AP Physics C: Mechanics Notes

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1 Kinematics

1.1 Scalars and Vectors

Scalars are quantities described by magnitude only, vectors are quantities described by both magnitude and direction.

Vectors can be visually modeled as arrows with appropriate direction and lengths proportional to their magnitudes.

Vectors can be expressed in unit vector notation or as a magnitude and a direction.

- Unit vector notation can be used to represent vectors as the sum of their constituent components in the x, y, and z directions, denoted by $\hat{i}, \hat{j}$ and $\hat{k}$.

$$\vec{r} = A\hat{i} + B\hat{j} + C\hat{k}$$

- The position vector of a point is given by $\vec{r}$ and the unit vector in the direction of the position vector is denoted $\hat{r}$.
- A resultant vector is the vector sum of the addend vectors' components.

$$\vec{C} = \vec{A} + \vec{B} = (A_x + B_y)\hat{i} + (A_y + B_y)\hat{j}$$

In a given one-dimensional coordinate system, opposite directions are denoted by opposite signs.

Example

If $\vec{a} = 3\hat{i} + 4\hat{j} - \hat{k}$ and $\vec{b} = -2\hat{i} + \hat{y} + 2\hat{k}$, what is (a) $\vec{c} = \vec{a} + \vec{b}$

$\vec{c} = (3 + -2)\hat{i} + (4 + 1)\hat{j} + (-1 + 2)\hat{k} = \hat{i} + 5\hat{j} + \hat{k}$

(b) $\vec{c} = \vec{a} - \vec{b}$

$\vec{c} = (3 + 2)\hat{i} + (4 - 1)\hat{j} + (-1 - 2)\hat{k} = 5\hat{i} + 3\hat{j} - 3\hat{k}$

(c) $\vec{c} = \vec{b} - \vec{a}$

$\vec{c} = (-2 - 3)\hat{i} + (1 - 4)\hat{j} + (2 + 1)\hat{k} = -5\hat{i} - 3\hat{j} + 3\hat{k}$

Exercise An object moves in the xy-plane a distance A at an angle θ measured counterclockwise from the positive x-direction, where $0 < \theta < 90^\circ$. The object then moves a distance B in the positive x-direction. The change in the x-component of the object's position is equal to the change in the y-component of its positions. What is B in terms of A and θ?

Exercise An object is moving with an initial velocity $\vec{v}_1 = (3.00\hat{i} + 4.00\hat{j})$ m/s. After a certain time interval, its velocity is $\vec{v}_2 = (-8.00\hat{i} + 15.0\hat{j})$ m/s. What is the magnitude of the change in the velocity of the object over this time interval?

1.2 Displacement, Velocity, and Acceleration

When using the object model, the size, shape and internal configuration are ignored.

- The object may be treated as a single point with extensive properties such as mass and charge.

Displacement is the change in an object's position: $\Delta x = x - x_0$

Averages of velocity and acceleration are calculated considering the initial and final states of an object over an interval of time.

Average velocity is the displacement of an object divided by the interval of time in which that displacement occurs:

$$\vec{v}_{avg} = \frac{\Delta\vec{x}}{\Delta t}$$

Average acceleration is the change in velocity divided by the interval of time in which that change in velocity occurs.

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t}$$

As the time interval used to calculate the average value of a quantity approaches zero, the average value of that quantity approaches the value of the quantity that is instant, called the instantaneous value.

- $\vec{v} = \frac{\mathrm{d}x}{\mathrm{d}t}$
- $\vec{a} = \frac{\mathrm{d}v}{\mathrm{d}t}$

Time dependent functions and instantaneous values of position, velocity and acceleration can be determined using differentiation and integration.

Example

A particle moves along the x-axis with an acceleration of $a = 18t$, where a has units of m/s^2. If the particle at $t = 0$ is at the origin with a velocity of -12 m/s, what is its position at $t = 4.0$ s?

The velocity is the integral of acceleration: $v = \int 18t\mathrm{d}t = 9t^2 + C$. Substituting the initial conditions gives $v(0) = -12$.

Integrating the velocity function: $\int 9t^2 - 12\mathrm{d}t = 3t^3 - 12t + C$.

Since we know $x(0) = 0$, plug this in and we find that $C = 0$. Therefore, $x(4) = 144$ m.

Exercise An object moves in one dimension along the x-axis. At time $t = 0$, the object is located at position $x = 1$ m and has a velocity of $v = 1$ m/s. The object's acceleration varies as $a = 3t$, where a is in m/s^2 and t is time in seconds. A student incorrectly derives the equation for the object's position as $x = \frac{1}{2}t^3 + 1$ where x is in meters. What is a possible error that could have resulted in the incorrect equation?

Exercise Two objects, Object 1 and Object 2, have velocities $\vec{v}_1 = (3t^2\hat{\imath} + 5t\hat{\jmath})$ m/s and $\vec{v}_2 = (5t^2\hat{\imath} - 3t\hat{\jmath})$ m/s, respectively, where t is time in seconds. What is the relationship between the magnitudes of the acceleration a_1 of Object 1 at $t = 1$ s and the acceleration a_2 of Object 2 at $t = 1$ s?

1.3 Representing Motion

Motion can be represented by motion diagrams, figures, graphs, equations and narrative descriptions.

For constant acceleration, three kinematics equations can used to describe the instantaneous linear motion in one dimension:

- $v = v_0 + at$
- $x = x_0 + v_0t + \frac{1}{2}at^2$
- $v^2 = v_0^2 + 2a\Delta x$

Near the surface of the Earth, the vertical acceleration caused by the force of gravity is downward, constant and has a measured value of $g = 9.8$ m/s^2 or $g = 10$ m/s^2.

Graphs of position, velocity and acceleration as functions of time can be used to find the relationships between those quantities.

Example

A large cat, running at a constant velocity of 5.0 m/s in the positive x direction, runs past a small dog that is initially at rest. Just as the cat passes the dog, the dog begins accelerating at 0.5 m/s^2 in the positive x direction. (a) How much time passes before the dog catches up to the cat?

We know from the formula $x = v_0 t + \frac{1}{2}at^2$ that the cat will have position $5t$ and the dog $0.25t^2$.

Setting these equal to each other gives $t = 20$ s.

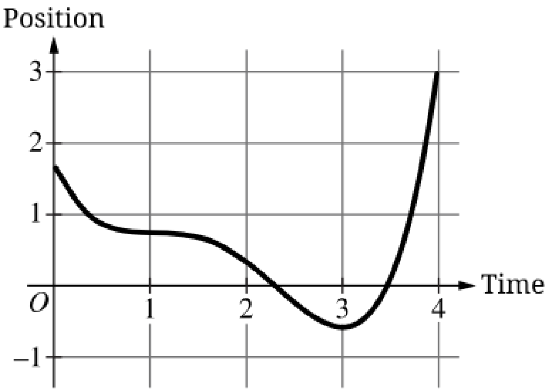
(b) How far has the dog traveled at this point?

Plug in $t = 20$ s to get 100 m.

(c) How fast is the dog traveling at this point?

The formula $v = v_0 + at$ can be used to get 10 m/s.

Exercise



An object moves in one dimension along the x-axis as described by the position versus time graph shown in the figure. During the time interval of the graph, how many times does the object change direction and what feature or features of the graph justifies this response?

Exercise On a distant planet where the acceleration due to gravity is g_P, an object takes a time t_P to reach the ground when dropped from a height h_0. On a small moon of the planet, the acceleration due to gravity is $\frac{g_P}{16}$. How long does it take an object to reach the ground when it is dropped from the same height on the moon?

1.4 Reference Frames and Relative Motion

The choice of reference frame will determine the direction and magnitude of quantities measured by an observer in that reference frame.

Measurements from a given reference frame may be converted to measurements from another reference frame.

The observed velocity of an object results from the combination of the object's velocity and the velocity of the observer's reference frame.

- Combining the motion of an object and the motion of an observer in a given reference frame involves the addition or subtraction of vectors.
- The acceleration of any object is the same as measured from all inertial reference frames.

Example

A cat passes a dog, traveling in the positive x-direction at 5.0 m/s. As the cat passes, the dog begins accelerating at 0.5 m/s^2 in the positive y-direction.

(a) What is the cat's acceleration relative to the dog?

The acceleration of the cat relative to the ground is $0\hat{\imath}$ and the acceleration of the dog relative to the ground is $0.5\hat{\jmath}$ so the acceleration of the cat relative to the dog is $0\hat{\imath} - 0.5\hat{\jmath}$.

(b) What is the cat's velocity relative to the dog at time $t = 5.0$ seconds after the dog begins running?

We have $v = v_{cat} + a_{CD}t = (5\hat{\imath} - 2.5\hat{\jmath})$ m/s

(c) What is the cat's position relative to the dog at $t = 5.0$ seconds after the dog begins running?

From $\Delta r = v_{CD}t + \frac{1}{2}a_{CD}t^2$, we get that $\Delta r = 25\hat{\imath} - 6.25\hat{\jmath}$.

Exercise A toy plane which maintains an airspeed v_p flies between points A and B in a time t_0 when there is negligible wind. When the air is moving at a velocity of $\frac{1}{2}v_p$ from point B to point A, the toy plane can make the trip from A to B in t_{AB}, and the return trip from B to A in t_{BA}. How do the three travel times compare?

Exercise Car A is traveling east with a speed of 30 m/s. Car B is traveling north with speed of 40 m/s. What is the direction of the velocity of Car B relative to Car A?

1.5 Motion in Two or Three Dimensions

Motion in two or three dimensions can be analyzed using one-dimensional kinematic relationships if the motion is separated into componenets.

Velocity and acceleration may be different in each dimension and be nonuniform.

Motion in one dimension may be changed without causing a change in the perpendicular dimension.

Projectile motion is a special case of two-dimensional motion that has zero acceleration in one dimension and constant, nonzero acceleration in the second dimension.

Example

The motin of an object can be described by the equations

- $x(t) = 4t^2 - 3t$
- $y(t) = 3t^3 - 2t^2 - 9t$

(a) What is the objects displacement after 2.5 s?

Plug in $t = 2.5$ for both equations to get $\Delta\vec{r} = (17.5\hat{\imath} + 11.9\hat{\jmath})$ m.

(b) Find the two equations that describe the object's velocity in the x and y directions.

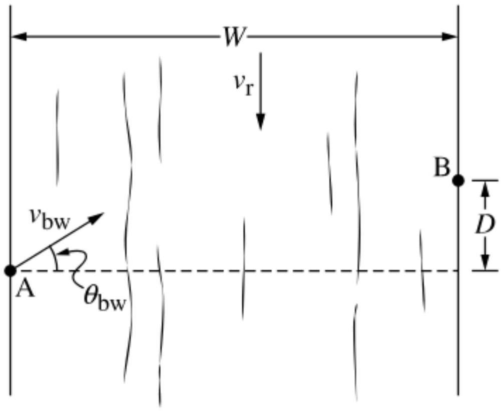
Take the derivative of both equations to get $v_x = 8t - 3$ and $v_y = 9t^2 - 4t - 9$.

(c) What is the object's velocity after 2.5 s?

Plug in 2.5 to both equations from part (b) to get $v = (17\hat{\imath} + 37.25\hat{\jmath})$ m/s.

Exercise Object 1 is launched at an initial speed v_0 at an angle θ above the horizontal and reaches a maximum height of y_1. Object 2 is launched at an initial speed $2v_0$ at the same angle θ, reaching a maximum height of y_2. What is the relationship between y_1 and y_2?

Exercise



A river has width W and a current v_r. A boat maintains a constant velocity as it travels from Point A to Point B. Point B is located at a distance D upstream from point A. The boat's water speed is v_{bw} with a heading of angle θ_{bw}, as shown in the figure. What is a correct expression for D?

2 Force and Translational Dynamics

2.1 Systems and Center of Mass

- System properties are determined by the interactions between objects within the system.
- If the properties or interactions of the constituent objects within a system are not important in modeling the behavior of a macroscopic system, the system can itself be treated as a single object.
- Systems may allow interactions between constituent parts of the system and the environment, which may result in the transfer of energy or mass.
- For objects with symmetrical mass distributions, the center of mass is located on lines of symmetry.
- The location of a system's center of mass along a given axis can be calculated using the equation:

$$x_{cm} = \frac{\sum m_i x_i}{M}$$

- For a nonuniform solid that can be considered as a collection of differential masses, $\mathrm{d}m$, the solid's center of mass can be calculated using

$$x_{cm} = \int x \mathrm{d}m / M$$

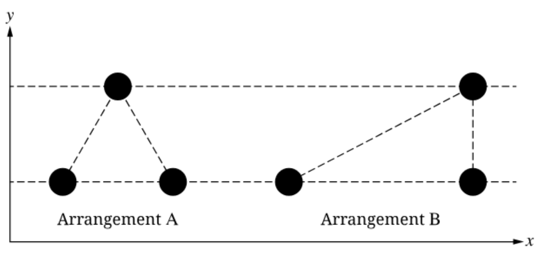
- A system can be modeled as a singular object that is located at the system's center of mass.

Example

A triangular rod of length L and mass M has a nonuniform linear mass density given by the equation $\lambda = \gamma x^2$ where $\gamma = \frac{3M}{L^2}$ and x is the distance from the left end of the rod. Determine the horizontal location of the center of the mass relative to point P. Express your answer in terms of L.

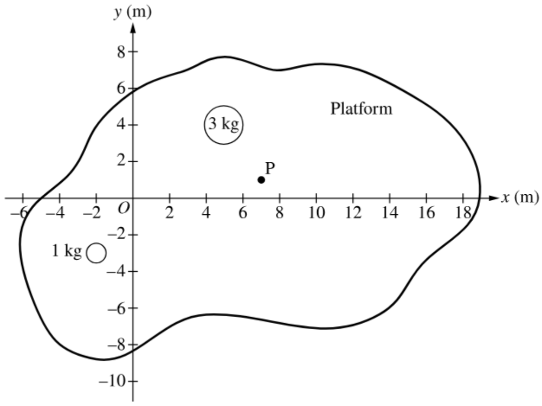
From $\lambda = \frac{\mathrm{d}m}{\mathrm{d}l}$ we know that $\mathrm{d}m = \lambda \mathrm{d}l$. Plugging this into the center of mass formula $x_{cm} = \frac{\int \mathrm{d}m}{M}$ gives us $x_{cm} = \frac{\gamma \int_0^L x^3 \mathrm{d}x}{M} = \frac{3}{4}L$.

Exercise



Six identical uniform spheres are arranged on a set of coordinate axes in two different triangular arrangements, A and B, as shown. How does the y-coordinate of the center of mass of the three spheres in arrangement A, $y_{cm,A}$ compare to the y-coordinate of the center of mass of the three spheres in arrangement B, $y_{cm,B}$?

Exercise



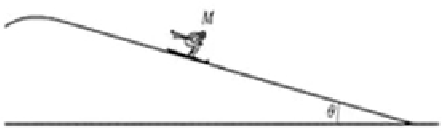
Top View

The center of mass of an irregularly shaped platform is balanced on a pivot Point P with coordinates (7.0 m, 1.0 m). Two rocks are then placed on top of the platform, as shown in the top view. One rock has a mass of 1.0 kg and is located at (-2.0 m, -3.0 m), and the second rock has a mass of 3.0 kg and is located at (5.0 m, 4.0 m). At what coordinates should a third rock of mass 4.0 kg be placed such that the three rock-platform system is balanced.

2.2 Forces and Free-Body Diagrams

- Forces are vector quantitites that describe interactions between objects or systems.
- Contact forces describe the interaction of an object or system touching another object or system.
- Free-body diagrams (FBDs) are useful tools for visualizing forces exerted on a single object or system and for determining the equations that represent a physical situation.
- The FBD of an object or system shows each of the forces exerted on the object or system by the environment.
- Forces exerted on an object or system are represented as vector originating from the center of mass, such as a dot.
- Choose a coordinate system such that one axis is parallel to the acceleration of the object or system.

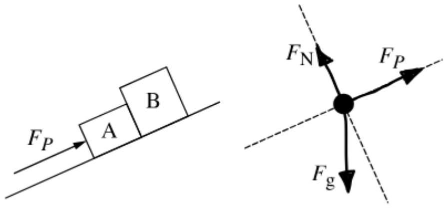
Example



A skier of mass M is skiing down a frictionless hill that makes an angle θ with the horizontal, as shown in the diagram. The skier starts from rest at time $t = 0$ and is subject to a velocity-dependent drag force due to air resistance of the form $F = -bv$, where v is the velocity of the skier and b is a positive constant. Express all algebraic answer in terms of M, b, θ, and fundamental constants. Draw a dot that represents the skier, and draw a free-body diagram indicating and labeling all of the forces that act on the skier while the skier descends the hill.

The correct answer will be $F_g = mg$ pointing downwards, a normal force an angle and the force $-bv$ perpendicular to this force.

Exercise



Two different blocks, A and B, are next to each other on an inclined smooth surface which has negligible friction. An applies force, F_P, pushes Block A as shown and the blocks move up the ramp. A student sketch of the free-body diagram representing the forces is given. What changes should be made to this free-body diagram?

Exercise A block is at rest on a desk's horizontal surface. A student correctly identifies the force exerted on the block as the force of Earth on the block and the force of the desk on the block. A book then is placed between the block and the desk. Which objects exert forces of equal magnitude on the block after the book has been introduced?

2.3 Newton's Third Law

Newton's third law describes the interaction of two objects or systems in terms of the paired forces that exerts on the other.

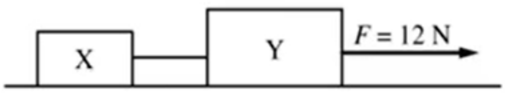
$$\vec{F}_{\text{A on B}} = -\vec{F}_{\text{B on A}}$$

Interactions between objects within a system do not influence the motion of a system's center of mass.

Tension is the macroscopic net results of forces that infinitesimal segments of a string, cable, chain or similar systme exert on each other in response to an external force.

- An ideal string has negligible mass and does not stretch when under tension.
- The tension in an ideal string is the same at all points within the string.
- In a string with nonneglibible mass, tension may not be the same at all points within the string.
- An ideal pulley that has negligible mass and rotates about an axle through its center of mass with negligible friction.

Example



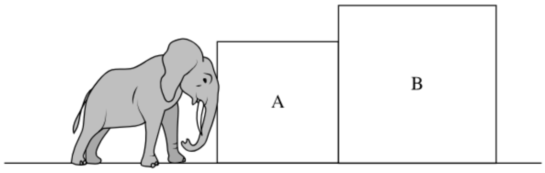
Blocks X and Y of masses 3.0 kg and 5.0 kg, respectively, are connected by a light string and are both on a level horizontal surface of negligible friction. A force $F = 12$ N is exerted on Block Y, as shown in the figure above. What is the tension in the string connecting the two blocks?

After drawing a free body diagram, we see that the $\sum F_x = ma_x$ and we can find that $a_x = 1.5$ m/s^2.

We alsk now that $F_T = ma_x$, so $F_T = 4.5$ N.

Exercise A cart moving to the right collides with a stationary block, resulting in the two objects sliding together along the horizontal surface until coming to a stop. During the collision, the cart exerts a force F_1 on the block, the surface exerts a force of friction F_2 on the block, and the block exerts a force F_3 on the cart. Which two forces are equal during the collision?

Exercise



An elephant pushes two heavy boxes across a rough surface. The force that Box A exerts on Box B is F_{AB} and the force that Box B exerts on Box A is F_{BA}. What must be true of the two boxes to support that $|F_{AB}| = |F_{BA}|$?

2.4 Newton's First Law

The net force on a system is the vector sum of all forces exerted on the system.

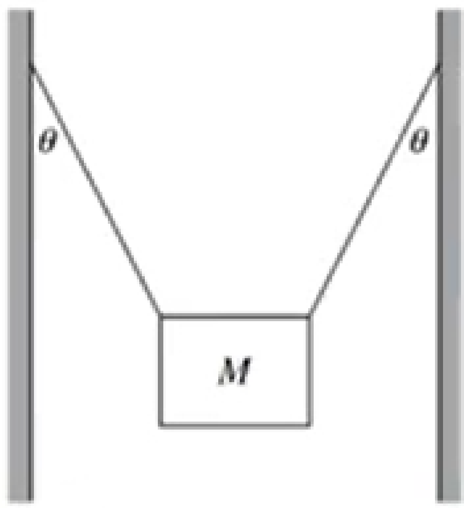
Translational equilibrium is the configuration of forces that the net force exerted on a system is zero.

$$\sum F = 0$$

Newton's first law states that if the net force exerted on a system is zero, the velocity of that system will remain constant.

Forces may be balanced in one dimension but unbalanced in another.

Example



A heavy sign of mass M is held at rest by two supporting wires between two buildings, with each wire making an angle θ with the vertical, as shown in the figure. What is the tension in each wire?

Drawing the free body diagram of the system results in the following:

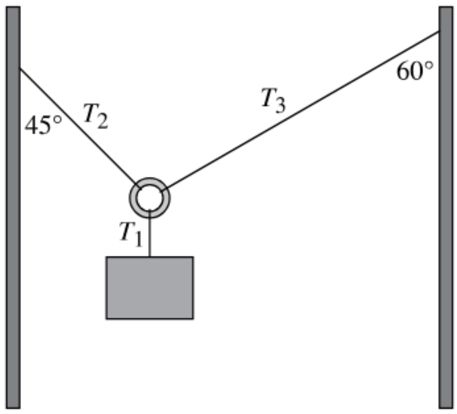
In the x-direction, we get $T = T$.

In the y-direction we get $2T\cos\theta = Mg$.

Solving this for T gives $T = \frac{Mg}{2\cos\theta}$

Exercise An object is moving while a constant force is exerted on it. Could the addition of a force of the same magnitude cause the object to move with a constant velocity? Why or why not?

Exercise



A heavy block is suspended by a string which is attached to a plastic ring. The ring is attached to two other strings which are tied to vertical supports at the angles shown. The masses of the ring and strings are negligible. Compare the magnitudes of the tensions in the strings T_1, T_2, and T_3.

2.5 Newton's Second Law

Unbalanced forces are a configuration of forces such that the net force exerted on a system is not equal to zero.

Newton's second law of motion states that the acceleration of a system's center of mass has a magnitude proportional to the magnitude of the net force exerted on the system and is in the same direction of the force.

$$\sum F = ma = 0$$

The velocity of a system's center of mass will only change if a nonzero net external force is exerted on that system/

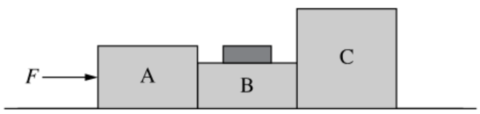
Example

An object of mass 10 kg starts from rest at time $t = 0$ and moves in a straight line. For time $t > 0$, the object's velocity as a function of time t is given by $v = 2t + 3t^2$, where v is in m/s and t is in seconds. What is the instantaneous net force that acts on the object at $t = 2$ s?

The acceleration function is given by $2 + 6t$, so $a(2) = 14$ m/s^2.

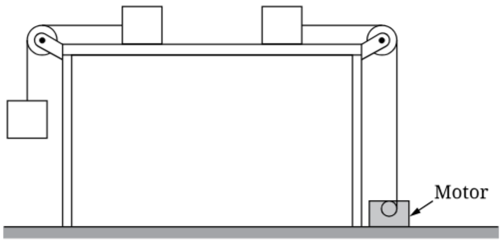
$F = ma$, so plugging in numbers gives 140 N.

Exercise



Three large blocks, A, B, and C, and a small block attached to Block B slide across a horizontal surface as a constant force F is exerted on Block A, as shown in the figure. There is negligible friction between the blocks and the horizontal surface. Block A pushes Block B with a force F_{AB}. The small block is then removed from Block B and attached to Block C and the same force F is exerted on Block A. How does F_{AB} compare in the second situation to the first situation and why?

Exercise



Two identical blocks are placed on a table as shown in the figure. The block on the left is attached to another identical block hanging over the edge of the table. The block on the right is attached to a motor pulling downward with a constant tension equal to the weight of one block. The mass of the strings and friction between the blocks and table are negligible and the pulleys are ideal. How do the magnitudes of the acceleration of the blocks compare and why?

2.6 Gravitational Force

Newton's law of universal gravitation describes the gravitational force between two objects as directly proportional to each of their masses and inversely proportional to the square of the distance between their centers.

$$F_G = \frac{Gm_1m_2}{d^2}$$

A field models the effects of a noncontact force exerted on an object at various positions in space.

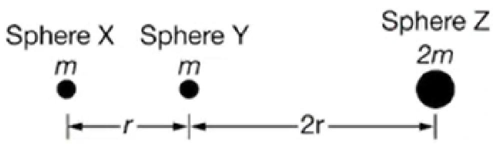
The magnitude of the gravitational field created by a system of mass M at a point in space is equal to the ratio of the gravitational force exerted by the system on a test object of mass m to the mass of the test object.

$$\vec{g} = \frac{\vec{F}_g}{m}$$

If a system is accelerating, the apparent weight of the system is not equal to the magnitude of the gravitational force exerted on the system.

Newton's shell law theorem describes the net gravitational force exerted on an object by a uniform spherical shell of mass.

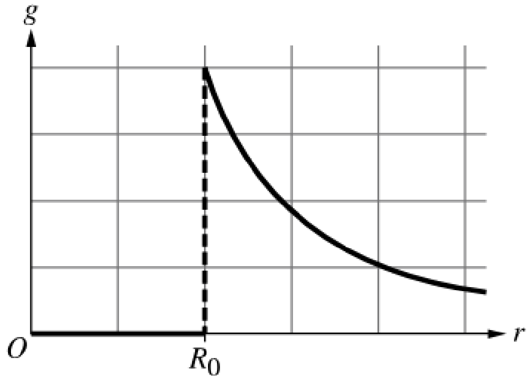
Example



Spheres X, Y, and Z have the masses and locations indicated in the figure above. What is the magnitude of the net gravitational force on sphere X due to the other two spheres?

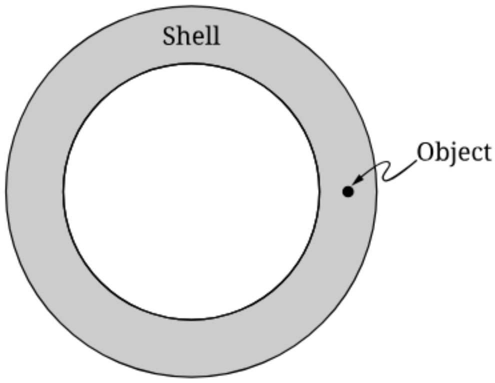
We have that $F_y = \frac{Gm^2}{r^2}$ and $F_z = \frac{1}{2}\frac{Gm^2}{r^2}$ so adding these two together gives $\frac{3}{2}\frac{Gm^2}{r^2}$.

Exercise



The gravitational field g of a spherically symmetric object of radius R_0 as a function of distance r from the object's center is shown in the graph. What best describes the object?

Exercise



A large spherical shell with a uniform mass distribution contains a small object within the thickness of the shell, as shown in the figure. At which locations could the object be moved to increase the magnitude of the gravitational force exerted on the object by the shell?

2.7 Kinetic and Static Friction

Kinetic friction occurs when two surfaces in contact move relative to each other.

- It opposes the direction of motion.
- The surface area of contact is not a factor.

The magnitude of the kinetic friction force exerted on an object is the product of the normal force the surface exerts on the object and the coefficient of kinetic friction.

$$f_k = \mu_k F_N$$

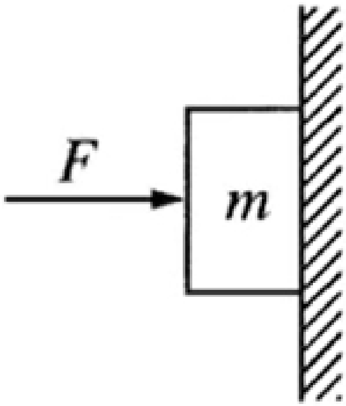
Static friction may occur between the contacting surfaces of two objects that are not moving relative to each other.

Static friction adopts the value and direction required to prevent an object from slipping or sliding on a surface.

$$f_s \leq \mu_s F_N$$

The coefficient of static friction is typically greater than the coefficient of kinetic friction for a given pair of surfaces.

Example

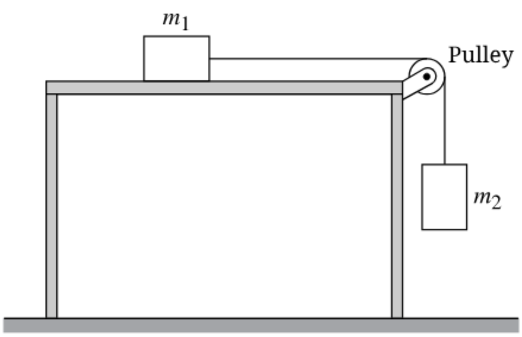


A horizontal force F pushes a block of mass m against a vertical wall. The coefficient of friction between the block and the wall is μ. What value of F is necessary to keep the block from slipping down the wall?

In the x direction the forces result in $F_N = F$.

In the y direction the forces end up with $f = F_g$ or $\mu F_N = mg = \mu F = mg$. The force is therefore $F = \frac{mg}{\mu}$.

Exercise



A block of mass m_1 rests on a rough horizontal tabletop, as shown in the figure. The block is connected by a string to a second block of mass m_2, which hangs below a pulley at the edge of the table. The coefficient of static friction between the tabletop and the first block is μ_s. The masses of the string and the pulley are negligible, and the pulley can rotate with negligible friction on its axle. What is the minimum mass m_2 that will cause the blocks to start moving?

Exercise A rectangular block is pushed by a constant force and accelerates along a rough horizontal surface. The block can be oriented to slide along any of three different sides, A, B, and C. Sides A, B, and C have surface areas S_A, S_B, and S_C, respectively where $S_A < S_B < S_C$. On which side should the block be placed to have the greatest magnitude of acceleration?

2.8 Spring Forces

An ideal spring has negligible mass and exerts a force that is proportional to the change in its length as measured from its relaxed length.

The magnitude of the force exerted by an ideal spring on an object is given by Hooke's Law:

$$F_{sp} = -k\Delta x$$

The force exerted on an object by a spring is always directed toward the equilibrium position of the object-spring system.

A collection of springs that exert forces on an object may behave as though they were a single spring with an equivalent spring constant.

- Springs in series: $\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots$
- Springs in parallel: $k_{eff} = k_1 + k_2 + \ldots$

Example

To illustrate a human soft tissue deformation, a science teacher uses two ideal springs and a small sphere. The sphere of mass m_s is attached to the free ends of the two springs. Then, the system is suspended vertically. The upper string has an equilibrium L_u and a spring constant k_u. The lower spring has an equilibrium length L_l and a spring constant k_l. The teacher fixes an additional small block of mass m_b to the free end of the lower spring. Find the expression of the system's total length.

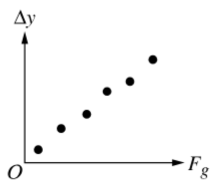
The upper string is given as $F_{sp} = k_u \Delta x_u$. Plugging in total mass and gravity we get $(m_b + m_s)g = k_u \Delta x_u$. Solving for Δx_u gives $\Delta x_u = \frac{(m_b+m_s)g}{k_u}$

The lower string is given by a similar approach and gives us $\Delta x_l = \frac{m_b g}{k_l}$.

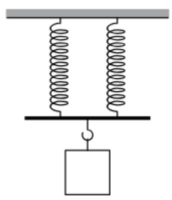
The total length is therefore $L_T = L_l + L_u + \frac{(m_b+m_s)g}{k_u} + \frac{m_b g}{k_l}$.

Exercise When a block of mass M is hung vertically from a spring, the spring is stretched by a distance D compared to its unstretched length. If a second identical spring is connected in series with the first spring and a larger block of mass $2M$ is then hung vertically from the two-spring combination, by how much is the combination stretched compared to its unstretched length?

Exercise



Data for Single Spring



Two-Spring Arrangement

Some students attach a single spring to a clamp and let the spring hang vertically. Objects of different mass are attached to the free end of the spring and allowed to hang at rest. The students measure the distance Δy the spring stretches from its equilibrium length for each object. The students produce the graph of Δy as a function of the weight F_g of the objects shown in the figure, and the slope of the best-fit line to the data is determined to be S_1. Next, the students take a second spring that is identical to the first and arrange the two springs as shown in the two-spring arrangement next to the graph. Once again, the objects of different mass are attached to the two-spring arrangement, Δy is measured, and the data is plotted on another graph showing Δy as a function of F_g. What best describes the slope of the best-fit line to the data collected for the two-spring arrangement?

2.9 Resistive Forces

A resistive force is defined as a velocity-dependent force in the opposite direction of an object's velocity.

$$F_R = -kv[F_R = -bv^2]$$

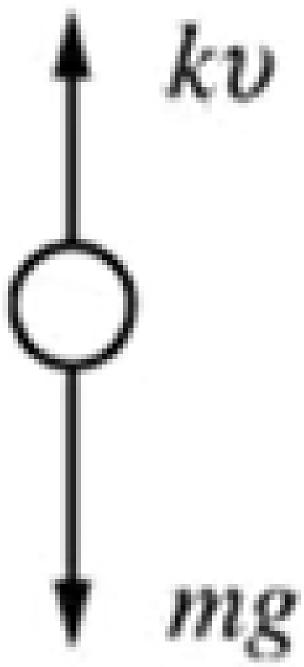
Applying Newton's second law to an object upon which a resistive force is exerted results in a differential equation for velocity.

- The differential portion of a=the equation comes from substituting in $a = \frac{\mathrm{d}v}{\mathrm{d}t}$

Terminal velocity is defined as the maximum speed achieved by an object moving under the influence of a constant force and a resistive force that are exerted on the object in opposite directions.

- For a falling object, this occurs when the air resistance equals the weight of the object.

Example



The object of mass m shown above is dropped from rest near Earth's surface and experiences a resistive force of magnitude kv, where v is the speed of the object and k is a constant. Derive an expression for the velocity of the object at any point in time. (Assume that the direction of the gravitational force is positive.)

We have that $\Delta F = ma$ so we have $mg - kv = ma$. We also have $mg - kv = m\frac{\mathrm{d}v}{\mathrm{d}t}$ as well as $mg - kv_T = 0$, so $v_T = \frac{mg}{k}$.

From $mg - kv = m\frac{\mathrm{d}v}{\mathrm{d}t}$ we can simplify this to $\int_0^t \mathrm{d}t = \int_0^{v(t)} \frac{\mathrm{d}v}{g - \frac{kv}{m}}$.

Solving this gives $t = -\frac{m}{k}\ln(1 - \frac{kv}{mg})$.

Simplifying for $v(t)$ gives $v(t) = \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right)$.

Exercise An object is released from rest and falls to the ground near Earth's surface. The resistive force exerted on the object is directly proportional to the speed of the object which results in a velocity function which includes the term $e^{-\frac{t}{\beta}}$, where β is a positive constant. What best describes the motion of the object if it falls for a time equal to β?

Exercise Two spheres, A and B, of identical size and surface material, but different masses, are dropped from rest near the surface of Earth. While falling, each sphere experiences a resistive force which is proportional to the sphere's velocity. What are the relationships of the magnitude of the initial acceleration a_0 of each sphere and of the terminal speed v_T of each sphere if $m_A < m_B$?

2.10 Circular Motion

Centripetal acceleration is the component of an object's acceleration directed toward the center of the object's circular path.

- The magnitude of the acceleration for an object moving in a circular path is the ratio of the object's tangential speed squared to the radius of the circular path.

$$a_c = v^2/r$$

Centripetal acceleration can result from a single force, more than one force, or components of forces that are exerted on an object in circular motion.

Tangential acceleration is the rate at which an object's speed changes and is directed tangent to the object's circular path.

$$a = \sqrt{a_c^2 + a_T^2}$$

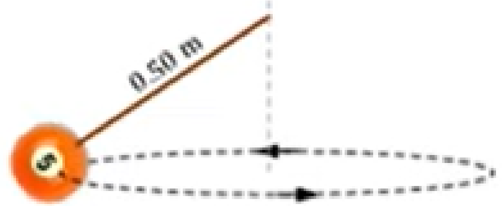
The net acceleration of an object moving in a circle is the vector sum of the centripetal acceleration and tangential acceleration.

The revolution of an object traveling in a circular path at a constant speed (UCM) can be described using period and frequency.

$$v = \frac{2\pi r}{T} = 2\pi r f \qquad T = \frac{1}{f}$$

Example

A billiard ball (mass $m = 0.150$ kg) is attached to a light string that is 0.50 meters long and swung so that it travels in a horizontal, circular path of radius 0.40 m, as shown.



a. On the diagram, draw a free-body diagram of the forces acting on the billiard ball.

There will be a force T in the direction of the string, a_c pointing right from the billiard ball and F_g pointing downwards.

b. Calculate the force of tension in the string as the ball swings in a horizontal circle.

We know that $T \sin\theta = F_g$. From this we can determine that $T = 2.5$ N.

c. Determine the magnitude of the centripetal acceleration of the ball as it travels in the horizontal circle.

We know that $T\cos\theta = ma_c$, so solving for a_c gives us 13.3 m/s^2.

d. Calculate the period T (time for one revolution) of the ball's motion.

We know that $a_c = \frac{v^2}{r}$ so we can find that that $v =$ 2.30 m/s. We also know that $v = \frac{2\pi r}{T}$, so solving for T gives 1.15.

Exercise An object of mass m is attached to the end of a spring. The string is spun around in a vertical circle of radius r. When the object is at the top of its path, the speed of the object is v and the string has a tension F_T. Write an expression for v at the top of the circular path.

Exercise Two small blocks, P and Q rotate without slipping on a horizontal disk with Block P being twice as far from the rotational axis of the disk as Block Q. The blocks are made of the same material and Block P is half the mass of Block Q. As the disk increases in speed, which block will be the first to begin to slide on the disk's surface?

3 Work, Energy and Power

3.1 Translational Kinetic Energy

An object's translational kinetic energy is given by the equation

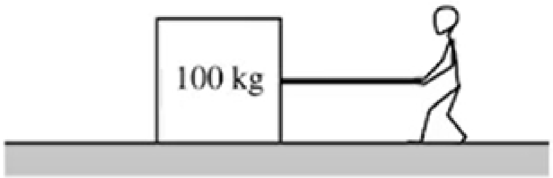
$$K = \frac{1}{2}mv^2$$

Translational kinetic energy is a scalar quantity.

Different observers may measure different values of the translational kinetic energy of an object, depending on the observer's frame of reference.

Example

A 100 kg box shown is being pulled along the x-axis by a student. The box slides across a rough surface, and its position x varies with time according to the equation $x = 0.5t^3 + 2t$, where x is in meters and t is in seconds.



(a) Determine the speed of the box at time $t = 0$. The derivative of the position function is $1.5t^2 + 2$, so $v(0) = 2$ m/s

(b) Determine the kinetic energy of the box as a function of time.

We know that $K = \frac{1}{2}mv^2$. Plugging in m and v, we get $K(t) = 50(1.5t^2 + 2)^2$.

Exercise Two identical blocks, Block A and Block B, slide across a horizontal surface. Block A has a speed v, and kinetic energy K_A. Block B has a speed $2v$ and a kinetic energy K_B. What is the ratio $K_A : K_B$ with a correct justification?

Exercise

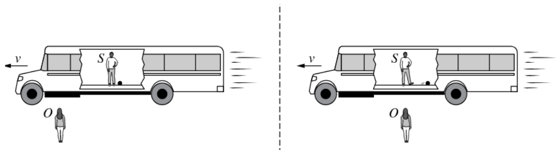


Figure 1 Figure 2

A student stands on a bus moving with a constant speed v to the left as shown in Figure 1. The student S is at rest relative to the bus and a ball sits at rest on the floor of the bus next to the student. Outside the bus standing at rest relative to the ground is an observer O. The kinetic energies of the ball as measured by the student and the observer at this moment are K_{S1} and K_{O1} respectively. As the bus passes the observer, the student kicks the ball toward the back of the bus with a constant speed less than v as shown in Figure 2. The kinetic energies of the ball as measured by the student and the observer after the ball is kicked are K_{S2} and K_{O2}, respectively. How do the kinetic energies measured by the student and observer compare before and after the ball is kicked?

3.2 Work

Work is the amount of energy transferred into or out of a system by a force exerted on that system over a distance.

- The work done by a conservative force is path-independent and only depends on the initial and final configurations of that system.
- The work done by a nonconservative force is path-dependent.

Work is scalar quantity that may be positive, negative or zero.

The work done on an object by a variable force is calculated as

$$W = \int_a^b \vec{F}(r) \cdot \mathrm{d}\vec{r} \qquad [W = Fd\cos\theta]$$

The work-energy theorem states that the change in an object's kinetic energy is equal to the sum of the work being done by all forces exerted on the object.

$$\delta K = W$$

Work is equal to the area under the curve of a graph of F as a function of displacement.

Example

A skier of mass m will be pulled up by a hill by a rope, as shown. The magnitude of the acceleration as a function of time t can be modeled by

- $a = a_{max} \sin \frac{\pi t}{T} (0 < t < T)$
- $a = 0 (t \geq T)$

Where a_{max} and T are constants. The hill is inclined at an angle θ above the horizontal, and friction between the skis and the snow is negligible. Express your answers in terms of given quantities and fundamental constants.

(a) Derive an expression for the velocity of the skier as a function of time during the acceleration. Assume the skier starts from rest.

We have $v = \int a(t)\mathrm{d}t = \int_0^t a_{max} \sin \frac{\pi t}{T} \mathrm{d}t$,

Integrating this and applying the limits of integration give $-\frac{a_{max}T}{\pi}\left(1 - \cos\frac{\pi t}{T}\right)$ for v.

(b) Derive an expression for work done by the net force from the skier from rest until terminal speed is reached.

We know $W = \Delta k$ and that $\Delta k = \frac{1}{2}mv^2$ in this case.

Plugging in the v just derived gives $W = -2a_{max}m\frac{T^2}{\pi^2}$.

Exercise A block of mass m slides with an initial velocity v_0 along a rough surface where the coefficient of kinetic friction between the block and the surface is μ. The box comes to rest after sliding a distance d_0. A new block of unknown mass slides with an initial velocity of $2v_0$ across a surface where the coefficient of kinetic friction between the new block and the surface is $\frac{\mu}{2}$. Write an expression that represents the distance the new block slides before coming to rest in terms of d_0.

Exercise A force F is exerted on an obejct which is initially at rest. The force varies with positions x and can be described by the equation $\vec{F} = (Ax - B)\hat{i}$, where A and B are constants with appropriate units. After moving a distance D_0, the block again comes to rest. An identical object, also initially at rest, experiences a force $2F$. The second object comes to rest again after moving a distance D_1. Describe the relationship between D_0 and D_1.

3.3 Potential Energy

A system composed of two or more objects has potential energy if the objects within that system only interact with each other through conservative forces.

Potential energy is a scalar quantity associated with the position of objects within a system.

The definition of zero potential energy for a given system is a decision made by the observer considering the situation to simplify or otherwise assist in analysis.

The relationship between conservative forces exerted on a system and the system's potential energy is

$$\delta U = -\int \vec{F}(r) \cdot \mathrm{d}\vec{r}$$

The conservative forces exerted on a single dimension can be determined using the slope of a system's potential energy with respect to position in that dimension, these forces point in the direction of decreasing potential energy.

$$F_x = -\mathrm{d}u(x)/\mathrm{d}x$$

The potential energy of common physical systems can be described using the physical properties of that system.

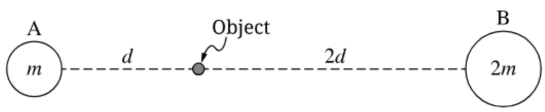
Example

When a certain spring is stretched by an amount x, it produces a restoring force of $F(x) = -ax + bx^2$, where a and b are constants. How much work is done by an external force in stretching the spring by an amount D from its equilibrium length?

Integrate this function with bounds 0 to D to get $\frac{bD^3}{3} - \frac{aD^2}{2}$.

Exercise The force exerted by a non-linear spring is given by $F(x) = -kx^{\left(\frac{4}{3}\right)}$. Write an expression that correctly models the potential energy stored in the spring when it is compressed a distance of D.

Exercise



Note: Figure not drawn to scale.

An object in space is placed between two planets as shown. Planet A has a mass m and the distance from the center of Planet A to the object is d. Planet B has a mass $2m$ and the distance from the center of

Planet B to the object is $2d$. Upon releasing the object from rest, where will the object move and how will the potential energy of the two-planet object system change?

3.4 Conservation of Energy

A system that contains objects that interact via conservative forces or that can change its shape reversibly may have both kinetic and potential energies.

Mechanical energy is the sum of a system's kinetic and potential energy.

A system may be selected so that the total energy of the system is constant.

If the total energy of a system changes, that change will be equivalent to the energy transferred into or out of the system.

Energy is conserved in all interactions.

If the work done on a selected system is zero and there are no nonconservative interactions within the system, the total mechanical energy of the system is constant.

If the work done on a selected system is nonzero, the energy is transferred between the system and the environment.

Example

A horizontal spring with spring constant k is compressed by x and then used to launch a m box across the floor. The coefficient of kinetic friction between the box and the floor is μ_k. Derive an expression for the box's launch speed v.

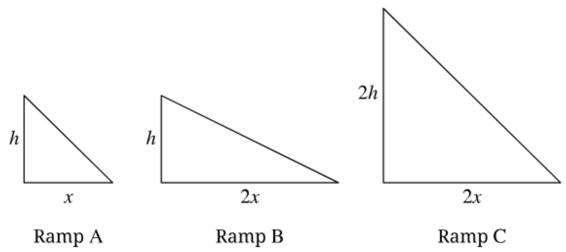
We can determine the conservatino equation to be $-\frac{1}{2}kx^2 = \frac{1}{2}mv^2 + W_f$.

$W_f = \mu mgx$, so $kx^2 = mv^2 + 2\mu mgx$.

Solving this equation for v gives $v = \sqrt{\frac{kx^2}{m} - 2\mu gx}$.

Exercise In an experiment, an object of mass m slides along a horizontal surface where friction between the object and the surface is negligible. The object has an initial speed v and then collides with a spring of spring constant k, and the object compresses the spring a total distance x from the equilibrium length of the spring before coming to rest. Students observing the experiment expected the object to compress the spring a distance D from equilibrium, where $D > x$. Assuming the students' expectation is correct, write an expression that can shown the change in the mechanical energy of the object-spring system during the experiment.

Exercise



Three ramps, A, B, and C, have the dimensions shown in the figure. Identical blocks are released from rest at the top of each of the three ramps and slide to the bottom. The coefficient of kinetic friction between each of the blocks and the ramp is the same. Rank the speeds of the blocks at the bottom of the ramps.

3.5 Power

Power is the rate at which energy changes with respect to time, either by transfer into or out of a system or by converstion from one type to another within the system.

Average power is the amount of energy being transferred or converted, divided by the time it took for that transfer to happen.

The instantaneous power delivered to an object by a force is given by the equation:

$$p_{ins} = \frac{\mathrm{d}E}{\mathrm{d}t}$$

The instantaneous power delivered to an object by the component of a constant force parallel to the object's velocity can be described with the derived equation:

$$p_{ins} = Fv\cos\theta$$

Example

A factory uses a motor and a cable to drag a 300 kg machine to the proper place on the factory floor. What power must the motor supply to drag the machine at a speed of 0.50 m/s? The coefficient of friction between the machine and the floor is 0.60.

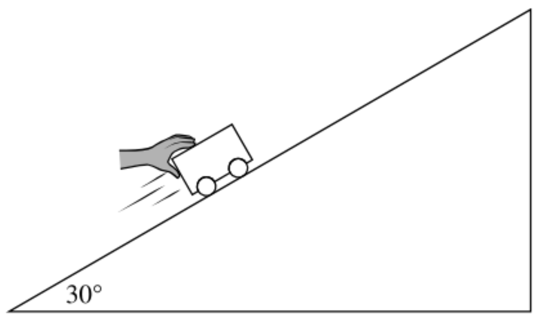
Let $F_{NET} = 0$, so $F = \mu mg$.

This gives us $F = 1764$N.

The power formula is $P_{ins} = F \| v = 882$W.

Exercise A box is pushed across a surface where friction between the box and the surface is negligible. There is a resistive force from the air exerted on the box equal to $F_R = -kv$. Draw a graph that correctly models the relationship between the power P required to move the box at a constant speed v to the speed of the box.

Exercise



A 0.250 kg cart is being pushed up a track that is inclined at $30°$ above the horizontal as shown. Friction between the cart and the track is negligible. Calculate the minimum power required to push the cart up the track at a constant speed of 2.4 m/s.

4 Linear Momentum

4.1 Linear Momentum

Linear momentum is defined by the equation:

$$\vec{p} = m\vec{v}$$

Momentum is a vector quantity and has the same direction as the velocity.

Momentum can be used to analyze collisions and explosions.

- A collision is a model for an interaction where the forces exerted between the involved objects in the system are much larger than the net external force exerted on those objects during the interaction.
- An explosion is a model for an interaction in which forces internal to the system move objects within that system apart.

Example

(a) Derive an expression for the kinetic energy of an object in terms of momentum.

Substitute $p = mv$ into $K = \frac{1}{2}mv^2$ to get $K = \frac{p^2}{2m}$.

(b) Derive an expression for the momentum of an object in terms of kinetic energy.

We get that $v = \sqrt{\frac{2k}{m}}$. Substituting this into $p = mv$ gives $p = \sqrt{2mk}$.

Exercise A moving block of mass m_0 has kinetic energy K_0 and momentum of magnitude p_0. A second block of mass $2m_0$ is moving with the same kinetic energy K_0. What is the magnitude of the momentum of the second blocK?

Exercise A student is pulling a cart full of water horizontally on flat ground at a constant velocity. The cart has a small hole through which the water slowly leaks out of the cart. For the system consisting of the cart and the water within the cart, how is the magnitude of the momentum of the system changing, if at all?

4.2 Change in Momentum and Impulse

The rate of change of a system's momentum is equal to the net external force exerted on that system.

$$F_{NET} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$$

Impulse is defined as the integral of a force exerted on an object or system over a time interval.

$$J = \int F(t)\mathrm{d}t$$

Impulse is a vector quantity and has the same direction as the net force exerted on the system.

Change in momentum is the difference between a system's final momentum and its initial momentum.

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i$$

The impulse-momentum theorem relates impulse delivered to an object and the object's change in momentum.

$$\vec{J} = \Delta\vec{p} \qquad \int F(t)\mathrm{d}t = m\Delta\vec{v}$$

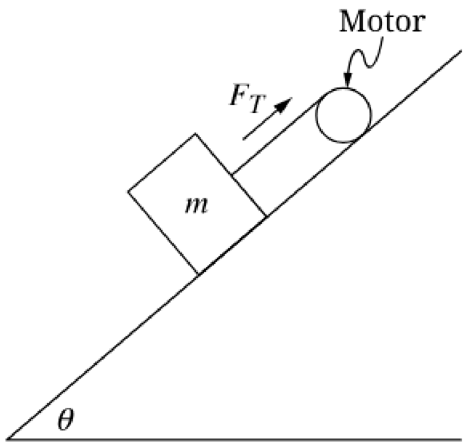
Example

The net force F acting on an object that moves along a straight line is given as a function of time t by $F(t) = \kappa t^2 + \tau$, where $\kappa = 2\text{N/s}^2$ and $\tau = 4\text{N}$. What is the change in momentum of the object from $t = 0$ s to $t = 3$ s?

Integrate $F(t)\mathrm{d}t$ from 0 to 3 to get $\Delta\vec{p} = 59$ N·m.

Exercise A 50-kg student sitting in a rolling chair at rest pushes against a wall, which applies a 10 N·s horizontal impulse to the student. Later, a 40-kg student is at rest in the same rolling chair and catches a 10-kg ball while applying a 10 N·s horizontal impulse to the ball. Describe the students' final speeds and provide a valid reasoning to support this claim.

Exercise



A block of mass m can move with negligible friction on a ramp that is at an angle θ from the horizontal, as shown in the figure. Starting at time $t = 0$, the motor creates a varying string tension F_T given by $F_T = 2kt$, where k is a positive constant. If the positive direction is taken to be up the ramp, what is the magnitude of the impulse exerted on the block between $t_1 > 0$ and a later time t_2?

4.3 Conservation of Linear Momentum

A collection of objects with individual momenta can be described as one system with one center of mass velocity.

- The velocity of a system's center of mass can be calculated using the equation

$$v_{cm} = \frac{\sum m_i v_i}{\sum m}$$

The total momentum of a system is the sum of the momenta of the system's constituent parts.

In the absence of net external forces, any change to the momentum of an object within a system must be balanced by an equivalent and opposite change of momentum elsewhere within the system.

Momentum is conserved in all interactions.

If the net external force on the selected system is zero, the total momentum of the system is constant.

If the net external force on the selected system is nonzero, momentum is not transferred between the system and the environment.

Example

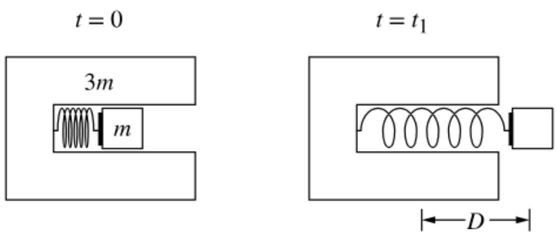
An object of mass m is moving with speed v_0 to the right on a horizontal frictionless surface, when it explodes into two points. Subsequently, one piece of mass $(2/5)m$ moves with a speed $v_0/2$ to the left. What is the speed of the other piece of the object?

We know momentum before has to equal the momentum after.

So we get $mv_0 = \frac{3}{5}mv_f - \frac{2}{5}m\frac{v_0}{2}$. Solving for v_f gives $v_f = 2v_0$.

Exercise In three different scenarios, a student of mass $3m$ pulls on a lightweight rope attached to a block of mass m. The student and the block are both initially at rest in each scenario. In Scenario 1, the student stands on the shore of a frozen lake and the block is on the ice. After the student starts pulling on the rope in Scenario 1, the block slides with negligible friction while the student remains in the same location on the shore. In Scenario 2, both the student and the block are on the ice. After the student starts pulling the rope in Scenario 2, both the student and the block slide with negligible friction. In Scenario 3, the block is at rest on the shore while the student is on the ice. After the student starts pulling on the rope in Scenario 3, the student slides with negligible friction while the block remains in the same location on the shore. The final velocity of the block relative to the student is the same in each scenario. Which scenario, if any, is the final speed of the center of mass of the block-student system the greatest?

Exercise



Note: Figure not drawn to scale.

A small block of mass m is held in place inside a larger, U-shaped block of mass $3m$, as shown in the figure. Initially, the centers of mass of the two blocks are located at the same position, and a spring attached to the larger block is compressed a distance D from its relaxed length by the smaller block. At time $t = 0$, the two blocks are released from rest. At time $t_1 > 0$, the spring is at its relaxed length, the centers of mass of the two blocks are a distance D apart, and the small block is moving to the right with a velocity v just as it loses contact with the spring. If there are no external forces exerted on the block-spring system, what is the distance of separation of the two centers of mass at a later time $t > t_1$?

4.4 Elastic and Inelastic Collisions

An elastic collision between objects is one in which the initial kinetic energy of the system is equal to the final kinetic energy of the system.

In an elastic collision, the final kinetic energies of each of the objects within the system may be different from their initial kinetic energies.

An inelastic collision between objects is one in which the total kinetic energy of the system decreases.

In an inelastic collision, some of the initial kinetic energy is not restored to kinetic energy but it is transformed by nonconservative forces into other forms of energy.

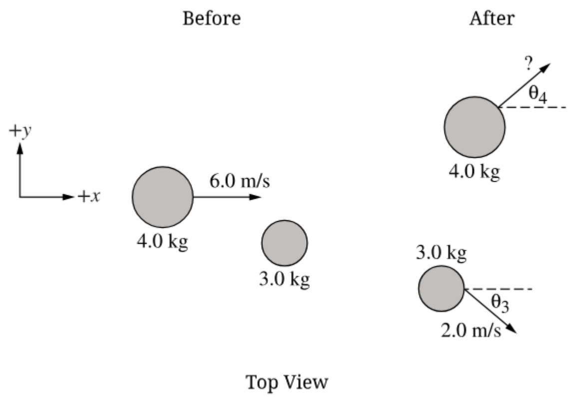
In a perfectly inelastic collision, the objects stick together and move the same velocity after the collision.

Example

A child of mass 20 kg who is running at a speed of 4.0 m/s jumps onto a stationary sled of mass 5.0 kg on a frozen lake. What is the speed at which the child and sled begin to slide across the ice?

From inelastic collision formula, we get $m_1 v_{1B} = (m_1 + m_2) v_f$. So, we get $v_f = \frac{m_1 v_{1B}}{m_1 + m_2} = \frac{80}{25} = 3.2$ m/s.

Exercise



Two pucks are free to slide on a horizontal surface with negligible friction. The figure shows a top view of the pucks, one of mass 4.0 kg and one of mass 3.0 kg, before and after they undergo an elastic collision with each other. Before the collision, the 4.0-kg puck slides at 6.0 m/s in the $+x$-direction and the 3.0-kg puck is at rest. After the collision, the 3.0-kg puck moves with a speed of 2.0 m/s at an unknown angle θ_3 measured clockwise from the $+x$-direction, as indicated, while the 4.0-kg puck moves at an unknown speed and at an unknown angle θ_4 measured counterclockwise from the $+x$-direction, as indicated. What is the speed of the 4.0-kg puck after the elastic collision.

Exercise Students perform an experiment with mutiple trials using blocks that can slide with negligible friction on a straight, horizontal track. In each trial, a block of mass m_1 slides with speed v_1 on the track and collides with a stationary block of mass m_2. The blocks stick together and move with speed v_f after the collision. In each trial, m_1 and v_1 are kept constant but the mass m_2 of the stationary block is varied, and v_f is recorded. The students graph v_f as a function of m_2. Graph a best-fit curve to the data collected by the students.

5 Torque and Rotational Dynamics

5.1 Rotational Kinematics

Angular displacement is the measurement of the angle, in radians, through which a point on a rigid system rotates about a specific axis.

- In general, the counterclockwise motion is positive, and the clockwise motion is negative.

Angular velocity is the rate at which angular displacement position changes with respect to time.

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

Angular acceleration is the rate at which angular velocity changes with respect to time.

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t}$$

Angular displacement, angular velocity, and angular acceleration around one axis are analogous to linear displacement, velocity, and acceleration in one dimension and demonstrate the same mathematical relationships.

Graphs of angular displacement, angular velocity, and angular acceleration as functions of time can be used to find the relationships between the above quantitites.

Example

A windmill is spinning because of the nonuniform force of the wind. The windmill is originally spinning at a speed of ω_0, and a crosswind slows it with an angular acceleration of $-A\omega^2$. What will be the angular speed of the windmill at time $t = T$?

From $\alpha = -A\omega^2$, we get that $\frac{1}{\omega^2} = -A$.

Integrating this from ω_0 to ω gives us $\omega = \frac{\omega_0}{\omega_0 AT+1}$.

Exercise Two disks, Disk 1 and Disk 2, are initially at rest and begin to rotate with a constant angular acceleration. Disk 1 has an angular acceleration α_1 and rotates through an angle θ_1 in a time Δt. Disk 2 has an angular acceleration $\alpha_2 = 2a_1$ and rotates through an angle θ_2 in the same amount of time Δt. What is θ_2 in terms of θ_1?

Exercise A wheel spins with an initial angular velocity of 18 rad/s in the clockwise direction and a constant angular acceleration. After 3 seconds the wheel is spinning at 6 rad/s in the counterclockwise direction. What is the magnitude and direction of the angular acceleration?

5.2 Connecting Linear and Rotational Motion

For a point at a distance r from a fixed axis of motion, the linear distance s traveled by the point as the system rotates through an angle $\Delta\theta$ is given by the equation

$$s = r\Delta\theta$$

Derived relationships of linear velocity and of the tangential component of acceleration to their respective angular quantities are given by the following equations

$$x = r\theta$$

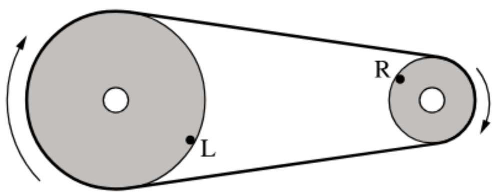
$$v = r\omega$$

$$a = r\alpha$$

For a rigid system, all points within that system have the same angular velocity and angular acceleration.

Exercise A car with tires of radius r_0 is moving along a straight road. Each tire's angular displacement is given by the equation $\theta = At^4 + Bt$, where t is time and A and B are constants with appropriate units. The tires do not slip on the road's surface. What is the car's acceleration as a function of time?

Exercise



Two disks each rotate about axes through their centers, and are connected by a belt that does not slip as the disks rotate, as shown in the figure. The disk on the left has a larger radius than the disk on the right. Points L and R are points at the edges of the disks on the left and right, respectively. Are the angular speeds and linear speeds of points L and R the same or different?

5.3 Torque

Torque results only from the force component perpendicular to the position to the position vector from the axis of rotation to the point of the application of the force.

The lever arm is perpendicular distance from the axis of rotation to the line of action of the exerted forces.

Torques can be described using force diagrams.

- These are sometimes referred to as a rigid body diagrams.
- The forces are placed on the object at the point of application in relation to the axis of rotation.

The torque exerted on a rigid system about a chosen pivot point by a given force is described by

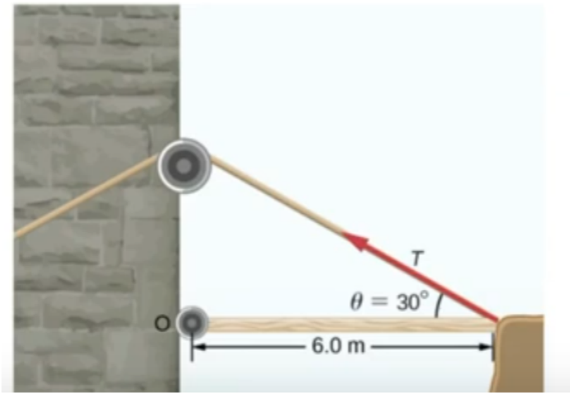
$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$||\tau|| = rF \sin\theta$$

The direction of the torque is determined by using the right hand curl rule.

Example

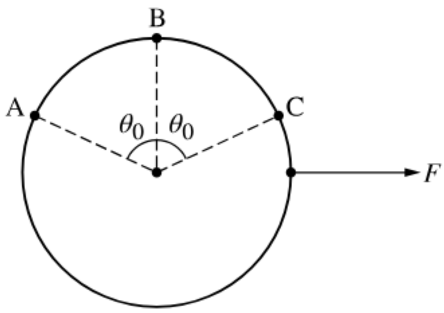
A torque of 5.00×10^3 N·m is required to raise a drawbridge (see the following figure). What is the tension necessary to produce this toruqe? Would it be easier to raise the drawbridge if the angle θ were larger or smaller?



Just plug into the formula. $\tau = rF\sin\theta \implies T = \frac{\tau}{r\sin\theta} = 1700\text{N}$.

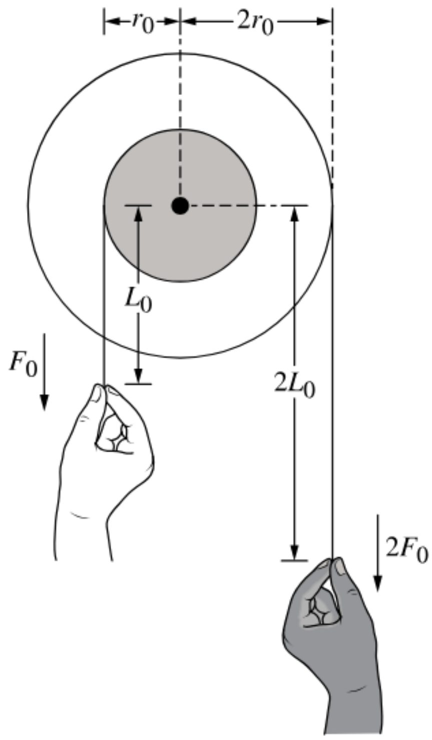
If you increase θ, the tension would be smaller.

Exercise



A force F, directed to the right, is exerted on a disk, as shown in the figure. The disk may pivot in the plane of the page about any one of the three labeled points, and the force results in lever arms of length L_A, L_B, and L_C for pivots at points A, B, and C, respectively. Rank L_A, L_B, and L_C.

Exercise



Two pulley wheels are rigidly connected so that they can rotate together around a common center, as shown in the figure. The wheels have radii r_0 and $2r_0$, and strings wrapped around each wheel are pulled by two students. One student pulls downward on the smaller wheel's string with a force of magnitude F_0 exerted a vetical distance L_0 below the wheels' center. The other student pulls downward on the larger wheel's string with a force of magnitude $2F_0$ exerted a distance $2L_0$ below the wheels' center. The students make the correct claim that the force $2F_0$ results in a greater torque with respect to the wheels' center. Provide the evidence needed to justify this claim.

5.4 Rotational Inertia

Rotational inertia measures a rigid system's resistance to changes in rotation and is related to the mass of the system and the distribution of the mass relative to the axis of rotation.

The rotational inertia of an object rotating a perpendicular distance r from an axis is described by the equation

$$I = mr^2$$

The total rotational inertia of a collection of objects about an axis is the sum of the rotational inertias of each about that that axis.

$$I = \sum m_i r_i^2$$

For a solid that can be considered as a collection of differential masses, $\mathrm{d}m$, the solid's rotational inertia can be calculated using the equation

$$I = \int r^2 \mathrm{d}m$$

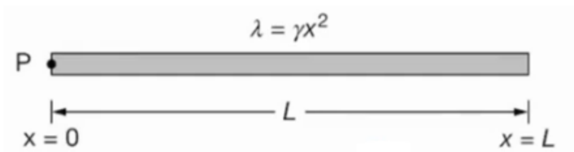
A rigid system's rotational inertia in a given plane is at minimum when the rotational axis passes through the system's center of mass.

The parallel axis theorem uses the following equation to relate the rotational inertia of a rigid system about any axis that is parallel to an axis through its center of mass.

$$I' = I_{cm} + md^2$$

Example

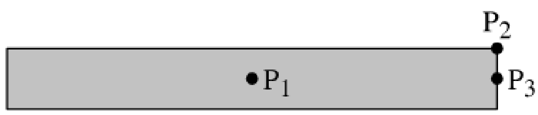
A triangular rod of length L and mass M has a nonuniform linear mass density given by the equation $\lambda = \gamma x^2$, where $\gamma = \frac{3M}{L^3}$ and x is the distance from point P at the left end of the rod. Using integral calculus, show that the rotational inertia I of the rod about an axis perpendicular to the page and through point P is $\frac{3}{5}ML^2$.



First integrate $\int_0^L x^2 \mathrm{d}m$. We can substitute $\mathrm{d}m = \gamma x^2 \mathrm{d}x$, and simplifying, we get $I = \gamma \int_0^L x^4 \mathrm{d}x$.

Integrating this will give you $I = \frac{3}{5}ML^2$

Exercise



The figure shows a narrow, uniform rectangular plate. The place can rotate about any of three axes that are each perpendicular to the plane of the figure and pass through one of the points P_1, P_2, or P_3. Correctly compare the rotational inertias I_1, I_2, and I_3 of the plate about points P_1, P_2, and P_3 respectively.

Exercise

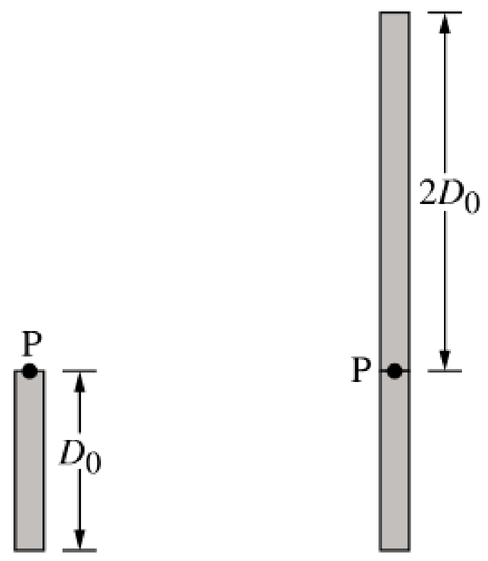


Figure 1 Figure 2

Figure 1 shows a uniform rod of mass M_0 and length D_0. Its rotational inertia when rotated about its end at point P is given by $I_0 = \frac{1}{3}M_0D_0^2$. A second uniform rod with twice the length and the same width is made of the same material, and the two rods are connected at their ends at point P, as shown in Figure 2. What is the rotational inertia about Point P for the two-rod system in Figure 2?

5.5 Newton's First and Second Law in Rotational Form

A system may exhibit rotational equilibrium (constant angular velocity) without being in translational equilibrium, and vice versa.

- FBD and force diagrams describe the nature of the forces and torques exerted on an object or rigid system.
- Rotational equilibrium is a configuration of torques such that the net torque exerted on the system is zero.
- The rotational analog of Newton's 1st law is that a system will have a constant angular velocity only if the net torque exerted on the system is zero.

A rotational collolary to Newton's 2nd law states that if the torques exerted on a rigid system are not balanced, the system's angular velocity must be changing.

Angular velocity changes when the net torque exerted on the object or system is not equal to zero.

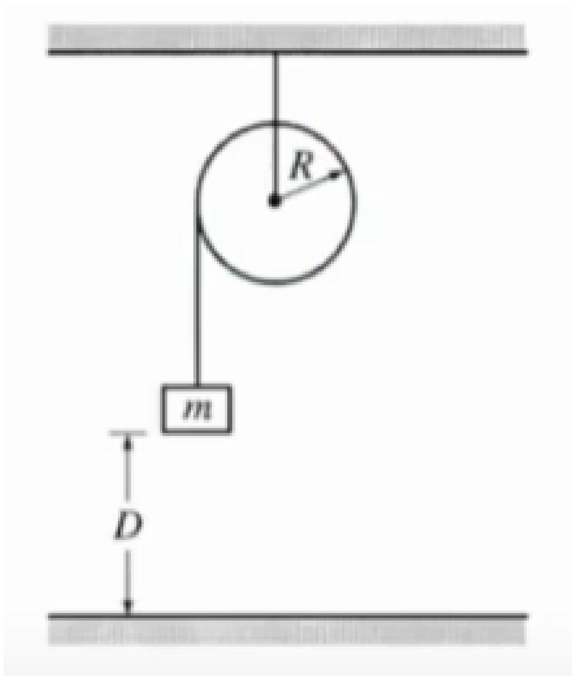
The rate at which angular velocity of a rigid system changes is directly proportional to the net torque exerted on the rigid system as in the same direction.

$$\sum \tau = I\alpha$$

To fully a describe a rotating rigid system, linear and rotational analyses may need to be performed independently.

Example

A solid disk of unknown mass and known radius R is used as a pulley in a lab experiment, as shown. A small block of mass m is attached to a string, the other end of which is attached to the pulley and wrapped around it several times. The block of mass m is released from rest and takes a time t to fall the distance D to the floor. Calculate the linear acceleration a of the falling block in terms of the given quantitites.



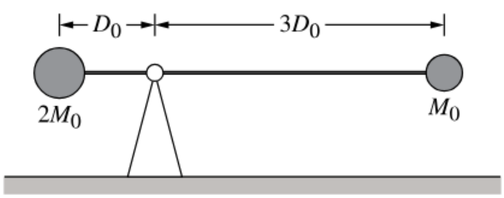
We first must find T, the tension.

To find this, we can determine that $T = \frac{1}{2}ma$, from plugging in known quantities and the fact that $I = \frac{1}{2}mr^2$.

Now the sum of the forces is $\sum F = ma$, so $T = mg - ma$.

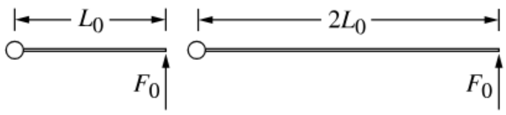
Solving for a after plugging in everything should give $\frac{2}{3}g$.

Exercise



A thin rod of negligible mass has spheres of different masses attached to its ends, as shown in the figure. The rod is free to rotate about an axis perpendicular to the plane of the figure and located at a triangular fulcrum that supports the rod. If the rod is held in place horizontally and then released from rest, what is the magnitude of the angular acceleration of the rod-spheres system immediately after the rod is released?

Exercise



Top View

A uniform rod of length L_0 is free to rotate about an axis through its left end and perpendicular to the horizontal plane of the figure shown. When a force with magnitude F_0 is exerted perpendicular to the rod at the right end, the angular acceleration of the rod is α_1. The same force is then exerted on the right end of a second uniform rod with the same linear mass density and twice the length as the original rod, as shown on the right side of the figure. The rod of length $2L_0$ has an angular acceleration of α_2. What is the ratio $\alpha_2 : \alpha_1$?

6 Energy and Momentum of Rotating Systems

6.1 Rotational Kinetic Energy

The rotational kinetic energy of an object or rigid system is related to the rotational intertia and angular velocity of the rigid system and is given by the equation

$$K_R = \frac{1}{2}I\omega^2$$

The total kinetic energy of a rigid system is the sum of its center of mass and the translational kinetic energy due to the linear motion of the center of mass.

$$K_{TOT} = K_T + K_R = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

A rigid system can have rotational kinetic energy while its center of mass is at rest due to the individual points within the rigid system having linear speed, and therefore, kinetic energy.

Example

A wheel with rotational inertia I is mounted on a fixed, frictionless axle. The angular speed ω of the wheel is increased from zero to ω_f in a time interval T.

What is the average power input to the wheel during this time interval?

Use power formula: $P = \frac{\Delta k}{t} = \frac{1/2I\omega_f^2 - \frac{1}{2}I\omega_0^2}{t}$.

The power is therefore $P = \frac{I\omega_f^2}{2t}$.

Exercise Three identical uniform disks, A, B, and C, can each slide along a horizontal surface with negligible friction. Disk A is rotating with angular speed ω_A while its center of mass remains in place. Disk B is moving with speed v_B without rotating. Disk C is rotating with the same angular speed ω_A as Disk A while its center of mass is moving with the same speed v_B as Disk B. Which disk has the greatest total kinetic energy?

Exercise A solid, uniform disk is spinning about an axis through its center while its center of mass remains at rest. Describe the disk's kinetic energy K with supporting reasoning.

6.2 Torque and Work

A torque can transfer energy into and out of an object or rigid system if the torque is exerted over an angular displacement.

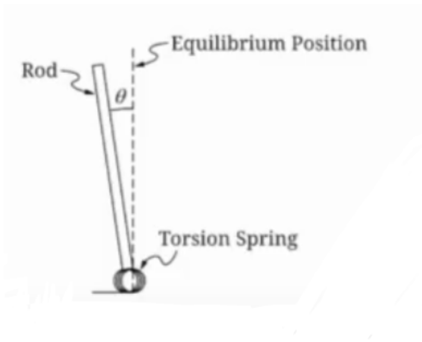
$$\tau = rF$$

The amount of work done on a rigid system by a torque is related to the magnitude of that torque and the angular displacement through which the rigid system rotates during the interval in which that torque is exerted.

$$W = \int_{\theta_0}^{\theta} \tau \mathrm{d}\theta$$

Work done on a rigid system by a given torque can be found from the area under the curve of a graph of the torque as a function of angular position.

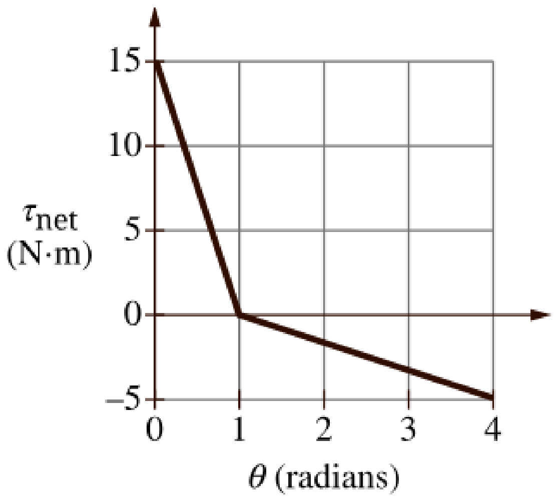
Example



A torsion spring is fixed to the end of rod of rotational inertia I_R. The torsion spring is fixed to a horizontal table with negligible friction, as shown in the image. WHen the rod is displaced an angle θ from equilibrium the torsion spring exerts a restoring torque of magnitude $\tau = 2\kappa\theta^2$, κ is a positive constant with appropriate units. After being displaced by an angle θ_0, the rod is released and rotates through its equilibruim position with angular speed ω_0. How much work was done in moving the rod to this position?

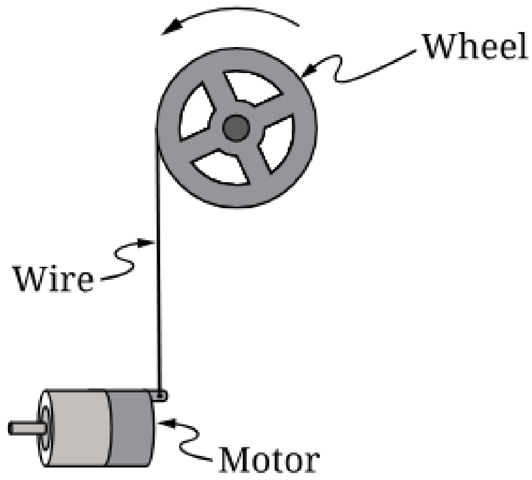
Integrate $2\kappa\theta^2$ from 0 to θ_0 to get $W = \frac{2\kappa}{3}\theta_0^3$.

Exercise



An object can rotate about an axis passing through its center of mass. At $t = 0$, the object is spinning in the positive direction and has an initial angular position of zero. The graph represents the net torque τ_{net} exerted on the object as a function of angular position θ. What is the total work done on the object and the maximum rotational kinetic energy of the object as it rotates from 0 radians to 4 radians?

Exercise



A wheel can rotate about an axis that passes through its center and is perpendicular to the page. THe edge of the wheel is attached to one end of a wire that is connected to a motor, which is fixed in place. The motor exerts a force on the wire such that the wire exerts a counterclockwise torque of magnitude τ_0 about the center of the wheel when the wheel is at an angular position $\theta_0 = 0$. As the wheel rotates, the torque exerted on the wheel changes as a function of θ that is given by $\tau = \tau_0 e^{-\frac{\theta}{k}}$, where k is a positive constant. If the positive direction for θ is counterclockwise, how much work is done by the wire on the wheel as the wheel rotates from θ_0 to an angular position θ_f?

6.3 Angular Momentum and Angular Impulse

The magnitude of the angular momentum of a rigid system about a specific axis can be described with the equation:

$$L = I\omega$$

The angular momentum of an object about a given point is

$$L = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \implies L = mvr\sin\theta$$

Angular impulse is defined as the product of the torque exerted on an object or rigid system at the time interval during which the torque is exerted.

$$J_{ang} = \int \tau \mathrm{d}t$$

The magnitude of the change in angular momentum can be described by comparing the magnitudes of the final and initial momenta of the object or rigid system.

$$\Delta L = \int_{t_0}^{t_1} \tau \mathrm{d}t$$

A rotational form of the impulse-momentum theorem relates the angular impulse delivered to an object or rigid system and the change in the angular momentum of that object or rigid system.

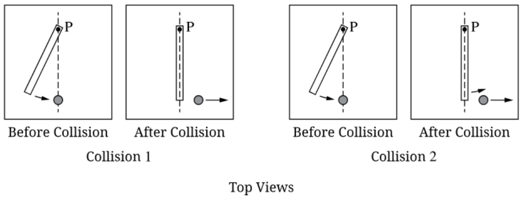
$$\tau_{NBT} = \frac{\mathrm{d}L}{\mathrm{d}t}$$

Example

A particle of mass 2.0 kg is moving in the xy-plane at a constant speed of 0.80 m/s in the $+x$-direction along the line $y = 4$ m. As the particle travels from $x = -3$ m to $x = +3$ m, what is the magnitude of its angular momentum with respect to the origin?

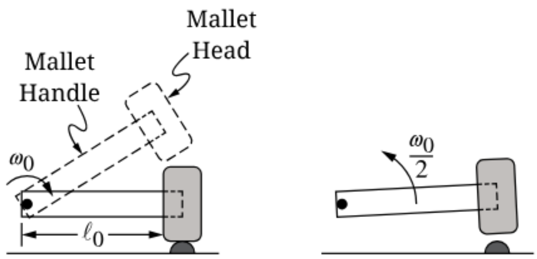
$L = mvr = 6.4$ kg·m^2/s, from the formulas given above.

Exercise



In the two collisions depicted in the figures, a bar is rotating about a pivot at one end, labeled Point P, on top of a horizontal surface before striking a disk that is initially at rest. The two collisions use the same bar but different disks, where the two disks are the same mass and size but are made of different materials. After each collision, the disk moves to the right at a constant velocity. After Collision 1, the bar stops rotating. After Collision 2, the bar is still rotating in the same direction, and the disk is moving more slowly than after Collision 1. There is negligible friction between the bar and the pivot, between the bar and the surface, and between each disk and the surface. Compare the magnitudes of J_1 and J_2 of the angular impulse delivered to each disk during their respective collisions, and provide evidence that supports this claim.

Exercise



Note: Figure not drawn to scale.

A mallet rotates about a pivot near one end of its handle and is moving clockwise with an angular speed ω_0 as it strikes a small, stationary rubber bumper, as shown in the figure. Immediately after the impact, the mallet is rotating counterclockwise about the pivot with an angular speed $\frac{\omega_0}{2}$. The handle of the mallet has length l_0, mass m_0, and rotational inertia $I_0 = \frac{1}{3}m_0 l_0^2$ about the pivot. The head of the mallet is small compared to the length of the handle, has the same mass m_0 as the handle, and is located a distance l_0 from the pivot. If the mallet is in contact with the bumper for an amount of time Δt, what is the magnitude of the average force that the mallet exerts on the rubber bumper during the contact time?

6.4 Conservation of Angular Momentum

The total angular momentum of a system about a rotational axis is the sum of the angular momenta of the system's constituent parts about that axis of rotation.

Any change to a system's angular momentum must be due to an interaction between the system and its surroundings.

Angular momentum is conserved in all interactions.

If the net external torque is exerted on a selected object or rigid system is:

- Zero: the total angular momentum is constant
- Nonzero: angular momentum is transferred between the system and the environment

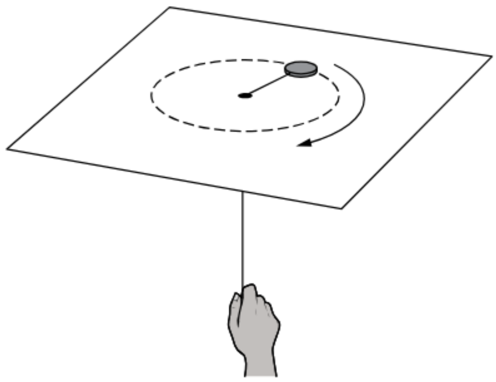
Example

A circular platform has a radius R and rotational inertia I. The platform rotates about a fixed pivot at its center with negligible friction and an initial angular velocity ω. A child of mass m rungs tangentially with speed v and jumps on the outer edge of the platform. When the child is standing on the outer edge of the platform, what is the system's new angular velocity?

We have $L_{c_i} + L_{p_i} = L_T$, so we get $mvr + I\omega = (I_c + I_p)\omega_f$.

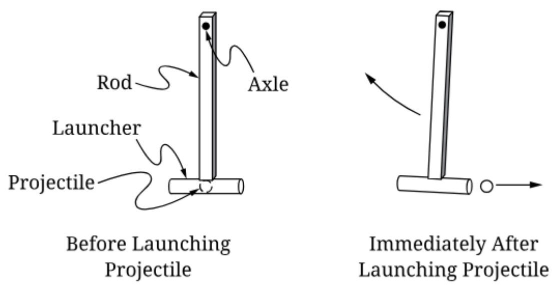
Solving for ω_f gives $\frac{mvr+I\omega}{I+mr^2}$

Exercise



A small disk can move on a horizontal surface with negligible friction. The disk is attached to a string that passes through a small hole in the surface and is held in place at the other end, under the surface. Initially, the disk is moving in a circle of radius 0.10 m about the hole with an angular speed of 4.0 rad/s. The bottom end of the string is then raised upward so that the disk travels in a new circle of radius 0.20 m. What is the angular speed of the disk at the new radius?

Exercise



A projectile launcher is attached to one end of a rod, as shown in the figure. The other end of the rod can pivot freely about an axle with negligible friction. Initially, the rod and launcher, with a projectile inside, are all at rest. The projectile is then launched to the right, causing the rod and launcher to recoil in the clockwise direction. Immediately after the projectile is launched, what is true regarding both the angular momenta of the rod-launcher system and the projectile, and the kinetic energies of the rod-launcher system and the projectile? Angular momentum is to be taken about the axle.

6.5 Rolling

The total kinetic energy of a system is the sum of the system's translational and rotational kinetic energies.

While rolling without slipping, the translational motion of a system's center of mass is related to the rotational motion of the system itself with the following equations:

$$x_{cm} = r\theta$$

$$v_{cm} = r\omega$$

$$a_{cm} = r\alpha$$

Rolling without slipping implies that the frictional force does not dissipate any energy from the system.

If the rolling object is slipping, the force of kinetic friction moves with respect to the surface, so the force of kinetic friction will dissipate energy from the system.

Example

A sphere of mass M, radius r, and rotational inertia I is released from rest at the top of an inclined plane of height h. If the plane has friction so that the sphere rolls without slipping, whta is the speed v_{cm} of the center of mass at the bottom of the incline?

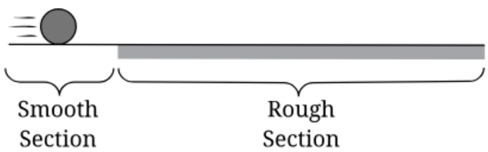
The inertia is $\frac{2}{5}mr^2$.

We know that the there is no kinetic energy initially and all the energy is kinetic at the bottom of the plane, so $U_i = K_f$.

Plugging numbers gives us $mgh = K_T + K_R = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$.

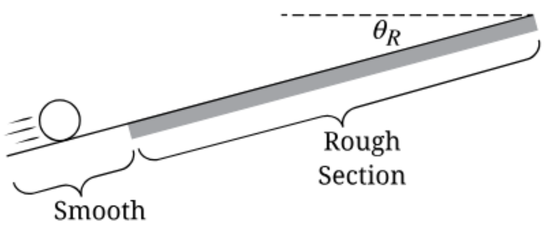
When we solve for v, we get $\sqrt{\frac{10}{7}gh} = v$.

Exercise



A solid sphere is initially moving at a constant speed v_0 on a horizontal surface, as shown in the figure. At first, the sphere is sliding without rotating and moves with negligible friction on a smooth section of the surface. The sphere then reaches a rough section of the surface where the coefficient of kinetic friction is μ_k. Sometime later, the sphere is rolling without slipping on the rough section. The sphere has a mass m_S, a radius r_S, and a rotational inertia about its center $I_S = \frac{2}{5}m_S r_S^2$. What is the final angular speed of the sphere?

Exercise



A hollow cylinder is initially sliding without rotating up a smooth section of a ramp that makes an angle θ_R with the horizontal, as shown in the figure. The cylinder then reaches a rouch section of the ramp where the coefficient of kinetic friction between the cylinder and the ramp is μ_k. The cylinder, which has mass m_C, radius r_C, and rotational inertia about its center of $I_C = m_C r_C^2$, starts to rotate while slipping on the rough section of the ramp. When the cylinder is rolling upward while slipping on the rough section, what is the magnitude a of the cylinder's translational acceleration in terms of the magnitude α of its angular acceleration and the given quantities?

6.6 Motion of Orbiting Satellites

In a system consisting only of a massive central object and an orbiting satellite with mass that is negligible to the central object's mass, the motion of the central object itself is negligible.

The motion of satellites in orbits is constrained by conservation laws.

- In circular orbits, mechanical energy, potential energy, kinetic energy, and angular momentum are conserved.
- In elliptical orbits, mechanical energy and angular momentum are conserved, but potential and kinetic energies are not.
- The gravitational PE is defined to be zero when a satellite is an infinite distance from the central object.

The total energy of a system with a central object and an orbiting satellite can be written in terms of GPE.

The escape velocity of a satellite is the satellite's velocity such that the ME of the satellite-central object system is equal to zero.

Example

A rocket of mass m is launched from the surface of Earth with an initial speed equal to one-half the escape speed. The mass and the radius of Earth are 6.0×10^{24} kg and 6.4×10^6 m, respectively. What is the maximum altitude achieved by the rocket? Assume air resistance is negligible.

Start with $V_E = \sqrt{\frac{2GM}{R}}$ and plug this into $K = \frac{1}{2}mv^2$ to get $\Delta K = \frac{1}{4}\frac{GMm}{R}$.

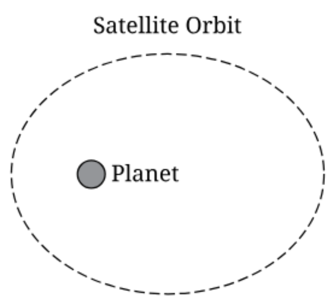
From this, we know that $\Delta U = U_f - U_i$.

This gives us $\frac{1}{4}\frac{GMm}{R} = -\frac{GMm}{R+h} + \frac{GMm}{R}$.

From this, solving for h gives $h = \frac{1}{5}R$.

Exercise Two satellies with the same mass, Satellite X and Satllite Y, are in different circular orbits around a planet. Satellite Y orbits with twice the speed of Satllite X. If U_X is the gravitational potential energy of the Satellite X-planet system, what is the gravitational potential energy of the Satellite Y-planet system?

Exercise



Note: Figure not drawn to scale.

A satellite is in an elliptical orbit around a planet, as shown in the figure. Why does the satellite's kinetic energy change as it orbits the planet?

7 Oscillations

7.1 Defining Simple Harmonic Motion (SHM)

7.2 Frequency and Period of SHM

7.3 Representing and Analyzing SHM

7.4 Energy of Simple Harmonic Oscillators

7.5 Simple and Physical Pendulums