

# Differential Equations Notes

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# 1 Introduction to Differential Equations

## 1.1 Background

In a variety of subject areas, mathematical models are developed to aid in understanding. These models often yield an equation that contains derivatives of an unknown function. Such an equation is called a differential equation.

One example is free fall of a body. An object is released from a certain height above the ground and falls under the force of gravity. Newton's second law states that an object's mass times its acceleration equals the total force acting on it.

$$m \frac{d^2 h}{dt^2} = -mg$$

We have  $h(t)$  as position,  $\frac{dh}{dt}$  as velocity and  $\frac{d^2 h}{dt^2}$  as acceleration. The independent variable is  $t$  and the dependent variable is  $h$ .

$m \frac{d^2 h}{dt^2} = -mg$  is a differential equation and  $h(t)$  is the unknown function that we are trying to find.

From this we have  $\frac{d^2 h}{dt^2} = -g$  and the integral of this is  $\frac{dh}{dt} = -gt + C_1$ . To find  $h$  we integrate again and we get  $h(t) = -\frac{gt^2}{2} + C_1 t + C_2$ .

Another example is the decay of a radioactive substance. The rate of decay is proportional to the amount of radioactive substance present.

$$\frac{dA}{dt} = -kA, \quad k > 0$$

where  $A$  is the unknown amount of radioactive substance present at time  $t$  and  $k$  is the proportionality constant.

We are looking for  $A(t)$  that satisfies this equation. We can solve this from  $\frac{1}{A} dA = -k dt$  and integrating both sides we get that  $\ln |A| + C_1 = -kt + C_2$ . We can rewrite this as  $\ln |A| = -kt + C$ . So,  $e^{-kt+C} = A$ .

So  $A(t) = e^{-kt} + e^C$ , so  $A(t) = Ce^{-kt}$ . Remember  $A$  is the dependent variable and  $t$  is the independent variable.

Notice that the solution of a differential equation is a function, not merely a number.

When a mathematical model involves the rate of change of one variable with respect to another, a differential equation is apt to appear.

### Terminology

If an equation involves the derivative of one variable with respect to another, then the former is called a dependent variable and the latter an independent variable.

In  $\frac{dh}{dt}$ ,  $h$  is dependent and  $t$  is independent.

A differential equation involving only ordinary derivatives with respect to a single independent variable is called an ordinary differential equation. A differential equation involving partial derivatives with respect to more than one independent variable is a partial differential equation.

For example we have  $z = f(x, y) = 4x^2 + 5xy$ , so  $\frac{\partial z}{\partial x} = 8x + 5y$  and that is partial differentiation.

The order of a differential equation is the order of the highest-order derivatives present in the equation.

For example,  $\frac{d^2 h}{dt^2} = -g$  has a order of 2.

A linear differential equation is one in which the dependent variable  $y$  and its derivatives appear in additive combinations of their first powers. A differential equation is linear if it has the format.

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = F(x)$$

$2x + 3y = 7$  is linear,  $2x^2 + 5xy + 7y + 8y = 1$  is second-degree. Nothing can have a second degree for this to be linear.

You are just looking at the dependent variable and the derivatives and adding their powers.

If an ordinary differential equation is not linear, we call it nonlinear.

### Example

For each differential equation, classify as ODE or PDE, linear or nonlinear, and indicate the dependent/independent variables and order.

(a)  $\frac{d^2 x}{dt^2} + a \frac{dx}{dt} + kx = 0$

Dependent is  $x$ , independent is  $t$  and the order is 2. This is an ODE and linear.

(b)  $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x - 2y$

The dependent variable is  $u$  and the independent variables are  $x, y$ , so this is a PDE. The order is 1.

(c)  $\frac{d^2 y}{dx^2} + y^3 = 0$

The dependent variable is  $y$ , the independent variable is  $x$ , the order is 2 and this is an ODE and this is nonlinear.

(d)  $t^3 \frac{dx}{dt} = t^3 + x$

Dependent is  $x$ , independent is  $t$ , order is 1, this is an ODE. We can rewrite this as  $t^3 \frac{dx}{dt} - 1x = t^3$ , and this matches the form of the linear equation so this is linear.

(e)  $\frac{d^2 y}{dx^2} - y \frac{dy}{dx} = \cos x$

The dependent is  $y$ , the independent is  $x$ , the order is 2 and this is an ODE and this is nonlinear because of  $y \frac{dy}{dx}$ .

## 1.2 Solutions and Initial Value Problems

An  $n$ th-order ordinary differential equation is an equality relating the independent variable to the  $n$ th derivative (and usually lower-order derivatives as well) of the dependent variable.

### Example

Identify the order, independent and dependent variable.

(a)  $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = x^3$ . Independent:  $x$ , dependent:  $y$ , order: 2

(b)  $\sqrt{1 - \left(\frac{d^2 y}{dt^2}\right)} - y = 0$ . Independent:  $t$ , dependent:  $y$ , order: 2

(c)  $\frac{d^4 x}{dt^4} = xt$ . Independent:  $t$ , dependent:  $x$ , order: 4. (This is also linear.)

A general form for an  $n$ th-order equation with  $x$  independent,  $y$  dependent can be expressed as

$$F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right) = 0$$

where  $F$  is a function that depends on  $x, y$ , and the derivatives of  $y$  up to order  $n$ . We assume the equations holds for all  $x$  in an open interval  $I$ . In many cases, we can isolate the highest-order term and write the

previous equation as

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1}y}{dx^{n-1}}\right)$$

This is called the normal form.

A function  $\phi(x)$  that when substituted for  $y$  in either the previous two equations satisfies the equation for all  $x$  in the interval  $I$  is called an explicit solution to the equation on  $I$ .

### Example

Show that  $\phi(x) = x^2 - x^{-1}$  is an explicit solution to the linear equation  $\frac{d^2 y}{dx^2} - \frac{2}{x^2}y = 0$  but  $\psi(x) = x^3$  is not.

So we have  $y = x^2 - x^{-1}$ . The first derivative of this is  $2x + 1x^{-2}$ . The second derivative is  $y'' = 2 - 2x^{-3}$ . If we plug in the values we end up getting from the derivatives, we get that  $2 - 2x^{-3} - 2x + 2x^{-3} = 0$ , so this is satisfied.

For the second part, the first derivative is  $3x^2$  and the second derivative is  $6x$ . Plugging this in, we get  $4x$  which is not 0, so  $\psi(x)$  is not a solution.

### Example

Show that for any choice of the constants  $c_1$  and  $c_2$ , the function  $\phi(x) = c_1 e^{-x} + c_2 e^{2x}$  is an explicit solution to the linear equation  $y'' - y' - 2y = 0$ .

We have that the first derivative is  $-c_1 e^{-x} + 2c_2 e^{2x}$  and the second derivative is  $c_1 e^{-x} + 4c_2 e^{2x}$ . When we plug this in, we find that this does satisfy the solution for the differential equation.

Methods for solving differential equations do not always yield an explicit solution for the equation. A solution may be defined implicitly.

### Example

Show that the relation  $y^2 - x^3 + 8 = 0$  implicitly defines a solution to the nonlinear equation  $\frac{dy}{dx} = \frac{3x^2}{2y}$  on the interval  $(2, \infty)$ .

We have from the given that  $y = \pm\sqrt{x^3 - 8}$ . The derivative (of the positive version) of this is  $\frac{3x^2}{2\sqrt{x^3 - 8}}$ . This is the same and defined on the interval.

A relation  $G(x, y) = 0$  is said to be an implicit solution to the previous equation on the interval  $I$  if it defines one or more explicit solutions on  $I$ .

### Example

Show that  $x + y + e^{xy} = 0$  is an implicit solution to the nonlinear equation  $(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0$ .

Taking the derivative of both sides gets us that  $1 + \frac{dy}{dx} + e^{xy} \frac{d}{dx}(xy) = 0$ . This does simplify to what was given in the problem.

### Example

Verify that for every constant  $C$  the relation  $4x^2 - y^2 = C$  is an implicit solution to  $y\frac{dy}{dx} - 4x = 0$ . Graph the solution curves for  $C = 0, \pm 1, \pm 4$ .

The derivative of what is given is  $8x - 2y\frac{dy}{dx} = 0$ . This simplifies to what is given, so it is clearly an implicit solution.

For  $C = 0$ , the solution curves for this is  $2x = y$  and  $-2x = y$ .

For  $C = \pm 4$ , the solution curves is given by a hyperbola  $\frac{x^2}{4} - \frac{y^2}{4} = 1$ .

The collection of all solutions in the previous example is called a one-parameter family of solutions.

In general, the methods for solving  $n$ th-order differential equations evoke  $n$  arbitrary constants. We often can evaluate these constants if we are given  $n$  initial values  $y(x_0), y'(x_0), \dots, y^{(n-1)}(x_0)$ .

### Definition

By an initial value problem for an  $n$ th-order differential equation

$$F(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}) = 0$$

we mean: Find a solution so the differential equation on an interval  $I$  that satisfies at  $x_0$  the  $n$  initial conditions

$$y(x_0) = y_0, \quad \frac{dy}{dx}(x_0) = y_1 \cdots \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1}$$

where  $x_0 \in I$  and  $y_0, y_1, \dots, y_{n-1}$  are constants.

### Example

Show that  $\phi(x) = \sin x - \cos x$  is a solution to the initial value problem

$$\frac{d^2 y}{dx^2} + y = 0; \quad y(0) = -1 \quad \frac{dy}{dx}(0) = 1$$

We have  $y = \sin x - \cos x$ ,  $y' = \cos x + \sin x$ , and  $y'' = -\sin x + \cos x$ . These satisfy the conditions.

### Theorem 1.1: Existence and Uniqueness of Solution

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

If  $f$  and  $\partial f / \partial y$  are continuous functions in some rectangle

$$R = \{(x, y) : a < x < b, c < y < d\}$$

that contains the point  $(x_0, y_0)$ , then the initial value problem has a unique solution  $\phi(x)$  in some interval  $x_0 - \delta < x < x_0 + \delta$ , where  $\delta$  is a positive number.

### Example

Does the theorem above imply the existence for this problem.

$$3 \frac{dy}{dx} = x^2 - xy^3, \quad y(1) = 6$$

The derivative exists for all  $(x, y)$  and is continuous in all intervals containing  $x = 1$  and all rectangular regions containing  $(1, 6)$ .

When we consider the partial derivatives,  $\partial f / \partial y = -xy^2$ , and this exists and is continuous for all rectangular regions in the  $xy$  plane.

## 1.3 Direction Fields

One technique useful in visualizing (graphing) the solutions to a first-order differential equation is to sketch the direction field for the equation. A first-order equation

$$\frac{dy}{dx} = f(x, y)$$

specifies a slope at each point in the  $xy$ -plane where  $f$  is defined.

### Definition

A plot of short line segments drawn at various points in the  $xy$ -plane showing the slope of the solution curve here is called a direction field for the differential equation.

For example, consider the equation  $\frac{dy}{dx} = x^2 - y$ . When we plug in the point  $(1, 0)$ , the slope is 1. When the point is  $(0, 1)$  the slope is  $-1$ . Notice the further right we get, the steeper the graph goes. When we look at this function,  $f(x, y) = x^2 - y$ , so taking the partial of  $y$  results in  $-1$  which is continuous so there is a solution curve at any point.

Note that we are basically just drawing slope fields from AP Calculus BC.

When we consider an equation  $\frac{dy}{dx} = -\frac{y}{x}$ , we have a unique solution when  $x \neq 0$  because  $f(x, y)$  is continuous if  $x \neq 0$  and  $\frac{\partial f}{\partial y} = -\frac{1}{x}$ , so if  $x \neq 0$ , then there is a unique solution.

### Example

Consider the direction field for  $\frac{dy}{dx} = 3y^{2/3}$ . Is there a unique solution passing through  $(2, 0)$ ?

$\frac{\partial f}{\partial y} = \frac{2}{\sqrt[3]{y}}$  is not continuous when  $y = 0$ .

### Example

The logistic equation of the population  $p$  (in thousands) at time  $t$  of a certain species is given by  $\frac{dp}{dt} = p(2 - p)$ . Use its direction field to answer the following questions.

(a) If the initial population is 3000 [ $p(0) = 3$ ], what can you say about the limiting population  $\lim_{t \rightarrow \infty} p(t)$ ?

We start at the point  $(0, 3)$  and as the field approaches  $p = 2$ , the rate of change becomes 0 so the limit is equal to 2.

(b) Can a population of 1000 ever decline to 500?

No

(c) Can a population of 1000 ever increase to 3000.

No

A differential equation  $\frac{dy}{dt} = f(t, y)$  is autonomous if the independent variable  $t$  does not appear explicitly:  $\frac{dy}{dt} = f(y)$ . An autonomous equation has the following properties

- The slopes in the direction field are all identical among horizontal lines
- New solutions can be generated from old ones by time shifting [i.e., replacing  $y(t)$  with  $y(t - t_0)$ ]

The constant, or equilibrium, solutions  $y(t) = c$  for autonomous equations are of particular interest. The equilibrium  $y = c$  is called a stable equilibrium, or sink, if neighboring solutions are attracted to it as  $t \rightarrow \infty$ . Equilibria that repel neighboring solutions, are known as sources; all other equilibria are called nodes. Sources and nodes are unstable equilibria. (Nodes are sometimes called semi stable).

A phase line indicates the zeros and signs of  $f(y)$  to describe the nature of the equilibrium solutions for an autonomous equation.



**Example**

Sketch the phase line for  $y' = -(y-1)(y-3)(y-5)^2$  and state the nature of its equilibria.

We have  $y' = 0$  and  $y = 1, 3, 5$ . When  $y = 0$ , then  $y' < 0$  that means that it is decreasing for values below 1. When we let  $y = 2$  then  $y' > 0$ , so any value between 1 and 3 is increasing. When we put  $y = 4$ , then  $y' < 0$  so any values between 3 and 5 are decreasing. When  $y = 6$  then  $y' < 0$  so it is decreasing.

At  $y = 5$ , above it is attracting but repelling below, so it is semi-stable or a node. At  $y = 3$ , it is stable or an attractor because it is approaching on both sides. At  $y = 1$ , it is a repeller.

Hand sketching the direction field for a differential equation is often tedious. Fortunately, several software programs are available for this task. When hand sketching is necessary, the method of isoclines can be helpful reducing the work.

A isocline for the differential equation

$$y' = f(x, y)$$

is a set of points in the  $xy$ -plane where all the solutions have the same slope  $\frac{dy}{dx}$ ; thus, it is a level curve for the function  $f(x, y)$ .

**Example**

Find isocline curves of  $y' = f(x, y) = x + y$  for a few select values of  $c$ . Use the isoclines to draw hash marks with slope  $c$  along the isocline  $f(x, y) = c$ .

When  $c = 0$ ,  $y' = 0$ ,  $x + y = 0$  and  $y = -x$ . When  $c = 1$ ,  $y' = 1$ ,  $x + y = 1$  and  $y = -x + 1$ . When  $c = 2$ ,  $y' = 2$ ,  $x + y = 2$  and  $y = -x + 2$ .

## 1.4 The Approximation Method of Euler

Euler's method (or the tangent-line method) is a procedure for constructing approximate solutions to an initial value problem for a first-order differential equation

$$y' = f(x, y), \quad y(x_0) = y_0$$

Euler's method can be summarized by the recursive formulas  $x_{n+1} = x_n + h$  and  $y_{n+1} = y_n + hf(x_n, y_n)$ , where  $n = 0, 1, 2, \dots$ .

$h$  is the step size,  $y'$  is the  $m$  of the tangent line. Remember that  $y - y_0 = m(x - x_0)$  and that  $y - y_0$  is just  $f(x_0, y_0)(x - x_0)$  so  $y = y_0 + f(x_0, y_0) \cdot h$ .

**Example**

Use Euler's method with step size  $h = 0.1$  to approximate the solution to the initial value problem

$$y' = x\sqrt{y}, \quad y(1) = 4$$

at the points  $x = 1.1, 1.2, 1.3, 1.4$  and  $1.5$ .

We know the  $x$  points so we can find the  $y$  values from  $y_n = y_{n-1} + f(x_{n-1}, y_{n-1})(0.1)$ .

We have  $(1, 4)$  then  $(1.1, 4.2)$  and continuing the calculations, we get  $(1.2, 4.43)$ ,  $(1.3, 4.68)$ ,  $(1.4, 4.96)$  and  $(1.5, 5.27)$ .

**Example**

Use Euler's method to find approximations to the solution of the initial value problem

$$y' = y, \quad y(0) = 1$$

at  $x = 1$ , taking 1, 2, 4, 8 and 16 steps.

Let  $y = e^x$  and a point  $(0, 1)$ . The recursion formula is  $y_n = y_{n-1} + f(x_{n-1}, y_{n-1})h$ .

If we use a step size of 1, then  $y(1) = 2$ .

If we use a step size of 0.5 then  $y(1) = 2.25$

If we use a step size of 0.25 then  $y(1) = 2.44$

Using technology we can see with a step size of 0.125 that  $y(1) = 2.57$  and with a step size of 0.0625,  $y(1) = 2.64$ .

## 2 First-Order Differential Equations

### 2.1 Separable Equations

#### Definition

If the right-hand side of the equation

$$\frac{dy}{dx} = f(x, y)$$

can be expressed as a function  $g(x)$  that depends only on  $x$  times a function  $p(y)$  that depends only on  $y$ , then the differential equation is called separable.

To solve the equation

$$\frac{dy}{dx} = g(x)p(y)$$

multiply by  $dx$  and by  $h(y) = 1/p(y)$  to obtain

$$h(y)dy = g(x)dx$$

Then integrate both sides and you end up getting  $H(y) = G(x) + C$ , where we have merged the two constants of integration into a single symbol  $C$ . The last equation gives an implicit solution to the differential equation.

#### Example

Solve the nonlinear equation

$$\frac{dy}{dx} = \frac{x-5}{y^2}$$

This can be rewritten as  $y^2 dy = (x-5)dx$ . Integrating both sides results in  $\frac{y^3}{3} = \frac{x^2}{2} - 5x + C$ .

To get the explicit form just solve for  $y$ , which is trivial.

#### Example

Solve the initial value problem

$$\frac{dy}{dx} = \frac{y-1}{x+3} \quad y(-1) = 0$$

Doing Calc BC stuff gives us  $y = 1 - \frac{1}{2}(x+3)$ .

Be careful because you can be losing solutions. Ok bye!

### 2.2 Linear Equations

Remember a linear first-order equation is an equation that can be expressed in the form

$$a_1(x) \frac{dy}{dx} + a_0 y = b(x)$$

where  $a_1(x)$ ,  $a_0(x)$ , and  $b(x)$  depend only on the independent variable  $x$ , not on  $y$ .

Method for solving linear equation:

- Write the equation in the standard form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

- Calculate the integrating factor  $\mu(x)$  by the formula

$$\mu(x) = \exp \left[ \int P(x) dx \right]$$

- Multiply the equation in standard form by  $\mu(x)$  and, recalling that the left-hand side is just  $\frac{d}{dx}[\mu(x)y]$ , obtain

$$\begin{aligned} \mu(x) \frac{dy}{dx} + P(x)\mu(x)y &= \mu(x)Q(x) \\ \frac{d}{dx}[\mu(x)y] &= \mu(x)Q(x) \end{aligned}$$

- Integrate the last equation and solve for  $y$  by dividing by  $\mu(x)$  to obtain.

### Example

Find the general solution to

$$\frac{1}{x} dy - \frac{2y}{x^2} = x \cos x \quad x > 0$$

We have  $\frac{dy}{dx} - \frac{2}{x}y = x^2 \cos x$ .

The integrating factor  $\mu(x) = e^{\int P(x) dx}$  which in this case is  $e^{-2 \int \frac{1}{x} dx}$  and this is equivalent to  $\frac{1}{x^2}$ .

Using this we can multiply through in standard form then we have  $\frac{1}{x^2} \frac{dy}{dx} - \frac{2}{x^3}y = \cos x$ .

The left side is just  $\frac{d}{dx} \left( \frac{1}{x^2} y \right) = \cos x$ .

Integrating and solving for  $y$  we get that  $y = x^2 \sin x + Cx^2$ .

### Example

For the initial value problem

$$y' + y = \sqrt{1 + \cos^2 x} \quad y(1) = 4$$

find the value of  $y(2)$ .

Our  $P(x)$  is 1 here, so  $\mu = e^x$ .

So the equation after multiplying through by it gives us that  $\mu y' + \mu y = \mu \sqrt{1 + \cos^2 x}$ , or  $e^x y' + e^x y = e^x \sqrt{1 + \cos^2 x}$ .

This is equivalent to basically  $\frac{d}{dx}(e^x y) = e^x \sqrt{1 + \cos^2 x}$ .

This is  $e^x y = \int e^x \sqrt{1 + \cos^2 x} dx$ .

Using a calculator  $y(2) = 2.127$ .

### Theorem 2.1: Existence and Uniqueness of Solution

Suppose  $P(x)$  and  $Q(x)$  are continuous on an interval  $(a, b)$  that contains the point  $x_0$ . Then for any choice of initial value  $y_0$ , there exists a unique solution  $y(x)$  on  $(a, b)$  to the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x) \quad y(x_0) = y_0$$

In fact the solution is given for a suitable value of  $C$ .

## 2.3 Exact Equations

### Definition: Exact Differential Form

The differential form  $M(x, y)dx + N(x, y)dy$  is said to be exact in a rectangle  $R$  if there is a function  $F(x, y)$  such that

$$\frac{\partial F}{\partial x}(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x, y) = N(x, y)$$

for all  $(x, y)$  in  $R$ . That is, the total differential of  $F(x, y)$  satisfies

$$dF(x, y) = M(x, y)dx + N(x, y)dy$$

If  $M(x, y)dx + N(x, y)dy$  is an exact differential form, then the equation

$$M(x, y)dx + N(x, y)dy = 0$$

is called an exact equation.

### Theorem 2.2: Test for Exactness

Suppose the first partial derivatives of  $M(x, y)$  and  $N(x, y)$  are continuous in a rectangle  $R$ . Then

$$M(x, y)dx + N(x, y)dy = 0$$

is an exact equation in  $R$  if and only if the compatibility condition

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$$

holds for all  $(x, y)$  in  $R$ .

### Example

Solve the differential equation

$$\frac{dy}{dx} = -\frac{2xy^2 + 1}{2x^2y}$$

Ok so this is not separable or linear, so we use exactness.

$$\frac{dy}{dx} + \frac{2xy^2 + 1}{2x^2y} = 0$$

$$dy + \frac{2xy^2 + 1}{2x^2y} dx = 0$$

$$\frac{2xy^2 + 1}{2x^2y} dx + 1 dy = 0$$

This is the same form we want.

Another form we can get is  $(2xy^2 + 1)dx + 2x^2y dy = 0$ .

Another form we can get is  $1dx + \frac{2x^2y}{2xy^2 + 1} dy = 0$ .

We are now looking for a  $F(x, y) = c$  and we know this is true when  $\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$ .

So the second one of these is probably the best, so we now have  $m = 2xy^2 + 1$  and  $n = 2x^2y$ .

Doing the partial of  $m$  with respect to  $y$  we get  $4xy$  and the partial of  $n$  with respect to  $x$  is  $4xy$  and these are the same.

Let  $F(x, y) = x^2y^2 + x = C$ . The partial of this function with respect to  $x$  is  $2xy^2 + 1$  and the partial of this function with respect to  $y$  is  $2x^2y$  and this is the same as previous.

Method for Solving Exact Equations:

- If  $Mdx + Ndy = 0$  is exact, then  $\partial F/\partial x = M$ . Integrate this last equation with respect to  $x$  to get

$$F(x, y) = \int M(x, y)dx + g(y)$$

- To determine  $g(y)$ , take the partial derivative with respect to  $y$  of both sides of the above equation and substitute  $N$  for  $\partial F/\partial y$ . We can now solve for  $g'(y)$ .
- Integrate  $g'(y)$  to obtain  $g(y)$  up to a numerical constant. Substituting  $g(y)$  into the equation from step 1 gives  $F(x, y)$
- The solution to  $Mdx + Ndy = 0$  is given implicitly by

$$F(x, y) = C$$

(Alternatively, starting with  $\partial F/\partial y = N$ , the implicit solution can be found by first integrating with respect to  $y$ .)

### Example

Solve

$$(2xy - \sec^2 x)dx + (x^2 + 2y)dy = 0$$

Let  $m$  be the first term and  $n$  be the second term, and the partial derivatives of these are the same, so they are exact.

Let  $F(x, y) = \int 2xy - \sec^2 x dx$ . When we integrate this, we get  $yx^2 - \tan x$ . In this case, the constant is anything with  $y$ , so the integral is equivalent to  $yx^2 - \tan x + g(y)$ .

Now we take the  $\frac{\partial F}{\partial y} = x^2 - 0 + g'(y)$ . These two are  $n$  so  $x^2 + 2y = x^2 + g'(y)$ , so solving for  $g(y)$  we get that this is equal to  $y^2 + C$ .

So  $F(x, y) = xy^2 - \tan x + y^2 = C$ .

**Exercise** Solve  $(1 + e^xy + xe^xy)dx + (xe^x + 2)dy = 0$ .

Solution:  $x + xye^x + 2y = C$ .

### Example

Solve

$$(x + 3x^3 \sin y)dx + (x^4 \cos y)dy = 0$$

Doing the partials originally makes them not equal to each other.

We can get this to exact form by multiplying through by  $x^{-1}$ . When we do this we get  $(1 + 3x^2 \sin y)dx + x^3 \cos y dy = 0$  and the partials of these are the same.

$x^{-1}$  is called an integrating factor.

Integrating  $m$  with respect to  $x$ , we get that  $F(x, y) = \int 1 + 3x^2 \sin y dx = x + \sin y \cdot x^3 + g(y)$ .

Doing the partial of  $F$  with respect to  $y$ , we get  $\frac{\partial F}{\partial y} = x^3 \cos y = 0 + x^3 \cos y + g'(y)$ , and this gets that  $g(y) = C$ .

So the answer is  $x + x^3 \sin y = C$ .

## 2.4 Special Integrating Factors

### Definition

If the equation

$$M(x, y)dx + N(x, y)dy = 0$$

is not exact, but the equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

which results from multiplying the first equation by the function  $\mu(x, y)$ , is exact, then  $\mu(x, y)$  is called an integrating factor of the first equation.

### Theorem 2.3: Special Integrating Factors

If  $(\partial M/\partial y - \partial N/\partial x)/N$  is continuous and depends only on  $x$ , then

$$\mu(x) = \exp \left[ \int \left( \frac{\partial M/\partial y - \partial N/\partial x}{N} \right) dx \right]$$

is an integrating factor for an equation. If  $(\partial N/\partial x - \partial M/\partial y)/M$  is continuous and depends only on  $y$ , then

$$\mu(y) = \exp \left[ \int \left( \frac{\partial N/\partial x - \partial M/\partial y}{M} \right) dy \right]$$

is an integrating factor for the same equation.

Method for Finding Special Integrating Factors:

If  $Mdx + Ndy = 0$  is neither separable nor linear, compute  $\partial M/\partial y$  and  $\partial N/\partial x$ . If  $\partial M/\partial y = \partial N/\partial x$ , then the equation is exact. If it is not exact, consider

$$\frac{\partial M/\partial y - \partial N/\partial x}{N}$$

If this is a function of just  $x$ , then an integrating factor is given by the formula above of  $\mu(x)$ . If not consider

$$\frac{\partial N/\partial x - \partial M/\partial y}{M}$$

If this is a function of just  $y$ , then an integrating factor is given by above of  $\mu(y)$ .

### Example

Solve  $(2x^2 + y)dx + (x^2y - x)dy = 0$

When we do the partials, we get that  $1 \neq 2xy - 1$ .

So let's look at  $\frac{\partial m/\partial y - \partial n/\partial x}{N}$ , which is  $\frac{1 - (2xy - 1)}{x^2y - x} = \frac{-2}{x}$  which is just a function of  $x$ . So we have that  $\mu = e^{\int -\frac{2}{x} dx}$ , so we don't have to look at the one in terms of  $y$ .

Doing the integral of all this gives us that  $e^{-2 \ln x} = x^{-2}$ . So when we multiply through by  $x^{-2}$ , we get that  $(2 + x^{-2}y)dx + (y - x^{-1})dy = 0$ .

The partials are equal to each other, so this equation is now exact.

Now we find  $F(x, y)$  by integrating  $m$ , so  $\int (2 + x^{-2}y)dx = 2x + -x^{-1}y + g(y) = F(x)$

Now we differentiate with respect to  $y$  so  $\frac{\partial F}{\partial y} = y - x^{-1} = -x^{-1} + g'(y)$ , so  $g(y) = \frac{y^2}{2}$

The solution is therefore  $2x - x^{-1}y + \frac{y^2}{2} = C$ .

## 2.5 Substitutions and Transformations

Substitution Procedure:

- Identify the type of equation and determine the appropriate substitution or transformation
- Rewrite the original equation in terms of new variables
- Solve the transformed equation
- Express the solution in terms of the original variables

### Definition: Homogeneous Equation

If the right-hand side of the equation

$$\frac{dy}{dx} = f(x, y)$$

can be expressed as a function of the ratio  $y/x$  alone, then we say the equation is homogeneous.

To solve a homogeneous equation, use the substitution  $v = \frac{y}{x}$ ;  $\frac{dy}{dx} = v + x \frac{dv}{dx}$  to transform the equation into a separable equation.

### Example

Solve  $(xy + y^2 + x^2)dx - x^2dy = 0$ .

Solving for  $\frac{dy}{dx}$  we get that this is equal to  $\frac{-x^2 - y^2 - xy}{-x^2}$  and this simplifies to  $1 + \left(\frac{y}{x}\right)^2 + \frac{y}{x}$ .

This is equivalent to  $v + x \frac{dv}{dx} = 1 + v^2 + v$ . We end up getting that  $\frac{dv}{dx} = \frac{v^2 + 1}{x}$  and this can be done by separation. The solution is  $y = x \tan(\ln|x| + C)$  after solving.

To solve an equation of the form  $\frac{dy}{dx} = G(ax + by)$ , use the substitution  $z = ax + by$  to transform the equation into a separable equation.

### Example

Solve  $\frac{dy}{dx} = y - x - 1 + (x - y + 2)^{-1}$

First we have  $\frac{dy}{dx} = -(x - y) - 1 + (x - y + 2)^{-1}$

So substituting with  $z = x - y$ , we have that  $\frac{dz}{dx} = 1 - \frac{dz}{dx}$ . Knowing this, the equation is equal to  $1 - \frac{dz}{dx} = -z - 1 + (z + 2)^{-1}$ . From this this simplifies to  $\frac{dz}{dx} = z + 2 - (z + 2)^{-1}$ .

So now we write this into a separable equation with  $\frac{(z+2)dz}{(z+2)^2 - 1} = dx$ .

Separating by parts and substituting gives  $(x - y + 2)^2 = ce^{2x} + 1$

### Definition: Bernoulli Equation

A first-order equation that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where  $P(x)$  and  $Q(x)$  are continuous on the interval  $(a, b)$  and  $n$  is a real number, is called a Bernoulli equation.

To solve a Bernoulli equation use the substitution  $v = y^{1-n}$  to transform the equation into a linear equation.



**Example**

Solve  $\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3$ .

From above, we have  $v = y^{-2}$  and  $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$  and  $-\frac{1}{2}\frac{dv}{dx} = y^{-3}\frac{dy}{dx}$ .

So multiplying through by  $y^{-3}$  and substituting, we get that  $-\frac{1}{2}\frac{dv}{dx} - 5v = -\frac{5}{2}x$ .

This is equal to  $\frac{dv}{dx} + 10v = 5x$ . The integrating factor here is  $\mu = e^{\int P(x)dx}$ , which is  $e^{10x}$  in this case.

Multiplying through by  $\mu$ , we get that  $e^{10x}\frac{dv}{dx} + 10ve^{10x} = 5xe^{10x}$  and the LHS should be equal to  $\frac{d}{dx}(e^{10x}v) = 5xe^{10x}$ .

Using elementary integration techniques the answer is  $y^{-2} = \frac{x}{2} - \frac{1}{20} + Ce^{-10x}$ .

# 3 Mathematical Models and Numerical Methods Involving First-Order Equations

## 3.1 Compartmental Analysis

We assume that the growth rate is proportional to the population present. A mathematical model called the Malthusian, or exponential law of population growth, model is given by

$$\frac{dp}{dt} = kp, \quad p(0) = p_0$$

where  $k > 0$  is the proportionality constant of the growth rate and  $p_0$  is the population at time  $t = 0$ .

### Example

In 1790, the population of the United States was 3.93 million, and in 1890 it was 62.98 million. Assuming the Malthusian model, estimate the U.S. population as a function of time.

Let  $t = 0$  be year 1790. Using separation of variables and integration from  $\frac{dp}{dt} = kp$ , we get that  $\ln P = kt + C$ . Solving for  $P$  we get that  $e^{\ln P} = P = e^{kt+C}$ , so this is equal to  $P = Ce^{kt}$ .

At  $P(0) = 3.93$ , and using this we can find  $k$ .  $3.93 = Ce^0$ , so  $C = 3.93$ . So we have  $P(t) = P_0 e^{kt}$ . We can now find  $k$  by plugging in  $P(100) = 62.98$  from this, and solving for  $k$ , we get that  $k \approx 0.027742$ .

So  $P(t) = 3.93e^{0.027742t}$

The Malthusian model considered only death by natural causes. Other factors such as premature deaths due to malnutrition, inadequate medical supplies, communicable diseases, violent crimes, etc involve a competition within the population. The logistic model is given by

$$\frac{dp}{dt} = -Ap(p - p_1), \quad p(0) = p_0$$

Note that this equilibrium has two equilibrium solutions  $p(t) = p_1$  and  $p(t) = 0$ .

### Example

Taking the population of 3.93 million as the initial population and given the 1840 and 1890 populations of 17.07 and 62.98 million respectively, use the logistic model to estimate the population at time  $t$ .

We have  $P(50) = 17.07$  and  $P(100) = 62.98$ . Using a previously derived formula,  $p(t) = \frac{p_0 p_1}{p_0 + (p_1 - p_0)e^{-Ap_1 t}}$ , we get  $17.07 = \frac{3.93 p_1}{3.93 + (p_1 - 3.93)e^{-50Ap_1}}$  and  $62.98 = \frac{3.93 p_1}{3.93 + (p_1 - 3.93)e^{-100Ap_1}}$ .

The answers are that  $p_1 \approx 251.78$  and  $A \approx 0.0001210$ , so the logistic equation ended up being  $p(t) = \frac{989.5}{3.93 + 247.85e^{-0.0304637t}}$ .

### Example

Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number  $x$  of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days,  $x(4) = 50$ .

We are solving the initial value problem  $\frac{dx}{dt} = kx(1000 - x)$ ,  $x(0) = 1$ . Doing some stuff we get that  $\ln \frac{x}{1000-x} = 1000kt + C$ .

After doing some more things we get that  $x = \frac{1000Ce^{1000kt}}{1+Ce^{1000kt}}$  and from this we can get that  $C = \frac{1}{999}$ .

So the function is now  $x(t) = \frac{\frac{1000}{999}e^{1000kt}}{1+\frac{1}{999}e^{1000kt}}$ , so simplifying we get that  $x(t) = \frac{1000}{999e^{-1000kt}+1}$ .

Doing some plugging in stuff we get that  $k \approx -.9906$ , so after 6 days approximately 276 students are infected.

## 3.2 Numerical Methods: A Closer Look At Euler's Algorithm

The numerical method defined by the formula

$$y_{n+1} = y_n + h \frac{f(x_n, y_n) + f(x_{n+1}, y_{n+1}^*)}{2}$$

where

$$y_{n+1}^* = y_n + hf(x_n, y_n)$$

is known as the improved Euler's method.

The improved Euler's method is an example of a predictor-corrector method.

It is recommended you use technology to find the answer.

## 3.3 Higher-Order Numerical Methods: Taylor and Runge-Kutta

We wish to obtain a numerical approximation of the solution  $\phi(x)$  to the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

To derive the Taylor methods, let  $\phi_n(x)$  be the exact solution to the initial value problem

$$\phi_n'(x) = f(x, \phi_n), \quad \phi_n(\text{problem } x_n) = y_n$$

The Taylor series for  $\phi_n(x)$  about the point  $x_n$  is

$$\phi_n(x) = \phi_n(x_n) + h\phi_n'(x_n) + \frac{h^2}{2!}\phi_n''(x_n) + \dots$$

where  $h = x - x_n$ . Since  $\phi_n$  satisfies the initial value, we can write this series in the form

$$\phi_n(x) = y_n + hf(x_n, y_n) + \frac{h^2}{2!}\phi_n''(x_n) + \dots$$

### Example

Determine the recursive formula for the Taylor method of order 2 for the initial value

$$y' = \sin(xy), y(0) = \pi$$

We know that  $\phi_n''(x_n) = \frac{\partial f}{\partial x}(x_n, y_n) + \frac{\partial f}{\partial y}(x_n, y_n) \cdot \frac{dy}{dx}$ .

So  $\frac{\partial f}{\partial x} = \cos(xy) \cdot y$  and  $\frac{\partial f}{\partial y} = \cos(xy) \cdot x$ .

Putting this all together we get that  $\phi_n(x) = y_{n+1} = y_n + h \sin(x_n y_n) + \frac{h^2}{2} [y_n \cos(x_n y_n) + x_n \cos(x_n y_n) \cdot \sin(x_n y_n)]$ .

Doing some magic,  $x_{n+1} = x_n + h$  and  $y_{n+1} = y_n + h \sin(x_n y_n) + \frac{h^2}{2} [y_n \cos(x_n y_n) + \frac{x_n}{2} \sin(2x_n y_n)]$ .

# 4 Linear Second-Order Equations

## 4.1 Introduction: The Mass-Spring Oscillator

A damped mass-spring oscillator consists of a mass  $m$  attached to a spring fixed at one end. Devise a differential equation that governs the motion of this oscillator, taking into account the forces acting on it due to the spring elasticity, damping friction, and possible external influences.

Newton's second law - force equals mass times acceleration ( $F = ma$ ) - is the most commonly encountered differential equation. It is an ordinary differential equation of the second order since acceleration is the second derivative of position ( $y$ ) with respect to time ( $a = d^2y/dt^2$ ).

If the spring is unstretched and the inertial mass  $m$  is still, the system is at equilibrium. We stretch the coordinate  $y$  of the mass by its displacement from the equilibrium position.

When the mass  $m$  is displaced from equilibrium, the spring is stretched or compressed and it exerts a force that resists the displacement. For most springs this force is directly proportional to the displacement  $y$  and is given by Hooke's law.

$$F_{\text{spring}} = -ky$$

where the positive constant  $k$  is known as the stiffness (spring constant) and the negative sign reflects the opposing nature of the force. Hooke's law is only valid for sufficiently small displacements.

Usually all mechanical systems also experience friction. For vibrational motion this force is usually modeled accurately by a term proportional to velocity:

$$F_{\text{friction}} = -b \frac{dy}{dt} = -by'$$

where  $b \geq 0$  is the damping coefficient and the negative sign reflects the opposing nature of the force.

The other forces on the oscillator are usually regarded as external to the system. Although they may be gravitational, electrical, or magnetic, commonly the most important external forces are transmitted to the mass by shaking the supports holding the system. For now we refer to the combined external forces by a single known function  $F_{\text{ext}}(t)$ . Newton's law provides the differential equation for the mass-spring oscillator:

$$my'' = -ky - by' + F_{\text{ext}}(t)$$

or

$$my'' + by' + ky = F_{\text{ext}}(t)$$

### Example

Verify that if  $b = 0$  and  $F_{\text{ext}} = 0$ , that the above equation has a solution of the form  $y(t) = \cos(\omega t)$  and the angular frequency  $\omega$  increases with  $k$  and decreases with  $m$ .

The differential equation is  $my'' + by' + ky = F_{\text{ext}}$  and that  $my'' + ky = 0$ .

Since we are given what  $y(t)$  is, taking the derivative of the differential equation gives that  $-m\omega^2 \cos(\omega t) + k \cos(\omega t) = 0$ , and solving for  $\omega$ , we get that  $\omega = \sqrt{\frac{k}{m}}$ .

If  $k$  increases,  $\omega$  increases, and if  $m$  increases,  $\omega$  decreases.

### Example

Verify that the exponentially damped sinusoid given by  $y(t) = e^{-3t} \cos 4t$  is a solution to the above differential equation if  $F_{\text{ext}} = 0$ ,  $m = 1$ ,  $k = 25$ , and  $b = 6$ .

From the differential equation  $my'' + by' + ky = F_{\text{ext}}$ , we can plug in stuff and we get that  $y'' + 6y' + 25y =$

0.

Since we are given  $y(t)$  we can find the derivatives and substitute. What we are given is quite long, but essentially it cancels out to  $0 = 0$ , which means it is a solution to the system.

*Exercise* Verify that the simple exponential function  $y(t) = e^{-5t}$  is a solution to the above differential equation if  $F_{\text{ext}} = 0$ ,  $m = 1$ ,  $k = 25$ , and  $b = 10$ .

Sometimes the external force will make the system look somewhat erratic. There are many real world examples where the external force must definitely be taken into account.

## 4.2 Homogeneous Linear Equations: The General Solution

A second-order constant-coefficient differential equation has the form

$$ay'' + by' + cy = f(t) \quad (a \neq 0)$$

A homogeneous second-order constant-coefficient differential equation is the special case with  $f(t) = 0$ .

$$ay'' + by' + cy = 0 \quad (a \neq 0)$$

A solution of this equation has the form  $y = e^{rt}$ . The resulting equation  $ar^2 + br + c = 0$  is called the auxiliary equation (or characteristic equation) associated with the homogeneous equation.

### Example

Find a pair of solutions to

$$y'' + 5y' - 6y = 0$$

Plugging in  $y = e^{rt}$ , we get that  $e^{rt}(r^2 + 5r - 6)$  after substituting. Solving the quadratic  $r^2 + 5r - 6 = 0$ , we get that  $r = -6$  and  $r = 1$ , which is  $y = e^{-6t}$  and  $y = e^t$ .

Note that the zero function,  $y(t) = 0$  is always a solution to an equation above (figure this out later). In addition when we have a pair of solutions  $y_1(t)$  and  $y_2(t)$ , we can construct an infinite number of other solutions by forming linear combinations:

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for any choice of the constants  $c_1$  and  $c_2$ . This is a two-parameter solution form since there are two unknown constants. To find a specific solution, two initial conditions are needed.

### Example

Solve the initial value problem

$$y'' + 2y' - y = 0 \quad y(0) = 0, \quad y'(0) = -1$$

Doing the default substitution, our auxiliary equation is  $r^2 + 2r - 1 = 0$ .

The solutions from this are  $r = -1 + \sqrt{2}$  and  $r = -1 - \sqrt{2}$ . Remember  $y = e^{rt}$ . Using the initial conditions, we get that  $c_1 = -\frac{\sqrt{2}}{4}$  and  $c_2 = \frac{\sqrt{2}}{4}$ .

### Theorem 4.1

For any real numbers  $a(\neq 0)$ ,  $b$ ,  $c$ ,  $t_0$ ,  $Y_0$ , and  $Y_1$ , there exists a unique solution to the initial value problem.

$$ay'' + by' + cy = 0 \quad y(t_0) = Y_0 \quad y'(t_0) = Y_1$$

The solution is valid for all  $t$  in  $(-\infty, \infty)$

**Definition**

A pair of functions  $y_1(t)$  and  $y_2(t)$  is said to be linearly independent on the interval  $t$  if and only if neither of them is a constant multiple of the other on all of  $t$ . We say that  $y_1$  and  $y_2$  are linearly dependent on  $t$  if one of them is a constant multiple of the other on all of  $t$ .

**Theorem 4.2**

If  $y_1(t)$  and  $y_2(t)$  are any two solutions to the differential equation that are linearly independent on  $(-\infty, \infty)$ , then unique constants  $c_1$  and  $c_2$  can always be found so that  $c_1y_1(t) + c_2y_2(t)$  satisfies the initial value problem on  $(-\infty, \infty)$ .

**Definition: Wronskian**

Suppose each of the functions  $f_1(x), f_2(x), \dots, f_n(x)$  possess at least  $n - 1$  derivatives.

The determinant

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of the functions.

**Theorem 4.3**

Let  $y_1, y_2, \dots, y_n$  be  $n$  solutions of the homogeneous linear  $n$ th-order differential equation on an interval  $I$ . Then the set of solutions is linearly independent on  $I$  if and only if  $W(y_1, y_2, \dots, y_n) \neq 0$  for every  $x$  in the interval.

**Distinct real roots:** If the auxiliary equation has distinct real roots  $r_1$  and  $r_2$ , then both  $y_1(t) = e^{r_1 t}$  and  $y_2(t) = e^{r_2 t}$  are solutions to the above differential equation and  $y(t) = e_1 e^{r_1 t} + e_2 e^{r_2 t}$  is a general solution.

**Repeated root:** if the auxiliary equation has a repeated root  $r$ , then both  $y_1(t) = e^{rt}$  and  $y_2(t) = te^{rt}$  are solutions to the differential equation and  $y(t) = e_1 e^{rt} + e_2 te^{rt}$  is a general solution.

A homogeneous linear  $n$ th-order equation has a general solution of the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \dots + c_n y_n(t)$$

where the individual solutions  $y_i(t)$  are linearly independent, i.e. no  $y_i(t)$  is expressible as a linear combination of the others.

### 4.3 Auxiliary Equations with Complex Roots

The simple harmonic equation  $y'' + y = 0$  so called because of its relation to the fundamental vibration of a musical tone, has as solutions  $y_1(t) = \cos t$  and  $y_2(t) = \sin t$ .

When  $b^2 - 4ac < 0$ , the roots of the auxiliary equation  $ar^2 + br + c = 0$  associated with the homogeneous equation  $ay'' + by' + cy = 0$  are the complex conjugate numbers  $r_1 = \alpha + i\beta$  and  $r_2 = \alpha - i\beta$  where  $\alpha = -\frac{b}{2a}$  and  $\beta = \frac{\sqrt{4ac - b^2}}{2a}$ .

Combining the solutions  $e^{r_1 t}$  and  $e^{r_2 t}$  with Euler's formula  $e^{i\theta} = \cos \theta + i \sin \theta$ , yields complex function solutions

$$e^{(\alpha + i\beta)t} = e^{\alpha t}(\cos \beta t + i \sin \beta t) \text{ and } e^{(\alpha - i\beta)t} = e^{\alpha t}(\cos \beta t - i \sin \beta t)$$

**Example**

Solve the initial value problem  $y'' + 2y' + 2y = 0$  given  $y(0) = 0$  and  $y'(0) = 2$ .

Using the auxiliary form of the equation we have  $r^2 + 2r + 2 = 0$ . From the quadratic formula,  $r = -1 \pm i$ , the two roots are  $r_1 = -1 + i$  and  $r_2 = -1 - i$ .

The solution is therefore  $y_1 = e^{(-1+i)t}$  and  $y_2 = e^{(-1-i)t}$

From the form given from euler's formula earlier,  $y_1 = e^{-t}(\cos t + i \sin t)$  and  $y_2 = e^{-t}(\cos t - i \sin t)$ , so our general solution is  $y = c_1 e^{-t}(\cos t + i \sin t) + c_2 e^{-t}(\cos t - i \sin t)$ .

Plugging the initial conditions, we get that  $0 = c_1 + c_2$ .

The derivative of the general solution is  $y' = c_1 e^{-t}(-\sin t + i \cos t) + (\cos t + i \sin t) \cdot c_1(-1)e^{-t} + c_2 e^{-t}(-\sin t - i \cos t) + (\cos t - i \sin t) \cdot c_2(-1)e^{-t}$ . Plugging in the initial conditions gives  $2 = c_1 i - c_1 - c_2 i - c_2$ .

Factoring we get  $2 = c_1(i - 1) + c_2(-i - 1)$ . Using some substitution  $c_2 = i$  and  $c_1 = -i$ . Plug this in the general solution to solve.

If the auxiliary equation has complex conjugate roots  $a \pm i\beta$ , then two linearly independent solutions to the equation are

$$e^{\alpha t} \cos \beta t \text{ and } e^{\alpha t} \sin \beta t$$

and a general solution is

$$y(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$$

where  $c_1$  and  $c_2$  are arbitrary constants.

**Example**

Find a general solution to  $y'' + 2y' + 4y = 0$ .

Using the auxiliary equation the roots are  $-1 \pm \sqrt{3}i$ . Using what was given above, the general solution is  $y = c_1 e^{-t} \cos(\sqrt{3}t) + c_2 e^{-t} \sin(\sqrt{3}t)$ .

**Example**

Newton's second law implies the position  $y(t)$  of the mass  $m$  is governed by the second-order differential equation  $my''(t) + by'(t) + ky(t) = 0$  where the terms are physically identified as  $my$  being inertial,  $by$  is damping and  $ky$  is stiffness. Determine the equation of motion for a spring system when  $m = 36$  kg,  $b = 12$  kg/sec (which is equivalent to 12 N - sec/m),  $k = 37$  kg/sec<sup>2</sup>,  $y(0) = 0.7$  m and  $y'(0) = 0.1$  m/sec. Also find  $y(10)$ , the displacement after 10 sec.

The differential equation is  $36y'' + 12y' + 37y = 0$ , so the roots are  $-\frac{1}{6} \pm i$ .

Doing the methods explained above, the solution is  $y = .7e^{-t/6} \cos t + \frac{13}{60}e^{-t/6} \sin t$ , and  $y(10) \approx -.13$  m.

*Exercise* Interpret the equation  $y'' + 5y' - 6y = 0$  in terms of the mass-spring system.

## 4.4 Nonhomogeneous Equations: the Method of Undetermined Coefficients

The method of Undetermined Coefficients is the technique used to guess a solution's form based on the form of the nonhomogeneous function  $f(t)$  in a linear equation with constant coefficients such as  $ay'' + by' + cy = f(t)$ .

For example the particular solution to  $ay'' + by' + cy = Ct^m$  is of the form  $y_p(t) = A_m t^m + \dots + A_1 t + A_0$ .

**Example**

Find a particular solution to  $y'' + 3y' + 2y = 10e^{3t}$ .

Our guess based on the form of  $f(t) = 10e^{3t}$  is that  $y = Ae^{3t}$  is the guess form of the particular solution, so we know that  $y' = 3Ae^{3t}$  and  $y'' = 9Ae^{3t}$ .

Substituting this in gives us  $9Ae^{3t} + 3(3Ae^{3t}) + 2(Ae^{3t}) = 10e^{3t}$ . Simplifying this gives us  $20Ae^{3t} = 10e^{3t}$  so  $A = 1/2$ .

So our particular solution is  $\frac{1}{2}e^{3t}$ .

**Example**

Find a particular solution to  $y'' + 3y' + 2y = \sin t$ .

Let  $y = A \sin t + B \cos t$  as the form of the particular solution.

Substituting and solving should result in  $A = 1/10$  and  $B = -3/10$ .

This example suggests an equation of the form  $ay'' + by' + cy = C \sin \beta t$  (or  $C \cos \beta t$ ) will have a particular solution of the form  $y_p(t) = A \cos \beta t + B \sin \beta t$ .

**Example**

Find a particular solution to  $y'' + 4y = 5t^2e^t$ .

Let  $y = At^2e^t + Bte^t + Ce^t = e^t(At^2 + Bt + C)$ . The result should be  $A = 1, B = -4/5, C = -2/25$ .

To find a particular solution to the differential equation

$$ay'' + by' + cy = Ct^m e^{rt}$$

where  $m$  is a nonnegative integer, use the form

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt}$$

with

1.  $s = 0$  if  $r$  is not a root of the associated auxiliary equation;
2.  $s = 1$  if  $r$  is a simple root of the associated auxiliary equation;
3.  $s = 2$  if  $r$  is a double root of the associated auxiliary equation.

To find a solution to the differential equation  $ay'' + by' + cy = Ct^m e^{\alpha t} \cos \beta t$  or equal to  $Ct^m e^{\alpha t} \sin \beta t$  for  $\beta \neq 0$ , use the form  $y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{\alpha t} \cos \beta t + t^s(B_m t^m + \cdots + B_1 t + B_0)e^{\alpha t} \sin \beta t$ , with

1.  $s = 0$  if  $\alpha + i\beta$  is not a root of the associated auxiliary equation; and
2.  $s = 1$  if  $\alpha + i\beta$  is a root of the associated auxiliary equation

## 4.5 The Superposition Principle and Undetermined Coefficients Revisited

**Theorem 4.4: Superposition Principle**

Let  $y_1$  be a solution to the differential equation

$$ay'' + by' + cy = f_1(t)$$

and  $y_2$  is a solution to

$$ay'' + by' + cy = f_2(t)$$



then for any constants  $k_1$  and  $k_2$ , the function  $k_1y_1 + k_2y_2$  is a solution to the differential equation

$$ay'' + by' + cy = k_1f_1(t) + k_2f_2(t)$$

### Example

Find a particular solution to

$$y'' + 3y' + 2y = 3t + 10e^{3t}$$

The solution for equal to  $3t$  is  $y = \frac{3t}{2} - \frac{9}{4}$  and for  $10e^{3t}$  is  $y = \frac{e^{3t}}{2}$  so the solution is  $y = \frac{3t}{2} - \frac{9}{4} + \frac{e^{3t}}{2}$ .

*Exercise* Find a particular solution to  $y'' + 3y' + 2y = -9t + 20e^{3t}$ .

General solution for Nonhomogeneous Differential Equations: Let  $y_p$  be a particular solution to

$$ay'' + by' + cy = f(t)$$

and  $c_1y_1 + c_2y_2$  be the general solution to the homogeneous equation

$$ay'' + by' + cy = 0$$

Then the general solution to the nonhomogeneous equation is given by

$$y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t)$$

### Theorem 4.5

For any real numbers  $a(\neq 0)$ ,  $b, c, t_0, Y_0$ , and  $Y_1$ , suppose  $y_p(t)$  is a particular solution to above in an interval  $I$  containing  $t_0$  and that  $y_1(t)$  and  $y_2(t)$  are linearly independent solutions to the associated homogeneous equation in  $I$ . Then there exists a unique solution in  $I$  to the initial value problem.

$$ay'' + by' + cy = f(t) \quad y(t_0) = Y_0 \quad y'(t_0) = Y_1$$

### Example

Given that  $y_p(t) = t^2$  is a particular solution to

$$y'' - y = 2 - t^2$$

Find a general solution and a solution satisfying  $y(0) = 1, y'(0) = 0$ .

Our general solution using the auxiliary equation is  $y = c_1e^t + c_2e^{-t}$ .

Our particular solution will be in  $At^2 + By + C$ .

So  $y = t^2 + c_1e^t + c_2e^{-t}$ . Solving for  $c_1$  and  $c_2$  by finding the derivative of this and using the initial conditions, the specific solution is  $y = t^2 + \frac{1}{2}e^t + \frac{1}{2}e^{-t}$ .

### Example

A mass-spring system is driven by a sinusoidal external force  $5\sin t + 5\cos t$ . The mass equals 1, the spring constant equals 2, and the damping coefficient equals 2 (in appropriate units), so the motion is governed by the differential equation

$$y'' + 2y' + y = 5\sin t + 5\cos t$$

If the mass is initially located at  $y(0) = 1$ , with a velocity  $y'(0) = 2$ , find its equation of motion.

Finding the general solution to this we get that  $y = c_1e^{-t}\cos t + c_2e^{-t}\sin t$ .

From  $y_p = A \sin t + B \cos t$ , we should solve that

$$y = 3 \sin t - \cos t + 2e^{-t} \cos t + e^{-t} \sin t$$

### Example

Find a particular solution to

$$y'' - y = 8te^t + 2e^t$$

The general solution for this is  $y = c_1 e^t + c_2 e^{-t}$ .  $y_p$  is equal to  $(At + B)e^t$ . Doing some calculations should result in  $y_p = (2t^2 - t)e^t$ .

### Method of Undetermined Coefficients (Revisited)

To find a particular solution to the differential equation

$$ay'' + by' + cy = P_m(t)e^{rt}$$

where  $P_m(t)$  is a polynomial of degree  $m$ , use the form

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt}$$

if  $r$  is not a root of the associated auxiliary equation, take  $s = 0$ ; if  $r$  is a simple root of the associated auxiliary equation, take  $s = 1$ ; and if  $r$  is a double root of the associated auxiliary equation, take  $s = 2$ .

To find a particular solution to the differential equation

$$ay'' + by' + cy = P_m(t)e^{\alpha t} \cos \beta t + Q_n(t)e^{\alpha t} \sin \beta t, \beta \neq 0$$

where  $P_m(t)$  is a polynomial of degree  $m$  and  $Q_n(t)$  is a polynomial of degree  $n$ , use the form  $y_p(t) = t^s(A_k t^k + \cdots + A_1 t + A_0)e^{\alpha t} \cos \beta t + t^s(B_k t^k + \cdots + B_1 t + B_0)e^{\alpha t} \sin \beta t$ , where  $k$  is the larger of  $m$  and  $n$ . If  $\alpha + i\beta$  is not a root of the associated auxiliary equation, take  $s = 0$ ; if  $\alpha + i\beta$  is a root of the associated auxiliary equation, take  $s = 1$ .

*Exercise* Write down the form of a particular solution to the equation  $y'' + 2y' + 2y = 5e^{-t} \sin t + 5t^3 e^{-t} \cos t$ .

*Exercise* Write down the form of a particular solution to the equation  $y''' + 2y'' + y' = 5e^{-t} \sin t + 3 + 7te^{-t}$ .

## 4.6 Variation of Parameters

The Method of Undetermined Coefficients is a procedure for determining a particular solution when the equation has constant coefficients and the nonhomogeneous term is of a special type.

Variation of Parameters is a more general method for finding a particular solution.

Consider a linear second-order equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

in the standard form

$$y'' + P(x)y' + Q(x)y = f(x)$$

Obtain the solution to the associated homogeneous equation

$$y = c_1 y_1(x) + c_2 y_2(x)$$

And replace the constants with functions

$$y = u_1 y_1(x) + u_2 y_2(x)$$

Substituting into the DE yields the system:

$$\begin{aligned} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1 + y_2' u_2 &= f(x) \end{aligned}$$

By Cramer's Rule, the solution can be expressed in terms of determinants-

$$u_1' = \frac{W_1}{W} = \frac{-y_2 f(x)}{W} \quad u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W}$$

The functions  $u_1$  and  $u_2$  are found by integrating.

A particular solution is  $y_p = u_1 y_1 + u_2 y_2$ .

### Example

Find a general solution on  $(-\frac{\pi}{2}, \frac{\pi}{2})$  to  $\frac{d^2 y}{dt^2} + y = \tan t$ .

The standard form is  $y'' + y = \tan t$ .

The solutions to the homogeneous equation is  $r = \pm i$ , so  $\alpha = 0$  and  $\beta = 1$ , so  $y_1 = \cos t$  and  $y_2 = \sin t$ .

Using what was previously introduced, we end up with  $y = c_1 \cos t + c_2 \sin t - \cos t \ln |\sec t + \tan t|$

*Exercise* Find a general solution on  $(-\frac{\pi}{2}, \frac{\pi}{2})$  to  $\frac{d^2 y}{dt^2} + y = \tan t + 3t - 1$ .

## 4.7 Variable-Coefficient Equations

We now consider equations with variable coefficients of the form

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$$

Typically, the equation is divided by the nonzero coefficient  $a_2(t)$  and is expressed in standard form

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)$$

### Theorem 4.6

Suppose  $p(t)$ ,  $q(t)$ , and  $g(t)$  are continuous on an interval  $(a, b)$  that contains the point  $t_0$ . The, for any choice of the initial values  $Y_0$  and  $Y_1$ , there exists a unique solution  $y(t)$  on the same interval  $(a, b)$  to the initial value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \quad y(t_0) = Y_0, \quad y'(t_0) = Y_1$$

### Example

Determine the largest interval for which the above theorem ensures the existence and uniqueness of a solution to the initial value problem

$$(t-3)y'' + y' + \sqrt{t}y = \ln t \quad y(1) = 3, \quad y'(1) = -5$$

First, let's put this into standard form.  $y'' + \frac{1}{t-3}y' + \frac{\sqrt{t}}{t-3}y = \frac{\ln t}{t-3}$ .

From this we can see that this is only continuous from  $(0, 3)$ .

### Definition

A linear second-order equation that can be expressed in the form

$$at^2y''(t) + bty'(t) + cy = f(t)$$

where  $a$ ,  $b$ , and  $c$  are constants, is called a Cauchy-Euler, or equidimensional, equation.

To solve a Cauchy-Euler equation, for  $t > 0$  look for solutions of the form

$$y = t^r$$

and substitute into the homogeneous form

$$at^2y''(t) + bty'(t) + cy = 0$$

the resulting equation  $ar^2 + (b - a)r + c = 0$  is called the associated characteristic equation.

### Example

Find two linearly independent solutions to the equation

$$3t^2y'' + 11ty' - 3y = 0 \quad t > 0$$

The solutions are  $y = t^r$  and plugging this into the equation results in  $3r^2 + 8r - 3$ , which factors to  $r = \frac{1}{3}$  and  $r = -3$ .

The solutions are  $y = t^{1/3}$  and  $y = t^{-3}$ .

If the roots of the associated characteristic equation  $r$  are equal, then independent solutions of the Cauchy-Euler equation on  $(0, \infty)$  are given by

$$y_1 = t^r \quad \text{and} \quad y_2 = t^r \ln t$$

If the roots are complex,  $r = \alpha \pm \beta i$ , then the independent solutions are given by

$$y_1 = t^\alpha \cos(\beta \ln t) \quad \text{and} \quad y_2 = t^\alpha \sin(\beta \ln t)$$

### Example

Find a pair of linearly independent solutions to the Cauchy-Euler equations for  $t > 0$ .

$$t^2y'' + 5ty' + 5y = 0$$

Answer:  $y_1 = t^{-2} \cos(\ln t)$ ,  $y_2 = t^{-2} \sin(\ln t)$

*Exercise* Do the same thing for  $t^2y'' + ty' = 0$ .

### Lemma 4.7

If  $y_1(t)$  and  $y_2(t)$  are any two solutions to the homogeneous differential equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0$$

on an interval  $I$  where the functions  $p(t)$  and  $q(t)$  are continuous and if the Wronskian is zero at any point  $t$  of  $I$ , then  $y_1$  and  $y_2$  are linearly dependent on  $I$ .

### Theorem 4.8

If  $y_1(t)$  and  $y_2(t)$  are any two solutions to the homogeneous differential equation that are linearly independent on an interval  $I$  containing  $t_0$ , then unique constants  $c_1$  and  $c_2$  can always be found so that  $c_1y_1(t) + c_2y_2(t)$  satisfies the initial conditions  $y(t_0) = Y_0$ ,  $y'(t_0) = Y_1$  for any  $Y_0$  and  $Y_1$ .

$y_h = c_1y_1 + c_2y_2$  is called a general solution to the homogeneous differential equation  $y''(t) + p(t)y'(t) + q(t)y(t) = 0$  if  $y_1$  and  $y_2$  are linearly independent solutions on  $I$ .

For the nonhomogeneous equation  $y''(t) + p(t)y'(t) + q(t)y(t) = g(t)$  a general solution on  $I$  is given by  $y = y_p + y_h$  where  $y_h = c_1y_1 + c_2y_2$  is a general solution to the corresponding homogeneous equation on  $I$  and  $y_p$  is a particular solution on  $I$ .

If linearly independent solutions to the homogeneous equation are known,  $y_p$  can be determined by the variation of parameters method.

**Theorem 4.9**

If  $y_1$  and  $y_2$  are two linearly independent solutions to the homogeneous equation on an interval  $I$  where  $p(t)$ ,  $q(t)$ , and  $g(t)$  are continuous, then a particular solution is given by  $y_p = u_1 y_1 + u_2 y_2$ , where  $u_1$  and  $u_2$  are determined up to a constant by the pair of equations,  $y_1 v_1' + y_2 v_2' = 0$ ,  $y_1' v_1' + y_2' v_2' = g$ , which have the solutions

$$v_1(t) = \int \frac{-g(t)y_2(t)}{W(y_1, y_2)(t)} dt$$

and

$$v_2(t) = \int \frac{g(t)y_1(t)}{W(y_1, y_2)(t)} dt$$

Note that the formulation above presumes that the differential equation has been put into standard form (that is divided by  $a_2(t)$ ).

**Theorem 4.10**

Let  $y_1(t)$  be a solution, not identically zero, to the homogeneous equation in an interval  $I$ . Then

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t)dt}}{y_1(t)^2} dt$$

is a second, linearly independent solution.

**Example**

Given that  $y_1(t) = t$  is a solution to

$$y'' - \frac{1}{t}y' + \frac{1}{t^2}y = 0$$

use the Reduction of Order formula to determine a second linearly independent solution for  $t > 0$ .

$y_2$  is equal to  $t \int \frac{e^{\frac{\ln t}{t^2}}}{t^2} dt = t \ln t$  from the formula, so the general solution is  $y = c_1 t + c_2 t \ln t$ .

*Exercise* The following equation arises in the mathematical modeling of reverse osmosis.

$$(\sin t)y'' - 2(\cos t)y' - (\sin t)y = 0, \quad 0 < t < \pi$$

Find a general solution.

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