AP Calculus AB Notes

Stasya

Fall 2023 & Spring 2024

Contents

1 Limits and Continuity

1.1 Introduction to Limits

Let's start with an example.

Given the function $f(x) = \frac{x^2-1}{x-1}$, find $f(1)$.

We will end up getting 0/0. This is a difficulty! 0/0 is called indeterminate, so we need another way to answer this.

Let's start by approaching $x = 1$ from the left and approach $x = 1$ from the right. If we put this in tabular data, we end up approaching 2.

Now we can see that as x gets close to 1, then the function $f(x) = \frac{x^2-1}{x-1}$ gets close to 2.

We are now faced with an interesting situation: When $x = 1$, the answer is undefined, but we can see that it is going to be 2.

We want to give the answer "2", but we can't, so instead mathematicians say exactly what is going on by using the special world "limit".

The limit of $\frac{x^2-1}{x-1}$ is 2.

Symbolically, this is written as $lim_{x\to 1}\frac{x^2-1}{x-1} = 2$. It is a special way of saying, "ignore what happens when we get there, but as we get closer and closer, the answer gets closer and closer to 2".

Definition

If when the x values are approaching $x = c$ from either side $f(x)$ becomes arbitrarily close to a single number $y = L$, then the limit $f(x)$ as x approaches c is L.

$$\lim_{x\to c} f(x) = L$$

Limits can also be used even when we know the value when we get there. Nobody said they are only for difficult functions.

Problem 1: What is the limit of $x^2 - 3x + 4$ as x approaches 2?

Problem 2: What is the limit of $\frac{(x+h)^2-x^2}{h}$ as h approaches 0?

1.2 Limits Properties

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

1. scalar multiple: $\lim_{x\to c}[bf(x)] = b[\lim_{x\to c} f(x)] = bL$
2. sum or difference: $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x) = L \pm K$
3. product: $\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = L \cdot K$
4. quotient: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{K}$
5. power: $\lim_{x\to c}[f(x)]^n = [\lim_{x\to c} f(x)]^n = L^n$
6. composition: If f is a continuous function, $\lim_{x\to c} f(g(x)) = f[\lim_{x\to c} g(x)] = f(K)$

Problem 1: If the limit of $f(x)$ as x approaches c is 9, what is the limit of $4f(x)$ as x approaches c?

Problem 2: Find A so that the limit as x approaches 2 for $\frac{x^2+Ax-10}{x-2}$ exists.

1.3 One-Sided Limits

Some basic notation:

$$\lim_{x \to a^-} f(x) \text{means the limit from the left side and } \lim_{x \to a^+} \text{ means the limit from the right side.}$$

Given horizontal asymptotes, given the power of the highest degree in the numerator and denominator - when the power is higher in the numerator, it will approach infinity, when the power is higher in the denominator, it will approach zero, and when the degrees are the same, then the asymptote is the coefficients of the highest degrees divided by each other.

Problem 1: Find the indicated limit.

$$\lim_{x \to -4^+} \left(\frac{3x-1}{x+4} \right)$$

1.4 Continuity

Continuity means that you draw a graph without picking up a pencil.

- Removable discontinuity - hole, algebraically, you can find this when a factor in the top cancels out a factor at the bottom.
- Jump discontinuity - break, algebraically, it is a piecewise function.
- Infinite discontinuity - vertical asymptotes, algebraically, you set the denominator equal to zero.
- Mix discontinuity - a mix of any of the above three.

Definition

A function $f(x)$ is continuous at $x = c$ if and only if $\lim_{x \to c} f(x) = f(c)$.

In order to prove continuity, you must show three things:

1. $\lim_{x \to c} f(x)$ exists.
2. $f(c)$ is defined.
3. $\lim_{x \to c} f(x) = f(c)$.

Problem 1: Given $f(x) = \begin{cases} 6 + cx & x < 1, \\ 9 + 2\ln x & x \geq 1. \end{cases}$ Find c.

1.5 Limits with Infinity

To find a limit that goes to infinity - in general, a function does not have a limit when the degree of the exponent in the numerator is higher than the denominator. It will have a limit of zero when the degree of the exponent in the denominator is higher than the numerator. It would be the ratio of the coefficients of the highest degree of exponent if the highest exponent degrees are the same.

Problem 1: Find

$$\lim_{x \to \infty} \frac{3^x - 3}{3^x + 1}$$

1.6 Limits with Trig

Know that the special limit

$$\lim_{x \to 0} \frac{\sin x}{x}$$

yields a value of 1.

Problem 1: Find

$$\lim_{x \to 0} \frac{\sin 4x}{2x}$$

1.7 Intermediate Value Theorem (IVT)

The Intermediate Value Theorem states that if f is continuous on $[a, b]$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

Problem 1: Use the Intermediate Value Theorem to show that there exists a solution to $\cos x = x$ on the interval $\left[0, \frac{\pi}{2}\right]$.

2 Differentiation: Definition and Fundamental Properties

2.1 Average Rate of Change and Secant Lines

The average rate of change is known as the secant line or slope. Represented as a formula this can be written as

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a}$$

When a function is linear, we often refer to the average rate of change as simply the rate of change.

A tangent line is the instantaneous rate of change.

Problem 1: Suppose an object's position at time t is described by $s(t) = t^2 - 5t + 1$. What is the object's average velocity between time 0 and 3 seconds later?

Problem 2: A rock is thrown straight up, with an initial velocity of 20 meters per second and from an initial height of 2 meters. The height h of the rock after t seconds by the equation: $h(t) = 2 + 20t - 4.9t^2$. What is the rock's average velocity during the first two seconds of its flight?

2.2 Definition of Derivative

When finding the average rate of change we need two points.

The formula to find the instantaneous rate of change is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is called the difference quotient. A derivative is a common word used to mean instantaneous rate of change.

Symbolically we can write:

- The derivative of $f(x)$ is $f'(x)$.
- The derivative of y is y'.
- The derivative of y is $\frac{dy}{dx}$.

Given a specific x-value where $x = a$, then

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Problem 1: Find the instantaneous rate of change of the function $f(x) = \frac{1}{2}x^2 - 1$ on $[x, x+h]$.

Problem 2: What function is the definition of the derivative being applied to in

$$\lim_{h \to 0} \frac{\sqrt{2(x+h)} - \sqrt{2x}}{h}$$

Problem 3: Use the limit defintiion to find the derivative of the function $f(x) = -x^2 + 2x - 3$.

2.3 Derivative Rules

The definition of the derivative explains why the derivative represents an instantaneous slope. However, there are some quick and easy rules that make finding derivatives much less time consuming.

Basic Derivative Rules:

- Constant Rule: $\frac{d}{dx}(c) = 0$
- Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$
- Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$
- Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
- Difference Rule: $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Extra Derivatives:

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\csc x) = -\csc x \cot x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Problem 1: Differentiate $f(x) = \sqrt[3]{x}$.

Problem 2: If $f(x) = 5x^3$, then $f'(2) =$?

Problem 3: A projectile starts at time $t = 0$ and moves along the x-axis so that its velocity at any time $t \geq 0$ is $v(t) = t^3 - 6\csc t + e^t$. What is the acceleration of the particle at $t = 1$?

2.4 Product Rule

The product rule states

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

Problem 1: Differentiate $g(z) = \sqrt[3]{z^2}\sin z$.

Problem 2: For what values does $f(t) = -e^t(2t + 1)$ have a horizontal tangent?

Problem 3: Let $s(t) = \frac{1}{\pi} + 3\sin t$ represent the position of an object moving on a line. At what time(s) is the object at rest?

2.5 Quotient Rule

The quotient rule states

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Problem 1: Differentiate $\frac{3(1-\sin z)}{e^z}$.

Problem 2: For what value(s) of x does the function $f(x) = \frac{x^2}{x+1}$ have a horizontal tangent?

Problem 3: Let $s(t) = \frac{1}{\pi} + 3\sin t$ represent the position of an object moving on a line. What is the velocity of the object when the acceleration is 3 on $[0, 2\pi]$?

2.6 Tangent Lines

Problem 1: Let f be the function defined by $f(x) = 4x^3 - 5x + 3$. Write the equation of the line tangent to the graph of f at the point where $x = -1$.

Problem 2: Write the equation of the line tangent to the graph of $y = \frac{3x-2}{2x-3}$ at the point $(-1, 1)$.

2.7 Linear Approximation

Problem 1: Use the differential equation $\frac{dy}{dx} = \frac{3x^2+1}{2y}$ to write an equation for the line tangent to the graph of f at $f(1) = -1$ and use it to approximate $f(1.2)$.

2.8 Continuity & Differentiability

Now that you know about continuity and differentiability, let's expand the idea of differentiability and continuity. A function to be differentiable requires that the function cannot have a sharp turn or a vertical tangent. A function must be continuous to be differentiable.

Problem 1: Let f be the function defined, where c and d are constants. If f is differentiable at $x = 2$, what is the value of $c + d$? $f(x) = \begin{cases} cx + d & x \leq 2 \\ x^2 - cx & x > 2 \end{cases}$

3 Differentiation: Composite, Implicit, and Inverse Functions

3.1 Chain Rule

The chain rule formula is

$$y = f(g(h(x))) \to y' = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x) \cdot 1$$

Problem 1: Differentiate $h(x) = \sec^2 x$.

Problem 2: Differentiate $h(x) = \frac{4}{\sqrt{x^3+2}}$.

Problem 3: Differentiate e^{-x^2}.

3.2 L'Hopital's Rule

L'Hopital's Rule:

Suppose $f(a) = 0$ and $g(a) = 0$ and $lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{``0''}{0}$ or $\frac{``\infty''}{\infty}$. L'Hopital's Rule allows you to apply the following

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Problem 1: Evaluate the following. $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.

Problem 2: Evaluate the following: $\lim_{x\to 2} \frac{e^{2x}-e^4}{x-2}$.

3.3 Implicit Differentiation

When you cannot isolate y in terms of x or one variable, then you want to take the derivative implicitly.

The steps are essentially to take the derivative normally and then move anything containing $\frac{dy}{dx}$ or y' to one side. Then you want to create only one $\frac{dy}{dx}$ or y' term by factoring and then solve for this term.

Problem 1: Solve for $\frac{dy}{dx}$ for $3x^2 + 4xy^2 - 5x^3 = 10$.

Problem 2: Find the equation of the tangent line for $x^2 + y^2 = 4$ at $x = 1$.

Problem 3: Find $\frac{d^2y}{dx^2}$ for $y^2 - 3y = 2x^2 + x$ in terms of x and y.

Problem 4: A curve is generated by the equation $x^2 + 4y^2 = 16$. How many points on this curve have tangent lines that are horizontal?

3.4 Inverse Functions & Derivatives

Recall for inverse functions that domain and range are switched and slopes are reciprocals.

The derivative of an inverse function:

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(x)}$$

Note the inverse trig derivatives:

- $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}$
- $\frac{d}{dx}\cot^{-1}x = -\frac{1}{x^2+1}$

Problem 1: If $f(x) = \cos x - 6x$ and $f^{-1}(-9\pi) = \frac{3\pi}{2}$, find the derivative of $f^{-1}(x)$ at $x = -9\pi$.

Problem 2: Find the derivative of $\sec^{-1} 5x^6$.

4 Contextual Applications of Differentiation

4.1 Related Rates

Questions that ask for the calculation of the rate at which one variable changes, based on the rate at which another variable is known to change, are usually called related rates. Solutions are found by writing an equation that relates the variables of the problem, then differentiating them with respect to another variable. Since time is rarely a variable in the equation you write, you will have to differentiate implicitly with respect to time.

Process:

1. Draw a picture.
2. Make a list of known and unknown rates and quantities. Translate the given information in the problem into "calculus-speak".
3. Write a formula or equation relating to the variables from step #2.
4. Differentiate implicitly with respect to time.
5. Now you can plug in numbers and do calculations.
6. Translate from "calculus-speak" back to English and answer the question that is being asked.

Formulas:

- Perimeter of a rectangle: $P = 2l + 2w$.
- Circumference of a circle: $C = 2\pi r$.
- Area of a rectangle: $A = lw$ or $A = bh$.
- Area of a circle: $A = \pi r^2$.
- Area of a triangle: $A = \frac{1}{2}bh$.
- Pythagorean Theorem: $a^2 + b^2 = c^2$.
- Volume of a cylinder: $V = \pi r^2 h$.
- Volume of a cone: $V = \frac{1}{3}\pi r^2 h$.
- Volume of a sphere: $V = \frac{4}{3}\pi r^3$.

Problem 1: A cube has an edge of 40 feet at $t = 0$, and the edge is decreasing at a constant rate of 4 feet per minute. After 2 minutes, what would the rate of change of the volume in cubic feet per minute be?

4.2 Intro to Position, Velocity, & Acceleration

4.3 Numerical Position, Velocity, & Acceleration

4.4 Graphing Position, Velocity, & Acceleration

4.5 Analytical Positions, Velocity, & Acceleration

5 Analytical Applications of Differentiation

5.1 Extreme Values

5.2 1st Derivative Test

5.3 2nd Derivative Test

5.4 Mean Value Theorem and Rolle's Theorem

5.5 Optimization

5.6 Curve Sketching

6 Integration and Accumulation of Change

6.1 Left and Right Riemann Sums

6.2 Midpoint Sums

6.3 Trapezoidal Sums

6.4 Definite Integrals

6.5 Fundamental Theorem of Calculus

6.6 Anti-Derivatives

6.7 2nd Fundamental Theorem of Calculus

6.8 Trig Integrals

6.9 U-Sub

6.10 Definite U-Sub

6.11 Complex U-Sub

7 Differential Equations

7.1 Slope Fields

7.2 Separation of Variables

8 Applications of Integration

8.1 Average Value

8.2 Net Change

8.3 Area between Two Curves

8.4 Volume - Disk & Washer

8.5 Volume - Cross Sections