1 Vector-Valued Functions

1.1 Vector-Valued Functions

A vector valued function has the form $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

A set of parametric equations $x = x(t), y = y(t), z = z(t)$ can describe a curve in space.

It can also be viewed as a vector function where each variable varies with respect to an independent variable t.

A point $(x(t), y(t), z(t))$ on the curve is the head of the vector $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

We can consider the vector-valued function of the form:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$$

or

$$f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

where f, g, h are defined on some interval a to b.

The positive orientation of a curve is the direction the curve is generated as the parameter increases.

Example

Find the domain of $\mathbf{r}(t) = \frac{4}{\sqrt{1-t}}\mathbf{i} + \frac{2}{t+3}\mathbf{j}$

The domain is the largest set of values of t on which both f and g are defined.

The first component has $1 - t > 0$, therefore the domain is $(-\infty, 1)$.

The second component's domain is $(-\infty, -3) \cup (-3, \infty)$.

We now find the intersection which is $(-\infty, -3) \cup (-3, 1)$

Definition

A vector valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$$

, provided

$$\lim_{t \to a} |\mathbf{r}(t) - \mathbf{L}| = 0$$

A function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is continuous at **a** provided $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$.

1.2 Calculus of Vector-Valued Functions

We can define the derivative of a vector-valued function as:

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

The result of the derivative is a vector-valued function.

Geometrically, as $\Delta t \to 0$, $\frac{\Delta \mathbf{r}}{\Delta t} \to \mathbf{r}'(t)$, which is a tangent vector at point P.

Much like single variable calculus, we can simply use the power rule as we know, rather than the limit definition.

The unit tangent vector for a particular value t is:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

A vector-valued function is smooth if it is differentiable and the derivative is not equal to $\langle 0, 0, 0 \rangle$.

We can also integrate vector-valued functions. The rules remain the same as in single variable calculus, just breaking it into three vector components.

1.3 Motion in Space

The derivative of the position vector function is the velocity vector function.

The speed of a scalar function is

$$|v(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

Acceleration is the derivative of the velocity vector function.

Straight-line motion has a uniform velocity. Given:

$$\mathbf{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

the velocity $\mathbf{v}(t) = \langle a, b, c \rangle$

Circular motion has constant $|r(t)|$. We let $\mathbf{r}(t) = \langle A\cos t, A\sin t \rangle$.

$\mathbf{r}(t)$ describes a circular trajectory counter-clockwise around a circle with radius A and center at the origin.

We have:

$$|\mathbf{r}(t)| = \sqrt{(A\cos t)^2 + (A\sin t)^2} = A$$

and

$$\mathbf{v}(t) = \langle -A\sin t, A\cos t \rangle$$

as well as

$$\mathbf{a}(t) = \langle -A\cos t, -A\sin t \rangle = -\mathbf{r}(t)$$

Also there are some important properties - the position and acceleration vectors are both orthogonal to the velocity vector as seen:

- $\mathbf{r}(t) \cdot \mathbf{v}(t) = -A^2\cos t\sin t + A^2\sin t\cos t = 0$
- $\mathbf{a}(t) \cdot \mathbf{v}(t) = A^2\sin t\cos t - A^2\cos t\sin t = 0$

Let $\mathbf{r}$ describe a path on which $|\mathbf{r}|$ is constant.

We can show that $\mathbf{r} \cdot \mathbf{v} = 0$, showing that the position and velocity vectors are always orthogonal.

For two-dimensional motion in a gravitational field:

The gravitational force is $\mathbf{F} = \langle 0, -mg \rangle$.

Therefore: $\mathbf{F} = m\mathbf{a}(t) = \langle 0, -mg \rangle$.

This shows that $\mathbf{a}(t) = \langle 0, -g \rangle$.

We can summarize this:

The velocity of the object is

$$\mathbf{v}(t) = \langle x'(t), y'(t) \rangle = \langle u_0, -gt + v_0 \rangle$$

where $\mathbf{v}(0) = \langle u_0, v_0 \rangle$ and $\mathbf{r}(0) = \langle x_0, y_0 \rangle$.

The position is:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle u_0 t + x_0, -\frac{1}{2}gt^2 + v_0 t + y_0 \rangle$$

1.4 Length of Curves

We know the arc length of a parametric equation is:

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\mathrm{d}t$$

Now we can consider this equation in three dimensions.

The arc length of a parametrized curve is:

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\mathrm{d}t = \int_a^b |\mathbf{r}'(t)|\mathrm{d}t$$

To find the arc length of a curve given $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ for $t \geq a$, we have:

$$s(t) = \int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2}\mathrm{d}u = \int_a^t |\mathbf{v}(u)|\mathrm{d}u$$

Suppose $|\mathbf{v}(t)| = 1$, then we have $t - a$. This shows the parameter t corresponds to arc length.

1.5 Curvature and Normal Vectors

Curvature will be a measure of how fast a curve $\mathbf{r}(t)$ turns at a point.

Recall the unit tangent vector:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The curvature is

$$\kappa(s) = \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right|$$

where s denotes arc length, and $\mathbf{T}$ denotes the tangent vector.

Lines have zero curvature.

We can write the curvature in terms of arclength:

$$\kappa(t) = \frac{1}{|\mathbf{v}|}\left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right| \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

Circles have constant curvature of $\frac{1}{R}$.

An alternative curvature formula is used for trajectories of moving objects in three-space:

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

where $\mathbf{v} = \mathbf{r'}$ is the velocity and $\mathbf{a} = \mathbf{v'}$ is the acceleration.

The principal unit normal vector will determine the direction in which the curve turns.

The principal unit normal vector at point P on the curve at which $kappa \neq 0$ is:

$$\mathbf{N}(s) = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}s}{|\mathrm{d}\mathbf{T}/\mathrm{d}s|} = \frac{1}{\kappa}\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$$

For other parameters t, we use the equivalent formula:

$$\mathbf{N}(s) = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}t}{|\mathrm{d}\mathbf{T}/\mathrm{d}t|}$$

There are two ways to change the velocity of an object or accelerate - to change its speed or its direction of motion.

The acceleration vector of an object moving in space along a smooth curve has the following representation of its tangential component a_T (in the direction of $\mathbf{T}$) and its normal component a_N (in the direction of $\mathbf{N}$):

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$$

where $a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}}{|\mathbf{v}|}$ and $a_T = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$.

Alternatively $\mathbf{a}\cdot\mathbf{N} = a_N$ and $\mathbf{a}\cdot\mathbf{T} = a_T$.

If we have the unit tangent and principal unit vectors $\mathbf{T}$ and $\mathbf{N}$. The unit binormal vector at each point in the curve is:

$$\mathbf{B} = \mathbf{T}\times\mathbf{N}$$

and the torsion is:

$$\tau = -\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\cdot\mathbf{N}$$

The binormal vector is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$.

$|\tau| = \left|\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\right|$ and the torsion gives the rate at which the curve moves out of the osculating plane formed by $\mathbf{T}$ and $\mathbf{N}$.