 problem1

problem (a):



problem (b):

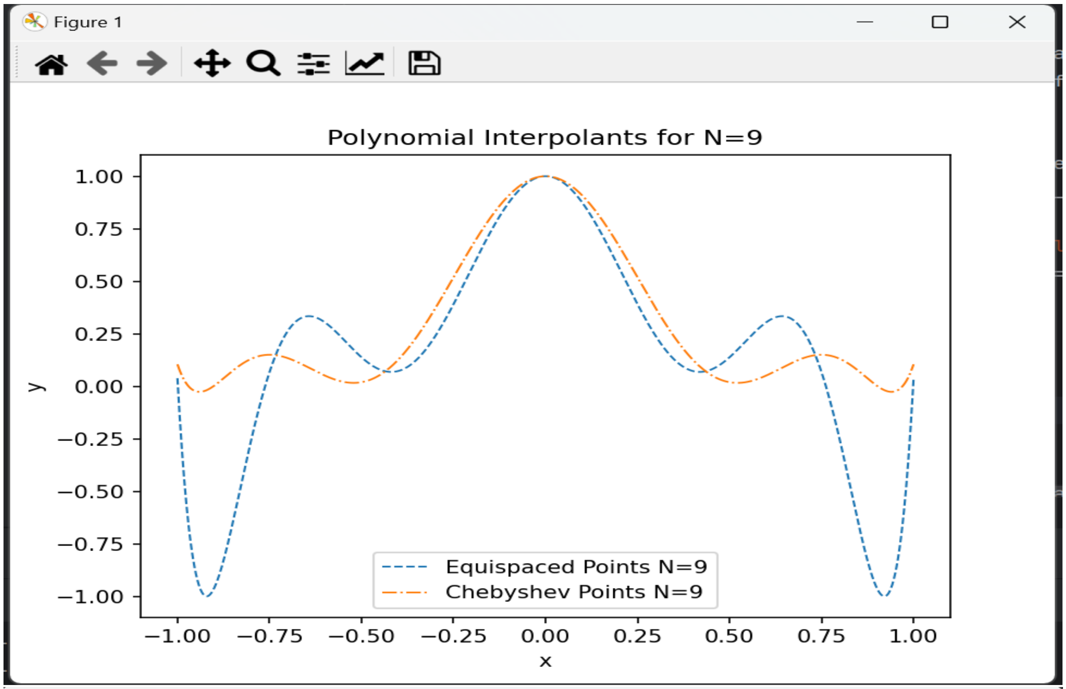


Figure 1
Polynomial Interpolants for N=9
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
x
Equispaced Points N=9
Chebyshev Points N=9

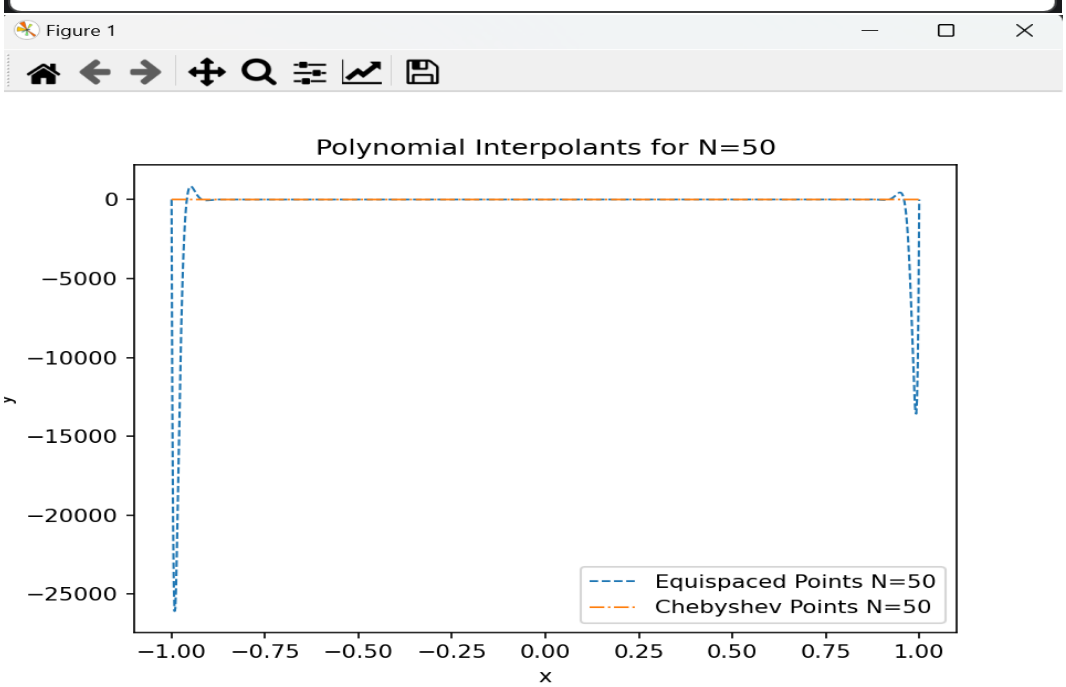


Figure 1
Polynomial Interpolants for N=50
0
−5000
−10000
−15000
−20000
−25000
y
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
x
Equispaced Points N=50
Chebyshev Points N=50

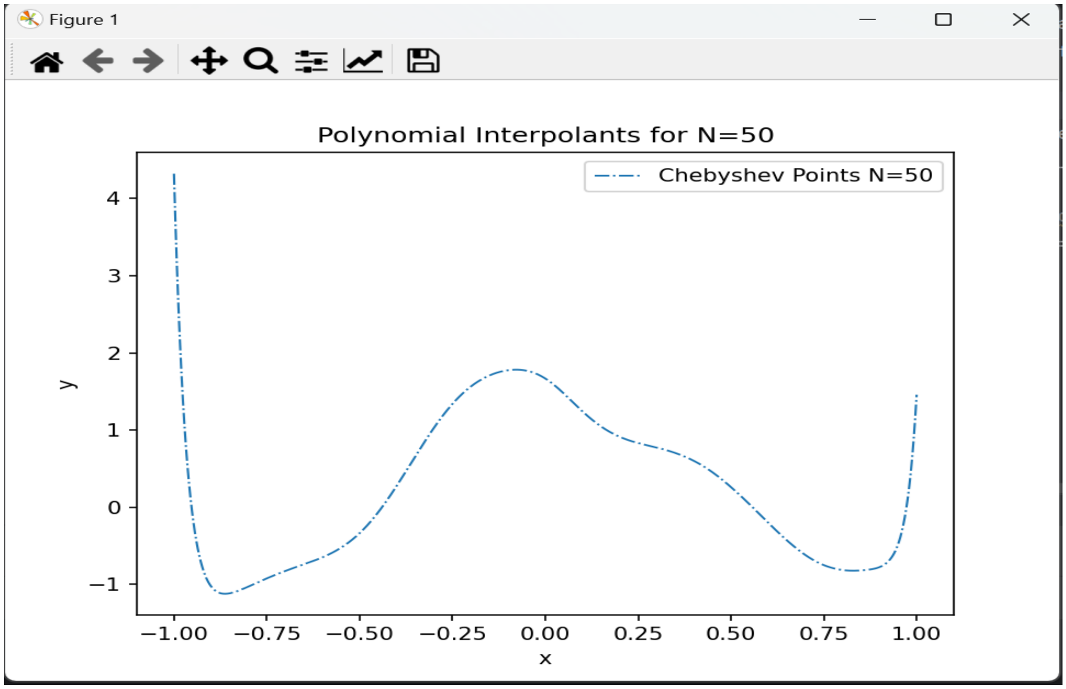


Figure 1
Polynomial Interpolants for N=50
Chebyshev Points N=50
y
x
4
3
2
1
0
−1
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00

problem (c):

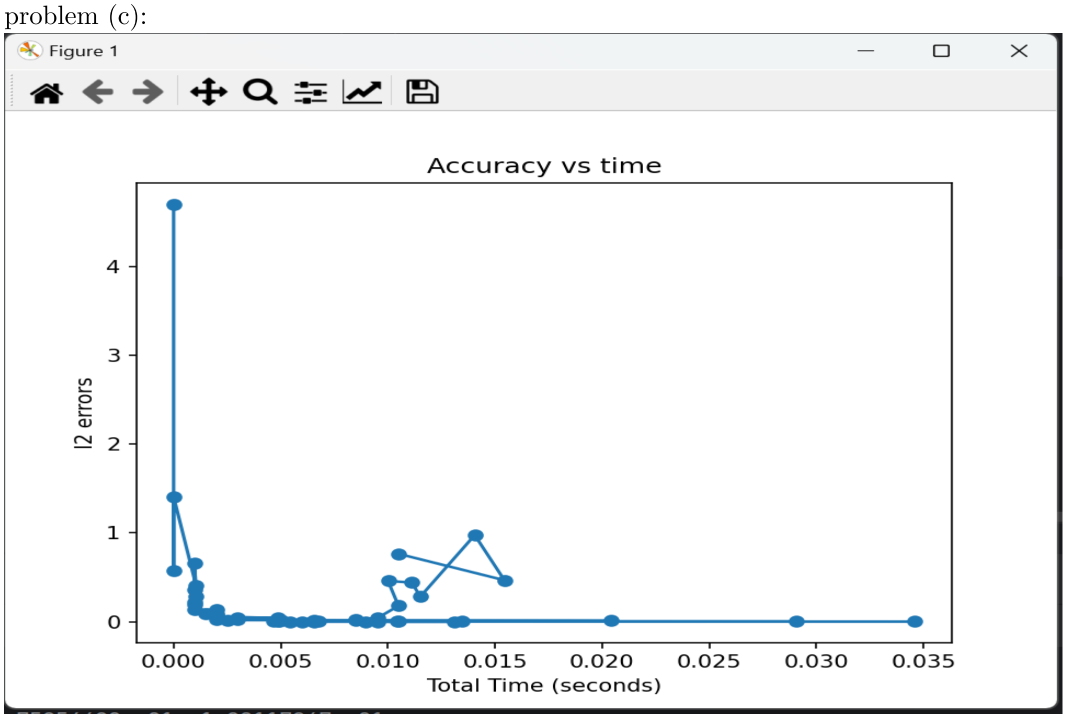


Figure 1
Accuracy vs time
l2 errors
Total Time (seconds)
4
3
2
1
0
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0.035

2 problem2

First, derive the Lagrange form polynomial:
as we know: $p''(x) = y_0 l_0''(x) + y_1 l_1''(x) + y_2 l_2''(x)$
and:
$l_0 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$
$l_1 = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$
$l_2 = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

By driving these three Lagrange basis functions:
$l_0' = \frac{\partial x(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$
$= \frac{(x-x_1)\partial x(x-x_2)+(x-x_2)\partial x(x-x_1)}{(x_0-x_1)(x_0-x_2)}$
$= \frac{x-x_2+x-x_1}{(x_0-x_1)(x_0-x_2)}$
$= \frac{2x-x_2-x_1}{(x_0-x_1)(x_0-x_2)}$
similarly, we can have:
$l_1' = \frac{2x-x_2-x_0}{(x_1-x_0)(x_1-x_2)}$
$l_2' = \frac{2x-x_1-x_0}{(x_2-x_0)(x_2-x_1)}$

By driving these three Lagrange basis functions again, we can get a second derivative:
$l_0'' = \frac{\partial x(2x-x_2-x_1)}{(x_0-x_1)(x_0-x_2)}$
$= \frac{2}{(x_0-x_1)(x_0-x_2)}$
similarly,we can have:
$l_1'' = \frac{2}{(x_1-x_0)(x_1-x_2)}$
$l_2'' = \frac{2}{(x_2-x_0)(x_2-x_1)}$

Therefore, $p''(x_1) = y0\frac{2}{(x_0-x_1)(x_0-x_2)} + y_1\frac{2}{(x_1-x_0)(x_1-x_2)} + y_2\frac{2}{(x_2-x_0)(x_2-x_1)}$
$= \frac{2y_0}{(-h)(-2h)} + \frac{2y_1}{(h)(-h)} + \frac{2y_2}{(2h)(h)}$
$= \frac{y_0}{h^2} - \frac{2y_1}{h^2} + \frac{y_2}{h^2}$
$= \frac{y_0-2y_1+y_2}{h^2}$

Then $f''(x_1) = \frac{y_0-2y_1+y_2}{h^2} + \mathcal{O}(h^2)$

Second, derive the error estimate.
$\mathcal{O}(h^2) = \frac{1}{3*2}f'''(\xi)(x_1-x_0)(x_1-x_2)$
$= \frac{1}{6}f'''(\xi)(x_1-x_0)(x_1-x_2)$
Therefore; we have
$f''(x_1) = \frac{y_0-2y_1+y_2}{h^2} + \frac{1}{6}f'''(\xi)(x_1-x_0)(x_1-x_2)$

3 problem3

First, since we have three points between a and b, we have:
$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx$
for the first term:
$\int_{x_0}^{x_1} f(x)dx \approx y_0 \int_{x_0}^{x_1} l_0(x)dx + y_1 \int_{x_0}^{x_1} l_1(x)dx$
$l_0 = \frac{x-x_1}{x_0-x_1}$
$l_1 = \frac{x-x_0}{x_1-x_0}$
$y_0 \int_{x_0}^{x_1} l_0(x)dx + y_1 \int_{x_0}^{x_1} l_1(x)dx = \frac{y_0}{x_0-x_1}(\frac{x^2}{2} - x_0 x)_{x_0}^{x_1} + \frac{y_1}{x_1-x_0}(\frac{x^2}{2} - x_1 x)_{x_0}^{x_1}$
$= \frac{x_1-x_0}{2} y_0 - \frac{x_0-x_1}{2} y_1$
$= \frac{1}{2}(x_1 - x_0)(y_0 + y_1)$

Similarly, for the second term, we have:
$\int_{x_1}^{x_2} f(x)dx \approx y_1 \int_{x_1}^{x_2} l_1(x)dx + y_2 \int_{x_1}^{x_2} l_2(x)dx$
$= \frac{1}{2}(x_2 - x_1)(y_1 + y_2)$ Therefore, we have:
$\int_a^b f(x)dx \approx \frac{1}{2}(x_1 - x_0)(y_0 + y_1) + \frac{1}{2}(x_2 - x_1)(y_1 + y_2)$
by substitute a, $(a + b)/2$, and b in to x_0,x_1,x_2:
$\int_a^b f(x)dx \approx \frac{1}{2}(\frac{b-a}{2})(y_0 + y_1) + \frac{1}{2}(\frac{b-a}{2})(y_1 + y_2)$
$\approx (\frac{b-a}{4})(y_0 + 2y_1 + y_2)$

Then, for the corresponding error estimate:
for the first term above:
$\int_{x_0}^{x_1} f(x)dx - y_0 \int_{x_0}^{x_1} l_0(x)dx + y_1 \int_{x_0}^{x_1} l_1(x)dx$
$\leq \frac{M_2}{2} \int_{x_0}^{x_1} e(x)dx$
$\leq \frac{M_2}{2} \int_{x_0}^{x_1} (x - x_0)(x - x_1)dx$
$\leq \frac{M_2}{12}(x_1 - x_0)^3$

similarly, for the second term above:
$\int_{x_1}^{x_2} f(x)dx - y_1 \int_{x_1}^{x_2} l_1(x)dx + y_2 \int_{x_1}^{x_2} l_2(x)dx$
$\leq \frac{M_2}{12}(x_2 - x_1)^3$
by substitute a, $(a + b)/2$, and b in to x_0,x_1,x_2:
add two error term up:
$\frac{M_2}{12}(x_1 - x_0)^3 + \frac{M_2}{12}(x_2 - x_1)^3$
$= \frac{M_2}{48}(b - a)^3$
Therefore,
$\int_a^b f(x)dx - \int_a^b p(x)dx \leq \frac{M_2}{48}(b - a)^3$