MATH 2415: Calculus 3
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1 Vectors and the Geometry of Space

1.1 Vectors in the Plane

Vectors are quantities with both magnitude and direction. The vector who's tail is at point P and head at point Q is denoted as $\vec{PQ}$.

Two vectors, **u** and **v** are equal if they have equal length and point in the same direction. They do not necessarily have to be in the same location.

Scalar multiplication happens when a c is multiplied to vector **v**. If $c < 0$, then vector cv and v will point in opposite directions, otherwise they will point in the same direction. Two vectors are parallel if they are scalar multiples of each other.

If you place the tail of a vector **v** at the head of another vector **u**, the sum **u+v** is the vector that extends from the tail of **u** to the head of **v**.

The vector difference **u-v** is defined as **u+(-v)**.

In order to do calculations with vectors, we must introduce a cartesian plane. Angle brackets $\langle a, b \rangle$ show the components of a vector.

The magnitude of a vector is simply its length. Given points $P(x_1, y_1)$ and $Q(x_1, y_1)$, the magnitude of the vector $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$ is denoted as $|\vec{PQ}|$, is equal to:

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We can also now do vector addition with components. Given two vectors **u** and **v**, the vector sum is:

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$$

For a scalar c and a vector **u**, the scalar multiple is $c\mathbf{u}$. i.e $|c\mathbf{u}| = |c||\mathbf{u}|$

A unit vector is any vector with length 1. **i** is a unit vector in the x-direction and **j** is a unit vector in the y-direction.

> **Example**
>
> Determine the necessary air speed and heading that a pilot must maintain in order to fly her commercial jet north at a speed of 480 mi/hr relative to the ground in a crosswind that is blowing $60\deg$ south of east at 20 mi/hr.
>
> Let $\vec{p}$ be the velocity vector we are trying to find.
>
> We have ground vector $\langle 0, 480 \rangle$ and a crosswind vector $\langle 10, -10\sqrt{3} \rangle$. Note we got the x-component and the y-component of the crosswind vector from the formula: $\vec{v} = \langle a\cos\theta, b\cos\theta \rangle$
>
> We can find the vector $\vec{p}$ from adding this vector to the crosswind vector resulting in: $\vec{p} = \vec{g} - \vec{c} = \langle -10, 480 + 10\sqrt{3} \rangle$.
>
> The magnitude of this vector is the speed and is equal to 497.2 mi/hr roughly.

1.2 Vectors in Three Dimensions

We can create a z-axis to create a three dimensional system.

The xyz-plane is divided into octants and has 3 planes, the xy-plane, the xz-plane, and the yz-plane.

We can also extend the distance formula to 3 dimensions. It is similar to the distance formula in two dimensions, with the z-component added, essentially:

$$|PQ| = \sqrt{|PR|^2 + |RQ|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The midpoint formula works the same way:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2} + \frac{y_1 + y_2}{2} + \frac{z_1 + z_2}{2} \right)$$

The normal form of the circle equation is: $(x - h)^2 + (y - k)^2 = r^2$.

For a disk we have $(x - h)^2 + (y - k)^2 \leq r^2$

We can generalize this to a sphere. A sphere centered at (a, b, c) with radius r is the set of points satisfying:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

A ball centered at (a, b, c) with radius r is the set of points satisfying:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2$$

Example

Find an equation of the sphere passing through $P(-4, 2, 3)$ and $Q(0, 2, 7)$ with its center at the midpoint of PQ.

We can find the midpoint from the midpoint formula and it is equal to $(-2, 2, 5)$.

The radius can be found through the distance formula and is equal to $\sqrt{8}$.

The equation is $(x + 2)^2 + (y - 2)^2 + (z - 5)^2 = 8$

All the vector operations from two-dimensions work in three-dimensions.

Example

A model airplane is flying horizontally due east at 10 mi/hr when it encounters a horizontal crosswind blowing south at 5 mi/hr and an updraft blowing vertically upward at 5 mi/hr.

- Find the position vector that represents the velocity of the plane relative to the ground.
- Find the speed of the plane relative to the ground.

The velocity vector of the model plane $\vec{p}$ is equal to $\langle 10, 0, 0 \rangle$

The velocity vector of the horizontal crosswind $\vec{w}$ is equal to $\langle 0, -5, 0 \rangle$

The velocity vector of the updraft $\vec{u}$ is $\langle 0, 0, 5 \rangle$

Adding the three vectors results in the speed: $\langle 10, -5, 5 \rangle$

The magnitude of this is roughly 12.25.

1.3   Dot Products

Definition

Given two nonzero vectors **u** and **v** in two or three dimensions, the dot product is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$$

The dot product is 0 when $\theta = \frac{\pi}{2}$, negative when $\theta > \frac{\pi}{2}$ and positive when $\theta < \frac{\pi}{2}$.

Two vectors are parallel if and only if $\mathbf{u}\cdot\mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$

When the dot product is zero, we call $\vec{u}$ and $\vec{v}$ orthogonal.

Theorem 1.1: Dot Product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

We now apply the dot product to vector projections.

Definition

The orthogonal projection of $\mathbf{u}$ on $\mathbf{v}$, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$ is:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right)$$

The orthogonal projections can also be computed with the formulas:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{scal}_{\mathbf{v}}\mathbf{u} \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v}$$

where the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is

$$\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$$

1.4 Cross Products

Definition

Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the cross product $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$$

Note that $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

There are some useful properties of the cross product.

- The cross product $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
- The cross product is zero when $\sin(\theta) = 0$.
- Two vectors are parallel if the cross product between them is zero.

We define the determinant of a 2×2 array $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ as $ad - bc$.

For a matrix $\begin{vmatrix} a & b & c \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ the determinant is $a \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - b \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + c \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$

or $a(u_2 v_3 - u_3 v_2) - b(u_1 v_3 - u_3 v_1) + c(u_1 v_2 - u_2 v_1)$.

1.5 Lines and Planes in Space

Recall that in two dimensions, we needed a point and a slope to write an equation for a line.

We can write an equation in three dimensions as well.

A vector equation of a line passing through the point $P_0(x_0, y_0, z_0)$ in the direction of vector $\mathbf{v} = \langle a, b, c \rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ or:

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$

The corresponding parametric equations of the line also are:

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct$$

The general equation of a plane in R^3 with the plane passing through $P_0(x_0, y_0, z_0)$ with vector $\mathbf{v} = \langle a, b, c \rangle$ is described by:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

1.6 Cylinders and Quadric Surfaces

2   Vector-Valued Functions

2.1   Vector-Valued Functions

2.2   Calculus of Vector-Valued Functions

2.3   Motion in Space

2.4   Length of Curves

2.5   Curvature and Normal Vectors

3 Functions of Several Variables

3.1 Graphs and Level Curves

3.2 Limits and Continuity

3.3 Partial Derivatives

3.4 The Chain Rule

3.5 Directional Derivatives and the Gradient

3.6 Tangent Planes and Linear Approximation

3.7 Maximum/Minimum Problems

3.8 Lagrange Multipliers

4 Multiple Integration

4.1 Double Integrals over Rectangular Regions

4.2 Double Integrals over General Regions

4.3 Double Integrals in Polar Coordinates

4.4 Triple Integrals

4.5 Triple Integrals in Cylindrical and Spherical Coordinates

4.6 Integrals for Mass Calculations

4.7 Change of Variables in Multiple Integrals

5 Vector Calculus

5.1 Vector Fields

5.2 Line Integrals

5.3 Conservative Vector Fields

5.4 Green's Theorem

5.5 Divergence and Curl

5.6 Surface Integrals

5.7 Stokes' Theorem

5.8 Divergence Theorem