# gibbs_sampler

January 22, 2023

#### 0.0.1 Imports

[1]:
```python
import numpy as np
import pandas as pd
from matplotlib import pyplot as plt
from tqdm import tqdm
from typing import Tuple, Callable, Optional, TypedDict
```

#### 0.0.2 Main part

**Task 1** Calculations needed for task 1 and 9 can be found at the end of pdf with report written out by hand.

**Task 2**

[2]:
```python
def gibbs_sampler(num_samples: int, previous: Tuple[int, int] = (0, 0), thin_out:
↪ int = 1, burn_in: int = 0) -> pd.DataFrame:
    ret = [previous]  # Since these are Markov chains we only need to remember
↪the last generated item
    for t in range(burn_in + num_samples * thin_out):
        idx = int(np.random.rand() > .5)
        cloud_prob = 4./9. if previous[0] else 1./21.
        rain_prob = .815 if previous[1] else .216
        previous = (np.random.choice((0, 1), p=[1 - rain_prob, rain_prob]),
↪previous[1]) if idx == 0 else (previous[0], np.random.choice((0, 1), p=[1 -
↪cloud_prob, cloud_prob]))
        if t > burn_in:
            if (t - burn_in) % thin_out == 0:
                ret.append(previous)

    return pd.DataFrame(ret, columns=["Rain", "Cloudy"])
```

[3]:
```python
N = 100
gibbs_sampler(N)
```

[3]:

| | Rain | Cloudy |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 0 |
| 2 | 0 | 0 |

```
3       0       1
4       1       1
..    ...     ...
95      1       1
96      1       0
97      0       0
98      0       0
99      0       0

[100 rows x 2 columns]
```

**Task 3**

[4]:
```python
probs = gibbs_sampler(N).mean()
print(f"P(R = T | S = T, W = T) is approximately: {probs['Rain']}")
```

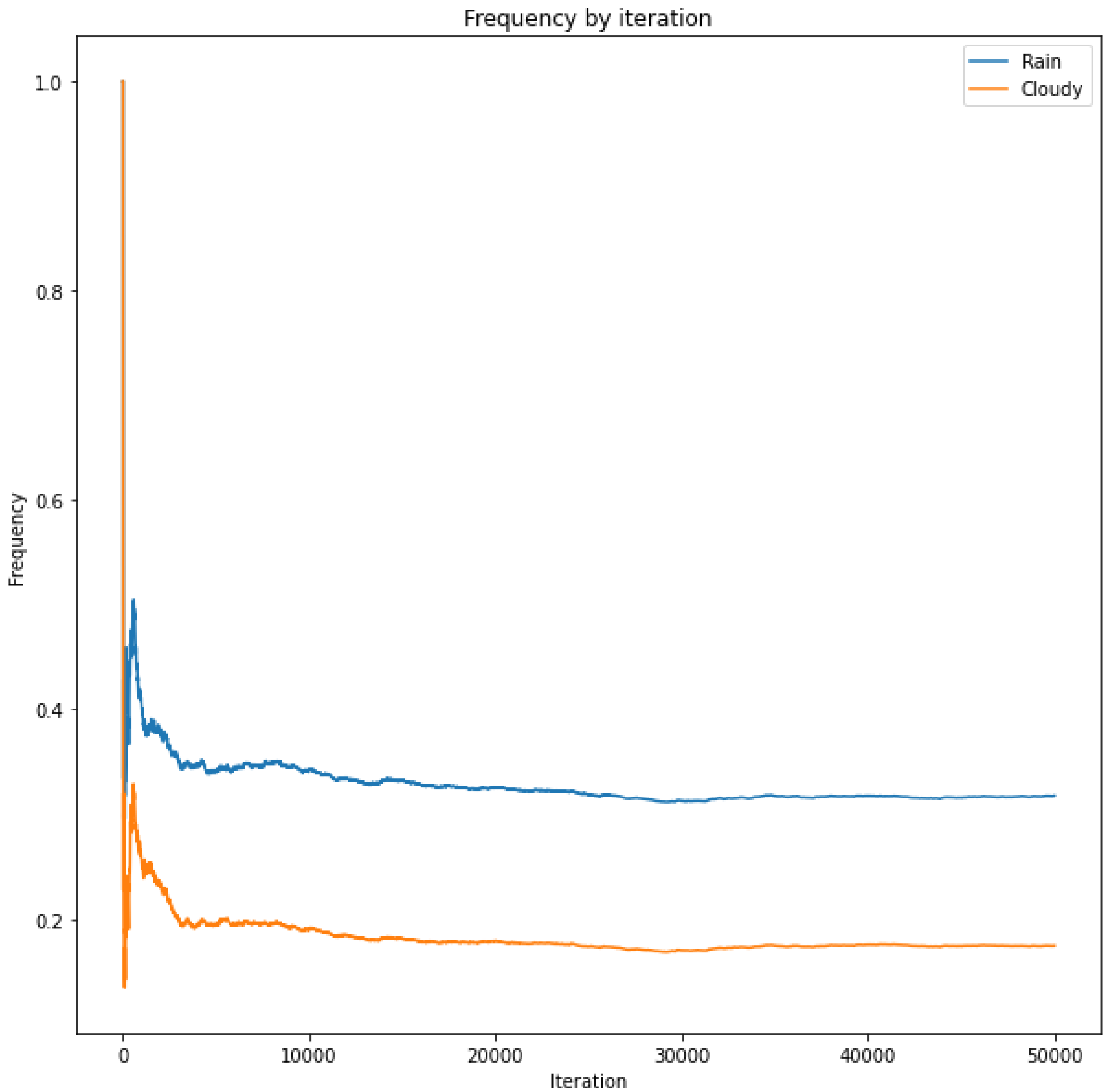
```
P(R = T | S = T, W = T) is approximately: 0.21
```

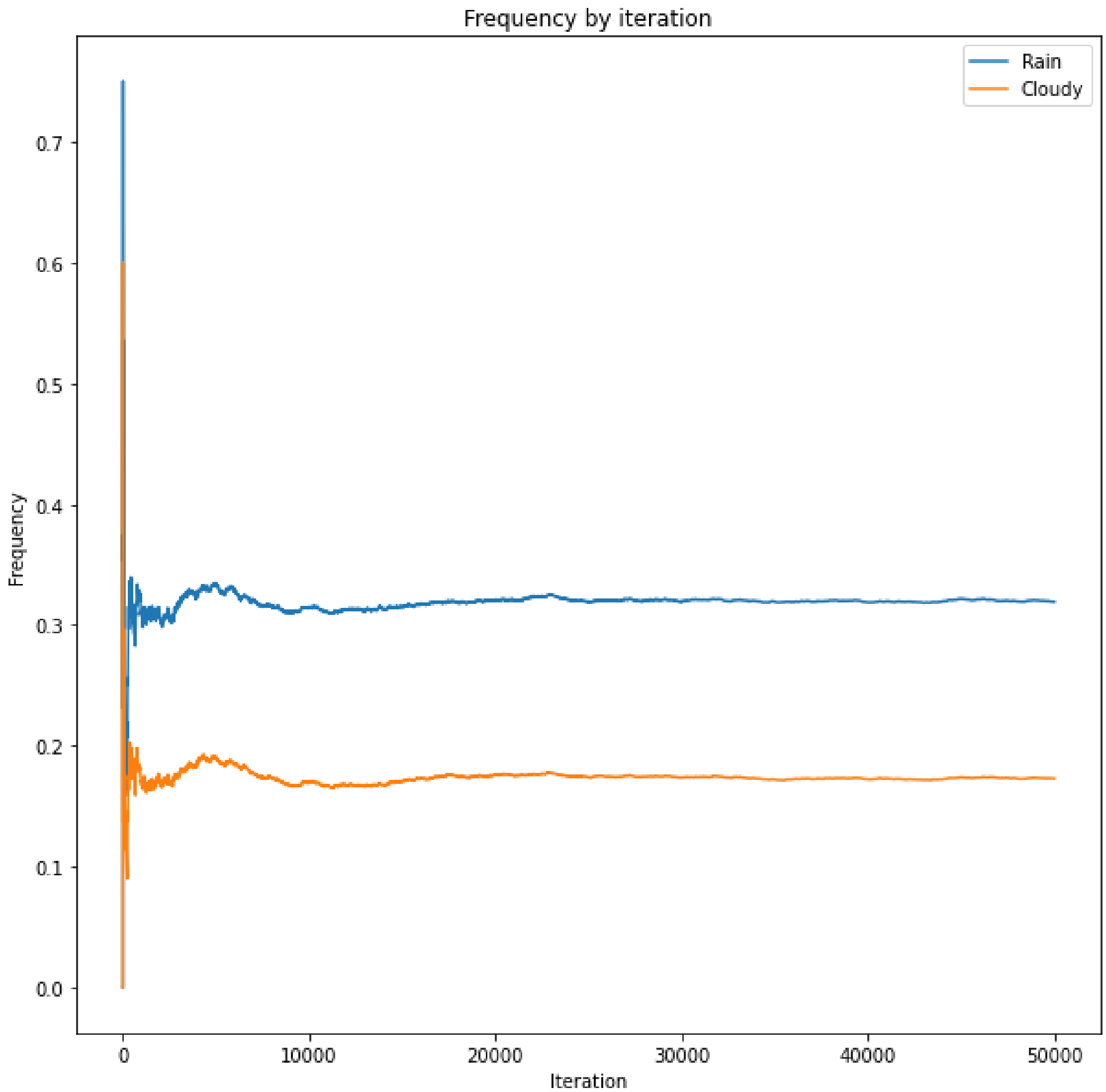
**Task 4 & 5**

[5]:
```python
def freq_by_iter(num_iter: int, **kwargs ) -> pd.DataFrame:
    gibbs = gibbs_sampler(num_iter, **kwargs)
    return gibbs.cumsum().divide(gibbs["Rain"].index.values, axis=0)

N = 50000
print(f"P(R = T | S = T, W = T) approximated by sampling 50 000 samples:␣
↪{gibbs_sampler(N).mean()['Rain']}")
for _ in range(2):
    freq_by_iter(N).plot(title="Frequency by iteration", ylabel="Frequency",␣
↪xlabel="Iteration", figsize=(10, 10))
    plt.show()
```

```
P(R = T | S = T, W = T) approximated by sampling 50 000 samples: 0.32046
```


Frequency by iteration
Rain
Cloudy
1.0
0.8
0.6
0.4
0.2
Frequency
0
10000
20000
30000
40000
50000
Iteration




Based on the plot above I would suggest burn-in time of at least 7500 iterations. However, if we want to be sure the burn-in time is sufficient 10000 iterations may be better idea.

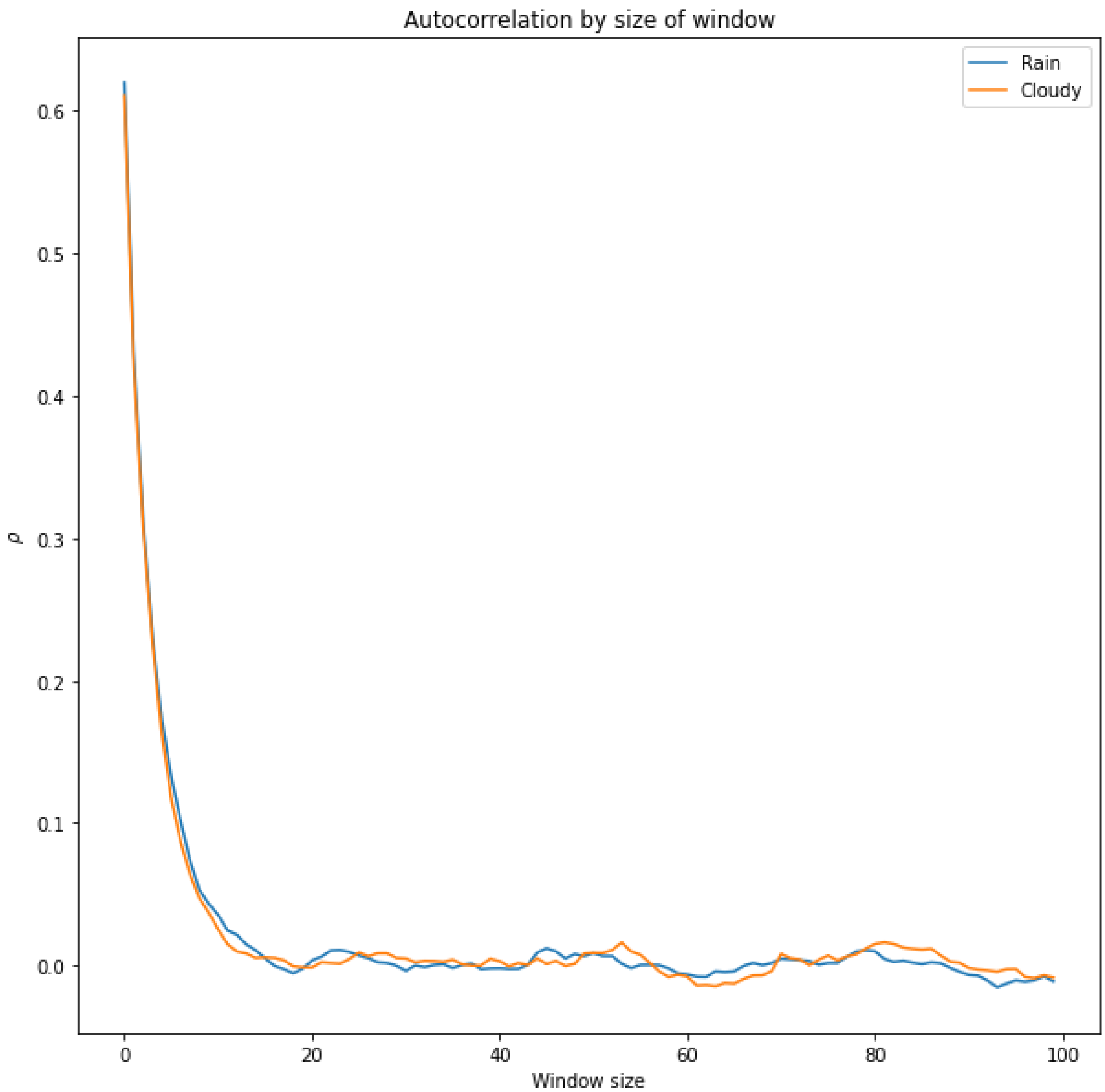
**Task 6**

[6]:
```
def autocorr_plot(samples: pd.DataFrame, max_lag: int = 100):
    autocorrs = [(samples["Rain"].autocorr(i), samples["Cloudy"].autocorr(i))␣
 ↪for i in range(1, max_lag + 1)]

    plt.figure(figsize=(10, 10))
    plt.plot(autocorrs, label=["Rain", "Cloudy"])
    plt.title("Autocorrelation by size of window")
    plt.xlabel("Window size")
    plt.ylabel("$\\rho$")
```

```
    plt.legend()
    plt.show()

autocorr_plot(gibbs_sampler(N), 100)
```




Based on this plot I would suggest interval of 20 iterations for drawing approximately independent samples.

**Task 7** See implementaion of `gibbs_sampler` above (`Task 2`).

**Task 8**

[7]:

```
```

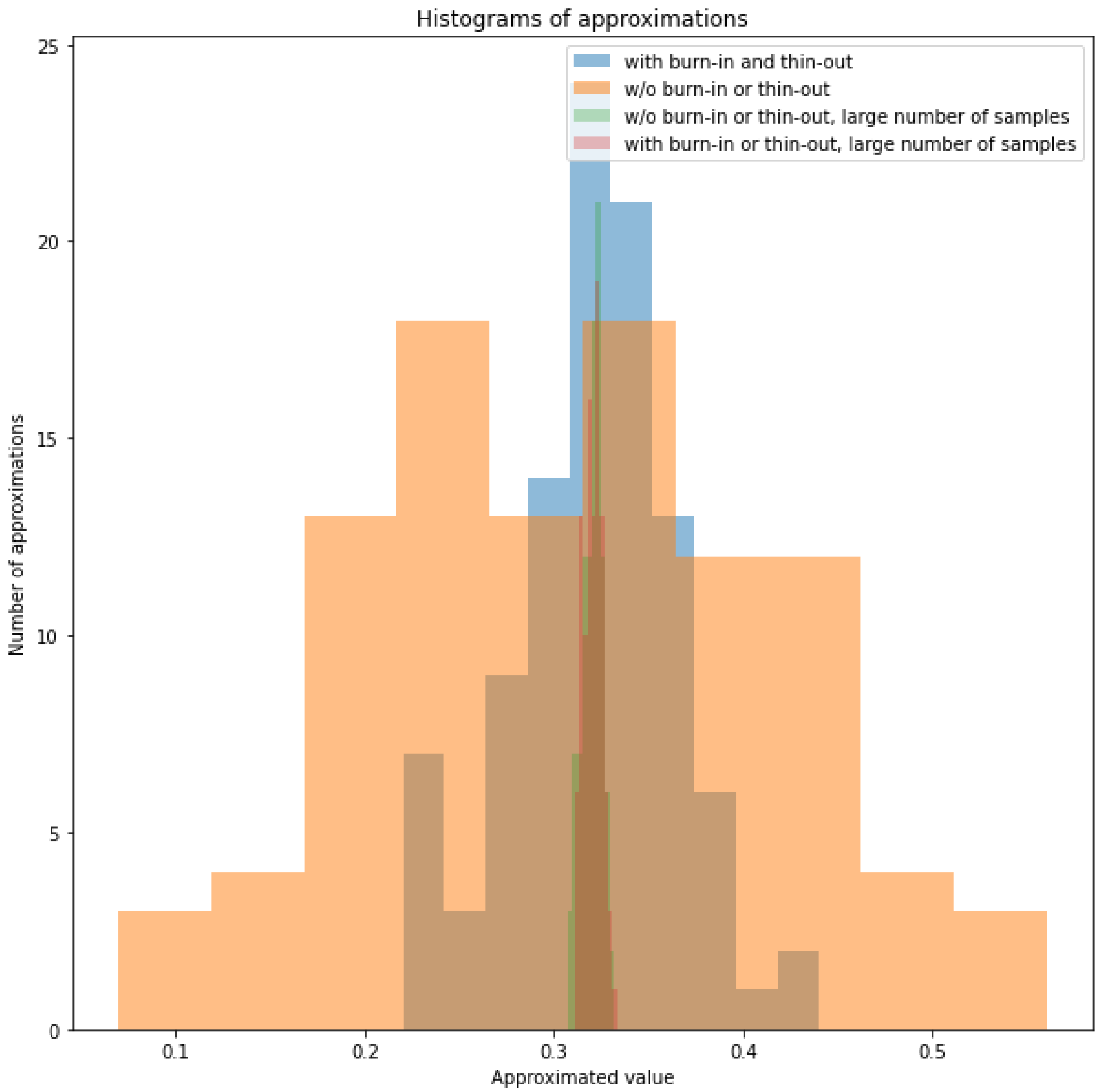
```
print(f"P(R = T | S = T, W = T) approximated with burn-in and thinning-out:␣
↪{gibbs_sampler(100, burn_in=10000, thin_out=20).mean()['Rain']},␣
↪\napproximation in Task 3 was: {probs['Rain']}")
```

```
P(R = T | S = T, W = T) approximated with burn-in and thinning-out: 0.37,
approximation in Task 3 was: 0.21
```

Due to small number of samples both approximations are quite random, however approximation with burn-in and thinning-out is more likely to be close to the real value of this probability which is 0.32. See histograms below:

[8]:
```
vanila = []
burn_thin = []
large = []
large_burn = []
for _ in range(100):
    vanila.append(gibbs_sampler(100).mean()["Rain"])
    burn_thin.append(gibbs_sampler(100, burn_in=10000, thin_out=20).
↪mean()["Rain"])
    large.append(gibbs_sampler(50000).mean()["Rain"])
    large_burn.append(gibbs_sampler(50000).mean()["Rain"])

plt.figure(figsize=(10, 10))
plt.hist(burn_thin, label="with burn-in and thin-out", alpha=.5)
plt.hist(vanila, label="w/o burn-in or thin-out", alpha=.5)
plt.hist(large, label='w/o burn-in or thin-out, large number of samples', alpha=.
↪3)
plt.hist(large_burn, label='with burn-in or thin-out, large number of samples',␣
↪alpha=.3)
plt.ylabel("Number of approximations")
plt.xlabel("Approximated value")
plt.title("Histograms of approximations")
plt.legend()
plt.show()
```




In those we can see that approximation by sampling with burn-in and thin-out is concentrated around real value `0.32` while the one without burn-in and thin-out is much more diverse. As we can see taking larger number of samples gives better approximation even without burn-in or thinning out (and even better with them).

**Task 10 - Convergence diagnostics**

[9]:
```
def gelman_rubin_samples(samples: np.array) -> np.array:
    """
    Calculates Gelman-Rubin statistic for chain from which came samples.
    :param samples: array of shape (m, n, k) where m - number of chains, n-␣
↪number of samples, k- number of batched chains in sample (i.e. 2 when we work␣
↪with sampler from SAD project)
```

```
    :return: array of length k containing Gelman-Rubin statistics for each of␣
↪batched chains.
    """
    m, n, k = samples.shape
    W = np.empty(k)
    B = np.empty(k)
    sigma_hat = np.empty(k)
    for i in range(k):
        sample = samples[:, :, i]
        W[i] = np.var(sample, axis=1).mean()
        B[i] = np.var(sample.mean(axis=1))
        sigma_hat[i] = (n-1)/n * W[i] + B[i]
    V_hat = sigma_hat + B/(n * m)
    return np.sqrt(V_hat/W)




def gelman_rubin(sampler: Callable[[int, Optional[Tuple[int, int]],␣
↪Optional[int], Optional[int]], pd.DataFrame], num_iter: int, num_samples: int␣
↪= 3, **kwargs: TypedDict) -> float:
    samples = np.stack([sampler(num_iter, **kwargs).to_numpy() for _ in␣
↪range(num_samples)])
    return gelman_rubin_samples(samples)
```

[10]:

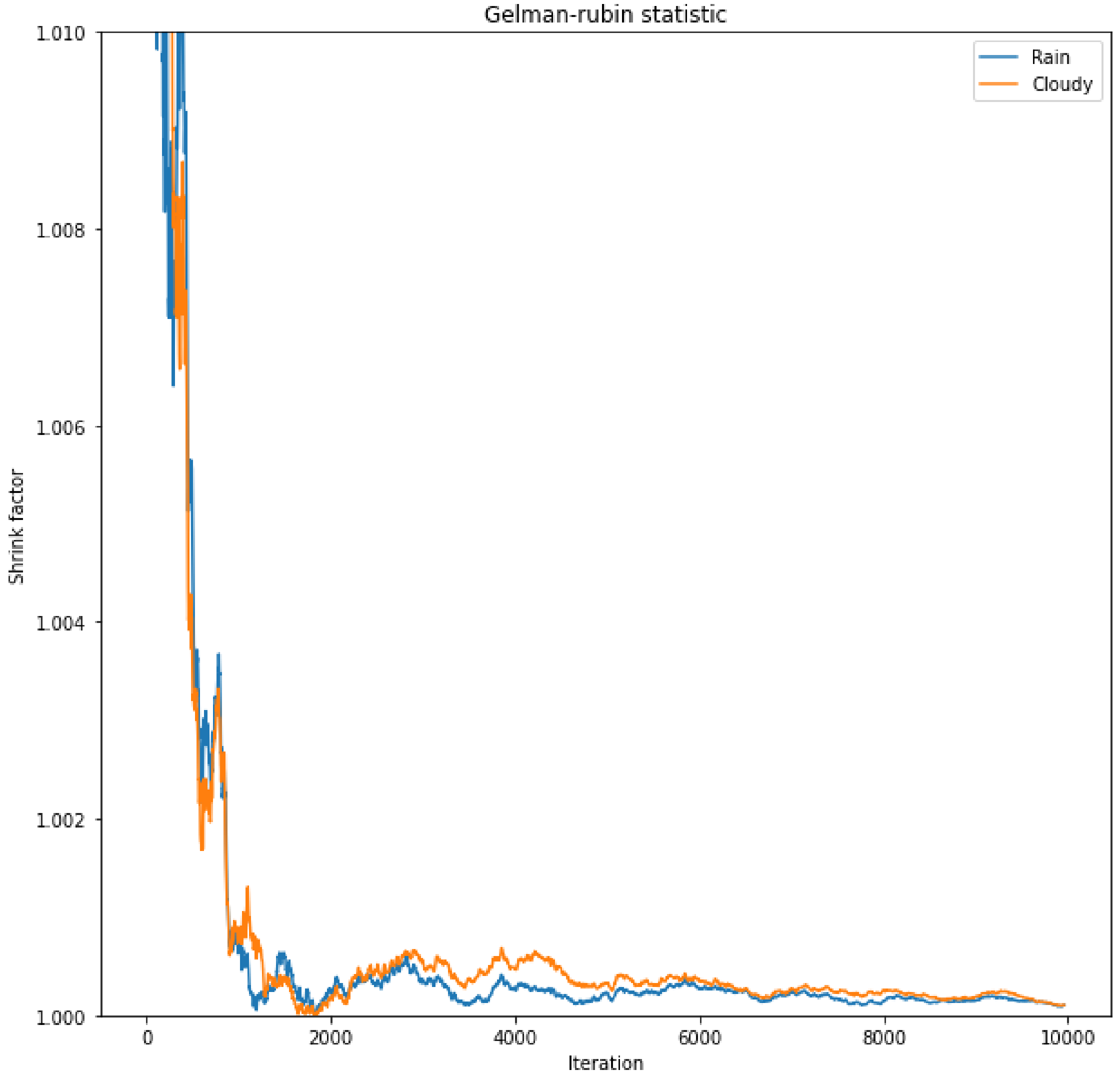
```
def gelman_plot(samples: np.array):
    """
    Generates plot of Gelman-Rubin statistic. Function needs chains of length␣
↪greater than 50.
    :param samples: Samples from m markov chains
    """
    assert samples.shape[1] > 50, "More than 50 samples needed from each chain"
    ret = []
    for i in tqdm(range(40, samples.shape[1])):
        ret.append(gelman_rubin_samples(samples[:, :i, :]))

    plt.figure(figsize=(10, 10))
    plt.plot(ret, label=["Rain", "Cloudy"])
    plt.ylabel("Shrink factor")
    plt.xlabel("Iteration")
    plt.title("Gelman-rubin statistic")
    plt.legend()
    plt.ylim(bottom=1., top=1.01)
    plt.show()
```

[11]:
```
samples = 10000
sample = np.stack([gibbs_sampler(samples, previous=(1, 1)).to_numpy() for _ in␣
 ↪range(7)])

gelman_plot(sample)
```

```
100%|| 9960/9960 [00:04<00:00, 2246.47it/s]
```




In this plot we can see that with number of iterations going up value of statistic is converging to 1, so we can assume that the chains are convergent.

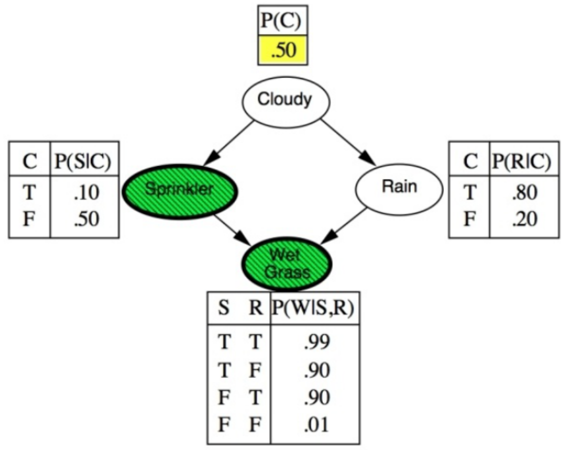



Figure 1: The *Rain Bayesian network*

$$P(C=T \mid R=T, S=T, W=T) \overset{MB}{=} P(C=T \mid R=T, S=T) =$$

$$= \frac{P(C=T, R=T, S=T)}{P(R=T, S=T)} = \frac{P(R=T \mid C=T)\, P(S=T \mid C=T)\, P(C=T)}{P(R=T, S=T)} =$$

$$= \frac{P(R=T \mid C=T)\, P(S=T \mid C=T)\, P(C=T)}{P(C=T)\, P(R=T \mid C=T) P(S=T \mid C=T) + P(C=F) P(R=T \mid C=F) P(S=T \mid C=F)} =$$

$$= \frac{0{,}8 \cdot 0{,}1 \cdot 0{,}5}{0{,}5 \cdot 0{,}8 \cdot 0{,}1 + 0{,}5 \cdot 0{,}2 \cdot 0{,}5} = \frac{0{,}04}{0{,}09} = \frac{4}{9}$$

$$P(C=T \mid R=F, S=T, W=T) = \frac{P(R=F \mid C=T)\, P(S=T \mid C=T)\, P(C=T)}{P(C=T)\, P(R=F \mid C=T) P(S=T \mid C=T) + P(C=F) P(R=F \mid C=F) P(S=T \mid C=F)}$$

$$= \frac{0{,}2 \cdot 0{,}1 \cdot 0{,}5}{0{,}5 \cdot 0{,}2 \cdot 0{,}1 + 0{,}5 \cdot 0{,}8 \cdot 0{,}5} = \frac{0{,}01}{0{,}21} = \frac{1}{21}$$

$$P(R=T \mid C=T, S=T, W=T) = \frac{P(R=T, C=T, S=T, W=T)}{P(C=T, S=T, W=T)} =$$

$$= \frac{P(W=T \mid S=T, R=T)\, \cancel{P(S=T \mid C=T)}\, P(R=T \mid C=T)\, \cancel{P(C=T)}}{P(W=T \mid S=T, C=T)\, \cancel{P(S=T \mid C=T)}\, \cancel{P(C=T)}} =$$

$$= \frac{P(W=T \mid S=T, R=T)\, P(R=T \mid C=T)}{P(W=T \mid S=T, C=T)}$$

If there is no clear indication otherwise, then $P(X)$ is the same as $P(X=T)$, except variables that are being summed out.

$$P(W=T \mid S=T, C=T) = \frac{P(W=T, S=T, C=T)}{P(S=T, C=T)} = \frac{\sum_R P(W, S, C, R)}{\sum_R P(S, C, R)} =$$

$$= \frac{\big(P(W \mid S, R=T)\, P(S \mid C)\, P(R=T \mid C)\, P(C)\big) + \big(P(W \mid S, R=F)\, P(S \mid C)\, P(R=F \mid C)\, P(C)\big)}{\big(P(S \mid C)\, P(C)\big)\big(P(R=T \mid C) + P(R=F \mid C)\big)}$$

$$P(R=T \mid C=T, S=T, W=T) = \frac{P(W=T \mid S=T, R=T)\, P(R=T \mid C=T)}{P(W=T \mid S=T, C=T)} \approx 0.815$$

Same process as above

$$P(R=T \mid C=F, S=T, W=T) \approx 0.216$$

$$P(R=T \mid S=T, W=T) = \frac{P(R,S,W)}{P(S,W)} = \frac{\overbrace{\sum_C P(R,S,W,C)}^{L}}{\underbrace{\sum_C P(S,C,W)}_{M}}$$

$$L = \underbrace{P(W \mid R,S)}_{0.99}\,\underbrace{P(S \mid C=T)}_{0.1}\,\underbrace{P(R \mid C=T)}_{0.8}\,\underbrace{P(C=T)}_{0.5} +$$

$$\underbrace{P(W \mid S,R)}_{0.99}\,\underbrace{P(S \mid C=F)}_{0.5}\,\underbrace{P(R \mid C=F)}_{0.2}\,\underbrace{P(C=F)}_{0.5} = 0{,}0891$$

$$M = \underbrace{P(S=T, R=T, W=T)}_{L} + P(S=T, R=F, W=T) =$$

$$= L + \sum_C P(S=T, R=F, W=T, C)$$

$$\underbrace{P(W \mid S, R=F)}_{0.9}\,\underbrace{P(S \mid C=T)}_{0.1}\,\underbrace{P(R=F \mid C=T)}_{0.2}\,\underbrace{P(C=T)}_{0.5} +$$

$$\underbrace{P(W \mid S, R=F)}_{0.9}\,\underbrace{P(S \mid C=F)}_{0.5}\,\underbrace{P(R=F \mid C=F)}_{0.8}\,\underbrace{P(C=F)}_{0.5} + L \approx 0{,}28$$

$$P(R=T \mid S=T, W=T) = \frac{L}{M} = \frac{0{,}0891}{0{,}28} \approx 0{,}32$$