1 Three-Dimensional Coordinate Systems

1.1 Vectors in the Plane

The first goal for calculus 3 is to define the rectangular coordinate plane in 3-space.

In 2-space, this is the normal coordinate axis with coordinates (x, y).

In 3-space, you now have (x, y, z). We have an xy-plane, a yz-plane, and an xz-plane. Also we have octants in 3-space similar to quadrants in 2-space.

The z-axis is determined by the right-hand rule: if you curl the fingers on your right hand from the positive x-axis to the y-axis, your thumb points in the direction of the positive z-axis.

Exercise Graph $(2, 3, 4)$.

When graphing points in 3-space, it might be easier to draw a prism to see the three-dimensional components easier.

Exercise Graph $(-3, 2, -6)$.

In 3-space the distance formula is simliar to 2-space. It is the following:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Example

Find the distance between $(2, 3, 4)$ and $(-3, 2, -6)$.

Plug in the numbers into the formula and the answer gives $d = \sqrt{126}$

Similar to circles in 2-space, there are spheres in 3-space. The equation of a sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Example

Write the equation of a sphere with a diameter having endpoints $(-1, 2, 1)$ and $(0, 2, 3)$.

We can find the center with the midpoint formula, and the center is $(-\frac{1}{2}, 2, 2)$.

Using the $(-1, 2, 1)$ endpoint and the center calculated above, we can determine the radius to be $\sqrt{5/4}$.

Therefore the equation of the sphere is $(x + \frac{1}{2})^2 + (y - 2)^2 + (z - 2)^2 = \frac{5}{4}$.

Example

Find the center and radius of the sphere $x^2 + y^2 + z^2 - 2x - 4y + 8z + 17 = 0$.

When we complete the square for this, we end up getting $(x - 1)^2 + (y - 2)^2 + (z + 4)^2 = 4$.

Therefore, the center is $(1, 2, -4)$ and the radius is 2.

Theorem 1.1

An equation of the form $x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$ represents a sphere, a point, or has no graph.

We would get a point if the radius gives you 0, and if the radius is negative, then there is no graph.

Exercise Graph $y = 3$ in $\mathbb{R}^3$.

Exercise Describe and sketch the surface in $\mathbb{R}^3$ represented by $y = x$.

Exercise Graph $x^2 + z^2 = 1$.

Example

Describe the graph of $1 \leq x^2 + y^2 + z^2 \leq 4$. What if $z \leq 0$?

This will represent the region between the spheres and centered at the origin with radii of 1 and 2. (This is a sphere with center cut out).

The condition $z \leq 0$ will give us a hemisphere, it will only give the lower half of the figure.

1.2 Vectors

Definition: Vector

A vector indicates a quantity that has both magnitude and direction.

Examples include displacement, velocity, or force.

Some ways to notate vectors are **u**, $\overrightarrow{u}$, $\vec{u}$, $\overrightarrow{AB}$, $\vec{AB}$. For the last two of these, these are read as a vector starting at A, heading towards B.

We say that 2 vectors are equivalent (or equal) if they have the same length and direction.

0 vector (**0**) has a length of 0 and no direction. (Note that **0** is a vector because it is bolded.)

Definition: Vector Addition

If **u** and **v** are positioned so that the initial point of **v** is at the terminal point of **u**, then **u** $+$ **v** is the vector from the initial point of **u** to the terminal point of **v**.

Note that $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

Definition: Scalar Multiplication

If c is a scalar and **v** is a vector, then c**v** is the vector whose length is $|c|$ times the length of **v** and whose direction is the same as **v** if $c > 0$ and is opposite if $c < 0$.

Using the two definitions above, we can find the difference **u** $-$ **v**. From the above definitions, we can rewrite this as $\vec{u} + (-\vec{v})$, which is equivalent to $\vec{u} - \vec{v}$.

Exercise Sketch **a** $- 2$**b** and define **a** and **b** as you wish.

Coordinate Systems: Generally we place the initial point at the origin and then express a vector as **a** $= \langle a_1, a_2 \rangle$ or **a** $= \langle a_1, a_2, a_3 \rangle$ where (a_1, a_2) and (a_1, a_2, a_3) are terminal points.

Given points $A(x_1, y_1)$ and $B(x_2, y_2)$, vector **a** $=$ **AB** is **a** $= \langle x_2 - x_1, y_2 - y_1 \rangle$. (In 3-space the idea is similar.)

Example

Express vector $\overrightarrow{P_1P_2}$ in bracket notation if $P_1(1,3)$ and $P_2(4,-2)$.

Note that we start at the terminal point. Therefore we have $\overrightarrow{P_1P_2} = \langle 4-1, -2-3\rangle = \langle 3, -5\rangle$.

Note that the vector between the two points and the vector found have the same direction and same length.

Arithmetic Operations: If $\vec{v} = \langle v_1, v_2\rangle$ and $\vec{w} = \langle w_1, w_2\rangle$, then

- $\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2\rangle$
- $\vec{v} - \vec{w} = \langle v_1 - w_1, v_2 - w_2\rangle$
- $k\vec{v} = \langle kv_1, kv_2\rangle$

Example

If $\vec{a} = \langle -2, 0, 1\rangle$ and $\vec{b} = \langle 3, 5, -4\rangle$, find $\vec{a} + \vec{b}$ and $\vec{b} - 2\vec{a}$.

Finding $\vec{a} + \vec{b}$ is simple just add them together to get $\langle 1, 5, -3\rangle$.

To find $\vec{b} - 2\vec{a}$, multiply $\vec{a}$ by 2 to get $\langle -4, 0, 2\rangle$. Then subtracting gives you $\langle 7, 5, -6\rangle$.

Properties of Vectors:

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3. $\vec{a} + \mathbf{0} = \vec{a}$
4. $\vec{a} + (-\vec{a}) = \mathbf{0}$ (Additive Inverse)
5. $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
6. $(c+d)\vec{a} = c\vec{a} + d\vec{a}$
7. $(cd)\vec{a} = c(d\vec{a})$
8. $1\vec{a} = \vec{a}$

Example

Prove property #2 from above.

Let $\vec{a} = \langle a_1, a_2\rangle$, $\vec{b} = \langle b_1, b_2\rangle$, and $\vec{c} = \langle c_1, c_2\rangle$.

From the left side of property 2, we have $\langle a_1, a_2\rangle + (\langle b_1, b_2\rangle + \langle c_1, c_2\rangle)$.

From this, we can simplify to get $\langle a_1 + a_2\rangle + \langle b_1 + c_1, b_2 + c_2\rangle$.

This gives us $\langle a_1 + (b_1 + c_1), a_2 + (b_2 + c_2)\rangle$.

From the associative law we can rewrite this as $\langle (a_1 + b_1) + c_1, (a_2 + b_2) + c_2\rangle$.

This is $\langle a_1 + b_1, a_2 + b_2\rangle + \langle c_1 + c_2\rangle$, which is equivalent to $(\vec{a} + \vec{b}) + \vec{c}$. $\square$

Unit Vectors: A unit vector is a vector with a length of 1.

In 2-space, define $\vec{i} = \mathbf{i} = \langle 1, 0\rangle$ and $\vec{j} = \mathbf{j} = \langle 0, 1\rangle$ and 3-space, $\vec{i} = \mathbf{i} = \langle 1, 0, 0\rangle$, $\vec{j} = \mathbf{j} = \langle 0, 1, 0\rangle$, and $\vec{k} = \mathbf{k} = \langle 0, 0, 1\rangle$.

$\vec{i}, \vec{j}$, and $\vec{k}$ are called unit or standard basis vectors. All have length 1 and point in the positive direction on the $x-$,$y-$, and $z-$ axes.

For example, $\vec{a} = \langle 1, 3, -4\rangle$ can be expressed as $\vec{a} = \vec{i} + 3\vec{j} - 4\vec{k}$.

Example

If $\vec{a} = \vec{i} + 2\vec{j} - 3\vec{k}$, and $\vec{b} = 4\vec{i} + 7\vec{k}$, find $2\vec{a} + 3\vec{b}$.

Adding them together gives $14\vec{i} + 4\vec{j} + 15\vec{k}$.

The norm (or magnitude or length) of a vector is defined as

$$|\vec{v}| = ||\vec{v}|| = \sqrt{v_1^2 + v_2^2}$$

Example

Let $\vec{u} = \vec{i} - 3\vec{j} + 2\vec{k}$ and $\vec{v} = \vec{i} + \vec{j}$. Find:

- $||\vec{u}|| + ||\vec{v}||$ Use the above formula to get $\sqrt{14} + \sqrt{2}$.
- $||\vec{u} + \vec{v}||$. First add the two vectors to get $\langle 2, -2, 2\rangle$. The norm of this is $2\sqrt{3}$.
- $\frac{1}{||\vec{v}||}\vec{v}$ We previously found the magnitude of $\vec{v}$ to be $\sqrt{2}$. So we simply have $\frac{1}{\sqrt{2}}\langle 1, 1, 0\rangle$. This is $\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$.
- $||\frac{1}{||\vec{v}||}\vec{v}||$. The norm of $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\rangle$ is 1.

The vector found in the third part of the previous example is known as a unit vector because it has a norm of 1.

The process of obtaining a unit vector with the same direction is called normalizing $\vec{v}$.

Example

Find the unit vector in the direction of $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$.

The norm of a is $||\vec{a}|| = \sqrt{4 + 1 + 4} = 3$. Then normalizing $\vec{a}$ gives $\langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\rangle$.

Vectors in Polar Form: Any vector can be written in the following form

$$\vec{v} = ||\vec{v}||\langle \cos\theta, \sin\theta\rangle$$

We know that $||\vec{v}||$ is the magnitude of the vector and $\langle cos\theta, \sin\theta\rangle$ is the direction.

Example

Find the angle that $\vec{v} = \langle -\sqrt{3}, 1\rangle$ makes with the positive x-axis.

We know we can write this as $\langle -\sqrt{3}, 1\rangle = ||\vec{v}||\langle \cos\theta, \sin\theta\rangle$.

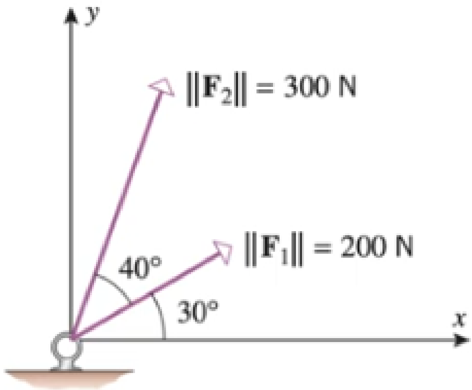
The magnitude of $\vec{v}$ is 2, so simplifying a little gives us $\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}\rangle = \langle \cos\theta, \sin\theta\rangle$.

From this, we can see that θ is the θ when $\cos\theta = -\frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$. So, $\theta = \frac{5\pi}{6}$.

Forces are often represented by vectors because they have a length and direction. If two forces are applied to the same point, they are concurrent. The two forces together form the resultant force, $\vec{F_1} + \vec{F_2}$.

Example

Suppose two forces are applied to an eye bracket. Find the magnitude of the resultant and the angle that it makes with the positive x-axis.



From the diagram, we can see that $\vec{F_1} = 200\langle\cos 30^\circ, \sin 30^\circ\rangle = \langle 100\sqrt{3}, 100\rangle$.

For $\vec{F_2}$, we get $\vec{F_2} = \langle 300\cos 70^\circ, 300\sin 70^\circ\rangle$.

Adding them gives $\vec{F} = \langle 100\sqrt{3} + 300\cos 70^\circ, 100 + 300\sin 70^\circ\rangle$.

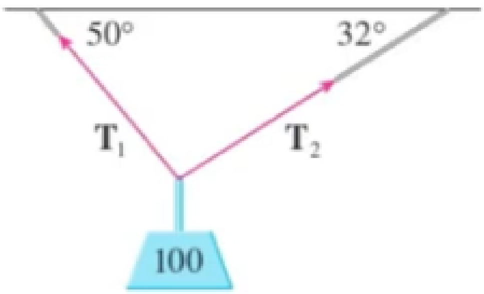
This approximates to $\langle 275.8, 381.9\rangle$. The magnitude of this approximates to $||\vec{F}|| \approx 471$ N.

If we let $100\sqrt{3} + 300\cos 70^\circ$ be $||\vec{F}||\cos\theta$, then we can determine the angle.

When we find θ in $\cos\theta = \frac{100\sqrt{3}+300\cos 70^\circ}{471}$, we get $\theta \approx 54.2^\circ$.

Example

A 100-lb weight hangs from two wires. Find the forces (tensions) $\vec{T_1}$ and $\vec{T_2}$ in both wires and the magnitude of those tensions.



Note that $\vec{T_1}$ does not have an angle of 50°, rather it has a degree of 130° from the point.

Therefore, we have $\vec{T_1} = |\vec{T_1}|\langle\cos 130^\circ, \sin 130^\circ\rangle$.

Note we can rewrite this with to be in the first quadrant as $|\vec{T_1}|\langle -cos 50^\circ, \sin 50^\circ\rangle$.

We also have $\vec{T_2} = |\vec{T_2}|\langle\cos 32^\circ, \sin 32^\circ\rangle$.

We also see that the weight of the block is $\vec{W} = 100\langle 0, -1\rangle = \langle 0, -100\rangle$, so we see to balance this out, both tension forces need to equal $\langle 0, 100\rangle$.

Therefore, we see that $-|\vec{T_1}|\cos 50^\circ + |\vec{T_2}|\cos 32^\circ = 0$ (addition of the $\vec{i}$ components).

We also have $|\vec{T_1}\sin 50^\circ + |\vec{T_2}|\sin 32^\circ = 100$ ($\vec{j}$ components).

Solving for $|\vec{T_1}|$ and $|\vec{T_2}|$ from this gives us 85.64 lbs and 64.91 lbs respectively.

Plugging these values back in gives us the tension vectors.

$\vec{T_1} \approx \langle -55.05, 65.60\rangle$

$\vec{T}_2 \approx \langle 55.05, 34.40\rangle$

1.3 The Dot Product

Definition: The Dot Product

If $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$, then the dot product $\vec{a} \cdot \vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b + 1 + a_2 b_2 + a_3 b_3$$

This is called the scalar or inner product. (Note: 2-space is a similar idea)

Example

$(\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (2\vec{j} - \vec{k})$

Using the dot product formula gives you 7.

Properties of dot products:

1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$
5. $\vec{0} \cdot \vec{a} = 0$

Example

Prove the first property from above.

Let $\vec{a} = \langle a_1, a_2, a_3\rangle$.

The dot product of $\vec{a}$ and $\vec{a}$ gives us $a_1^2 + a_2^2 + a_3^2$.

The magnitude of this is $\sqrt{a_1^2 + a_2^2 + a_3^2}^2$ which is equal to $|\vec{a}|^2$ □

Example

Prove the third property from above.

Let $\vec{a} = \langle a_1, a_2, a_3\rangle$, likewise for $\vec{b}$ and $\vec{c}$.

Then when we do $\vec{a} \cdot (\vec{b} + \vec{c})$ we get

$$\begin{aligned}
&\langle a_1, a_2, a_3\rangle \cdot (\langle b_1, b_2, b_3\rangle + \langle c_1, c_2, c_3\rangle) \\
&= \langle a_1, a_2, a_3\rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3\rangle \\
&= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\
&= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\
&= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\
&= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}
\end{aligned}$$

□

Theorem 1.2

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

Corollary 1.3

$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

Example

Find the angle between $\vec{u} = \vec{i} - 2\vec{j} + 2\vec{k}$ and $\vec{v} = -3\vec{i} + 6\vec{j} + 2\vec{k}$.

The dot product of the two gives -11.

The magnitude of $\vec{u}$ is 3 and the magnitude of $\vec{v}$ is 7.

We see that $\cos\theta = -\frac{11}{3 \cdot 7}$, so $\theta = 2.12$ radians or 121.6°.

Recall, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$. Since $|\vec{a}||\vec{b}|$ is always positive, the sign of the dot product is determined by $\cos\theta$.

- If $\vec{a} \cdot \vec{b} > 0$, then the angle is acute.
- If $\vec{a} \cdot \vec{b} < 0$, then the angle is obtuse.
- If $\vec{a} \cdot \vec{b} = 0$, then the vectors are orthogonal.

To determine if two vectors are parallel, the vectors have to be scalar multiples of each other.

Definition: Direction Angles

The direction angles α, β, and $\gamma([0, \pi])$ are the angles that $\vec{a}$ makes with the positive $x-$, $y-$, and $z-$ axes. Their cosines are called direction cosines.

$\cos\alpha = \frac{\vec{a} \cdot \vec{i}}{\Longrightarrow} \cos\alpha = \frac{a_1}{|\vec{a}|}$. Likewise, $\cos\beta = \frac{a_2}{|\vec{a}|}$ and $\cos\gamma = \frac{a_3}{|\vec{a}|}$.

Notice that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Also $\vec{a} = |\vec{a}|\langle \cos\alpha, \cos\beta, \cos\gamma \rangle$, which can be expressed as $\frac{\vec{a}}{|\vec{a}|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$.

So, the direction cosines form the unit vector in the direction of $\vec{a}$.

Example

Find the direction cosines of $\vec{a} = \langle 2, -4, 4 \rangle$ and approximate the direction angles to the nearest degree.

The magnitude of $\vec{a}$ is 6.

From the formulas, we can find that $\cos\alpha = \frac{1}{3}$, $\cos\beta = -\frac{2}{3}$, and $\cos\gamma = \frac{2}{3}$.

Finding the angles gives us $\alpha \approx 71^\circ$, $\beta \approx 132^\circ$, and $\gamma \approx 48^\circ$.

Example

Find the angle between a diagonal of a cube and one of its edges.

Let's call the vector from the diagonal to the edge as $\vec{d}$ and the length of the edge be a.

We get $\vec{d} = \langle a, a, a \rangle$ as a result. The magnitude of $\vec{a}$ ends up being $\sqrt{3a^2}$.

We get that $\cos\alpha = \frac{a}{\sqrt{3a^2}} = \frac{1}{\sqrt{3}}$.

This approximates $\alpha \approx 0.955$ radians or 54.7°.

proj$_{\vec{a}}\vec{b}$ is the vector projection of $\vec{b}$ onto $\vec{a}$

comp$_{\vec{a}}\vec{b}$ is the scalar projection of $\vec{b}$ onto $\vec{a}$ (a signed magnitude of the vector projection).

If we look at the scalar projection, comp$_{\vec{a}}\vec{b} = |\vec{b}|\cos\theta$ and remember that $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$. Therefore comp$_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

The projection will just be the magnitude (which is the scalar projection) multiplied by the unit vector: $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\left(\frac{\vec{a}}{|\vec{a}|}\right)$.

This is equal to $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$.

Example

Find the scalar and vector projections of $\vec{b} = \langle 1,1,2\rangle$ onto $\vec{a} = \langle -2,3,1\rangle$.

The dot product of the vectors is 3, the magnitude of $\vec{a} = \sqrt{14}$.

From the formula above, the scalar projection is $\frac{3}{\sqrt{14}}$.

The vector projection gives $\frac{3}{(\sqrt{14})^2}\langle -2,3,1\rangle$, simplifying to $\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\rangle$.

Previously, you learnt work is defined as $W = F \cdot d$ (this only applies when the force is directed along the line of motion).

Now we can define work as $W = (|\vec{F}|\cos\theta)|\vec{d}|$. Moving stuff around gives $W = |\vec{F}||\vec{D}|\cos\theta$.

Now we see that $W = \vec{F}\cdot\vec{d}$. (Work is constant which means it should result in a scalar.)

Example

A wagon is pulled horizontally by exerting a constant force of 10 lb on the handle at an angle of $60°$ with the horizontal. How much work is done in moving the wagon 50 feet?

We can easily see that the displacement vector is $\vec{d} = \langle 50,0\rangle$.

The force vector we must divide into components, so we get $\vec{F} = 10\langle\cos 60°, \sin 60°\rangle = \langle 5, 5\sqrt{3}\rangle$.

So the dot product of both vectors gives us 250 ft-lb.