MATH 2415: Calculus 3
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1 Vectors and the Geometry of Space

1.1 Vectors in the Plane

Vectors are quantities with both magnitude and direction. The vector who's tail is at point P and head at point Q is denoted as $\overrightarrow{PQ}$.

Two vectors, **u** and **v** are equal if they have equal length and point in the same direction. They do not necessarily have to be in the same location.

Scalar multiplication happens when a c is multiplied to vector **v**. If $c < 0$, then vector cv and v will point in opposite directions, otherwise they will point in the same direction. Two vectors are parallel if they are scalar multiples of each other.

If you place the tail of a vector **v** at the head of another vector **u**, the sum **u+v** is the vector that extends from the tail of **u** to the head of **v**.

The vector difference **u-v** is defined as **u+(-v)**.

In order to do calculations with vectors, we must introduce a cartesian plane. Angle brackets $\langle a, b\rangle$ show the components of a vector.

The magnitude of a vector is simply its length. Given points $P(x_1, y_1)$ and $Q(x_1, y_1)$, the magnitude of the vector $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1\rangle$ is denoted as $|\vec{PQ}|$, is equal to:

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We can also now do vector addition with components. Given two vectors **u** and **v**, the vector sum is:

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2\rangle$$

For a scalar c and a vector **u**, the scalar multiple is $c\mathbf{u}$. i.e $|c\mathbf{u}| = |c||\mathbf{u}|$

A unit vector is any vector with length 1. **i** is a unit vector in the x-direction and **j** is a unit vector in the y-direction.

Example

Determine the necessary air speed and heading that a pilot must maintain in order to fly her commercial jet north at a speed of 480 mi/hr relative to the ground in a crosswind that is blowing $60\deg$ south of east at 20 mi/hr.

Let $\vec{p}$ be the velocity vector we are trying to find.

We have ground vector $\langle 0, 480\rangle$ and a crosswind vector $\langle 10, -10\sqrt{3}\rangle$. Note we got the x-component and the y-component of the crosswind vector from the formula: $\vec{v} = \langle a\cos\theta, b\cos\theta\rangle$

We can find the vector $\vec{p}$ from adding this vector to the crosswind vector resulting in: $\vec{p} = \vec{g} - \vec{c} = \langle -10, 480 + 10\sqrt{3}\rangle$.

The magnitude of this vector is the speed and is equal to 497.2 mi/hr roughly.

1.2 Vectors in Three Dimensions

We can create a z-axis to create a three dimensional system.

The xyz-plane is divided into octants and has 3 planes, the xy-plane, the xz-plane, and the yz-plane.

We can also extend the distance formula to 3 dimensions. It is similar to the distance formula in two dimensions, with the z-component added, essentially:

$$|PQ| = \sqrt{|PR|^2 + |RQ|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The midpoint formula works the same way:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2} + \frac{y_1 + y_2}{2} + \frac{z_1 + z_2}{2}\right)$$

The normal form of the circle equation is:

$$(x - h)^2 + (y - k)^2 = r^2$$

For a disk we have

$$(x - h)^2 + (y - k)^2 \leq r^2$$

We can generalize this to a sphere. A sphere centered at (a, b, c) with radius r is the set of points satisfying:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

A ball centered at (a, b, c) with radius r is the set of points satisfying:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2$$

Example

Find an equation of the sphere passing through $P(-4, 2, 3)$ and $Q(0, 2, 7)$ with its center at the midpoint of PQ.

We can find the midpoint from the midpoint formula and it is equal to $(-2, 2, 5)$.

The radius can be found through the distance formula and is equal to $\sqrt{8}$.

The equation is $(x + 2)^2 + (y - 2)^2 + (z - 5)^2 = 8$

All the vector operations from two-dimensions work in three-dimensions.

Example

A model airplane is flying horizontally due east at 10 mi/hr when it encounters a horizontal crosswind blowing south at 5 mi/hr and an updraft blowing vertically upward at 5 mi/hr.

- Find the position vector that represents the velocity of the plane relative to the ground.
- Find the speed of the plane relative to the ground.

The velocity vector of the model plane $\vec{p}$ is equal to $\langle 10, 0, 0\rangle$

The velocity vector of the horizontal crosswind $\vec{w}$ is equal to $\langle 0, -5, 0\rangle$

The velocity vector of the updraft $\vec{u}$ is $\langle 0, 0, 5\rangle$

Adding the three vectors results in the speed: $\langle 10, -5, 5\rangle$

The magnitude of this is roughly 12.25.

1.3 Dot Products

Definition

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in two or three dimensions, the dot product is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$$

The dot product is 0 when $\theta = \frac{\pi}{2}$, negative when $\theta > \frac{\pi}{2}$ and positive when $\theta < \frac{\pi}{2}$.

Two vectors are parallel if and only if $\mathbf{u}\cdot\mathbf{v}=\pm|\mathbf{u}||\mathbf{v}|$

When the dot product is zero, we call $\vec{u}$ and $\vec{v}$ orthogonal.

Theorem 1.1: Dot Product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

We now apply the dot product to vector projections.

Definition

The orthogonal projection of $\mathbf{u}$ on $\mathbf{v}$, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$ is:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)$$

The orthogonal projections can also be computed with the formulas:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{scal}_{\mathbf{v}}\mathbf{u} \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$$

where the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is

$$\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$$

1.4 Cross Products

Definition

Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the cross product $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$$

Note that $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

There are some useful properties of the cross product.

- The cross product $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
- The cross product is zero when $\sin(\theta) = 0$.
- Two vectors are parallel if the cross product between them is zero.

We define the determinant of a 2×2 array $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ as $ad - bc$.

For a matrix $\begin{vmatrix} a & b & c \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

the determinant is $a\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - b\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + c\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$ or:

$$a(u_2v_3 - u_3v_2) - b(u_1v_3 - u_3v_1) + c(u_1v_2 - u_2v_1)$$

1.5 Lines and Planes in Space

Recall that in two dimensions, we needed a point and a slope to write an equation for a line.

We can write an equation in three dimensions as well.

A vector equation of a line passing through the point $P_0(x_0, y_0, z_0)$ in the direction of vector $\mathbf{v} = \langle a, b, c\rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ or:

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

The corresponding parametric equations of the line also are:

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct$$

The general equation of a plane in R^3 with the plane passing through $P_0(x_0, y_0, z_0)$ with vector $\mathbf{v} = \langle a, b, c\rangle$ is described by:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

1.6 Cylinders and Quadric Surfaces

A cylinder is a surface that is parallel to a line.

A trace of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the xy-trace, the yz-trace, and the xz-trace.

To sketch quadric surfaces:

- Determine the points where the surface intersects the coordinate axes.
- Finding traces of the surface helps visualize the surface
- Sketch at least two traces in parallel planes

Example

For:

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 1$$

We set certain variables to zero to find the x-intercept to be ± 3, the y-intercept to be ± 4 and the z-intercept to be ± 5.

We can also find the traces:

- xy: $\frac{x^2}{9} + \frac{y^2}{16} = 1$
- xz: $\frac{x^2}{9} + \frac{z^2}{25} = 1$
- yz: $\frac{y^2}{16} + \frac{z^2}{25} = 1$

The resulting shape is a ellipsoid.

In general the equation of an ellipsoid is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

In this all traces are ellipses.

For an elliptic cone the equation is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.

For an elliptic paraboloid the equation is:

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.

For a hyperbolic paraboloid:

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ or $y = y_0$ are parabolas.

For a hyperboloid of one sheet:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Traces with $z = z_0$ are ellipses for all z_0. Traces with $x = x_0$ or $y = y_0$ are hyperbolas.

For a hyperboloid of two sheets:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Traces with $z = z_0$ with $|z_0| > |c|$ are ellipses. Traces with $x = x_0$ and $y = y_0$ are hyperbolas.

2 Vector-Valued Functions

2.1 Vector-Valued Functions

A vector valued function has the form $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

A set of parametric equations $x = x(t), y = y(t), z = z(t)$ can describe a curve in space.

It can also be viewed as a vector function where each variable varies with respect to an independent variable t.

A point $(x(t), y(t), z(t))$ on the curve is the head of the vector $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

We can consider the vector-valued function of the form:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$$

or

$$f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

where f, g, h are defined on some interval a to b.

The positive orientation of a curve is the direction the curve is generated as the parameter increases.

Example

Find the domain of $\mathbf{r}(t) = \frac{4}{\sqrt{1-t}}\mathbf{i} + \frac{2}{t+3}\mathbf{j}$

The domain is the largest set of values of t on which both f and g are defined.

The first component has $1 - t > 0$, therefore the domain is $(-\infty, 1)$.

The second component's domain is $(-\infty, -3) \cup (-3, \infty)$.

We now find the intersection which is $(-\infty, -3) \cup (-3, 1)$

Definition

A vector valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$$

, provided

$$\lim_{t \to a} |\mathbf{r}(t) - \mathbf{L}| = 0$$

A function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is continuous at $\mathbf{a}$ provided $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$.

2.2 Calculus of Vector-Valued Functions

We can define the derivative of a vector-valued function as:

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

The result of the derivative is a vector-valued function.

Geometrically, as $\Delta t \to 0$, $\frac{\Delta \mathbf{r}}{\Delta t} \to \mathbf{r}'(t)$, which is a tangent vector at point P.

Much like single variable calculus, we can simply use the power rule as we know, rather than the limit definition.

The unit tangent vector for a particular value t is:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

A vector-valued function is smooth if it is differentiable and the derivative is not equal to $\langle 0, 0, 0\rangle$.

We can also integrate vector-valued functions. The rules remain the same as in single variable calculus, just breaking it into three vector components.

2.3 Motion in Space

The derivative of the position vector function is the velocity vector function.

The speed of a scalar function is

$$|v(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

Acceleration is the derivative of the velocity vector function.

Straight-line motion has a uniform velocity. Given:

$$\mathbf{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle$$

the velocity $\mathbf{v}(t) = \langle a, b, c\rangle$

Circular motion has constant $|r(t)|$. We let $\mathbf{r}(t) = \langle A\cos t, A\sin t\rangle$.

$\mathbf{r}(t)$ describes a circular trajectory counter-clockwise around a circle with radius A and center at the origin.

We have:

$$|\mathbf{r}(t)| = \sqrt{(A\cos t)^2 + (A\sin t)^2} = A$$

and

$$\mathbf{v}(t) = \langle -A\sin t, A\cos t\rangle$$

as well as

$$\mathbf{a}(t) = \langle -A\cos t, -A\sin t\rangle = -\mathbf{r}(t)$$

Also there are some important properties - the position and acceleration vectors are both orthogonal to the velocity vector as seen:

- $\mathbf{r}(t) \cdot \mathbf{v}(t) = -A^2 \cos t \sin t + A^2 \sin t \cos t = 0$
- $\mathbf{a}(t) \cdot \mathbf{v}(t) = A^2 \sin t \cos t - A^2 \cos t \sin t = 0$

Let $\mathbf{r}$ describe a path on which $|\mathbf{r}|$ is constant.

We can show that $\mathbf{r} \cdot \mathbf{v} = 0$, showing that the position and velocity vectors are always orthogonal.

For two-dimensional motion in a gravitational field:

The gravitational force is $\mathbf{F} = \langle 0, -mg\rangle$.

Therefore: $\mathbf{F} = m\mathbf{a}(t) = \langle 0, -mg\rangle$.

This shows that $\mathbf{a}(t) = \langle 0, -g\rangle$.

We can summarize this:

The velocity of the object is

$$\mathbf{v}(t) = \langle x'(t), y'(t)\rangle = \langle u_0, -gt + v_0\rangle$$

where $\mathbf{v}(0) = \langle u_0, v_0\rangle$ and $\mathbf{r}(0) = \langle x_0, y_0\rangle$.

The position is:

$$\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle u_0 t + x_0, -\frac{1}{2}gt^2 + v_0 t + y_0\rangle$$

2.4 Length of Curves

We know the arc length of a parametric equation is:

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\mathrm{d}t$$

Now we can consider this equation in three dimensions.

The arc length of a parametrized curve is:

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\mathrm{d}t = \int_a^b |\mathbf{r}'(t)|\mathrm{d}t$$

To find the arc length of a curve given $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ for $t \geq a$, we have:

$$s(t) = \int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2}\mathrm{d}u = \int_a^t |\mathbf{v}(u)|\mathrm{d}u$$

Suppose $|\mathbf{v}(t)| = 1$, then we have $t - a$. This shows the parameter t corresponds to arc length.

2.5 Curvature and Normal Vectors

Curvature will be a measure of how fast a curve $\mathbf{r}(t)$ turns at a point.

Recall the unit tangent vector:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The curvature is

$$\kappa(s) = \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right|$$

where s denotes arc length, and $\mathbf{T}$ denotes the tangent vector.

Lines have zero curvature.

We can write the curvature in terms of arclength:

$$\kappa(t) = \frac{1}{|\mathbf{v}|}\left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right| \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

Circles have constant curvature of $\frac{1}{R}$.

An alternative curvature formula is used for trajectories of moving objects in three-space:

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

where $\mathbf{v} = \mathbf{r'}$ is the velocity and $\mathbf{a} = \mathbf{v'}$ is the acceleration.

The principal unit normal vector will determine the direction in which the curve turns.

The principal unit normal vector at point P on the curve at which $kappa \neq 0$ is:

$$\mathbf{N}(s) = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}s}{|\mathrm{d}\mathbf{T}/\mathrm{d}s|} = \frac{1}{\kappa}\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$$

For other parameters t, we use the equivalent formula:

$$\mathbf{N}(s) = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}t}{|\mathrm{d}\mathbf{T}/\mathrm{d}t|}$$

There are two ways to change the velocity of an object or accelerate - to change its speed or its direction of motion.

The acceleration vector of an object moving in space along a smooth curve has the following representation of its tangential component a_T (in the direction of $\mathbf{T}$) and its normal component a_N (in the direction of $\mathbf{N}$):

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$$

where $a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}}{|\mathbf{v}|}$ and $a_T = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$.

Alternatively $\mathbf{a}\cdot\mathbf{N} = a_N$ and $\mathbf{a}\cdot\mathbf{T} = a_T$.

If we have the unit tangent and principal unit vectors $\mathbf{T}$ and $\mathbf{N}$. The unit binormal vector at each point in the curve is:

$$\mathbf{B} = \mathbf{T}\times\mathbf{N}$$

and the torsion is:

$$\tau = -\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\cdot\mathbf{N}$$

The binormal vector is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$.

$|\tau| = \left|\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\right|$ and the torsion gives the rate at which the curve moves out of the osculating plane formed by $\mathbf{T}$ and $\mathbf{N}$.

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