1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

A linear equation in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the coefficients $a_1, \ldots, a_n$ are real or complex numbers, usually known in advance.

A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables - say, $x_1, \ldots, x_n$.

A solution of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.

The set of all possible solutions is called the solution set of a linear system.

Two linear systems are called equivalent if they have the same solution set.

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system of linear equations is said to be consistent if it has either one solution or infinitely many solutions.

A system is inconsistent if it has no solution.

The essential information of a linear system can be recorded compactly in a rectangular array called a matrix. Given a system,

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

, with the coefficients of each variable aligned in columns, the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

is called the coefficient matrix (or matrix of coefficients) of the system.

An augmented matrix of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.

For the given system of equations, $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$

is called the augmented matrix of the system.

The size of a matrix tells how many rows and columns it has. If m and n are positive integers, an $m \times n$ matrix is a rectangular array of numbers with m rows and n columns. (The number of rows always comes first.)

For example, $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$ is a 2×4 matrix.

The basic strategy for solving a linear system is to replace one system with an equivalent system (i.e., one with the same solution set) that is easier to solve.

Elementary row operations include the following:

1. (Replacement) Replace one row by the sum of itself and a multiple of another row. For example, $2R_1 + R_3 \to R_3$.
2. (Interchange) Interchange two rows. For example: $R_1 \leftrightarrow R_4$.
3. (Scaling) Multiply all entries in a row by a nonzero constant. For example: $\frac{1}{3}R_3 \to R_3$.

Two matrices are called row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other.

It is important to note that row operations are reversible.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Two fundamental equations about a linear system are as follos:

1. Is the system consistent: that is, does at least one solution exist?
2. If a solution exists, is it the only one, that is, is it unique?

Example

Determine if the following system is consistent:

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

First interchange row 1 and 2 to get

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Do the operation $-\frac{5}{2}R_1 + R_3 \to R_3$:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

Do the operation $2R_3 + R_2 \to R_3$:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

That bottom row says that no x_1, x_2 or x_3 will equal 5, or $0 = 5$. This is not true, so the system is inconsistent.

1.2 Row Reduction and Echelon Forms

Leading entry of a row: the first (counting from left to right) non-zero entry (in a nonzero row)

Echelon Form: upper right-hand stair-case, triangle

1. all rows that consist entirely of zeros are grouped together at the bottom of the matrix

2. the first (counting from left to right) non-zero entry in the $(i+1)$st row must appear in a column to the right of the first non-zero entry in the ith row.
3. all entries in a column below a leading entry are zeros

Reduced Echelon Form: an echelon Form matrix that also has the following properties:

1. the leading entry in each nonzero row is 1
2. each leading one is the only nonzero entry in its column

A pivot point: a location in a matrix A that corresponds to a leading 1 in a reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

Steps to solving a system of linear equations:

1. begin with the leftmost nonzero column. This is the pivot column, the pivot position is at the top.
2. select a non zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
3. Use row replacement operations to create zeros in all positions below the pivot.
4. Cover (or ignore) the row containing the pivot position and cover all rows (if any) above it. Apply previous steps to the sub matrix that remains. Repeat until there are no more nonzero rows to modify.
5. Beginning with the rightomst pivot, create zeros above each pivot. Make each pivot equal to 1 by scaling.

Example

Determine the existence and uniqueness of the solution to

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

We can find that $x_5 = 4$, and x_3, x_4 are of infinite number of solutions.

Example

Find the general solution of the linear system whose augmented matrix has been reduced to $\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$

We have that $x_5 = 7$, $x_3 = 1/2(3 + 8x_4 + 7)$ and $x_1 = -4 - 6x_2 - 2(1/2)(3 + 8x_4 + 7) + 14$

Theorem 1.1: Existence and Uniqueness

A linear system is consistent if and only if the right most column of the augmented matrix is not a pivot column, that is if and only if an echelon form of the augmented matrix has no row of the form $[0, \ldots, 0b]$ with b nonzero. If the system if consistent in the solution contains either a unique solution when there are no free variables or infinitely many solutions when there is at least one free variable.

1.3 Vector Equations

Vectors in $\mathbf{R}^2$: a matrix with only 1 column is called a vector. The set of all vectors with 2 entries is $\mathbf{R}^2$

$\vec{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ for example.

The vectors are odered pairs of real numbers:

$\vec{u}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (3, -1) \neq (-1, 3) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Vector addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

Geometric interpretation: parallelogram law.

$(2, 4) + (3, 1) = (2 + 3, 4 + 1) = (5, 5)$. We can create a parallelogram (google for review).

Let's say we subtract, $-a$ has the same magnitude as a but points in the opposite direction in this case. $(1, 3) - (2, 1) = (-1, 2)$. Drawing the line from the origin to the tip of the vectors give you what you get algebraically.

Vector multiplication: scalar multiplication: $a(x, y) = (ax, ay)$ where a is a scalar.

Geometric interpretation: $a(x, y, z)$ points in the same direction as (x, y, z) but is scaled by a factor of a.

Example

Let $\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ Find $4\vec{u}$, $-3\vec{v}$ and $4\vec{u} + (-3)\vec{v}/$

$4\vec{u} = (4, -8)$, $-3\vec{v} = (-6, 15)$ and $4\vec{u} + (-3)\vec{v} = (-2, 7)$

Vectors start at the origin and have magnitude and direction.

Representing vectors in $\mathbf{R}^3$. We add the z-axis.

Vectors in $\mathbf{R}^n$ we have that $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = (u_1, u_2, \ldots, u_n)$ where $u_1, u_2, \dots \in \mathbb{R}$.

The $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is the zero vector.

Algebraic Properties of $\mathbf{R}^n$: for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbf{R}^n$ and all scalars c and d:

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\mathbf{u} + 0 = 0 + \mathbf{u} = \mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = 0$
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $c(d\mathbf{u}) = (cd)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

Linear Combinations: given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbf{R}^n$ and scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$ is called a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p$ with weights $c_1, c_2, \ldots, c_p$.

For example, if we have $3^{1/2}\mathbf{v}_1 + \mathbf{v}_2$ this can be written as $\vec{y} = \sqrt{3}\vec{v_1} + \vec{v_2}$ with $c_1 = \sqrt{3}$ and $c_2 = 1$.

Example

If $\vec{a_1} = (1, -2, -5), \vec{a_2} = (2, 5, 6)$, and $\vec{a_3} = (7, 4, -3)$ then determine if $\vec{b}$ can be written as a linear combination of $\vec{a_1}$ and $\vec{a_2}$. That is determine if there exists weights x_1, x_2 such that $x_1\vec{a_1} + x_2\vec{a_2} = \vec{b}$.

Using elementary row operations, we can determine that $x_1 = 3$, $x_2 = 2$ which is the linear combination of $\vec{a_1}$ and $\vec{a_2}$.

A vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots x_n\mathbf{x}_n = \mathbf{b}$ has the same solution set as the linear system with augmented matrix $[\mathbf{a}_1\mathbf{a}_2 \dots \mathbf{a}_n\mathbf{b}]$. In particular $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \dots \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix $[\mathbf{a}_1\mathbf{a}_2 \dots \mathbf{a}_n\mathbf{b}]$

Span: if $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p$ are in $\mathbf{R}^n$ then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ is denoted Span$\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_p\}$ and is called the subset of $\mathbf{R}^n$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p$. That is, Span$\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_n\}$ is the collection of all vectors that can be written in the form: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$ with $c_1, c_2, \dots, c_p$ scalars.

Asking if a vector $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_p\}$ amounts to asking whether the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_p = \mathbf{b}$ has a solution, or equivalently whether the linear system with augmented matrix $[\mathbf{v}_1\mathbf{v}_2\mathbf{v}_p\mathbf{b}]$ has a solution.

Note Span$\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_p\}$ contains every scalar multiple of $\mathbf{v}_1$.

The span of a single vector is a line. The span of 2 linearly independent vectors is a plane (not scalar multiples of each other).

1.4 The Matrix Equation Ax = b

The Matrix Equation: if A is an $m \times n$ matrix with columns, $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ and if $\mathbf{x}$ is in $\mathbf{R}^n$, then the product of A and $\mathbf{x}$ denoted $A\mathbf{x}$ is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights.

Note: $\mathbf{A}x$ is defined only if the number of columns of A equals the numbers of entries in x.

For example: $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4(1,0) + 3(2,-5) + 7(-1,3) = (3,6)$

Theorem 1.2: Matrix Equation, Vector Equation, System of Linear Equations

If A is an $m \times n$ matrix, with columns, $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ and if $\mathbf{b}$ is in $\mathbf{R}^m$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution as the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$ which in turn has the same solution as the system of linear equations represented by the augmented matrix $[\mathbf{a}_1\mathbf{a}_2 \dots \mathbf{a}_n\mathbf{b}]$.

Existence of Solutions: the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Example

Is $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & 7 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ Is the equation $A\vec{x} = \vec{b}$ consistent for all $\vec{b}$. Using rref, we get that $0 = -2b_1 + b_2 - 2b_3$, so it is not consistent for every $\vec{b}$. It is only consistent if $b_2 = 2b_1 + 2b_3$.

So let $\vec{b} = (1, 4, 1)$ and then do rref again and we get that x_3 is free, $x_2 = 1/7(4 - 5x_3)$ and $x_1 = 1 - 3(x_2) - 4x_3$ and this basically gives us $(1, 4, 1)$ too.

Theorem 1.3: Existence of soultion for $A\mathbf{x} = \mathbf{b}$

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, they are all true statements or they are all false:

1. for each $\mathbf{b}$ in $\mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution
2. each $\mathbf{b}$ in $\mathbf{R}^m$ is a linear combination of the columns of A
3. The columns of A span $\mathbf{R}^m$

4. A has a pivot position in every row. Note: A is a coefficient matrix, not an augmented matrix.

Computation of $A\mathbf{x}$ - an efficient method (matrix multiplication): if the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of the corresponding entries from row i of A and from vector $\mathbf{x}$.

The above is trivial.

Properties of the Matrix-Vector Product $A\mathbf{x}$

Theorem 1.4

if A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbf{R}^n$, and c is a scalar, then :

1. $A(\mathbf{u} + \mathbf{v})$
2. $A(c\mathbf{u}) = c(A\mathbf{u})$

Algebraic Properties of $\mathbf{R}^n$: for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbf{R}^n$ and all scalars c and d. (This was from an above topic)

1.5 Solution Sets of Linear Systems

Parametric Vector Form of Solutions:

- Parametric Vector Form of a Plane: a plane can be expressed in explicit form, such as $10x_1 - 3x_2 - 2x_3 = 0$ or implicit form: $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$, for s and t scalars.
- Parametric Form of a Line containing point $\mathbf{p}$ in direction of $\mathbf{v}$: $l(t) = \mathbf{p} + t\mathbf{v}$

The parametric equation of a plane in $\mathbf{R}^2$: $\mathbf{x} = a\mathbf{v} + b\mathbf{s}$.

The span of 2 non-colinear vectors is a plane. Span$\{\mathbf{v}, \mathbf{s}\}$ = the $\mathbf{R}^2$ plane.

In $\mathbf{R}^3$, the Span is still a plane, just in $\mathbf{R}^3$.

The parametric equation of a line in $\mathbf{R}^2$: $\mathbf{l} = \mathbf{p} + t\mathbf{v}$.

Homogeneous Linear Equation - $A\mathbf{x} = 0$.

The homogeneous equation always has at least 1 solution, $\mathbf{x} = 0$ (the trivial solution).

Recall that a system of linear equation either has infinitely many solutions, no solution, or a unique solution.

The question is whether there exists a nontrivial solution (in which case there are infinitely many solutions).

- The homogeneous equation $A\mathbf{x} = 0$ has nontrivial solution if and only if the equation has at least 1 free variable

Description of solutions: if the solutions consists of:

- the 0 vector: Span$\{0\}$
- 1 free variable: Span$\{\mathbf{v}\}$, the solutions are a line through the origin
- 2 free variables, Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane through the origin

Example

Determine if the following system has a nontrivial solution. Then describe the solution set

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

x_3 is free. And everything is in the form $(4/3, 0, 1)$.

Nonhomogeneous Equation: $A\mathbf{x} = \mathbf{b}$.

For example, in the previous example when we let $x_3 = 0$ we get $(-1, 2, 0)$.

Theorem 1.5

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$ and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the Form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = 0$.

1.6 Applications of Linear Systems

There are three examples here: economics, chemical equations and network flow.

Start with economics. There exist equilibrium prices that can be assigned to the total outputs of the various sectors in an economy in such a way that the income of each sector exactly balances its expenses.

You can use row operations to find an equilibrium price.

Other examples run similarly (sorry for bad note taking today I'm sick)

1.7 Linear Independence

Linear Independence: an indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is said to be

- linearly independent if the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = 0$ has only the trivial solution
- linearly dependent if there exists weights $c_1, c_2, \ldots, c_p$ not all zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = 0$

I'm too lazy to write matrices so much.

Linear Independence of Matrix Columns: the columns of matrix A are linearly independent if and only if the equation $A\mathbf{x} = 0$ has only the trivial solution.

Sets of One or Two Vectors

- A set with 1 vector is linearly independent iff $\mathbf{v}$ is not the 0 vector because $x_1\mathbf{v} = 0$ has only the trivial solution
- the zero vector, $\mathbf{0}$ is linearly dependent because $x_1\mathbf{0} = \mathbf{0}$ has many nontrivial solutions
- two vector $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly dependent iff at least one of the vectors is a multiple of the other

Theorem 1.6

Characterization of Linearly Dependent Sets: An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of 2 or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1$ is not $\mathbf{0}$, then some $\mathbf{v}_j$ with $j > 1$ is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

Note: the theorem does not say every vector in a linearly dependent set is a linear combination of preceding vectors

Theorem 1.7

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is linearly dependent if $p > n$.

Theorem 1.8

If a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ contains the zero vector, then the set is linearly dependent.

1.8 Introduction to Linear Transformations

Linear Transformations: we can view $A\mathbf{x} = \mathbf{b}$ as a mapping: the $m \times n$ matrix A is the transform, $A : \mathbf{R}^n \to \mathbf{R}^m$.

From this point of view, solving the equation $A\mathbf{x} = \mathbf{b}$ amounts to finding all the vectors $\mathbf{x}$ in $\mathbf{R}^n$ that are transformed to $\mathbf{b}$ in $\mathbf{R}^m$. The correspondence from $\mathbf{x}$ to $A\mathbf{x}$ is a function from one set of vectors to another.

Definition

A transform (or function or mapping) T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$. The set $\mathbf{R}^n$ is called the domain of T, and $\mathbf{R}^m$ is called the codomain. The notion $T : \mathbf{R}^n \to \mathbf{R}^m$ indicates that the domain of T is $\mathbf{R}^n$ and the codomain is $\mathbf{R}^m$. For $\mathbf{x}$ in $\mathbf{R}^n T(\mathbf{x})$ in $\mathbf{R}^m$ is called the image of $\mathbf{x}$. The set of all images $T(\mathbf{x})$ is called the range of T.

Matrix Transformations: $T(\mathbf{x})$ is computed as $A\mathbf{x}$ where A is an $m \times n$ matrix. Note: the domain of T is $\mathbf{R}^n$ and the codomain of T is $\mathbf{R}^m$. The range of T is the set of all linear combinations of the columns of A.

Linear Transformations: a transformation (or mapping) T is linear if

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in the domain of T
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

- every matrix transformation is a linear transformation: $A(\mathbf{u} + \mathbf{v}) = A(\mathbf{u}) + A\mathbf{v}$ and $A(c\mathbf{u}) = cA(\mathbf{u})$
- linear transformations preserve the operations of vector addition and scalar multiplication

If T is a linear transformation, then

1. $T(\mathbf{0}) = \mathbf{0}$
2. $T(c\mathbf{u}+d\mathbf{v}) = cT(\mathbf{u})+dT(\mathbf{v})$ for all scalars c, d and all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T. The generalization $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_pT(\mathbf{v}_p)$ is known in engineering as the superposition principle: whenever an input is expressed as a linear combination of signals the systems response is the same linear combination of the responses to the individual signals.

1.9 The Matrix of a Linear Transformation

Goal: given a geometric desciprtion of a transformation, T, we want to find a "formula" for T

- Every linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$ can be represented by a matrix transformation $A(\mathbf{x})$.
- The key to finding matrix A is to that T is completely determined by what it does to the columns of the $n \times n$ identiy matrix, I_n.

Theorem 1.9

Standard Matrix for a Linear Transformation: let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^n$. And, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$ where $\mathbf{e}_j$ is jth column of the identity matrix in $\mathbf{R}^n$. $A = [T(\mathbf{e}_1)T(\mathbf{e}_2)\ldots T(\mathbf{e}_n)]$. A is caleld the standard matrix for the linear transformation T.

Onto/Existence: a mapping $T : \mathbf{R}^n \to \mathbf{R}^m$ is said to be onto $\mathbf{R}^m$ if each $\mathbf{b}$ in $\mathbf{R}^m$ is the image of at least 1 $\mathbf{x}$ in $\mathbf{R}^n$.

- T is onto $\mathbf{R}^m$ when the range of T is all of the codomain $\mathbf{R}^m$; for each **b** in $\mathbf{R}^m$, there exists at least one solution of $T(\mathbf{x}) = \mathbf{b}$. The mapping T is not onto when there is some **b** in $\mathbf{R}^m$ for which $T(\mathbf{x}) = \mathbf{b}$ has no solution.

T is one-to-one if for each **b** in $\mathbf{R}^n$, the equation $T(\mathbf{x}) = \mathbf{b}$ has either unique solution or no solution. The mapping is not one-to-one when some **b** in $\mathbf{R}^m$ is the image of more than one vector in $\mathbf{R}^n$.

Theorem 1.10

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation, then T is one-to-one iff $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem 1.11

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

1. T maps $\mathbf{R}^n$ onto $\mathbf{R}^m$ iff the columns of A span $\mathbf{R}^m$
2. T is one-to-one iff the columns of A are linearly independent

1.10 Linear Models in Business, Science, and Engineering

Linear Equations can be done in electrical networks.

Current flow in a simple electrical network can be described by a system of linear equations. Consider Ohm's Law, $V = IR$, which describes the current which passes through a resistor. The algebraic sum of IR voltage drops in one direction around a loop equals the algebraic of the voltage sources in the same direction around the loop.

The model for current flow is linear since the voltage drop across a resistor is proportional to the current flowing through it, and the sum of the voltage drops in a loop equals the sum of the voltage sources in the loop.

For difference equations, if there is a matrix A such that $\mathbf{x}_1 = A\mathbf{x}_0, x_2 = A\mathbf{x}_1$, and in general $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for $k = 0, 1, 2, \ldots$. then this is called a linear difference equation (or recurrence relation).

Ok whatever just use logic.