AP Calculus AB Notes

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Contents

1 Limits and Continuity

1.1 Introduction to Limits

Let's start with an example.

Given the function $f(x) = \frac{x^2-1}{x-1}$, find $f(1)$.

We will end up getting 0/0. This is a difficulty! 0/0 is called indeterminate, so we need another way to answer this.

Let's start by approaching $x = 1$ from the left and approach $x = 1$ from the right. If we put this in tabular data, we end up approaching 2.

Now we can see that as x gets close to 1, then the function $f(x) = \frac{x^2-1}{x-1}$ gets close to 2.

We are now faced with an interesting situation: When $x = 1$, the answer is undefined, but we can see that it is going to be 2.

We want to give the answer "2", but we can't, so instead mathematicians say exactly what is going on by using the special world "limit".

The limit of $\frac{x^2-1}{x-1}$ is 2.

Symbolically, this is written as $lim_{x\to 1}\frac{x^2-1}{x-1} = 2$. It is a special way of saying, "ignore what happens when we get there, but as we get closer and closer, the answer gets closer and closer to 2".

Definition

If when the x values are approaching $x = c$ from either side $f(x)$ becomes arbitrarily close to a single number $y = L$, then the limit $f(x)$ as x approaches c is L.

$$\lim_{x\to c} f(x) = L$$

Limits can also be used even when we know the value when we get there. Nobody said they are only for difficult functions.

Problem 1: What is the limit of $x^2 - 3x + 4$ as x approaches 2?

Problem 2: What is the limit of $\frac{(x+h)^2-x^2}{h}$ as h approaches 0?

1.2 Finding Limits and Limit Properties

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

1. scalar multiple: $\lim_{x\to c}[bf(x)] = b[\lim_{x\to c} f(x)] = bL$
2. sum or difference: $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x) = L \pm K$
3. product: $\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = L \cdot K$
4. quotient: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{K}$
5. power: $\lim_{x\to c}[f(x)]^n = [\lim_{x\to c} f(x)]^n = L^n$
6. composition: If f is a continuous function, $\lim_{x\to c} f(g(x)) = f[\lim_{x\to c} g(x)] = f(K)$

Problem 1: If the limit of $f(x)$ as x approaches c is 9, what is the limit of $4f(x)$ as x approaches c?

Problem 2: Find A so that the limit as x approaches 2 for $\frac{x^2+Ax-10}{x-2}$ exists.

1.3 One Sided Limits

Some basic notation:

$$\lim_{x \to a^-} f(x) \text{means the limit from the left side and } \lim_{x \to a^+} \text{ means the limit from the right side.}$$

Given horizontal asymptotes, given the power of the highest degree in the numerator and denominator - when the power is higher in the numerator, it will approach infinity, when the power is higher in the denominator, it will approach zero, and when the degrees are the same, then the asymptote is the coefficients of the highest degrees divided by each other.

Problem 1: Find the indicated limit.

$$\lim_{x \to -4^+} \left(\frac{3x-1}{x+4} \right)$$

1.4 Continuity

Continuity means that you draw a graph without picking up a pencil.

- Removable discontinuity - hole, algebraically, you can find this when a factor in the top cancels out a factor at the bottom.
- Jump discontinuity - break, algebraically, it is a piecewise function.
- Infinite discontinuity - vertical asymptotes, algebraically, you set the denominator equal to zero.
- Mix discontinuity - a mix of any of the above three.

Definition

A function $f(x)$ is continuous at $x = c$ if and only if $\lim_{x \to c} f(x) = f(c)$.

In order to prove continuity, you must show three things:

1. $\lim_{x \to c} f(x)$ exists.
2. $f(c)$ is defined.
3. $\lim_{x \to c} f(x) = f(c)$.

Problem 1: Given $f(x) = \begin{cases} 6 + cx & x < 1, \\ 9 + 2\ln x & x \geq 1. \end{cases}$ Find c.

1.5 Limits with Infinity

To find a limit that goes to infinity - in general, a function does not have a limit when the degree of the exponent in the numerator is higher than the denominator. It will have a limit of zero when the degree of the exponent in the denominator is higher than the numerator. It would be the ratio of the coefficients of the highest degree of exponent if the highest exponent degrees are the same.

Problem 1: Find

$$\lim_{x \to \infty} \frac{3^x - 3}{3^x + 1}$$

1.6 Special Trig Limits

Know that the special limit

$$\lim_{x \to 0} \frac{\sin x}{x}$$

yields a value of 1.

Problem 1: Find

$$\lim_{x \to 0} \frac{\sin 4x}{2x}$$

1.7 Intermediate Value Theorem (IVT)

The Intermediate Value Theorem states that if f is continuous on $[a, b]$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

Problem 1: Use the Intermediate Value Theorem to show that there exists a solution to $\cos x = x$ on the interval $\left[0, \frac{\pi}{2}\right]$.

2 Differentiation: Definition and Fundamental Properties

2.1 Average Rate of Change and Secant Lines

The average rate of change is known as the secant line or slope. Represented as a formula this can be written as

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a}$$

When a function is linear, we often refer to the average rate of change as simply the rate of change.

A tangent line is the instantaneous rate of change.

Problem 1: Suppose an object's position at time t is described by $s(t) = t^2 - 5t + 1$. What is the object's average velocity between time 0 and 3 seconds later?

Problem 2: A rock is thrown straight up, with an initial velocity of 20 meters per second and from an initial height of 2 meters. The height h of the rock after t seconds by the equation: $h(t) = 2 + 20t - 4.9t^2$. What is the rock's average velocity during the first two seconds of its flight?

2.2 Definition of Derivative

When finding the average rate of change we need two points.

The formula to find the instantaneous rate of change is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is called the difference quotient. A derivative is a common word used to mean instantaneous rate of change.

Symbolically we can write:

- The derivative of $f(x)$ is $f'(x)$.
- The derivative of y is y'.
- The derivative of y is $\frac{dy}{dx}$.

Given a specific x-value where $x = a$, then

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Problem 1: Find the instantaneous rate of change of the function $f(x) = \frac{1}{2}x^2 - 1$ on $[x, x+h]$.

Problem 2: What function is the definition of the derivative being applied to in

$$\lim_{h \to 0} \frac{\sqrt{2(x+h)} - \sqrt{2x}}{h}$$

Problem 3: Use the limit defintiion to find the derivative of the function $f(x) = -x^2 + 2x - 3$.

2.3 Derivative Rules

The definition of the derivative explains why the derivative represents an instantaneous slope. However, there are some quick and easy rules that make finding derivatives much less time consuming.

Basic Derivative Rules:

- Constant Rule: $\frac{d}{dx}(c) = 0$
- Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$
- Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$
- Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
- Difference Rule: $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Extra Derivatives:

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\csc x) = -\csc x \cot x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Problem 1: Differentiate $f(x) = \sqrt[3]{x}$.

Problem 2: If $f(x) = 5x^3$, then $f'(2) =$?

Problem 3: A projectile starts at time $t = 0$ and moves along the x-axis so that its velocity at any time $t \geq 0$ is $v(t) = t^3 - 6\csc t + e^t$. What is the acceleration of the particle at $t = 1$?

2.4 Product Rule

The product rule states

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

Problem 1: Differentiate $g(z) = \sqrt[3]{z^2}\sin z$.

Problem 2: For what values does $f(t) = -e^t(2t+1)$ have a horizontal tangent?

Problem 3: Let $s(t) = \frac{1}{\pi} + 3\sin t$ represent the position of an object moving on a line. At what time(s) is the object at rest?

2.5 Quotient Rule

The quotient rule states

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Problem 1: Differentiate $\frac{3(1-\sin z)}{e^z}$.

Problem 2: For what value(s) of x does the function $f(x) = \frac{x^2}{x+1}$ have a horizontal tangent?

Problem 3: Let $s(t) = \frac{1}{\pi} + 3\sin t$ represent the position of an object moving on a line. What is the velocity of the object when the acceleration is 3 on $[0, 2\pi]$?

2.6 Tangent Lines

Problem 1: Let f be the function defined by $f(x) = 4x^3 - 5x + 3$. Write the equation of the line tangent to the graph of f at the point where $x = -1$.

Problem 2: Write the equation of the line tangent to the graph of $y = \frac{3x-2}{2x-3}$ at the point $(-1, 1)$.

2.7 Linear Approximation

Problem 1: Use the differential equation $\frac{dy}{dx} = \frac{3x^2+1}{2y}$ to write an equation for the line tangent to the graph of f at $f(1) = -1$ and use it to approximate $f(1.2)$.

2.8 Differentiability & Continuity

Now that you know about continuity and differentiability, let's expand the idea of differentiability and continuity. A function to be differentiable requires that the function cannot have a sharp turn or a vertical tangent. A function must be continuous to be differentiable.

Problem 1: Let f be the function defined, where c and d are constants. If f is differentiable at $x = 2$, what is the value of $c + d$? $f(x) = \begin{cases} cx + d & x \leq 2 \\ x^2 - cx & x > 2 \end{cases}$

3 Differentiation: Composite, Implicit, and Inverse Functions

3.1 The Chain Rule

The chain rule formula is

$$y = f(g(h(x))) \to y' = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x) \cdot 1$$

Problem 1: Differentiate $h(x) = \sec^2 x$.

Problem 2: Differentiate $h(x) = \frac{4}{\sqrt{x^3+2}}$.

Problem 3: Differentiate e^{-x^2}.

3.2 L'Hopital's Rule

L'Hopital's Rule:

Suppose $f(a) = 0$ and $g(a) = 0$ and $lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{``0''}{0}$ or $\frac{``\infty''}{\infty}$. L'Hopital's Rule allows you to apply the following

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Problem 1: Evaluate the following. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

Problem 2: Evaluate the following: $\lim_{x\to 2}\frac{e^{2x}-e^4}{x-2}$.

3.3 Implicit Differentiation

When you cannot isolate y in terms of x or one variable, then you want to take the derivative implicitly.

The steps are essentially to take the derivative normally and then move anything containing $\frac{dy}{dx}$ or y' to one side. Then you want to create only one $\frac{dy}{dx}$ or y' term by factoring and then solve for this term.

Problem 1: Solve for $\frac{dy}{dx}$ for $3x^2 + 4xy^2 - 5x^3 = 10$.

Problem 2: Find the equation of the tangent line for $x^2 + y^2 = 4$ at $x = 1$.

Problem 3: Find $\frac{d^2y}{dx^2}$ for $y^2 - 3y = 2x^2 + x$ in terms of x and y.

Problem 4: A curve is generated by the equation $x^2+4y^2 = 16$. How many points on this curve have tangent lines that are horizontal?

3.4 Inverse Functions & their Derivatives

Recall for inverse functions that domain and range are switched and slopes are reciprocals.

The derivative of an inverse function:

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(x)}$$

Note the inverse trig derivatives:

- $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}$
- $\frac{d}{dx}\cot^{-1}x = -\frac{1}{x^2+1}$

Problem 1: If $f(x) = \cos x - 6x$ and $f^{-1}(-9\pi) = \frac{3\pi}{2}$, find the derivative of $f^{-1}(x)$ at $x = -9\pi$.

Problem 2: Find the derivative of $\sec^{-1} 5x^6$.

4 Contextual Applications of Differentiation

4.1 Related Rates

Questions that ask for the calculation of the rate at which one variable changes, based on the rate at which another variable is known to change, are usually called related rates. Solutions are found by writing an equation that relates the variables of the problem, then differentiating them with respect to another variable. Since time is rarely a variable in the equation you write, you will have to differentiate implicitly with respect to time.

Process:

1. Draw a picture.
2. Make a list of known and unknown rates and quantities. Translate the given information in the problem into "calculus-speak".
3. Write a formula or equation relating to the variables from step #2.
4. Differentiate implicitly with respect to time.
5. Now you can plug in numbers and do calculations.
6. Translate from "calculus-speak" back to English and answer the question that is being asked.

Formulas:

- Perimeter of a rectangle: $P = 2l + 2w$.
- Circumference of a circle: $C = 2\pi r$.
- Area of a rectangle: $A = lw$ or $A = bh$.
- Area of a circle: $A = \pi r^2$.
- Area of a triangle: $A = \frac{1}{2}bh$.
- Pythagorean Theorem: $a^2 + b^2 = c^2$.
- Volume of a cylinder: $V = \pi r^2 h$.
- Volume of a cone: $V = \frac{1}{3}\pi r^2 h$.
- Volume of a sphere: $V = \frac{4}{3}\pi r^3$.

Problem 1: A cube has an edge of 40 feet at $t = 0$, and the edge is decreasing at a constant rate of 4 feet per minute. After 2 minutes, what would the rate of change of the volume in cubic feet per minute be?

Problem 2: A camera is filming the progress as a daredevil attempts to scale the wall of a skyscraper. The climber is moving vertically at a constant rate of 16 feet per minute, and the camera if 400 feet from the base of the skyscraper. Through how many radians per minute is the camera angle changing when the climber is 300 feet up the building?

Problem 3: Water is poured into a conical tank that is 24 feet tall and has a diameter at the top of 20 feet. The radius of the surface of the water in the tank is increasing at 0.75 feet per minute. At what rate is the area of the surface changing when the radius is 4.2 feet?

4.2 Particle Motion - Position, Velocity, & Acceleration

If $x(t)$ represents the position of a particle along the x-axis at any time t, then the following statements are true.

1. "Initially" means when $t = 0$.
2. "At the origin" means $x(t) = 0$.
3. "At rest" means $v(t) = 0$.
4. If the velocity of the particle is positive, then the particle is moving to the right.
5. If the velocity of the particle is negative, then the particle is moving to the left.
6. To find the average velocity over a time interval, divide the change in velocity by the change in time.
7. Instantaneous velocity is the velocity at a single moment in time.
8. If the acceleration of the particle is positive, then the velocity is increasing.
9. If the acceleration of the particle is negative, then the velocity is decreasing.
10. In order for a particle to change direction, the velocity must change sigs.

Problem 1: Draw a position, velocity and acceleration graph for the following scenario: Start at the 10 yard mark of the measuring tape. Walk forward quickly for 8 seconds. Then, turn around and walk slowly back toward the beginning of the tape for 8 seconds. Turn around again and walk away at a medium rate from the beginning of the tape until time runs out.

5 Analytical Applications of Differentiation

5.1 Extrema on an Interval

Definition

Let f be defined on an interval I containing c.

1. $f(c)$ is the minimum of f on I if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the maximum of f on I if $f(c) \geq f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the extreme values or extrema of the function on the interval. The minimum and maximum of a function on an interval are also caleld the absolute minimum and absolute maximum, or the global minimum and global maximum, on the interval.

Definition

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $(c, f(c))$ is called a relative maximum of f, or you can say that f has a relative maximum at $(c, f(c))$.
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $(c, f(c))$ is called a relative minimum of f, or you can say that f has a relative minimum at $(c, f(c))$.

The relative maximum and relative minimum points are sometimes called local maximum and local minimum points, respectively.

Definition

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a critical number of f and the point $(c, f(c))$ is a critical point of f.

Relative extrema occur only at critical numbers.

Theorem 5.1

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ or an absolute minimum value $f(d)$ for some numbers c and d in $[a, b]$.

To find the extrema of a continuous function f on a closed interval $[a, b]$ use the following steps:

1. Find $f'(x)$ and the critical numbers of f in $[a, b]$.
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint in $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

Problem 1: Find the absolute maximums and minimums of f on the given closed interval, and state where these values occur.

$f(x) = \sin^2 x + \cos x \quad [0, 2\pi]$.

5.2 1st Derivative Test

- When the function is increasing, the derivative is positive.
- When the function is decreasing, the derivative is negative.
- When the function changes from increasing to decreasing (or vice versa), the derivative is zero or undefined.

The first derivative test:

1. Find the critical points.
2. Draw a number line with those critical points.
3. Identify the intervals to consider.
4. Choose a test value in each interval.
5. Plug the test value into the derivative to find the sign.
6. Make conclusions about the function and relative extrema.

When a function changes concavity that is called a point of inflection. A point of inflection occurs whenever the second derivative is zero or undefined.

Problem 1: Find the interval(s) where the function $f(x) = \frac{1}{2}x - \sin x$ is increasing and decreasing and the interval(s) where the function is concave up and down and has point(s) of inflection on the interval $[0, 2\pi]$.

5.3 2nd Derivative Test

- If the function has horizontal tangent $(f' = 0)$ and is concave up $(f'' > 0)$, then the function has a relative minimum at that x-value.
- If the function has horizontal tangent $(f' = 0)$ and is concave down $(f'' < 0)$, then the function has a relative maximum at that x-value.

The second derivative test:

1. Find the possible point(s) of inflection.
2. Draw a number line with those point(s) of inflection.
3. Identify the intervals to consider.
4. Choose a test value in each interval.
5. Plug the test value into the second derivative to find the sign.
6. Make conclusions about the function and point(s) of inflection.

Problem 1: Find the interval in which $f'(x) = x \ln x$ is concave down given that the domain of this function f is $x > 0$.

5.4 Mean Value Theorem and Rolle's Theorem

Theorem 5.2

Rolle's Theorem: Let f be continuous on the closed interval $[a, b]$ and differentiable in the open interval (a, b). If $f(a) = f(b)$, then there is at least one number in c in (a, b) such that $f'(c) = 0$.

Theorem 5.3

Mean Value Theorem: If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Problem 1: Let f be a twice-differentiable function such that $f(2) = 5$ and $f(5) = 2$. Let g be the function given by $g(x) = f(f(x))$.

- Explain why there must be a value c for $2 < c < 5$ such that $g'(c) = 1$.
- Show that $g'(2) = g'(5)$. Use this result to explain why there must be a value k for $2 < k < 5$ such that $g''(k) = 0$.

5.5 Optimization

Optimization is the process of finding the "optimal" value of a quantity. Optimal values are often either the maximum or minimum values of a certain functino.

1. Determine what you are trying to maximize of minimize.
2. Determine what your answer must look like.
3. Draw a visual if possible.
4. Write the equation you must maximize or minimize.
5. Use information from the problem to get the equation from #4 in terms of one variable.
6. Take the derivative of the new equation from #5.
7. Set the derivative equal to zero to identify critical values.
8. Choose the critical value that is also the particular extreme value you need from #1 and justify your choice.
9. Answer the question identified in #2.

Problem 1: What point on the graph $y = \sqrt{x}$ is closest to $(5, 0)$?

Problem 2: A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?

Problem 3: A power station is on onside of a river that is 1/2 mile wide, and a factory is 6 miles downstream on the other side. It costs $60,000 per mile to run power lines over land and $85,000 per mile to run them underwater. FInd the most economical path for the transmission line from the power station to the factory.

5.6 Sketching Graphs and Their Derivatives

Problem 1: Sketch the graph of a function f with the given characteristics

- $f(2) = f(4) = 0$
- $f'(x) > 0$ if $x < 3$
- $f'(3)$ does not exist
- $f'(x) < 0$ if $x > 3$
- $f''(x) > 0, x \neq 3$

6 Integration and Accumulation of Change

6.1 Left and Right Riemann Sums

Problem 1: Let $f(x) = x^2 + x$. Consider the region bounded by the graph of f, the x-axis and the line $x = 2$. Divide the interval $[0, 2]$ into 4 equal subintervals.

- Obtain an estimate for the region using a right Riemann sum. Is this an over or underestimate. Why?
- Obtain an estimate for the region using a left Riemann sum. Is this an over or underestimate. Why?

6.2 Midpoint Sums

Problem 1: Given the function $f(x) = x^2 + 1$, estimate the area bounded by the graph of the curve and the x-axis on $[0, 2]$ by using a midpoint Riemann sum with $n = 2$ equal subintervals.

6.3 Trapezoidal Sums

You jump out of an airplane. Before your parachute opens, you fall faster and faster, but your acceleration decreases as you fall because of air resistance. The following data gives you acceleration in m/sec^2 after t seconds.

- Time: 0, Acceleration: 9.81
- Time: 2, Acceleration: 8.03
- Time: 4, Acceleration: 6.53
- Time: 6, Acceleration: 5.38
- Time: 8, Acceleration: 4.41
- Time: 10, Acceleration: 3.61

Use the trapezoid method with 5 subintervals of equal length to estimate your speed after ten seconds. Is this an underestimate of overestimate? Why?

6.4 Definite Integrals

The notation of a definite integral is $\int_a^b f(x)\mathrm{d}x$.

The defintion of a definite integral is the exact area between a curve $f(x)$ and the x-axis on an interval (a, b).

Properties of definite integrals:

- Zero Integral: $\int_a^a f(x)\mathrm{d}x = 0$
- Negation: $\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x$
- Multiply by a constant: $\int_a^b kf(x)\mathrm{d}x = k\int_a^b f(x)\mathrm{d}x$
- Decomposition: $\int_a^c f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x$
- Addition and Subtraction: $\int_a^b f(x) \pm g(x) = \int_a^b f(x) \pm \int_a^b g(x)$

Problem 1: Given $\int_{-2}^1 f(x)\mathrm{d}x = 4 \quad \int_1^5 f(x)\mathrm{d}x = -3 \quad \int_{-2}^1 g(x)\mathrm{d}x = 8$, find $\int_{-2}^1 [f(x) + 2g(x)]\mathrm{d}x$.

6.5 Fundamental Theorem of Calculus

The basic rule of integration is $\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C$.

Indefinite integral: $\int f'(x)\mathrm{d}x = f(x) + C$

Theorem 6.1

If a function f' is continuous on the closed interval $[a, b]$ and f is an antiderivative of f' on the interval $[a, b]$, then

$$\int_b^a f'(x)\mathrm{d}x = f(x)|_a^b = f(b) - f(a)$$

Problem 1: Evaluate the definite integral $\int_0^3 |x-2|\mathrm{d}x$.

6.6 Antiderivatives and Specific Solutions

Memorize the following integrals:

- $\int \frac{1}{x}\mathrm{d}x = |\ln x| + C$
- $\int e^x \mathrm{d}x = e^x + C$
- $\int \cos x \mathrm{d}x = \sin x + C$
- $\int \sin x \mathrm{d}x = -\cos x + C$

Problem 1: Find y, if $y'' = \cos x$ and $y'\left(\frac{\pi}{2}\right) = 2$ and $y\left(\frac{\pi}{2}\right) = 3\pi$.

Problem 2: A particle moves along the x-axis with an acceleration of $a(t) = 12t - 4$. The particle's velocity is 18 centimeters per second at $t = 2$. The initial position of the particle is 8 cm. What is the position of the particle at $t = 3$?

Problem 3: Evaluate $\int \frac{1-\sin^2 x}{\cos x}\mathrm{d}x$.

6.7 2nd Fundamental Theorem of Calculus

Theorem 6.2

If $f(x) = \int_a^b f'(t)\mathrm{d}t$, f is a continuous function, and g, h are differentiable functions, then

$$f'(x) = f(b) \cdot b' - f(a) \cdot a'$$

Break down:

- Plug in the top bound and multiply by the derivative of the top bound
- Plug in the bottom bound and multiply by the derivative of the bottom bound
- Derivative of an Integral = #1 - #2

Use this theorem if you have an integral with a variable on at least one bound or are taking the derivative of an integral.

Problem 1: Find $f'(x) : f(x) = \int_1^{4x} h(t)\mathrm{d}t$

Problem 2: Take the derivative with respect to x: $\int_{-x}^{x} 5t\mathrm{d}t$

6.8 Trigonometric Integrals

Trig Integrals:

- $\int \cos x \mathrm{d}x = \sin x + C$
- $\int -\sin x \mathrm{d}x = \cos x + C$
- $\int \sec x \tan x \mathrm{d}x = \sec x + C$
- $\int -\csc x \cot x \mathrm{d}x = \csc x + C$
- $\int \sec^2 x \mathrm{d}x = \tan x + C$
- $\int -\csc^2 x \mathrm{d}x = \cot x + C$

Problem 1: $\int_{-\pi/16}^{0} \sec 4x \tan 4x \mathrm{d}x$

Problem 2: $\int \frac{20x^3}{\sqrt{1-25x^8}} \mathrm{d}x$

6.9 Integration with U-Substitution

- Find the outside function and the inside function. The outside function should be the derivative of the inside.
- Set u equal to the inside function.
- Find $\frac{\mathrm{d}u}{\mathrm{d}x}$. This is the derivative of u with respect to x.
- Solve for $\mathrm{d}u$ Make sure this matches the outside function $+\mathrm{d}x$.
- Substitute the u variables into the integral for the x variables.
- Integrate.
- Substitute x back into the answer.

Problem 1: $\int x(1+2x^2)^2 \mathrm{d}x$

Problem 2: $\int \tan^4 x \sec^2 x \mathrm{d}x$

Problem 3: $\int \frac{\csc^2 x}{\cot x} \mathrm{d}x$

6.10 Definite Integrals with U Substitution

Problem 1: $\int_0^1 \frac{x}{x^2+1}^3 \mathrm{d}x$

Problem 2: $\int_1^e \frac{\ln x}{x} \mathrm{d}x$

6.11 Complex U Substitution

Problem 1: $\int (x+1)\sqrt{2-x} \mathrm{d}x$

Problem 2: $\int x^2 \sqrt{1-x} \mathrm{d}x$

7 Differential Equations

A differentiable equation is an equation that involves a derivative.

Steps to draw a slope field:

- Choose a (x, y) point to graph.
- PLug that point into $\frac{\mathrm{d}y}{\mathrm{d}x}$
- Draw a line fragment that represents the slope found in #2 at the point chosen in #1.

Problem 1: Consider the differential equation given by $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy}{2}$.

(a) Draw a slope field for the given differential equation.

(b) Let f be the function that satisfies the given differential equation. Write an equation for the tangent line to the curve $y = f(x)$ through the point $(1, 1)$. Then use your tangent line equation to estimate the value of $f(1.2)$.

Problem 2: Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x + y - 1$.

(a) Sketch a slope field for the given differential equation.

(b) Find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of x and y. Describe the region in the xy-plane in which all solution curves to the differential equation are concave up.

(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condiction $f(0) = 1$. Does f have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.

8 Applications of Integration

8.1 Average Value of a Function

Theorem 8.1

If f is continuous on $[a, b]$, then there exists some number c in the open interval (a, b) such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$

Problem 1: Given the function $r(x) = 2\sin x - 1$ where r is the rate at which Mr. Brust's waistline is changing (inches per month) and x is time (months):

(a) What is the average rate that Mr. Brust's waistline changes from the 10th month to the 12th month?

(b) What is the average change of this rate during the first 5 months?

8.2 Net Change

The integral of the rate of change is the next change.

- $\int v(t)\mathrm{d}t$ is the displacement.
- $\int |v(t)|\mathrm{d}t$ is the total distance.

Problem 1: A particle's velocity is given by $v(t) = t^3 - 2t^2 + 1$. If $x(t)$ represents the position of the particle along the x-axis, find the following:

(a) The position of the particle after 3 seconds if $x(0) = 5$.

(b) The position of the particle after 2 seconds if $x(1) = -2$.

Problem 2: If $H(-1) = 12$ and $H'(t) = cos(\pi t)$, what is $H\left(\frac{3}{2}\right)$?

8.3 Area between Two Curves

Steps to find the area between curves:

- Sketch the graph.
- Determine if the graph is perpendicular to x or y.
- Set up the integral.
- Solve the integral.

If it is perpendicular to the x axis the formula is $A = \int_{x_1}^{x_2}[(\text{Top Curve}) - (\text{Bottom Curve})]\mathrm{d}x$

If it is perpendicular to the y-axis the formula is $A = \int_{x_1}^{x_2}[(\text{Right Curve}) - (\text{Left Curve})]\mathrm{d}y$

Problem 1: Find the area enclosed by the curves $y = x^2 + 3$ and $y = 4x^2$.

Problem 2: Find the area enclosed by the curves $y = |x|$ and $y = x^2 - 2$.

8.4 Volume - Disk & Washer

If a region in a plane is revolved about a line, the resulting figure is a solid of revolution, and the line is called the axis of revolution.

Horizontal Axis of Revolution: $V = \pi \int_a^b [R(x)]^2 \mathrm{d}x$

Vertical Axis of Revolution: $V = \pi \int_c^d [R(y)]^2 \mathrm{d}y$

For the washer method if there is a gap in the shape:

Horizontally: $V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2\right) \mathrm{d}x$

Vertically: $V = \pi \int_c^d \left([R(y)]^2 - [r(y)]^2\right) \mathrm{d}y$

Problem 1: Set up the integral to find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the line $x = 6$. $y = x \quad y = 0 \quad y = 4 \quad x = 6$

Problem 2: Set up the integral to find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the x-axis. $y = \sin x \quad y = 0 \quad x = 0 \quad x = \pi$

8.5 Volume - Cross Sections

- Volume for a rectangle: $\int_a^b \frac{(T-B)}{b} h \mathrm{d}x$.
- Volume for a square: $\int_a^b \frac{(T-B)^2}{s} \mathrm{d}x$.
- Volume for a semi-circle: $\frac{1}{8}\pi \int_a^b (T-B)^2 \mathrm{d}x$.
- Volume for a isosceles triangle: $\frac{1}{2} \int_a^b (T-B)^2 \mathrm{d}x$.
- Volume for an equilaterial triangle: $\frac{\sqrt{3}}{4} \int_a^b (T-B)^2 \mathrm{d}x$.

Problem 1: Set up an integral to find the volume of the solid whose base is bounded by the graphs of $y = x^3$ and $y = 0$ and $x = 1$ with the indicated cross sections taken perpendicular to the y-axis.

(a) Squares

(b) Rectangles of height 1

(c) Semicircles