Linear Algebra Notes

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Contents

1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

A linear equation in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ where b and the coefficients $a_1, a_2, \ldots, a_n$ are real or complex numbers.

Example: $4x_1 - 5x_2 + 2 = x_1, x_2 = 2(6^{1/2} - x_1) + x_3$ are both linear. Not linear examples are $4x_1 - 5x_2 = x_1x_2, x_2 = 2(x_1^{1/2}) - 6$, and $2x_1^{-1} + \sin x_2 = 0$.

Systems of Linear Equations: an $(m \times n)$ system of linear equations is a system of m linear equations with n unknowns.

Example: the 2×3 system of equations below has a solution $x_1 = 5, x_2 = 6.5$, and $x_3 = 3$:

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 - 4x_3 &= -7 \end{aligned}$$

Two linear systems are called equivalent if they have the same solution set.

A system of linear equations can have: infinitely many solutions, no solution, or a unique solution. Coincident lines have infinitely many solutions, parallel lines have no solution, and intersecting lines have a unique solution.

Matrix Notation:

Let's say we have $x_1 - 2x_2 + x_3 = 0, 2x_2 - 8x_3 = 8$, and $5x_1 - 5x_3 = 10$.

The coefficient matrix is $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$ and the augmented matrix is $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$. The augmented matrix is 3×4 (3 rows, 4 columns).

To solve a linear system: if one of the following elementary operations is applied to a system of linear equations, the resulting system is equivalent, that is the resulting system has the same set of solutions as the original:

1. interchange two equations
2. multiply an equation by a non-zero scalar
3. add a constant multiple of one equation to another

Let's use the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

So using row operations and rearranging the rows, we can do $R_1 \leftrightarrow R_2$ to swap the first and second row. Now we can multiply the second line by $1/2$ so $1/2R_2$. The next step is to do $R_2 + R_1$.

The matrix that results from this is $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$

Doing the operations $R_3 - 5R_1$, $R_3 - 5R_2$, and $1/2R_2$ and $1/30R_3$, we end up getting $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, so from this we have the equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned}$$

, so the solution set is $(1, 0, -1)$. Since a solution exists, the system is consistent.

Let's see if this one is consistent.

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned}$$

Doing the row operations $R_1 \leftrightarrow R_2$, $R_3 - 2R_1$, $R_3 + 2R_2$ we get that $0 = 15$, so this is inconsistent.

This last example has infinitely many solutions:

$$\begin{aligned} x_1 - 2x_2 - x_3 &= -2 \\ 2x_1 + x_2 + 3x_3 &= 1 \\ -3x_1 + x_2 - 2x_3 &= 1 \end{aligned}$$

Doing the row operations $R_2 - 2R_1$ and $R + 3 + 3R_1$, then $R_3 + R_2$ and then $1/5R_2$ results in $\begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solving this, we have x_3 with no restrictions, it is a "free parameter" and $x_2 = 1 - x_3$ and $x_1 = -x_3$ so x_1 and x_2 are parameterized by x_3.

1.2 Row Reduction and Echelon Forms

Leading entry of a row: the first (counting from left to right) non-zero entry (in a nonzero row)

Echelon Form: upper right-hand stair-case, triangle

1. all rows that consist entirely of zeros are grouped together at the bottom of the matrix
2. the first (counting from left to right) non-zero entry in the $(i + 1)$st row must appear in a column to the right of the first non-zero entry in the ith row.
3. all entries in a column below a leading entry are zeros

Reduced Echelon Form: an echelon Form matrix that also has the following properties:

1. the leading entry in each nonzero row is 1
2. each leading one is the only nonzero entry in its column

A pivot point: a location in a matrix A that corresponds to a leading 1 in a reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

Steps to solving a system of linear equations:

1. begin with the leftmost nonzero column. This is the pivot column, the pivot position is at the top.
2. select a non zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
3. Use row replacement operations to create zeros in all positions below the pivot.
4. Cover (or ignore) the row containing the pivot position and cover all rows (if any) above it. Apply previous steps to the sub matrix that remains. Repeat until there are no more nonzero rows to modify.
5. Beginning with the rightomst pivot, create zeros above each pivot. Make each pivot equal to 1 by scaling.

Example

Determine the existence and uniqueness of the solution to

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

We can find that $x_5 = 4$, and x_3, x_4 are of infinite number of solutions.

Example

Find the general solution of the linear system whose augmented matrix has been reduced to $\left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$

We have that $x_5 = 7$, $x_3 = 1/2(3 + 8x_4 + 7)$ and $x_1 = -4 - 6x_2 - 2(1/2)(3 + 8x_4 + 7) + 14$

Theorem 1.1: Existence and Uniqueness

A linear system is consistent if and only if the right most column of the augmented matrix is not a pivot column, that is if and only if an echelon form of the augmented matrix has no row of the form $[0, \ldots, 0b]$ with b nonzero. If the system if consistent in the solution contains either a unique solution when there are no free variables or infinitely many solutions when there is at least one free variable.

1.3 Vector Equations

Vectors in $\mathbf{R}^2$: a matrix with only 1 column is called a vector. The set of all vectors with 2 entries is $\mathbf{R}^2$

$\vec{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ for example.

The vectors are odered pairs of real numbers:

$\vec{u}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (3, -1) \neq (-1, 3) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Vector addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

Geometric interpretation: parallelogram law.

$(2, 4) + (3, 1) = (2 + 3, 4 + 1) = (5, 5)$. We can create a parallelogram (google for review).

Let's say we subtract, $-a$ has the same magnitude as a but points in the opposite direction in this case. $(1, 3) - (2, 1) = (-1, 2)$. Drawing the line from the origin to the tip of the vectors give you what you get algebraically.

Vector multiplication: scalar multiplication: $a(x, y) = (ax, ay)$ where a is a scalar.

Geometric interpretation: $a(x, y, z)$ points in the same direction as (x, y, z) but is scaled by a factor of a.

Example

Let $\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ Find $4\vec{u}$, $-3\vec{v}$ and $4\vec{u} + (-3)\vec{v}/$

$4\vec{u} = (4, -8)$, $-3\vec{v} = (-6, 15)$ and $4\vec{u} + (-3)\vec{v} = (-2, 7)$

Vectors start at the origin and have magnitude and direction.

Representing vectors in $\mathbf{R}^3$. We add the z-axis.

Vectors in $\mathbf{R}^n$ we have that $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = (u_1, u_2, \dots, u_n)$ where $u_1, u_2, \dots \in \mathbb{R}$.

The $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is the zero vector.

Algebraic Properties of $\mathbf{R}^n$: for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbf{R}^n$ and all scalars c and d:

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\mathbf{u} + 0 = 0 + \mathbf{u} = \mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = 0$
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $c(d\mathbf{u}) = (cd)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

Linear Combinations: given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in $\mathbf{R}^n$ and scalars $c_1, c_2, \dots, c_p$, the vector $\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$ is called a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p$ with weights $c_1, c_2, \dots, c_p$.

For example, if we have $3^{1/2}\mathbf{v}_1 + \mathbf{v}_2$ this can be written as $\vec{y} = \sqrt{3}\vec{v_1} + \vec{v_2}$ with $c_1 = \sqrt{3}$ and $c_2 = 1$.

Example

If $\vec{a_1} = (1, -2, -5), \vec{a_2} = (2, 5, 6)$, and $\vec{a_3} = (7, 4, -3)$ then determine if $\vec{b}$ can be written as a linear combination of $\vec{a_1}$ and $\vec{a_2}$. That is determine if there exists weights x_1, x_2 such that $x_1\vec{a_1} + x_2\vec{a_2} = \vec{b}$.

Using elementary row operations, we can determine that $x_1 = 3$, $x_2 = 2$ which is the linear combination of $\vec{a_1}$ and $\vec{a_2}$.

A vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots x_n\mathbf{x}_n = \mathbf{b}$ has the same solution set as the linear system with augmented matrix $[\mathbf{a}_1\mathbf{a}_2 \dots \mathbf{a}_n\mathbf{b}]$. In particular $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \dots \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix $[\mathbf{a}_1\mathbf{a}_2 \dots \mathbf{a}_n\mathbf{b}]$

Span: if $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p$ are in $\mathbf{R}^n$ then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ is denoted Span$\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_p\}$ and is called the subset of $\mathbf{R}^n$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p$. That is, Span$\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_n\}$ is the collection of all vectors that can be written in the form: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$ with $c_1, c_2, \dots, c_p$ scalars.

Asking if a vector $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_p\}$ amounts to asking whether the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_p = \mathbf{b}$ has a solution, or equivalently whether the linear system with augmented matrix $[\mathbf{v}_1\mathbf{v}_2\mathbf{v}_p\mathbf{b}]$ has a solution.

Note Span$\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_p\}$ contains every scalar multiple of $\mathbf{v}_1$.

The span of a single vector is a line. The span of 2 linearly independent vectors is a plane (not scalar multiples of each other).

1.4 The Matrix Equation Ax = b

The Matrix Equation: if A is an $m \times n$ matrix with columns, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and if $\mathbf{x}$ is in $\mathbf{R}^n$, then the product of A and $\mathbf{x}$ denoted $A\mathbf{x}$ is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights.

Note: $\mathbf{A}x$ is defined only if the number of columns of A equals the numbers of entries in x.

For example: $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4(1,0) + 3(2,-5) + 7(-1,3) = (3,6)$

Theorem 1.2: Matrix Equation, Vector Equation, System of Linear Equations

If A is an $m \times n$ matrix, with columns, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and if $\mathbf{b}$ is in $\mathbf{R}^m$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution as the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$ which in turn has the same solution as the system of linear equations represented by the augmented matrix $[\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_n \mathbf{b}]$.

Existence of Solutions: the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Example

Is $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & 7 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ Is the equation $A\vec{x} = \vec{b}$ consistent for all $\vec{b}$. Using rref, we get that $0 = -2b_1 + b_2 - 2b_3$, so it is not consistent for every $\vec{b}$. It is only consistent if $b_2 = 2b_1 + 2b_3$.

So let $\vec{b} = (1,4,1)$ and then do rref again and we get that x_3 is free, $x_2 = 1/7(4 - 5x_3)$ and $x_1 = 1 - 3(x_2) - 4x_3$ and this basically gives us $(1,4,1)$ too.

Theorem 1.3: Existence of soultion for $A\mathbf{x} = \mathbf{b}$

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, they are all true statements or they are all false:

1. for each $\mathbf{b}$ in $\mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution
2. each $\mathbf{b}$ in $\mathbf{R}^m$ is a linear combination of the columns of A
3. The columns of A span $\mathbf{R}^m$
4. A has a pivot position in every row. Note: A is a coefficient matrix, not an augmented matrix.

Computation of $A\mathbf{x}$ - an efficient method (matrix multiplication): if the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of the corresponding entries from row i of A and from vector $\mathbf{x}$.

The above is trivial.

Properties of the Matrix-Vector Product $A\mathbf{x}$

Theorem 1.4

if A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbf{R}^n$, and c is a scalar, then :

1. $A(\mathbf{u} + \mathbf{v})$
2. $A(c\mathbf{u}) = c(A\mathbf{u})$

Algebraic Properties of $\mathbf{R}^n$: for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbf{R}^n$ and all scalars c and d. (This was from an above topic)

1.5 Solution Sets of Linear Systems

Parametric Vector Form of Solutions:

- Parametric Vector Form of a Plane: a plane can be expressed in explicit form, such as $10x_1 - 3x_2 - 2x_3 = 0$ or implicit form: $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$, for s and t scalars.
- Parametric Form of a Line containing point $\mathbf{p}$ in direction of $\mathbf{v}$: $l(t) = \mathbf{p} + t\mathbf{v}$

The parametric equation of a plane in $\mathbf{R}^2$: $\mathbf{x} = a\mathbf{v} + b\mathbf{s}$.

The span of 2 non-colinear vectors is a plane. Span$\{\mathbf{v}, \mathbf{s}\}$ = the $\mathbf{R}^2$ plane.

In $\mathbf{R}^3$, the Span is still a plane, just in $\mathbf{R}^3$.

The parametric equation of a line in $\mathbf{R}^2$: $\mathbf{l} = \mathbf{p} + t\mathbf{v}$.

Homogeneous Linear Equation - $A\mathbf{x} = 0$.

The homogeneous equation always has at least 1 solution, $\mathbf{x} = 0$ (the trivial solution).

Recall that a system of linear equation either has infinitely many solutions, no solution, or a unique solution.

The question is whether there exists a nontrivial solution (in which case there are infinitely many solutions).

- The homogeneous equation $A\mathbf{x} = 0$ has nontrivial solution if and only if the equation has at least 1 free variable

Description of solutions: if the solutions consists of:

- the 0 vector: Span$\{0\}$
- 1 free variable: Span$\{\mathbf{v}\}$, the solutions are a line through the origin
- 2 free variables, Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane through the origin

Example

Determine if the following system has a nontrivial solution. Then describe the solution set

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

x_3 is free. And everything is in the form $(4/3, 0, 1)$.

Nonhomogeneous Equation: $A\mathbf{x} = \mathbf{b}$.

For example, in the previous example when we let $x_3 = 0$ we get $(-1, 2, 0)$.

Theorem 1.5

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$ and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the Form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = 0$.

1.6 Applications of Linear Systems

There are three examples here: economics, chemical equations and network flow.

Start with economics. There exist equilibrium prices that can be assigned to the total outputs of the various sectors in an economy in such a way that the income of each sector exactly balances its expenses.

You can use row operations to find an equilibrium price.

Other examples run similarly (sorry for bad note taking today I'm sick)

1.7 Linear Independence

Linear Independence: an indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is said to be

- linearly independent if the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = 0$ has only the trivial solution
- linearly dependent if there exists weights $c_1, c_2, \ldots, c_p$ not all zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = 0$

I'm too lazy to write matrices so much.

Linear Independence of Matrix Columns: the columns of matrix A are linearly independent if and only if the equation $A\mathbf{x} = 0$ has only the trivial solution.

Sets of One or Two Vectors

- A set with 1 vector is linearly independent iff $\mathbf{v}$ is not the 0 vector because $x_1\mathbf{v} = 0$ has only the trivial solution
- the zero vector, $\mathbf{0}$ is linearly dependent because $x_1\mathbf{0} = \mathbf{0}$ has many nontrivial solutions
- two vector $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly dependent iff at least one of the vectors is a multiple of the other

Theorem 1.6

Characterization of Linearly Dependent Sets: An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of 2 or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1$ is not $\mathbf{0}$, then some $\mathbf{v}_j$ with $j > 1$ is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

Note: the theorem does not say every vector in a linearly dependent set is a linear combination of preceding vectors

Theorem 1.7

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is linearly dependent if $p > n$.

Theorem 1.8

If a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ contains the zero vector, then the set is linearly dependent.

1.8 Introduction to Linear Transformations

Linear Transformations: we can view $A\mathbf{x} = \mathbf{b}$ as a mapping: the $m \times n$ matrix A is the transform, $A : \mathbf{R}^n \to \mathbf{R}^m$.

From this point of view, solving the equation $A\mathbf{x} = \mathbf{b}$ amounts to finding all the vectors $\mathbf{x}$ in $\mathbf{R}^n$ that are transformed to $\mathbf{b}$ in $\mathbf{R}^m$. The correspondence from $\mathbf{x}$ to $A\mathbf{x}$ is a function from one set of vectors to another.

Definition

A transform (or function or mapping) T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$. The set $\mathbf{R}^n$ is called the domain of T, and $\mathbf{R}^m$ is called the codomain. The notion $T : \mathbf{R}^n \to \mathbf{R}^m$ indicates that the domain of T is $\mathbf{R}^n$ and the codomain is $\mathbf{R}^m$. For $\mathbf{x}$ in $\mathbf{R}^n T(\mathbf{x})$ in $\mathbf{R}^m$ is called the image of $\mathbf{x}$. The set of all images $T(\mathbf{x})$ is called the range of T.

Matrix Transformations: $T(\mathbf{x})$ is computed as $A\mathbf{x}$ where A is an $m \times n$ matrix. Note: the domain of T is $\mathbf{R}^n$ and the codomain of T is $\mathbf{R}^m$. The range of T is the set of all linear combinations of the columns of A.

Linear Transformations: a transformation (or mapping) T is linear if

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in the domain of T
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

- every matrix transformation is a linear transformation: $A(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}) + A\mathbf{v}$ and $A(c\mathbf{u}) = cA(\mathbf{u})$
- linear transformations preserve the operations of vector addition and scalar multiplication

If T is a linear transformation, then

1. $T(\mathbf{0}) = \mathbf{0}$
2. $T(c\mathbf{u}+d\mathbf{v}) = cT(\mathbf{u})+dT(\mathbf{v})$ for all scalars c, d and all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T. The generalization $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_pT(\mathbf{v}_p)$ is known in engineering as the superposition principle: whenever an input is expressed as a linear combination of signals the systems response is the same linear combination of the responses to the individual signals.

1.9 The Matrix of a Linear Transformation

Goal: given a geometric desciprtion of a transformation, T, we want to find a "formula" for T

- Every linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$ can be represented by a matrix transformation $A(\mathbf{x})$.
- The key to finding matrix A is to that T is completely determined by what it does to the columns of the $n \times n$ identiy matrix, I_n.

Theorem 1.9

Standard Matrix for a Linear Transformation: let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^n$. And, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$ where $\mathbf{e}_j$ is jth column of the identity matrix in $\mathbf{R}^n$. $A = [T(\mathbf{e}_1)T(\mathbf{e}_2)\ldots T(\mathbf{e}_n)]$. A is caleld the standard matrix for the linear transformation T.

Onto/Existence: a mapping $T : \mathbf{R}^n \to \mathbf{R}^m$ is said to be onto $\mathbf{R}^m$ if each $\mathbf{b}$ in $\mathbf{R}^m$ is the image of at least 1 $\mathbf{x}$ in $\mathbf{R}^n$.

- T is onto $\mathbf{R}^m$ when the range of T is all of the codomain $\mathbf{R}^m$; for each $\mathbf{b}$ in $\mathbf{R}^m$, there exists at least one solution of $T(\mathbf{x}) = \mathbf{b}$. The mapping T is not onto when there is some $\mathbf{b}$ in $\mathbf{R}^m$ for which $T(\mathbf{x}) = \mathbf{b}$ has no solution.

T is one-to-one if for each $\mathbf{b}$ in $\mathbf{R}^n$, the equation $T(\mathbf{x}) = \mathbf{b}$ has either unique solution or no solution. The mapping is not one-to-one when some $\mathbf{b}$ in $\mathbf{R}^m$ is the image of more than one vector in $\mathbf{R}^n$.

Theorem 1.10

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation, then T is one-to-one iff $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem 1.11

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

1. T maps $\mathbf{R}^n$ onto $\mathbf{R}^m$ iff the columns of A span $\mathbf{R}^m$
2. T is one-to-one iff the columns of A are linearly independent

1.10 Linear Models in Business, Science, and Engineering

Linear Equations can be done in electrical networks.

Current flow in a simple electrical network can be described by a system of linear equations. Consider Ohm's Law, $V = IR$, which describes the current which passes through a resistor. The algebraic sum of IR voltage drops in one direction around a loop equals the algebraic of the voltage sources in the same direction around the loop.

The model for current flow is linear since the voltage drop across a resistor is proportional to the current flowing through it, and the sum of the voltage drops in a loop equals the sum of the voltage sources in the loop.

For difference equations, if there is a matrix A such that $\mathbf{x}_1 = A\mathbf{x}_0, x_2 = A\mathbf{x}_1$, and in general $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for $k = 0, 1, 2, \ldots.$ then this is called a linear difference equation (or recurrence relation).

Ok whatever just use logic.

2 Matrix Algebra

2.1 Matrix Operations

Sums and Scalar Multiples of Matrices: if A and B are $m \times n$ matrices, $A + B$ is the $m \times n$ whose columns are the sums of the corresponding columns in A and B, the scalar multiple rA is the matrix whose columns are r times the corresponding columns in A.

Theorem 2.1

Matrix addition and scalar multiplication: Let A, B, and C be matrices of the same size, and let r and s be scalars.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$
4. $r(A + B) = rA = rB$
5. $(r + s)A = rA = sA$
6. $r(sA) = (rs)A$

Matrix Multiplication: if A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{b}_1, \mathbf{b}_2, \ldots \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_p$, i.e, $AB = A[\mathbf{b}_1 \mathbf{b}_2 \ldots \mathbf{b}_p] = [A\mathbf{b}_1 A\mathbf{b}_2 \ldots A\mathbf{b}_p]$. Matrix multiplication corresponds to composition of linear transformations.

- An efficient Matrix Multpilcation: if the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)th entry in AB, and if A is $m \times n$, then $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

Properties of Matrix Multpilcation: Let A be $m \times n$ and let B, C have sizes such that the sums and products are defined:

1. $A(BC) = (AB)C$ associative law
2. $A(B + C) = AB + AC$ left distributive law
3. $(B + C)A = BA + CA$ right distributive law
4. $r(AB) = (rA)B = A(rB)$ for any scalar r
5. $I_mA = A = AI_n$ Identity matrix for multiplication

- Matrix mutiplication is not commutative. In general AB does not equal BA.
- Cancellation laws do not hold for matrix multiplication.
- If $AB = 0$, you cannot conclude either $A = 0$ or $B = 0$.

Powers of a Matrix: If A is $n \times n$ and k is a positive integer, A^k denote the product of k copies of A.

The Transpose of a Matrix: given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix whose columns are formed from the corresponding rows of A.

Theorem 2.2: Transpose

Let A, B denote matrices whose sizes are appropriate for the following:

1. $(A^T)^T = A$

2. $(A + B)^T = A^T + B^T$
3. for any scalar r, $(rA)^T = r(A)^T$
4. $(AB)^T = B^T A^T$

2.2 The Inverse of a Matrix

The Matrix Inverse is the matrix analogue of the multiplicative inverse of in real numbers.

- Invertible: an $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix A^{-1} such that $A^{-1}A = AA^{-1} = I_n$. In this case A^{-1} is said to be the unique inverse of A.

Notice: because matrix multipilcation is not commutative, both equations are needed.

Singular Matrix: A matrix that is not invertible is a single matrix. An invertible matrix is nonsingular.

Theorem 2.3

Inverse of a 2×2: Let A be the 2×2 matrix shown. If $ab - dc$ is not zero, then A is invertible with A^{-1} as shown:

$$A = \begin{bmatrix} a & d \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinant: det $A = ad - bc$. The theorem says that a 2×2 matrix is invertible iff det A is not zero.

Theorem 2.4

If A is an invertible matrix, then for each $\mathbf{b}$ in $\mathbf{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique soultion $\mathbf{x} = A^{-1}\mathbf{b}$.

Theorem 2.5

1. If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$
2. If A and B are $n \times n$ invertible matrices, then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$. Generalization: the product of $n \times n$ invertible matrices is invertible, and the inverse is the product of the their inverse in the reverse order.
3. If A is an invertible matrix, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$

Theorem 2.6

An $n \times n$ matrix is invertible iff it is row equivalent to I_n, and any sequence of elementary row operations that reduces A to I_n also transforms I_n to A^{-1}.

2.3 Characterizations of Invertible Matrices

The Invertible Matrix Theorem: let A be an $n \times n$ matrix. Then the following statements are equivalent.

- A is an invertible matrix.
- A is row equivalent to the $n \times n$ identity matrix.
- A has n pivot positions
- the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
- the columns of A form a linearly independent set
- the linear transform $\mathbf{x} \to A\mathbf{x}$ is one-to-one

- the equation $A\mathbf{x} = \mathbf{b}$ has at least 1 soln for each $\mathbf{b}$ in $\mathbf{R}^n$
- the columns of A span $\mathbf{R}^n$
- the linear transformation $\mathbf{x} \to A\mathbf{x}$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$
- there is an $n \times n$ matrix C such that $CA = I$
- there is an $n \times n$ matrix D such that $AD = I$
- A^T is an invertible matrix

Note that this only applies to square matrices.

Theorem 2.7: Inverse Transformation

Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique solution satisfying $S(T(\mathbf{x})) = \mathbf{x}$ and $T(S(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^n$.

Recall that matrix multipilcation corresponds to composition of linear transformations. When a matrix A is invertible, the equation $A^{-1}A\mathbf{x} = \mathbf{x}$ can be viewed as a statement about linear transformations. A linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ is said to be invertible if there exists a function $S : \mathbf{R}^n \to \mathbf{R}^n$ such that $S(T(\mathbf{x})) = \mathbf{x}$ and $T(S(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^n$.

2.4 Matrix Factorizations

A factorization of a matrix A is an equation that expresses A as a product of two or more matrices.

Whereas matrix multiplication involves a synthesis of data (combining the effects of two or more linear transformations into a single matrix), matrix factorization is an analysis of data.

The LU factorization:

At first assume that A is an $m \times n$ matrix that can be row reduced to echelon form, without row interchanges. Then A can be written in the form $A = LU$, where L is an $m \times m$ lower triangular matrix with 1's on the diagonal and U is an $m \times n$ echelon form of A.

Suppose A can be reduced to an echelon form U using only row replacements that add a multiple of one row to another below it. In this case, there exist unit lower triangular elementary matrices, $E_1, \ldots, E_p$ such that $E_p \cdots E_1 A = U$.

Then $A = (E_p \cdots E_1)^{-1} U = LU$ where $L = (E_p \cdots E_1)^{-1}$.

It can be shown that products and inverses of unit lower triangular matrices are also unit lower triangular. Thus L is unit lower triangular.

Algorithm:

- Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
- Place entries in L such that the same sequence of row operations reduces L to I.

3 Determinants

3.1 Introduction to Determinants

Definition of the Determinant: given an $n \times n$ matrix $A = [a_{ij}]$, the determinant is det A $= a_{11}a_{22} - a_{12}a_{21}$.

For $n \geq 2$, the determinant of an A is the sum of n terms of the form $+/-a_{1j}A_{1j}$ with the plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A.

The (i, j)-cofactor $= C_{ij} = (-1)^{i+j}$ det A$_{ij}$ s.t. det A $= a_{11}C_{11} + a_{12} + C_{12} + \cdots + a_{1n} + C_{1n}$.

Theorem 3.1

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column/

The expansion across the ith row using the cofactors is: det A $= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}Cin$

The expansion across the jth column is: det A $= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + n_{nj}C_{nj}$.

This theorem is helpful for computing determinants of a matrix that contains many zeros. For example, if 1 row contains many zeros, than a cofactor expansion across that row will be easier to calculate.

Theorem 3.2

If a is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

3.2 Properties of Determinants

"The secret of determinants lies in how they change when row operations are performed"

Theorem 3.3

Let A be a square matrix.

- A multiple of one row of A is added to another row to produce a matrix B, then det $B =$ det A
- if two rows of A are interchanged to produce B then det $B =$ -det A
- if 1 row of A is multiplied by k to produce B, then det $B = k$det A

we can use a strategy to reduce a matrix to echelon form and then use the fact that the determinant of a triangular matrix is the product of its diagonal entries.

Theorem 3.4

A square matrix A is invertible if and only if det A is not zero. If A is an $n \times n$ matrix, then det $A^T =$ det A.

Theorem 3.5

If A and B are $n \times n$ matrices, then det $AB =$ (det A)(det B)

3.3 Cramer's Rule, Volume, and Linear Transformations

Cramer's Rule: Cramers rule is needed for a variety of theoretical calculations. However the formula is inefficient for hand calculations except for 2×2.

For any $n \times n$ matrix A and any **b** in $\mathbf{R}^n$ let $A_1(\mathbf{b})$ be the matrix obtained from A by replacing the ith column by **b**, $A_i(\mathbf{b}) = [\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_{i-1} \mathbf{b} \mathbf{a}_{i+1} \ldots \mathbf{a}_n]$

Theorem 3.6

Let A be an intertible invertible $n \times n$ matrix. For any **b** in $\mathbf{R}^n$, the unique solution **x** of $A\mathbf{x} = \mathbf{b}$ has entries given by: $\mathbf{x}_i = (\det A_i(\mathbf{B})/(\det A))$ $i = 1, 2, \ldots, n$.

Application to engineering: A number of important engineering problems can be analyzed by Laplace transformations. This approach converts an appropriate system of linear differential equations into a system of linear algebraic equations whose coefficient involve a parameter.

Formula for A^{-1}: Cramer's Rule leads to a general formula for the inverse of an $n \times n$ matrix.

Theorem 3.7

Let A be an invertible $n \times n$ matrix. Then A^{-1} is given by: (Google This)

Determinants as Area of Volume

Theorem 3.8

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is —det A—. If A is a 3×3 matrix, the volume of the paralellepiped determined by the columns is —det A—.

Theorem 3.9

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear the transformation obtained by a 2×2 matrix A. If S is a parallelogram in $\mathbf{R}^2$ then the area of $T(S)$ is equal to —det A— times the area of S.

If T is determined by a 3×3 matrix A, and S is a paralellepiped in $\mathbf{R}^3$, then the volume of $T(S)$ is equal to the area of —det A— times the volume of S.

4 Vector Spaces

4.1 Vector Spaces and Subspaces

A vector space is a nonempety set V of objects, called vectors, on which are defined two operations called addition and scalar multiplication to the 10 axioms listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars a and b.

Closure Properties:

- $\mathbf{u} + \mathbf{v}$ is a vector in V
- $a\mathbf{v}$ is a vector in V

Properties of Addition:

- commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- associative: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- additive identity: there is a vector $\mathbf{0}$ in V such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$ in V
- additive inverse: given a vector $\mathbf{v}$ in V, there is a vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$

Properties of Scalar Multiplication:

- associative: $a(b\mathbf{v}) = (ab)\mathbf{v}$
- distributive: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
- distributive: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$
- multiplicative identity: $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v}$ in V

A subspace of a vector space, V, is a subset H of V that has three properties:

1. the zero vector of V is H
2. H is closed under vector addition
3. H is closed under scalar multiplication

A subspace H of V itself is a vector space. The other properties of a vector space are "inherited" since H is a subset of V.

Theorem 4.1

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a subspace of V.

4.2 Null Spaces, Column Spaces, Row Spaces, and Linear Transformations

The Null Space of a $m \times n$ matrix A, denoted Nul A, is the set of solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation: Nul $A = \{\mathbf{x} : \mathbf{x}$ is in $\mathbf{R}^n$ and $A\mathbf{x} = \mathbf{0}\}$.

Theorem 4.2

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknown is a subspace of $\mathbf{R}^n$.

The Column Space of a $m \times n$ matrix A, denoted Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ then Col $A = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

Theorem 4.3

The column space of a $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

The Row Space of an $m \times n$ matrix A, denoted Row A, is the set of all linear combinations of the row vectors. Each row has n entries, so the Row A is a subspace of $\mathbf{R}^n$.

The null and column space are related. They are very different: When A Is not square, the column space and the null space exist entirely in different "universes". For an $m \times n$ matrix, the Column Space is in m-dimensional space, where the Null space is in n-dimensional space.

Kernel and Range if a Linear Transformation

- a Linear Transformation T from a vector space V to be vector space W is a rule that assigns each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W such that $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}, \mathbf{v}$ in V and c in $\mathbf{R}$.
- The Kernel of T is the set of all $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$
- The Range of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V
- The kernel and range of T are subspaces of V

4.3 Linearly Independent Sets; Bases

A basis spans a vector space as "efficiently" as possible.

- linearly independent: an indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in V is linearly independent if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ has only the trivial solution.
- linearly dependent: an indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in V is linearly dependent if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ has a nontrivial solution

Theorem 4.4

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of 2 or more vectors with $\mathbf{v}_1$ not zero, is linearly dependent if and only if some $\mathbf{v}_j (j > 1)$ is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

This definition is for general vectors where we may be unable to write the vectors as columns of a matrix A.

Basis: Let H be a subspace of a vector space V. A set of vectors $\mathcal{B}$ in V is a basis of H if:

1. $\mathcal{B}$ is a linearly independent set and
2. the subspace spanned by $\mathcal{B} = H$ (or H = span $\mathcal{B}$)

Theorem 4.5

$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a set in a vector space, and let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.

- if one of the vectors in S, say $\mathbf{c}_k$, is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.
- if H is not equal to $\{\mathbf{0}\}$, some subset of S is a basis for H

The basis if the smallest possible spanning set (because all vectors are linearly independent)

The basis is the largest possible linearly independent set that spans (an additional vector will make the set dependent)

4.4 Coordinate Systems

Theorem 4.6

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, c_2, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$.

Coordinates: suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. The coordaintes of $\mathbf{x}$ relative to the basis $\mathcal{B}$ are the weights $c_1, c_2, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$.

The change of coordinates matrix, $P_{\mathcal{B}} = [\mathbf{b}_1\mathbf{b}_2 \ldots \mathbf{b}_n]$ change coordinates from $\mathcal{B}$ to the standard basis in $\mathbf{R}^n$. $P_{\mathcal{B}}^{-1}$ transforms $\mathbf{x}$ to $[\mathbf{x}]_{\mathcal{B}}$.

Theorem 4.7

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} - > [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one, onto linear transformation from V to $\mathbf{R}^n$.

The coordinate mapping in this theorem is an important example of isomorphism from V onto $\mathbf{R}^n$. The notation and terminology for the two vector space may be different, but the two spaces are indistinguishable. Every vector space calculation in V is accurately reproduced in $\mathbf{R}^n$ and vice versa.

4.5 The Dimension of a Vector Space

Theorem 4.8

If a vector space V has a basis $\mathcal{B} = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n]$ then any set in V containing more than n vectors must be linearly dependent.

Theorem 4.9

If a vector space V has a basis $\mathcal{B} = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n]$ then every basis of V must contain exactly n vectors.

Dimension: if a vector space V is spanned by a finite space, then V is said to be finite-dimensional, and the dimension of V, dim V is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be infinite-dimensional.

Theorem 4.10

Let H be a subspace of a finite dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to be a basis for H. Also, H is finite dimensional and dim H $\leq$ dim V

Theorem 4.11

Let V be a p-dimensional vector space with $p \geq 1$. Any linearly independent set of exactly p elements in V is a basis for V. Any set of exactly p elements that spans V is a basis for V

The rank of an $m \times n$ matrix A is the dimension of the column space, and the nullity of A is the dimension of the null space.

Theorem 4.12

The dimension of the column space and the null space of an $m \times n$ matrix A satisfies the equation: rank A + nullity A = number of columns in A

The rank of an $m \times n$ matrix A is the number of pivot columns and nullity of A is the number of free variables. Since the dimension of the row space is the number of pivot rows, dim row space = rank A.

The following are equivalent to the statement A is an invertible matrix:

- the columns of A for a basis for $\mathbf{R}^n$
- Col $A = \mathbf{R}^n$
- rank $A = n$
- nullity $A = 0$
- Nul A $= \{0\}$

4.6 Change of Basis

Theorem 4.13

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{B}' = \{\mathbf{b}'_1, \mathbf{b}'_2, \ldots, \mathbf{b}'_n\}$ be bases of a vector space V.

Then there is a unique $n \times n$ matrix $\mathbf{P}_{\mathcal{B} \leftarrow \mathcal{B}}$ such that $[\mathbf{x}]_{\mathcal{B}} = \mathbf{P}_{\mathcal{B} \leftarrow \mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$. The columns of $\mathbf{P}_{\mathcal{B} \leftarrow \mathcal{B}}$ are the $\mathcal{B}'$ coordinates vectors of the vectors in the basis $\mathcal{B}$.

If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{E}$ is the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ then $[\mathbf{b}_1]_{\mathcal{E}} = \mathbf{b}_1$, and likewise for the other vectors in $\mathcal{B}$.

5 Eigenvalues and Eigenvectors

5.1 Eigenvectors and Eigenvalues

5.2 The Characteristic Equation

5.3 Diagonalization

5.4 Applications to Differential Equations

6 Orthogonality and Least Sqaures

6.1 Inner Product, Length, and Orthogonality

6.2 Orthogonal Sets

6.3 Orthogonal Projections

6.4 The Gram-Schmidt Process

6.5 Least-Squares Problems

6.6 Machine Learning and Linear Models

6.7 Inner Product Spaces

7 Symmetric Matrices and Quadratic Forms

7.1 Diagonalization of Symmetric Matrices

7.2 Quadratic Forms