

# Linear Algebra Notes

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# 1 Linear Equations in Linear Algebra

## 1.1 Systems of Linear Equations

A linear equation in the variables  $x_1, x_2, \dots, x_n$  is an equation that can be written in the form  $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$  where  $b$  and the coefficients  $a_1, a_2, \dots, a_n$  are real or complex numbers.

Example:  $4x_1 - 5x_2 + 2 = x_1$ ,  $x_2 = 2(6^{1/2} - x_1) + x_3$  are both linear. Not linear examples are  $4x_1 - 5x_2 = x_1x_2$ ,  $x_2 = 2(x_1^{1/2}) - 6$ , and  $2x_1^{-1} + \sin x_2 = 0$ .

Systems of Linear Equations: an  $(m \times n)$  system of linear equations is a system of  $m$  linear equations with  $n$  unknowns.

Example: the  $2 \times 3$  system of equations below has a solution  $x_1 = 5$ ,  $x_2 = 6.5$ , and  $x_3 = 3$ :

$$\begin{aligned}2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 - 4x_3 &= -7\end{aligned}$$

Two linear systems are called equivalent if they have the same solution set.

A system of linear equations can have: infinitely many solutions, no solution, or a unique solution. Coincident lines have infinitely many solutions, parallel lines have no solution, and intersecting lines have a unique solution.

Matrix Notation:

Let's say we have  $x_1 - 2x_2 + x_3 = 0$ ,  $2x_2 - 8x_3 = 8$ , and  $5x_1 - 5x_3 = 10$ .

The coefficient matrix is  $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$  and the augmented matrix is  $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$ . The augmented matrix is  $3 \times 4$  (3 rows, 4 columns).

To solve a linear system: if one of the following elementary operations is applied to a system of linear equations, the resulting system is equivalent, that is the resulting system has the same set of solutions as the original:

1. interchange two equations
2. multiply an equation by a non-zero scalar
3. add a constant multiple of one equation to another

Let's use the system

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10\end{aligned}$$

So using row operations and rearranging the rows, we can do  $R_1 \leftrightarrow R_2$  to swap the first and second row. Now we can multiply the second line by  $1/2$  so  $1/2R_2$ . The next step is to do  $R_2 + R_1$ .

The matrix that results from this is  $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$

Doing the operations  $R_3 - 5R_1$ ,  $R_3 - 5R_2$ , and  $1/2R_2$  and  $1/30R_3$ , we end up getting  $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ ,

so from this we have the equations

$$x_1 - 2x_2 + x_3 = 0$$

$$x_2 - 4x_3 = 4$$

$$x_3 = -1$$

, so the solution set is  $(1, 0, -1)$ . Since a solution exists, the system is consistent.

Let's see if this one is consistent.

$$x_2 - 4x_3 = 8$$

$$2x_1 - 3x_2 + 2x_3 = 1$$

$$4x_1 - 8x_2 + 12x_3 = 1$$

Doing the row operations  $R_1 \leftrightarrow R_2$ ,  $R_3 - 2R_1$ ,  $R_3 + 2R_2$  we get that  $0 = 15$ , so this is inconsistent.

This last example has infinitely many solutions:

$$x_1 - 2x_2 - x_3 = -2$$

$$2x_1 + x_2 + 3x_3 = 1$$

$$-3x_1 + x_2 - 2x_3 = 1$$

Doing the row operations  $R_2 - 2R_1$  and  $R_3 + 3R_1$ , then  $R_3 + R_2$  and then  $1/5R_2$  results in

$$\begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solving this, we have  $x_3$  with no restrictions, it is a "free parameter" and  $x_2 = 1 - x_3$  and  $x_1 = -x_3$  so  $x_1$  and  $x_2$  are parameterized by  $x_3$ .

## 1.2 Row Reduction and Echelon Forms

Leading entry of a row: the first (counting from left to right) non-zero entry (in a nonzero row)

Echelon Form: upper right-hand stair-case, triangle

1. all rows that consist entirely of zeros are grouped together at the bottom of the matrix
2. the first (counting from left to right) non-zero entry in the  $(i + 1)$ st row must appear in a column to the right of the first non-zero entry in the  $i$ th row.
3. all entries in a column below a leading entry are zeros

Reduced Echelon Form: an echelon Form matrix that also has the following properties:

1. the leading entry in each nonzero row is 1
2. each leading one is the only nonzero entry in its column

A pivot point: a location in a matrix  $A$  that corresponds to a leading 1 in a reduced echelon form of  $A$ . A pivot column is a column of  $A$  that contains a pivot position.

Steps to solving a system of linear equations:

1. begin with the leftmost nonzero column. This is the pivot column, the pivot position is at the top.
2. select a non zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
3. Use row replacement operations to create zeros in all positions below the pivot.
4. Cover (or ignore) the row containing the pivot position and cover all rows (if any) above it. Apply previous steps to the sub matrix that remains. Repeat until there are no more nonzero rows to modify.
5. Beginning with the rightmost pivot, create zeros above each pivot. Make each pivot equal to 1 by scaling.

**Example**

Determine the existence and uniqueness of the solution to

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

We can find that  $x_5 = 4$ , and  $x_3, x_4$  are of infinite number of solutions.

**Example**

Find the general solution of the linear system whose augmented matrix has been reduced to  $\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$

We have that  $x_5 = 7$ ,  $x_3 = 1/2(3 + 8x_4 + 7)$  and  $x_1 = -4 - 6x_2 - 2(1/2)(3 + 8x_4 + 7) + 14$

**Theorem 1.1: Existence and Uniqueness**

A linear system is consistent if and only if the right most column of the augmented matrix is not a pivot column, that is if and only if an echelon form of the augmented matrix has no row of the form  $[0, \dots, 0b]$  with  $b$  nonzero. If the system is consistent in the solution contains either a unique solution when there are no free variables or infinitely many solutions when there is at least one free variable.

## 1.3 Vector Equations

Vectors in  $\mathbf{R}^2$ : a matrix with only 1 column is called a vector. The set of all vectors with 2 entries is  $\mathbf{R}^2$

$\vec{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$  for example.

The vectors are ordered pairs of real numbers:

$$\vec{u}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (3, -1) \neq (-1, 3) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Vector addition:  $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

Geometric interpretation: parallelogram law.

$(2, 4) + (3, 1) = (2 + 3, 4 + 1) = (5, 5)$ . We can create a parallelogram (google for review).

Let's say we subtract,  $-a$  has the same magnitude as  $a$  but points in the opposite direction in this case.  $(1, 3) - (2, 1) = (-1, 2)$ . Drawing the line from the origin to the tip of the vectors give you what you get algebraically.

Vector multiplication: scalar multiplication:  $a(x, y) = (ax, ay)$  where  $a$  is a scalar.

Geometric interpretation:  $a(x, y, z)$  points in the same direction as  $(x, y, z)$  but is scaled by a factor of  $a$ .

**Example**

Let  $\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$  Find  $4\vec{u}$ ,  $-3\vec{v}$  and  $4\vec{u} + (-3)\vec{v}$

$$4\vec{u} = (4, -8), -3\vec{v} = (-6, 15) \text{ and } 4\vec{u} + (-3)\vec{v} = (-2, 7)$$

Vectors start at the origin and have magnitude and direction.

Representing vectors in  $\mathbf{R}^3$ . We add the  $z$ -axis.

Vectors in  $\mathbf{R}^n$  we have that  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = (u_1, u_2, \dots, u_n)$  where  $u_1, u_2, \dots \in \mathbb{R}$ .

The  $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$  is the zero vector.

Algebraic Properties of  $\mathbf{R}^n$ : for all  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  in  $\mathbf{R}^n$  and all scalars  $c$  and  $d$ :

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $c(d\mathbf{u}) = (cd)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

Linear Combinations: given vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$  in  $\mathbf{R}^n$  and scalars  $c_1, c_2, \dots, c_p$ , the vector  $\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p$  is called a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$  with weights  $c_1, c_2, \dots, c_p$ .

For example, if we have  $3^{1/2}\mathbf{v}_1 + \mathbf{v}_2$  this can be written as  $\vec{y} = \sqrt{3}\vec{v}_1 + \vec{v}_2$  with  $c_1 = \sqrt{3}$  and  $c_2 = 1$ .

### Example

If  $\vec{a}_1 = (1, -2, -5)$ ,  $\vec{a}_2 = (2, 5, 6)$ , and  $\vec{a}_3 = (7, 4, -3)$  then determine if  $\vec{b}$  can be written as a linear combination of  $\vec{a}_1$  and  $\vec{a}_2$ . That is determine if there exists weights  $x_1, x_2$  such that  $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$ .

Using elementary row operations, we can determine that  $x_1 = 3$ ,  $x_2 = 2$  which is the linear combination of  $\vec{a}_1$  and  $\vec{a}_2$ .

A vector equation  $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{b}$  has the same solution set as the linear system with augmented matrix  $[\mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n \mathbf{b}]$ . In particular  $\mathbf{b}$  can be generated by a linear combination of  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$  if and only if there exists a solution to the linear system corresponding to the matrix  $[\mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n \mathbf{b}]$ .

Span: if  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$  are in  $\mathbf{R}^n$  then the set of all linear combinations of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$  is denoted  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_p\}$  and is called the subset of  $\mathbf{R}^n$  spanned by  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ . That is,  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_p\}$  is the collection of all vectors that can be written in the form:  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p$  with  $c_1, c_2, \dots, c_p$  scalars.

Asking if a vector  $\mathbf{b}$  is in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_p\}$  amounts to asking whether the vector equation  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{b}$  has a solution, or equivalently whether the linear system with augmented matrix  $[\mathbf{v}_1 \mathbf{v}_2 \dots \mathbf{v}_p \mathbf{b}]$  has a solution.

Note  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_p\}$  contains every scalar multiple of  $\mathbf{v}_1$ .

The span of a single vector is a line. The span of 2 linearly independent vectors is a plane (not scalar multiples of each other).

## 1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

The Matrix Equation: if  $A$  is an  $m \times n$  matrix with columns,  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$  and if  $\mathbf{x}$  is in  $\mathbf{R}^n$ , then the product of  $A$  and  $\mathbf{x}$  denoted  $A\mathbf{x}$  is the linear combination of the columns of  $A$  using the corresponding entries in  $\mathbf{x}$  as weights.

Note:  $A\mathbf{x}$  is defined only if the number of columns of  $A$  equals the numbers of entries in  $\mathbf{x}$ .

For example:  $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4(1, 0) + 3(2, -5) + 7(-1, 3) = (3, 6)$

### Theorem 1.2: Matrix Equation, Vector Equation, System of Linear Equations

If  $A$  is an  $m \times n$  matrix, with columns,  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$  and if  $\mathbf{b}$  is in  $\mathbf{R}^m$ , the matrix equation  $A\mathbf{x} = \mathbf{b}$  has the same solution as the vector equation  $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{b}$  which in turn has the same solution as the system of linear equations represented by the augmented matrix  $[\mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n \mathbf{b}]$ .

Existence of Solutions: the equation  $A\mathbf{x} = \mathbf{b}$  has a solution if and only if  $\mathbf{b}$  is a linear combination of the columns of  $A$ .

### Example

Is  $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & 7 \end{bmatrix} \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$  Is the equation  $A\vec{x} = \vec{b}$  consistent for all  $\vec{b}$ . Using rref, we get that

$0 = -2b_1 + b_2 - 2b_3$ , so it is not consistent for every  $\vec{b}$ . It is only consistent if  $b_2 = 2b_1 + 2b_3$ .

So let  $\vec{b} = (1, 4, 1)$  and then do rref again and we get that  $x_3$  is free,  $x_2 = 1/7(4 - 5x_3)$  and  $x_1 = 1 - 3(x_2) - 4x_3$  and this basically gives us  $(1, 4, 1)$  too.

### Theorem 1.3: Existence of solution for $A\mathbf{x} = \mathbf{b}$

Let  $A$  be an  $m \times n$  matrix. Then the following statements are logically equivalent. That is, for a particular  $A$ , they are all true statements or they are all false:

1. for each  $\mathbf{b}$  in  $\mathbf{R}^m$ , the equation  $A\mathbf{x} = \mathbf{b}$  has a solution
2. each  $\mathbf{b}$  in  $\mathbf{R}^m$  is a linear combination of the columns of  $A$
3. The columns of  $A$  span  $\mathbf{R}^m$
4.  $A$  has a pivot position in every row. Note:  $A$  is a coefficient matrix, not an augmented matrix.

Computation of  $A\mathbf{x}$  - an efficient method (matrix multiplication): if the product  $A\mathbf{x}$  is defined, then the  $i$ th entry in  $A\mathbf{x}$  is the sum of the products of the corresponding entries from row  $i$  of  $A$  and from vector  $\mathbf{x}$ .

The above is trivial.

Properties of the Matrix-Vector Product  $A\mathbf{x}$

### Theorem 1.4

if  $A$  is an  $m \times n$  matrix,  $\mathbf{u}$  and  $\mathbf{v}$  are vectors in  $\mathbf{R}^n$ , and  $c$  is a scalar, then :

1.  $A(\mathbf{u} + \mathbf{v})$
2.  $A(c\mathbf{u}) = c(A\mathbf{u})$

Algebraic Properties of  $\mathbf{R}^n$ : for all  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  in  $\mathbf{R}^n$  and all scalars  $c$  and  $d$ . (This was from an above topic)

## 1.5 Solution Sets of Linear Systems

Parametric Vector Form of Solutions:

- Parametric Vector Form of a Plane: a plane can be expressed in explicit form, such as  $10x_1 - 3x_2 - 2x_3 = 0$  or implicit form:  $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ , for  $s$  and  $t$  scalars.
- Parametric Form of a Line containing point  $\mathbf{p}$  in direction of  $\mathbf{v}$ :  $l(t) = \mathbf{p} + t\mathbf{v}$

The parametric equation of a plane in  $\mathbf{R}^2$ :  $\mathbf{x} = a\mathbf{v} + b\mathbf{s}$ .

The span of 2 non-colinear vectors is a plane.  $\text{Span}\{\mathbf{v}, \mathbf{s}\} = \text{the } \mathbf{R}^2 \text{ plane.}$

In  $\mathbf{R}^3$ , the Span is still a plane, just in  $\mathbf{R}^3$ .

The parametric equation of a line in  $\mathbf{R}^2$ :  $\mathbf{l} = \mathbf{p} + t\mathbf{v}$ .

Homogeneous Linear Equation -  $A\mathbf{x} = 0$ .

The homogeneous equation always has at least 1 solution,  $\mathbf{x} = 0$  (the trivial solution).

Recall that a system of linear equation either has infinitely many solutions, no solution, or a unique solution.

The question is whether there exists a nontrivial solution (in which case there are infinitely many solutions).

- The homogeneous equation  $A\mathbf{x} = 0$  has nontrivial solution if and only if the equation has at least 1 free variable

Description of solutions: if the solutions consists of:

- the 0 vector:  $\text{Span}\{0\}$
- 1 free variable:  $\text{Span}\{\mathbf{v}\}$ , the solutions are a line through the origin
- 2 free variables,  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$  is a plane through the origin

### Example

Determine if the following system has a nontrivial solution. Then describe the solution set

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

$x_3$  is free. And everything is in the form  $(4/3, 0, 1)$ .

Nonhomogeneous Equation:  $A\mathbf{x} = \mathbf{b}$ .

For example, in the previous example when we let  $x_3 = 0$  we get  $(-1, 2, 0)$ .

### Theorem 1.5

Suppose the equation  $A\mathbf{x} = \mathbf{b}$  is consistent for some given  $\mathbf{b}$  and let  $\mathbf{p}$  be a solution. Then the solution set of  $A\mathbf{x} = \mathbf{b}$  is the set of all vectors of the Form  $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$  where  $\mathbf{v}_h$  is any solution of the homogeneous equation  $A\mathbf{x} = 0$ .

## 1.6 Applications of Linear Systems

There are three examples here: economics, chemical equations and network flow.

Start with economics. There exist equilibrium prices that can be assigned to the total outputs of the various sectors in an economy in such a way that the income of each sector exactly balances its expenses.

You can use row operations to find an equilibrium price.



Other examples run similarly (sorry for bad note taking today I'm sick)

## 1.7 Linear Independence

Linear Independence: an indexed set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  in  $\mathbf{R}^n$  is said to be

- linearly independent if the vector equation  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{0}$  has only the trivial solution
- linearly dependent if there exists weights  $c_1, c_2, \dots, c_p$  not all zero such that  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p = \mathbf{0}$

I'm too lazy to write matrices so much.

Linear Independence of Matrix Columns: the columns of matrix  $A$  are linearly independent if and only if the equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.

Sets of One or Two Vectors

- A set with 1 vector is linearly independent iff  $\mathbf{v}$  is not the  $\mathbf{0}$  vector because  $x_1\mathbf{v} = \mathbf{0}$  has only the trivial solution
- the zero vector,  $\mathbf{0}$  is linearly dependent because  $x_1\mathbf{0} = \mathbf{0}$  has many nontrivial solutions
- two vector  $\{\mathbf{v}_1, \mathbf{v}_2\}$  are linearly dependent iff at least one of the vectors is a multiple of the other

### Theorem 1.6

Characterization of Linearly Dependent Sets: An indexed set  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  of 2 or more vectors is linearly dependent if and only if at least one of the vectors in  $S$  is a linear combination of the others. In fact, if  $S$  is linearly dependent and  $\mathbf{v}_1$  is not  $\mathbf{0}$ , then some  $\mathbf{v}_j$  with  $j > 1$  is a linear combination of the preceding vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1}$ .

Note: the theorem does not say every vector in a linearly dependent set is a linear combination of preceding vectors

### Theorem 1.7

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  in  $\mathbf{R}^n$  is linearly dependent if  $p > n$ .

### Theorem 1.8

If a set  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  in  $\mathbf{R}^n$  contains the zero vector, then the set is linearly dependent.

## 1.8 Introduction to Linear Transformations

Linear Transformations: we can view  $A\mathbf{x} = \mathbf{b}$  as a mapping: the  $m \times n$  matrix  $A$  is the transform,  $A : \mathbf{R}^n \rightarrow \mathbf{R}^m$ .

From this point of view, solving the equation  $A\mathbf{x} = \mathbf{b}$  amounts to finding all the vectors  $\mathbf{x}$  in  $\mathbf{R}^n$  that are transformed to  $\mathbf{b}$  in  $\mathbf{R}^m$ . The correspondence from  $\mathbf{x}$  to  $A\mathbf{x}$  is a function from one set of vectors to another.

### Definition

A transform (or function or mapping)  $T$  from  $\mathbf{R}^n$  to  $\mathbf{R}^m$  is a rule that assigns to each vector  $\mathbf{x}$  in  $\mathbf{R}^n$  a vector  $T(\mathbf{x})$  in  $\mathbf{R}^m$ . The set  $\mathbf{R}^n$  is called the domain of  $T$ , and  $\mathbf{R}^m$  is called the codomain. The notion  $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$  indicates that the domain of  $T$  is  $\mathbf{R}^n$  and the codomain is  $\mathbf{R}^m$ . For  $\mathbf{x}$  in  $\mathbf{R}^n$ ,  $T(\mathbf{x})$  in  $\mathbf{R}^m$  is called the image of  $\mathbf{x}$ . The set of all images  $T(\mathbf{x})$  is called the range of  $T$ .

Matrix Transformations:  $T(\mathbf{x})$  is computed as  $A\mathbf{x}$  where  $A$  is an  $m \times n$  matrix. Note: the domain of  $T$  is  $\mathbf{R}^n$  and the codomain of  $T$  is  $\mathbf{R}^m$ . The range of  $T$  is the set of all linear combinations of the columns of  $A$ .

Linear Transformations: a transformation (or mapping)  $T$  is linear if

1.  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$  for all  $\mathbf{u}$  and  $\mathbf{v}$  in the domain of  $T$
2.  $T(c\mathbf{u}) = cT(\mathbf{u})$  for all scalars  $c$  and all  $\mathbf{u}$  in the domain of  $T$ .
  - every matrix transformation is a linear transformation:  $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$  and  $A(c\mathbf{u}) = cA\mathbf{u}$
  - linear transformations preserve the operations of vector addition and scalar multiplication

If  $T$  is a linear transformation, then

1.  $T(\mathbf{0}) = \mathbf{0}$
2.  $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$  for all scalars  $c, d$  and all vectors  $\mathbf{u}, \mathbf{v}$  in the domain of  $T$ . The generalization  $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_pT(\mathbf{v}_p)$  is known in engineering as the superposition principle: whenever an input is expressed as a linear combination of signals the systems response is the same linear combination of the responses to the individual signals.

## 1.9 The Matrix of a Linear Transformation

Goal: given a geometric description of a transformation,  $T$ , we want to find a “formula” for  $T$

- Every linear transformation from  $\mathbf{R}^n$  to  $\mathbf{R}^m$  can be represented by a matrix transformation  $A(\mathbf{x})$ .
- The key to finding matrix  $A$  is to that  $T$  is completely determined by what it does to the columns of the  $n \times n$  identity matrix,  $I_n$ .

### Theorem 1.9

Standard Matrix for a Linear Transformation: let  $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$  be a linear transformation. Then there exists a unique matrix  $A$  such that  $T(\mathbf{x}) = A\mathbf{x}$  for all  $\mathbf{x}$  in  $\mathbf{R}^n$ . And,  $A$  is the  $m \times n$  matrix whose  $j$ th column is the vector  $T(\mathbf{e}_j)$  where  $\mathbf{e}_j$  is  $j$ th column of the identity matrix in  $\mathbf{R}^n$ .  $A = [T(\mathbf{e}_1)T(\mathbf{e}_2)\dots T(\mathbf{e}_n)]$ .  $A$  is called the standard matrix for the linear transformation  $T$ .

Onto/Existence: a mapping  $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$  is said to be onto  $\mathbf{R}^m$  if each  $\mathbf{b}$  in  $\mathbf{R}^m$  is the image of at least 1  $\mathbf{x}$  in  $\mathbf{R}^n$ .

- $T$  is onto  $\mathbf{R}^m$  when the range of  $T$  is all of the codomain  $\mathbf{R}^m$ ; for each  $\mathbf{b}$  in  $\mathbf{R}^m$ , there exists at least one solution of  $T(\mathbf{x}) = \mathbf{b}$ . The mapping  $T$  is not onto when there is some  $\mathbf{b}$  in  $\mathbf{R}^m$  for which  $T(\mathbf{x}) = \mathbf{b}$  has no solution.

$T$  is one-to-one if for each  $\mathbf{b}$  in  $\mathbf{R}^m$ , the equation  $T(\mathbf{x}) = \mathbf{b}$  has either unique solution or no solution. The mapping is not one-to-one when some  $\mathbf{b}$  in  $\mathbf{R}^m$  is the image of more than one vector in  $\mathbf{R}^n$ .

### Theorem 1.10

Let  $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$  be a linear transformation, then  $T$  is one-to-one iff  $T(\mathbf{x}) = \mathbf{0}$  has only the trivial solution.

### Theorem 1.11

Let  $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$  be a linear transformation and let  $A$  be the standard matrix for  $T$ . Then:

1.  $T$  maps  $\mathbf{R}^n$  onto  $\mathbf{R}^m$  iff the columns of  $A$  span  $\mathbf{R}^m$
2.  $T$  is one-to-one iff the columns of  $A$  are linearly independent

## 1.10 Linear Models in Business, Science, and Engineering

Linear Equations can be done in electrical networks.

Current flow in a simple electrical network can be described by a system of linear equations. Consider Ohm's Law,  $V = IR$ , which describes the current which passes through a resistor. The algebraic sum of  $IR$  voltage drops in one direction around a loop equals the algebraic of the voltage sources in the same direction around the loop.

The model for current flow is linear since the voltage drop across a resistor is proportional to the current flowing through it, and the sum of the voltage drops in a loop equals the sum of the voltage sources in the loop.

For difference equations, if there is a matrix  $A$  such that  $\mathbf{x}_1 = A\mathbf{x}_0$ ,  $x_2 = A\mathbf{x}_1$ , and in general  $\mathbf{x}_{k+1} = A\mathbf{x}_k$  for  $k = 0, 1, 2, \dots$  then this is called a linear difference equation (or recurrence relation).

Ok whatever just use logic.

## 2 Matrix Algebra

### 2.1 Matrix Operations

Sums and Scalar Multiples of Matrices: if  $A$  and  $B$  are  $m \times n$  matrices,  $A + B$  is the  $m \times n$  whose columns are the sums of the corresponding columns in  $A$  and  $B$ , the scalar multiple  $rA$  is the matrix whose columns are  $r$  times the corresponding columns in  $A$ .

#### Theorem 2.1

Matrix addition and scalar multiplication: Let  $A$ ,  $B$ , and  $C$  be matrices of the same size, and let  $r$  and  $s$  be scalars.

1.  $A + B = B + A$
2.  $(A + B) + C = A + (B + C)$
3.  $A + 0 = A$
4.  $r(A + B) = rA + rB$
5.  $(r + s)A = rA + sA$
6.  $r(sA) = (rs)A$

Matrix Multiplication: if  $A$  is an  $m \times n$  matrix and  $B$  is an  $n \times p$  matrix with columns  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$ , then the product  $AB$  is the  $m \times p$  matrix whose columns are  $A\mathbf{b}_1, A\mathbf{b}_2, \dots, A\mathbf{b}_p$ , i.e.,  $AB = A[\mathbf{b}_1 \mathbf{b}_2 \dots \mathbf{b}_p] = [A\mathbf{b}_1 A\mathbf{b}_2 \dots A\mathbf{b}_p]$ . Matrix multiplication corresponds to composition of linear transformations.

- An efficient Matrix Multiplication: if the product  $AB$  is defined, then the entry in row  $i$  and column  $j$  of  $AB$  is the sum of the products of corresponding entries from row  $i$  of  $A$  and column  $j$  of  $B$ . If  $(AB)_{ij}$  denotes the  $(i, j)$ th entry in  $AB$ , and if  $A$  is  $m \times n$ , then  $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$ .

Properties of Matrix Multiplication: Let  $A$  be  $m \times n$  and let  $B$ ,  $C$  have sizes such that the sums and products are defined:

1.  $A(BC) = (AB)C$  associative law
  2.  $A(B + C) = AB + AC$  left distributive law
  3.  $(B + C)A = BA + CA$  right distributive law
  4.  $r(AB) = (rA)B = A(rB)$  for any scalar  $r$
  5.  $I_m A = A = A I_n$  Identity matrix for multiplication
- Matrix multiplication is not commutative. In general  $AB$  does not equal  $BA$ .
  - Cancellation laws do not hold for matrix multiplication.
  - If  $AB = 0$ , you cannot conclude either  $A = 0$  or  $B = 0$ .

Powers of a Matrix: If  $A$  is  $n \times n$  and  $k$  is a positive integer,  $A^k$  denote the product of  $k$  copies of  $A$ .

The Transpose of a Matrix: given an  $m \times n$  matrix  $A$ , the transpose of  $A$  is the  $n \times m$  matrix whose columns are formed from the corresponding rows of  $A$ .

**Theorem 2.2: Transpose**

Let  $A, B$  denote matrices whose sizes are appropriate for the following:

1.  $(A^T)^T = A$

2.  $(A + B)^T = A^T + B^T$
3. for any scalar  $r$ ,  $(rA)^T = r(A)^T$
4.  $(AB)^T = B^T A^T$

## 2.2 The Inverse of a Matrix

The Matrix Inverse is the matrix analogue of the multiplicative inverse of in real numbers.

- Invertible: an  $n \times n$  matrix  $A$  is said to be invertible if there is an  $n \times n$  matrix  $A^{-1}$  such that  $A^{-1}A = AA^{-1} = I_n$ . In this case  $A^{-1}$  is said to be the unique inverse of  $A$ .

Notice: because matrix multiplication is not commutative, both equations are needed.

Singular Matrix: A matrix that is not invertible is a singular matrix. An invertible matrix is nonsingular.

### Theorem 2.3

Inverse of a  $2 \times 2$ : Let  $A$  be the  $2 \times 2$  matrix shown. If  $ab - dc$  is not zero, then  $A$  is invertible with  $A^{-1}$  as shown:

$$A = \begin{bmatrix} a & d \\ c & b \end{bmatrix} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinant:  $\det A = ad - bc$ . The theorem says that a  $2 \times 2$  matrix is invertible iff  $\det A$  is not zero.

### Theorem 2.4

If  $A$  is an invertible matrix, then for each  $\mathbf{b}$  in  $\mathbf{R}^n$ , the equation  $A\mathbf{x} = \mathbf{b}$  has a unique solution  $\mathbf{x} = A^{-1}\mathbf{b}$ .

### Theorem 2.5

1. If  $A$  is an invertible matrix, then  $A^{-1}$  is invertible and  $(A^{-1})^{-1} = A$
2. If  $A$  and  $B$  are  $n \times n$  invertible matrices, then so is  $AB$  and  $(AB)^{-1} = B^{-1}A^{-1}$ . Generalization: the product of  $n \times n$  invertible matrices is invertible, and the inverse is the product of the their inverse in the reverse order.
3. If  $A$  is an invertible matrix, then so is  $A^T$ , and  $(A^T)^{-1} = (A^{-1})^T$

### Theorem 2.6

An  $n \times n$  matrix is invertible iff it is row equivalent to  $I_n$ , and any sequence of elementary row operations that reduces  $A$  to  $I_n$  also transforms  $I_n$  to  $A^{-1}$ .

## 2.3 Characterizations of Invertible Matrices

The Invertible Matrix Theorem: let  $A$  be an  $n \times n$  matrix. Then the following statements are equivalent.

- $A$  is an invertible matrix.
- $A$  is row equivalent to the  $n \times n$  identity matrix.
- $A$  has  $n$  pivot positions
- the equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution
- the columns of  $A$  form a linearly independent set
- the linear transform  $\mathbf{x} \rightarrow A\mathbf{x}$  is one-to-one

- the equation  $A\mathbf{x} = \mathbf{b}$  has at least 1 soln for each  $\mathbf{b}$  in  $\mathbf{R}^n$
- the columns of  $A$  span  $\mathbf{R}^n$
- the linear transformation  $\mathbf{x} \rightarrow A\mathbf{x}$  maps  $\mathbf{R}^n$  onto  $\mathbf{R}^n$
- there is an  $n \times n$  matrix  $C$  such that  $CA = I$
- there is an  $n \times n$  matrix  $D$  such that  $AD = I$
- $A^T$  is an invertible matrix

Note that this only applies to square matrices.

### Theorem 2.7: Inverse Transformation

Let  $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a linear transformation and let  $A$  be the standard matrix for  $T$ . Then  $T$  is invertible if and only if  $A$  is an invertible matrix. In that case, the linear transformation  $S$  given by  $S(\mathbf{x}) = A^{-1}\mathbf{x}$  is the unique solution satisfying  $S(T(\mathbf{x})) = \mathbf{x}$  and  $T(S(\mathbf{x})) = \mathbf{x}$  for all  $\mathbf{x}$  in  $\mathbf{R}^n$ .

Recall that matrix multiplication corresponds to composition of linear transformations. When a matrix  $A$  is invertible, the equation  $A^{-1}A\mathbf{x} = \mathbf{x}$  can be viewed as a statement about linear transformations. A linear transformation  $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$  is said to be invertible if there exists a function  $S : \mathbf{R}^n \rightarrow \mathbf{R}^n$  such that  $S(T(\mathbf{x})) = \mathbf{x}$  and  $T(S(\mathbf{x})) = \mathbf{x}$  for all  $\mathbf{x}$  in  $\mathbf{R}^n$ .

## 2.4 Matrix Factorizations

A factorization of a matrix  $A$  is an equation that expresses  $A$  as a product of two or more matrices.

Whereas matrix multiplication involves a synthesis of data (combining the effects of two or more linear transformations into a single matrix), matrix factorization is an analysis of data.

The LU factorization:

At first assume that  $A$  is an  $m \times n$  matrix that can be row reduced to echelon form, without row interchanges. Then  $A$  can be written in the form  $A = LU$ , where  $L$  is an  $m \times m$  lower triangular matrix with 1's on the diagonal and  $U$  is an  $m \times n$  echelon form of  $A$ .

Suppose  $A$  can be reduced to an echelon form  $U$  using only row replacements that add a multiple of one row to another below it. In this case, there exist unit lower triangular elementary matrices,  $E_1, \dots, E_p$  such that  $E_p \cdots E_1 A = U$ .

Then  $A = (E_p \cdots E_1)^{-1}U = LU$  where  $L = (E_p \cdots E_1)^{-1}$ .

It can be shown that products and inverses of unit lower triangular matrices are also unit lower triangular. Thus  $L$  is unit lower triangular.

Algorithm:

- Reduce  $A$  to an echelon form  $U$  by a sequence of row replacement operations, if possible.
- Place entries in  $L$  such that the same sequence of row operations reduces  $L$  to  $I$ .

# **3 Determinants**

**3.1 Introduction to Determinants**

**3.2 Properties of Determinants**

**3.3 Cramer's Rule, Volume, and Linear Transformations**



# **4 Vector Spaces**

## **4.1 Vector Spaces and Subspaces**

## **4.2 Null Spaces, Column Spaces, Row Spaces, and Linear Transformations**

## **4.3 Linearly Independent Sets; Bases**

## **4.4 Coordinate Systems**

## **4.5 The Dimension of a Vector Space**

## **4.6 Change of Basis**

# **5 Eigenvalues and Eigenvectors**

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**5.2 The Characteristic Equation**

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# **6 Orthogonality and Least Squares**

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**6.4 The Gram-Schmidt Process**

**6.5 Least-Squares Problems**

**6.6 Machine Learning and Linear Models**

**6.7 Inner Product Spaces**

# **7 Symmetric Matrices and Quadratic Forms**

## **7.1 Diagonalization of Symmetric Matrices**

## **7.2 Quadratic Forms**