Contest Math Problems

anastasia

Contents

1 AMC/AHSME

1.1 2011-2020

2020 AMC 12A/8

December 17, 2024

What is the median of the following list of 4040 numbers?

$$1, 2, 3, ..., 2020, 1^2, 2^2, 3^2, ..., 2020^2$$

Solution:

First, we notice that there are an even number of elements - indeed, 4040 of them, to be precise. This means we must determine the average of the 2020th and 2021st elements.

We know that the sequence $1^2, 2^2, \ldots, 2020^2$ will contribute a certain number of elements than 2020, so our task is to determine how many of these are smaller than 2020.

By testing a few values, we observe that $45^2 = 2025$, which tells us that only the squares up to 44^2 need be considered.

From this, it follows that $2020 - 44$ will correspond to the 2020th element, as we are including $1^2, 2^2, \ldots, 44^2$ in the list. Thus, the 2020th element is 1976, and similarly, the 2021st element is 1977. Therefore, the median is given by $\boxed{1976.5}$.

2020 AMC 12B/8

December 12, 2024

How many ordered pairs of integers (x, y) satisfy the equation

$$x^{2020} + y^2 = 2y?$$

Solution:

By completing the square we get that $x^{2020} + (y-1)^2 = 1$.

Observe that $x = \pm 1$ or 0 for this to work, otherwise $(y-1)^2$ will be negative which is kind of hard to do.

Therefore, when $x = \pm 1$, we have $1 + y^2 = 2y$, and this is equivalent when $y = 1$, so we have $(-1, 1)$ and $(1, 1)$ as solutions.

When $x = 0$, we have that $y^2 = 2y$, which has two solutions, when $y = 0$ or when $y = 2$, so we have $(0, 0)$ and $(0, 2)$ as solutions.

Therefore, there are $\boxed{4}$ pairs of integers that satisfy this equation.

2019 AMC 12A/7

November 28, 2024

Melanie computes the mean μ, the median M, and the modes of the 365 values that are the dates in the months of 2019. Thus her data consist of 12 1s, 12 2s, . . . , 12 28s, 11 29s, 11 30s, and 7 31s. Let d be the median of the modes. Which of the following statements is true?

(A) $\mu < d < M$ **(B)** $M < d < \mu$ **(C)** $d = M = \mu$ **(D)** $d < M < \mu$ **(E)** $d < \mu < M$

Solution:

I first find the mode of the numbers. The mode will be all the numbers from 1 to 28 since every month has those. Therefore, the median of this will be 14.5.

The median of all the numbers will be in the 183rd spot of all the data. There will be 12 of each number from 1 to 28, and since $12(15) = 180$, we can assume that the 183rd spot will have a 16 there.

For the mean, if we assume that each month as 31 days, then the mean of all these values will be 16. However, since we counted more days than actually in the year, we assume that the mean is less than 16.

Therefore the answer is $\boxed{\textbf{E}}$.

2019 AMC 12B/7

November 26, 2024

What is the sum of all real numbers x for which the median of the numbers $4, 6, 8, 17,$ and x is equal to the mean of those five numbers?

Solution:

Ok the mean of this is simple to calculate, it is just $\frac{4+6+8+17+x}{x} = \frac{35+x}{5}$.

There are three possible medians, they are 6, 8, and x.

Case 1: median is 6.

This only works when $x \leq 6$, so for this to work, we need $\frac{35+x}{5} = 6$. This is possible when $x = -5$.

Case 2: median is 8.

This only works when $x \geq 8$, so for this to work, we need $\frac{35+x}{5} = 8$. The only value of x that this works for is $x = 5$, so this is not a solution.

Case 3: median is x.

This case is only for $6 < x < 8$. We need for $\frac{35+x}{5} = x$. Simplifying this, we get $x = \frac{35}{4}$. This value is not less than 8, so we banish this to the shadow realm!! Ok the answer is $\boxed{-5}$ btw.

2018 AMC 12A/6

December 17, 2024

For positive integers m and n such that $m + 10 < n + 1$, both the mean and the median of the set $\{m, m+4, m+10, n+1, n+2, 2n\}$ are equal to n. What is $m + n$?

Solution:

Since there are an even number of elements, the median is

$$\frac{m + 10 + n + 1}{2} = n$$

Simplifying this, we have $n = 11 + m$, or $m = n - 11$.

When we plug this back and try to find the mean we will get that $\frac{n-11+n-11+4+n-11+10+n+1+n+2+2n}{6} = \frac{7n-16}{6} = n$.

From this, we get that $7n - 16 = 6n$ and that $n = 16$.

Plugging this back into $m = n - 11$, we find that $m = 16 - 11 = 5$.

Our answer now is $16 + 5 = \boxed{21}$.

2018 AMC 12A/16

December 19, 2024

Which of the following describes the set of values of a for which the curves $x^2 + y^2 = a^2$ and $y = x^2 - a$ in the real xy-plane intersect at exactly 3 points?

(A) $a = \frac{1}{4}$ **(B)** $\frac{1}{4} < a < \frac{1}{2}$ **(C)** $a > \frac{1}{4}$ **(D)** $a = \frac{1}{2}$ **(E)** $a > \frac{1}{2}$

Solution:

When we square $y = x^2 - a$, we get $y^2 = (x^2 - a)(x^2 - a) = x^4 - 2ax^2 + a^2$. Since we also know that $y^2 = a^2 - x^2$, we can set them equal to each other.

Now we have $x^4 - 2ax^2 + a^2 = a^2 - x^2$. Simplifying we get that $x^4 - 2ax^2 + x^2 = 0$.

Factoring this we have $x^2(x^2 - 2a + 1)$. From this we know that we can get three intersections, by getting $x = 0$ first. The other two roots can be found from $x^2 - 2a + 1$. From this we are looking for a and we have $x^2 - 2a + 1 = 0$. So solving for a we have $x^2 = 2a - 1$. We know that $2a - 1$ has to be greater than 0 because you cannot square root a negative. From $2a - 1 > 0$, we can see that $a > \frac{1}{2}$ or $\boxed{\textbf{(E)}}$.

2018 AMC 12B/2

November 28, 2024

Sam drove 96 miles in 90 minutes. His average speed during the first 30 minutes was 60 mph (miles per hour), and his average speed during the second 30 minutes was 65 mph. What was his average speed, in mph, during the last 30 minutes?

Solution:

Using $d = rt$ we can find that in the first 30 minutes, Sam drove $60 \cdot 0.5 = 30$ miles.

In the next 30 minutes, Sam drove $65 \cdot 0.5 = 32.5$ miles.

Therefore in the last 30 minutes, he drove $96 - 32.5 - 30 = 33.5$ miles. Since this is the rate for 30 minutes, simply multiply this value by two using the power of dimensional analysis in chemistry to get $\boxed{67}$ mph.

2017 AMC 12B/21

November 27, 2024

Last year Isabella took 7 math tests and received 7 different scores, each an integer between 91 and 100, inclusive. After each test she noticed that the average of her test scores was an integer. Her score on the seventh test was 95. What was her score on the sixth test?

(A) 92 **(B)** 94 **(C)** 96 **(D)** 98 **(E)** 100

Solution:

The smallest possible sum and the largest possible sum that is divisible by 7 just happens to be the lowest 7 and highest 7 numbers added together.

For example the highest possible sum is just $100 + 99 + 98 + 97 + 96 + 95 + 94 = 679 = 7(97)$, and the lowest possible sum is $91 + 92 + 93 + 94 + 95 + 96 + 97 = 658 = 7(94)$. Therefore, the only two other sums that can be $7(95)$ and $7(96)$, or 665 and 672.

Now we can subtract 95 from all of them to find the sum of the first 6 test scores.

After we subtract 95 from all the numbers, we get $563, 570, 577, 584$. The only one of these divisible by 6 is 570, so we know that this is the sum of the first six test scores.

The sum of the first five test scores will have to be divisible by 5, so note that because of this, the only numbers you can subtract from 570 that will result in a number that can be divisible by 5 are 95 and 100. 95 cannot be subtracted from this because note the condition in the question that states that each score has to be different, so the answer is $\boxed{100}$.

2016 AMC 12A/7

December 10, 2024

Which of these describes the graph of $x^2(x+y+1) = y^2(x+y+1)$?

(A) two parallel lines
(B) two intersecting lines
(C) three lines that all pass through a common point
(D) three lines that do not all pass through a common point
(E) a line and a parabola

Solution:

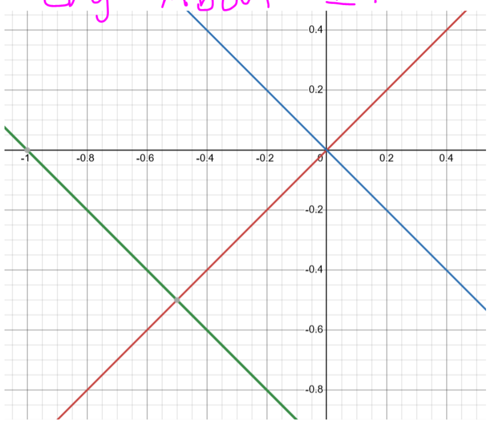
Case 1: $x + y + 1 = 0$

In this case, then they are equal, so $y = -x - 1$.

Case 2: $x + y + 1 \neq 0$

In this case, we have $x^2 = y^2$, so $y = \pm x$.

Cry About It



Therefore from this, we have $\boxed{\text{three lines that do not all pass through a common point}}$

2016 AMC 12B/18

December 21, 2024

What is the area of the region enclosed by the graph of the equation $x^2 + y^2 = |x| + |y|$?

Solution:

One can see the four cases from the absolute value first.

These are

$$x^2 + y^2 = x + y$$
$$x^2 + y^2 = x - y$$
$$x^2 + y^2 = -x + y$$
$$x^2 + y^2 = -x - y$$

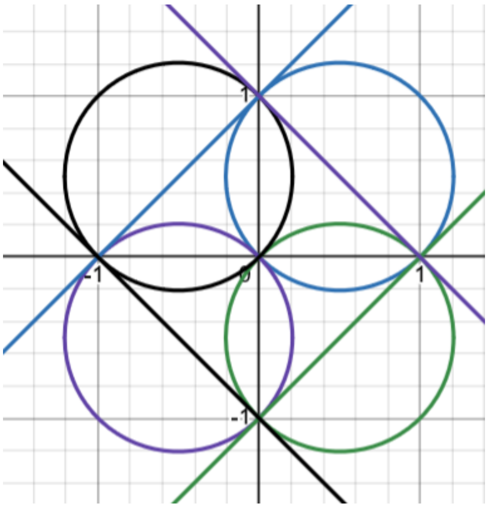
Now by completing the square on each of these, we can see that there are four circles, each with radius of $\frac{\sqrt{2}}{2}$ and with a center of $\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$.

Now when we draw the diameters of all the circles, we can see that this makes a square. The diamater is $\sqrt{2}$, so the area of this square is 2.

We are left with the four semicircles now, we can just find the area of one of these semicircles and just multiply by 4.

The area of one of these is $\frac{1}{2}\pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{4}\pi$. Multiplying by 4 we get that the areas of the semicircles combined is π.

The area is $\boxed{\pi + 2}$.



2014 AMC 12A/19

December 5, 2024

There are exactly N distinct rational numbers k such that $|k| < 200$ and

$$5x^2 + kx + 12 = 0$$

has at least one integer solution for x. What is N?

Solution:

First, rearrange the equation as $5x^2 + kx = -12$. Factoring out x we get $x(5x + k) = -12$. Solving for k we get $\frac{-12}{x} - 5x$.

From here we can see that when x is a positive value, and then k will be negative. So, we can see that $|-5x| < 200$, since the same will be for negative values of x.

Solving this inequality yields $-200 < -5x < 200$ or $40 > x > -40$, where $x \neq 0$. So we can see that the values from 1 to 39 and -39 to -1 will be the only values that work.

Therefore $39 + 39 = \boxed{78}$.

Note that the value 0 will not work because the $\frac{-12}{x}$ is undefined.

2013 AMC 12A/16

November 21, 2024

A, B, C are three piles of rocks. The mean weight of the rocks in A is 40 pounds, the mean weight of the rocks in B is 50 pounds, the mean weight of the rocks in the combined piles A and B is 43 pounds, and the mean weight of the rocks in the combined piles A and C is 44 pounds. What is the greatest possible integer value for the mean in pounds of the rocks in the combined piles B and C?

(A) 55 **(B)** 56 **(C)** 57 **(D)** 58 **(E)** 59

Solution:

We can write a few equations from the information given:

$$\frac{A}{a} = 40 \qquad \frac{B}{b} = 50 \qquad \frac{A+B}{a+b} = 43 \qquad \frac{A+C}{a+c} \qquad \frac{B+C}{b+c} = x$$

where a, b, c are the number of rocks in piles A, B, C respectively.

$A = 40b$ and $B = 50b$ as a result.

We can solve for a in terms of b now:

$A + B = 43(a+b) = 43a + 43b$.

Substituting $A + B = 40b + 50b$, we get $40a + 50b = 43a + 43b$.

Solving for a, we get $7b = 3a$, and $a = \frac{7}{3}b$.

We can also solve for C.

$$\begin{aligned} A + C &= 44a + 44c \\ 40a + C &= 44a + 44c \\ C &= 44c + 4a \end{aligned}$$

Also, we can write C in terms of b as well:

$$\begin{aligned} B + C &= x(b+c) \\ 50b + C &= x(b+c) \\ C &= xc + b(x-50) \end{aligned}$$

Putting these equations equal to each other:

$$\begin{aligned} 44c + 4a &= xc + b(x-50) \\ (44-x)c + 4\left(\frac{7}{3}\right)b &= (x-50)b \\ (44-x)c + \frac{28}{3}b &= (x-50)b \\ (44-x)c &= \left(x - 50 - \frac{28}{3}\right)b \\ c &= \frac{\left(x - 50 - \frac{28}{3}\right)b}{44-x} \end{aligned}$$

Now we can see from the answer choices that the denominator will be negative, which therefore means we need the numerator to be negative as well to result in a positive value for c.

$50 + \frac{28}{3}$ is equal to $59\frac{1}{3}$, so the largest value x that can result in the equation resulting in a positive number is $\boxed{59}$.

2011 AMC 12A/18

December 1, 2024

Suppose that $|x+y| + |x-y| = 2$. What is the maximum possible value of $x^2 - 6x + y^2$?

Solution:

There are four cases:

$(x+y) + (x-y)$ yields $x = 1$.

$-(x+y) + (x-y)$ yields $y = -1$.

$(x+y) - (x-y)$ yields $y = 1$.

$-(x+y) - (x-y)$ yields $x = -1$.

This expression is maximized when the $-6x$ term is positive, and x^2 and y^2 are always going to be 1 each, so it is maximized at $(-1, \pm 1)$, or $1^2 - 6(-1) + 1^2 = \boxed{8}$.

1.2 2000-2010

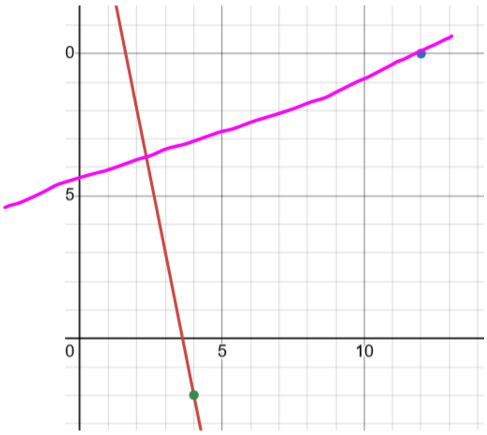
2007 AMC 12A/13

December 6, 2024

A piece of cheese is located at $(12, 10)$ in a coordinate plane. A mouse is at $(4, -2)$ and is running up the line $y = -5x + 18$. At the point (a, b) the mouse starts getting farther from the cheese rather than closer to it. What is $a + b$?

Solution:

First we should graph the line $y = -5x + 18$ and label the points.



The mouse will be closest to the cheese when it is at the foot of the perpendicular line from $(12, 10)$, so we need to find the perpendicular line of $y = -5x + 18$ that includes the point $(12, 10)$.

From the point-slope formula we have

$$y - 10 = \frac{1}{5}(x - 12)$$

Putting this in a system with the other equation, we have

$$\begin{aligned} y - 10 &= \frac{1}{5}x - \frac{12}{5} \\ y &= -5x + 18 \end{aligned}$$

When we put both of these equations equal to y and equal them to each other, we have

$$\begin{aligned} -5x + 18 &= \frac{1}{5}x + \frac{38}{5} \\ -25x + 90 &= x + 38 \\ -26x &= -52 \\ x &= 2 \end{aligned}$$

Plugging this back into $y = -5x + 18$, we get $y = 8$.

The point at which the mouse is closest to the cheese is at $(2, 8)$, so $2 + 8 = \boxed{10}$.

2006 AMC 12A/11

December 5, 2024

Which of the following describes the graph of the equation $(x+y)^2 = x^2 + y^2$?

(A) the empty set (B) one point (C) two lines (D) a circle (E) the entire plane

Solution:

Simplifying the equation we have:

$$\begin{aligned} (x+y)^2 &= x^2 + y^2 \\ x^2 + y^2 + 2xy &= x^2 + y^2 \\ 2xy &= 0 \end{aligned}$$

From this we can see that $x = 0$ or $y = 0$, so the graphs will just be the x-axis and the y-axis which is $\boxed{\text{two lines}}$.

2006 AMC 12B/8

December 9, 2024

The lines $x = \frac{1}{4}y + a$ and $y = \frac{1}{4}x + b$ intersect at the point $(1, 2)$. What is $a + b$?

Solution:

Simply plug in x and y and we should get

$$\begin{aligned} 1 &= \frac{1}{2} + a \\ 2 &= \frac{1}{4} + b \end{aligned}$$

Adding the two equations together, we get $3 = \frac{3}{4} + a + b$, so $a + b = 3 - \frac{3}{4} = \boxed{\frac{9}{4}}$.

2005 AMC 12B/7

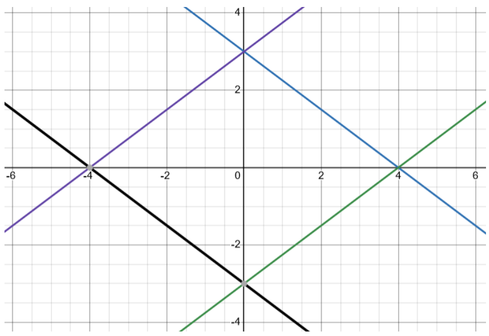
December 4, 2024

What is the area enclosed by the graph of $|3x| + |4y| = 12$?

Solution:

The graph produced will be a rhombus bounded by 4 lines:

$3x + 4y = 12$ $3x - 4y = 12$ $-3x + 4y = 12$ $-3x - 4y = 12$



Now the area of a rhombus is $\frac{1}{2}$ the product of the diagonals. So, $\frac{1}{2} \cdot 6 \cdot 8 = \boxed{24}$.

2004 AMC 10A/4

November 30, 2024

What is the value of x if $|x-1| = |x-2|$?

Solution:

Case 1: The absolute values are less than or greater than zero.

This case does not work since $x - 1 \neq x - 2$.

Case 2: One is positive, one is negative.

Therefore, we have $1 - x = x - 2$. Solving we get $x = \boxed{\frac{3}{2}}$.

2002 AMC 10A/21

December 18, 2024

The mean, median, unique mode, and range of a collection of eight integers are all equal to 8. The largest integer that can be an element of this collection is

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

Solution:

Let's write the list first as $a, b, c, 8, 8, d, e, f$. Since there are an even amount of numbers in the list, then the middle two will be 8.

Let's start by trying 15, since it is the largest listed in the answer choices. We can write the mean of this as $\frac{a+b+c+8+8+d+e+f}{8} = 8$, or $a+b+c+d+e+f+16 = 64$. Since we know that $f = 15$, then $a = 15 - 8 = 7$. Now we have the list $7, b, c, 8, 8, d, e, 15$. Plugging this into the mean we see that $b+c+d+e = 26$. For this to work, we would need at least one number less than 7, so this would not work since the range has to be 8.

Now we can try 14. Using the same setup as before we can see that we would end up with the list $6, b, c, 8, 8, d, e, 14$. From this we can see that $b + c + d + e = 28$. Ok so there are four numbers now. Now if I assume that they are all 7, it will work, but it wouldn't satisfy the mode being 8. But we can see that $7 + 7 + 7 + 7 = 6 + 6 + 8 + 8$, and this satisfies the list, so the answer is $\boxed{14}$.

2000 AMC 12/5

December 2, 2024

If $|x-2| = p$, where $x < 2$, then $x - p =$

Solution:

Ok so $x < 2$ so that means $x - 2$ is negative, so $-(x-2) = p$, or $-x + 2 = p$, so $x = 2 - p$.

Therefore $x - p = 2 - p - p = \boxed{2 - 2p}$.

2000 AMC 12/14

November 28, 2024

When the mean, median, and mode of the list

$$10, 2, 5, 2, 4, 2, x$$

are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real values of x?

Solution:

The list can be written as $2, 2, 2, 4, 5, 10$ with x somewhere.

The mode of this list is 2 regardless of anything.

The mean of this list is $\frac{2+2+2+4+5+10+x}{7} = \frac{25+x}{7}$.

The medians can be 2, 4, or x.

If the median is 2 and the mode is 2, the mean has to be 2, this is kinda erm constant.

If the median is 4 we have $\frac{25+x}{7}$ equal to either 0, 3, or 6 as the only ways to make this an arithmetic sequence.

$\frac{25+x}{7} = 0$ results in a negative number, putting this as x will make the median no longer equal 4 so it is a bad number. The same applies for $\frac{25+x}{7} = 3$. However $\frac{25+x}{7} = 6$ results in a number greater than 4, and so $x = 17$.

Now the last case is when x is the median. In this case, there is an arithmetic sequence of $2, x, \frac{25+x}{7}$ because the median is x which is greater than 2 and the mean of the numbers will be greater than the median.

The differences between each term have to be the same, so

$$\begin{aligned}
x - 2 &= \frac{25+x}{7} - x \\
2x &= \frac{25+x}{7} + \frac{14}{7} \\
2x &= \frac{39+x}{7} \\
14x &= 39 + x \\
13x &= 39 \\
x &= 3
\end{aligned}$$

Since this number for x is a valid median $x = 3$ is also a solution.

Therefore $17 + 3 = \boxed{20}$.

2000 AMC 12/20

November 24, 2024

If $x, y,$ and z are positive numbers satisfying

$$x + \frac{1}{y} = 4, \qquad y + \frac{1}{z} = 1, \qquad \text{and} \qquad z + \frac{1}{x} = \frac{7}{3}$$

Then what is the value of xyz ?

Solution:

I noticed first that we can get xyz by multiplying the equations together.

$$\begin{gathered}
\left(x + \frac{1}{y}\right)\left(y + \frac{1}{z}\right) = 4 \\
\left(xy + \frac{x}{z} + 1 + \frac{1}{yz}\right)\left(z + \frac{1}{x}\right) = \frac{28}{3} \\
xyz + y + x + \frac{1}{z} + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{xyz} = \frac{28}{3}
\end{gathered}$$

Substituting values in, we get

$$\begin{gathered}
xyz + \frac{1}{xyz} + 4 + 1 + \frac{7}{3} = \frac{28}{3} \\
xyz + \frac{1}{xyz} = \frac{28}{3} - \frac{22}{3} = 2
\end{gathered}$$

We can see that $\boxed{xyz = 1}$ is a solution to this.

1.3 Before 2000

1987 AHSME/15

November 24, 2024

If (x, y) is a solution to the system

$$xy = 6 \qquad \text{and} \qquad x^2y + xy^2 + x + y = 63,$$

find $x^2 + y^2$.

Solution:

Factoring the second equation, we get $xy(x + y) + x + y = 63$.

Substituting in xy, we get $6(x + y) + x + y = 6x + 6y + x + y = 7x + 7y = 63$ or $x + y = 9$.

Squaring both sides, we get $(x + y)^2 = x^2 + y^2 + 2xy = 81$. Substituting in xy once again, we get $x^2 + y^2 + 2(6) = 81$, so $x^2 + y^2 = 81 - 12 = 69$.

1963 AHSME/3

December 8, 2024

If the graphs of $2y + x + 3 = 0$ and $3y + ax + 2 = 0$ are to meet at right angles, the value of a is:

Solution:

The dot product of two vectors must be 0.

So let the two vectors be $\langle 1, 2\rangle$ and $\langle a, 3\rangle$. The dot product of these two vectors, $\langle 1, 2\rangle \cdot \langle a, 3\rangle = (1)(a) + 2(3) = a + 6$. So, when $a + 6 = 0$, the two vectors will be perpendicular, so solving for a, we get $a = \boxed{-6}$.

1951 AHSME/50

November 29, 2024 Tom, Dick and Harry started out on a 100-mile journey. Tom and Harry went by automobile at the rate of 25 mph, while Dick walked at the rate of 5 mph. After a certain distance, Harry got off and walked on at 5 mph, while Tom went back for Dick and got him to the destination at the same time that Harry arrived. The number of hours required for the trip was:

Solution:

Let x be the amount of miles that Harry travels on the car and y be the amount that Tom backtracks to get Dick (old ah names I think).

Tom's equation will be $\frac{x}{25} + \frac{y}{25} + \left(\frac{100}{25} - \frac{x-y}{25}\right)$

Dick's equation will be $\frac{x}{5} - \frac{y}{5} + \left(\frac{100}{25} - \frac{x-y}{25}\right)$

Harry's equation will (not be so hairy) $\frac{x}{25} + \left(\frac{100}{5} - \frac{x}{5}\right)$

Note that Tom and Dick's last terms in parenthesis are the same because they will be traveling the same distance at this point in time.

Since all these equations give the time that it takes for each of these members to get to the destination, we can set them all equal to each other :-D

Aight so

$$\frac{x+y+100-x+y}{25}=\frac{5x-5y+100-x+y}{25}=\frac{x+500-5x}{25}$$
$$2y+100=4x-4y+100=500-4x$$

We can solve for y in terms of x now:

We have $2y+100=4x-4y+100$, so cancelling out the 100 and adding the respective terms to both sides, we get $6y=4x$ or $y=\frac{2}{3}x$.

Now substituting this value for y back in to the $2y+100=500-4x$, we get that $\frac{4}{3}x+100=500-4x$. So after doing some calculations (i hate calculations), we get $x=75$.

Plugging this value for x back into $y=\frac{2}{3}x$, we get $y=50$.

Now since we have both values of x and y, we can now plug it back to one of the original equations we defined. Since Harry's little equation looks the last harriest, I plug it back into this.

$\frac{75}{25}+\frac{100-75}{5}=3+5=\boxed{8}$

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2 AIME

2.1 2011-2020

2016 AIME I/1

December 4, 2024

For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \cdots .$$

Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.

Solution:

We can see that $S(a) = 12 + 12a + 12a^2 + 12a^3 + \cdots = \frac{12}{1-a}$.

We can also see that $S(-a) = 12 + 12(-a) + 12(-a)^2 + 12(-a)^3 + \cdots = \frac{12}{1+a}$.

So, $\frac{12}{1-a} \cdot \frac{12}{1+a} = \frac{144}{(1-a)(1+a)} = 2016$.

Now note that we are looking for $S(a) + S(-a) = \frac{12}{1-a} + \frac{12}{1+a}$. Simplifying this, we get $\frac{12(1+a)+12(1-a)}{(1-a)(1+a)} = \frac{24}{(1-a)(1+a)}$.

This looks similar to the equation above that equals 2016, and we can see all we have to do is divide the equation by 6, so $\frac{\frac{144}{(1-a)(1+a)}}{6} = \frac{2016}{6} = \boxed{336}$.

2015 AIME II/14

November 18, 2024

Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.

Solution:

We let $a = xy$ and $b = x + y$.

Factoring $x^4y^4 + y^4x^5 = 810$ we get $(xy)^4(x + y) = 810$ and substituting we get $(a^4)(b) = 810$.

Factoring the other takes a little more steps.

When we factor $x^3y^6 + y^3x^6$ we get $(xy)^3 + (x^3 + y^3)$.

Note that $(x + y)^3 = x^3 + y^3 + 3xy(x + y) \implies b^3 = x^3 + y^3 + 3ab$, so $x^3 + y^3 = b^3 - 3ab$.

Therefore, factoring $x^3y^6 + y^3x^6 = 945$ gets $(a^3)(b^3 - 3ab) = 945$.

Note we are also looking for the value of $2(x^3 + y^3) + (xy)^3$ or $2(b^3 - 3ab) + a^3$ or $2b(b^2 - 3a) + a^3$.

Dividing the two factored equations we get:

$$\begin{aligned}
\frac{(a^3)(b^3-3ab)}{(a^4)(b)} &= \frac{945}{810}\\
\frac{a^3b^3-3a^4b}{a^4b} &= \frac{945}{810}\\
\frac{b^2}{a}-3 &= \frac{945}{810}\\
\frac{b^2}{a}-3 &= \frac{7}{6}\\
\frac{b^2}{a} &= \frac{25}{6}
\end{aligned}$$

Now for some manipulations.

Solving for a we get $a = \frac{6}{25}b^2$.

I didn't know a better way to go from here, so I plugged it into the equation defined earlier, $(a^4)(b) = 810$ to get $\left(\frac{6}{25}b^2\right)^4(b) = 810$.

After cheating with some desmos calculations this resulted in $\frac{1296}{390625}b^9 = 810$.

Solving for b we get $b^9 = \frac{890\cdot 390625}{1296}$

Using some more online calculators for simplifying fractions, I got $b^9 = \frac{1953125}{8}$. Now the cube root of 1953125. Now 1953125 looked like some power of 5 so I assumed it was 5^9 and desmos said it was correct! Happy Stasya!

So $b^3 = \sqrt[3]{\frac{1953125}{8}} = \frac{125}{2}$.

Now from earlier, there was the equation $a = \frac{6}{25}b^2$. We are looking for a^3 (we will look for $2b(b^2-3a)$ momentarily).

So we can take what we know and plug it into a^3:

$$\begin{aligned}
a^3 &= \left(\frac{6}{25}b^2\right)^3\\
&= \frac{6^3}{25^3}b^6\\
&= \frac{6^3}{25^3}b^3b^3\\
&= \frac{6^3}{25^3}\frac{125}{2}\frac{125}{2} = \frac{6^3}{4} = 54
\end{aligned}$$

We have found that $a^3 = 54$!

Ok now we need to find out $2b(b^2-3a)$ to finish the problem.

We can rewrite this as $2b^3 - 6ab$ to make this easier actually since we already know b^3.

We know that $a^3 = 54$, so we can plug that into one of the first equations we defined at the top, $(a^4)(b) = 810$.

This results in $54ab = 810$, so $ab = \frac{810}{54} = 15$.

Now we only need to plug these values in: $2\left(\frac{125}{2}\right) - 6(15) = 125 - 90 = 35$.

Adding $a^3 = 54$ with what I just found, we get $54 + 35 = \boxed{89}$.

2015 AIME I/10

December 8, 2024

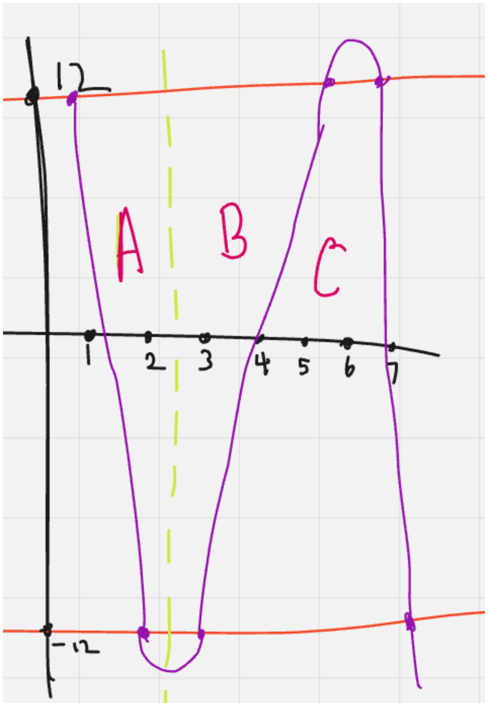
Let $f(x)$ be a third-degree polynomial with real coefficients satisfying

$$|f(1)| = |f(2)| = |f(3)| = |f(5)| = |f(6)| = |f(7)| = 12.$$

Find $|f(0)|$.

Solution:

We have this graph:



From this, we see that for 3 of these points $f(x) = 12$ and the for the other three points we will have $f(x) = -12$.

In the regions A, B, and C, each region has a point where $f(x) = 12$ and $f(x) = -12$, and we can see from the visual that $f(1) = f(5) = f(6) = 12$ and $f(2) = f(3) = f(7) = -12$.

Now knowing that a cubic polynomial is in the form $f(x) = ax^3 + bx^2 + cx + d$, we can find what $f(0)$ is. Plugging in 0, we find that $f(0) = d$.

We can now set up a system of equations from the knowledge we know above.

$$\begin{aligned} a + b + c + d &= 12 \\ 8a + 4b + 2c + d &= -12 \\ 27a + 9b + 3c + d &= -12 \\ 125a + 25b + 5c + d &= 12 \\ 216a + 36b + 6c + d &= 12 \\ 343 + 49b + 7c + d &= -12 \end{aligned}$$

Now since we have 4 variables, we can just use the first four of these equations.

Subtracting the first equation from the second equation we get $7a + 3b + c = -24$. Let's call this equation 5.

Subtracting the first equation from the third equation we get $26a + 8b + 2c = -24$. Let's call this equation 6.

Subtracting the first equation from the fourth equation we get $124a + 24b + 4c = 0$. Let's call this equation 7.

Now multiplying equation 5 by 2, we get $14a + 6b + 2c = -48$ and subtracting this from equation 6, we get $6a + b = 12$

Multiplying equation 6 by 2, we get $52a + 16b + 4c = -48$ and subtracting this from equation 7, we get $9a + b = 6$.

Now subtracting the two equations we just got, we can get $9a + b - (6a + b) = 3a = -6$, so $a = -2$.

Now plugging this back into $6a + b$, we get $-12 + b = 12$, so $b = 24$.

Plugging a and b into equation 5, we get $58 + c = -24$. So $c = -82$.

Plugging this back into the first original equation, we get $-2 + 24 - 82 + d = 12$ so $d = 12 + 60 = \boxed{72}$.

2012 AIME II/2

December 12, 2024

Two geometric sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ have the same common ratio, with $a_1 = 27$, $b_1 = 99$, and $a_{15} = b_{11}$. Find a_9.

Solution:

We have the two sequences $27, a_2, a_3, \ldots$ and $99, b_2, b_3, \ldots$.

Since they have the same common ratio, we can just do $27r^{14} = 99r^{10}$, and we are looking for $27r^8$.

When we divide the two, we get that $27r^4 = 99$, so $r^4 = \frac{99}{27} = \frac{11}{3}$.

The answer is therefore $27\left(\frac{11}{3}\right)\left(\frac{11}{3}\right) = 3 \cdot 121 = \boxed{363}$.

2012 AIME I/2

December 6, 2024

The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence.

Solution:

Let the arithmetic sequence be $a_1 + a_2 + a_3 + \cdots + a_n = 715$.

We also have $(a_1 + 1) + (a_2 + 3) + (a_3 + 5) + (a_n + (2n - 1)) = 836$.

Notice that the first arithmetic sequence is in the second arithmetic sequence. We can rewrite the second arithmetic sequence as

$$(a_1 + a_2 + a_3 + \cdots + a_n) + (1 + 3 + 5 + \cdots + (2n - 1))$$

Which is also $715 + (1 + 3 + 5 + \cdots + (2n - 1)) = 836$.

Note that the sum $(1 + 3 + 5 + \cdots + (2n - 1)) = n^2$. Therefore we have $n^2 = 836 - 715 = 121$. We now have $n = 11$.

Now we are looking for the sum of $a_1 + a_6 + a_{11}$. We know that a_6 will be the average of a_1 and a_{11}, so from that we know that $a_1 + a_{11} = 2a_6$.

Now if we continue the average of a_2 and a_{10} will also be a_6 so the sum of a_2 and a_{10} is $2a_6$.

If we continue this, we should see that the sum of $a_1 + a_2 + a_3 + \cdots + a_{11} = 735$ will just be equivalent to $11a_6 = 735$.

Note that $a_1 + a_{11} + a_6 = 2a_6 + a_6 = 3a_6$, so we can just multiply 735 by $11a_6 = 735$ by $\frac{3}{11}$ to get $3a_6 = 715\left(\frac{3}{11}\right) = \boxed{195}$.

2010 AIME I/9

November 25, 2024

Let (a, b, c) be a real solution of the system of equations $x^3 - xyz = 2$, $y^3 - xyz = 6$, $z^3 - xyz = 20$. The greatest possible value of $a^3 + b^3 + c^3$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Ok pretty much let $a = x$, $b = y$, and $c = z$.

Then we have

$$a^3 = 2 + abc \qquad b^3 = 6 + abc \qquad c^3 = 20 + abc$$

Also adding these we have $a^3 + b^3 + c^3 = 28 + 3(abc)$.

So we can like substitute $r = abc$ and then multiply the first rewritten equations above so:

$$\begin{aligned} a^3 &= 2 + r \\ b^3 &= 6 + r \\ c^3 &= 20 + r \end{aligned}$$

Multiplying these, we get $(abc)^3 = r^3 = (2 + r)(6 + r)(20 + r)$.

Ok now we can just expand:

$$\begin{aligned} r^3 &= (12 + 2r + 6r + r^2)(20 + r) \\ r^3 &= 240 + 12r + 160r + 8r^2 + 20r^2 + r^3 \\ 0 &= 240 + 12r + 160r + 8r^2 + 20r^2 \\ 0 &= 28r^2 + 172r + 240 \\ 0 &= 7r^2 + 43r + 60 \end{aligned}$$

The factors that multiply to 420 that add up to 43 are 28 and 15, so this can be factored to $(7r + 15)(r + 4)$.

Therefore, $r = -\frac{15}{7}$ and $r = -4$.

Above, we see that $a^3 + b^3 + c^3 = 28 + 3r$, so since both values of r are negative, the negative number closer to 0 will maximize the value.

Therefore, $r = -\frac{15}{7}$.

Plugging this in, we get $a^3 + b^3 + c^3 = 28 + 3\left(-\frac{15}{7}\right)$.

Expanding this, we get $\frac{151}{7}$, so $151 + 7 = \boxed{158}$.

2.2 2000-2010

2007 AIME I/2

November 29, 2024

A 100 foot long moving walkway moves at a constant rate of 6 feet per second. Al steps onto the start of the walkway and stands. Bob steps onto the start of the walkway two seconds later and strolls forward along the walkway at a constant rate of 4 feet per second. Two seconds after that, Cy reaches the start of the walkway and walks briskly forward beside the walkway at a constant rate of 8 feet per second. At a certain time, one of these three persons is exactly halfway between the other two. At that time, find the distance in feet between the start of the walkway and the middle person.

Solution:

Ah well Al ends up going 6 ft/sec, Cy goes 8 ft/sec and Bob ends up going 10 ft/sec.

So like pretty early on, we can find that Al will end up somewhere in between Cy and Bob, eventually Al will fall behind, but at earlier times it can be assumed that Al will end up halfway between Cy and Bob.

Therefore we have $\frac{8(t-4)+10(t-2)}{2} = 6t$.

Solving for t we get $8t - 32 + 10t - 20 = 12t$ and $6t = 52$, so $t = \frac{52}{6} = \frac{26}{3}$.

Substituting this into $6t$ for Al, we get $6\left(\frac{26}{3}\right) = \boxed{52}$.

2006 AIME II/1

October 25, 2024

In convex hexagon $ABCDEF$, all six sides are congruent, $\angle A$ and $\angle D$ are right angles, and $\angle B, \angle C, \angle E$, and $\angle F$ are congruent. The area of the hexagonal region is $2116(\sqrt{2}+1)$. Find AB.

Solution:

When we draw the diagram, we get a 45-45-90 triangle with a rectangle and a 45-45-90 triangle again on the top of the rectangle.

From this, we can find the area of the rectangle - so the area of the rectangle is $x(x\sqrt{2})$. The two 45-45-90 triangles create a square, so the area is x^2 so the area of convex hexagon is $x(x\sqrt{2}) + x^2$.

Simplifying, we get $x^2(\sqrt{2}+1)$. From the given statement in the question, we can find that $x^2 = 2116$, so $x = \boxed{46}$.

2005 AIME II/3

December 12, 2024

An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has 10 times the sum of the original series. The common ratio of the original series is $\frac{m}{n}$ where m and n are relatively prime integers. Find $m + n$.

Solution:

We let the first sequence be $a, ar, ar^2, \ldots$ and the second sequence be $(a)^2, (ar)^2, (ar^2)^2, \ldots$. The second sequence can be written as $a^2, a^2r^2, a^2r^4, \ldots$ now.

From this we can see that for the first sequence the sum is $\frac{a}{1-r} = 2005$, and for the second sequence it is $\frac{a^2}{1-r^2} = 2005 \cdot 10 = 20050$.

Now it is much easier to think about both of these in terms of just r, since that is what we are trying to find.

We can actually simplify $\frac{a^2}{1-r^2}$ by substituting the sum of the first sequence. So we have $a \cdot a(1+r)(1-r)$. From this we can see we can substitute 2005 in, so $\frac{2005a}{1+r} = 20050$. And now dividing, we get that $\frac{a}{1+r} = \frac{20050}{2005} = 10$.

Now with these two equations we can set them equal to each other. First we have $a = 2005(1-r) = 2005 - 2005r$ from the first sequence and $a = 10(1+r) = 10 + 10r$ from the second sequence.

Setting these equal to each other we get that $2005 - 2005r = 10 + 10r$ or $2015r = 1995$. Therefore $r = \frac{1995}{2015} = \frac{399}{403}$. The answer is therefore $399 + 403 = \boxed{802}$.

2001 AIME I/2

December 17, 2024

A finite set $\mathcal{S}$ of distinct real numbers has the following properties: the mean of $\mathcal{S} \cup \{1\}$ is 13 less than the mean of $\mathcal{S}$, and the mean of $\mathcal{S} \cup \{2001\}$ is 27 more than the mean of $\mathcal{S}$. Find the mean of $\mathcal{S}$.

Solution:

We can let the mean of $\mathcal{S}$ to be x and the amount of elements originally in $\mathcal{S}$ to be y.

Then we have $\frac{xy+1}{y+1} = x - 13$ and $\frac{xy+2001}{y+1} = x + 27$. It would be really nice to get rid of the x term, which we can do by subtracting!

So now we have $\frac{xy+2001-(xy+1)}{y+1} = x + 27 - (x - 13)$. Simplifying this, we get that $\frac{2000}{y+1} = 40$.

We can now solve for y pretty easily, this is $40y + 40 = 2000$, so $40y = 1960$ and dividing we get that $y = 49$.

Substituting this back into $\frac{xy+1}{y+1} = x - 13$, we get that $\frac{49x+1}{50} = x - 13$. Multiplying by 50 we get that $49x + 1 = 50(x - 13) = 50x - 650$. Solving for x we get that $x = \boxed{651}$.

2001 AIME I/3

November 23, 2024

Find the sum of the roots, real and non-real, of the equation $x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0$, given that there are no multiple roots.

Solution:

Let $a = x - \frac{1}{4}$.

Then we have

$$\left(\frac{1}{4} + a\right)^{2001} + \left(\frac{1}{4} - a\right)^{2001} = 0$$

The binomial theorem states that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Expanding both terms we get

$$\left(\frac{1}{4} + a\right)^{2001} = \sum_{k=0}^{2001} \binom{2001}{k} \left(\frac{1}{4}\right)^{2001-k} a^k$$

and

$$\left(\frac{1}{4} - a\right)^{2001} = \sum_{k=0}^{2001} \binom{2001}{k} \left(\frac{1}{4}\right)^{2001-k} (-a)^k$$

Adding these two together, we get

$$\sum_{k=0}^{2001} \binom{2001}{k} \left(\frac{1}{4}\right)^{2001-k} (a^k + (-a)^k)$$

Note that $a^k + (-a)^k$ will be equal to zero when k is an odd number, so only when k is an even number will there be a non-zero value and that it will result in $2a^k$ given an even k.

We therefore get:

$$2\sum_{k=0}^{2001} \binom{2001}{k} \left(\frac{1}{4}\right)^{2001-k} a^k = 0$$

Therefore the second term will have an odd power, and therefore a coefficient of 0. Therefore by Vieta's, the sum of the roots will be 0 divided by something, or 0.

Because the highest power will be 2000, then there are 2000 roots. Substituting back to the original equation, we can see that $a = x - \frac{1}{4}$. The sum of the roots of x will be the sum of the roots of $a + 2000 \cdot \frac{1}{4}$ or $\boxed{500}$.

2.3 Before 2000

1998 AIME/3

December 9, 2024

The graph of $y^2 + 2xy + 40|x| = 400$ partitions the plane into several regions. What is the area of the bounded region?

Solution: Isolating the $40|x|$ term, we get that $400 - y^2 - 2xy = 40|x|$.

Then we have two cases:

Case 1: $40x = 400 - y^2 - 2xy$

Case 2: $40x = -400 + y^2 + 2xy$

In the first case -

$$\begin{aligned} 40x &= 400 - y^2 - 2xy \\ 40x + 2xy &= 400 - y^2 \\ 2x(20 + y) &= (20 - y)(20 + y) \end{aligned}$$

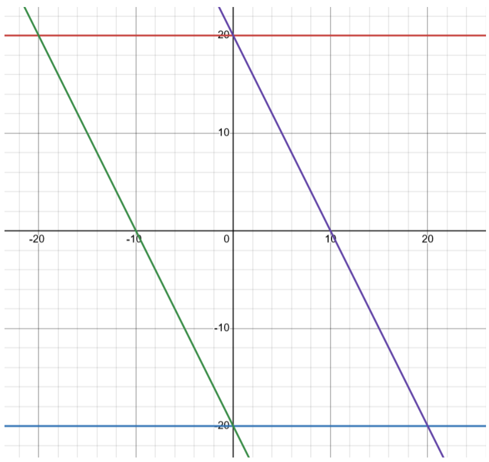
We can see that when $y = -20$, then the two equations are equal. When we cancel out the $(20 + y)$ term, we can see that $2x = 20 - y$, or $y = -2x + 20$.

For the second case -

$$\begin{aligned} 40x &= -400 + y^2 + 2xy \\ 40x - 2xy &= -400 + y^2 \\ -2x(y - 20) &= (y - 20)(y + 20) \end{aligned}$$

Here, we can see that $y = 20$ will result in both equations being the same. When cancelling out the $(y - 20)$ term, then we have $-2x = y + 20$, so $y = -2x - 20$.

When we graph the 4 lines, we have the following graph



The height of this shape is 40 and the base is 20, so $40 \cdot 20 = \boxed{800}$.

1991 AIME/1

November 21, 2024

Find $x^2 + y^2$ if x and y are positive integers such that

$$\begin{aligned} xy + x + y &= 71, \\ x^2y + xy^2 &= 880. \end{aligned}$$

Solution:

Let $a = xy$ and $b = x + y$.

By substituting these variables in, we get $a + b = 71$ and $ab = 880$.

We can solve for a by rearranging $a + b = 71$ to $b = 71 - a$.

Substituting this into the second equation, we get $a(71 - a) = 880$.

This simplifies to $a^2 - 71a - 880$.

Factoring this, we get $(a - 16)(a - 55)$, so $a = 16, 55$.

We can show that $a \neq 16$ because in this case, it would mean that $x + y = 55$, and there are no two factors that add up to 55 and can multiply to 16.

So, $a = 55$ or $xy = 55$, so (x, y) is $(11, 5)$ or $(5, 11)$.

Therefore, $5^2 + 11^2 = \boxed{146}$.

1989 AIME/8

December 22, 2024

Assume that $x_1, x_2, \ldots, x_7$ are real numbers such that

$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1 \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12 \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123. \end{aligned}$$

Find the value of

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7.$$

Solution:

So just do systems stuff

Subtracting the second equation and first equation you get

$$3x_1 + 5x_2 + 7x_3 + 9x_4 + 11x_5 + 13x_6 + 15x_7 = 11$$

Subtracting the third equation and second equation you get

$$5x_1 + 7x_2 + 9x_3 + 11x_4 + 13x_5 + 15x_6 + 17x_7 = 111$$

Ok now subtract these two equations and you get

$$2(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7) = 100$$

Cool!

Now we can do the same thing with the value we are trying to find.

Subtracting the fourth equation (the one starting with $16x_1$) with the third equation we get

$$7x_1 + 9x_2 + 11x_3 + 13x_4 + 15x_5 + 17x_6 + 19x_7 = x - 123$$

Now subtracting this from the $5x_1 + 7x_2 + 9x_3 + 11x_4 + 13x_5 + 15x_6 + 17x_7 = 111$ equation, we get

$$2(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7) = x - 123 - 111$$

Now we have $x - 234 = 100$, so $x = \boxed{334}$.

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3 Other Computational Contests

3.1 HMMT

2013 HMMT Algebra/4

November 28, 2024

Determine all real values of A for which there exist distinct complex numbers x_1, x_2 such that the following three equations hold:

$$\begin{aligned} x_1(x_1+1) &= A \\ x_2(x_2+1) &= A \\ x_1^4+3x_1^3+5x_1 &= x_2^4+3x_2^3+5x_2. \end{aligned}$$

Solution:

Ok I'm going to let $a = x_1$ and $b = x_2$ for this, because for the life of me I can't type underscores.

So we can rewrite the given equations as

$$\begin{aligned} a^2+a &= A \\ b^2+b &= A \\ a^4+3a^3+5a-b^4-3b^3-5b &= 0 \end{aligned}$$

Note that for the third equation I just subtracted stuff.

So let's go factor that third equation

$$\begin{aligned} a^4-b^2+3a^3-3b^3+5a-5b \\ (a^2+b^2)(a^2-b^2)+3(a^3-b^3)+5(a-b) \\ (a^2+b^2)(a+b)(a-b)+3(a-b)(a^2+ab+b^2)+5(a-b) = 0 \end{aligned}$$

Note that we can factor out $(a-b)$ from this, therefore we can get

$$(a^2+b^2)(a+b)+3(a^2+ab+b^2)+5=0$$

Now we can find out what $a+b$ is.

We can make $a^2+a=b^2+b$, so we can get a^2-b^2+a-b or $(a+b)(a-b)+(a-b)=0$. Similar to above where we factor $a-b$ out, we get $a+b+1=0$, or $a+b=-1$.

Now substituting this back in, we can get

$$\begin{aligned} -(a^2+b^2)+3a^2+3ab+3b^2+5 &= 0 \\ -a^2-b^2+3a^2+3ab+3b^2+5 &= 0 \\ 2a^2+2b^2+3ab+5 &= 0 \end{aligned}$$

Now using the above where $a+b=-1$, we can notice that we can actually find a way to write similar to the terms of the other equation we got. Multiplying both sides by 2 and squaring, we can get $2(a+b)^2=2$. This is equal to $2=2a^2+2b^2+4ab$. So substituting some values in, we can get

$$2-ab=2a^2+2b^2+3ab+5$$

. So we can rewrite this now as

$$2-ab-(2a^2+2b^2+ab)$$

Note that term in parenthesis is just equal to -5, so we get $2 - ab + 5 = 0$, therefore $ab = 7$.

Now we can find A by just using the equations given originally.

Adding both $a^2 + a$ and $b^2 + b$, we get $a^2 + b^2 + a + b = 2A$.

$a^2 + b^2$ is equal to $(a + b)^2 - 2ab$, or $1 - 14 = -13$.

Therefore, we get $-13 - 1 = -14 = 2A$, or $A = \boxed{-7}$.

2010 HMMT Algebra/1

December 15, 2024

Suppose that x and y are positive reals such that

$$x - y^2 = 3, \qquad x^2 + y^4 = 13.$$

Find x.

Solution:

First square the first equation.

$$\begin{aligned}(x - y)^2 &= 3^2 \\ (x - y^2)(x - y^2) &= 9 \\ x^2 + y^4 - 2xy^2 &= 9\end{aligned}$$

Now note that it would be really nice if we could find out what $x + y^2$ is to cancel out the y terms, so we are trying to find $\sqrt{x^2 + y^4 + 2xy^2}$ or $\sqrt{(x + y^2)^2}$.

We can do this by doubling the second equation to get $2x^2 + 2y^4$ and now subtracting $x^2 + y^4 - 2xy^2$ from this, we get

$$\begin{aligned}2x^2 + 2y^4 - x^2 - y^4 + 2xy^2 &= 26 - 9 \\ x^2 + y^4 + 2xy^2 &= 17\end{aligned}$$

Now we have $(x + y^2)^2 = 17$, so $x + y^2 = \pm\sqrt{17}$.

Now adding this with the first equation, we get that $x - y^2 + x + y^2 = 2x = 3 \pm \sqrt{17}$.

Note that since x is positive, then it will be $3 + \sqrt{17}$, since $3 - \sqrt{17}$ is less than 0.

Dividing by 2, we get that $x = \boxed{\dfrac{3 + \sqrt{17}}{2}}$.

2008 HMMT Algebra/1

December 17, 2024

Positive real numbers x, y satisfy the equations $x^2 + y^2 = 1$ and $x^4 + y^4 = \frac{17}{18}$. Find xy.

Solution:

Squaring the first equation, we get that $x^4 + 2(xy)^2 + y^4 = 1$.

Substituting in $\frac{17}{18}$, we ge that $\frac{17}{18} + 2(xy)^2 = 1$.

Solving for xy we get that $(xy)^2 = \frac{1}{36}$, or $xy = \boxed{\dfrac{1}{6}}$. (Note that this is positive because it was indicated in the question.)

2004 HMMT Algebra/4

November 21, 2024

Find all real solutions to $x^4 + (2-x)^4 = 34$.

Solution:

Note that $2-x$ and x are symmetric around 1, so substituting $x = t+1$, we can get $x = t+1$ and $2-x = 1-t$.

We can write $(1+t)^4 + (1-t)^4 = 34$ as a result of this.

Expanding, the two we get

$$(1+2t+t^2)^2 + (1-2t+t^2)^2$$
$$(1+2t+t^2+2t+4t^2+2t^3+t^2+2t^3+4)+(1-2t+t^2-2t+4t^2-2t^3+t^2-2t^3+t^4)$$
$$2t^4+12t^2 = 32$$
$$2t^4+12t^2-32 = 0$$

Dividing by 2, we get $t^4 + 6t^2 - 16 = 0$.

Substituting $t^2 = a$, we can rewrite this as $a^2 + 6a - 16 = 0$.

Factoring results in $a = -8$ and $a = 2$. Since t^2 cannot be equal to -8, the only number for a that works is 2. Therefore, $t = \pm\sqrt{2}$.

Substituting this back to the original substitution of $x = t+1$, we get the answer as $\boxed{1 \pm \sqrt{2}}$.

3.2 Math Prize for Girls

2018 Math Prize for Girls/19

December 13, 2024

Consider the sum

$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{2k-1}}\,.$$

Determine $\lfloor S_{4901} \rfloor$. Recall that if x is a real number, then $\lfloor x \rfloor$ (the floor of x) is the greatest integer that is less than or equal to x.

Solution:

Just by plugging in 1, we can see that $S_1 = 1$, but after this it is hard to add square roots.

So, we can let $T_n = \sum_{k=2}^{n} \frac{1}{\sqrt{2k-1}}$ and we can see that $S_n = 1 + T_n$.

So now we can bound the solution by writing

$$\sqrt{2k-1} + \sqrt{2(k-1)-1} < 2\sqrt{2k-1} < \sqrt{2k-1} + \sqrt{2(k+1)-1}$$

To put this in a form similar to the original equation, we can find the reciprocal of each. From this we get

$$\frac{1}{\sqrt{2k-1} + \sqrt{2(k-1)-1}} < \frac{1}{2\sqrt{2k-1}} < \frac{1}{\sqrt{2k-1} + \sqrt{2(k+1)-1}}$$

Rationalising the denominator, we get that

$$\frac{\sqrt{2k-1} - \sqrt{2(k-1)-1}}{2} < \frac{1}{2\sqrt{2k-1}} < \frac{\sqrt{2(k+1)-1} - \sqrt{2k-1}}{2}$$

So we are actually now finding that T_{4901} is bounded between

$$\sum_{k=2}^{4901} \sqrt{2k-1} - \sqrt{2(k-1)-1} < T_{4901} < \sum_{k=2}^{4901} \sqrt{2(k+1)-1} - \sqrt{2k-1}$$

We see that many terms actually cancel out on both sides of the inequality, so when we sum all the terms, the lower bound ends up being $\sqrt{9801} - 1$. The square root of 9801 is actually 99, so this bound equals 98.

The other bound is $\sqrt{9803} - \sqrt{3}$. We can assume that $\sqrt{9803}$ is a little larger than 99 and that the square root of 3 is a little under 2, so we can assume that it is around 97.

So, flooring T_{4901}, we should get 97. So $S_{4901} = 1 + 97 = \boxed{98}$.
