AP Physics C: Electricity & Magnetism
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1 Electric Charges, Fields and Gauss's Law

Brief Calculus Review

The derivative of a function at some point characterizes the rate of change of the function at that point; The rate of change of the function is basically the slope at that point.

Because the derivative is a slope, the notation can be written as $f'(x) = \frac{\mathrm{d}x}{\mathrm{d}t}$.

There are some derivative rules to know.

- $\frac{\mathrm{d}}{\mathrm{d}x} = 0$
- $\frac{\mathrm{d}}{\mathrm{d}x}(x) = C$
- $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$
- $\frac{\mathrm{d}}{\mathrm{d}x}(f(x) \pm g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}f(x) \pm \frac{\mathrm{d}}{\mathrm{d}x}g(x)$

An integral is simply finding the area under a curve.

The integral notation is $f(x) = \int f'(x)\mathrm{d}x$

In physics, we use the definite integral, where the area is found over an interval $[a, b]$.

The notation for this is $A = \int_b^a f'(x)\mathrm{d}x = f(b) - f(a)$

There are some integration rules to know as well.

- $\int \mathrm{d}x = x + C$
- $\int x^n \mathrm{d}x = \frac{1}{n+1}x^{n+1}, n \neq -1$
- $\int f(x) \pm g(x)\mathrm{d}x = \int f(x)\mathrm{d}x \pm \int g(x)\mathrm{d}x$

A differential equation is an equation involving one or more derivatives of an unknown function. The order of a differential equation is defined to be the order of the highest derivative it contains.

All differential equations are considered to be separable and can be solved by integration. This process is called separation of parts.

Much like derivatives there are set of integrals that don't follow the basic power rule of integration. There are some special integral rules.

- $\int e^{ax}\mathrm{d}x = \frac{1}{a}e^{ax} + C$
- $\int \frac{\mathrm{d}x}{x+n} = \ln|x + a| + C$
- $\int \cos(ax)\mathrm{d}x = \frac{1}{a}\sin(ax)$
- $\int \sin(ax)\mathrm{d}x = \frac{-1}{a}\cos(ax)$

Integration by substitution is a way of undoing the derivative's chain rule. You need the integral to look like this:

$\int f(g(x))g'(x)\mathrm{d}x = \int f(u)\mathrm{d}u$

Here are some special derivatives:

- $\frac{\mathrm{d}}{\mathrm{d}x}e^{ax} = ae^{ax}$
- $\frac{\mathrm{d}}{\mathrm{d}x}\ln ax = \frac{1}{x}$

- $\frac{\mathrm{d}}{\mathrm{d}x}\sin ax = a\cos ax$
- $\frac{\mathrm{d}}{\mathrm{d}x}\cos ax = -a\sin ax$

All derivatives on the formula sheet are written using the chain rule.

The chain rule follows this general rule: $f(g(x)) = f'(g(x)) \cdot g'(x)$

A vector is a quantity that has both magnitude and direction. The length of the line sohws its magnitude and the arrowhead points in the direction. To add vectors, place the tip of the first vector to the tail of the second vector. The resultant is the arrow drawn from the tail of the first vector to the tip of the second vector.

The goal of subtracting vectors is to turn it into addition by finding the inverse of the second vector. Basically: $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

When adding or subtracting vectors algebraically, the first thing you need to do is to resolve the vectors into components.

- $A_x = A\cos\theta$
- $A_y = A\sin\theta$

Once all the vectors are broken down, you can add the horizontal and vertical components. This will give you the horizontal and vertical components of the resultant.

To find the magnitude of the resultant, you can find the hypotenuse: $R = \sqrt{R_x^2 + R_y^2}$. The direction can be found from $\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$

There are times when you need to "scale" up or down a vector. To do so, you multiply the magnitude of a vector, but not the direction, by a scalar.

A unit vector has a magnitude of 1 and a direction that goes along one of the axes.

The dot product is the process of multiplying two vectors and getting a scalar answer in return. There are two ways to find this:

- $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y$
- $\vec{a} \cdot \vec{b} = |a||b|\cos\theta$

The second method also helps determine if the vectors are orthogonal, or perpendicular to each other.

The cross product is the process of multiplying two vectors and getting a vector in return. The answer is a vector that is at a right angle to the two original vectors. The magnitude of the cross product equals the area of the parallelogram with the two original vectors as sides.

The cross product is zero in length when the original vectors point in the same or opposite directions. It reaches maximum length when the original vectors are at right angles to each other.

There are two ways to calculate the cross product.

The first is $\vec{a}$x$\vec{b} = [|a||b|\sin\theta]\hat{n}$

This method does not give you the direction of the vector.

The second way is to use a set of formulas to find the components:

- $C_x = a_y b_2 - a_2 b_y$
- $C_y = a_2 b_x - a_x b_2$
- $C_z = a_x b_y - a_y b_x$

The direction is determined by the right hand rule. Your index finger points in the direction of vector a, your middle points in the direction of b, and your thumb points in the direction of the answer.

1.1 Electric Charge and Electric Force

Electric charge is a fundamental property of all matter.

Charge is scalar value, which means it has no direction, and is described as either positive or negative.

The magnitude of charge on a single electron is the elementary charge which is $e = 1.6 \times 10^{-19}$ C (coulomb). The coulomb is the unit of charge.

Coulomb's Law describes the electrostatic force between two charges objects. The equation for this is:

$$F_{\mathsf{E}} = \frac{kq_1q_2}{r^2}$$

This equation is similar to the universal gravitation formula. Note that r can be written sometimes as d, it is the distance between the centers. k is the electrostatic constant and is equal to 9×10^9N·m^2/C^2. k is sometimes written as $\frac{1}{4\pi\epsilon_0}$.

The direction of the electrostatic force depends on the signs. Opposite charges attract and like charges repel. Electrostatic force can also cause other forces like tension, friction, and normal force.

Electrostatic force can be attractive (different signs) or repulsive (same signs), while gravitational force, which is similar, can only be attractive.

The electrostatic force has a much larger magnitude than gravitational force, but gravitational force acts on a larger scale in that the electrostatic force works at a microscropic scale, while gravitational force will be on a planetary scale.

Free space (a region where there is no electromagnetic or gravitational fields) has a constant value of electric (or vacuum) permittivity which is equal to $\epsilon_0 = 8.85 \times 10^{-12}$C^2/(N·m^2).

Example

Point charges Q_1 = 2.0μC and Q_2 = -4.0μC are located at $\vec{r_1} = (4.0\hat{i} - 2.0\hat{j} + 5.0\hat{k})$m and $\vec{r_2} = (8.0\hat{i} + 5.0\hat{j} - 9.0\hat{k})$m. What is the force of Q_2 on Q_1?

We have the equation

$$F_{\mathsf{E}} = \frac{kq_1q_2}{r^2}$$

We first have to find the distance between the charges. We can use the distance formula for this:

$$\begin{aligned} r &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z^1)^2} \\ r &= \sqrt{(8-4)^2 + (5+2)^2 + (-9-5)^2} \\ r &= 16.2\text{m} \end{aligned}$$

Now we can plug this into the equation.

$$\begin{aligned} F_{\mathsf{E}} &= \frac{kq_1q_2}{r^2} \\ F_{\mathsf{E}} &= \frac{(9 \times 10^9)(2 \times 10^{-6})(-4 \times 10^{-6})}{(16.2)^2} \\ F_{\mathsf{E}} &= -2.74 \times 10^{-4}\text{N} \end{aligned}$$

Note that since both point charges have opposite signs, they will try and attract each other, which means the resulting force calculated will be negative.

1.2 Conservation of Electric Charge and the Process of Charging

The net charge or charge distribution of a system can change in response to the prescence of, or changes in, the net charge or charge distribution of other systems. For example, the net charge can change due to friction or contact between systems.

Induced charge separation occurs when electrostatic force between two systems alters the distribution or charges within the systems, resulting in the polarization of one or both systems. Induced charge separation can only occur in neutral systems.

Any change to a system's net charge is due to a transfer of charge between the system and its surroundings. Most of the time, this is the result of a transfer of electrons.

An application of this is grounding, which involves electricially connecting a charged object to a much larger and approximately neutral system (such as the Earth).

Example

There are two identical metal spheres on insulating stands. Sphere 1 has a charge of -1.02×10^{-16}C. Sphere 2 has a deficit of 841 electrons. The two spheres are brought together so that they touch each other. They are then separated again so that they are no longer touching. What charge does each sphere have after they have touched? Consider the new charge on Sphere 2. Does this correspond to a deficit or an excess of electrons? How many electrons is the deficit/excess?

First, we must find the charge on Sphere 2. Because the problem states that there is a deficit of electrons, the charge is positive. The charge on Sphere 2 is:

$$841 \times (1.6 \times 10^{-19}) = 1.35 \times 10^{-16}\text{C}$$

Now we find the net charge on the system by adding it to the charge on Sphere 1:

$$1.35 \times 10^{-16} + (-1.02 \times 10^{-16}) = 3.3 \times 10^{-17}\text{C}$$

Now we simply divide this number by two since the charge is divded evenly between the two spheres.

$$\frac{3.3 \times 10^{-17}\text{C}}{2} = 1.6 \times 10^{-17}\text{C}$$

This answers the first part of the question.

The new charge is positive, which means there is an deficit of electrons. To find the amount of electrons in this deficit we must convert to electrons:

$$1.65 \times 10^{-17}\text{C} \times \frac{1e}{1.6 \times 10^{-19}\text{C}} = 103 \text{ electrons}$$

1.3 Electric Fields

Electric fields may originate from charged particles.

The electric field at a given point is the ratio of the electric force exerted on a test charge at the point to the charge and the charge on the test charge itself. Mathematically this is:

$$E = \frac{F_E}{q}[N/C]$$

Another way of writing this is:

$$F_E = qE$$

The E-field points away from an isolated positive charge towards an isolated negative charge. Therefore, if the test charge is negative, the electric force points opposite the direction of the electric field.

Example

(a) Find the direction and magnitude of an electric field that exerts a 4.80×10^{-17}N westward force on an electron.

(b) What magnitude and direction force does this field exert on a proton?

For part (a), we use the formula:

$$E = \frac{F_E}{q} = \frac{4.80 \times 10^{-17}\text{ N}}{1.6 \times 10^{-19}\text{ C}} = 300\text{ N/C East}$$

For part (b), the proton and electron have the same charge, so they experience the same electric field. The force and the E-field because of the positive charge results in the force pointing in the same direction, so 300 N/C West.

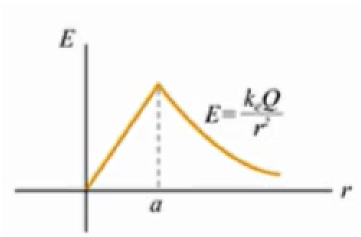
1.4 Electrostatic Equilibrium

Many problems on the FRQ section will involve conductors and insulators.

A conductor is an object or type of material that allows the flow or charge (electric current) in one or more directions. An insulator is a material in which electric current does not flow freely.

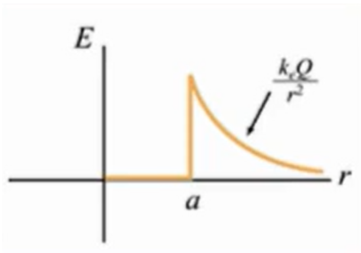
Electrostatic equilibrium occurs when there is no net motion of charge in an insulator or conductor.

When an insulator is in equilibrium, the excess charge of an insulator is distributed throughout the interior of the insulator as well as the surface. The electric field within the insulator may have a nonzero value.



In this, the peak is the surface of the insulator.

When in equilibrium, the excess charge on a conductor lies on the surface of the conductor, making the electric field equal to zero inside. The electric field is perpendicular to the surface of the conductor.



Example

A particle of charge 2.0×10^{-8}C experiences an upward force of magnitude 4.0×10^{-6}N when it is placed in a particular point in an electric field. (a) What is the electric field at that point? (b) If a charge $q = -1.0 \times 10^{-8}$C is placed there, what is the force on it?

For part A:

$$E = \frac{F_E}{q} = \frac{4.0 \times 10^{-6}\text{N}}{2.0 \times 10^{-8}\text{C}} = 200\text{N/C upward}$$

For part B:

$$F_E = gE = (1.0 \times 10^{-8}\text{C})(200\text{N/C}) = 2.0 \times 10^{-6}\text{N downwards}$$

1.5 Electric Fields of Charge Distributions

The expressions for the electric field of specified charge distributions can be found using integration and the principle of superposition.

Mathematically:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r^2}\hat{r}$$

Symmetry considerations of certain charge distributions can simplify analysis of the electric field resulting from those charge distributions.

Some common distributions are: Infinite Wire, Finite Wire, Semicircle (Arc), Ring of Charge

There are three densities - linear, area, and volume.

Linear: $\lambda = \frac{Q}{L} \longrightarrow \lambda = \frac{dQ}{dL} \longrightarrow dQ = \lambda dL$

Area: $\sigma = \frac{Q}{A} \longrightarrow \sigma = \frac{dQ}{dA} \longrightarrow dQ = \sigma dA$

Volume: $\rho = \frac{Q}{V} \longrightarrow \rho = \frac{dQ}{dV} \longrightarrow DQ = \rho dV$

$E_{PC} = \frac{kQ}{r^2} \therefore E = \frac{1}{4\pi\epsilon_0}\int \frac{dQ}{r^2}\hat{r}$

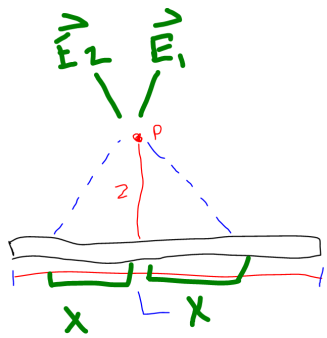
Example

Find the electric field a distance z above the midpoint of a straight line segment of length L that carries a uniform line charge density λ.

There will be two electric fields, therefore $\vec{E_{\text{net}}} = \vec{E_1} + \vec{E_2}$.

Therefore, $E_{1x}\hat{i} + E_{1z}\hat{k} + E_{2x}\hat{(} - \hat{i}) + E_{2z}\hat{k}$ and the x-components cancel in this case and gives us:

$$E_{1z}\hat{k} + E_{2z}\hat{k}$$



Now we use calculus:

$$\vec{E}_{\text{net}} = \int \frac{k\cos\theta dQ}{r^2} + \int \frac{k\cos\theta dQ}{r^2} = 2\int \frac{k\cos\theta dQ}{r^2}$$

Now we have:

$$2\int \frac{k\cos\theta\lambda dL}{r^2}$$

Because we know $\cos\theta = \frac{z}{\sqrt{z^2+x^2}}$, we can plug that in for $\cos\theta$:

$$\begin{aligned} &2\int \frac{k\cos\theta\lambda dL}{r^2} \\ &= 2\int k\lambda\left[\frac{z}{\sqrt{x^2+z^2}}\right]\frac{1}{(\sqrt{x^2+z^2})^2}dx \\ &= 2k\lambda z\int_0^{L/2}\frac{1}{(x^2+z^2)^{3/2}}dx \end{aligned}$$

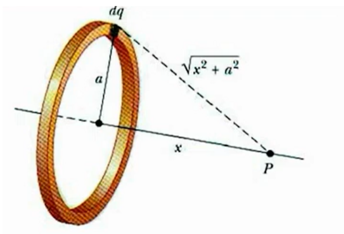
When we integrate this we get:

$$\begin{aligned} E &= 2k\lambda z\left[\frac{x}{z^2\sqrt{x^2+z^2}}\right]_0^{L/2} \\ &= 2k\lambda z\left[\frac{L/2}{z^2\sqrt{(L/2)^2+z^2-0}}\right] \\ &= \frac{2k\lambda z}{z^2}\left[\frac{L}{2\sqrt{\frac{L^2}{4}+z^2}}\right] \\ &= \frac{kQ}{Lz}\left[\frac{L}{\sqrt{L^2/4+z^2}}\right] \\ &= \frac{kQ}{z\sqrt{L^2/4+z^2}} \end{aligned}$$

and we are done.

Example

Derive an expression for the electric field at the center of a ring of charge with radius a a distance P away from the center.



Note that $\sqrt{x^2+a^2}$, a, and x are constant.

Looking at point P, there will be an infinite amount of electric fields caused by dq at point P.

All the y-components will cancel, so we will integrate through x.

$$\begin{aligned} E_{net} &= dE_x = dE\cos\theta \\ E_x &= \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r^2} \end{aligned}$$

We want to substitute $\frac{x}{\sqrt{x^2+a^2}}$ for $\cos\theta$.

$$\begin{aligned} E_x &= \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r^2}\cos\theta \\ &= \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\sqrt{x^2+a^2}}\cdot\frac{x}{\sqrt{x^2+a^2}} \\ &= \frac{x}{4\pi\epsilon_0(x^2+a^2)^{3/2}}\int dQ \\ &= \frac{qx}{4\pi\epsilon_0(x^2+a^2)^{3/2}} \end{aligned}$$

This is the expression for the ring of charge.

1.6 Electric Flux

Flux describes the amount of a given quantity that passes through a given area.

For an electric field that is constant across an area, the electric flux through the area is defined as:

$$\phi_E = \vec{E}\cdot\vec{A} = EA\cos\theta$$

The direction of the area vector is defined as perpendicular to the plane of the surface and outward from closed surface.

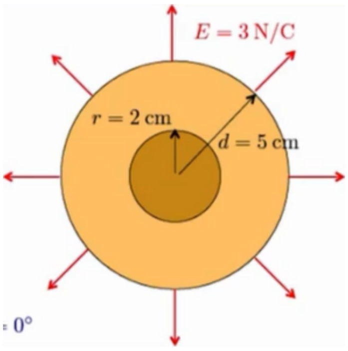
The sign of the flux is given by the dot product of the electric field vector and the area vector.

The total electric flux passing through a surface is defined by the surface integral of the electric field over the surface:

$$\phi_E = \int \vec{E}\cdot \mathrm{d}\vec{A} = EA$$

Example

A sphere of radius $r = 2$ cm creates an electric field $E = 3$N/C at a distance $d = 5$ cm from the center of the sphere. What is the electric flux through the surface of the sphere drawn at a distance $d = 5$cm?



The surface area of a sphere is $4\pi r^2$, so

$$\phi_E = \int \vec{E}\cdot \mathrm{d}\vec{A} = EA = E(4\pi r^2) = (3)(4\pi(0.05)^2) = 0.094\text{N}\cdot\text{m}^2/\text{C}$$

1.7 Gauss's Law

Gauss's law relates electric flux to a Gaussian surface to the charge enclosed by that surface:

$$\phi_E = \frac{q_{\text{enc}}}{\epsilon_0} = \oint \vec{E}\cdot \mathrm{d}\vec{A} = EA$$

A gaussian surface is a three-dimensional, closed surface.

The total electric flux through the surface is independent of the size of the Gaussian surface if the amount of enclosed charge remains constant.

Surfaces are constructed such that the electric field generated by the enclosed charge is either perpendicular or parallel to different regions of the Gaussian surface.

If a function of charge density is given for a charge distribution, the total charge can be determined by integrating the charge density of the length (1D), area (2D), or volume (3D).

Maxwell's equations are the collection of equations that fully describe electromagnetism. The first of these is Gauss's Law.

Example

A spherical cloud of charge radius R contains a total charge $+Q$ with a nonuniform charge density that varies according to the equation:

$$\rho(r) = \rho_0 \left(1 - \frac{r}{R}\right) \text{ for } r \leq R \text{ and}$$
$$\rho = 0 \text{ for } r > R,$$

where r is the distance from the center of the cloud. Express all algebraic answers in terms of Q, R, and fundamental constants. Determine the magnitude E of the electric field when $r > R$.

$$\frac{q_{\text{enc}}}{\epsilon_0} = \int \vec{E} \cdot \mathrm{d}\vec{A} = EA = E(4\pi r^2)$$
$$\implies E = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r^2}$$

2 Electric Potential

3 Conductors and Capacitors

4 Electric Circuits

5 Magnetic Fields and Electromagnetism

6 Electromagnetic Induction