Calculus III Notes

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Contents

1 Vectors and the Geometry of Space

1.1 Vectors in the Plane

Vectors are quantities with both magnitude and direction. The vector who's tail is at point P and head at point Q is denoted as $\overrightarrow{PQ}$.

Two vectors, **u** and **v** are equal if they have equal length and point in the same direction. They do not necessarily have to be in the same location.

Scalar multiplication happens when a c is multiplied to vector **v**. If $c < 0$, then vector cv and v will point in opposite directions, otherwise they will point in the same direction. Two vectors are parallel if they are scalar multiples of each other.

If you place the tail of a vector **v** at the head of another vector **u**, the sum **u+v** is the vector that extends from the tail of **u** to the head of **v**.

The vector difference **u-v** is defined as **u+(-v)**.

In order to do calculations with vectors, we must introduce a cartesian plane. Angle brackets $\langle a, b\rangle$ show the components of a vector.

The magnitude of a vector is simply its length. Given points $P(x_1, y_1)$ and $Q(x_1, y_1)$, the magnitude of the vector $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1\rangle$ is denoted as $|\vec{PQ}|$, is equal to:

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We can also now do vector addition with components. Given two vectors **u** and **v**, the vector sum is:

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2\rangle$$

For a scalar c and a vector **u**, the scalar multiple is c**u**. i.e $|c\mathbf{u}| = |c||\mathbf{u}|$

A unit vector is any vector with length 1. **i** is a unit vector in the x-direction and **j** is a unit vector in the y-direction.

Example

Determine the necessary air speed and heading that a pilot must maintain in order to fly her commercial jet north at a speed of 480 mi/hr relative to the ground in a crosswind that is blowing 60$\deg$ south of east at 20 mi/hr.

Let $\vec{p}$ be the velocity vector we are trying to find.

We have ground vector $\langle 0, 480\rangle$ and a crosswind vector $\langle 10, -10\sqrt{3}\rangle$. Note we got the x-component and the y-component of the crosswind vector from the formula: $\vec{v} = \langle a\cos\theta, b\cos\theta\rangle$

We can find the vector $\vec{p}$ from adding this vector to the crosswind vector resulting in: $\vec{p} = \vec{g} - \vec{c} = \langle -10, 480 + 10\sqrt{3}\rangle$.

The magnitude of this vector is the speed and is equal to 497.2 mi/hr roughly.

1.2 Vectors in Three Dimensions

We can create a z-axis to create a three dimensional system.

The xyz-plane is divided into octants and has 3 planes, the xy-plane, the xz-plane, and the yz-plane.

We can also extend the distance formula to 3 dimensions. It is similar to the distance formula in two dimensions, with the z-component added, essentially:

$$|PQ| = \sqrt{|PR|^2 + |RQ|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The midpoint formula works the same way:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2} + \frac{y_1 + y_2}{2} + \frac{z_1 + z_2}{2}\right)$$

The normal form of the circle equation is:

$$(x - h)^2 + (y - k)^2 = r^2$$

For a disk we have

$$(x - h)^2 + (y - k)^2 \leq r^2$$

We can generalize this to a sphere. A sphere centered at (a, b, c) with radius r is the set of points satisfying:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

A ball centered at (a, b, c) with radius r is the set of points satisfying:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2$$

Example

Find an equation of the sphere passing through $P(-4, 2, 3)$ and $Q(0, 2, 7)$ with its center at the midpoint of PQ.

We can find the midpoint from the midpoint formula and it is equal to $(-2, 2, 5)$.

The radius can be found through the distance formula and is equal to $\sqrt{8}$.

The equation is $(x + 2)^2 + (y - 2)^2 + (z - 5)^2 = 8$

All the vector operations from two-dimensions work in three-dimensions.

Example

A model airplane is flying horizontally due east at 10 mi/hr when it encounters a horizontal crosswind blowing south at 5 mi/hr and an updraft blowing vertically upward at 5 mi/hr.

- Find the position vector that represents the velocity of the plane relative to the ground.
- Find the speed of the plane relative to the ground.

The velocity vector of the model plane $\vec{p}$ is equal to $\langle 10, 0, 0\rangle$

The velocity vector of the horizontal crosswind $\vec{w}$ is equal to $\langle 0, -5, 0\rangle$

The velocity vector of the updraft $\vec{u}$ is $\langle 0, 0, 5\rangle$

Adding the three vectors results in the speed: $\langle 10, -5, 5\rangle$

The magnitude of this is roughly 12.25.

1.3 Dot Products

Definition

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in two or three dimensions, the dot product is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$$

The dot product is 0 when $\theta = \frac{\pi}{2}$, negative when $\theta > \frac{\pi}{2}$ and positive when $\theta < \frac{\pi}{2}$.

Two vectors are parallel if and only if $\mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$

When the dot product is zero, we call $\vec{u}$ and $\vec{v}$ orthogonal.

Theorem 1.1: Dot Product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

We now apply the dot product to vector projections.

Definition

The orthogonal projection of $\mathbf{u}$ on $\mathbf{v}$, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$ is:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)$$

The orthogonal projections can also be computed with the formulas:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{scal}_{\mathbf{v}}\mathbf{u} \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v}$$

where the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is

$$\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$$

1.4 Cross Products

Definition

Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the cross product $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$$

Note that $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

There are some useful properties of the cross product.

- The cross product $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
- The cross product is zero when $\sin(\theta) = 0$.
- Two vectors are parallel if the cross product between them is zero.

We define the determinant of a 2×2 array $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ as $ad - bc$.

For a matrix $\begin{vmatrix} a & b & c \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

the determinant is $a\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - b\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + c\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$ or:

$$a(u_2v_3 - u_3v_2) - b(u_1v_3 - u_3v_1) + c(u_1v_2 - u_2v_1)$$

1.5 Lines and Planes in Space

Recall that in two dimensions, we needed a point and a slope to write an equation for a line.

We can write an equation in three dimensions as well.

A vector equation of a line passing through the point $P_0(x_0, y_0, z_0)$ in the direction of vector $\mathbf{v} = \langle a, b, c\rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ or:

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

The corresponding parametric equations of the line also are:

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct$$

The general equation of a plane in R^3 with the plane passing through $P_0(x_0, y_0, z_0)$ with vector $\mathbf{v} = \langle a, b, c\rangle$ is described by:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

1.6 Cylinders and Quadric Surfaces

A cylinder is a surface that is parallel to a line.

A trace of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the xy-trace, the yz-trace, and the xz-trace.

To sketch quadric surfaces:

- Determine the points where the surface intersects the coordinate axes.
- Finding traces of the surface helps visualize the surface
- Sketch at least two traces in parallel planes

Example

For:

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 1$$

We set certain variables to zero to find the x-intercept to be ± 3, the y-intercept to be ± 4 and the z-intercept to be ± 5.

We can also find the traces:

- xy: $\frac{x^2}{9} + \frac{y^2}{16} = 1$
- xz: $\frac{x^2}{9} + \frac{z^2}{25} = 1$
- yz: $\frac{y^2}{16} + \frac{z^2}{25} = 1$

The resulting shape is a ellipsoid.

In general the equation of an ellipsoid is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

In this all traces are ellipses.

For an elliptic cone the equation is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.

For an elliptic paraboloid the equation is:

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.

For a hyperbolic paraboloid:

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ or $y = y_0$ are parabolas.

For a hyperboloid of one sheet:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Traces with $z = z_0$ are ellipses for all z_0. Traces with $x = x_0$ or $y = y_0$ are hyperbolas.

For a hyperboloid of two sheets:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Traces with $z = z_0$ with $|z_0| > |c|$ are ellipses. Traces with $x = x_0$ and $y = y_0$ are hyperbolas.

2 Vector-Valued Functions

2.1 Vector-Valued Functions

A vector valued function has the form $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

A set of parametric equations $x = x(t), y = y(t), z = z(t)$ can describe a curve in space.

It can also be viewed as a vector function where each variable varies with respect to an independent variable t.

A point $(x(t), y(t), z(t))$ on the curve is the head of the vector $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

We can consider the vector-valued function of the form:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

or

$$f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

where f, g, h are defined on some interval a to b.

The positive orientation of a curve is the direction the curve is generated as the parameter increases.

Example

Find the domain of $\mathbf{r}(t) = \frac{4}{\sqrt{1-t}}\mathbf{i} + \frac{2}{t+3}\mathbf{j}$

The domain is the largest set of values of t on which both f and g are defined.

The first component has $1 - t > 0$, therefore the domain is $(-\infty, 1)$.

The second component's domain is $(-\infty, -3) \cup (-3, \infty)$.

We now find the intersection which is $(-\infty, -3) \cup (-3, 1)$

Definition

A vector valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$$

, provided

$$\lim_{t \to a} |\mathbf{r}(t) - \mathbf{L}| = 0$$

A function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is continuous at $\mathbf{a}$ provided $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$.

2.2 Calculus of Vector-Valued Functions

We can define the derivative of a vector-valued function as:

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

The result of the derivative is a vector-valued function.

Geometrically, as $\Delta t \to 0$, $\frac{\Delta \mathbf{r}}{\Delta t} \to \mathbf{r'}(t)$, which is a tangent vector at point P.

Much like single variable calculus, we can simply use the power rule as we know, rather than the limit definition.

The unit tangent vector for a particular value t is:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

A vector-valued function is smooth if it is differentiable and the derivative is not equal to $\langle 0, 0, 0 \rangle$.

We can also integrate vector-valued functions. The rules remain the same as in single variable calculus, just breaking it into three vector components.

2.3 Motion in Space

The derivative of the position vector function is the velocity vector function.

The speed of a scalar function is

$$|v(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

Acceleration is the derivative of the velocity vector function.

Straight-line motion has a uniform velocity. Given:

$$\mathbf{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

the velocity $\mathbf{v}(t) = \langle a, b, c \rangle$

Circular motion has constant $|r(t)|$. We let $\mathbf{r}(t) = \langle A\cos t, A\sin t \rangle$.

$\mathbf{r}(t)$ describes a circular trajectory counter-clockwise around a circle with radius A and center at the origin.

We have:

$$|\mathbf{r}(t)| = \sqrt{(A\cos t)^2 + (A\sin t)^2} = A$$

and

$$\mathbf{v}(t) = \langle -A\sin t, A\cos t \rangle$$

as well as

$$\mathbf{a}(t) = \langle -A\cos t, -A\sin t \rangle = -\mathbf{r}(t)$$

Also there are some important properties - the position and acceleration vectors are both orthogonal to the velocity vector as seen:

- $\mathbf{r}(t) \cdot \mathbf{v}(t) = -A^2\cos t\sin t + A^2\sin t\cos t = 0$
- $\mathbf{a}(t) \cdot \mathbf{v}(t) = A^2\sin t\cos t - A^2\cos t\sin t = 0$

Let $\mathbf{r}$ describe a path on which $|\mathbf{r}|$ is constant.

We can show that $\mathbf{r} \cdot \mathbf{v} = 0$, showing that the position and velocity vectors are always orthogonal.

For two-dimensional motion in a gravitational field:

The gravitational force is $\mathbf{F} = \langle 0, -mg \rangle$.

Therefore: $\mathbf{F} = m\mathbf{a}(t) = \langle 0, -mg \rangle$.

This shows that $\mathbf{a}(t) = \langle 0, -g \rangle$.

We can summarize this:

The velocity of the object is

$$\mathbf{v}(t) = \langle x'(t), y'(t) \rangle = \langle u_0, -gt + v_0 \rangle$$

where $\mathbf{v}(0) = \langle u_0, v_0 \rangle$ and $\mathbf{r}(0) = \langle x_0, y_0 \rangle$.

The position is:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle u_0 t + x_0, -\frac{1}{2}gt^2 + v_0 t + y_0 \rangle$$

2.4 Length of Curves

We know the arc length of a parametric equation is:

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2} \mathrm{d}t$$

Now we can consider this equation in three dimensions.

The arc length of a parametrized curve is:

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} \mathrm{d}t = \int_a^b |\mathbf{r}'(t)| \mathrm{d}t$$

To find the arc length of a curve given $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ for $t \geq a$, we have:

$$s(t) = \int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2} \mathrm{d}u = \int_a^t |\mathbf{v}(u)| \mathrm{d}u$$

Suppose $|\mathbf{v}(t)| = 1$, then we have $t - a$. This shows the parameter t corresponds to arc length.

2.5 Curvature and Normal Vectors

Curvature will be a measure of how fast a curve $\mathbf{r}(t)$ turns at a point.

Recall the unit tangent vector:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The curvature is

$$\kappa(s) = \left| \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \right|$$

where s denotes arc length, and $\mathbf{T}$ denotes the tangent vector.

Lines have zero curvature.

We can write the curvature in terms of arclength:

$$\kappa(t) = \frac{1}{|\mathbf{v}|} \left| \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t} \right| \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

Circles have constant curvature of $\frac{1}{R}$.

An alternative curvature formula is used for trajectories of moving objects in three-space:

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

where $\mathbf{v} = \mathbf{r'}$ is the velocity and $\mathbf{a} = \mathbf{v'}$ is the acceleration.

The principal unit normal vector will determine the direction in which the curve turns.

The principal unit normal vector at point P on the curve at which $kappa \neq 0$ is:

$$\mathbf{N}(s) = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}s}{|\mathrm{d}\mathbf{T}/\mathrm{d}s|} = \frac{1}{\kappa} \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$$

For other parameters t, we use the equivalent formula:

$$\mathbf{N}(s) = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}t}{|\mathrm{d}\mathbf{T}/\mathrm{d}t|}$$

There are two ways to change the velocity of an object or accelerate - to change its speed or its direction of motion.

The acceleration vector of an object moving in space along a smooth curve has the following representation of its tangential component a_T (in the direction of **T**) and its normal component a_N (in the direction of **N**):

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$$

where $a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}}{|\mathbf{v}|}$ and $a_T = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$.

Alternatively $\mathbf{a}\cdot\mathbf{N} = a_N$ and $\mathbf{a}\cdot\mathbf{T} = a_T$.

If we have the unit tangent and principal unit vectors **T** and **N**. The unit binormal vector at each point in the curve is:

$$\mathbf{B} = \mathbf{T}\times\mathbf{N}$$

and the torsion is:

$$\tau = -\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\cdot\mathbf{N}$$

The binormal vector is orthogonal to both **T** and **N**.

$|\tau| = \left|\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\right|$ and the torsion gives the rate at which the curve moves out of the osculating plane formed by **T** and **N**.

3 Functions of Several Variables

3.1 Graphs and Level Curves

Definition

A function $z = f(x, y)$ assigns to each point (x, y) in a set D in $\mathbb{R}^2$ a unique real number z in a subset of $\mathbb{R}$. The set D is the domain of f. The range of f is the set of all real numbers z that are assumed as the points (x, y) vary over the domain.

Example

Find the domain of

$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2 - 25}}$$

Note that the denominator cannot be zero or is negative.

Therefore, the domain is $D = (x, y) : x^2 + y^2 > 25$.

Definition

For a function of two variables $f(x, y)$ a level curve is the set of points (x, y) in the xy-plane where $f(x, y)$ is equal to a constant z_0.

We can extend our defintions to functions of more than two variables.

Definition

For a function of three variables $f(x, y, z)$ a level surface is the set of points (x, y, z) in xyz-space where $f(x, y, z)$ is equal to a constant w_0.

We can describe level surfaces as well.

3.2 Limits and Continuity

We can write the limit of two variables as:

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L$$

There are some laws of limits.

Theorem 3.1

Let a, b, and c be real numbers.

- Constant function $f(x, y) = c$:

$$\lim_{(x,y)\to(a,b)} c = c$$

- Linear function $f(x, y) = x$:

$$\lim_{(x,y)\to(a,b)} x = a$$

- Linear function $f(x, y) = y$:

$$\lim_{(x,y)\to(a,b)} y = b$$

If we let the limit of a function $f(x, y) = L$ and the limit of a function $g(x, y) = M$, we can see:

- The sum of the limits is $L + M$.
- The difference of the limits is $L - M$.
- If we multiply a function by a constant c, the limit is cL.
- The product of the limits is LM.
- The difference of the limits is $\frac{L}{M}$.
- Putting a limit to a power n is L^n.

Definition

Let R be a region in $\mathbb{R}^2$. An interior point P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R.

A boundary point Q of R lies on the edge of R in the sense that every disk centered at Q contains at least one point in R and at least one point not in R.

Definition

A region is open if it consists entirely of interior points. A region is closed if it contains all its boundary points.

The two-path test for nonexistence of limits:

If $f(x, y)$ approaches the two different values as (x, y) approaches (a, b) along two different paths in the domain of f, then the limit does not exist.

Definition

The function f is continuous at the point (a, b) provided:

- f is defined at (a, b).
- The limit for f exists.
- The two quantities must be equal.

3.3 Partial Derivatives

The idea of a partial derivative is to differentiate f with respect to one variable, treating the other variable as a constant.

Definition

The partial derivative of f with respect to x at the point (a, b) is:

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

The partial derivative of f with respect to y at the point (a, b) is:

$$f_y(a, b) = \lim_{h\to 0} \frac{f(a + h) - f(a, b)}{h}$$

provided these limits exist.

The notation for these are:

$$f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

We can first partially differentiate f with respect to x and then with respect to x denoted as:

$$\frac{\partial^2 f}{\partial x^2}$$

We can do it first with respect to x and then with respect to y denoted as:

$$\frac{\partial^2 f}{\partial y \partial x}$$

We can also differentiate with respect to y and then with respect to x:

$$\frac{\partial^2 f}{\partial x \partial y}$$

Lastly, we can differentiate with respect to y twice:

$$\frac{\partial^2 f}{\partial y^2}$$

Theorem 3.2: Equality of Mixed Partial Derivatives

Assume f is defined on an open set D of $\mathbb{R}^2$, and that f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

3.4 The Chain Rule

Theorem 3.3

Let z be a differentiable function on x and y on its domain, where x and y are differentiable functions of t on an interval I. Then

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Let's suppose that we have a function $w = f(x, y, z)$ where x, y, z depend on t.

Then we have:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial w}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

Theorem 3.4

Let z be a differentiable function of x and y, where x and y are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Theorem 3.5

Let F be differentiable on its domain and suppose $F(x, y) = 0$ defines y as a differentiable function of x. Provided $F_y \neq 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$$

3.5 Directional Derivatives and the Gradient

Suppose we have the function $f(x, y)$.

$f_x(x, y)$ measures the rate of change in the direction of positive x-axis and $f_y(x, y)$ measures the rate of change in the direction of positive y-axis.

Definition

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The directional derivative of f at (a, b) in the direction of u is

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

provided the limit exists.

The directional derivative $D_{\mathbf{u}} f(a, b)$ measures the rate of change of f at the point (a, b) in the direction of $\mathbf{u}$.

Theorem 3.6

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The directional derivative of f at (a, b) in the direction of u is

$$D_{\mathbf{u}} f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle$$

Definition

Let f be differentiable at the point (x, y). The gradient of f at (x, y) is the vector-valued function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

The applications of this are:

Directional derivatives: $D_{\mathbf{u}} f(a, b) = \nabla f(a, b) \cdot \mathbf{u}$

The gradient gives the direction of maximum increase of a function.

Gradient is normal to level curves.

Theorem 3.7

Let f be differentiable at (a, b) with $\nabla(a, b) \neq 0$.

f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $|\nabla f(a, b)|$. (This will be maximized if $\theta = 0$.)

f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-|\nabla f(a, b)|$. (This will be minimized if $\theta = \pi$.)

The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

Theorem 3.8

Given a function f differentiable at (a, b), the line tangent to the level curve of f at (a, b) is orthogonal to the gradient $\nabla f(a, b)$, provided $\nabla f(a, b) \neq 0$.

We can now consider gradients of functions of three variables.

Theorem 3.9

Let f be differentiable at (a, b, c) and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector. The directional derivative of f at (a, b, c) in the direction of $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}} f(a, b, c) &= \nabla(a, b, c) \cdot \mathbf{u} \\ &= \langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \rangle \cdot \langle u_1, u_2, u_3 \rangle \end{aligned}$$

Assuming $\nabla f(a, b, c) \neq 0$, the gradient in three dimensions has the following properties.

f has its maximum rate of increase at (a, b, c) in the direction of the gradient $\nabla f(a, b, c)$, and the rate of change in this direction is $|\nabla f(a, b, c)|$.

f has its maximum rate of decrease at (a, b, c) in the direction of $-\nabla f(a, b, c)$, and the rate of change in this direction is $-|\nabla f(a, b, c)|$.

The directional derivative is zero in any direction orthogonal to $\nabla f(a, b, c)$.

3.6 Tangent Planes and Linear Approximation

Definition

Let F be differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq 0$. The plane tangent to the surface $F(x, y, z) = 0$ at P_0, called the tangent plane, is the plane passing through P_0 orthogonal to $\nabla F(a, b, c)$. An equation of the tangent plane is

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

Definition

Let f be differentiable at the point (a, b). An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

One of the main applications of tangent planes is linear approximation for a function near a point.

Definition

Let f be differentiable at (a, b). The linear approximation to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the tangent plane at that point, given by the equation

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

For a function of three variables, the linear approximation to $w = f(x, y, z)$ at the point $(a, b, c, f(a, b, c))$ is given by

$$L(x, y, z) = f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) + f(a, b, c)$$

The idea for linear approximations of functions gives the defintion of the differential.

Definition

Let f be differentiable at the point (x, y). The change in $z = f(x, y)$ as the independent variables change from (x, y) to $(x + \mathrm{d}x, y + \mathrm{d}y)$ is denoted Δz and is approximated by the differential $\mathrm{d}z$.

$$\Delta z \approx \mathrm{d}z = f_x(x, y)\mathrm{d}x + f_y(x, y)\mathrm{d}y$$

Note that Δz is the change in f on the surface and is equal to $f(x, y) - f(a, b)$.

Note that $\mathrm{d}z$ is the change in l on the tangent plane and is equal to $L(x, y) - f(a, b)$.

3.7 Maximum/Minimum Problems

Definition

Suppose (a, b) is a point in a region R on which $f(x, y)$ is defined. If $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a local maximum value of f. If $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a local minimum value of f. Local maximum and local minimum values are also cllaed local extreme values or local extrema.

Theorem 3.10

If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.

Definition

An interior point (a, b) in the domain of f is a critical point of f is either:

- $f_x(a, b) = f_y(a, b) = 0$ or
- at least one of the partial derivatives f_x and f_y does not exist at (a, b).

Theorem 3.11

Suppose the second partial derivatives of f are continuous throughout an open disk centered at the point (a, b), where $f_x(a, b) = f_y(a, b) = 0$. Let $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$.

- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b).
- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b).
- If $D(a, b) < 0$, then f has a saddle point at (a, b).
- If $D(a, b) = 0$, then the test is inconclusive.

Definition

Let f be defined on a set R in $\mathbb{R}^2$ containing the point (a, b). If $f(a, b) \geq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an absolute maximum value of f on R. If $f(a, b) \leq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an absolute minimum value of f on R.

We can describe the procedure for finding the absolute maximum/minimum values on closed bounded sets:

Let f be continuous on a closed bounded set R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R:

- Determine the values of f at all critical points in R.
- Find the maximum and minimum values of f on the boudnary of R.
- The greatest function value found in the previous steps is the absolute maximum value of f on R, and the least function value found is the absolute minimum value of f on R.

3.8 Lagrange Multipliers

The goal is to maximize (or minimize) a function $f(x,y)$ subject to the constraint $g(x,y) = 0$.

To find find absolute extrema on closed and bounded constraint curves using Lagrange Multipliers:

Let the objective function f and the constraint function g be differentiable on a region of $\mathbb{R}^2$ with $\nabla g(x,y) \neq 0$ on the curve $g(x,y) = 0$. To locate the absolute maximum and minimum values of f subject to the constraint $g(x,y) = 0$, carry out the following steps:

1. Find the values of x, y, and λ (if they exist) that satisfy the equations

$$\nabla f(x,y) = \lambda \nabla g(x,y) \qquad \text{and} \qquad g(x,y) = 0$$

2. Evalute f at the values (x,y) found in Step 1 and at the endpoints of the constraint curve (if they exist). Select the largest and smallest corresponding function values. These values are the absolute maximum and minimum values of f subject to the constraint.

(This section is kinda annoying, I don't like the amount of steps each question has) Basically, I hate optimization.

4 Multiple Integration

4.1 Double Integrals over Rectangular Regions

Definition

A function f defined on a rectangular region R in the xy-plane is integrable on R if

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} f\left(x_k^*, y_k^*\right) \Delta A_k$$

exists for all partitions of R and for all choices of (x_k^*, y_k^*) within those partitions. The limit is the double integral of f over R, which we write

$$\iint_R f(x, y) \mathrm{d}A = \lim_{\Delta \to 0} \sum_{k=1}^{n} f\left(x_k^*, y_k^*\right) \Delta A_k$$

We usually use Fubini's theorem.

Theorem 4.1

Let f be continuous on the rectangular region $R = (x, y) : a \leq x \leq b, c \leq y \leq d$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint_R f(x, y) \mathrm{d}A = \int_c^d \int_a^b f(x, y) \mathrm{d}x \mathrm{d}y = \int_a^b \int_c^d f(x, y) \mathrm{d}y \mathrm{d}x$$

Definition

The average value of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R} \iint_R f(x, y) \mathrm{d}A$$

The average height of $f(x, y)$ is $\frac{1}{Area(R)}$(Volume under $f(x, y)$).

4.2 Double Integrals over General Regions

4.3 Double Integrals in Polar Coordinates

4.4 Triple Integrals

4.5 Triple Integrals in Cylindrical and Spherical Coordinates

4.6 Integrals for Mass Calculations

4.7 Change of Variables in Multiple Integrals

5 Vector Calculus

5.1 Vector Fields

5.2 Line Integrals

5.3 Conservative Vector Fields

5.4 Green's Theorem

5.5 Divergence and Curl

5.6 Surface Integrals

5.7 Stokes' Theorem

5.8 Divergence Theorem