6220HW1

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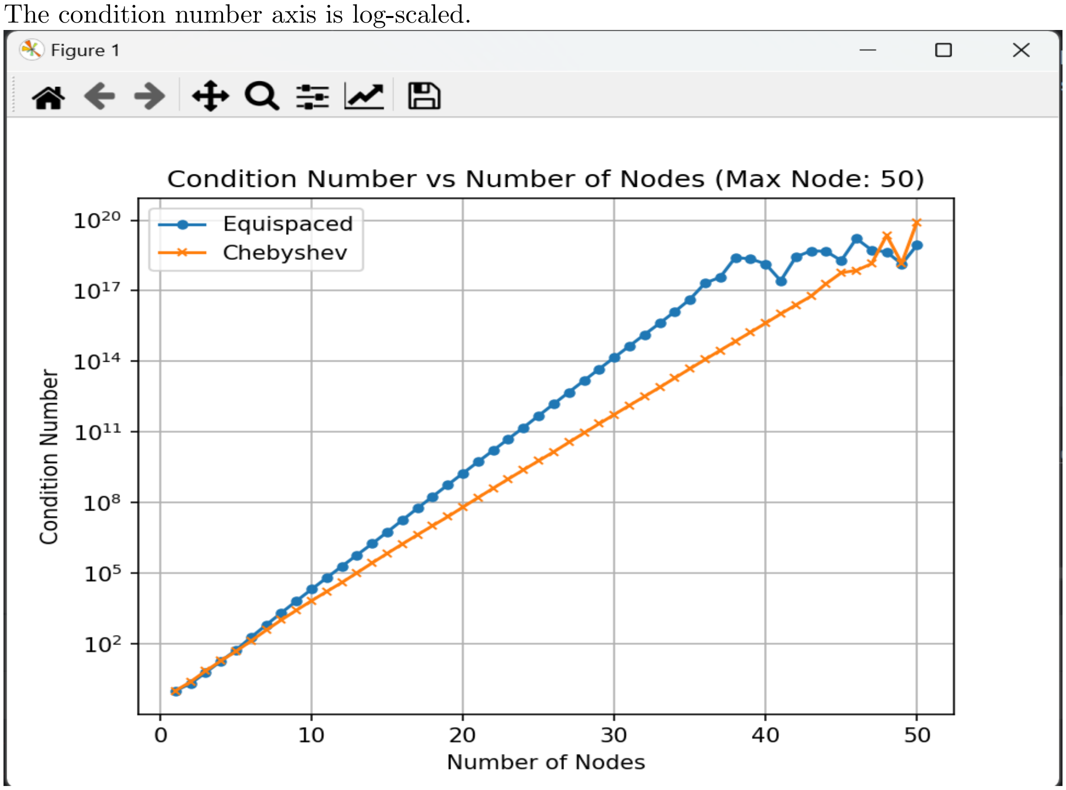
1 problem1

problem (a):
The condition number axis is log-scaled.



problem (b):

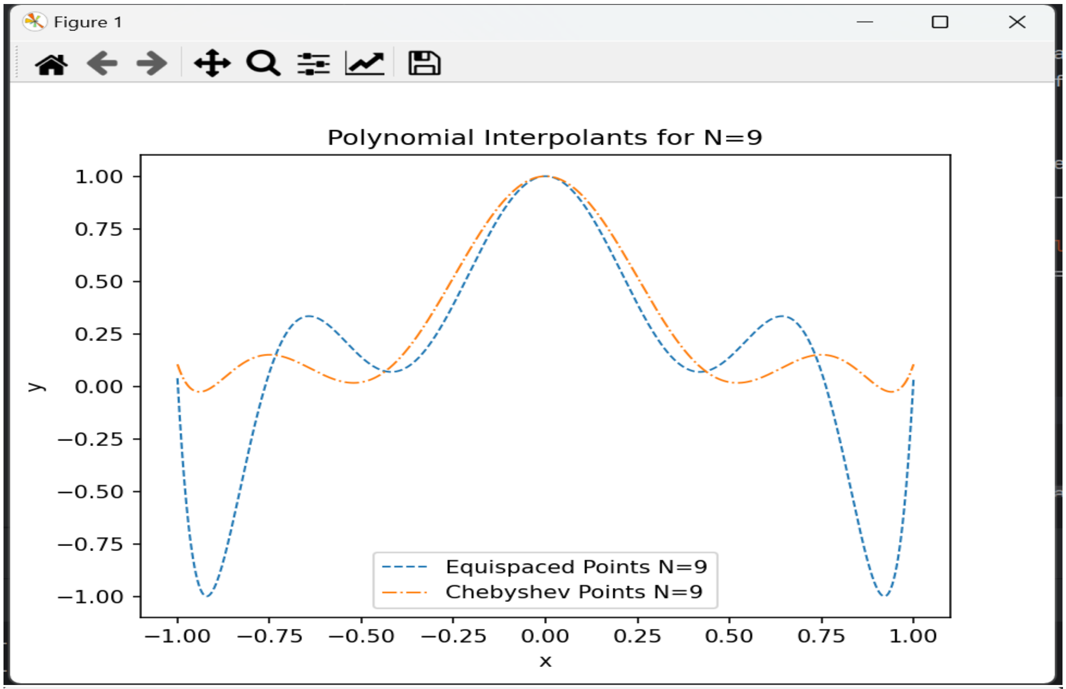


Figure 1
Polynomial Interpolants for N=9
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y
−1.00 −0.75 −0.50 −0.25 0.00 0.25 0.50 0.75 1.00
x
Equispaced Points N=9
Chebyshev Points N=9

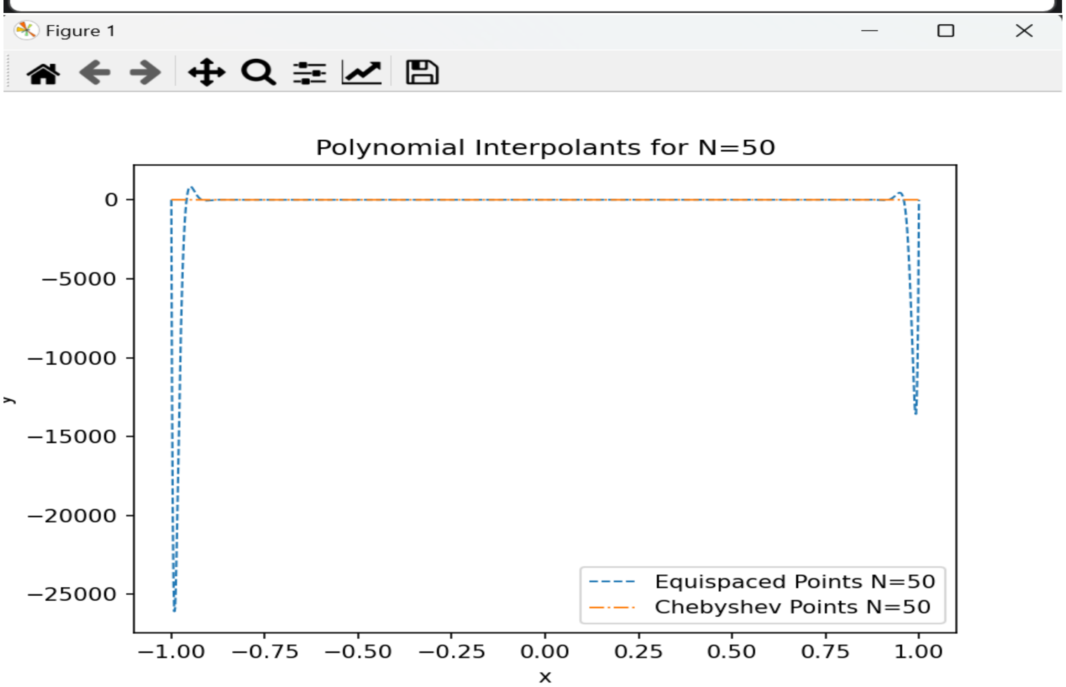


Figure 1
Polynomial Interpolants for N=50
0
−5000
−10000
−15000
−20000
−25000
y
−1.00 −0.75 −0.50 −0.25 0.00 0.25 0.50 0.75 1.00
x
Equispaced Points N=50
Chebyshev Points N=50

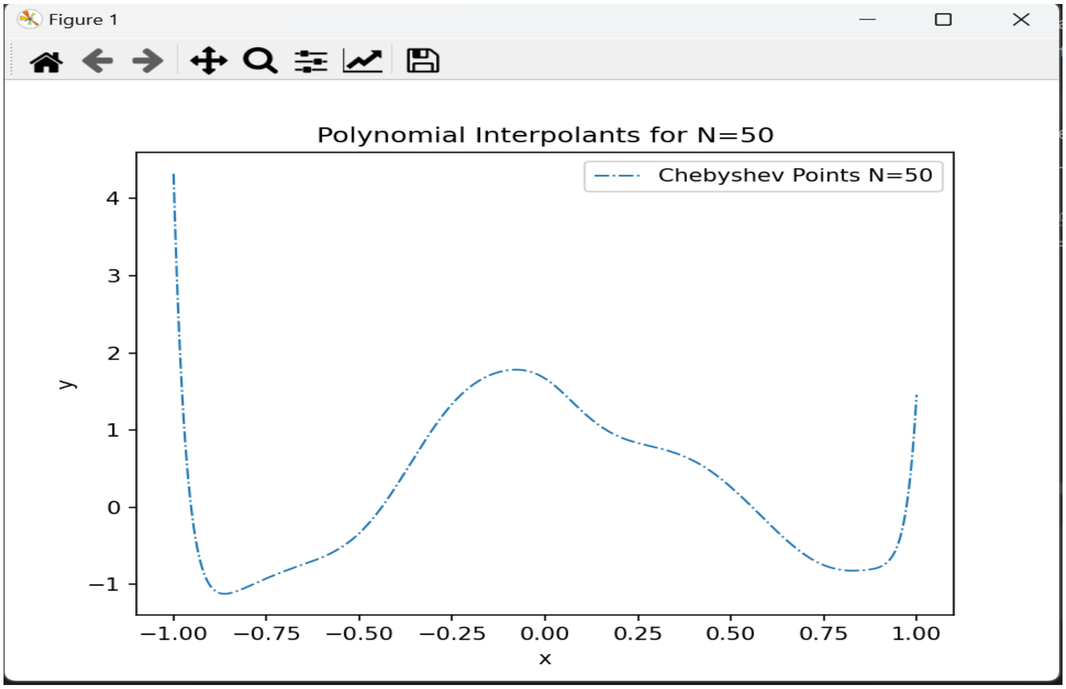


Figure 1
Polynomial Interpolants for N=50
Chebyshev Points N=50
y
x
4
3
2
1
0
−1
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00

problem (c):

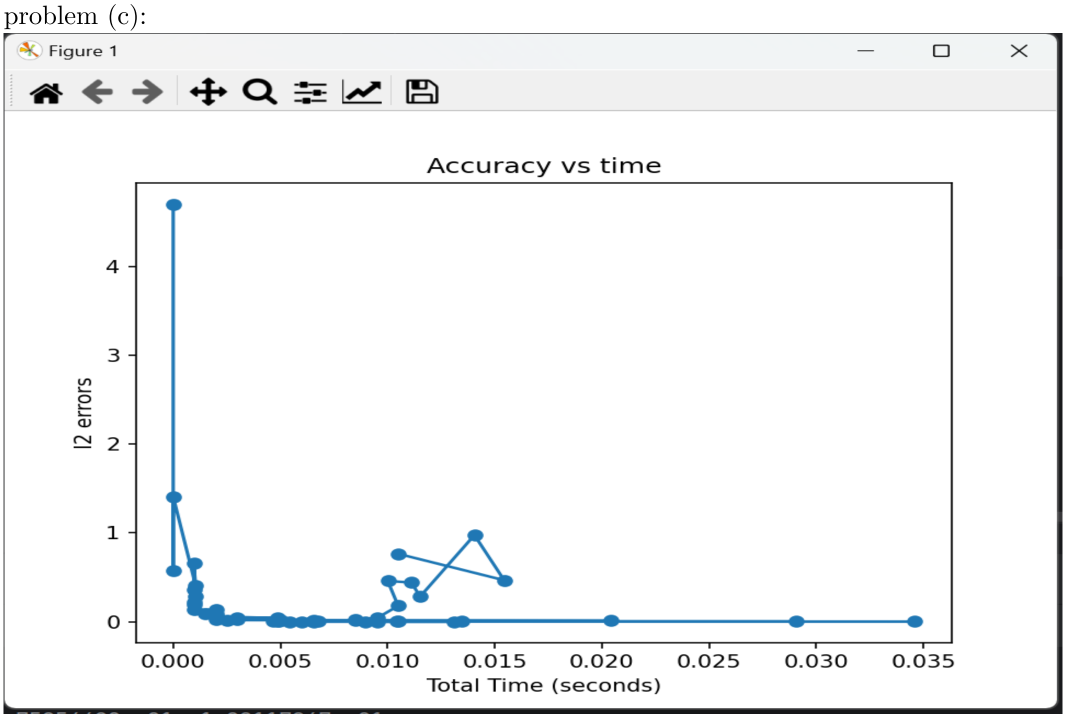


Figure 1
Accuracy vs time
l2 errors
Total Time (seconds)
4
3
2
1
0
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0.035

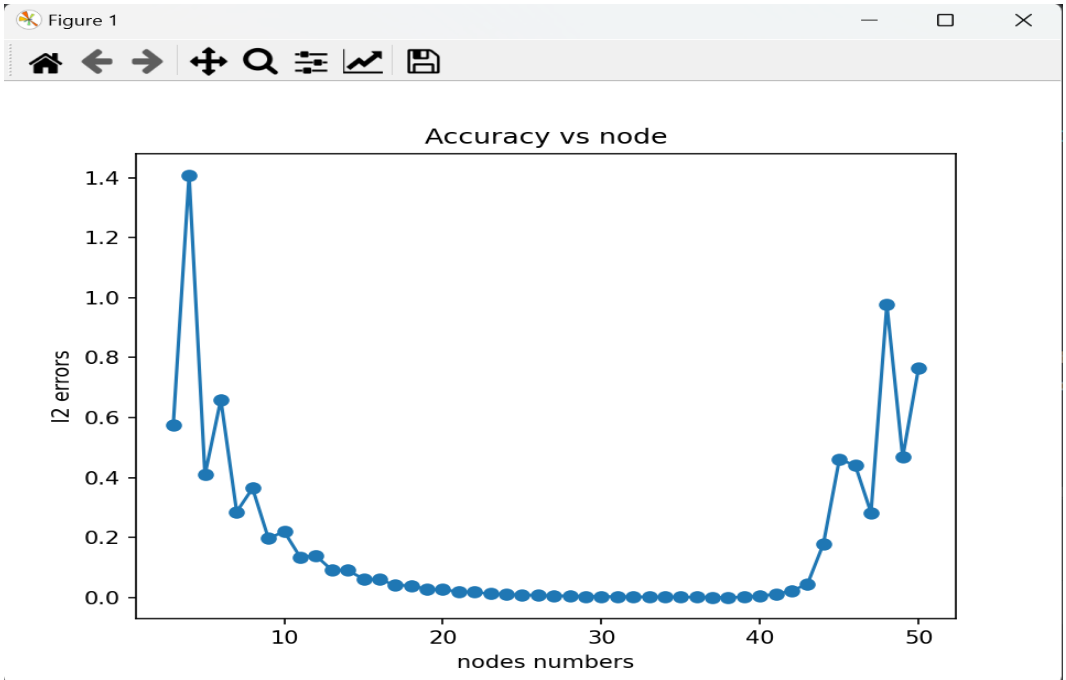


Figure 1
Accuracy vs node
l2 errors
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
10
20
30
40
50
nodes numbers

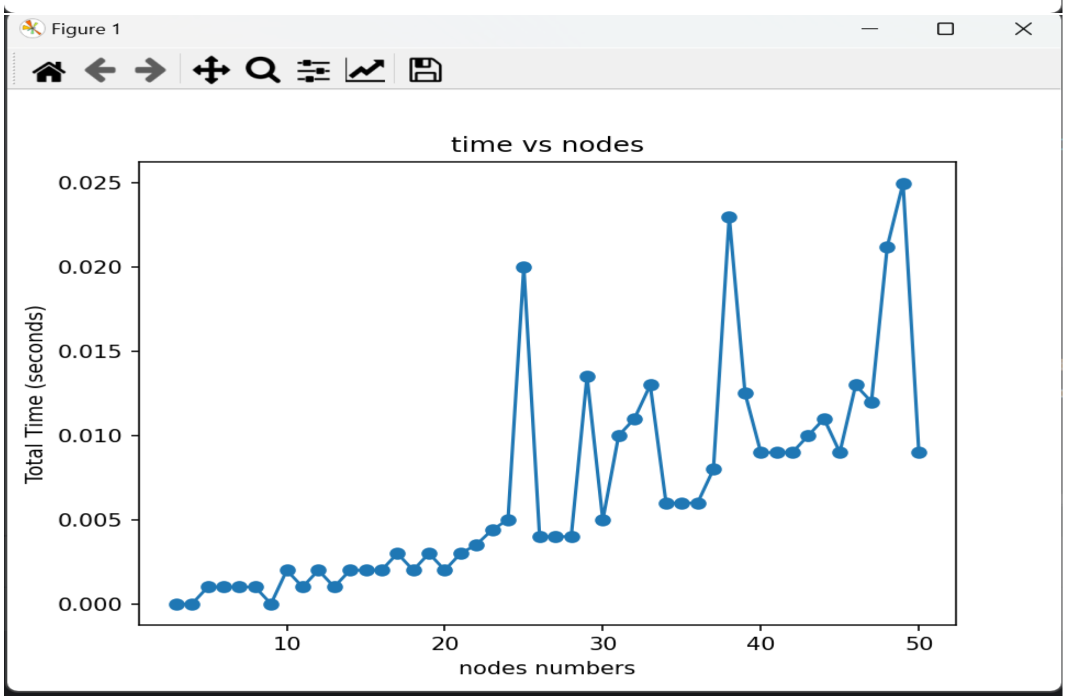


Figure 1
time vs nodes
Total Time (seconds)
0.025
0.020
0.015
0.010
0.005
0.000
10
20
30
40
50
nodes numbers

2 problem2

First, derive the Lagrange form polynomial:
as we know:$p''(x) = y_0 l_0''(x) + y_1 l_1''(x) + y_2 l_2''(x)$
and:
$l_0 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$
$l_1 = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$
$l_2 = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

By driving these three Lagrange basis functions:
$l_0' = \frac{\partial x(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$
$= \frac{(x-x_1)\partial x(x-x_2)+(x-x_2)\partial x(x-x_1)}{(x_0-x_1)(x_0-x_2)}$
$= \frac{x-x_2+x-x_1}{(x_0-x_1)(x_0-x_2)}$
$= \frac{2x-x_2-x_1}{(x_0-x_1)(x_0-x_2)}$
similarly, we can have:
$l_1' = \frac{2x-x_2-x_0}{(x_1-x_0)(x_1-x_2)}$
$l_2' = \frac{2x-x_1-x_0}{(x_2-x_0)(x_2-x_1)}$

By driving these three Lagrange basis functions again, we can get a second derivative:
$l_0'' = \frac{\partial x(2x-x_2-x_1)}{(x_0-x_1)(x_0-x_2)}$
$= \frac{2}{(x_0-x_1)(x_0-x_2)}$
similarly,we can have:
$l_1'' = \frac{2}{(x_1-x_0)(x_1-x_2)}$
$l_2'' = \frac{2}{(x_2-x_0)(x_2-x_1)}$

Therefore, $p''(x_1) = y0\frac{2}{(x_0-x_1)(x_0-x_2)} + y_1\frac{2}{(x_1-x_0)(x_1-x_2)} + y_2\frac{2}{(x_2-x_0)(x_2-x_1)}$
$= \frac{2y_0}{(-h)(-2h)} + \frac{2y_1}{(h)(-h)} + \frac{2y_2}{(2h)(h)}$
$= \frac{y_0}{h^2} - \frac{2y_1}{h^2} + \frac{y_2}{h^2}$
$= \frac{y_0-2y_1+y_2}{h^2}$

Then $f''(x_1) = \frac{y_0-2y_1+y_2}{h^2} + \mathcal{O}(h^2)$

Second, derive the error estimate.
$\mathcal{O}(h^2) = \frac{1}{3*2}f'''(\xi)e''(x_1) + 2\frac{1}{3*2}f''''(\xi)e'(x_1)\frac{d\xi}{dx} + \frac{1}{3*2}f'''''(\xi)e(x_1)\frac{d\xi}{dx}$
$= \frac{1}{6}f'''(\xi)(x_1-x_0)(x_1-x_2) + 0 + 0$
$= \frac{1}{6}f'''(\xi)(x_1-x_0)(x_1-x_2)$
Therefore; we have
$f''(x_1) = \frac{y_0-2y_1+y_2}{h^2} + \frac{1}{6}f'''(\xi)(x_1-x_0)(x_1-x_2)$

3 problem3

First, since we have three points between a and b, we have:
$\int_a^b p(x)dx = y_0 \int_a^b l_0(x)dx + y_1 \int_a^b l_1(x)dx + y_2 \int_a^b l_2(x)dx$
$l_0 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$
$l_1 = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$
$l_2 = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$
for the first term:
$y_0 \int_a^b l_0(x)dx = \frac{y_0}{(x_0-x_1)(x_0-x_2)}((x-x_1)(\frac{1}{2}x^2 - x*x_2)_a^b - \int_a^b(\frac{1}{2}x^2 - x*x_2))$
$= \frac{y_0}{(-\frac{(b-a)}{2})(-2\frac{(b-a)}{2})}((-\frac{1}{2}b^2 * \frac{(b-a)}{2}) + \frac{(b-a)}{2}(\frac{1}{2}(b-2*\frac{(b-a)}{2})) - b(b-2*\frac{(b-a)}{2}))$
$= \frac{y_0}{(-\frac{(b-a)}{2})(-2\frac{(b-a)}{2})} * \frac{2(\frac{(b-a)}{2})^3}{3}$
$= \frac{(b-a)}{6} * y_0$

Similarly, for the second term, we have:
$y_1 \int_a^b l_1(x)dx = \frac{2(b-a)}{3} * y_1$
for the third term, we have:
$y_2 \int_a^b l_2(x)dx = \frac{(b-a)}{6} * y_2$
Therefore, we have:
$\int_a^b f(x)dx \approx y_0 \int_a^b l_0(x)dx + y_1 \int_a^b l_1(x)dx + y_2 \int_a^b l_2(x)dx$
$\approx \frac{(b-a)}{6}(y_0 + 4y_1 + y_2)$
$\approx \frac{(b-a)}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b))$

it is also Simpson's rule

Then, for the corresponding error estimate:
$\int_a^b(f(x) - p(x)) = \int_a^b \frac{f^{(3)}\xi(x)}{3*2*1}(x-a)(x-\frac{a+b}{2})(x-b)$
$= \frac{f^{(3)}\xi(x)}{3*2*1}\int_a^b(x-a)(x-\frac{a+b}{2})(x-b) - \int_a^b \int(x-a)(x-\frac{a+b}{2})(x-b)\frac{f^{(4)}\xi(x)}{3*2*1}\frac{d\xi}{dx}$
$= 0 - \frac{f^{(4)}\xi(x)}{3*2*1}\int_a^b \int(x-a)(x-\frac{a+b}{2})(x-b)$
$= -\frac{f^{(4)}\xi(x)}{3*2*1}\int_a^b \frac{1}{4}x(a^2(x-2b) - 2a(b-x)^2 + x(b-x)^2)dx$
$= -\frac{f^{(4)}\xi(x)}{3*2*1}(\frac{1}{20}(b^5-a^5) + \frac{1}{8}(a^5 + a^4b - b^4a - b^5) + \frac{1}{2}(-a^5 - 4a^4b - a^3b^2 + a^2b^3 +$
$4ab^4 + b^5) + \frac{1}{4}ab(a-b)(a+b^2))$
$= -\frac{f^{(4)}\xi(x)}{3*2*1}(a-b)^5$
$= \frac{f^{(4)}\xi(x)}{3*2*1}(b-a)^5$