

Linear Algebra Notes

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1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

A linear equation in the variables x_1, x_2, \dots, x_n is an equation that can be written in the form $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ where b and the coefficients a_1, a_2, \dots, a_n are real or complex numbers.

Example: $4x_1 - 5x_2 + 2 = x_1$, $x_2 = 2(6^{1/2} - x_1) + x_3$ are both linear. Not linear examples are $4x_1 - 5x_2 = x_1x_2$, $x_2 = 2(x_1^{1/2}) - 6$, and $2x_1^{-1} + \sin x_2 = 0$.

Systems of Linear Equations: an $(m \times n)$ system of linear equations is a system of m linear equations with n unknowns.

Example: the 2×3 system of equations below has a solution $x_1 = 5$, $x_2 = 6.5$, and $x_3 = 3$:

$$\begin{aligned}2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 - 4x_3 &= -7\end{aligned}$$

Two linear systems are called equivalent if they have the same solution set.

A system of linear equations can have: infinitely many solutions, no solution, or a unique solution. Coincident lines have infinitely many solutions, parallel lines have no solution, and intersecting lines have a unique solution.

Matrix Notation:

Let's say we have $x_1 - 2x_2 + x_3 = 0$, $2x_2 - 8x_3 = 8$, and $5x_1 - 5x_3 = 10$.

The coefficient matrix is $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$ and the augmented matrix is $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$. The augmented matrix is 3×4 (3 rows, 4 columns).

To solve a linear system: if one of the following elementary operations is applied to a system of linear equations, the resulting system is equivalent, that is the resulting system has the same set of solutions as the original:

1. interchange two equations
2. multiply an equation by a non-zero scalar
3. add a constant multiple of one equation to another

Let's use the system

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10\end{aligned}$$

So using row operations and rearranging the rows, we can do $R_1 \leftrightarrow R_2$ to swap the first and second row. Now we can multiply the second line by $1/2$ so $1/2R_2$. The next step is to do $R_2 + R_1$.

The matrix that results from this is $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$

Doing the operations $R_3 - 5R_1$, $R_3 - 5R_2$, and $1/2R_2$ and $1/30R_3$, we end up getting $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$,

so from this we have the equations

$$x_1 - 2x_2 + x_3 = 0$$

$$x_2 - 4x_3 = 4$$

$$x_3 = -1$$

, so the solution set is $(1, 0, -1)$. Since a solution exists, the system is consistent.

Let's see if this one is consistent.

$$x_2 - 4x_3 = 8$$

$$2x_1 - 3x_2 + 2x_3 = 1$$

$$4x_1 - 8x_2 + 12x_3 = 1$$

Doing the row operations $R_1 \leftrightarrow R_2$, $R_3 - 2R_1$, $R_3 + 2R_2$ we get that $0 = 15$, so this is inconsistent.

This last example has infinitely many solutions:

$$x_1 - 2x_2 - x_3 = -2$$

$$2x_1 + x_2 + 3x_3 = 1$$

$$-3x_1 + x_2 - 2x_3 = 1$$

Doing the row operations $R_2 - 2R_1$ and $R_3 + 3R_1$, then $R_3 + R_2$ and then $1/5R_2$ results in

$$\begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solving this, we have x_3 with no restrictions, it is a "free parameter" and $x_2 = 1 - x_3$ and $x_1 = -x_3$ so x_1 and x_2 are parameterized by x_3 .

1.2 Row Reduction and Echelon Forms

Leading entry of a row: the first (counting from left to right) non-zero entry (in a nonzero row)

Echelon Form: upper right-hand stair-case, triangle

1. all rows that consist entirely of zeros are grouped together at the bottom of the matrix
2. the first (counting from left to right) non-zero entry in the $(i + 1)$ st row must appear in a column to the right of the first non-zero entry in the i th row.
3. all entries in a column below a leading entry are zeros

Reduced Echelon Form: an echelon Form matrix that also has the following properties:

1. the leading entry in each nonzero row is 1
2. each leading one is the only nonzero entry in its column

A pivot point: a location in a matrix A that corresponds to a leading 1 in a reduced echelon form of A . A pivot column is a column of A that contains a pivot position.

Steps to solving a system of linear equations:

1. begin with the leftmost nonzero column. This is the pivot column, the pivot position is at the top.
2. select a non zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
3. Use row replacement operations to create zeros in all positions below the pivot.
4. Cover (or ignore) the row containing the pivot position and cover all rows (if any) above it. Apply previous steps to the sub matrix that remains. Repeat until there are no more nonzero rows to modify.
5. Beginning with the rightmost pivot, create zeros above each pivot. Make each pivot equal to 1 by scaling.

Example

Determine the existence and uniqueness of the solution to

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

We can find that $x_5 = 4$, and x_3, x_4 are of infinite number of solutions.

Example

Find the general solution of the linear system whose augmented matrix has been reduced to $\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$

We have that $x_5 = 7$, $x_3 = 1/2(3 + 8x_4 + 7)$ and $x_1 = -4 - 6x_2 - 2(1/2)(3 + 8x_4 + 7) + 14$

Theorem 1.1: Existence and Uniqueness

A linear system is consistent if and only if the right most column of the augmented matrix is not a pivot column, that is if and only if an echelon form of the augmented matrix has no row of the form $[0, \dots, 0b]$ with b nonzero. If the system is consistent in the solution contains either a unique solution when there are no free variables or infinitely many solutions when there is at least one free variable.

1.3 Vector Equations

Vectors in \mathbf{R}^2 : a matrix with only 1 column is called a vector. The set of all vectors with 2 entries is \mathbf{R}^2

$\vec{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ for example.

The vectors are ordered pairs of real numbers:

$$\vec{u}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (3, -1) \neq (-1, 3) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Vector addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

Geometric interpretation: parallelogram law.

$(2, 4) + (3, 1) = (2 + 3, 4 + 1) = (5, 5)$. We can create a parallelogram (google for review).

Let's say we subtract, $-a$ has the same magnitude as a but points in the opposite direction in this case. $(1, 3) - (2, 1) = (-1, 2)$. Drawing the line from the origin to the tip of the vectors give you what you get algebraically.

Vector multiplication: scalar multiplication: $a(x, y) = (ax, ay)$ where a is a scalar.

Geometric interpretation: $a(x, y, z)$ points in the same direction as (x, y, z) but is scaled by a factor of a .

Example

Let $\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ Find $4\vec{u}$, $-3\vec{v}$ and $4\vec{u} + (-3)\vec{v}$

$$4\vec{u} = (4, -8), -3\vec{v} = (-6, 15) \text{ and } 4\vec{u} + (-3)\vec{v} = (-2, 7)$$

Vectors start at the origin and have magnitude and direction.

Representing vectors in \mathbf{R}^3 . We add the z -axis.

Vectors in \mathbf{R}^n we have that $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = (u_1, u_2, \dots, u_n)$ where $u_1, u_2, \dots \in \mathbb{R}$.

The $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is the zero vector.

Algebraic Properties of \mathbf{R}^n : for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in \mathbf{R}^n and all scalars c and d :

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $c(d\mathbf{u}) = (cd)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

Linear Combinations: given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in \mathbf{R}^n and scalars c_1, c_2, \dots, c_p , the vector $\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p$ is called a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ with weights c_1, c_2, \dots, c_p .

For example, if we have $3^{1/2}\mathbf{v}_1 + \mathbf{v}_2$ this can be written as $\vec{y} = \sqrt{3}\vec{v}_1 + \vec{v}_2$ with $c_1 = \sqrt{3}$ and $c_2 = 1$.

Example

If $\vec{a}_1 = (1, -2, -5)$, $\vec{a}_2 = (2, 5, 6)$, and $\vec{a}_3 = (7, 4, -3)$ then determine if \vec{b} can be written as a linear combination of \vec{a}_1 and \vec{a}_2 . That is determine if there exists weights x_1, x_2 such that $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$.

Using elementary row operations, we can determine that $x_1 = 3$, $x_2 = 2$ which is the linear combination of \vec{a}_1 and \vec{a}_2 .

A vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{b}$ has the same solution set as the linear system with augmented matrix $[\mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n \mathbf{b}]$. In particular \mathbf{b} can be generated by a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix $[\mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n \mathbf{b}]$.

Span: if $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are in \mathbf{R}^n then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ is denoted $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_p\}$ and is called the subset of \mathbf{R}^n spanned by $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$. That is, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p$ with c_1, c_2, \dots, c_p scalars.

Asking if a vector \mathbf{b} is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_p\}$ amounts to asking whether the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{b}$ has a solution, or equivalently whether the linear system with augmented matrix $[\mathbf{v}_1 \mathbf{v}_2 \dots \mathbf{v}_p \mathbf{b}]$ has a solution.

Note $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_p\}$ contains every scalar multiple of \mathbf{v}_1 .

The span of a single vector is a line. The span of 2 linearly independent vectors is a plane (not scalar multiples of each other).

1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

The Matrix Equation: if A is an $m \times n$ matrix with columns, $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ and if \mathbf{x} is in \mathbf{R}^n , then the product of A and \mathbf{x} denoted $A\mathbf{x}$ is the linear combination of the columns of A using the corresponding entries in \mathbf{x} as weights.

Note: $A\mathbf{x}$ is defined only if the number of columns of A equals the numbers of entries in \mathbf{x} .

For example: $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4(1, 0) + 3(2, -5) + 7(-1, 3) = (3, 6)$

Theorem 1.2: Matrix Equation, Vector Equation, System of Linear Equations

If A is an $m \times n$ matrix, with columns, $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ and if \mathbf{b} is in \mathbf{R}^m , the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution as the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{b}$ which in turn has the same solution as the system of linear equations represented by the augmented matrix $[\mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n \mathbf{b}]$.

Existence of Solutions: the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if \mathbf{b} is a linear combination of the columns of A .

Example

Is $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & 7 \end{bmatrix} \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ Is the equation $A\vec{x} = \vec{b}$ consistent for all \vec{b} . Using rref, we get that

$0 = -2b_1 + b_2 - 2b_3$, so it is not consistent for every \vec{b} . It is only consistent if $b_2 = 2b_1 + 2b_3$.

So let $\vec{b} = (1, 4, 1)$ and then do rref again and we get that x_3 is free, $x_2 = 1/7(4 - 5x_3)$ and $x_1 = 1 - 3(x_2) - 4x_3$ and this basically gives us $(1, 4, 1)$ too.

Theorem 1.3: Existence of solution for $A\mathbf{x} = \mathbf{b}$

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A , they are all true statements or they are all false:

1. for each \mathbf{b} in \mathbf{R}^m , the equation $A\mathbf{x} = \mathbf{b}$ has a solution
2. each \mathbf{b} in \mathbf{R}^m is a linear combination of the columns of A
3. The columns of A span \mathbf{R}^m
4. A has a pivot position in every row. Note: A is a coefficient matrix, not an augmented matrix.

Computation of $A\mathbf{x}$ - an efficient method (matrix multiplication): if the product $A\mathbf{x}$ is defined, then the i th entry in $A\mathbf{x}$ is the sum of the products of the corresponding entries from row i of A and from vector \mathbf{x} .

The above is trivial.

Properties of the Matrix-Vector Product $A\mathbf{x}$

Theorem 1.4

if A is an $m \times n$ matrix, \mathbf{u} and \mathbf{v} are vectors in \mathbf{R}^n , and c is a scalar, then :

1. $A(\mathbf{u} + \mathbf{v})$
2. $A(c\mathbf{u}) = c(A\mathbf{u})$

Algebraic Properties of \mathbf{R}^n : for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in \mathbf{R}^n and all scalars c and d . (This was from an above topic)

1.5 Solution Sets of Linear Systems

Parametric Vector Form of Solutions:

- Parametric Vector Form of a Plane: a plane can be expressed in explicit form, such as $10x_1 - 3x_2 - 2x_3 = 0$ or implicit form: $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$, for s and t scalars.
- Parametric Form of a Line containing point \mathbf{p} in direction of \mathbf{v} : $l(t) = \mathbf{p} + t\mathbf{v}$

The parametric equation of a plane in \mathbf{R}^2 : $\mathbf{x} = a\mathbf{v} + b\mathbf{s}$.

The span of 2 non-colinear vectors is a plane. $\text{Span}\{\mathbf{v}, \mathbf{s}\} = \text{the } \mathbf{R}^2 \text{ plane.}$

In \mathbf{R}^3 , the Span is still a plane, just in \mathbf{R}^3 .

The parametric equation of a line in \mathbf{R}^2 : $\mathbf{l} = \mathbf{p} + t\mathbf{v}$.

Homogeneous Linear Equation - $A\mathbf{x} = 0$.

The homogeneous equation always has at least 1 solution, $\mathbf{x} = 0$ (the trivial solution).

Recall that a system of linear equation either has infinitely many solutions, no solution, or a unique solution.

The question is whether there exists a nontrivial solution (in which case there are infinitely many solutions).

- The homogeneous equation $A\mathbf{x} = 0$ has nontrivial solution if and only if the equation has at least 1 free variable

Description of solutions: if the solutions consists of:

- the 0 vector: $\text{Span}\{0\}$
- 1 free variable: $\text{Span}\{\mathbf{v}\}$, the solutions are a line through the origin
- 2 free variables, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane through the origin

Example

Determine if the following system has a nontrivial solution. Then describe the solution set

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

x_3 is free. And everything is in the form $(4/3, 0, 1)$.

Nonhomogeneous Equation: $A\mathbf{x} = \mathbf{b}$.

For example, in the previous example when we let $x_3 = 0$ we get $(-1, 2, 0)$.

Theorem 1.5

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given \mathbf{b} and let \mathbf{p} be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the Form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ where \mathbf{v}_h is any solution of the homogeneous equation $A\mathbf{x} = 0$.

1.6 Applications of Linear Systems

There are three examples here: economics, chemical equations and network flow.

Start with economics. There exist equilibrium prices that can be assigned to the total outputs of the various sectors in an economy in such a way that the income of each sector exactly balances its expenses.

You can use row operations to find an equilibrium price.

Other examples run similarly (sorry for bad note taking today I'm sick)

1.7 Linear Independence

1.8 Introduction to Linear Transformations

1.9 The Matrix of a Linear Transformation

1.10 Linear Models in Business, Science, and Engineering

2 Matrix Algebra

2.1 Matrix Operations

2.2 The Inverse of a Matrix

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6.1 Inner Product, Length, and Orthogonality

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6.4 The Gram-Schmidt Process

6.5 Least-Squares Problems

6.6 Machine Learning and Linear Models

6.7 Inner Product Spaces

7 Symmetric Matrices and Quadratic Forms

7.1 Diagonalization of Symmetric Matrices

7.2 Quadratic Forms