# 1 Functions of Several Variables

## 1.1 Graphs and Level Curves

**Definition**

A function $z = f(x, y)$ assigns to each point $(x, y)$ in a set $D$ in $\mathbb{R}^2$ a unique real number $z$ in a subset of $\mathbb{R}$. The set $D$ is the domain of $f$. The range of $f$ is the set of all real numbers $z$ that are assumed as the points $(x, y)$ vary over the domain.

**Example**

Find the domain of

$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2 - 25}}$$

Note that the denominator cannot be zero or is negative.

Therefore, the domain is $D = (x, y) : x^2 + y^2 > 25$.

**Definition**

For a function of two variables $f(x, y)$ a level curve is the set of points $(x, y)$ in the $xy$-plane where $f(x, y)$ is equal to a constant $z_0$.

We can extend our defintions to functions of more than two variables.

**Definition**

For a function of three variables $f(x, y, z)$ a level surface is the set of points $(x, y, z)$ in $xyz$-space where $f(x, y, z)$ is equal to a constant $w_0$.

We can describe level surfaces as well.

## 1.2 Limits and Continuity

We can write the limit of two variables as:

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L$$

There are some laws of limits.

**Theorem 1.1**

Let $a, b$, and $c$ be real numbers.

- Constant function $f(x, y) = c$:

$$\lim_{(x,y)\to(a,b)} c = c$$

- Linear function $f(x, y) = x$:

$$\lim_{(x,y)\to(a,b)} x = a$$

- Linear function $f(x, y) = y$:

$$\lim_{(x,y)\to(a,b)} y = b$$

If we let the limit of a function $f(x, y) = L$ and the limit of a function $g(x, y) = M$, we can see:

- The sum of the limits is $L + M$.
- The difference of the limits is $L - M$.
- If we multiply a function by a constant $c$, the limit is $cL$.
- The product of the limits is $LM$.
- The difference of the limits is $\frac{L}{M}$.
- Putting a limit to a power $n$ is $L^n$.

**Definition**

Let $R$ be a region in $\mathbb{R}^2$. An interior point $P$ of $R$ lies entirely within $R$, which means it is possible to find a disk centered at $P$ that contains only points of $R$.

A boundary point $Q$ of $R$ lies on the edge of $R$ in the sense that every disk centered at $Q$ contains at least one point in $R$ and at least one point not in $R$.

**Definition**

A region is open if it consists entirely of interior points. A region is closed if it contains all its boundary points.

The two-path test for nonexistence of limits:

If $f(x, y)$ approaches the two different values as $(x, y)$ approaches $(a, b)$ along two different paths in the domain of $f$, then the limit does not exist.

**Definition**

The function $f$ is continuous at the point $(a, b)$ provided:

- $f$ is defined at $(a, b)$.
- The limit for $f$ exists.
- The two quantities must be equal.

## 1.3 Partial Derivatives

The idea of a partial derivative is to differentiate $f$ with respect to one variable, treating the other variable as a constant.

**Definition**

The partial derivative of $f$ with respect to $x$ at the point $(a, b)$ is:

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

The partial derivative of $f$ with respect to $y$ at the point $(a, b)$ is:

$$f_y(a, b) = \lim_{h\to 0} \frac{f(a + h) - f(a, b)}{h}$$

provided these limits exist.

The notation for these are:

$$f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

We can first partially differentiate $f$ with respect to $x$ and then with respect to $x$ denoted as:

$$\frac{\partial^2 f}{\partial x^2}$$

We can do it first with respect to $x$ and then with respect to $y$ denoted as:

$$\frac{\partial^2 f}{\partial y \partial x}$$

We can also differentiate with respect to $y$ and then with respect to $x$:

$$\frac{\partial^2 f}{\partial x \partial y}$$

Lastly, we can differentiate with respect to $y$ twice:

$$\frac{\partial^2 f}{\partial y^2}$$

**Theorem 1.2: Equality of Mixed Partial Derivatives**

Assume $f$ is defined on an open set $D$ of $\mathbb{R}^2$, and that $f_{xy}$ and $f_{yx}$ are continuous throughout $D$. Then $f_{xy} = f_{yx}$ at all points of $D$.

## 1.4 The Chain Rule

**Theorem 1.3**

Let $z$ be a differentiable function on $x$ and $y$ on its domain, where $x$ and $y$ are differentiable functions of $t$ on an interval $I$. Then

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Let's suppose that we have a function $w = f(x, y, z)$ where $x, y, z$ depend on $t$.

Then we have:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial w}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

**Theorem 1.4**

Let $z$ be a differentiable function of $x$ and $y$, where $x$ and $y$ are differentiable functions of $s$ and $t$. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

**Theorem 1.5**

Let $F$ be differentiable on its domain and suppose $F(x,y) = 0$ defines $y$ as a differentiable function of $x$. Provided $F_y \neq 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$$

## 1.5 Directional Derivatives and the Gradient

Suppose we have the function $f(x,y)$.

$f_x(x,y)$ measures the rate of change in the direction of positive $x$-axis and $f_y(x,y)$ measures the rate of change in the direction of positive $y$-axis.

**Definition**

Let $f$ be differentiable at $(a,b)$ and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the $xy$-plane. The directional derivative of $f$ at $(a,b)$ in the direction of $u$ is

$$D_{\mathbf{u}}f(a,b) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

provided the limit exists.

The directional derivative $D_{\mathbf{u}}f(a,b)$ measures the rate of change of $f$ at the point $(a,b)$ in the direction of $\mathbf{u}$.

**Theorem 1.6**

Let $f$ be differentiable at $(a,b)$ and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the $xy$-plane. The directional derivative of $f$ at $(a,b)$ in the direction of $u$ is

$$D_{\mathbf{u}}f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle$$

**Definition**

Let $f$ be differentiable at the point $(x,y)$. The gradient of $f$ at $(x,y)$ is the vector-valued function

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

The applications of this are:

Directional derivatives: $D_{\mathbf{u}}f(a,b) = \nabla f(a,b) \cdot \mathbf{u}$

The gradient gives the direction of maximum increase of a function.

Gradient is normal to level curves.

**Theorem 1.7**

Let $f$ be differentiable at $(a,b)$ with $\nabla(a,b) \neq 0$.

$f$ has its maximum rate of increase at $(a,b)$ in the direction of the gradient $\nabla f(a,b)$. The rate of change in this direction is $|\nabla f(a,b)|$. (This will be maximized if $\theta = 0$.)

$f$ has its maximum rate of decrease at $(a,b)$ in the direction of $-\nabla f(a,b)$. The rate of change in this direction is $-|\nabla f(a,b)|$. (This will be minimized if $\theta = \pi$.)

The directional derivative is zero in any direction orthogonal to $\nabla f(a,b)$.

**Theorem 1.8**

Given a function $f$ differentiable at $(a, b)$, the line tangent to the level curve of $f$ at $(a, b)$ is orthogonal to the gradient $\nabla f(a, b)$, provided $\nabla f(a, b) \neq 0$.

We can now consider gradients of functions of three variables.

**Theorem 1.9**

Let $f$ be differentiable at $(a, b, c)$ and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector. The directional derivative of $f$ at $(a, b, c)$ in the direction of $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}} f(a, b, c) &= \nabla(a, b, c) \cdot \mathbf{u} \\ &= \langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \rangle \cdot \langle u_1, u_2, u_3 \rangle \end{aligned}$$

Assuming $\nabla f(a, b, c) \neq 0$, the gradient in three dimensions has the following properties.

$f$ has its maximum rate of increase at $(a, b, c)$ in the direction of the gradient $\nabla f(a, b, c)$, and the rate of change in this direction is $|\nabla f(a, b, c)|$.

$f$ has its maximum rate of decrease at $(a, b, c)$ in the direction of $-\nabla f(a, b, c)$, and the rate of change in this direction is $-|\nabla f(a, b, c)|$.

The directional derivative is zero in any direction orthogonal to $\nabla f(a, b, c)$.

## 1.6 Tangent Planes and Linear Approximation

**Definition**

Let $F$ be differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq 0$. The plane tangent to the surface $F(x, y, z) = 0$ at $P_0$, called the tangent plane, is the plane passing through $P_0$ orthogonal to $\nabla F(a, b, c)$. An equation of the tangent plane is

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

**Definition**

Let $f$ be differentiable at the point $(a, b)$. An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

One of the main applications of tangent planes is linear approximation for a function near a point.

**Definition**

Let $f$ be differentiable at $(a, b)$. The linear approximation to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the tangent plane at that point, given by the equation

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

For a function of three variables, the linear approximation to $w = f(x, y, z)$ at the point $(a, b, c, f(a, b, c))$ is given by

$$L(x, y, z) = f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) + f(a, b, c)$$

The idea for linear approximations of functions gives the defintion of the differential.

**Definition**

Let $f$ be differentiable at the point $(x, y)$. The change in $z = f(x, y)$ as the independent variables change from $(x, y)$ to $(x + \mathrm{d}x, y + \mathrm{d}y)$ is denoted $\Delta z$ and is approximated by the differential $\mathrm{d}z$.

$$\Delta z \approx \mathrm{d}z = f_x(x, y)\mathrm{d}x + f_y(x, y)\mathrm{d}y$$

Note that $\Delta z$ is the change in $f$ on the surface and is equal to $f(x, y) - f(a, b)$.

Note that $\mathrm{d}z$ is the change in $l$ on the tangent plane and is equal to $L(x, y) - f(a, b)$.

## 1.7 Maximum/Minimum Problems

**Definition**

Suppose $(a, b)$ is a point in a region $R$ on which $f(x, y)$ is defined. If $f(x, y) \leq f(a, b)$ for all $(x, y)$ in the domain of $f$ and in some open disk centered at $(a, b)$, then $f(a, b)$ is a local maximum value of $f$. If $f(x, y) \geq f(a, b)$ for all $(x, y)$ in the domain of $f$ and in some open disk centered at $(a, b)$, then $f(a, b)$ is a local minimum value of $f$. Local maximum and local minimum values are also cllaed local extreme values or local extrema.

**Theorem 1.10**

If $f$ has a local maximum or minimum value at $(a, b)$ and the partial derivatives $f_x$ and $f_y$ exist at $(a, b)$, then $f_x(a, b) = f_y(a, b) = 0$.

**Definition**

An interior point $(a, b)$ in the domain of $f$ is a critical point of $f$ is either:

- $f_x(a, b) = f_y(a, b) = 0$ or
- at least one of the partial derivatives $f_x$ and $f_y$ does not exist at $(a, b)$.

**Theorem 1.11**

Suppose the second partial derivatives of $f$ are continuous throughout an open disk centered at the point $(a, b)$, where $f_x(a, b) = f_y(a, b) = 0$. Let $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$.

- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f$ has a local maximum value at $(a, b)$.
- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f$ has a local minimum value at $(a, b)$.
- If $D(a, b) < 0$, then $f$ has a saddle point at $(a, b)$.
- If $D(a, b) = 0$, then the test is inconclusive.

**Definition**

Let $f$ be defined on a set $R$ in $\mathbb{R}^2$ containing the point $(a, b)$. If $f(a, b) \geq f(x, y)$ for every $(x, y)$ in $R$, then $f(a, b)$ is an absolute maximum value of $f$ on $R$. If $f(a, b) \leq f(x, y)$ for every $(x, y)$ in $R$, then $f(a, b)$ is an absolute minimum value of $f$ on $R$.

We can describe the procedure for finding the absolute maximum/minimum values on closed bounded sets:

Let $f$ be continuous on a closed bounded set $R$ in $\mathbb{R}^2$. To find the absolute maximum and minimum values of $f$ on $R$:

- Determine the values of $f$ at all critical points in $R$.
- Find the maximum and minimum values of $f$ on the boudnary of $R$.
- The greatest function value found in the previous steps is the absolute maximum value of $f$ on $R$, and the least function value found is the absolute minimum value of $f$ on $R$.

## 1.8 Lagrange Multipliers

The goal is to maximize (or minimize) a function $f(x, y)$ subject to the constraint $g(x, y) = 0$.

To find find absolute extrema on closed and bounded constraint curves using Lagrange Multipliers:

Let the objective function $f$ and the constraint function $g$ be differentiable on a region of $\mathbb{R}^2$ with $\nabla g(x, y) \neq 0$ on the curve $g(x, y) = 0$. To locate the absolute maximum and minimum values of $f$ subject to the constraint $g(x, y) = 0$, carry out the following steps:

1. Find the values of $x$, $y$, and $\lambda$ (if they exist) that satisfy the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y) \qquad \text{and} \qquad g(x, y) = 0$$

2. Evalute $f$ at the values $(x, y)$ found in Step 1 and at the endpoints of the constraint curve (if they exist). Select the largest and smallest corresponding function values. These values are the absolute maximum and minimum values of $f$ subject to the constraint.

(This section is kinda annoying, I don't like the amount of steps each question has) Basically, I hate optimization.