AP Calculus BC Notes

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Contents

1 Limits and Continuity

1.1 Limits and Their Properties

Properties of Limits:

If L, M, c, and k are real numbers and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

1. Sum Rule: $\lim_{x \to c} (f(x) + g(x)) = L + M$
2. Difference Rule: $\lim_{x \to c} (f(x) - g(x)) = L - M$
3. Product Rule: $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$
4. Quotient Rule: $\lim_{x \to c} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$ where $M \neq 0$
5. Constant Multiple Rule: $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$
6. Power Rule: If r and s are integers, $s \neq 0$, then $\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$ provided that $L^{r/s}$ is a real number.
7. Limit of a Composite Function Rule: If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to c} f(x) = f(L)$, then $\lim_{x \to c} f(g(x)) = f(\lim_{x \to c} g(x)) = f(L)$.

Example

Given that $\lim_{x \to a} f(x) = 2$ and $\lim_{x \to a} g(x) = 3$, find the limit if it exists.

(a) $\lim_{x \to a} (5g(x)) = 15$

(b) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{2}{3}$

Two special trig limits:

$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$

Example

$\lim_{x \to 0} \frac{\sin(5x)}{4x}$

This can be written as $\frac{1}{4} \lim_{x \to 0} \frac{\sin(5x)}{x} = \frac{5}{4}$

Example

$\lim_{x \to 0} \frac{2x + \sin x}{x}$

We can split it up, $\lim_{x \to 0} \frac{2x}{x} + \lim_{x \leq 0} \frac{\sin x}{x} = 2 + 1 = 3$

Theorem 1.1: Squeeze Theorem

If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possible at c itself, and if $\lim_{x \to c} h(x) = L$ and $\lim_{x \to c} g(x) = L$, then $\lim_{x \to c} f(x) = L$.

Example

If $2 \leq f(x) \leq x^2 + 2$ for all x, find $\lim_{x \to 0} f(x)$.

The answer is 2.

1.2 Continuity

Definition: Continuity

A function f is said to be continuous at $x = c$ if and only if:

1. $f(c)$ defined
2. $\lim_{x\to c} f(x)$ exists
3. $f(c) = \lim_{x\to c} f(x)$

Essentially, no gaps or holes.

Exercise Sketch a function f so that $f(c)$ is not defined.

Exercise Sketch a function f so that $\lim_{x\to c} f(x)$ does not exist.

Exercise Sketch a function where $f(c)$ is defined and $\lim_{x\to c} f(x)$ exists but $\lim_{x\to c} f(x) \neq f(c)$

Exercise Sketch a function where f is continuous at $x = c$

If a function f is not continuous at $x = c$, the discontinuity may be one of three types:

1. point discontinuity (essentially a hole)
2. jump discontinuity (gap)
3. asymptotic discontinuity (asymptote)

A point discontinuity is said to be a removable discontinuity because the function can be redefined at the point in such as way as to make the function continuous there. Jump discontinuites and asymptotic discontinuites are non-removable because the functions which contain them cannot be redefined at a point to make them continuous.

Example

Find the value(s) of x at which the given function is discontinuous. Identify each value as a point, jump, or asymptotic discontinuity. Identify each value as a removable or non-removable discontinuity. If it is removable, redefine the function at that value so that it will be continuous.

(a) $f(x) = \frac{x^2-x-6}{x-3}$

Factoring this gives gives $x = 3$ a hole, so the function is redefined as $f(x) = x + 2$. There is a removable discontinuity in this.

(b) $f(x) = \frac{1}{x-3}$

This is asymptotic at $x = 3$.

Exercise Do the same for the example above with the function $f(x) = \begin{cases} x+2 \text{ if } x < 1 \\ 2-x \text{ if } x > 1 \end{cases}$

Example

Find k so that f will be continuous at $x = 2$ given $f(x) = \begin{cases} x+3 \text{ if } x \leq 2 \\ kx+6 \text{ if } x > 2 \end{cases}$

Since we know that $x + 3 = kx + 6$ when $x = 2$, plug in 2 for x, to get $-1/2 = k$.

1.3 Intermediate Value Theorem

Since a person's height is a continuous function of time, if a child had a height of 58 inches at age 11 and a height of 64 inches at age 14, then the child's height took on every value between 58 and 64 inches for some time between the age of 11 and the age of 14. This is a simple example of an important theorem in calculus called the Intermediate Value Theorem.

Theorem 1.2: Intermediate Value Theorem

If

1. $f(x)$ is continuous on the closed interval $[a, b]$
2. if $f(a) \neq f(b)$
3. if k is between $f(a)$ and $f(b)$

then there exists a number c between a and b for which $f(c) = k$.

In other words, a function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. If k is between $f(a)$ and $f(b)$, then there is at least one value c in (a, b) for which $f(c) = k$.

Example

Determine if the Intermediate Value Theorem holds for the given values of k. If the theorem holds, find a number c for which $f(c) = k$. If the theorem does not hold, give the reason.

$f(x) = \frac{1}{x-2}$, $[a, b] = \left[2\frac{1}{2}, 7\right], k = \frac{1}{4}$

We have $\frac{1}{x-2} = \frac{1}{4}$, so $x = 6$.

Exercise Do the same for $f(x) = x^2 + 5x - 6, [a, b] = [-1, 2], k = 4$

2 Differentiation

2.1 Derivatives

Rates of change play a role whenever we study the relationship between two changing quantities. A familiar example is velocity, which is the rate of change of position with respect to time.

If an object is traveling in a straight line, the average velocity over a given time interval from t_1 to t_2 is defined as

$$\text{Average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{\Delta s}{\Delta t} = \frac{s(t_2)-s(t_1)}{t_2-t_1}$$

For example, if a car travels 200 miles in 4 hours, then its average velocity during this 4-hour period is 50 miles per hour. At any given moment, the car may be going faster or slower than the average.

We cannot define instantaneous velocity as a ratio because we would have to divide by the length of the time interval, which is zero. However we can estimate the instantaneous velocity by computing the average velocity over successively smaller intervals and then determine the limit of the average velocity as the lengths of the time intervals get closer and closer to zero.

If we look at a graph of the position of an object, a secent line can be drawn through two arbitrary points $(t_1, s(t_1))$ and $(t_2, s(t_2))$. The average velocity of the object would be the slope of the secant line through these two points. If we move the secant line so that t_2 gets closer and closer to t_1, the secant line gets closer and closer to becoming a tangent line to the graph at $(t_1, s(t_1))$. The slope of the tangent line is the instantaneous velocity of the object when the time is t_1.

Velocity is only one of many examples of a rate of change. Our same reasoning applies to any quantity y that depends on a variable x. The average rate of change of y with respect to x over the interval x_1 to x_2 is the ratio

$$\text{Average rate of change} = \frac{\text{change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x} = \frac{y(x_2)-y(x_1)}{x_2-x_1}$$

The instantaneous rate of change at $x = x_1$ is the limit of the average rates of change as x_2 gets closer and closer to x_1. We must consider values of x_2 on both the left side of x_1 and on the right side of x_1 as we take the limit.

An expression such as $\frac{s(t_2)-s(t_1)}{t_2-t_1}$ or $\frac{f(x)-f(c)}{x-c}$ is called a difference quotient.

The instantaneous rate of change of a function is called its derivative and is defined as follows.

Definition

The derivative of $f(x)$ at $x = c$ is the limit of the difference quotients (if it exists):

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

When the limit exists, we say that f is differentiable at $x = c$. An equivalent definition of the derivative is called the alternative form of the definition of the derivative.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Example

A diver jumps from a diving board that is 32 feet above the water. The position of the diver from the water is given by $s(t) = -16t^2 + 16t + 32$, where t is measured in seconds.

(a) Find the average velocity at which the diver is moving for the time interval $[1, 1.5]$.

This is the slope of the secant line between those two values which is equal to -24 ft/sec.

(b) Estimate the instantaneous velocity by finding the average velocity

betewen the intervals $[0.9, 1], [0.99, 1], [0.999, 1], [1, 1.1], [1, 1.01], [1, 1.001]$.

We can see that between $[0.999, 1]$, the slope is -15.984, for $[1, 1.001]$ it is -16.016. It is reasonable to assume that the diver is moving -16 ft/sec at $t = 1$ second.

Example

Given $f(x) = x^2 - 4x$

(a) Graph the function and draw a tangent line at $x = 3$. Do you expect $f'(3)$ to be positive or negative?

Graph the line please. We expect $f'(3) > 0$

(b) Compute $f'(3)$ by using the definition of the derivative and the alternative form of the derivative.

Limit definition:

$$\frac{(x+h)^2 - 4(x+h) - (x^2 - 4x)}{h}$$
$$\frac{2xh + h^2 - 4h}{h}$$
$$\frac{2xh + h^2 - 4h}{h} = 2x + h - 4$$

Taking the limit of this gives $f'(x) = 2x - 4$, so $f'(3) = 2$.

The alternative form is up to the reader.

(c) Write the equation of the tangent line to the graph of $y = f(x)$ at $x = 3$. Leave your equation in point-slope form, $y - y_1 = m(x - x_1)$.

This is just $y + 3 = 2(x - 3)$

2.2 More on Derivatives

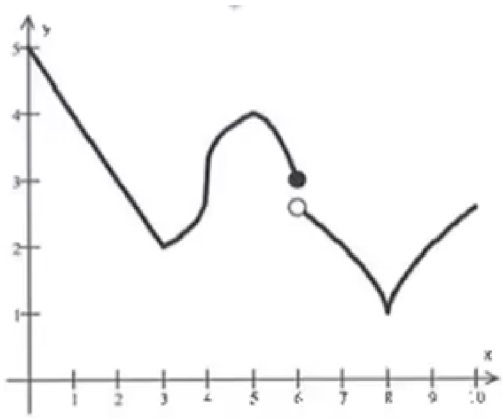
Example

List the x-values for which the function appears to be: (a) not continuous

Remember this is where there are holes or gaps are where the function will not be continuous, so $x = 6$.

(b) not differentiable

$x = 3, 4, 6, 8$, because these are sharp turns, vertical tangents, or places where the function is not continuous.



Example

At which labeled points is the slope of the graph:

(a) zero

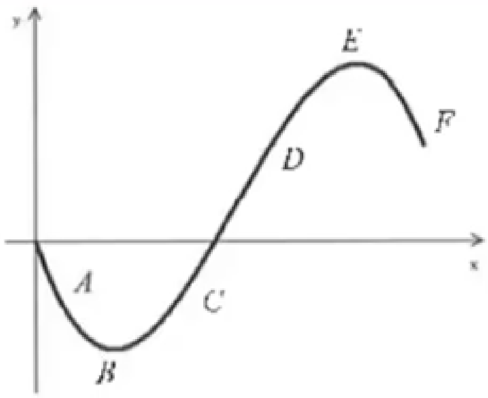
This is horizontal tangent lines. B and E.

(b) positive

C, D

(c) negative

A, F



Example

Let f be a function which satisfies $f(2+h)-f(2)=5h-3h^2+9h^3$ for all real numbers h. Find $f'(2)$.

We can find it by writing $f'(c)=\lim\limits_{h\to 0}\frac{f(c+h)-f(c)}{h}$, so $f'(2)=\lim\limits_{h\to 0}\frac{f(2+h)-f(2)}{h}$.

And we know these values, plug them in, and we get that $f'(2)=5$.

Example

Let f be a function which satisfies the property $f(x+y)=f(x)-4f(y)+9xy$ for all real numbers x and y and suppose that $\lim\limits_{h\to 0}\frac{f(h)}{h}=2$. Use the definition of the derivative to find $f'(x)$.

Using the limit definition of a derivative, we can simplify the limit to end up being $\frac{-4f(h)+9xh}{h}$, and the limit of this gives $f'(x)=-8+9x$.

Exercise Given $\lim\limits_{x\to 5}\frac{f(x)-f(5)}{x-5}=3$. Which of the following must be true, might be true, or can never be true?

1. $f'(5)=3$
2. $f'(5)=0$
3. $f(5)=3$
4. f is continuous at $x=0$
5. f is continuous at $x=5$
6. $\lim\limits_{x\to 5}f(x)=f(5)$

2.3 Derivatives Using Data

Example

Water is flowing into a tank over a 24-hour period. The amount of water in the tank is modeled by a differentiable function W for $0 \leq t \leq 24$, where t is measured in hours and $W(t)$ is measured in gallons. Values of $W(t)$ at selected values of time t are shown in the table below.

t (hours)	0	4	8	12	16	20	24
$W(t)$ (gallons)	150	184	221	257	294	327	357

(a) Use the data in the table to find $W(8)$. Using appropriate units, explain the meaning of your answer.

There are 221 gallons of water in the tank when $t = 8$.

(b) Use the data in the table to find $W^{-1}(257)$. Using appropriate units, explain the meaning of your answer.

There are 257 gallons of water in the tank when $t = 12$.

(c) Use the data in the table to find an approximation for $W'(15)$. Show the computations that lead to your answer. Using appropriate units, explain the meaning of your answer.

The secant line that contains $t = 15$ is 37/4 gallons/hr. At $t = 15$, the rate of change of the amount of water in the tank is $\approx \frac{37}{4}$ gallons/hr.

(d) Use the data in the table to find the average rate of change of $W(t)$ over the time period $4 \leq t \leq 20$ hours. Show the computations that lead to your answer.

143/16 gallons per hour

(e) For $0 < t < 24$ must there be a time t when the tank contains 265 gallons of water? Justify your answer.

IVT, yes, the function is continuous becase it's differentiable.

Exercise Continuing the following above FRQ:

- A model for the amount of water in the tank is given by $A(t) = \frac{1}{225}(-t^2 + 30t^2 + 1800t + 33750)$, where $A(t)$ is measured in gallons and t is measured in hours. Find $A'(15)$.
- Use the model given in the above question to find the average rate of change of $A(t)$ over the time period $4 \leq t \leq 20$ hours.

2.4 Basic Differentiation Rules

The power rule is $\frac{d}{dx}[x^n] = nx^{n-1}$.

Exercise Find the derivative of $f(x) = x^3$

Exercise Find the derivative of $5x^3$

Example

(a) Find the derivative of $y = \sqrt{x}$

This can be written as $x^{1/2}$, so the derivative is $\frac{1}{2}x^{-1/2}$.

(b) Find the derivative of $f(x) = 5$

This is equal to $5x^0$, the derivative is therefore 0.

Exercise Find the derivative of $g(t) = \frac{1}{t^4}$

If c is a constant, then $\frac{d}{dx}[c] = 0$

$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Exercise Find the derivative of $f(x) = 4x^3 - 5x^2 + 7x + 3$

Example

Find the derivative of $y = \frac{5x^5 - 3x^4 + 4}{x}$.

Rewrite this as $5x^4 - 3x^3 + \frac{4}{x}$ and take the derivative of this to get $20x^3 - 9x^2 - 4x^{-2}$.

Example

Determine the point(s) at which the given function has a horizontal tangent line. $f(x) = x^3 - 12x$

This is where the slope is 0, of $f'(x) = 0$.

So just plug in $0 = 3x^2 - 12$, and get the x values.

Plug those x values in to get y values, the points are $(2, -16)$ and $(-2, 16)$.

The derivative of $\sin x$ is $\cos x$, the derivative of $\cos x$ is $-\sin x$.

Exercise Find the derivative of $f(x) = 6x^2 + 5\cos x$

Example

$f(x) = x^3 - 3x^2 + 4$.

(a) Find the average rate of change of f on $[1, 5]$.

The slope, so 13.

(b) Find the instantaneous rate of change of f at $x = 3$.

The derivative, you should get 9.

Example

Find k so that the function $f(x) = x^2 + kx$ will be tangent to the line $y = 2x - 9$.

Let $x^2 + kx = 2x - 9$, so $k = 2 - 2x$.

Then we can have $x^2 + (2 - 2x)x = 2x - 9$ and $-x^2 = -9$. $x = \pm 3$, so $k = -4$ or $k = 8$.

Exercise Sketch the graph of a function f such that $f'(x) < 0$ for all x and the rate of change of the function is increasing.

Example

Find a and b so that f is differentiable everywhere.

$$f(x) = \begin{cases} ax^2 + 1, x \leq 2 \\ bx - 3, x > 2 \end{cases}$$

We have $ax^2 + 1 = bx - 3$ and when $x = 2$ we get $4a + 1 = 2b - 3$.

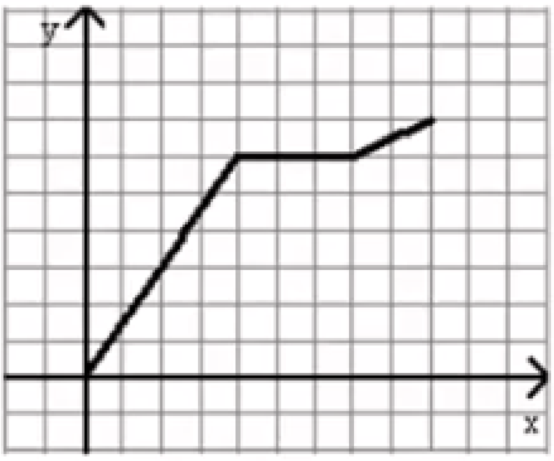
Solving for b we get $2ax$, so plugging this in we get $4a = b$. We have everything now.

Solve for a to get $a = 1$, and $b = 4$.

Exercise At time $t = 0$, a diver jumps from a diving board that is 32 ft above the water with an initial velocity of 16 ft/sec. Use the position function $s(t) = -16t^2 + v_0 t + s_0$, where v_0 is initial velocity and s_0 is initial velocity.

- (a) When the diver hit the water?
- (b) Find the instantaneous velocity when $t = 1$ seconds.
- (C) Find the average velocity on the interval $[1, 2]$.

Exercise The graph of a position function is shown. Sketch the velocity function on $(0, 9)$.



Position function

2.5 Product and Quotient Rules

Product Rule: $d[f(x) \cdot g(x)] = fg' + gf'$

Example

Find y' if $y = x^3 \cos x - 2\sin x$.

Do the product rule on the first term and then just do the derivative of the second term.

$y' = x^3(-\sin x) + \cos x(3x^2) - 2\cos x$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{gf' - fg'}{g^2}$.

Exercise Find the derivative of $y = \frac{\sin x}{x^3}$.

Example

Find the derivative of $f(x) = \frac{x^2+2}{\sqrt[4]{x}}$.

Don't do the quotient rule. The derivative is $\frac{7}{4}x^{3/4} - \frac{2}{4}x^{-5/4}$

If you have a quotient in which the numerator is a constant, it's easier to rewrite it as a function with a negative exponent and use the power rule.

Exercise Find the derivative of $y = \frac{9}{5x^2}$

You can also use the quotient rule to derive the derivative of $\tan x$: you get $\sec^2 x$.

Likewise a similar process can be used to find the derivative of $\sec x$: you get $\tan x \sec x$.

The derivative of $\cot x$ is $-\csc x$ and the derivative of $\csc x$ is $-\csc x \cot x$.

Exercise Find the derivative of $f(x) = 3x^2 \tan x$.

Exercise Given $f(x) = g(x)h(x)$. Find $f'(2)$ if $g(2) = 3, g'(2) = -4, h(2) = -1$, and $h'(2) = 5$.

2.6 Chain Rule

In algebra, you learned how to find a composition of two functions $f(x)$ and $g(x)$, which was written symbolically as $f(g(x))$.

In Calculus, we use the chain rule when we need to find the derivative of a composition of functions.

If $y = (u)^n$, then $y' = n(u)^{n-1}\frac{du}{dx}$.

Example

Find the derivative of $f(x) = (4x^3 + 3x - 2)^5$.

Chain Rule: $5(4x^3 + 3x - 2)^4(12x^2 + 3)$.

Exercise Find the derivative of $y = (x^4 + 2)^(2/3)$

Exercise Find the derivative of $y = \frac{-7}{(2x-3)^2}$

Example

Find the derivative of $g(x) = \frac{5x}{\sqrt[3]{x^2+2}}$

You have to do the quotient rule as well as the chain rule. You end up with

$\frac{(x^2+2)^{1/3}(5)-5(\frac{1}{3}(x^2+2)^{-2/3}(2x))}{((x^2+2)^{1/3})^2}$

Here are some more exercises for chain rule.

Exercise Find the derivative of $y = \cos(3x)$.

Exercise Find the derivative of $f(x) = sin(5x^2)$.

Exercise Find the derivative of $y = \csc(7x^5)$.

Example

Find the derivative of $y = \cos^4(5x)$.

This can be written as $(\cos(5x))^4$ and we can chain rule this.

We get $4(\cos(5x))^3(-\sin(5x)) \cdot 5$.

Exercise Find the derivative of $f(x) = \sec^2(4x^3 + 5)$.

Exercise Find the derivative of $y = \tan(\sin x)$

Exercise Find $h'(3)$ if $h(x) = f(g(x))$ if $f(3) = 2, g(3) = 4, f(4) = -6, f'(3) = -7, g'(3) = -5, f'(4) = 8.$

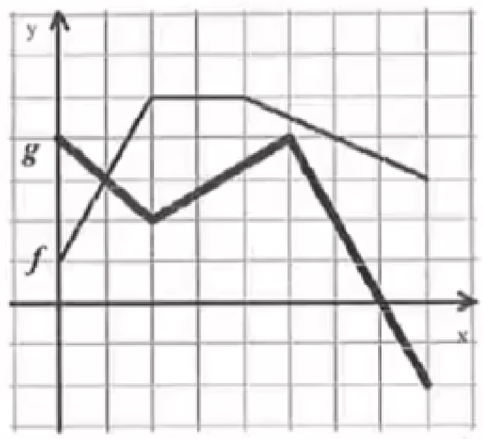
Exercise Find the second derivative of $f(x) = 4(x^2 - 5)^3$.

Exercise Find the point(s) where f has a horizontal tangent, when $f(x) = \frac{x}{\sqrt{2x-1}}$.

Example

Use the graphs of f and g to find the following, if they exist.

(a) $h(x) = f(x)g(x)$. Find $h'(1)$



$h'(1) = f(1)g'(1) + g(1)f'(1) = 3.$

(b) $k(x) = \frac{f(x)}{g(x)}$. Find $k'(6)$.

Use the quotient rule to get $9/4$.

Exercise These next two parts are a continuation of the above.

- (c) $p(x) = f(g(x))$. Find $p'(6)$.
- (d) $t(x) = g(f(x))$. Find $t'(7)$.

2.7 Implicit Differentiation

Examples of explicitly defined functions are

- $y = x^2 - 5x + 7$
- $f(x) = \sqrt{x^2 + 5}$

Examples of implicitly defined functions:

- $x^2 + xy - y^2 = 3$
- $\cos x \sin y = \frac{\sqrt{3}}{4}$
- $y^3 + y^2 - x^2 = -4$

Example

Given $y^3 + y^2 - x^2 = -4$

(a) Differentiate with respect to t. Since you must apply the Chain Rule, each derivative will a d/dt as part of the derivative.

This will end up being $3y^2 \frac{dy}{dt} + 2y \frac{dy}{dt} - 2x \frac{dx}{dt} = 0$

(b) Differentiate with respect to w.

The exact same, but with w on the bottom. $ey^2 \frac{dy}{dw} + 2y \frac{dy}{dw} - 2x \frac{dx}{dw} = 0$

(c) Now differentiate with respect to x

You get $3y^2 \frac{dy}{dx} + 2y \frac{dy}{dx} - 2x = 0$.

You can see above you can find $\frac{dy}{dx}$ easily from this, it is $\frac{2x}{3y^2+2y}$.

Exercise Find the derivative of $x^2 + xy - y^2 = 3$

Exercise Find the derivative of $\cos x \sin y = \frac{\sqrt{3}}{4}$

Example

Consider the curve given by $y^3 + y^2 - 5y - x^2 = -4$.

(a) Find $\frac{dy}{dx}$

You should get $y' = \frac{2x}{3y^2+2y-5}$

(b) Write the equation of the tangent line to the curve at the point $(1, -3)$.

Plug in what you have to get $y + 3 = \frac{1}{8}(x - 1)$.

(c) Find the coordinates of the point(s) on the curve where the line tangent to the curve is vertical.

The Bottom has to equal 0 of the derivative, and the top cannot be 0.

$3y^2 + 2y - 5$ will give you points for y values. Find your points from this.

3 Applications of Differentiation

3.1 Definition of the Derivative Meets Derivative Rules

Example

Evaluate the following by recognizing that the given limit represents a derivative.

$$\lim_{h \to 0} \frac{2(x+h)^3 - 2x^3}{h}$$

We know $f(x) = 2x^3$, so the derivative is trivial from this.

Exercise Evaluate the following by recognizing the given limit represents a derivative.

$\lim\limits_{h \to 0} \frac{\cos(5(x+h)) - \cos(5x)}{h}$

Exercise Evaluate the following by recognizing the given limit represents a derivative.

$\lim\limits_{h \to 0} \frac{\sin\left(\frac{\pi}{6}+h\right) - \frac{1}{2}}{h}$

3.2 Related Rates

We have previously learned the Chain Rule and this allows us to use implicit differentiation for related rates.

Example

Suppose $y = 5x^2 - 6x + 2$. Find $\frac{dy}{dt}$ when $x = 4$, given that $\frac{dx}{dt} = 2$ when $x = 4$.

We are taking the derivative of y with respect to t.

We get $\frac{dy}{dt}(y = 5x^2 - 6x + 2) = \frac{dy}{dt} = 10x\frac{dx}{dt} - 6\frac{dx}{dt}$.

Plug this in to get 68 as the answer.

Example

A pebble is dropped into a calm pond, causing ripples in the shape of concentric circles. The radius of the outer ripple is increasing at a constant rate of 1 ft/sec. When the radius is 4 ft, find the rate at which the area of the disturbed water is changing.

We have to do the derivative of $A = \pi r^2$, the circle formula.

This is $2\pi r\frac{dr}{dt}$. Plug in numbers to get 8π ft^2/sec

Exercise Water runs out of a conical tank at the constant rate of 2 cubic feet per minute. The radius at the top of the tank is 5 feet, and the height of the tank is 10 feet. How fast is the water level sinking when the water is 4 feet deep?

Example

A fish is reeled in at a rate of 2 ft/sec from a bridge 16 ft above the water. At what rate is the angle between the line and the water changing when there are 20 ft of line out?

If we let x be the distance from the fish to the person, then we know $\frac{dx}{dt} = -2$.

We are trying to find $\frac{d\theta}{dt}$.

Using trig, we can find $\tan\theta = 16x^{-1}$.

The derivative gives $\sec^2\theta\frac{d\theta}{dt} = -16x^{-2}\frac{dx}{dt}$.

Plugging in numbers and solving for $\frac{d\theta}{dt} = \frac{2}{25}$ rad/s.

Exercise A man 6 ft tall walks at a rate of 5 ft/sec away from a lightpole 16 ft tall.

(a) At what rate is the tip of his shadow moving when he is 10 ft from the base of the light?

(b) At what rate is the length of his shadow moving when he is 10 ft from the base of the light?

Exercise A trough is 10 ft long and 6 ft across the top. Its ends are isosceles triangles with an altitude of 4 ft. If water is being pumped into the trough at 9 ft^3/sec, how fast is the water level rising when the water is 2 ft deep?

3.3 Extrema on an Interval

Definition: Definition of Extrema

Let f be defined on an interval I containing c.

1. $f(c)$ is the minimum of f on I if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the maximum of f on I if $f(c) \geq f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the extreme values or extrema of the function on the interval. The minimum and maximum of a function on an interval are also called the absolute minimum and absolute maximum, or the global minimum and global maximum, on the interval.

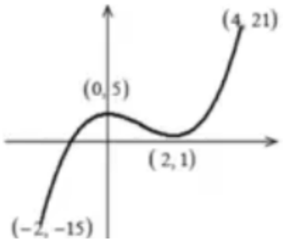
Definition: Relative Extrema

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $(c, f(c))$ is called a relative maximum of f, or you can say that f has a relative maximum at $(c, f(c))$.
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $(c, f(c))$ is called a relative minimum on f, or you can say that f has a relative minimum at $(c, f(c))$.

The relative maximum and relative minimum points are sometimes called local maximum and local minimum points, respectively.

Example

In the figure, wind where f has an absolute maximum, absolute minimum, relative maximum, and relative minimum on the interval $[-2, 4]$.



Absolute maximum at 4, absolute minimum at -2, relative maximum at 0, relative minimum at 2.

Definition: Critical Number and Critical Point

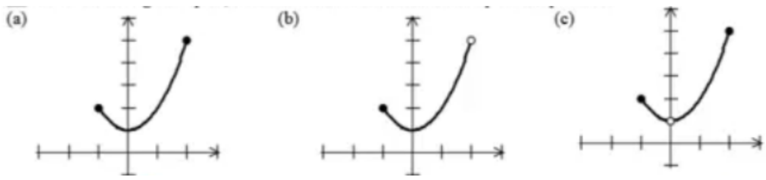
Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a critical number of f and the point $(c, f(c))$ is a critical opint of f.

Theorem 3.1

Relative extrama occur only at critical numbers.

Example

In the following, name the maximum and minimum points.



(a) Minimum at 0, maximum at 2

(b) Minimum at 0, no maximum

(c) no minimum, maximum at 2

Continuity is needed to guarantee a maximum and minimum.

Theorem 3.2: Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ or an absolute minimum value $f(d)$ for some numbers c and d in $[a, b]$.

Guidelines for Finding Extrema on a Closed Interval - Candidates Test

To find the extrema of a continuous function f on a closed interval $[a, b]$, we used the following steps:

1. Find $f'(x)$ and the critical numbers of f in $[a, b]$.
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint in $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

Example

Find the absolute maximums and minimums of f on the given closed interval, and state where these values occur.

(a) $f(x) = 3x^2 - 24x - 1 \qquad [-1, 5]$

$f'(x) = 0$ when $x = 4$.

f has an absolute maximum of 26 at $x = -1$ and f has an absolute minimum of -49 at $x = 4$.

(b) $f(x) = 6x^3 - 6x^4 + 5 \qquad [-1, 2]$.

$f'(x) = 18x^2 - 24x^3 = 0$.

$x = 0$ and $x = 3/4$.

f has an absolute maximum of 5.6328 at $x = 3/4$ and an absolute minimum of -43 at $x = 2$.

Exercise Same as above for $f(x) = 3x^{2/3} - 2x + 1 \qquad [-1, 8]$

Exercise Same as above for $f(x) = \sin^2 x + \cos x \qquad [0, 2\pi]$

3.4 Mean Value Theorem and Rolle's Theorem

Theorem 3.3: Mean Value Theorem

If a function f is:

1. continuous on $[a, b]$ and
2. differentiable on (a, b)

then there is at least one number c in (a, b) such that

1. $\frac{f(b)-f(a)}{b-a} = f'(c)$
2. $f(b) - f(a) = f'(c)(b - a)$
3. $f(b) = f(a) + f'(c)(b - a)$

If $f(a)$ and $f(b)$ are equal, this special case of the Mean Value Theorem is called Rolle's Theorem.

Theorem 3.4: Rolle's Theorem

If a function f is

1. continuous on $[a, b]$ and
2. differentiable on (a, b) and
3. $f(a) = f(b)$

then there is at least one number c in (a, b) such that

1. $\frac{f(b)-f(a)}{b-a} = f'(c)$
2. $\frac{0}{b-a} = f'(c)$
3. $0 = f'(c)$

Example

Show that the conditions of Rolle's Theorem are met, and find the c that Rolle's Theorem guarantees.

$$f(x) = x^2 - x - 6 \quad [-2, 3]$$

This is a continuous and differentiable function.

$f(3) = 0$ and $f(-2) = 0$, so $f'(x) = 2x - 1 = 0$, so $x = 1/2$.

Exercise Given the function $f(x) = 3 - \frac{5}{x}$ on the interval $[1, 5]$, show that the Mean Value Theorem applies, and find the c that the theorem guarantees.

Example

You are driving a car traveling on an interstate at 50 mph, and you pass a police car. Four minutes later, you pass a second police car, and you are again traveling at 50 mph. The speed limit on the interstate is 60 mph. The distance between the two police cars is five miles. The patrolman in the second police car gives you a speeding ticket for driving 75 mph. How can he prove that you were speeding?

$r = \frac{d}{t} = 75$ mph. Since speed is a differnetiable function of time and averaged 75 mph, the MVT says there is at least one time in those 4 minutes where the instantaneous rate of change is 75 mph.

3.5 Increasing and Decreasing Functions and the First Derivative Test

Definition

1. A function f is increasing on an interval if for any two numbers

   x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
2. A function f is decreasing on an interval if for any two numbers

   x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$

Test for Increasing and Decreasing Functions

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

First Derivative Test:

Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $(c, f(c))$ can be classified as follows.

1. If $f'(x)$ changes from negative to positive at $x = c$, then $(c, f(c))$ is a relative minimum of f.
2. If $f'(x)$ changes from positive to negative at $x = c$, then $(c, f(c))$ is a relative maximum of f.

Example

Find the intervals where f is increasing and decreasing, identify all points that are relative maximum and minimum points, and justify your answers.

$$f(x) = \frac{1}{3}x^3 - x^2 - 3x + 2$$

$f'(x) = x^2 - 2x - 3$ or $(x-3)(x+1) = 0$.

We can try points, $f'(-2) = 5$, $f'(0) = -3$, $f'(4) = 5$.

f is increasing on $(-\infty] \cup [3, \infty)$ because f' is positive and decreasing on $[-1, 3]$ because f' is negative.

f has a relative maximum at -1 because f' changes from positive to negative.

f has a relative minimum at $x = 3$ because f' changes from negative to positive.

Exercise Do the same as above for the function $f(x) = (x^2 - 4)^{2/3}$

Exercise Do the same above for the function $f(x) = \frac{x^2}{2x-1}$

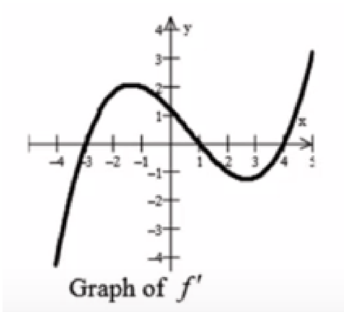
Exercise Do the same above for the function $f(x) = \begin{cases} 2x+1, & x \leq -1 \\ x^2 - 2, & x > -1 \end{cases}$

Example

Use the graph of f' to:

(a) identify the interval(s) on which the graph of f is increasing and decreasing

(b) estimate the value(s) of x at which the graph of f has a relative maximum or minimum. Justify your answers.



We can see critial points are $-3, 1, 4$ when they cross the x-axis.

Looking at the graph, $(-3, 1) \cup (4, \infty)$ f is increasing on this interval because $f' > 0$ and f has a relative maximum at $x = 1$ because f' goes positive to negative.

From the interval $(-\infty, -3) \cup (1, 4)$ f is decreasing because $f' < 0$ and f has a relative minimum at -3 and 4 because f' goes negative to positive.

Exercise Given $f'(x) < 0$ on $(-\infty, 3)$, $f'(x) > 0$ on $(-3, 2)$, $f'(x) < 0$ on $(2, \infty)$.

(a) If $g(x) = 2f(x)$, what is the sign of $g'(-1)$?

(b) If $g(x) = f(3x - 5)$, what is the sign of $g'(-1)$?

Example

The function $s(t) = t^2 - 7t + 10$ describes the motion of a particle along a line.

(a) Find the velocity function of the particle at any time $t \geq 0$.

$v(t) = 2t - 7$

(b) Identify the time interval(s) in which the particle is moving in a positive direction. Justify.

$(7/2, \infty)$ because $v(t) = s'(t) > 0$

(c) Identify the interval(s) in which the particle is moving in a negative direction. Justify.

$(-\infty, 7/2)$ because $v(t) = s'(t) < 0$

(d) Identify the time(s) at which the particle changes direction. Justify.

$t = 7/2$ because $v(t)$ changes from negative to positive.

3.6 Concavity and the Second Derivative

Definition: Concavity

Let f be differentiable on an open interval I.

1. The graph of f is concave upward on an interval I if f' is increasing on the interval.
2. The graph of f is concave downward on an interval I if f' is decreasing on the interval.

Test for Concavity

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave up in I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave down in I.

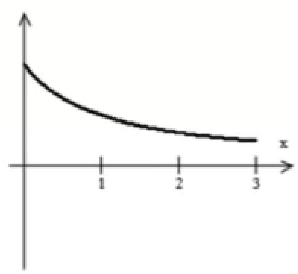
Definition: Inflection Point

A function f has an inflection point at $(c, f(c))$

1. if $f'' = 0$ or if f'' does not exist and
2. if f'' changes sign from negative to positive or positive to negative at $x = c$ OR if $f'(x)$ changes from increasing to decreasing or decreasing to increasing at $x = c$.

Example

The graph of f is shown below. State the signs of f' and f'' on the interval $(0, 3)$.



f is decreasing the whole time, so $f' < 0$, but f' is increasing so $f'' > 0$.

Example

Given the function $f(x) = x^4 - 4x^3$ find:

(a) the intervals where f is increasing and decreasing

f is decreasing on $(-\infty, 0) \cup (0, 3)$ because $f' < 0$.

f is increasing on $(3, \infty)$ because $f' > 0$.

(b) the relative extrema

f has a relative minimum at $x = 3$ because f' changes negative to positive.

(c) the intervals where f is concave up and concave down

Concave up: $(-\infty, 0) \cup (2, \infty)$.

Concave down: $(0, 2)$

(d) the inflection points

$x = 0$ and $x = 2$

Exercise Sketch a graph for the following characteristics

(a)

- $f(0) = f(2) = 0$
- $f'(x) > 0$ if $x < 1$
- $f'(1) = 0$
- $f'(x) < 0$ if $x > 1$
- $f''(x) < 0$

(b)

- $f(0) = f(2) = 0$
- $f'(x) > 0$ if $x < 1$
- $f'(1)$ is undefined
- $f'(x) < 0$ if $x > 1$
- $f''(x) > 0$

3.7 Second Derivative Test

Second Derivative Test:

Let f be a function such that the second derivative of f exists on an open interval containing c.

1. If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a relative minimum of f.
2. If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a relative maximum of f.

Example

Use the Second Derivative Test, if possible, to find the relative extrema of $f(x) = -3x^5 + 5x^3$. Justify your answer.

The first derivative is $-15x^4 + 15x^2$. We can factor this to get $x = 0, \pm 1$.

$f''(x) = -60x^3 + 30x$, $f''(1) = -30 < 0$, $f''(-1) = 30 > 0$ and $f''(0) = 0$.

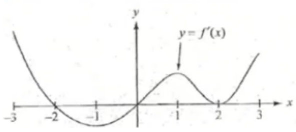
So we can see there is a relative maximum at $x = 1$ because $f' = 0$ and $f'' < 0$.

There is a relative minimum at $x = -1$ because $f' = 0$ and $f'' > 0$.

Exercise Suppose that the function f has a continuous second derivative for all x, and that $f(3) = -4, f'(3) = 1, f''(3) = -2$. Let g be a function whose derivative is given by $g'(x) = (x^2 - 9)(2f(x) + 5f'(x))$ for all x. Does g have a local maximum or a local minimum at $x = 3$. Justify your answer.

3.8 Graphs of f, f', and f''

Example



The graph of $y = f'(x)$ is given above. The domain of f is $[-3, 3]$.

(a) For what value(s) of $x, -3 < x < 3$, is the graph of f increasing? Justify your answer.

$(-3, -2) \cup (0, 2) \cup (2, 3)$ because $f' > 0$.

(b) For what value(s) of $x, -3 < x < 3$, does the graph of f have a relative maximum? Justify your answer.

f has a relative maximum at $x = -2$ because f' goes from positive to negative.

(c) For what value(s) of $x, -3 < x < 3$, is the graph of f concave down? Justify your answer.

f is concave down on $(-3, -1) \cup (1, 2)$ because f is decreasing.

(d) For what value(s) of $x, -3 < x < 3$, does the graph of f have an inflection point? Justify your answer.

f has a point of inflection at $x = -1, 1, 2$ because f' changes from decreasing to increasing at this point.

Exercise Using the information above, and the fact that $f(-3) = 0$ sketch a possible graph for f.

4 Particle Motion and Integration

4.1 Particle Motion

Example

A particle moves along a horizontal line so that its position at any time $t \geq 0$ is given by

$$s(t) = 2t^3 - 7t^2 + 4t + 5$$

where s is measured in meters and t in seconds.

(a) Find the velocity at time t and at $t = 1$ second.

$v(t) = s'(t) = 6t^2 - 14t + 4$, so $v(1) = -4$ m/s

(b) When is the particle at rest? Moving left? Moving right? Justify your answers.

We are looking for $v(t) = 0$, $v(t) < 0$ and $v(t) > 0$.

We know that $v(t) = 0$ at $t = 1/3$ and $t = 2$.

It is moving left at $(1/3, 2)$ because $v(t) < 0$ and right from $(0, 1/3) \cup (2, \infty)$ because $v(t) > 0$.

(c) Find the acceleration at time t and at $t = 1$ seconds.

$a(t) = v'(t) = s''(t) = 12t - 14$, so $a(1) = -2$ m/s^2

(d) Find the displacement of the particle between $t = 0$ and $t = 3$ seconds. Explain the meaning of your answer.

$s(3) - s(0) = 3$. Displacement was 3 meters to the right.

(e) Find the distance traveled by the particle between $t = 0$ and $t = 3$ seconds.

From 0 to $1/3$ it travels 0.6296 meters, from $1/3$ to 2 it travels an absolute value of 4.6296 meters, and from 2 to 3, it travels 7 meters, so adding these values gives us 12.2592 meters.

(f) When is the particle speeding up? Slowing down? Justify your answer.

Hint: Since speed is the absolute value of velocity, the particle is

1. Speeding up when the velocity and acceleration have the same signs.
2. Slowing down when the velocity and acceleration have opposite signs.

Speeding up at $(1/3, 7/6) \cup (2, \infty)$ because $v(t)$ and $a(t)$ have the same sign.

Slowing down $(0, 1/3) \cup (7/6, 2)$ because $v(t)$ and $a(t)$ have different signs.

Example

A particle moves along a horizontal line so that its position at any time $t \geq 0$ is given by

$$s(t) = t^3 - 5t^2 + 2t - 3$$

(a) When is the particle moving right? moving left? Justify your answer.

This is when $v(t) > 0$, or above the x-axis for moving right.

For moving left, it is $v(t) < 0$ or below the x-axis.

$v(t) = s'(t) = 3t^2 - 10t + 2.$

Using a calculator, we see that it is moving right from $(0, .2137) \cup (3.11963, \infty)$ because $v(t) > 0$.

It is moving left $(.2137, 3.11963)$ because $v(t) < 0$.

(b) Find the distance traveled by the particle from $t = 0$ to $t = 5$. Justify your answer.

We have 4 values of t, $0, .2317, 3.11963, 5$. We can find the $s(t)$ values for all of these and adding up the absolute value of all these gives a value 34.638.

(c) Find the intervals where the speed is increasing. Justify your answer.

This is when $v(t)$ and $a(t)$ are the same sign.

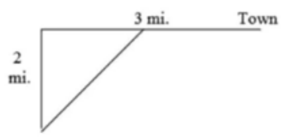
This is in the interval $(.2137, 5/3) \cup (3.11963, \infty)$ because $v(t)$ and $a(t)$ have the same sign.

Exercise A particle moves along the x-axis so that at any time $t \geq 0$, its velocity is given by $v(t) = 3 + 4.1\cos(0.9t)$. What is the acceleration of the particle at time $t = 4$?

4.2 Optimization

Example

A swimmer is 2 miles in the ocean and wishes to get to a town 3 miles down the coast which is very rocky. The swimmer needs to swim to the shore and then walk along the shore. He can swim at 2 mph and walk at 4 mph. To what point should he swim along the shoreline so that the time it takes to get to town is a minimum?



We can use the formula $d = rt$ or $t = \frac{d}{r}$ to find this.

The time is $\frac{\sqrt{x^2+4}}{2} + \frac{3-x}{4}$ and we are trying to find when $T' = 0$. The x value that this corresponds to is $x = 1.1547$.

Exercise Find the dimensions of a 12-oz. can that can be constructed with the least amount of metal. Justify your answer.

4.3 Antiderivatives and Indefinite Integrals

If you were given $f'(x) = 3x^2$ and asked what function $f(x)$ had this derivative, what would you say?

You would say $f(x) = x^3$ or $f(x) = x^3 - 7$ or any other constant.

$f(x)$ is called the antiderivative of $f'(x)$.

The symbol $\int g(x)$ is the indefinite integral. The term indefinite integral is a synonym for antiderivative.

We can get formulas for antiderivatives by reversing the differentiation rules.

Integration Rules:

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
- $\int \cos u du = \sin u + C$
- $\int \sin u du = -\cos u + C$
- $\int \sec^2 u du = \tan u + C$
- $\int \csc^2 u du = -\cot u + C$
- $\int \sec u \tan u du = \sec u + C$
- $\int \csc u \cot u du = -\csc u + C$

Properties of Indefinite Integrals:

- $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$
- $\int kf(x) = k \int f(x)dx$, where k is a constant

Note that $\int f(x) \cdot g(x)dx \neq \int f(x)dx \cdot \int g(x)dx$ and $\int \frac{f(x)}{g(x)} \neq \frac{\int f(x)dx}{\int g(x)dx}$

Example

$$\int (3x^2 - 5x + 4)dx =$$

Use the integration rules to get $\frac{3x^3}{3} - \frac{5x^2}{2} + 4x + C$ or $x^3 - \frac{5}{2}x^2 + 4x + C$.

There isn't a product rule of a quotient rule for antiderivatives so you must simplify first.

Example

$$\int (2x - 1)(x + 3)dx =$$

Simplifying gives $\int 2x^2 + 6x - x - 3dx = \int 2x^2 + 5x - 3dx$.

This is equal to $\frac{2x^3}{3} + \frac{5x^2}{2} - 3x + C$.

Exercise $\int \frac{x^2-2x+7}{\sqrt{x}} dx =$

Example

Solve the differential equation $f'(x) = 6x^2, f(1) = -3$.

When we integrate we get $f(x) = \frac{6x^3}{3} + C$.

$f(x) = 2x^3 + C$, and we have the condition $f(1) = -3$. so we can plug this in to get $-3 = 2(1)^3 + C$ which gives $C = -5$.

Therefore $f(x) = 2x^3 - 5$.

Exercise Solve the differential equation $f''(x) = \cos x, f'(0) = 3, f(0) = -2$.

Exercise A particle moves along the x-axis at a velocity of $v(t) = 4t^3 - 3t^2 + 5, t \geq 0$. At time $t = 2$, its position is $x = 3$. Find the acceleration function and find the position function.

4.4 Integration using U-Substitution

When we differentiated composite functions, we used the Chain Rule. The reverse process is called u-substitution.

Example

$$\int (x^2+1)^5(2x)dx =$$

If we let $u = x^2+1$, we can see that $\frac{du}{dx} = 2x$ which implies that $du = 2xdx$.

We can rewrite this function now as $\int u^5 du = \frac{u^6}{6} + C$.

Therefore, the integral comes out to $\frac{(x^2+1)^6}{6} + C$.

Exercise $\int x^2(2x^3+5)^4 dx =$

Example

$$\int x\sqrt{x^2+3}dx =$$

We let $u = x^2+3$ and $\frac{du}{dx} = 2x$. This gives $\frac{1}{2}du = xdx$.

So we can write the integral as $\frac{1}{2}\int u^{1/2}du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$.

The result is $\frac{1}{3}(x^2+3)^{3/2} + C$.

Exercise $\int \frac{x}{\sqrt{3x^2+4}}dx =$

Exercise $\int \cos(3x)dx =$

Exercise Solve the differential equation $\frac{dy}{dx} = \frac{9x^2}{\sqrt{1+x^3}} + 5x$

4.5 Integration and Area Under a Curve

Example

A car is traveling so that its speed is never decreasing during a 12-second interval. The speed at various moments in time is listed in the table below.

Time in Seconds	0	3	6	9	12
Speed in ft/sec	30	37	45	54	65

(a) Estimate the distance traveled by the car during the 12 seconds by finding the areas of four rectangles drawn at the heights of the left endpoints. This is called a left Riemann sum.

If you draw a rough sketch of the graph and use the rectangle heights, you can get four rectangles and find the areas.

You should get $3(30) + 3(37) + 3(45) + 3(54)$

(b) Estimate the distance traveled by the car during the 12 seconds by finding the areas of four rectangles drawn at the heights of the right endpoints. This is called a right Riemann sum.

Similar process: $3(37) + 3(45) + 3(54) + 3(65)$

(c) Estimate the distance traveled by the car during the 12 seconds by finding the areas of two rectangles drawn at the heights of the midpoints. This is called a midpoint Riemann sum.

You get $6(37) + 6(54)$.

Exercise Given the function $y = x^2 + 1$, estimate the area bounded by the graph of the curve and the x-axis on $[0, 2]$ by using:

(a) a left Riemann sum with $n = 4$ equal subintervals.

(b) a right Riemann sum with $n = 4$ equal subintervals

(c) a midpoint Riemann sum with $n = 4$ equal subintervals

4.6 Riemann Sums and the Definite Integral

To estimate the area bounded by the graph of $f(x)$ and the x-axis between the vertical lines $x = a$ and $x = b$, partition the area and divide it into subintervals. We previously drew rectangles with the height at the left endpoint or the right endpoint or at the midpoint of the interval. For this, we will draw rectangles at some general point within the subinterval, not necessarily at the left endpoint or the right endpoint or at the midpoint of the interval.

Let $x = c_k$ be any point in the kth subinterval. Draw a rectangle with a height of $f(c_k)$.

The area of this rectangle is $f(c_k)\Delta x$.

The sum of all the rectangles is $\sum\limits_{k=1}^{n} f(c_k)(\Delta x_k)$.

This sum is called a Riemann sum.

To find the exact area under the curve we use

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)(\Delta x_k)$$

Definition: Definite Integral

Defines as

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} (f(c_k))(\Delta x_k)$$

or

$$\int_a^b f(x)dx = \lim_{\Delta x_k\to 0} \sum_{k=1}^{n} (f(c_k))(\Delta x_k)$$

The area bounded by $y = f(x)$ and the x-axis on $[a, b] = \int_a^b f(x)dx$.

Properties of Definite Integrals

- $\int_a^a f(x)dx = 0$
- $\int_b^a f(x)dx = -\int_a^b f(x)dx$
- $\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx$
- If c lies between a and b, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
- $\int_a^b (f(x) - g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

Note: $\int_a^b (f(x) \cdot g(x))dx \neq \int_a^b f(x)dx \cdot \int_a^b g(x)dx$ and $\int_a^b \left(\frac{f(x)}{g(x)}\right) dx \neq \frac{\int_a^b f(x)dx}{\int_a^b g(x)dx}$

Example

Evaluate by using a geometric formula.

$$\int_1^6 4dx$$

If we draw a rectangle of this, we get the area of the rectangle as 20.

Example

Evaluate by using a geometric formula.

$$\int_1^3 (x+2)dx$$

This is a triangle, the area is 8.

Exercise $\int_{-2}^{2} \sqrt{4-x^2}dx =$

Exercise $\int_{-4}^{4} f(x)dx =$

Example

Given $\int_0^3 f(x)dx = 4$ and $\int_3^7 f(x)dx = -$. Find:

(a) $\int_0^7 f(x)dx$

Answer is 3.

(b) $\int_3^7 2f(x)dx$

This is $2\int_3^7 f(x)dx = -2$

(c) $\int_5^5 f(x)dx =$.

0

4.7 Fundamental Theorem of Calculus

Theorem 4.1: Fundamental Theorem of Calculus

$$\int_a^b f'(x)dx = [f(x)]_{x=a}^{x=b} = f(b) - f(a)$$

Example

$$\int_1^2 (x^2+3)dx =$$

We are just doing

$$\left[\frac{x^3}{3} + 3x\right]_1^2 = \left(\frac{8}{3}+6\right) - \left(\frac{1}{3}+3\right) = \frac{16}{3}$$

Exercise $\int_1^9 \frac{x-4}{\sqrt{x}}dx =$

Exercise $\int_0^{\pi/4} \sec^2 x dx =$

Exercise $\int_0^3 |2x-1|dx =$. Note, $|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0 \end{cases}$

Example

Find the area bounded by the graph of $y = 2x^2 - 3x + 2$, the x-axis, and the vertical lines $x = 0$ and $x = 2$.

The integral is $\int_0^2 2x^2 - 3x + 2dx$.

If we integrate this and use the bounds, we get the answer of $\left(\frac{16}{3} - \frac{12}{2} + 4\right) - (0)$.

4.8 U-Substitution with Definite Integrals

Example

$$\int_1^2 x(x^2+1)^3 dx =$$

We let $u = x^2+1$, and $\frac{du}{dx} = 2x$.

We also have $\frac{1}{2}du = xdx$.

When $x = 1$ then $u = 2$ and when $x = 2$ then $u = 5$.

So we know integrate $\frac{1}{2}\int_2^5 u^3 du$ to get the result $\frac{1}{2}\left[\frac{5^4}{4} - \frac{2^4}{4}\right]$.

Exercise $\int_0^2 \frac{x}{\sqrt{1+2x^2}}dx =$

Exercise $\int_{\pi/12}^{\pi/9} \sin(3x)dx =$

Exercise Find the area bounded by the graph of $y = x\sqrt{x^2+1}$ and the x-axis on the interval $[0, 2]$.

Example

Water is being pumped into a tank at a rate given by $R(t)$. A table of values of $R(t)$ is given.

t (min.)	0	5	9	15	20
$R(t)$ (gal/min)	14	18	20	27	32

(a) Use the data from the table and four subintervals to find a left Riemann sum to approximate $\int_0^{20} R(t)dt$.

$$14(5) + 18(4) + 20(6) + 27(5)$$

(b) Use data from the table and four subintervals to find a right Riemann sum to approximate $\int_0^{20} R(t)dt$.

$$18(5) + 20(4) + 27(6) + 32(5)$$

4.9 Another Kind of U-Substitution

Sometimes the u-substitution method we have learned does not work, and we need to do something different to integrate.

Example

$$\int x\sqrt{2x-1}dx =$$

We can let $u = 2x-1$ and $\frac{du}{dx} = 2$. We also see that $\frac{u+1}{2} = x$.

So we can write the integral as $\frac{1}{2}\int \frac{u+1}{2} \cdot u^{1/2}$.

The result is $\frac{1}{4}\left(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C$.

Exercise $\int_1^5 \frac{x}{\sqrt{2x-1}}dx =$

Exercise $\int_0^7 \frac{x}{\sqrt[3]{x+1}}dx =$

If a function f is even, then f has y-axis symmetry so $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$.

If a function f is odd, then f has origin symmetry so $\int_{-a}^{a} f(x)dx = 0$.

Example

Given $f(x)$ is even and $\int_0^5 f(x)dx = 3$. Find:

(a) $\int_{-5}^{5} f(x)dx$

$2(3) = 6$

(b) $\int_{-5}^{0} f(x)dx$

3

(c) $\int_{-5}^{5} 4f(x)dx$

$4(6) = 24.$

Exercise Given $f(x)$ is odd and $\int_0^5 f(x)dx = 3$. Find:

(a) $\int_{-5}^{5} f(x)dx$

(b) $\int_{-5}^{0} f(x)dx$

(c)$\int_{-5}^{5} 4f(x)dx$

4.10 Fundamental Theorem of Calculus

Example

Given $\frac{dy}{dx} = 3x^2 + 4x - 5$ with the initial condition $y(2) = -1$. Find $y(3)$.

Method 1: Integrate $y = \int(3x^2 + 4x - 5)dx$ and use the initial condition to find C. Then write the particular solution, and use your particular solution to find $y(3)$.

$y = x^3 + 2x^2 - 5x + C$. Plugging in the initial conditions gives $C = -7$.

Therefore $y = x^3 + 2x^2 - 5x - 7$, so $y(3) = 23$.

Method 2: Use the Fundamental Theorem of Calculus: $\int_a^b f'(X)dx = f(b) - f(a)$

This gives $\int_2^3 f'(x)dx = f(3) - f(2) \implies f(3) = -1 + \int_2^3 f'(x)dx$.

So $= 1 + [x^3 + 2x^2 - 5x]_2^3 = f(3) = 23$.

Sometimes there is no antiderivative, so you must use Method 2 and a graphing calculator.

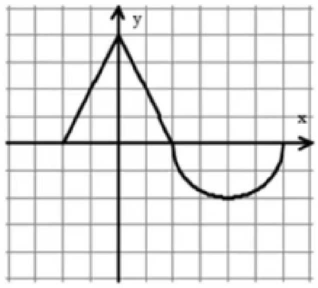
Example

$f'(x) = \sin(x^2)$ and $f(1) = -5$. Find $f(2)$.

$\int_1^2 \sin(x^2)dx = f(2) - f(1)$.

$f(2) = -5 + \int_1^2 \sin(x^2)dx = -4.50549$.

Example



Graph of f'

The graph of f' consists of two line segments and a semicircle as shown above. Given that $f(-2) = 5$, find

(a) $f(0)$

We are finding $\int_{-2}^{0} f'(x)dx = f(0) - f(-2)$. So $f(0) = 5 + \int_{-2}^{0} f'(x)dx = 9$.

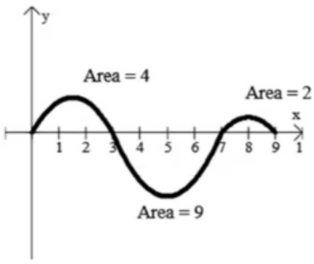
(b) $f(2)$

$f(2) = 5 + \int_{-2}^{2} f'(x)dx = 13$.

(c) $f(6)$

$f(6) = 5 + \int_{-2}^{6} f'(x)dx = 13 - 2\pi$.

Exercise The graph of f' is shown. Use the figure and the fact that $f(3) = 5$ to find



(a) $f(0)$

(b) $f(7)$

(c) $f(9)$

Exercise A pizza with a temperature of $95°$C is put into a $25°$C room when $t = 0$. The pizza's temperature is decreasing at a rate of $r(t) = 6e^{-0.1t}°$C per minute. Estimate the pizza's temperature when $t = 5$ minutes.

Example

If $f(3) = 5$, f' is continuous, and $\int_{3}^{8} f'(x)dx = 20$, find the value of $f(8)$.

$\int_{3}^{8} f'(x)dx = f(8) - f(3)$. We know $20 = f(8) - 5$, so $f(8) = 25$.

Exercise If $\int_{-1}^{4} (3f(x) + 2)dx = 28$, find $\int_{-1}^{4} f(x)dx$.

Exercise $f(x) = \begin{cases} 3x - 1, x \leq 2 \\ x^2 + 1, x > 2 \end{cases}$.

Evaluate $\int_{-1}^{4} f(x)dx$.

4.11 Average Value of a Continuous Function

Theorem 4.2: Mean Value Theorem for Integrals

If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that $\int_a^b f(x)dx = f(c)(b-a)$.

The geometric interpretation of the Mean Value Theorem for Integrals is that, for a positive function f, there is a number c between a and b such that the rectangle with base $[a,b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b. In other words, c is the value of x on $[a,b]$ where you can build a "perfect" rectangle - a rectangle whose area is exactly equal to the area of the region under the graph of f from a to b.

The value $f(c)$ is called the average value of the function f and is defined by:

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

Example

Given $f(x) = 1 + x^2$ and the interval $[-1,2]$.

(a) Find the average value of f on the given interval.

$f_{ave} = \frac{1}{3}\int_{-1}^2 1 + x^2 dx = 2.$

(b) Find c such that $f_{ave} = f(c)$.

$2 = 1 + c^2$ implies $c = 1$.

Example

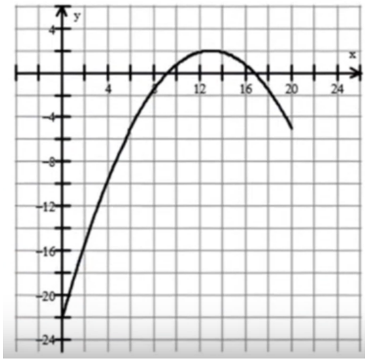
The table below gives values of a continuous function f. Use a left Riemann sum with three subintervals and values from the table to estimate the average value of f on $[5,17]$.

x	5	9	12	17
f(x)	23	29	36	27

This is $\frac{1}{17-5}\int_5^{17} f(x)dx = \frac{1}{12}[4(23) + 3(29) + 5(36)] = 29.9167$

Exercise A study suggests that between the hours of 1:00 PM and 4:00 PM on a normal weekday, the speed of the traffic on a certain freeway exit is modeled by the formula $S(t) = 2t^3 - 21t^2 + 60t + 20$ where the speed is measured in kilometers per hour and t is the number of hours past noon. Compute the average speed of the traffic between the hours of 1:00 PM and 4:00 PM.

Exercise Suppose that during a typical winter day in Minneapolis, the temperature (in degrees Celsius) x hours after midnight is shown in the figure below.



(a) Use a midpoint Riemann sum with four equal subintervals to approximate the average temperature over the time period from 4:00 AM to 8 PM.

(b) Use your answer from part (a) to estimate the time when the average temperature occurred.

Exercise Find the average value of the function on the given interval without integrating.

$$f(x) = \begin{cases} 3 - x \text{ if } -1 \leq x \leq 3 \\ 2x - 6 \text{ if } 3 < x \leq 6 \end{cases}$$

on $[-1, 6]$

5 Integration with Data, Functions Defined by Integrals, and Natural Logs

5.1 Integration Using Data

Example

Water is flowing into a tank over a 24-hour period. The rate at which water is flowing into the tank at various times is measured, and the results are given in the table below, where $R(t)$ is measured in gallons per hour and t is measured in hours. The tank contains 150 gallons of water when $t = 0$.

t (hours)	0	4	8	12	16	20	24
$R(t)$ (gal/hr)	8	8.8	9.3	9.2	8.9	8.1	6.7

(a) Estimate the number of gallons of water in the tank at the end of 24 hours by using a midpoint Riemann sum with three subintervals and values from the table. Show the computations that lea dto your answer.

We are estimating $150 + \int_0^{24} R(t)dt = W(24) - W(0)$.

So $150 + [3(8.8) + 8(9.2) + 8(8.1)] = 358.8$ gallons

(b) Estimate the number of gallons of water in the tank at the end of 24 hours by using a trapezoidal sum with three subintervals and values from the table. Show the computations that lead to your answer.

This is $150 + \left[8\left(\frac{8+9.3}{2}\right) + 8\left(\frac{9.3+8.9}{2}\right) + 8\left(\frac{8.9+6.7}{2}\right)\right] = 354.4$ gallons.

(c) A model for this function is given by $W(t) = \frac{1}{75}(600 + 20t - t^2)$. Use the model to find the number of gallons of water in the tank at the end of 24 hours.

$\int_0^{24} w(t)dt = W(24) - W(0)$.

$150 + \int_0^{24} W(t)dt = 357.36$

(d) Use the model given in (c) to find the average rate of water flow over the 24-hour period.

$\frac{1}{24-0}\int_0^{24} W(t)dt = 8.64$ gallons/hr

5.2 Second Fundamental Theorem of Calculus

Let us investigate first.

Find $\frac{d}{dx}\int_1^x t^2 dt$. This is equal to x^2.

Find $\frac{d}{dx}\int_{\pi/6}^x \cos t dt$. This is equal to $\cos x$.

See a pattern?

Theorem 5.1: Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Example

$$\frac{d}{dx}\int_x^4 t^2 dt$$

This is $-\frac{d}{dx}\int_4^x t^2 dt = -x^2$

In general, $\frac{d}{dx}\int_x^a f(t)dt = -f(x)$.

Example

$$\frac{d}{dx}\int_{\pi/6}^{x^2} \cos t dt$$

This is $\frac{d}{dx}[\sin t]$ with bounds $\pi/6$ to x^2.

We end up getting $\frac{d}{dx}[\sin(x^2) - \frac{1}{2}] = \cos(x^2) \cdot 2x$.

Theorem 5.2: Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x)) \cdot g'(x)$$

Example

Use the Second Fundamental Theorem to evaluate.

(a) $\frac{d}{dx}\int_3^x \sqrt{1+t^2}dt$

This is $\sqrt{1+x^2}$

(b) $\frac{d}{dx}\int_2^x \tan(t^3)dt$

This is $\tan(x^3)$.

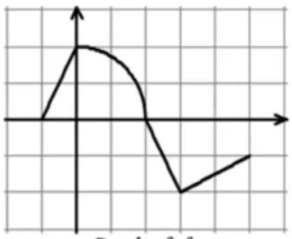
Exercise Same as above for $\frac{d}{dx}\int_{-1}^{x^3} \frac{1}{1+t}dt$

Exercise Same as above for $\frac{d}{dx}\int_2^{\sin x} \sqrt[3]{1+t^2}dt$

Example

The graph of a function f consists of a quarter circle and line segments. Let g be the function given by

$$g(x) = \int_0^x f(t)dt$$



Graph of f

(a) Find $g(0), g(-1), g(2), g(5)$

$g(0) = \int_0^0 f(t)dt = 0$

$g(-1) = \int_0^{-1} f(t)dt = -\int_{-1}^0 f(t)dt = -1$

$g(2) = \int_0^2 f(t)dt = \pi$

$g(5) = \int_0^5 f(t)dt = \pi - 4$

(b) Find all values of x on the open interval $(-1, 5)$ at which g has a relative maximum. Justify your answer.

$g'(x) = f(x)$ crosses the x-axis from positive to negative at $x = 2$.

Exercise Using the information above, (c) Find the absolute minimum value of g on $[-1, 5]$ and the value of x at which it occurs. Justify your answer.

Exercise Using the information above, (d) Find the x-coordinate of each point of inflection of the graph of g on $(-1, 5)$. Justify your answer.

5.3 Natural Logs and Differentiation

Definition: Natural Logarithmic Function

The natural logarithmic function is defined by $\ln x = \int_1^x \frac{1}{t}dt$ where $x > 0$.

The base of natural logs is the number e. e was named for a Swiss mathematician, Leonhard Euler.

By definition:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n \approx 2.7183\ldots$$

$y = \ln x$ and $y = e^x$ are inverses.

Properties of Natural Logs:

1. Domain of $y = \ln x$ is $(0, \infty)$. Range of $y = \ln x$ is $(-\infty, \infty)$.
2. The graph of $y = \ln x$ is continuous, increasing, and one-to-one
3. The graph of $y = \ln x$ is concave down.

Other properties:

If a and b are positive numbers and n is rational, then:

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln ab = \ln a + \ln b$
4. $\ln \frac{a}{b} = \ln a - \ln b$
5. $\ln a^n = n \ln a$

Exercise Write as a sum, difference, or multiple of logs: $\ln \frac{(x^2+3)^2}{\sqrt[3]{x^2+1}}$

Exercise Write as a single log: $2\ln(x+3) + \frac{1}{2}\ln(x-2)$

Definition

$$\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}, u > 0$$

Example

If $y = \ln(2x)$, what is y'.

$y' = \frac{1}{2x} \cdot 2 = \frac{1}{x}$.

Exercise Find $f'(x)$ if $f(x) = \ln(x^2+1)$

Exercise Find y' if $y = x \ln x$

Exercise Find $f'(x)$ if $f(x) = \ln \sqrt{x+1}$

Exercise Find y' if $y = \ln(\ln x)$

Exercise Find y' if $y = \ln(x^3)$

Exercise Find y' if $y = (\ln x)^3$

Example

Show that $y = x\ln x - 4x$ is a solution to the differential equation

$$x + y - xy' = 0$$

$y' = -3 + \ln x$, so plugging this in gives $x + (x\ln x - 4x) - x(-3 + \ln x) = 0$.

Everything cancels out and we see that $0 = 0$ which is true.

5.4 The Natural Log Function and Integration

We previously saw differentiation.

Now integration.

$$\int \frac{1}{u} du = \ln|u| + C$$

Example

$\int \frac{2}{x} dx$

Simple! This is $2\ln|x| + C$.

Exercise $\int_1^e \frac{2}{x} dx$

Exercise $\int \frac{1}{2x-1} dx$

Exercise $\int \frac{3x^2+1}{x^3+x} dx$

Exercise $\int_1^e \frac{(1+\ln x)^3}{x} dx$

Exercise $\int_e^{e^2} \frac{(\ln x)^4}{x} dx$

Exercise $\int_0^3 \frac{x^2-5}{x+2} dx$

If you are integrating a quotient and the power of the numerator is greater than or equal to the power of the denominator you must divide.

Four more integration formulas:

- $\int \tan u du = -\ln|\cos u| + C$
- $\int \cot u du = \ln|\sin u| + C$
- $\int \sec u du = \ln|\sec u + \tan u| + C$
- $\int \csc u du = -\ln|\csc u + \cot u| + C$

Example

Why is $\int \tan x dx = -\ln|cos x| + C$ true?

Let $\int \frac{\sin x}{\cos x} dx$ and this is equal to $-\ln|\cos x| + C$.

Exercise Show why $\int \sec x dx$ works.

Exercise $\int \tan(3x) dx$

5.5 Derivatives of Inverse Functions

A function g is the inverse of a function f if and only if

$f(g(x)) = x$ for each x in the domain of g and $g(f(x)) = x$ for each x in the domain of f.

The inverse of f is denoted f^{-1}.

Properties of inverses:

- If g is the inverse of f, then f is the inverse of g.
- The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
- Not every function has an inverse, but if a function does have an inverse, the inverse is unique.

Example

(a) Find the inverse function of f.

The inverse function is $x^2 + 1 = y$.

(b) State the domain and range of f and f^{-1}.

For $f(x)$ the domain is $x \geq 1$, range $y \geq 0$.

For $f^{-1}(x)$ the domain is $x \geq 0$, range is $y \geq 1$.

Example

Given $f(x) = x^3$ and $f^{-1}(x) = \sqrt[3]{x}$

(a) $f(2)$

8

(b) $f'(x)$

$3x^2$

(c) $f'(2)$

12

(d) $f^{-1}(8)$

2

(e) What is the derivative of $f^{-1}(x)$?

$\frac{1}{3}x^{-2/3}$

(f) $(f^{-1})'(8)$

$\frac{1}{12}$

In (c) and (f) notice they are reciprocals.

Theorem 5.3: Derivative of an Inverse Function

Let f be any function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$ and $g'(x) = \frac{1}{f'(g(x))}$ so that $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$.

Example

If $f(3) = 5$ and $f'(3) = \frac{7}{2}$, find $(f^{-1})'(5)$.

$(f^{-1})'(5) = \frac{1}{f'(3)} = \frac{1}{\frac{7}{2}} = \frac{2}{7}$.

Exercise Let $f(x) = x^3 + 2x - 1$. Find $(f^{-1})'(2)$.

Exercise Let $g(x) = \sqrt{x+1}$. Find $(g^{-1})'(2)$.

Exercise Let $f(x) = \cos x, 0 \leq x \leq \pi$. Find $(f^{-1})'\left(\frac{\sqrt{3}}{2}\right)$.

5.6 Exponential Functions

You learned in the past

$y = \log_b x$ means $x = b^y$ where $b > 0$ and $x > 0$

$y = \ln x$ means $x = e^y$ where $x > 0$.

Exercise Solve $e^{x+1} = 7$

Exercise Solve $\ln(2x - 3) = 5$.

Derivative of an exponential function: $\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$.

Example

Find the derivative.

(a) $y = e^{3x^2}$

$y' = 6xe^{3x^2}$

(b) $y = \sin^2(e^x)$

$y' = 2\sin(e^x)\cos(e^x) \cdot e^x$

Exercise Find the derivative of $y = \ln(4 + e^{3x})$

Exercise Find the derivative of $y = \ln(e^{x^3})$

Exercise Find the derivative of $f(x) = \ln\left(\frac{3+e^x}{3-e^x}\right)$.

Exercise Find the derivative of $y = x^2e^{-x}$

Exercise Use implicit differentiation to find the derivative $\frac{dy}{dx}$ of $e^{xy} + x^2 - y^2 = 10$.

Example

Find the relative extrema and the points of inflection for

$$f(x) = xe^x$$

The first derivative of this is $f' = xe^x + e^x$.

We can see that -1 is a relative minimum.

The second derivative is $xe^x + e^x + e^x$.

We can see that -2 is a point of inflection.

The integral of an exponential function is $\int e^u du = e^u + C$.

Example

$$\int e^{3x+1} dx$$

Let $u = 3x + 1$ then $\frac{1}{3}du = dx$.

$\frac{1}{3}\int e^u du = \frac{1}{3}e^{3x+1} + C$.

Exercise $\int 5xe^{-x^2} dx$

Exercise $\int \frac{e^{1/x}}{x^2} dx$

Exercise $\int \sin x e^{\cos x} dx$

Exercise $\int_0^1 \frac{e^x}{1+e^x} dx$

Exercise $\int_{-1}^0 e^x \cos(e^x) dx$

Example

Solve the differential equation

$$\frac{dy}{dx} = (e^x - e^{-x})^2$$

We have $\int dy = \int (e^x - e^{-x})^2 dx$

This gives $y = \frac{1}{2}e^{2x} - 2x - \frac{1}{2}$.

Exercise Find the particular solution of the differential equation that satisfies the initial conditions.

$$f''(x) = \sin x + e^{2x}, f(0) = \frac{1}{4}, f'(0) = \frac{1}{2}$$

Example

The rate at which water is being pumped into a tank is $r(t) = 20e^{0.02t}$ where t is in minutes and $r(t)$ is in gallons per minute. How many gallons of water have been pumped into the tank in the first five minutes?

$$\int_0^5 r(t)dt = 105.171 \text{ gallons}$$

5.7 Bases other than e

Remember that $y = \log_b x$ means $x = b^y$ where $b > 0$ and $x > 0$.

Exercise Solve $2^{3x} = 45$

Exercise Solve $\log_5(x - 2) = 3$.

Formulas:

- $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}$
- $\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$
- $\int e^u du = e^u + C$
- $\frac{d}{dx}[\log_a u] = \frac{1}{u \ln a}\frac{du}{dx}$
- $\frac{d}{dx}[a^u] = a^u \ln a \frac{du}{dx}$
- $\int a^u du = \frac{a^u}{\ln a} + C$

Example

Find the derivative of $y = 2^{x^3}$.

Let $u = x^3$ so $\frac{du}{dx} = 3x^2$.

So $y' = 2^{x^3} \cdot \ln 2 \cdot 3x^2$.

Exercise Differentiate $f(x) = \log_3(x^2 + 1)$

Exercise $\int 2^x dx$

Exercise $\int x^2 3^{x^3} dx$

If you are asked to differentiate a function that contains a variable raised to a power that contains a variable, we have no formula for this and must use a process called logarithmic differentiation.

Example

Find $\frac{dy}{dx}$ in terms of x.

$$y = (x+1)^{x-3}$$

Let $y = x^x$.

We can see logarithmic differentiation is needed.

$\frac{d}{dx}(\ln y = (x-3)\ln(x+1))$

We can see that $y' = \left(\frac{x-3}{x+1} + \ln(x+1)\right)$.

5.8 Inverse Trig Functions and Differentiation

In the past, you learned two notations for inverse trig functions. The inverse of cosine can be symbolized as $\arccos x$ or $\cos^{-1} x$. You were also taught restrictions for these.

Exercise $\arcsin\left(-\frac{1}{2}\right)$

Exercise $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Exercise $\arctan(-0.3)$

We can derive the formulas for the derivatives of the inverse trig functions by using implicit differentiation.

Example

Let $y = \arcsin x$

$x = \sin y$

$1 = \cos y \cdot y'$

$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$

A similar process can be done for $y = \arctan x$.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$$

$$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

Exercise $f(x) = \arcsin(2x)$. What is $f'(x)$?

Exercise $f(x) = \tan^{-1}(3x)$. What is $f'(x)$?

Exercise $f(x) = \cos(\arcsin(3x))$. What is $f'(x)$?

5.9 Inverse Trig Integration

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$
$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

Example

$$\int \frac{dx}{\sqrt{4 - x^2}}$$

$a = 2$, $u = x$, $\frac{du}{dx} = 1$, $du = dx$.

Integrate to get $\sin^{-1}\left(\frac{x}{2}\right) + C$

Exercise $\int \frac{dx}{\sqrt{4-25x^2}}$

Exercise $\int_{\sqrt{3}}^{3} \frac{1}{9+x^2}dx$

Exercise $\int \frac{dx}{x^2-4x+7}$

Exercise $\int_0^{\sqrt{2}/2} \frac{\arccos x}{\sqrt{1-x^2}}dx$

6 Differential Equations

6.1 Differential Equations

The process is the following:

1. Separate Variables (multiply or divide to get the x and y's on opposite sides. dx and dy must always be on top).
2. Integrate both sides
3. Add $+C$ with the x side.

Example

Use integration to find the general solution to the differential equation

$$\frac{dy}{dx} = 2x(x-4)$$

We are finding $dy = 2x^2 - 8xdx$

Integrate both sides to get $y = \frac{2}{3}x^3 - \frac{8x^2}{2} + C$.

$y = \frac{2}{3}x^3 - 4x^2 + C$ is the general solution.

Exercise Find the particular solution of $f'(x) = 7x - 6$ knowing thta $f(1) = \frac{3}{2}$.

Exercise Use integration to find the general solution to the differential equation $\frac{dy}{dx} = \frac{x-1}{y-6}$

6.2 Euler's Method

This is basically baby steps with tangent lines.

The general procedure is

$$\text{New } y = \text{Old } y + \text{Slope(step size)}$$

Example

Consider a function whose slope is given by $\frac{dy}{dx} = 2xy$ with initial condition $f(1) = 1$.

Use Euler's method with a step size of 0.1 to approximate $f(1.3)$.

$f(1.1) \approx 1 + 2(1)(1)(.1) = 1.2$

$f(1.2) \approx 1.2 + 2(1.1)(1.2)(.1)$

Keep doing this to find $f(1.3)$.

6.3 Exponential Growth and Decay

Solving problems where the rate of growth is proportional to the amount present.

$$\frac{dy}{dt} = kt \implies \ln|y| = kt + C$$
$$\implies y = Ce^{kt}$$

Example

Write an equation for the amount Q of a radioactive substance with a half-life of 30 days, if 10 grams are present when $t = 0$.

$Q(t) = Ce^{kt}$.

$Q(t) = 10e^{kt}$, so we need to find k.

$5 = 10e^{30k}$, so $k = \frac{\ln \frac{1}{2}}{30}$

Exercise The balance in an account triples in 30 years. Assuming that interest is compounded continuously, what is the annual percentage rate?

Exercise In 1990 the population of a village was 21,000 and in 2000 it was 20,000. Assuming the population decreases continuously at a constant rate proportional to the existing population, estimate the population in the year 2020.

Exercise A certain type of bacteria increases continuously at a rate proportional to the number present. If there are 500 present at a given time and 1,000 present 2 hours later, how many will there be 5 hours from the initial time given?

7 Area between Curves, Volume, and Arc Length

7.1 Area Between Two Curves

To find the area bounded by two functions $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$:

Area $= \int_a^b f(x) - g(x)dx$

To find the area bounded by two functions $x = f(y)$ and $x = g(y)$ on the interval $[a, b]$:

Area $= \int_a^b g(y) - f(y)dy$

Example

Find the area bounded by the graphs $y = 3 - x^2$ and $y = -x + 1$.

The intersections are when the two graphs are equal to each other.

Setting $3 - x^2 = -x + 1$ results in $x = -1$ and $x = 2$.

$A = \int_{-1}^{2} 3 - x^2 - (-x + 1)dx$

This is equal to $\int_{-1}^{2} 2 - x^2 + xdx$.

Simplifying this gives $A = 4.5$.

Exercise Find the area between the two graphs $x = 5 - y^2$ and $x = y - 1$.

7.2 Volume with Known Cross Sections

For this, we will find the volume of a solid whose cross sections are familiar geometric shapes, such as squares, rectangles, triangles, and semicircles.

For cross sections of area $A(x)$ taken perpendicular to the x-axis, the volume is $\int_a^b A(x)dx$

For cross sections of area $A(y)$ taken perpendicular to the y-axis, volume is $\int_{y=c}^{y=d} A(y)dy$

Example

Set up the integrals needed to find the volume of the solid whose base is the area bounded by the lines $y = x^2$ and $y = -2x + 3$ and whose cross sections perpendicular to the x-axis are the following shapes.

(a) Rectangles of height 4

$V = \int_{-3}^{1} -8x + 12 - 4x^2dx$

(b) Semicircles

$V = \frac{1}{2}\pi \int_{-3}^{1} \left(\frac{-2x+3-x^2}{2}\right)^2 dx$

Exercise Set up the integrals needed to find the volume of the solid whose base is the area bounded by the circle $x^2 + y^2 = 9$ and whose cross sections perpendicular to the x-axis are equilateral triangles. Note the area of an equilateral triangle is $\frac{s^2\sqrt{3}}{4}$ where s is a side of a triangle.

Exercise The base of a solid is bounded by $y = x^2$, $y = 0$, and $x = 2$. For this solid, each cross section perpendicular to the y-axis is square. Set up the integral needed to find the volume of this solid.

7.3 Volume: The Disc Method

If we revolve a figure around a line, a solid of revolution is formed. The line is called the axis of revolution. The simplest such solid is a right circular cylinder or disc.

To find the volume of the solid, we partition it into rectangles, which are revolved about the axis of revolution.

Each disc is a thin cylinder standing on its side. A volume of a cylinder is $\pi r^2 h$, a volume of a disc is $\pi(R(x))^2\Delta x$.

Adding the volumes of all of the discs together, we get the volume of a solid to be approximately $\sum_{i=1}^{n} \pi[R(x_1)]^2\Delta x_i$.

To get the exact volume, this is equal to

$$\lim_{n\to\infty}\sum_{i=1}^{n}\pi[R(x_i)]^2\Delta x_i = \pi\int_a^b [R(x)]^2 dx$$

Volume about horizontal axis by discs: $V = \pi\int_a^b[R(x)]^2dx$

Volume about vertical axis by discs: $V = \pi\int_c^d[R(y)]^2dy$

The disc method can be extended to cover solids of revolutions with a hole in them. This is called the washer method.

If $R(x)$ is the outer radius and $r(x)$ is the inner radius:

Volume about horizontal axis by washers: $V = \pi\int_a^b[R(x)]^2 - [r(x)]^2dx$

Volume about vertical axis by washers: $V = \pi\int_c^d[R(y)]^2 - [r(y)]^2dy$

Things to remember: In the disc or washer method:

1. The representative rectangle is always perpendicular to the axis of revolution.
2. If the representative rectangle is vertical, you will work in x's. If the representative rectangle is horizontal, you will work in y's.

Example

Find the volume of the solid formed by revolving the region bounded by the graphs of the given equations about the indicated axis.

$$y = 9 - x^2, x = 0, y = 0$$

(a) about the x-axis.

$\pi\int_0^3(9 - x^2)^2dx$

(b) about the line $y = -2$

$V = \pi\int_0^3(11 - x^2)^2 - 2^2dx$

(c) about the y-axis

$V = \pi\int_0^9(\sqrt{9-y})^2dy$

(d) about the line $x = -2$

$V = \pi\int_0^9(\sqrt{9-y} + 2)^2 - 2^2dy$

Exercise Find the volume of the solid former by revolving the region bounded by the graphs of $y = 2x - x^2$ and $y = x^2$ about the line $y = 3$.

7.4 Arc Length

Arc length of $f(x)$ from $x = a$ to $x = b$:

$$s = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Arc length of $f(y)$ from $y = c$ to $y = d$:

$$s = \int_c^d \sqrt{1 + (f'(y))^2} dy$$

Example

Find the arc length of the graph of the given function over the indicated interval.

$$y = x^{3/2} - 1 \quad [0, 4]$$

$s = \int_0^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2} dx$

Exercise Find the arc length of the graph of the function $y = 3x^{2/3} - 10$ on the interval $[8, 27]$.

8 Techniques of Integration

8.1 Integration by Parts

Integration by parts is used to integrate a product, such as the product of an algebraic and a transcendental function:

For example, $\int xe^x dx, \int x \sin x dx, \int x \ln x dx$.

Recall the product rule is $\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$.

Integrating both sides, we get $uv = \int udv + vdu$.

Rearranging we get the formula for integration by parts:

$$\int udv = uv - \int vdu$$

Example

$$\int x \sin x dx$$

Let $u = x$, $dv = \sin x dx$, then $du = dx$ and $v = -\cos x$.

We get $x \cos x + \int \cos dx$.

Simplifying, we get $-x \cos x + \sin x + C$

Exercise $\int x^2 e^x dx$

A tabular approach is helpful with these "repeated" integration by parts problems

Example

$$x^2 e^x dx$$

u	v
x^2	e^x
$2x$	e^x
2	e^x
0	e^x

Criss crossing gives you $x^2 e^x - 2xe^x + 2e^x$.

If you have limits of integration, first integrate without them.

Example

$$\int_0^{\pi/2} x \sin x dx$$

Using integration by parts you get $[-x \cos x + \sin x]$.

Using the limits of integration, you get 1.

Exercise $\int e^x \cos x dx$

8.2 Integration by Partial Fractions

Fractions which have a denominator that can be factored can be decomposed into a sum or difference of fractions.

Fractions which have a denominator that can be factored into distinct linear factors

$$\frac{4x+1}{x^2-5x+6} = \frac{4x+1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

Solving for A and B results in $A = 13$ and $B = -9$, so that the above equals

$$\frac{13}{x-3} - \frac{9}{x-2}$$

Example

$$\int \frac{4x+41}{x^2+3x-10} dx$$

This is $\frac{4x+41}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$ inside the integral.

$4x + 41 = A(x+5) + B(x-2)$. If we let $x = -5$, $B = -3$. Let $x = 2$, then $A = 7$.

We are now integrating $\int \frac{7}{x-2} - \frac{3}{x+5} dx$.

This is $7 \ln|x-2| - 3 \ln|x+5| + C$.

8.3 Logistic Growth

In exponential growth (or decay), we assume that the rate of increase (or decrease) of a population at any time t is directly proportional to the population P. In otehr words, $\frac{dP}{dt} = kP$. However, in many situations population growth levels off and approaches a limiting number L (the carrying capacity) because of limited resources. In this situation the rate of increase (or decrease) is directly proportional to both P and $L - P$. This type of growth is called logistic growth. It is modeleted by the differential equation $\frac{dP}{dt} = kPL - P$.

If we find $\frac{d^2P}{dt^2}$ we can find out an important fact about the time when P is growing the fastest. We will do this in the example below.

Example

The population $P(t)$ of fish in a lake satisfies the logistic differnetial equation $\frac{dP}{dt} = 3P - \frac{P^2}{6000}$ where t is measured in years and $P(0) = 4000$.

(a) $\lim_{t\to\infty} P(t) = ?$

18000

(b) What is the range of the solution curve?

$4000 \leq P(t) < 18000$

(c) For what values of P is the solution curve increasing? Decreasing? Justify your answer.

$P(t)$ is increasing because $\frac{dP}{dt} > 0$.

(d) For what values of P is the solution curve concave up? Concave down? Justify your answer.

Concave up from $(4000, 9000)$ and concave down $(9000, 18000)$.

(e) Does the solution curve have an inflection point? Justify your answer.

Yes because the second derivative changed signs.

Exercise The population $P(t)$ of fish in a lake satisfies the logistic differential equation $\frac{dP}{dt} = 3P - \frac{P^2}{6000}$ where t is measured in years and $P(0) = 10000$.

(a) $\lim\limits_{t\to\infty} P(t)$

(b) What is the range of the solution cuve?

(c) For what values of P is the solution curve increasing? Decreasing? Justify your answer.

(d) For what values of P is the solution curve concave up? Concave down? Justify your answer.

(e) Does the solution curve have an inflection point? Justify your answer.

Exercise The population $P(t)$ of fish in a lake satisfies the logistic differential equation $\frac{dP}{dt} = 3P - \frac{P^2}{6000}$ where t is measured in years and $P(0) = 20000$.

(a) $\lim\limits_{t\to\infty} P(t)$

(b) What is the range of the solution cuve?

(c) For what values of P is the solution curve increasing? Decreasing? Justify your answer.

(d) For what values of P is the solution curve concave up? Concave down? Justify your answer.

(e) Does the solution curve have an inflection point? Justify your answer.

9 Improper Integrals, Sequences and Series, Taylor Polynomials, and Taylor Series

9.1 Improper Integrals

Example

$$\int_1^\infty \frac{1}{x^2}dx$$

This can be written as

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left[-\frac{1}{b}-\frac{-1}{1}\right]$$

Which converges to 1.

Exercise $\int_{-1}^{0} \frac{1}{x^2}dx$

Exercise $\int_{-1}^{2} \frac{1}{x^3}dx$

9.2 nth Term Test

A sequence $\{a_n\} = a_1, a_2, a_3, a_4, \ldots, a_n, a_{n2}$.

$\{a_n\}$ converges if $\lim\limits_{n\to\infty} a_n = L$ and diverges if $L \to \infty$ or L does not exist.

A series $\sum\limits_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots a_n + \ldots$.

If $\sum\limits_{n=1}^{\infty} a_n$ becomes ∞ we say the series diverges.

If $\sum\limits_{n=1}^{\infty} a_n$ stays finite, we say it converges.

The 2 big questions are

1. Does $\sum\limits_{n=1}^{\infty} a_n$ converge?
2. If so, to what value?

Let's say we have a sum $\sum\limits_{n=1}^{\infty} 2n = 2 + 4 + 6 + 8 + \ldots$.

S_1 would be 2, S_2 would be $2 + 4 = 6$, S_3 would be $2 + 4 + 6 = 12$. We can see that the $\lim\limits_{n\to\infty} = \infty$ so it diverges.

Example

Does this diverge or converge?

$$\sum_{n=1}^{\infty} \frac{n}{2n-1}$$

Expanding we see that This ends up being $\frac{1}{1} + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} + \dots$

The limit $\lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}$.

$S_n \to \infty$ so diverges.

To "stand a chance" to converge

$$\lim_{n\to\infty} a_n = 0$$

nth term for divergence:

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Exercise Use the nth term test to determine whether the series diverges for $\sum_{n=1}^{\infty} \frac{1}{3n}$.

Exercise Use the nth term test to determine whether the series diverges for $\sum_{n=1}^{\infty} \frac{n!}{3n!+2}$.

Never use the nth term test to argue for convergence.

9.3 Geometric Series and Telescopic

Exercise Does $\sum_{n=1}^{\infty} \tan^{-1} n$ diverge? Use the nth term test.

Example

$$\sum_{n=1}^{\infty} \frac{e^n}{3^n}$$

The summation can be written as $\left(\frac{e}{3}\right)^n$. Using the nth term test, the limit goes to 0.

Whenever you have $\sum_{n=0}^{\infty} a_1(r)^n = \frac{a_1}{1-r}$ if $-1 < r < 1$ is what it converges to.

This converges if $-1 < r < 1$.

Converges to $\frac{a_1}{1-r}$.

In the above case, $a_1 = \frac{e}{3}$ and $r = \frac{e}{3}$ so it converges to $\frac{\frac{e}{3}}{1-\frac{e}{3}}$.

Exercise Can you use geometric series or nth term test on $\sum_{n=1}^{\infty} \frac{1}{10n+7}$?

Example

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

We can separate this into $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1$ becauase you can see that every term will cancel out except

$\frac{1}{1}$.

Converges to 1.

Exercise Does $\sum_{n=1}^{\infty} \frac{3^{n-2}}{2^n}$ diverge or converge?

Exercise Does $\sum_{n=1}^{\infty} \frac{3^n+1}{3^{n+1}}$ diverge or converge?

9.4 Integral Test and p-Series

Integral Test:

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either both converge or both diverge.

Example

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

$\lim_{n\to\infty} a_n = 0$, so we use a different test.

$\int_1^{\infty} \frac{x}{x^2+1}$ diverges, so the series will diverge as a result.

Exercise Determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$ and if $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ both converge, then the series converges to S, and the remainder, $R_N = S - S_N$ is bounded by $0 \leq R_N \leq \int_N^{\infty} f(x)dx$.

Example

Approximate the sum of the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ by using six terms. Include an estimate of the maximum error for your approximation.

Using the first six terms gives you 1.07209.

The error is error $\leq \int_6^{\infty} \frac{1}{x^4} dx = 0.0015$.

$1.07289 < \sum < 1.07289 + 0.0015$

p-Series

A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n_p} + \ldots$ is called a p-series, where p is a positive constant.

For $p = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \ldots$ is called the harmonic series.

The p-series diverges if $0 < p \leq 1$ and converges if $p > 1$. The harmonic series diverges.

Example

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$$

$p = 3/2$ for this, so the series converges.

9.5 Comparison of Series

Direct Comparison Test

If $a_n \geq 0$ and $b_n \geq 0$,

1. If $\sum\limits_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.
2. If $\sum\limits_{n=1}^{\infty} a_n$ diverges and $0 \leq a_n \leq b_n$, then $\sum\limits_{n=1}^{\infty} b_n$ diverges.

Example

Determine if this converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^3+1}$$

Compare this to $\sum\limits_{1}^{\infty} \frac{1}{n^3}$. This is a p-series that converges, so this series converges.

Exercise Determine whether the following series converges or diverges. $\sum\limits_{n=1}^{\infty} \frac{1}{3^n+2}$

Exercise Determine whether the following series converges or diverges. $\sum\limits_{n=4}^{\infty} \frac{1}{\sqrt{n}-1}$

Limit Comparison Test

Suppose $a_n > 0, b_n > 0$ and $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$, where L is both finite and positive. Then the two series $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ either both converge or both diverge.

Example

Determine whether the following converge or diverge.

$$\sum_{n=1}^{\infty} \frac{n^4+10}{4n^5-n^3+7}$$

Compare this to $\frac{1}{n}$ and this diverges. The limit of what is in the summation as $n \to \infty$ is $\frac{1}{4}$ so this diverges so they diverge.

Exercise Determine whether the series converges or diverges. $\sum\limits_{n=2}^{\infty} \frac{1}{n^3-2}$.

9.6 Alternating Series

An alternating series is a series whose terms are alternately positive and negative.

Examples are $\sum\limits_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$ or $\sum\limits_{n=1}^{\infty} (-1)^n \frac{1}{n!}$.

In general, just knowing that $\lim_{n\to\infty} a_n = 0$ tells us very little about the convergence of the series $\sum_{n=1}^{\infty} a_n$ however it turns out that an alternating series must converge if the terms have a limit of 0 and the terms decrease in magnitude.

Alternating Series Test

Let $a_n > 0$. The alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ and $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converge if the following two conditions are met

1. $\lim_{n\to\infty} a_n = 0$
2. $a_{n+1} < a_n$ for all n

In other words, a series converges if its terms

1. alternate in sign
2. decrease in magnitude
3. have a limit of 0

Note: This does not say that if $\lim_{n\to\infty} a_n \neq 0$, the series diverges by the Alternating Series Test. The Alternating Series Test can only be used to prove convergence. If $\lim_{n\to\infty} a_n \neq 0$ then the series diverges by the nth Term Test for Divergence not by the Alternating Series Test.

Example

Determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n-1}$$

The limit $\lim\limits_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}$ so this diverges by the nth term test.

Exercise Determine if the series converges or diverges. $\sum\limits_{n=1}^{\infty} \frac{(-1)^n n}{\ln(2n)}$

Exercise Determine if the series converges or diverges. $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$

The above series is called the alternating harmonic series. If an alternating series converges to a sum S, then its partial sums jump around S from side to side with decreasing distances from S.

Definition

$\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum\limits_{n=1}^{\infty} |a_n|$ converges.

$\sum\limits_{n=1}^{\infty} a_n$ is conditionally convergent if $\sum\limits_{n=1}^{\infty} a_n$ converges but $\sum\limits_{n=1}^{\infty} |a_n|$ diverges.

Example

Determine whether the given alternating series converges or diverges. If it converges, determine whether it is absolutely convergent or conditionally convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

The limit of $\frac{1}{\sqrt{n}} = 0$ as n goes to infinity.

It is an alternating series because terms are decreasing in magnitude.

So we see that $\left|\frac{(-1)^n}{\sqrt{n}}\right|$ converges, but $\frac{1}{\sqrt{n}}$ will diverge.

This is conditionally convergent.

Alternating Series Remainder

If a series has terms that are alternating, decreasing in magnitude, and having a limit of 0, then the series converges so that it has a sum S. If the sum S is approximated by the nth partial sum, S_n, then the error in the approximation, $|R_n|$, which equals $|S - S_n|$, will be less than the absolute value of the first omitted or trucnated term.

In other words, if the three conditions are met, you can approximate the sum of the series by using the nth partial sum, S_n, and your error will be bounded by the absolute value of the first truncated term.

Example

Given the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

(a) Approximate hte sum S of the series by using its first four terms.

First four terms comes out to 0.625

(b) Explain why the estimate found in (a) differs from the actual value by less than $\frac{1}{100}$.

error$< \left|\frac{(-1)^{5+1}}{5!}\right| = \frac{1}{120} < \frac{1}{100}$

(c) Use your results to explain why $S \neq 0.7$

$0.625 - \frac{1}{120} <$ sum$< 0.625 + \frac{1}{120}$.

0.7 is not in this interval.

Exercise How many terms are needed to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ so that the estimate differs from the actual sum by less than $\frac{1}{1000}$? Justify your answer.

9.7 Ratio Test

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms.

1. $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.
2. $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$.
3. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ the Ratio Test is inconclusive so another test would need to be used.

Example

Determine whether the following converge or diverge.

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

We do $\lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right|$.

This simplies to $\lim_{n\to\infty} \left|\frac{2}{n+1}\right| = 0 < 1$ converges.

Exercise Determine whether the following converges or diverges. $\sum_{n=1}^{\infty} \frac{n^2 3^{n+1}}{2^n}$

Exercise Determine whether the following converges or diverges. $\sum_{n=1}^{\infty} \frac{(n+1)!}{3^n}$

9.8 Taylor Polynomials

9.9 Radius and Interval of Convergence

9.10 Taylor Series

9.11 Lagrange Error Bound

9.12 More on Error

10 Parametrics, Vectors, and Polar

10.1 Parametric Equations: The Basics

10.2 Vectors and Motion along a Curve

10.3 Polar Coordinates and Polar Graphs

10.4 Area Bounded by a Polar Curve

10.5 Notes on Polar

10.6 More on Polar Graphs