1 Vector-Valued Functions

1.1 Vector Functions and Space Curves

Review: Parametric Curves

- $x = f(t)$
- $y = g(t)$
- $z = h(t)$

These represent a curve in 3-space (for 2-space, it is just x and y.)

The above represents a path in space that is traced in a specific direction as t increases (orientation). The domain is $(-\infty, \infty)$, unless specified otherwise.

Definition

$$\vec{r} = \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

At any given t value, $\vec{r}$ represents a vector whose initial point is at the origin and terminal point is $(f(t), g(t), h(t))$.

The domain is $(-\infty, \infty)$ and the range is the set of vectors.

Graphs of vector-valued functions: curve that is traced by connecting tips of "radius vectors".

Example

Graph $\vec{r}(t) = 2\cos t\vec{i} - 3\sin t\vec{j}$ for $0 \leq t \leq 2\pi$.

We could write this as $x = 2\cos t$ and $y = -3\sin t$ (parametric).

We could instead write a table.

t	x	y
0	2	0
$\pi/2$	0	-3
π	-2	0
$3\pi/2$	0	3
2π	2	0

As you draw this, you can see that this will be an ellipse.

Example

$$\vec{r}(t) = \langle 4\cos t, 4\sin t, t \rangle$$

We should know that since there are trig things in here, that we go from 0 to 2π, and if we put this on a table, we can see that x and y will give you a circle from the table. The z is moving up though, so basically the function will just be circling around a cylinder of radius 2.

Example

Find a vector and parametric equations for the line segment that joins $A(1, -3, 4)$ to $B(-5, 1, 7)$.

We have $\vec{r} = \vec{AB} = \langle -6, 4, 3\rangle$. So $\vec{r}(t) = \langle 1 - 6t, -3 + 4t, 4 + 3t\rangle$, and we want to put the bound $0 \leq t \leq 1$

The parametrics are $x(t) = 1 - 6t, y(t) = -3 + 4t$, and $z = 4 + 3t$, with $0 \leq t \leq 1$.

Example

Find a vector function that represents the curve of intersection of $x^2 + y^2 = 1$ and $y + z = 2$.

$x^2 + y^2 = 1$ is a cylinder and $y + z = 2$ is a plane.

We can represent $x^2 + y^2 = 1$ as $x = \cos t$ and $y = \sin t$, with bounds $0 \leq t \leq 2\pi$.

$y + z = 2$ can be represnetd as $z = 2 - y$ or $z = 2 - \sin t$ with $0 \leq t \leq 2\pi$.

So $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + (2 - \sin t)\vec{k} = \langle \cos t, \sin t, 2 - \sin t\rangle$ with $0 \leq t \leq 2\pi$.

Example

Find the domain of $\vec{r}(t) = \langle \ln|t - 1|, e^t, \sqrt{t}\rangle$.

The domain is all values of t for which $\vec{r}(t)$ is defined.

So we have $x = \ln|t - 1|$, $y = e^t$ and $z = \sqrt{t}$.

For x, we have the domain as $(-\infty, 1) \cup (1, \infty)$, for y we have the domain as $t \in \mathbb{R}$, and for z, we have $t \geq 0$, so combining them gives domain $[0, 1) \cup (1, \infty)$.

Definition

If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, then $\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle$ (as long as all 3 limits exist).

Example

Let $\vec{r}(t) = t^2\vec{i} + e^t\vec{j} - (2\cos \pi t)\vec{k}$. Find $\lim_{t\to 0} \vec{r}(t)$.

The limit of the $\vec{i}$ term is 0 as it goes to 0.

The limit of the $\vec{j}$ term is 1 as it approaches 0.

The limit of the $\vec{k}$ term is -2 as it approaches 0.

So the limit is $\lim_{t\to 0} \vec{r}(t) = \vec{j} - 2\vec{k}$

Example

Let $\vec{r}(t) = \left(\frac{4t^3+5}{3t^3+1}\right)\vec{i} + \left(\frac{1-\cos t}{t}\right)\vec{j} + \left(\frac{\ln(t+1)}{t}\right)\vec{k}$. Find $\lim_{t\to 0} \vec{r}(t)$.

For the first term, we get 5 as the limit.

For the other two, we will use L'Hopital's Rule.

Doing this and finding the limits should give that $\lim_{t\to 0} \vec{r}(t) = \langle 5, 0, 1\rangle$.

Continuity: A vector function $\vec{r}(t)$ is continuous at a if: $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$. (This is just AP Calculus BC)

1.2 Derivatives and Integrals of Vector Functions

Definition

If $\vec{r}(t)$ is a vector function, the derivative of $\vec{r}(t)$ with respect to t is

$$\vec{r}' = \vec{r}(t)' = \frac{d\vec{r}}{dt} = \frac{d}{dt}(\vec{r}(t)) = \lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$

Geometrically, this would have $\vec{r}(t)$ as a vector tangent to the curve at the tip of $\vec{r}(t)$. It points in the direction of increasing parameter.

Theorem 1.1

If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, where f, g, and h are differentiable functions, then

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$$

Proof. Let $\vec{r}(t) = \langle x(t), y(t)\rangle$

By definition, $\vec{r}'(t) = \lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$.

This is equal to $\lim_{h\to 0}\frac{[x(t+h)\vec{i}+y(t+h)\vec{j}]-[x(t)\vec{i}+y(t)\vec{j}]}{h}$.

Which is equal to

$$\left(\lim_{h\to 0}\frac{x(t+h)\vec{i}-x(t)\vec{i}}{h}\right)+\left(\lim_{h\to 0}\frac{y(t+h)\vec{j}-y(t)}{h}\right)$$

Taking out the $\vec{i}$ and $\vec{j}$, allows us to see that this equals to $x'(t)\vec{i}+y'(t)\vec{j}$. □□

Example

$\vec{r}(t) = \frac{1}{t}\vec{i} + e^{2t}\vec{j} - 2\cos\pi t\vec{k}$. Find $\vec{r}(t)$.

The derivative of this is simply $\langle\frac{-1}{t^2}, 2e^{2t}, 2\pi\sin\pi t\rangle$.

$\vec{r}'(t)$ refers to the tangent vector. The tangent line is the line through P that is parallel to $\vec{r}'(t)$.

Unit Tangent Vector: $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$.

Example

From the previous example, find the unit tangent vector at $t = 1$.

We know that $\vec{r}'(t) = \langle\frac{-1}{t^2}, 2e^{2t}, 2\pi\sin\pi t\rangle$.

From this, $\vec{r}'(1) = \langle -1, 2e^2, 0\rangle$, and the magnitude of this is $\sqrt{1+4e^4}$.

Therefore, $\vec{T}(1) = \langle\frac{-1}{\sqrt{1+4e^4}}, \frac{2e^2}{\sqrt{1+4e^4}}, 0\rangle$.

Exercise For the curve $\vec{r}(t) = \sqrt{t}\vec{i} + (2-t)\vec{j}$, find $\vec{r}'(t)$. Sketch $\vec{r}(1)$ and $\vec{r}'(1)$.

Example

Find parametric equations for the tangent line to the helix with equations $x = 2\cos t$, $y = \sin t$, and $z = t$ at the point $(0, 1, \pi/2)$.

We have $\vec{r}(t) = \langle 2\cos t, \sin t, t\rangle$, so $\vec{r}'(t) = \langle -2\sin t, \cos t, 1\rangle$.

We get $0 = 2\cos t$, $1 = \sin t$, and $\frac{\pi}{2} = t$, so we know that t is.

Plugging this in gives $\vec{r}'\left(\frac{\pi}{2}\right) = \langle -2, 0, 1\rangle$. This is the tangent vector.

So $\vec{r}(t) = \langle 0, 1, \frac{\pi}{2}\rangle + t\langle -2, 0, 1\rangle$.

Parametrically: $x = -2t$, $y = 1$, $z = \frac{\pi}{2} + t$.

Differentiation Rules:

1. $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$
2. $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$
3. $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
4. $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
5. $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$ (Order matters here)
6. $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\vec{u}'(f(t))$

Theorem 1.2

If $\vec{r}(t)$ is differentiable and $||\vec{r}(t)||$ is constant for all t, then $\vec{r}(t) \cdot \vec{r}'(t) = 0$.

This means they are orthogonal for all t.

1.3 Arc Length and Curvature

1.4 Motion in Space - Velocity and Acceleration