A Basic Guide to High School Mathematics

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Contents

1 Proof **9**

2 Algebra & Functions **13**
2.1 Indices 13
2.2 Surds 15
2.3 Quadratics 17
2.4 Simultaneous Equations 17
2.5 Inequalities 17
2.6 Polynomials & Rational Expressions 17
2.7 Graphs & Proportion 17
2.8 Functions 17
2.9 Graph Transformations 17
2.10 Algebraic Fractions 17
2.11 Modelling 17

3 Coordinate Geometry **19**
3.1 Coordinate Geometry 19
3.2 Circles 19
3.3 Parametric Equations 19
3.4 Parametric Equation Modelling 19

4 Sequences & Series **21**
4.1 Binomial Expansion 21
4.2 Sequences 21
4.3 Sigma Notation 21
4.4 Arithmetic Sequences 21
4.5 Geometric Sequences 21
4.6 Modelling with Sequences 21

5 Trigonometry **23**
5.1 Trigonometry 24
5.2 Small Angle Approximation 24
5.3 Trig Graphs 24
5.4 Further Trigonometry 24
5.5 Trigonometric Identities 24
5.6 Compound Angles & Equivalent Forms 24
5.7 Trig Equations 24
5.8 Proving Trigonometric Identities 24
5.9 Trigonometry in Context 24

6 Exponentials & Logarithms **25**
6.1 Exponentials 25
6.2 Exponential Models 25
6.3 Logarithms 25
6.4 Laws of Logarithms 25
6.5 Exponential & Logarithmic Equations 25
6.6 Reduction to Linear Form 25
6.7 Exponential Growth & Decay 25

7 Differentiation **27**
7.1 Differentiation from First Principles 28

7.2 Differentiation 28
7.3 Gradients 28
7.4 Further Differentiation 28
7.5 Implicit Differentiation & Parametric Differentiation 28
7.6 Forming Differential Equations 28

8 Integration **29**
8.1 Fundamental Theorem of Calculus 29
8.2 Indefinite Integrals 29
8.3 Definite Integrals & Parametric Integration 29
8.4 Integration as the Limit of a Sum 29
8.5 Further Integration 29
8.6 Integration with Partial Fractions 29
8.7 Differential Equations 29
8.8 Differential Equations in Context 29

9 Numerical Methods **31**
9.1 The Change of Sign Method 31
9.2 The x=g(x) Method & The Newton-Raphson Method 31
9.3 Numerical Integration 31
9.4 Numerical Methods in Context 31

10 Vectors **33**
10.1 Introducing Vectors 33
10.2 Magnitude & Direction of a Vector 33
10.3 Resultant & Parallel Vectors 33
10.4 Position Vectors 33
10.5 Vector Problems 33

11 Statistical Sampling **35**

12 Data Presentation & Interpretation **37**
12.1 Box Plots, Cumulative Frequency, & Histograms 37
12.2 Scatter Graphs 37
12.3 Central Tendency & Variation 37
12.4 Outliers & Cleaning Data 37

13 Probability **39**
13.1 Venn Diagrams, Tree Diagrams, & Two-Way Tables 39
13.2 Conditional Probability 39
13.3 Modelling with Probability 39

14 Statistical Distributions **41**
14.1 Discrete Random Variables & The Binomial Distribution 41
14.2 The Normal Distribution 41
14.3 Appropriate Distributions 41

15 Hypothesis Testing **43**
15.1 Introducing Hypothesis Testing 43
15.2 Binomial Hypothesis Testing 43
15.3 Sample Means Hypothesis Testing 43

16 Quantities & Units in Mechanics **45**

17 Kinematics **47**
17.1 Displacement, Velocity, & Acceleration 47

17.2 Graphs of Motion 47
17.3 SUVAT 47
17.4 Calculus in Kinematics 47
17.5 Projectiles 47

18 Forces & Newton's Laws **49**
18.1 Introducing Forces & Newton's First Law 49
18.2 Newton's Second Law 49
18.3 Weight & Tension 49
18.4 Newton's Third Law and Pulleys 49
18.5 F=ma & Differential Equations 49
18.6 The Coefficient of Friction 49

19 Moments **51**

20 Proof **53**

21 Complex Numbers **55**
21.1 Introducing Complex Numbers 56
21.2 Working with Complex Numbers 56
21.3 Complex Conjugates 56
21.4 Introducing the Argand Diagram 56
21.5 Introducing Modulus-Argument Form 56
21.6 Multiply and Divide in Modulus-Argument Form 56
21.7 Loci with Argand Diagrams 56
21.8 De Moivre's Theorem 56
21.9 $z = re^{(i\theta)}$ 56
21.10 nth Roots of Unity 56
21.11 Geometrical Problems 56

22 Matrices **57**
22.1 Introducing Matrices 58
22.2 The Zero & Identity Matrices 58
22.3 Matrix Transformations 58
22.4 Invariance 58
22.5 Determinants 58
22.6 Inverse Matrices 58
22.7 Simultaneous Equations 58
22.8 Geometrical Interpretation 58
22.9 Factorising Determinants 58
22.10 Eigenvalues and Eigenvectors 58
22.11 Diagonalisation 58
22.12 Cayley-Hamilton Theorem 58

23 Further Algebra & Functions **59**
23.1 Roots of Polynomials 60
23.2 Forming New Equations 60
23.3 Summations 60
23.4 Method of Differences 60
23.5 Introducing Maclaurin Series 60
23.6 Standard Maclaurin Series 60
23.7 Limits and l'Hospital's Rule 60
23.8 Polynomial Inequalities 60
23.9 Rational Function Inequalities 60
23.10 Modulus of Functions 60

23.11 Reciprocal Graphs 60
23.12 Linear Rational Graphs 60
23.13 Quadratic Rational Functions 60
23.14 Discriminants 60
23.15 Conic Sections 60
23.16 Transformations 60

24 Further Calculus **61**
24.1 Improper Integrals 61
24.2 Volumes of Revolution 61
24.3 Mean Value 61
24.4 Partial Fractions 61
24.5 Differentiating Inverse Trig 61
24.6 Integrals of the Form $\sqrt{a^2 - x^2}$ and $1/(a^2 + x^2)$ 61
24.7 Arc Length and Sector Area 61
24.8 Reduction Formulae 61
24.9 Limits 61

25 Further Vectors **63**
25.1 Equations of Lines 63
25.2 Equations of Planes 63
25.3 The Scalar Product 63
25.4 Perpendicular Vectors 63
25.5 Intersections 63
25.6 The Vector Product 63

26 Polar Coordinates **65**
26.1 Polar Coordinates 65
26.2 Polar Curves 65
26.3 Polar Integration 65

27 Hyperbolic Functions **67**
27.1 Hyperbolic Functions 67
27.2 Hyperbolic Calculus 67
27.3 Hyperbolic Inverse 67
27.4 Hyperbolic Inverse 67
27.5 Hyperbolic Integration 67
27.6 Hyperbolic Identities 67
27.7 Hyperbolic Identities 67

28 Differential Equations **69**
28.1 1st Order Differential Equations - Integrating Factors 69
28.2 1st Order Differential Equations - Particular Solutions 69
28.3 Modelling 69
28.4 2nd Order Homogeneous Differential Equations 69
28.5 2nd Order Non-Homogeneous Differential Equations 69
28.6 2nd Order Non-Homogeneous Differential Equations 69
28.7 Simple Harmonic Motion 69
28.8 Damped Oscillations 69
28.9 Systems of Differential Equations 69
28.10 Hooke's Law 69
28.11 Damping Force 69

29 Numerical Methods **71**
29.1 Mid-Ordinate Rule & Simpson's Rule 71

29.2 Euler's Step by Step Method . . . 71
29.3 Euler's Improved Step by Step Method . . . 71

30 Tracing an Algorithm **73**
30.1 Tracing an Algorithm . . . 73
30.2 Complexity . . . 73

31 Bin Packing **75**
31.1 Bin Packing . . . 75
31.2 Complexity . . . 75

32 Sorting Algorithms **77**
32.1 Introduction . . . 77
32.2 Quick Sort . . . 77
32.3 Bubble Sort . . . 77

33 Graph Theory **79**

34 Minimum Spanning Trees **81**
34.1 Introduction . . . 81
34.2 Kruskal's Algorithm . . . 81
34.3 Prim's Algorithm . . . 81
34.4 Prim's Algorithm with a Matrix . . . 81

35 Dijkstra's Algorithm **83**

36 Critical Path Analysis **85**
36.1 Critical Path Analysis (CPA) . . . 85
36.2 Precedence Tables . . . 85
36.3 Activity Networks . . . 85
36.4 Dummy Activities . . . 85

37 Network Flows **87**
37.1 Network Flows . . . 87
37.2 Cuts . . . 87
37.3 Supersinks & Supersources . . . 87

38 Linear Programming **89**
38.1 Drawing Inequalities & The Objective Function . . . 89
38.2 Formulating an LP Problem . . . 89
38.3 3-Variable to 2-Variable . . . 89

39 Simplex Algorithm **91**

40 LP Solvers **93**
40.1 Indicator Variables . . . 93
40.2 Shortest Path (Dijkstra's) . . . 93
40.3 Longest Path (CPA) . . . 93
40.4 Network Flows . . . 93
40.5 Critical Path Analysis (Alternative) . . . 93
40.6 Matching . . . 93
40.7 Allocation . . . 93
40.8 Transportation . . . 93
40.9 LINDO . . . 93

41 PMCC **95**

41.1 Bivariate Data 95
41.2 Correlation & Association 95
41.3 The PMCC 95

42 Linear Regression **97**
42.1 Introduction 97
42.2 Calculating Regression Lines 97
42.3 Interpreting 97

43 PMCC Hypothesis Testing **99**
43.1 PMCC Hypothesis Testing 99
43.2 Effect Sizes 99

44 Spearman's Rank **101**
44.1 Spearman's Rank Correlation Coefficient 101
44.2 Hypothesis Testing 101

45 Chi-Squared Contingency Table Tests **103**
45.1 The Chi-Squared Statistic 103
45.2 Hypothesis Testing 103

46 Discrete Random Variables **105**
46.1 Discrete Random Variables 105
46.2 The Expected Value E(X) 105
46.3 The Variance Var(X) 105
46.4 E(aX+b)=aE(X)+b 105
46.5 Var(aX+b)= a^2 Var(X) 105
46.6 E(X+Y) = E(X) + E(Y) and Var(X+Y) = Var(X) + Var(Y) 105

47 Discrete Uniform Distributions **107**

48 Geometric Distributions **109**

49 Binomial Distributions **111**

50 Poisson Distribution **113**

51 Goodness of Fit Tests **115**
51.1 Goodness of Fit Tests 115
51.2 The Uniform Distribution 115
51.3 The Poisson Distribution 115
51.4 The Binomial Distribution 115
51.5 The Left Hand Tail 115

52 Energy **117**
52.1 Introduction to Energy 117
52.2 Conservation of Mechanical Energy 117
52.3 The Work-Energy Principle 117

53 Power **119**
53.1 Introduction to Power 119
53.2 Horsepower 119
53.3 Maximum Speed 119
53.4 Work, Energy, & Power 119

54 Friction **121**

54.1 Introduction to Friction . . . 121
54.2 Block Sliding Down a Slope . . . 121
54.3 Friction Examples . . . 121
54.4 Exam-Style Question . . . 121

55 Momentum & Impulse **123**
55.1 Momentum . . . 123
55.2 Impulse . . . 123

56 Collisions **125**
56.1 Conservation of Linear Momentum . . . 125
56.2 The Coefficient of Restitution . . . 125
56.3 Hitting the Ground/Hitting the Wall . . . 125

57 Moments **127**
57.1 Moments - The Basics . . . 127
57.2 Couples . . . 127
57.3 Ladders . . . 127
57.4 Pivots/Hinges . . . 127
57.5 Sliding & Toppling . . . 127

58 Centre of Mass **129**
58.1 Introducing CoM . . . 129
58.2 Laminas . . . 129
58.3 Suspending a Lamina . . . 129
58.4 Triangles . . . 129
58.5 Other Shapes . . . 129

59 Dimensional Analysis **131**
59.1 Introducing Dimensional Analysis . . . 131
59.2 Dimensional Consistency . . . 131
59.3 Finding Formulae . . . 131
59.4 Triangles . . . 131
59.5 Other Shapes . . . 131

1 Proof

Introduction to Proof

Introduction to Proof

In this section we will working with these topics:

- Consequence and Equivalence
- Proof by Exhaustion
- Proof by Deduction
- Disproof by Counter-Example
- Proof by Contradiction

Introducing Consequence and Equivalence

When we look at consequence, we essentially say that "a implies b", or:

$$a \rightarrow b$$

If the arrow points the other way, we say that "b implies a", or:

$$a \leftarrow b$$

Let's say that statement a states that p is a prime number > 2.

Let's say that statement b states that p is an odd number.

For these statements, we see that a does imply b, so we can write that

$$a \rightarrow b$$

The other way however does not work, since because p is an odd number, it does not imply that p is a prime number.

However, if this was true, we can write that a implies b and b implies a, or:

$$a \leftrightarrow b$$

which is sometimes written as "a if and only b" or "a iff b".

Let's show a logical equivalence. Let a be the statement n^2 is odd and b be the statement n is odd.

We know that when n^2 is odd, that n is odd when we list out the odd squared numbers. We can see the converse is true as well in this statement since every time a number n is squared, we are given an odd number, therefore:

$$a \leftrightarrow b$$

Consequence and Equivalence Examples

Let's give some examples where we determine whether one of the statements implies the other statement.

Given that an object is a cube and an object has six faces. If an object is a cube, it definitely has six faces. Therefore, The object is a cube $\implies$ The the object has six faces. The opposite is not true, because it can be a cuboid, for example.

Given $x = 29$ and $x > 10$, then $x = 29 \implies x > 10$. The opposite is not true, since there are many more values where $x > 10$.

Given $x^3 = x$ and $x = -1$. We need to find the solutions of $x^3 = x$ first. By subtracting and obtaining $x^3 - x = 0$, we can factor this to $x(x^2 - 1) = 0$. Then we have $x(x-1)(x+1) = 0$, and the solution of this equation are $0, 1$, and -1. Therefore $x^3 = x$ does not imply $x = -1$. However, going the other way, $x = -1 \implies x^3 = x$.

Given n is a positive integer greater than 1, we are given the statements that n is a prime number and n has exactly two factors. n always has two factors if it is prime, then n is a prime number $\implies$ n has exactly two factors. If n has exactly two factors, then it must be prime, so we can see that n has exactly two factors $\implies$ n is a prime number, so n is a prime number $\leftrightarrow$ n has exactly two factors.

Proof by Exhaustion

Introducing Proof by Exhaustion

Proof by Exhaustion is trying all possible variations to prove a statement is true.

We are going to prove a conjecture, which is a statement that we believe to be correct but needs to be proved.

The conjecture is "97 is a prime number". To show this, we need to show that 97 has two factors, 1 and itself.

Let's try some numbers.

$97 \div 2$ is 48.5, clearly 2 is not a factor of 97. $97 \div 3$ is $32.\overline{3}$. Therefore, 3 is not a factor either. We wouldn't need to try 4 since 2 already isn't a factor. Let's try 5. $97 \div 5$ is 19.4, so 5 is also not a factor of 97. We don't need to try 6 since 3 and 2 are both not factors of 97. Now we try 7. $97 \div 7 = 13.85\ldots$, so 7 is not a factor either. It's clear we are just working through all the prime numbers now.

We don't need to go further than this because when we square root 97, we will get a number a little less than 10. Because the square root of 97 is a little less than 10, when we go beyond 10, if we are to find any factor above 10, then there would have to have been a factor less than 10 to multiply with to make 97.

In other words, because there were no factors below the square root of 97, this implies there are no factors larger than the square root of 97, indicating that 97 is a prime number.

Proof by Exhaustion Examples

Let's do three examples.

- No square number ends in an 8

  This problem looks at squaring each unit digit. If a number ends in a 1, the square one gets will end in a 1 as well. If the number ends in a 2, and I square it, then this number will end with a 4. If the number ends with a 3, the number will end with a 9. If the number ends with a 4, the squared number will end with a 6. If the number ends with a 5, the squared number will end with a 5. If the number ends with a 6, the squared number will end with a 6. If the number ends with a 7, the squared number will end with a 9. If the number ends with a 8, the squared number will end with a 4. If the number ends with a 9, the squared number will end with a 1. If the number ends with a 0, the squared number will end with up with a 0.

  As we can see, there are no numbers that can have a unit digit of 8.

- If n is an integer and $2 \leq n \leq 7$, then $A = n^2 + 2$ is not divisible by 4.

  To show this, lets consider all values of n.

n	n^2+2	divisible by 4?
2	6	no
3	11	no
4	18	no
5	27	no
6	38	no
7	51	no

so in none of these cases, none of these values of A are divisible by 4 and we have gone through every single part of this and show that this is never divisible by 4.

- Every integer that is a perfect cube is either a multiple of 9, is 1 more than a multiple of 9, or is 1 less than a multiple of 9.

  The first statement says that $n = 3k$, that the number is a multiple of 3, or $n = 3k - 1$, one less than a multiple of three, or $n = 3k - 2$, a number is two less than a multiple of 3.

  Let's start by cubing. $n^3 = 27k^3$. Because 27 is a multiple of 9, k is an integer and n^3 is a multiple of 9.

  Let's look at $n = 3k - 1$. $n^3 = 27k^3 - 27k^2 + 9k - 1$. If we factor a 9 out, we get $9(3k^3 - 3k^2 + k) - 1$. This is clearly 1 less than a multiple of 9.

  Now let's look at $n = 3k - 2$. $n^3 = 27k^3 - 54k^2 + 36k - 8$. If I write the 8 as a $-9 + 1$, we can factor out the 9 and get $9(3k^3 - 6k^2 + 4k - 1) + 1$, or one more than a multiple of 9.

Proof by Deduction

Introduction Proof by Deduction

Proof by deduction is all about going through a logical sequence of arguments where you will start with something you know to be true, and subsequently, the next thing is true, etc, until the conjecture is true.

Conjecture: "The sum of any two consecutive odd numbers is a multiple of 4."

We can start with an odd number $2n+1$, since $2n$ is always an even number, so adding 1 will make it odd. If we are looking for the next consecutive odd number, then we can see this as $2n+3$. The conjecture talks about the sum of the consecutive odd numbers. Adding them together, we get $4n+4$, which factors to $4(n+1)$, which is always a multiple of 4.

Proof by Deduction Example

Example

For any four consecutive integers, the difference between the prodcut of the last two and the product of the first two of these numbers is equal to their sum.

Let's first label four consecutive integers as $n, n+1, n+2, n+3$. We have to find the product of the last two and the product of the last two and to find the difference between the two things.

Therefore, we are finding $(n+2)(n+3) - n(n+1)$. Expanding this, we get $n^2 + 5n + 6 - n^2 - n$. Simplifying, we get $4n + 6$.

Adding the consecutive integers, we have $n + n + 1 + n + 2 + n + 3 = 4n + 6$. We have shown that the difference between the products of the last two and the first two is the same as the sum of the four numbers.

Example

$k^3 - k$ is divisible by 6 for all integers $k > 1$.

First we can factor $k^3 - k$ to $k(k^2 - 1)$. We can factor this further as $k(k-1)(k+1)$. Now if we write this in a slightly different order, as $(k-1)(k)(k+1)$. What we have here is the product of three consecutive integers. At least one of these integers therefore will be an even integer, so $k^3 - k$ is divisible by 2.

Now because we have three consecutive integers, precisely one of them will be a multiple of 3 because since $k > 1$, there will always be a number that is divisible by 3 when consecutively counting. Therefore $k^3 - k$ is also divisible by 3.

Because $k^3 - k$ is divisible by 2 and 3, then it is divisible by 6.

Disprove by Counter-Example

Introducing Disproof by Counter Example

Sometimes we are asked to find a single example where a conjecture fails.

Let's start with the conjecture "The value of $n^2 + n + 11$ is prime for all integers $n > 0$"

When $n = 11$, we can see that $11^2 + 11 + 11$ which is equal to $11(13)$ which is evidently not prime.

Disproof by Counter Example Examples

Example

If $x^2 > x$, then $x > 1$.

When we plug in $x = -2$, we can see that $4 > -2$, but -2 is not greater than 1.

Example

If n is prime, then $n^2 + n + 1$ is prime.

When we plug in $n = 7$, we get $n^2 + n + 1 = 57$, which is not prime, so this conjecture fails.

Example

The sum of n consecutive integers is divisible by n (where n is a positive integer).

We can easily disprove this in one example. $1 + 2 + 3 + 4 = 10$, which is not divisible by 4.

2 Algebra & Functions

2.1 Indices

Subsets of Real Numbers

Introducing Subsets of Real Numbers

Natural numbers are represented by $\mathbb{N}$. They are just the counting numbers - like $1, 2, 3, 4, 5, 6, \ldots$. This does not include 0 or negative numbers.

Integers are represented by $\mathbb{Z}$. This includes all the natural numbers and also includes $0, -1, -2, -3, \ldots$. It is twice the size of natural numbers plus a zero.

Rational numbers are represented by $\mathbb{Q}$. This would include $\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, -\frac{5}{7}, -\frac{7}{2}$ along with the natural numbers and integers.

The real numbers are represented by $\mathbb{R}$. This includes everything above, along with things such as $\sqrt{2}, \sqrt{3}, \pi, e$.

The complex numbers are based on if we allowed to square root -1. We define this as i. The complex numbers will include things such as $2i, 3+i$.

The Laws of Indices

The Laws of Indices

We should know that $x^2 = x \times x$, and $x^3 = x \times x \times x$. The index tells us how many times we are multiplying x by itself.

When we put the x as x^2, we can see that $x^2 \times x^2 = x \times x \times x \times x = x^4$ or $(x^2)^2$.

As we can see, when multiplying $x^p \times x^q = x^{p+q}$.

Also when we have $(x^p)^q = x^{pq}$. Of course we know that $pq = qp$, and ad can also see that $(x^q)^p = (x^p)^q$.

Now let's imagine what we have $x^5 \div x^3 = \frac{x \times x \times x \times x \times x}{x \times x \times x} = x \times x = x^2$.

When we are dividing, then $x^p \div x^q = x^{p-q}$.

Let's say we have $x^{3.5}$. As long as the power is a rational number (in this case $3.5 = \frac{7}{2}$), then we can have an idea on what it is. We can write $x^{\frac{7}{2}}$ as $x^{\frac{1}{2} \times 7}$. This is the same now as $(x^{\frac{1}{2}})^7$.

This shows us our next rule - $x^{\frac{1}{p}} = \sqrt[p]{x}$.

So the above equation can be written as $(\sqrt{x})^7$.

Now let's also consider x^0. If you think about writing this as x^{2-2}, this equals $\frac{x^2}{x^2} = 1$.

Therefore, $x^0 = 1$.

Now we can look at $x^{-1} = x^{4-5} = \frac{x^4}{x^5}$. So from this we get $\frac{x \times x \times x \times x}{x \times x \times x \times x \times x} = \frac{1}{x}$.

This means that $x^{-1} = \frac{1}{x}$.

We have the rule then that $x^{-p} = \frac{1}{x^p}$.

Examples of Negative Indices

Exercise $2^{-3} =$

Exercise $3^{-4} =$

Exercise $5^{-2} =$

Exercise $\left(\frac{1}{4}\right)^{-2} =$

Exercise $\left(\frac{2}{3}\right)^{-3} =$

Examples of Positive Rational Indices

Exercise $36^{\frac{1}{2}} =$

Exercise $81^{\frac{1}{4}} =$

Exercise $\left(\frac{1}{8}\right)^{\frac{1}{3}} =$

Exercise $25^{\frac{3}{2}} =$

Exercise $\left(\frac{8}{27}\right)^{\frac{2}{3}} =$

Examples of Negative Rational Indices

Exercise $8^{-\frac{1}{3}} =$

Exercise $16^{-\frac{3}{4}} =$

Exercise $4^{-\frac{5}{2}} =$

Exercise $\left(\frac{36}{49}\right)^{-\frac{1}{2}} =$

Exercise $\left(\frac{10000}{16}\right)^{-\frac{5}{4}} =$

More Complicated Examples

Example

$2^3 \times 8^{-\frac{5}{3}} \times \frac{1}{\sqrt{2}} = 2^k$. Find k.

For this problem, you want to write everything in terms of 2 to the power of something. We can rewrite this equation as

$$2^3 \times (2^3)^{-\frac{5}{3}} \times 2^{-\frac{1}{2}}$$

So this can be rewritten as $2^3 \times 2^{-5} \times 2^{-\frac{1}{2}}$, and using laws of indices, we can see that this is equivalent to $2^{-\frac{5}{2}}$. So $k = -\frac{5}{2}$.

Exercise Write $\frac{x^2y^5}{\sqrt{x}} \div \frac{x^{\frac{3}{2}}}{y^7}$ as a product of powers of x and y.

Examples of Simplifying Expressions

Exercise $5a^3b^2c \times 6a^8bc^{-3} =$

Exercise $(60a^4b^2c) \div (12a^8b^5c^{-4}) =$

Exercise $\frac{(3x)^3 \times (2x^3)^4}{(6x^8)^2} =$

Write in the form of 2^k

Example

Write $\frac{\sqrt{2}}{4^3}$ in the form 2^k.

We can rewrite $\sqrt{2} = 2^{\frac{1}{2}}$ and $4^3 = (2^2)^3 = 2^6$. So now we have $\frac{2^{\frac{1}{2}}}{2^6} = 2^{-\frac{11}{2}}$.

Exercise Write $8^4 \times \frac{2}{\sqrt[3]{16}}$ in terms of 2^k.

Write in the form of 3^k

Example

Write $\sqrt[3]{3} \times \sqrt[3]{9}$ in terms of 3^k.

We can rewrite this as $3^{\frac{1}{3}} \times (3^2)^{\frac{1}{3}} = 3^{\frac{1}{3}} \times 3^{\frac{2}{3}} = 3^1$.

Exercise Write $\frac{\sqrt[5]{27}}{\sqrt{3}} \times 81$ in terms of 3^k.

Write in the form of 4^k

Example

Write $\frac{16}{\sqrt[4]{5}}$ in terms of 4^k.

This can be rewritten as $\frac{4^2}{4^{\frac{1}{5}}}$, so this is equivalent to $4^{\frac{9}{5}}$.

Exercise Rewrite $2 \times \sqrt[3]{16} \times \sqrt[5]{64}$ in terms of 4^k.

Write in the form 5^k

Example

Rewrite $\frac{125}{\sqrt[3]{25}} \times \sqrt{5}$ in terms of 5^k.

We first start off with $\frac{5^3}{(5^2)^{\frac{1}{3}}} \times 5^{\frac{1}{2}}$.

This is equal to $5^{\frac{7}{3}} \times 5^{\frac{1}{2}} = 5^{\frac{17}{6}}$.

Exercise Rewrite $\frac{\sqrt[3]{50}}{\sqrt{625}} \times \sqrt[3]{12.5}$ in terms of 5^k.

2.2 Surds

Simplifying Surds

Introducing Surds and Simplifying Surds

You can quite easily build up a list of what you believe are surds - $\sqrt{1} = 1$, $\sqrt{2}$ and $\sqrt{3}$ are surds, $\sqrt{4} = 2$. If the number under the square root if a square number, then obviously it will not be a surd.

Let's get an example with $\sqrt{8}$. Using our indices knowledge, we can write this as $8^{\frac{1}{2}} = (4 \times 2)^{\frac{1}{2}} = 4^{\frac{1}{2}} \times 2^{\frac{1}{2}}$.

This can be written as $\sqrt{4} \times \sqrt{2} = 2\sqrt{2}$.

Now let's look at $\sqrt{12}$. Now we can write this as $\sqrt{6} \times \sqrt{2}$, but there is no real point in doing this, since none of these can be simplified. We are looking for any square numbers that can go in 12. So we can write this as $\sqrt{4} \times \sqrt{3}$ which is equivalent to $2\sqrt{3}$.

This leads us to the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.

Simplifying Surds Examples

Exercise $\sqrt{18}$

Exercise $\sqrt{200}$

Exercise $\sqrt{48}$

Exercise $\frac{\sqrt{12}}{\sqrt{300}}$

Exercise $\sqrt{24} \times \sqrt{150}$

Adding/Subtracting Surds

Let's start with an example.

If we are given $\sqrt{20} + \sqrt{180}$, we cannot add the two together. In most cases $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$.

We have $\sqrt{4} \times \sqrt{5} + \sqrt{36} \times \sqrt{5}$ when we simplify this. Now we can simplify this as $2\sqrt{5} + 6\sqrt{5}$. Surds can be combined like 'like' terms in algebra. So the answer so this expression is $8\sqrt{5}$.

Here's two more examples to try with the answer given.

Example 1: $\sqrt{63} - \sqrt{28} = \sqrt{7}$

Example 2: $\sqrt{108} + \sqrt{72} = \sqrt{2} + \sqrt{3} + \sqrt{5}$

Rationalising the Denominator

Problem Solving

2.3 Quadratics

The Difference of Two Squares

Factorising Quadratics

Sketching Quadratics from Factorised Form

Completing the Square

Sketching Quadratics from Completed Square Form

Solving Quadratics

Using the Discriminant

Using the Quadratic Formula

Sketching Quadratics Using the Quadratic Formula

Sketching Quadratic Using a Calculator

Using Quadratic Methods for Solving

2.4 Simultaneous Equations

The Elimination Method

The Substitution Method

Further Simultaneous Equations

2.5 Inequalities

Introducing Inequalities, Set Notation and Interval Notation

Linear Inequalities

Quadratic Inequalities

Discriminant Inequalities

More Inequalities

Double and Triple Inequalities

Representing Inequalities Graphically

2.6 Polynomials & Rational Expressions

Introducing Polynomials

Polynomial Division

The Factor Theorem

Simplifying Algebraic Fractions

Adding and Subtracting Algebraic Fractions

Simplifying using Polynomial Division

2.7 Graphs & Proportion

3 Coordinate Geometry

3.1 Coordinate Geometry

Introduction to Coordinate Geometry

Finding the Midpoint

Finding the Distance between Two Points

Finding the Gradient

The Equation of a Line

Parallel and Perpendicular Lines

Sketching Linear Graphs

Perpendicular Bisectors

Intersections of Lines

An Application of Linear Graphs

3.2 Circles

The Equation of a Circle

Sketching Circles

Circles: Completing the Square

Intersections with Circles

Circle Theorems

Circles: Pependicular Bisectors

Tangents and Normals

3.3 Parametric Equations

Introducing Parametric Equations

Cartesian to Parametric

Graphing Parametric Curves

Parametric to Cartesian

Ellipses

3.4 Parametric Equation Modelling

4 Sequences & Series

4.1 Binomial Expansion

The Factorial Function

Pascal's Triangle

Algebra Problems with nCr

Binomial Expansion

Finding a Coefficient

Approximating using Binomial Expansion

Further Binomial Expansion

The Range of Validity

4.2 Sequences

GCSE Sequences Revision

Inductive Definitions and Recurrence Relations

Describing Sequences

4.3 Sigma Notation

4.4 Arithmetic Sequences

Introducing Arithmetic Sequences

Arithmetic Series

Simultaneous Equation Problems

4.5 Geometric Sequences

Introducing Geometry Sequences

Geometric Series

Sum to Infinity

Simultaneous Equation Problems

4.6 Modelling with Sequences

5 Trigonometry

5.1 Trigonometry

SOHCAHTOA

The Sine Rule

The Cosine Rule

The Area of a Triangle

Radians

Arc Length

Area of a Sector

5.2 Small Angle Approximation

5.3 Trig Graphs

Sketching sin(x), cos(x), and tan(x)

Radians

5.4 Further Trigonometry

Cosec(x), Sec(x), Cot(x)

Sketching cosec(x), sec(x), and cot(x)

Inverse Trigonometric Functions

5.5 Trigonometric Identities

Trigonometric Identities

Further Trigonometric Identities

5.6 Compound Angles & Equivalent Forms

Compund Angle Formulae

Double Angle Formulae

Equivalent Forms

5.7 Trig Equations

Basic Trigonometric Equations

Quadratic Trigonometric Equations

Using tan(x)=sin(x)/cos(x)

Trigonometric Equations with Transformations

More Quadratic Trigonometric Equations

6 Exponentials & Logarithms

6.1 Exponentials

Introducing a^x

Introducing e

6.2 Exponential Models

6.3 Logarithms

Introducing Logarithms

Introducing Logarithmic Graphs

Sketching $y = \log_b(x + a)$

Sketching $y = \log_b(x + a) + c$

Introducing the Natural Logarithm

Sketching $y = \ln(x + a)$

SKetching $y = \ln(x + a) + b$

6.4 Laws of Logarithms

The Laws of Logarithms

The Natural Logarithm

6.5 Exponential & Logarithmic Equations

Solving $a^x = b$

Logging Both Sides

Inequalities

Hidden Quadratics

Solving $e^x = k$

Logarithmic Equations

Solving ln(x)=k

6.6 Reduction to Linear Form

6.7 Exponential Growth & Decay

7 Differentiation

7.1 Differentiation from First Principles

Gradient of a Straight Line

Differentiating Polynomials

Gradients of Gradient Functions

Second Derivatives

Differentiation from First Principles

Convex and Concave

7.2 Differentiation

Differentiating x^n

Differentiating Standard Functions

7.3 Gradients

Gradients of Functions

Tangents and Normals

Stationary Points

Increasing and Decreasing

The Second Derivative Test

Types of Stationary Point

Convex and Concave

Points of Inflection

Points of Inflection of the Normal Distribution

Optimisation

7.4 Further Differentiation

The Chain Rule

Connected Rates of Change

The Product Rule

The Quotient Rule

Choosing Between Rules

Differentiating an Inverse Function

7.5 Implicit Differentiation & Parametric Differentiation

8 Integration

8.1 Fundamental Theorem of Calculus

8.2 Indefinite Integrals

Integrating ax^n

Finding the Constant of Integration

Integrating Standard Functions

8.3 Definite Integrals & Parametric Integration

Finding Areas

Definite Integrals

Areas Between a Curve and a Line

Areas between Two Curves

Parametric Integration

8.4 Integration as the Limit of a Sum

8.5 Further Integration

Reversing the Chain Rule

Integrating by Substitution

Integration by Parts

Integrating ln(x)

Integration by Parts Twice

The Tabular Method for Integration by Parts

Further Integration

8.6 Integration with Partial Fractions

8.7 Differential Equations

8.8 Differential Equations in Context

9 Numerical Methods

9.1 The Change of Sign Method

The Need for Numerical Methods

The Change of Sign Method

9.2 The x=g(x) Method & The Newton-Raphson Method

The x=g(x) Method

The Newton-Raphson Method

9.3 Numerical Integration

Estimating Areas with Rectangles

The Trapezium Rule

9.4 Numerical Methods in Context

10 Vectors

10.1 Introducing Vectors

What is a Vector?

Finding the Vector between Two Points

Vectors in 3D

10.2 Magnitude & Direction of a Vector

The Magnitude & Direction of a 2D Vector

Finding the Angle Between two Vectors

The Magnitude of a 3D Vector

The Angle between two 3D Vectors

10.3 Resultant & Parallel Vectors

Resultant Vectors

Parallel Vectors

Collinear Points

10.4 Position Vectors

10.5 Vector Problems

11 Statistical Sampling

The Large Data Set

Types of Sample and Sampling Methods

12 Data Presentation & Interpretation

12.1 Box Plots, Cumulative Frequency, & Histograms

Introducing Data Representation

Box Plots/Box and Whisker Diagrams

Cumulative Frequency Curves

Histograms

12.2 Scatter Graphs

Bivariate Data

The Product Moment Correlation Coefficient

Regression Lines

Interpolation vs Extrapolation

12.3 Central Tendency & Variation

Ungrouped Data: Mean, Mode, Median & Quartiles

Grouped Data: Mean, Mode, Median & Quartiles

The Interquartile Range

The Midrange

Comparing Data Sets

Variance and Standard Deviation

Linear Coding

12.4 Outliers & Cleaning Data

13 Probability

13.1 Venn Diagrams, Tree Diagrams, & Two-Way Tables

Basic Probability Concepts

Venn Diagrams

Independent Events / Mutually Exclusive Events

Tree Diagrams

Two-Way Tables

Probability with a Histogram

13.2 Conditional Probability

13.3 Modelling with Probability

14 Statistical Distributions

14.1 Discrete Random Variables & The Binomial Distribution

Introducing Discrete Random Variables

Discrete Probability Distributions as Algebraic Functions

Discrete Uniform Distributions

Cumulative Distribution Functions

The Binomial Distribution

14.2 The Normal Distribution

Introducing the Normal Distribution

Finding Probabilities

The Inverse Normal

Normal to Binomial Problem

Normal to Histogram

Approximating the Binomial Distribution

14.3 Appropriate Distributions

15 Hypothesis Testing

15.1 Introducing Hypothesis Testing

Introducing Hypothesis Testing

Product Moment Correlation Coefficient Hypothesis Testing

Rank Correlation Coefficient Hypothesis Testing

15.2 Binomial Hypothesis Testing

Binomial Hypothesis Testing

Finding the Critical Region

The Critical Region Method

15.3 Sample Means Hypothesis Testing

Introducing Sample Means Hypothesis Testing

Example 1

Example 2

Example 3

16 Quantities & Units in Mechanics

17 Kinematics

17.1 Displacement, Velocity, & Acceleration

Position vs Displacement vs Distance

Velocity vs Speed

Acceleration and Deceleration

17.2 Graphs of Motion

Displacement / Time Graphs

Velocity / Time Graphs

Acceleration / Time Graphs

Graphs of Motion

17.3 SUVAT

Deriving the SUVAT Formulae

Using the SUVAT Formulae

Gravity

More Complicated SUVAT Problems

SUVAT in 2D

17.4 Calculus in Kinematics

General Motion in 1D

General Motion in 2D

17.5 Projectiles

Introducing Projectiles

Projectiles from the Grond

Projectiles from a Height

18 Forces & Newton's Laws

18.1 Introducing Forces & Newton's First Law

Introducing Forces

Force Diagrams

Resultant Forces

Newton's First Law

18.2 Newton's Second Law

Newton's Second Law

Working with the SUVAT Equations

18.3 Weight & Tension

18.4 Newton's Third Law and Pulleys

Newton's Third Law

Pulleys

Lifts and Scale Pans

18.5 F=ma & Differential Equations

F=ma in Two Dimensions

F=ma as Differential Equations

18.6 The Coefficient of Friction

19 Moments

Introducing Moments

Centre of Mass

Equilibrium of a Rigid Body

Tilting

Non-Parallel Forces with Pivots and Ladders

20 Proof

Introducing Proof by Induction

Sums of Series

Divisibility

Sequences

Matrices

Inequalities

Extras

21 Complex Numbers

21.1 Introducing Complex Numbers

Introducing Complex Numbers

Solving Polynomial Equations with Real Coefficients

21.2 Working with Complex Numbers

Real and Imaginary Parts

Working with Complex Numbers

21.3 Complex Conjugates

The Complex Conjugate

Complex Conjugate Pairs

21.4 Introducing the Argand Diagram

21.5 Introducing Modulus-Argument Form

Introducing the Modulus and Argument

Modulus-Argument Form

21.6 Multiply and Divide in Modulus-Argument Form

21.7 Loci with Argand Diagrams

Circles

Perpendicular Bisectors

Loci Problems with Circles & Perpendicular Bisectors

Half-Lines

Loci Problems with Circles, Perpendicular Bisectors and Half-Lines

21.8 De Moivre's Theorem

Introducing De Moivre's Theorem

Expansions of $\cos(n\theta)$ and $\sin(n\theta)$

21.9 $z = re^{(i\theta)}$

Introducing $z = re^{i\theta}$

Summing Series

21.10 nth Roots of Unity

22 Matrices

22.1 Introducing Matrices

Introducing Matrices

Multiplying Matrices

22.2 The Zero & Identity Matrices

The Zero Matrix

The Identity Matrix

22.3 Matrix Transformations

2D Transformations

3D Transformations

22.4 Invariance

22.5 Determinants

Introducing Determinants

2x2 Matrix Determinants

Negative Determinants and Orientation

3x3 Matrix Determinants

Determinant Problems

22.6 Inverse Matrices

Notation

2x2 Inverse Matrices

Singular Matrices

3x3 Inverse Matrices

22.7 Simultaneous Equations

Two-Variable Simultaneous Equations

Three-Variable Simultaneous Equations

22.8 Geometrical Interpretation

Two Dimensions

Three Dimensions

22.9 Factorising Determinants

23 Further Algebra & Functions

23.1 Roots of Polynomials

23.2 Forming New Equations

Quadratics

Cubics

Quartics

The Substitution Method

23.3 Summations

Introduction

Examples

23.4 Method of Differences

Method of Differences

Method of Differences with Partial Fractions

23.5 Introducing Maclaurin Series

23.6 Standard Maclaurin Series

23.7 Limits and l'Hospital's Rule

Finding a Limit using Maclaurin Series

l'Hopital's Rule

23.8 Polynomial Inequalities

Cubic Inequalities

Quartic Inequalities

23.9 Rational Function Inequalities

Introducing Rational Function Inequalities

Rational Function Inequality Examples

23.10 Modulus of Functions

Modulus of Functions

Solving Equations

Solving Inequalities

24 Further Calculus

24.1 Improper Integrals

Introducing Improper Integrals

Integration Techniques Part 1

Integration Techniques Part 2

24.2 Volumes of Revolution

Introducing Volumes of Revolution

Revolution about the x-axis

Parametric Equations

24.3 Mean Value

Introducing the Mean Value

Examples

24.4 Partial Fractions

Re-Introducing Partial Fractions

Quadratic Factors in the Denominator

24.5 Differentiating Inverse Trig

24.6 Integrals of the Form $\sqrt{a^2 - x^2}$ and $1/(a^2 + x^2)$

24.7 Arc Length and Sector Area

Arc Length

Surface Area

24.8 Reduction Formulae

24.9 Limits

25 Further Vectors

25.1 Equations of Lines

The Vector Equation of a Line

The Cartesian Equation of a Line

25.2 Equations of Planes

25.3 The Scalar Product

25.4 Perpendicular Vectors

25.5 Intersections

Two Lines Intersecting

Intersection of a Line and a Plane

Distance between Two Lines

Distance from a Point to a Line

Distance from a Point to a Plane

25.6 The Vector Product

Introducing the Vector Product

Using the Vector Product

Distances

26 Polar Coordinates

26.1 Polar Coordinates

Introducing Polar Coordinates

Converting between Polar and Cartesian Coordinates

26.2 Polar Curves

Polar Curves

Limacons

Rhodonea / Rose Curves

Further Polar Curves

26.3 Polar Integration

The Area enclosed by a Polar Curve

Polar Tangents

27 Hyperbolic Functions

27.1 Hyperbolic Functions

Introducing Hyperbolic Functions

Hyperbolic Identities & Equations

27.2 Hyperbolic Calculus

Differentiation & Integration

Differentiation

Integration

27.3 Hyperbolic Inverse

27.4 Hyperbolic Inverse

Logarithmic Forms

Differentiation

27.5 Hyperbolic Integration

Differentiating Standard Forms

Integration

27.6 Hyperbolic Identities

Proving "Double Angle" formulae

Using Identities

27.7 Hyperbolic Identities

28 Differential Equations

28.1 1st Order Differential Equations - Integrating Factors

Introduction

Integrating Factors

28.2 1st Order Differential Equations - Particular Solutions

28.3 Modelling

28.4 2nd Order Homogeneous Differential Equations

Introduction

The Auxiliary Equation

28.5 2nd Order Non-Homogeneous Differential Equations

28.6 2nd Order Non-Homogeneous Differential Equations

28.7 Simple Harmonic Motion

28.8 Damped Oscillations

28.9 Systems of Differential Equations

28.10 Hooke's Law

28.11 Damping Force

29 Numerical Methods

29.1 Mid-Ordinate Rule & Simpson's Rule

Mid-Ordinate Rule

Simpson's Rule

29.2 Euler's Step by Step Method

29.3 Euler's Improved Step by Step Method

30 Tracing an Algorithm

30.1 Tracing an Algorithm

30.2 Complexity

31 Bin Packing

31.1 Bin Packing

31.2 Complexity

32 Sorting Algorithms

32.1 Introduction

32.2 Quick Sort

32.3 Bubble Sort

33 Graph Theory

34 Minimum Spanning Trees

34.1 Introduction

34.2 Kruskal's Algorithm

34.3 Prim's Algorithm

34.4 Prim's Algorithm with a Matrix

35 Dijkstra's Algorithm

36 Critical Path Analysis

36.1 Critical Path Analysis (CPA)

36.2 Precedence Tables

36.3 Activity Networks

36.4 Dummy Activities

37 Network Flows

37.1 Network Flows

37.2 Cuts

37.3 Supersinks & Supersources

38 Linear Programming

38.1 Drawing Inequalities & The Objective Function

38.2 Formulating an LP Problem

38.3 3-Variable to 2-Variable

39 Simplex Algorithm

40 LP Solvers

40.1 Indicator Variables

40.2 Shortest Path (Dijkstra's)

40.3 Longest Path (CPA)

40.4 Network Flows

40.5 Critical Path Analysis (Alternative)

40.6 Matching

40.7 Allocation

40.8 Transportation

40.9 LINDO

41 PMCC

41.1 Bivariate Data

41.2 Correlation & Association

41.3 The PMCC

42 Linear Regression

42.1 Introduction

42.2 Calculating Regression Lines

42.3 Interpreting

43 PMCC Hypothesis Testing

43.1 PMCC Hypothesis Testing

43.2 Effect Sizes

44 Spearman's Rank

44.1 Spearman's Rank Correlation Coefficient

44.2 Hypothesis Testing

45 Chi-Squared Contingency Table Tests

45.1 The Chi-Squared Statistic

45.2 Hypothesis Testing

46 Discrete Random Variables

46.1 Discrete Random Variables

46.2 The Expected Value E(X)

46.3 The Variance Var(X)

46.4 E(aX+b)=aE(X)+b

46.5 Var(aX+b)= a^2 Var(X)

46.6 E(X+Y) = E(X) + E(Y) and Var(X+Y) = Var(X) + Var(Y)

47 Discrete Uniform Distributions

48 Geometric Distributions

49 Binomial Distributions

50 Poisson Distribution

51 Goodness of Fit Tests

51.1 Goodness of Fit Tests

51.2 The Uniform Distribution

51.3 The Poisson Distribution

51.4 The Binomial Distribution

51.5 The Left Hand Tail

52 Energy

52.1 Introduction to Energy

52.2 Conservation of Mechanical Energy

52.3 The Work-Energy Principle

53 Power

53.1 Introduction to Power

53.2 Horsepower

53.3 Maximum Speed

53.4 Work, Energy, & Power

54 Friction

54.1 Introduction to Friction

54.2 Block Sliding Down a Slope

54.3 Friction Examples

54.4 Exam-Style Question

55 Momentum & Impulse

55.1 Momentum

55.2 Impulse

56 Collisions

56.1 Conservation of Linear Momentum

56.2 The Coefficient of Restitution

56.3 Hitting the Ground/Hitting the Wall

57 Moments

57.1 Moments - The Basics

57.2 Couples

57.3 Ladders

57.4 Pivots/Hinges

57.5 Sliding & Toppling

58 Centre of Mass

58.1 Introducing CoM

58.2 Laminas

58.3 Suspending a Lamina

58.4 Triangles

58.5 Other Shapes

59 Dimensional Analysis

59.1 Introducing Dimensional Analysis

59.2 Dimensional Consistency

59.3 Finding Formulae

59.4 Triangles

59.5 Other Shapes