Differential Equations Notes

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1 Introduction to Differential Equations

1.1 Background

In a variety of subject areas, mathematical models are developed to aid in understanding. These models often yield an equation that contains derivatives of an unknown function. Such an equation is called a differential equation.

One example is free fall of a body. An object is released from a certain height above the ground and falls under the force of gravity. Newton's second law states that an object's mass times its acceleration equals the total force acting on it.

$$m\frac{\mathrm{d}^2h}{dt^2} = -mg$$

We have $h(t)$ as position, $\frac{\mathrm{d}h}{\mathrm{d}t}$ as velocity and $\frac{\mathrm{d}^2h}{\mathrm{d}t}$ as acceleration. The independent variable is t and the dependent variable is h.

$m\frac{\mathrm{d}^2h}{dt^2} = -mg$ is a differential equation and $h(t)$ is the unknown function that we are trying to find.

From this we have $\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = -g$ and the integral of this is $\frac{\mathrm{d}h}{\mathrm{d}t} = -gt + C_1$. To find h we integrate again and we get $h(t) = -\frac{gt^2}{2} + C_1t + C_2$.

Another example is the decay of a radioactive substance. The rate of decay is proportional to the amount of radioactive substance present.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -kA, \qquad k > 0$$

where A is the unknown amount of radioactive substance present at time t and k Is the proportionality constant.

We are looking for $A(t)$ that satisfies this equation. We can solve this from $\frac{1}{A}\mathrm{d}A = k\mathrm{d}t$ and integrating both sides we get that $\ln|A| + C_1 = -kt + C_2$. We can rewrite this as $\ln|A| = -kt + C$. So, $e^{-kt+C} = A$.

So $A(t) = e^{-kt} + e^C$, so $A(t) = Ce^{-kt}$. Remember A is the dependent variable and t is the independent variable.

Notice that the solution of a differential equation is a function, not merely a number.

When a mathematical model involves the rate of change of one variable with respect to another, a differential equation is apt to appear.

Terminology

If an equation involves the derivative of one variable with respect to another, then the former is called a dependent variable and the latter an independent variable.

In $\frac{\mathrm{d}h}{\mathrm{d}t}$, h is dependent and t is independent.

A differential equation involving only ordinary derivatives with respect to a single independent variable is called an ordinary differential equation. A differential equation involving partial derivatives with respect to more than one independent variable is a partial differential equation.

For example we have $z = f(x,y) = 4x^2 + 5xy$, so $\frac{\partial z}{\partial x} = 8x + 5y$ and that is partial differentiation.

The order of a differential equation is the order of the highest-order derivatives present in the equation.

For example, $\frac{\mathrm{d}^2h}{dt^2} = -g$ has a order of 2.

A linear differential equation is one in which the dependent variable y and its derivatives appear in additive combinations of their first powers. A differential equation is linear if it has the format.

$$a_n(x)\frac{\mathrm{d}^n y}{dx^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = F(x)$$

$2x + 3y = 7$ is linear, $2x^2 + 5xy + 7y + 8y = 1$ is second-degree. Nothing can have a second degree for this to be linear.

You are just looking at the dependent variable and the derivatives and adding their powers.

If an ordinary differential equation is not linear, we call it nonlinear.

Example

For each differential equation, classify as ODE or PDE, linear or nonlinear, and indicate the dependent/independent variables and order.

(a) $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + a\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0$

Dependent is x, independent is t and the order is 2. This is an ODE and linear.

(b) $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x - 2y$

The dependent variable is u and the independent variables are x, y, so this is a PDE. The order is 1.

(c) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^3 = 0$

The dependent variable is y, the independent variable is x, the order is 2 and this is an ODE and this is nonlinear.

(d) $t^3\frac{\mathrm{d}x}{\mathrm{d}t} = t^3 + x$

Dependent is x, independent is t, order is 1, this is an ODE. We can rewrite this as $t^3\frac{\mathrm{d}x}{\mathrm{d}t} - 1x = t^3$, and this matches the form of the linear equation so this is linear.

(e) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - y\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x$

The dependent is y, the independent is x, the order is 2 and this is an ODE and this is nonlinear because of $y\frac{\mathrm{d}y}{\mathrm{d}x}$.

1.2 Solutions and Initial Value Problems

An nth-order ordinary differential equation is an equality relating the independent variable to the nth derivative (and usually lower-order derivatives as well) of the dependent variable.

Example

Identify the order, independent and dependent variable.

(a) $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3$. Independent: x, dependent: y, order: 2

(b) $\sqrt{1 - \left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right)} - y = 0$. Independent: t, dependent: y, order: 2

(c) $\frac{\mathrm{d}^4x}{\mathrm{d}t^4} = xt$. Independent: t, dependent: x, order: 4. (This is also linear.)

A general form for an nth-order equation with x independent, y dependent can be expressed as

$$F(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}) = 0$$

where F is a function that depends on x, y, and the derivatives of y up to order n. We assume the equations holds for all x in an open interval I. In many cases, we can isolate the highest-order term and write the previous equation as

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}\right)$$

This is called the normal form.

A function $\phi(x)$ that when substituted for y in either the previous two equations satisfies the equation for all x in the interval I is called an explicit solution to the equation on I.

Example

Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to the linear equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - frac2x^2y = 0$ but $\psi(x) = x^3$ is not.

So we have $y = x^2 - x^{-1}$. The first derivative of this is $2x + 1x^{-2}$. The second derivative is $y'' = 2 - 2x^{-3}$. If we plug in the values we end up getting from the derivatives, we get that $2 - 2x^{-3} - 2x + 2x^{-3} = 0$, so this is satisfied.

For the second part, the first derivative is $3x^2$ and the second derivative is $6x$. Plugging this in, we get $4x$ which is not 0, so $\psi(x)$ is not a solution.

Example

Show that for any choice of the constants c_1 and c_2, the function $\phi(x) = c_1e^{-x} + c_2e^{2x}$ is an explicit solution to the linear equation $y'' - y' - 2y = 0$.

We have that the first derivative is $-c_1e^{-x} + 2c_2e^{2x}$ and the second derivative is $c_1e^{-x} + 4c_2e^{2x}$. When we plug this in, we find that this does satisfy the solution for the differential equation.

Methods for solving differential equations do not always yield an explicit solution for the equation. A solution may be defined implicitly.

Example

Show that the relation $y^2 - x^3 + 8 = 0$ implicitly defines a solution to the nonlinear equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{2y}$ on the interval $(2, \infty)$.

We have from the given that $y = \pm\sqrt{x^3 - 8}$. The derivative (of the positive version) of this is $\frac{3x^2}{2\sqrt{x^3-8}}$. This is the same and defined on the interval.

A relation $G(x, y) = 0$ is said to be an implicit solution to the previous equation on the interval I if it defines one or more explicit solutions on I.

Example

Show that $x + y + e^{xy} = 0$ is an implicit solution to the nonlinear equation $(1 + xe^{xy})\frac{\mathrm{d}y}{\mathrm{d}x} + 1 + ye^{xy} = 0$.

Taking the derivative of both sides gets us that $1 + \frac{\mathrm{d}y}{\mathrm{d}x} + e^{xy}\frac{\mathrm{d}}{\mathrm{d}x}(xy) = 0$. This does simplify to what was given in the problem.

Example

Verify that for every constant C the relation $4x^2 - y^2 = C$ is an implicit solution to $y\frac{\mathrm{d}y}{\mathrm{d}x} - 4x = 0$. Graph the solution curves for $C = 0, \pm1, \pm4$.

The derivative of what is given is $8x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. This simplifies to what is given, so it is clearly an implicit solution.

For $C = 0$, the solution curves for this is $2x = y$ and $-2x = y$.

For $C = \pm 4$, the solution curves is given by a hyperbola $\frac{x^2}{-}\frac{y^2}{4} = 1$.

The collection of all solutions in the previous example is called a one-parameter family of solutions.

In general, the methods for solving nth-order differential equations evoke n arbitrary constants. We often can evalute these constants if we are given n initial values $y(x_0), y'(x_0), \dots, y^{(n-1)}(x_0)$.

Definition

By an initial value problem for an nth-order differential equation

$$F(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \dots, \frac{\mathrm{d}^n y}{\mathrm{d}x^2}) = 0$$

we mean: Find a solution so the differential equation on an interval I that satisfies at x_0 the n initial conditions

$$y(x_0) = y_0, \quad \frac{\mathrm{d}y}{\mathrm{d}x}(x_0) = y_1 \cdots \frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}(x_0) = y_{n-1}$$

where $x_0 \in I$ and $y_0, y_1, \dots, y_{n-1}$ are constants.

Example

Show that $\phi(x) = \sin x - \cos x$ is a solution to the initial value problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0; \qquad y(0) = -1 \qquad \frac{\mathrm{d}y}{\mathrm{d}x}(0) = 1$$

We have $y = \sin x - \cos x$, $y' = \cos x + \sin x$, and $y'' = -\sin x + \cos x$. These satisfy the conditions.

Theorem 1.1: Existence and Uniqueness of Solution

Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \qquad y(x_0) = y_0$$

If f and $\partial f / \partial y$ are continuous functions in some rectangle

$$R = \{(x, y) : a < x < b, c < y < d\}$$

that contains the point (x_0, y_0), then the initial value problem has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 \delta$, where δ is a positive number.

Example

Does the theorem above imply the existence for this problem.

$3\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - xy^3, \qquad y(1) = 6$

The derivative exists for all (x, y) and is continuous in all intervals containing $x = 1$ and all rectangular regions containing $(1, 6)$.

When we consider the partial derivatives, $\partial f / \partial y = -xy^2$, and this exists and is continuous for all rectangular regions in the xy plane.

1.3 Direction Fields

One technique useful in visualizing (graphing) the solutions to a first-order differential equation is to sketch the direction field for the equation. A first-order equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

specifies a slope at each point in the xy-plane where f is defined.

Definition

A plot of short line segments drawn at various points in the xy-plane showing the slope of the solution curve htere is called a direction field for the differential equation.

For example, consider the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - y$. When we plug in the point $(1, 0)$, the slope is 1. When the point is $(0, 1)$ the slope is -1. Notice the further right we get, the steeper the graph goes. When we look at this function, $f(x, y) = x^2 - y$, so taking the partial of y results in -1 which is continuous so there is a solution curve at any point.

Note that we are basically just drawing slope fields from AP Calculus BC.

When we consider an equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}$, we have a unique solution when $x \neq 0$ because $f(x, y)$ is continuous if $x \neq 0$ and $\frac{\partial f}{\partial y} = -\frac{1}{x}$, so if $x \neq 0$, then there is a unique solution.

Example

Consider the direction field for $\frac{\mathrm{d}y}{\mathrm{d}x} = 3y^{2/3}$. Is there a unique solution passing through $(2, 0)$?

$\frac{\partial f}{\partial y} = \frac{2}{\sqrt[3]{y}}$ is not continuous when $y = 0$.

Example

The logistic equation of the population p (in thousands) at time t of a certain species is given by $\frac{\mathrm{d}p}{\mathrm{d}t} = p(2 - p)$. Use its direction field to answer the following questions.

(a) If the initial population is $3000[p(0) = 3]$, what can you say about the limiting population $\lim_{t\to\infty} p(t)$?

We start at the point $(0, 3)$ and as the field approaches $p = 2$, the rate of change becomes 0 so the limit is equal to 2.

(b) Can a population of 1000 ever decline to 500?

No

(c) Can a population of 1000 ever increase to 3000.

No

A differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$ is autonomous if the independent variable t does not appear explicitly: $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$. An autonomous equation has the following properties

- The slopes in the direction field are all identical among horizontal lines
- New solutions can be generated from old ones by time shifting [i.e., replacing $y(t)$ with $y(t - t_0)$]

The constant, or equilibrium, solutions $y(t) = c$ for autonomous equations are of particular interest. The equilibrium $y = c$ is called a stable equilibrium, or sink, if neighboring solutions are attracted to it as $t \to \infty$. Equilibria that repel neighboring solutions, are known as sources; all other equilibria are called nodes. Sources and nodes are unstable equilibria. (Nodes are sometimes called semi stable).

A phase line indicates the zeros and signs of $f(y)$ to describe the nature of the equilibrium solutions for an autonomous equation.

Example

SKetch the phase line for $y' = -(y-1)(y-3)(y-5)^2$ and state the nature of its equilibria.

We have $y' = 0$ and $y = 1, 3, 5$. When $y = 0$, then $y' < 0$ that means that it is decreasing for values below 1. When we let $y = 2$ then $y' > 0$, so any value between 1 and 3 is increasing. When we put $y = 4$, then $y' < 0$ so any values between 3 and 5 are decreasing. When $y = 6$ then $y' < 0$ so it is decreasing.

At $y = 5$, above it is is attracting but repeling below, so it is semi-stable or a node. At $y = 3$, it is stable or an attractor because it is approaching on both sides. At $y = 1$, it is a repeller.

Hand sketching the direction field for a differential equation is often tedious. Fortunately, several software programs are available for this task. When hand sketching is necessary, the method of isoclines can be helpful reducing the work.

A isocline for the differential equation

$$y' = f(x, y)$$

is a set of points in the xy-plane where all the solutions have the same slope $\frac{\mathrm{d}y}{\mathrm{d}x}$; thus, it is a level curve for the function $f(x, y)$.

Example

Find isocline curves of $y' = f(x, y) = x + y$ for a few select values of c. Use the isoclines to draw hash marks with slope c along the isocline $f(x, y) = c$.

When $c = 0, y' = 0, x + y = 0$ and $y = -x$. When $c = 1, y' = 1, x + y = 1$ and $y = -x + 1$. When $c = 2, y' = 2, x + y = 2$ and $y = -x + 2$.

1.4 The Approximation Method of Euler

Euler's method (or the tangent-line method) is a procedure for constructing approximate solutions to an initial value problem for a first-order differential equation

$$y' = f(x, y), \qquad y(x_0) = y_0$$

Euler's method can be summarized by the recursive formulas $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n, y_n)$, where $n = 0, 1, 2, \ldots$.

h is the step size, y' is the m of the tangent line. Remember that $y - y_0 = m(x - x_0)$ and that $y - y_0$ is just $f(x_0, y_0)(x - x_0)$ so $y = y_0 + f(x_0, y_0) \cdot h$.

Example

Use Euler's method with step size $h = 0.1$ to approximate the solution to the initial value problem

$$y' = x\sqrt{y}, \qquad y(1) = 4$$

at the points $x = 1.1, 1.2, 1.3, 1.4$ and 1.5.

We know the x points so we can find the y values from $y_n = y_{n-1} + f(x_{n-1}, y_{n-1})(0.1)$.

We have $(1, 4)$ then $(1.1, 4.2)$ and continuing the calculations, we get $(1.2, 4.43)$, $(1.3, 4.68)$, $(1.4, 4.96)$ and $(1.5, 5.27)$.

Example

Use Euler's method to find approximations to the solution of the initial value problem

$$y' = y, \qquad y(0) = 1$$

at $x = 1$, taking $1, 2, 4, 8$ and 16 steps.

Let $y = e^x$ and a point $(0, 1)$. The recursion formula is $y_n = y_{n-1} + f(x_{n-1} + y_{n-1})h$.

If we use a step size of 1, then $y(1) = 2$.

If we use a step size of 0.5 then $y(1) = 2.25$

If we use a step size of 0.25 then $y(1) = 2.44$

Using technology we can see with a step size of 0.125 that $y(1) = 2.57$ and with a step size of 0.0625, $y(1) = 2.64$.

2 First-Order Differential Equations

2.1 Separable Equations

Definition

If the right-hand side of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)$$

can be expressed as a function $g(x)$ that depends only on x times a function $p(y)$ that depends only on y, then the differential equation is called separable.

To solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)p(y)$$

multiply by $\mathrm{d}x$ and by $h(y) = 1/p(y)$ to obtain

$$h(y)\mathrm{d}y = g(x)\mathrm{d}x$$

Then integrate both sides and you end up getting $H(y) = G(x)+C$, where we have merged the two constants of integration into a single symbol C. The last equation gives an implicit solution to the differential equation.

Example

Solve the nonlinear equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-5}{y^2}$$

This can be rewritten as $y^2\mathrm{d}y = (x-5)\mathrm{d}x$. Integrating both sides results in $\frac{y^3}{3} = \frac{x^2}{2} - 5x + C$.

To get the explicit form just solve for y, which is trivial.

Example

Solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-1}{x+3} \qquad y(-1) = 0$$

Doing Calc BC stuff gives us $y = 1 - \frac{1}{2}(x+3)$.

Be careful because you can be losing solutions. Ok bye!

2.2 Linear Equations

Remember a linear first-order equation is an equation that can be expressed in the form

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = b(x)$$

where $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on the independent variable x, not on y.

Method for solving linear equation:

- Write the equation in the standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

- Calculate the integrating factor $\mu(x)$ by the formula

$$\mu(x) = \exp\left[\int P(x)\mathrm{d}x\right]$$

- Multiply the equation in standard form by $\mu(x)$ and, recalling that the left-hand side is just $\frac{\mathrm{d}}{\mathrm{d}x}[\mu(x)y]$, obtain

$$\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\mu(x)y = \mu(x)Q(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\mu(x)y] = \mu(x)Q(x)$$

- Integrate the last equation and solve for y by dividing by $\mu(x)$ to obtain.

Example

Find the general solution to

$$\frac{1}{x}\mathrm{d}y\mathrm{d}x - \frac{2y}{x^2} = x\cos x \qquad x > 0$$

We have $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x}y = x^2\cos x$.

The integrating factor $\mu(x) = e^{\int P(x)\mathrm{d}x}$ which in this case is $e^{-2\int \frac{1}{x}\mathrm{d}x}$ and this is equivalent to $\frac{1}{x^2}$.

Using this we can multiply through in standard form then we have $\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x^3}y = \cos x$.

The left side is just $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^2}\cdot\right) = \cos x$.

Integrating and solving for y we get that $y = x^2\sin x + Cx^2$.

Example

For the initial value problem

$$y' + y = \sqrt{1+\cos^2 x} \qquad y(1) = 4$$

find the value of $y(2)$.

Our $P(x)$ is 1 here, so $\mu = e^x$.

So the equation after multiplying through by it gives us that $\mu y' + \mu y = \mu\sqrt{1+\cos^2 x}$, or $e^x y' + e^x y = e^x\sqrt{1+\cos^2 x}$.

This is equivalent to basically $\frac{\mathrm{d}}{\mathrm{d}x}(e^x y) = e^x\sqrt{1+\cos^2 x}$.

This is $e^x y = \int e^x\sqrt{1+\cos^2 x}\mathrm{d}x$.

Using a calculator $y(2) = 2.127$.

Theorem 2.1: Existence and Uniqueness of Solution

Suppose $P(x)$ and $Q(x)$ are continuous on an interval (a,b) that contains the point x_0. Then for any choice of initial value y_0, there exists a unique solution $y(x)$ on (a,b) to the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Qx \qquad y(x_0) = y_0$$

In fact the solution is given for a suitable value of C.

2.3 Exact Equations

Definition: Exact Differential Form

The differential form $M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y$ is said to be exact in a rectangle R is there is a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x}(x,y) = M(x,y) \qquad \text{and} \qquad \frac{\partial F}{\partial y}(x,y) = N(x,y)$$

for all (x,y) in R. That is, the total differential of $F(x,y)$ satisfies

$$\mathrm{d}F(x,y) = M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y$$

If $M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y$ is an exact differential form, then the equation

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0$$

is called an exact equation.

Theorem 2.2: Test for Exactness

Suppose the first partial derivatives of $M(x,y)$ and $N(x,y)$ are continuous in a rectangle R. Then

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0$$

is an exact equation in R if and only if the compatibility condition

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$$

holds for all (x,y) in R.

Example

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy^2+1}{2x^2y}$$

Ok so this is not separable or linear, so we use exactness.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2xy^2+1}{2x^2y} &= 0 \\
\mathrm{d}y + \frac{2xy^2+1}{2x^2y}\mathrm{d}x &= 0 \\
\frac{2xy^2+1}{2x^2y}\mathrm{d}x + 1\mathrm{d}y &= 0
\end{aligned}$$

This is the same form we want.

Another form we can get is $(2xy^2+1)\mathrm{d}x + 2x^2y\mathrm{d}y = 0$.

Another form we can get is $1\mathrm{d}x + \frac{2x^2y}{2xy^2+1}\mathrm{d}y = 0$.

We are now looking for a $F(x,y) = c$ and we know this is true when $\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$.

So the second one of these is probably the best, so we now have $m = 2xy^2+1$ and $n = 2x^2y$.

Doing the partial of m with respect to y we get $4xy$ and the partial of n with respect to x is $4xy$ and these are the same.

Let $F(x,y) = x^2y^2 + x = C$. The partial of this function with respect to x is $2xy^2 + 1$ and the partial of this function with respect to y is $2x^2y$ and this is the same as previous.

Method for Solving Exact Equations:

- If $M\mathrm{d}x + N\mathrm{d}y = 0$ is exact, then $\partial F/\partial x = M$. Integrate this last equation with respect to x to get

$$F(x,y) = \int M(x,y)\mathrm{d}x + g(y)$$

- To determine $g(y)$, take the partial derivative with respect to y of both sides of the aboev equation and substitute N for $\partial F/\partial y$. We can now solve for $g'(y)$.
- Integrate $g'(y)$ to obtain $g(y)$ up to a numerical constant. Substituting $g(y)$ into the equation from step 1 gives $F(x,y)$
- The solution to $M\mathrm{d}x + N\mathrm{d}y = 0$ is given implicitly by

$$F(x,y) = C$$

(Alternatively, starting with $\partial F/\partial y = N$, the implicit solution can be foudn by first integrating with respect to y.)

Example

Solve

$$(2xy - \sec^2 x)\mathrm{d}x + (x^2 + 2y)\mathrm{d}y = 0$$

Let m be the first term and n be the second term, and the partial derivatives of these are the same, so they are exact.

Let $F(x,y) = \int 2xy - \sec^2 x\mathrm{d}x$. When we integrate this, we get $yx^2 - \tan x$. In this case, the constant is anything with y, so the integral is equivalent to $yx^2 - \tan x + g(y)$.

Now we take the $\frac{\partial F}{\partial y} = x^2 - 0 + g'(y)$. These two are n so $x^2 + 2y = x^2 + g'(y)$, so solving for $g(y)$ we get that this is equal to $y^2 + C$.

So $F(x,y) = xy^2 - \tan x + y^2 = C$.

Exercise Solve $(1 + e^x y + xe^x y)\mathrm{d}x + (xe^x + 2)\mathrm{d}y = 0$.

Solution: $x + xye^x + 2y = C$.

Example

Solve

$$(x + 3x^3 \sin y)\mathrm{d}x + (x^4 \cos y)\mathrm{d}y = 0$$

Doing the partials originally makes them not equal to each other.

We can get this to exact form by multiplying through by x^{-1}. When we do this we get $(1+3x^2 \sin y)\mathrm{d}x + x^3 \cos y\mathrm{d}y = 0$ and the partials of these are the same.

x^{-1} is called an integrating factor.

Integrating m with respect to x, we get that $F(x,y) = \int 1 + 3x^2 \sin y\mathrm{d}x = x + \sin y \cdot x^3 + g(y)$.

Doing the partial of F with respect to y, we get $\frac{\partial F}{\partial y} = x^3 \cos y = 0 + x^3 \cos y + g'(y)$, and this gets that $g(y) = C$.

So the answer is $x + x^3 \sin y = C$.

2.4 Special Integrating Factors

Definition

If the equation

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0$$

is not exact, but the equation

$$\mu(x,y)M(x,y)\mathrm{d}x + \mu(x,y)N(x,y)\mathrm{d}y = 0$$

which results from multiplying the first equation by the function $\mu(x,y)$, is exact, then $\mu(x,y)$ is called an integrating factor of the first equation.

Theorem 2.3: Special Integrating Factors

If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

$$\mu(x) = \exp\left[\int\left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right)\mathrm{d}x\right]$$

is an integrating factor for an equation. If $(\partial N/\partial x - \partial M/\partial y)/M$ is continuous and depends only on y, then

$$\mu(y) = \exp\left[\int\left(\frac{\partial M/\partial x - \partial N/\partial y}{M}\right)\mathrm{d}y\right]$$

is an integrating factor for the same equation.

Method for Finding Special Integrating Factors:

If $M\mathrm{d}x + N\mathrm{d}y = 0$ is neither separable nor linear, compute $\partial M/\partial y$ and $\partial N/\partial x$. If $\partial M/\partial y = \partial N/\partial x$, then the equation is exact. If it is not exact, consider

$$\frac{\partial M/\partial y - \partial N/\partial x}{N}$$

If this is a function of just x, then an integrating factor is given by the formula above of $\mu(x)$. If not consider

$$\frac{\partial N/\partial x - \partial M/\partial y}{M}$$

If this is a function of just y¡ then an integrating factor is given by above of $\mu(y)$.

Example

Solve $(2x^2+y)\mathrm{d}x + (x^2y - x)\mathrm{d}y = 0$

When we do the partials, we get that $1 \neq 2xy - 1$.

So lets look at $\frac{\partial m/\partial y - \partial n/\partial x}{N}$, which is $\frac{1-(2xy-1)}{x^2y-x} = \frac{-2}{x}$ which is just a function of x. So we have that $\mu = e^{\int -\frac{2}{x}\mathrm{d}x}$, so we don't have to look at the one in terms of y.

Doing the integral of all this gives us that $e^{-2\ln x} = x^{-2}$. So when we multiply through by x^{-2}, we get that $(2 + x^{-2}y)\mathrm{d}x + (y - x^{-1})\mathrm{d}y = 0$.

The partials are equal to each other, so this equation is now exact.

Now we find $F(x,y)$ by integrating m, so $\int(2+x^{-2}y)\mathrm{d}x = 2x + -x^{-1}y + g(y) = F(x)$

Now we differentiation with respect to y so $\frac{\partial F}{\partial y} = y - x^{-1} = -x^{-1} + g'(y)$, so $g(y) = \frac{y^2}{2}$

The solution is therefore $2x - x^{-1}y + \frac{y^2}{2} = C$.

2.5 Substitutions and Transformations

Substitution Procedure:

- Identify the type of equation and determine the appropriate substitution or transformation
- Rewrite the original equation in terms of new variables
- Solve the transformed equation
- Express the solution in terms of the original variables

Definition: Homogeneous Equation

If the right-hand side of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

can be expressed as a function of the ratio y/x alone, then we say the equation is homogeneous.

To solve a homogeneous equation, use the substitution $v = \frac{y}{x}$; $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ to transform the equation into a separable equation.

Example

Solve $(xy + y^2 + x^2)\mathrm{d}x - x^2\mathrm{d}y = 0$.

Solving for $\frac{\mathrm{d}y}{\mathrm{d}x}$ we get that this is equal to $\frac{-x^2-y^2-xy}{-x^2}$ and this simplifies to $1 + \left(\frac{y}{x}\right)^2 + \frac{y}{x}$.

This is equivalent to $v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v^2 + v$. We end up getting that $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v^2+1}{x}$ and this can be done by separation. The solution is $y = x\tan(\ln|x| + C)$ after solving.

To solve an equation of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = G(ax+by)$, use the substitution $z = ax+by$ to transform the equation into a separable equation.

Example

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = y - x - 1 + (x - y + 2)^{-1}$

First we have $\frac{\mathrm{d}y}{\mathrm{d}x} = -(x - y) - 1 + (x - y + 2)^{-1}$

So substituting with $z = x - y$, we have that $\frac{\mathrm{d}z}{=}1 - \frac{\mathrm{d}z}{\mathrm{d}x}$. Knowing this, the equation is equal to $1 - \frac{\mathrm{d}z}{\mathrm{d}x} = -z - 1 + (z + 2)^{-1}$. From this this simplifies to $\frac{\mathrm{d}z}{\mathrm{d}x} = z + 2 - (z + 2)^{-1}$.

So now we write this into a separable equation with $\frac{(z+2)\mathrm{d}z}{(z+2)^2-1} = \mathrm{d}x$.

Separating by parts and substituting gives $(x - y + 2)^2 = ce^{2x} + 1$

Definition: Bernoulli Equation

A first-order equation that can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

where $P(x)$ and $Q(x)$ are continuous on the interval (a, b) and n is a real number, is called a Bernoulli equation.

To solve a Bernoulli equation use the substitution $v = y^{1-n}$ to transform the equation into a linear equation.

Example

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = -\frac{5}{2}xy^3$.

From above, we have $v = y^{-2}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = -2y^{-3}\frac{\mathrm{d}u}{\mathrm{d}x}$ and $-\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}x} = y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}$.

So multiplying through by y^{-3} and substituting, we get that $-\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}x} - 5v = -\frac{5}{2}x$.

This is equal to $\frac{\mathrm{d}v}{\mathrm{d}x} + 10v = 5x$. The integrating factor here is $\mu = e^{\int P(x)\mathrm{d}x}$, which is e^{10x} in this case.

Multiplying through by μ, we get that $e^{10x}\frac{\mathrm{d}v}{\mathrm{d}x} + 10ve^{10x} = 5xe^{10}x$ and the LHS should be equal to $\frac{\mathrm{d}}{\mathrm{d}x}(e^{10x}v) = 5xe^{10x}$.

Using elementary integration techniques the answer is $y^{-2} = \frac{x}{2} - \frac{1}{20} + Ce^{-10x}$.

3 Mathematical Models and Numerical Methods Involving First-Order Equations

3.1 Compartmental Analysis

We assume that the growth rate is proportional to the population present. A mathematical model called the Malthusian, or exponential law of population growth, model is given by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = kp, \qquad p(0) = p_0$$

where $k > 0$ is the proportionality constant of the growth rate and p_0 is the population at time $t = 0$.

Example

In 1790, the population of the United States was 3.93 million, and in 1890 it was 62.98 million. Assuming the Mathusian model, estimate the U.S. population as a function of time.

Let $t = 0$ be year 1790. Using separation of variables and integration from $\frac{\mathrm{d}p}{\mathrm{d}t} = kp$, we get that $\ln P = kt + C$. Solving for P we get that $e^{\ln P} = P = e^{kt+C}$, so this is equal to $P = Ce^{kt}$.

At $P(0) = 3.93$, and using this we can find k. $3.93 = Ce^0$, so $C = 3.93$. So we have $P(t) = P_0e^{kt}$. We can now find k by plugging in $P(100) = 62.98$ from this, and solving for k, we get that $k \approx 0.027742$.

So $P(t) = 3.93e^{.027742t}$

The Malthusian model considered only death by natural casues. Other factors such as premature deaths due to malnutrition, inadequate medical supplies, communicable diseases, violent crimes, etc involve a competition within the population. The logistic model is given by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -Ap(p - p_1), \qquad p(0) = p_0$$

Note that this equilibrium has two equilibrium solutions $p(t) = p_1$ and $p(t) = 0$.

Example

Taking the population of 3.93 million as the initial population and given the 1840 and 1890 populations of 17.07 and 62.98 million respectively, use the logistic model to estimate the population at time t.

We have $P(50) = 17.07$ and $P(100) = 62.98$. Using a previously derived formula, $p(t) = \frac{p_0p_1}{p_0+(p_1-p_0)e^{-Ap_1t}}$, we get $17.07 = \frac{3.93p_1}{3.93+(p_1-3.93)e^{-50Ap_1}}$ and $62.98 = \frac{3.93p_1}{3.93+(p_1-3.93)e^{-100Ap_1}}$.

The answers are that $p_1 \approx 251.78$ and $A \approx 0.0001210$, so the logistic equation ended up being $p(t) = \frac{989.5}{3.93+247.85e^{-0.030463t}}$.

Example

Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days, $x(4) = 50$.

We are solving the initial value problem $\frac{\mathrm{d}x}{\mathrm{d}t} = kx(1000 - x), x(0) = 1$. Doing some stuff we get that $\ln \frac{x}{1000-x} = 1000kt + C$.

After doing some more things we get that $x = \frac{1000Ce^{1000kt}}{1+Ce^{1000kt}}$ and from this we can get that $C = \frac{1}{999}$.

So the function is now $x(t) = \frac{\frac{1000}{999}e^{1000kt}}{1+\frac{1}{999}e^{1000kt}}$, so simplifying we get that $x(t) = \frac{1000}{999e^{-1000kt}+1}$.

Doing some plugging in stuff we get that $k \approx -.9906$, so after 6 days approximately 276 students are infected.

3.2 Numerical Methods: A Closer Look At Euler's Algorithm

The numerical method defined by the formula

$$y_{n+1} = y_n + h\frac{f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})}{2}$$

where

$$y^*_{n+1} = y_n + hf(x_n, y_n)$$

is known as the improved Euler's method.

The improved Euler's method is an example of a predictor-corrector method.

It is recommended you use technology to find the answer.

3.3 Higher-Order Numerical Methods: Taylor and Runge-Kutta

We wish to obtain a numerical approximation of the solution $\phi(x)$ to the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0$$

To derive the Tayler methods, let $\phi_n(x)$ be the exact solution to the initial value problem

$$\phi_n'(x) = f(x, \phi_n), \qquad \phi_n(\text{problem}x_n) = y_n$$

The Taylor series for $\phi_n(x)$ about the point x_n is

$$\phi_n(x) = \phi_n(x_n) + h\phi_n'(x_n) + \frac{h}{2!}\phi_n''(x_n) + \dots$$

where $h = x - x_n$. Since ϕ_n satisfies the initial value, we can write this series in the form

$$\phi_n(x) = y_n + hf(x_n, y_n) + \frac{h^2}{2!}\phi_n''(x_n) + \dots$$

Example

Determine the recursive formula for the Taylor method of order 2 for the initial value

$$y' = \sin(xy), y(0) = \pi$$

We know that $\phi_n''(x_n) = \frac{\partial f}{\partial x}(x_n, y_n) + \frac{\partial f}{\partial y}(x_n, y_n) \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$.

So $\frac{\partial f}{\partial x} = \cos(xy) \cdot y$ and $\frac{\partial f}{\partial y} = \cos(xy) \cdot x$.

Putting this all together we get that $\phi_n(x) = y_{n+1} = y_n + h\sin(x_n y_n) + \frac{h^2}{2}[y\cos(xy) + x\cos(xy) \cdot \sin(xy)]$.

Doing some magic, $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h\sin(x_n y_n) + \frac{h^2}{2}[y_n\cos(x_n y_n) + \frac{x_n}{2}\sin(2x_n y_n)]$.

4 Linear Second-Order Equations

4.1 Introduction: The Mass-Spring Oscillator

A damped mass-spring oscillator consists of a mass m attached to a spring fixed at one end. Devise a differential equation that governs the motion of this oscillator, taking into account the forces acting on it due to the spring elasticity, damping friction, and possible external influences.

Newton's second law - force equals mass times acceleration ($F = ma$) - is the most commonly encountered differential equation. It is an ordinary differential equation of the second order since acceleration is the second derivative of position (y) with respect to time ($a = \mathrm{d}^2y/\mathrm{d}t^2$).

If the spring is unstretched and the inertial mass m is still, the system is at equilibrium. We stretch the coordinate y of the mass by its displacement from the equilibrium position.

When the mass m is displaced from equilibrium, the spring is stretched or compressed and it exerts a force that resists the displacement. For most springs this force is directly proportional to the displacement y and is given by Hooke's law.

$$F_{\text{spring}} = -ky$$

where the positive constant k is known as the stiffness (spring constant) and the negative sign reflects the opposing nature of the force. Hooke's law is only valid for sufficiently small displacements.

Usually all mechanical systems also experience friction. For vibrational motion this force is usually modeled accurately by a term proportional to velocity:

$$F_{\text{friction}} = -b\frac{\mathrm{d}y}{\mathrm{d}t} = -by'$$

where $b \geq 0$ is the damping coefficient and the negative sign reflects the opposing nature of the force.

The other forces on the oscillator are usually regarded as external to the system. Although they may be gravitational, electrical, or magnetic, commonly the most important external forces are transmitted to the mass by shaking the supports holding the system. For now we refer to the combined external forces by a single known function $F_{\text{ext}}(t)$. Newton's law provides the differential equation for the mass-spring oscillator:

$$my'' = -ky - by' + F_{\text{ext}}(t)$$

or

$$my'' + by' + ky = F_{\text{ext}}(t)$$

Example

Verify that if $b = 0$ and $F_{\text{ext}} = 0$, that the above equation has a solution of the form $y(t) = \cos(\omega t)$ and the angular frequency ω increases with k and decreases with m.

The differential equation is $my'' + by' + ky = F_{\text{ext}}$ and that $my'' + ky = 0$.

Since we are given what $y(t)$ is, taking the derivative of the differential equation gives that $-m\omega^2\cos(\omega t) + k\cos(\omega t) = 0$, and solving for ω, we get that $\omega = \sqrt{\frac{k}{m}}$.

If k increases, ω increases, and if m increases, ω decreases.

Example

Verify that the exponentially damped sinusoid given by $y(t) = e^{-3t}\cos 4t$ is a solution to the above differential equation if $F_{\text{ext}} = 0, m = 1, k = 25$, and $b = 6$.

From the differential equation $my'' + by' + ky = F_{\text{ext}}$, we can plug in stuff and we get that $y'' + by' + 25y =$

0.

Since we are given $y(t)$ we can find the derivatives and substitute. What we are given is quite long, but essentially it cancels out to $0 = 0$, which means it is a solution to the system.

Exercise Verify that the simple exponential function $y(t) = e^{-5t}$ is a solution to the above differential equation if $F_{\text{ext}} = 0$, $m = 1$, $k = 25$, and $b = 10$.

Sometimes the external force will make the system look somewhat erratic. There are many real world examples where the external force must defintely be taken into account.

4.2 Homogeneous Linear Equations: The General Solution

A second-order constant-coefficient differential equation has the form

$$ay'' + by' + cy = f(t) \qquad (a \neq 0)$$

A homogeneous second-order constant-coefficient differential equation is the special case with $f(t) = 0$.

$$ay'' + by' + cy = 0 \qquad (a \neq 0)$$

A solution of this equation has the form $y = e^{rt}$. The resulting equation $ar^2 + br + c = 0$ is called the auxiliary equation (or characteristic equation) associated with the homogeneous equation.

Example

Find a pair of solutions to

$$y'' + 5y' - 6y = 0$$

Plugging in $y = e^{rt}$, we get that $e^{rt}(r^2+5r-6)$ after substituting. Solving the quadratic $r^2+5r-6 = 0$, we get that $r = -6$ and $r = 1$, which is $y = e^{-6t}$ and $y = e^t$.

Note that the zero function, $y(t) = 0$ is always a solution to an equation above (figure this out later). In addition when we have a pair of solutions $y_1(t)$ and $y_2(t)$, we can construct an infinite number of other solutions by forming linear combinations:

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for any choice of the constants c_1 and c_2. This is a two-paramter solution form since there are two unknown constants. To find a specific solution, two initial conditions are needed.

Example

Solve the initial value problem

$$y'' + 2y' - y = 0 \qquad y(0) = 0, \qquad y'(0) = -1$$

Doing the default substitution, our auxiliary equation is $r^2 + 2r - 1 = 0$.

The solutions from this are $r = -1 + \sqrt{2}$ and $r = -1 - \sqrt{2}$. Remember $y = e^{rt}$. Using the initial conditions, we get that $c_1 = -\frac{\sqrt{2}}{4}$ and $c_2 = \frac{\sqrt{2}}{4}$.

Theorem 4.1

For any real numbers $a(\neq 0), b, c, t_0, Y_0$, and Y_1, there exists a unique soultion to the initial value problem.

$$ay'' + by' + cy = 0 \qquad y(t_0) = Y_0 \qquad y'(t_0) = y_1$$

The solution is valid for all t in $(-\infty, \infty)$

Definition

A pair of functions $y_1(t)$ and $y_2(t)$ is said to be linearly independent on the interval t if and only if neither of them is a constant multiple of the other on all of t. We say that y_1 and y_2 are linearly dependent on t if one of them is a constant multiple of the other on all of t.

Theorem 4.2

If $y_1(t)$ and $y_2(t)$ are any two solutions to the differential equation that are linearly independent on $(-\infty, \infty)$, then unique constants c_1 and c_2 can always be found so that $c_1 y_1(t) + c_2 y_2(t)$ satisfies the initial value problem on $(-\infty, \infty)$.

Definition: Wronksian

Suppose each of the functions $f_1(x), f_2(x), \ldots f_n(x)$ possess at least $n-1$ derivatives.

The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronksian of the functions.

Theorem 4.3

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear nth-order differential equation on an interval I. Then the set of solutions is linearly independent on I if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

Distinct real roots: If the auxiliary equation has distinct real roots r_1 and r_2, then both $y_1(t) = e^{rt}$ and $y_2(t) = e^{rt}$ are solutions to the above differential equation and $y(t) = e_1 e^{rt} + e_2 e^{rt}$ is a general solution.

Repeated root: if the auxiliary equation has a repeated root r, then both $y_1(t) = e^{rt}$ and $y_2(t) = te^{rt}$ are solutions to the differential equation and $y(t) = e_1 e^{rt} + e_2 te^{rt}$ is a general solution.

A homogeneous linear nth-order equation has a general solution of the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

where the individual solutions $y_i(t)$ are linearly independent, i.e. no $y_i(t)$ is expressible as a linear combination of the others.

4.3 Auxiliary Equations with Complex Roots

The simple harmonic equation $y'' + y = 0$ so called because of its relation to the fundamental vibration of a musical tone, has as solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$.

When $b^2 - 4ac < 0$, the roots of the auxiliary equation $ar^2 + br + c = 0$ associated with the homogeneous equation $ay'' + by' + cy = 0$ are the complex conjugate numbers $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ where $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{4ac-b^2}}{2a}$

Combing the solutions $e^{r_1 t}$ and $e^{r_2 t}$ with Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, yields complex function solutions

$$e^{(\alpha+i\beta)t} = e^{at}(\cos\beta t + i\sin\beta t) \text{ and } e^{(\alpha-i\beta)t} = e^{at}(\cos\beta t - i\sin\beta t)$$

Example

Solve the initial value problem $y'' + 2y' + 2y = 0$ given $y(0) = 0$ and $y'(0) = 2$.

Using the auxiliary form of the equation we have $r^2+2r+2 = 0$. From the quadratic formula, $r = -1\pm i$, the two roots are $r_1 = -1 + i$ and $r_2 = -1 - i$.

The solution is therefore $y_1 = e^{(-1+i)t}$ and $y_2 = e^{(-1-i)t}$

From the form given from euler's formula earlier, $y_1 = e^{-t}(\cos t + i \sin t)$ and $y_2 = e^{-t}(\cos t - i \sin t)$, so our general solution is $y = c_1e^{-t}(\cos t + i \sin t) + c_2e^{-t}(\cos t - i \sin t)$.

Plugging the initial conditions, we get that $0 = c_1 + c_2$.

The derivative of the general solution is $y' = c_1e^{-t}(-\sin t + i \cos t) + (\cos t + i \sin t) \cdot c_1(-1)e^{-t} + c_2e^{-t}(-\sin t - i \cos t) + (\cos t - i \sin t) \cdot c_2(-1)e^{-t}$. Plugging in the initial conditions gives $2 = c_1i - c_1 - c_2i - c_2$.

Factoring we get $2 = c_1(i - 1) + c_2(-i - 1)$. Using some substitution $c_2 = i$ and $c_1 = -i$. Plug this in the general solution to solve.

If the auxiliary equation has complex conjugate roots $a \pm i\beta$, then two linearly independent solutions to the equation are

$$e^{\alpha t} \cos \beta t \text{ and } e^{\alpha t} \sin \beta t$$

and a general solution is

$$y(t) = c_1e^{\alpha t} \cos \beta t + c_2e^{\alpha t} \sin \beta t$$

where c_1 and c_2 are arbitrary constants.

Example

Find a general solution to $y'' + 2y' + 4y = 0$.

Using the auxiliary equation the roots are $-1 \pm \sqrt{3}i$. Using what was given above, the general solution is $y = c_1e^{-t} \cos(\sqrt{3}t) + c_2e^{-t} \sin(\sqrt{3}t)$.

Example

Newton's second law implies the position $y(t)$ of the mass m is governed by the second-order differential equation $my''(t) + by'(t) + ky(t) = 0$ where the terms are physically identified as my being interial, by is damping and ky is stiffness. Determine the equation of motion for a spring system when $m = 36$ kg, $b = 12$ kg/sec (which is equivalent to 12 N - sec/m), $k = 37$ kg/sec^2, $y(0) = 0.7$ m and $y'(0) = 0.1$ m/sec. Also find $y(10)$, the displacement after 10 sec.

The differential equation is $36y'' + 12y' + 37$, so the roots are $-\frac{1}{6} \pm i$.

Doing the methods explained above, the solution is $y = .7e^{-t/6} \cos t + \frac{13}{60}e^{-t/6} \sin t$, and $y(10) \approx -.13$ m.

Exercise Interpret the equation $y'' + 5y' - 6y = 0$ in terms of the mass-spring system.

4.4 Nonhomogeneous Equations: the Method of Undetermined Coefficients

The method of Undetermined Coefficients is the technique used to guess a solution's form based on the form of the nonhomogeneous function $f(t)$ in a linear equation with constant coefficients such as $ay''+by'+cy = f(t)$.

For example the particular solution to $ay'' + by' + cy = Ct^m$ is of the form $y_p(t) = A_mt^m + \cdots + A_1t + A_0$.

Example

Find a particular solution to $y'' + 3y' + 2y = 10e^{3t}$.

Our guess based on the form of $f(t) = 10e^{3t}$ is that $y = Ae^{3t}$ is the guess form of the particular solution, so we know that $y' = 3A3^{3t}$ and $y'' = 9Ae^{3t}$.

Substituting this in gives us $9Ae^{3t} + 3(3Ae^{3t}) + 2(Ae^{3t}) = 10e^{3t}$. Simplifying this gives us $20Ae^{3t} = 10e^{3t}$ so $A = 1/2$.

So our particular solution is $\frac{1}{2}e^{3t}$.

Example

Find a particular solution to $y'' + 3y' + 2y = \sin t$.

Let $y = A\sin t + B\cos t$ as the form of the particualr solution.

Substituting and solving should result in $A = 1/10$ and $B = -3/10$.

This example suggests an equation of the form $ay'' + by' + cy = C\sin\beta t$ (or $C\cos\beta t$) will have a particular solution of the form $y_p(t) = A\cos\beta t + B\sin\beta t$.

Example

Find a particular solution to $y'' + 4y = 5t^2e^t$.

Let $y = At^2e^t + Bte^t + Ce^t = e^t(At^2 + Bt + C)$. The result should be $A = 1, B = -4/5, C = -2/25$.

To find a particular soultion to the differential equation

$$ay'' + by' + cy = Ct^m e^{rt}$$

where m is a nonnegative integer, use the form

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt}$$

with

1. $s = 0$ if r is not a root of the associated auxiliary equation;
2. $s = 1$ if r is a simple root of the associated auxiliary equation;
3. $s = 2$ if r is a double root of the associated auxiliary equation.

To find a solution to the differential equation $ay'' + by' + cy = Ct^m e^{\alpha t}\cos\beta t$ or equal to $Ct^m e^{\alpha t}\sin\beta t$ for $\beta \neq 0$, use the form $y_p(t) = t^s(A_m t^m + \ldots A_1 t + A_0)e^{\alpha t}\cos\beta t + t^s(B_m t^m + \cdots + B_1 t + B_0)e^{\alpha t}\sin\beta t$, with

1. $s = 0$ if $\alpha + i\beta$ is not a root of the associated auxiliary equation; and
2. $s = 1$ if $\alpha + i\beta$ is a root of the associated auxiliary equation

4.5 The Superposition Principle and Undetermined Coefficients Revisited

Theorem 4.4: Superposition Principle

Let y_1 be a solution to the differential equation

$$ay'' + by' + cy = f_1(t)$$

and y_2 is a solution to

$$ay'' + by' + cy = f_2(t)$$

then for any constants k_1 and k_2, the function $k_1y_1 + k_2y_2$ is a solution to the differential equation

$$ay'' + by' + cy = k_1f_1(t) + k_2f_2(t)$$

Example

Find a particular solution to

$$y'' + 3y' + 2y = 3t + 10e^{3t}$$

The solution for equal to $3t$ is $y = \frac{3t}{2} - \frac{9}{4}$ and for $10e^{3t}$ is $y = \frac{e^{3t}}{2}$ so the solution is $y = \frac{3t}{2} - \frac{9}{4} + \frac{e^{3t}}{2}$.

Exercise Find a particualr solution to $y'' + 3y' + 2y = -9t + 20e^{3t}$.

General solution for Nonhomogeneous Differential Equations: Let y_p be a particular solution to

$$ay'' + by' + cy = f(t)$$

and $c_1y_1 + c_2y_2$ be the general solution to the homogeneous equation

$$ay'' + by' + cy = 0$$

Then the general solution to the nonhomogeneous equation is given by

$$y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t)$$

Theorem 4.5

For any real numbers $a(\neq 0), b, c, t_0, Y_0$, and Y_1, suppose $y_p(t)$ is a particular solution to above in an interval I containing t_0 and that $y_1(t)$ and $y_2(t)$ are linearly independent solutions to the associated homogeneous equation in I. Then there exists a unique solution in I to the initial value problem.

$$ay'' + by' + cy = f(t) \qquad y(t_0) = Y_0 \qquad y'(t_0) = Y_1$$

Example

Given that $y_p(t) = t^2$ is a particular solution to

$$y'' - y = 2 - t^2$$

Find a general solution and a solution satisfying $y(0) = 1, y'(0) = 0$.

Our general solution using the auxiliary equation is $y = c_1e^t + c_2e^{-t}$.

Our particular solution will be in $At^2 + By + C$.

So $y = t^2 + c_1e^t + c_2e^{-t}$. Solving for c_1 and c_2 by finding the derivative of this and using the initial conditions, the specific solution is $y = t^2 + \frac{1}{2}e^t + \frac{1}{2}e^{-t}$.

Example

A mass-spring system is driven by a sinusodial external force $5\sin t + 5\cos t$. The mass equals 1, the spring constant equals 2, and the damping coefficient equals 2 (in appropriate units), so the motion is governed by the differential equation

$$y'' + 2y' + y = 5\sin t + 5\cos t$$

If the mass is initially located at $y(0) = 1$, with a velocity $y'(0) = 2$, find its equation of motion.

Finding the general solution to this we get that $y = c_1e^{-t}\cos t + c_2e^{-t}\sin t$.

From $y_p = A\sin t + B\cos t$, we should solve that

$$y = 3\sin t - \cos t + 2e^{-t}\cos t + e^{-t}\sin t$$

Example

Find a particular solution to

$$y'' - y = 8te^t + 2e^t$$

The general solution for this is $y = c_1e^t + c_2e^{-t}$. y_p is equal to $(At + B)e^t$. Doing some calculations should result in $y_p = (2t^2 - t)e^t$.

Method of Undetermined Coefficients (Revisited)

To find a particular solution to the differential equation

$$ay'' + by' + cy = P_m(t)e^{rt}$$

where $P_m(t)$ is a polynomial of degree m, use the form

$$y_p(t) = t^s(A_mt^m + \cdots + A_1t + A_0)e^{rt}$$

if r is not a root of the associated auxiliary equation, take $s = 0$; ir r is a simple root of the associated auxiliary equation, take $s = 1$; and if r is a double root of the associated auxiliary equation, take $s = 2$.

To find a particular solution to the differential equation

$$ay'' + by' + cy = P_m(t)e^{\alpha t}\cos\beta t + Q_n(t)e^{\alpha t}\sin\beta t, \beta \neq 0$$

where $P_m(t)$ is a polynomial of degree m and $Q_n(t)$ is a polynomial of degree n, use the form $y_p(t) = t^s(A_kt^k + \cdots + A_1t + A_0)e^{\alpha t}\cos\beta t + t^s(B_kt^k + \cdots + B_1t + B_0)e^{\alpha t}\sin\beta t$, where k is the larger of m and n. If $\alpha + i\beta$ is not a root of the associated auxiliary equation, take $s = 0$; if $\alpha + i\beta$ is a root of the associated auxiliary equation, take $s = 1$.

Exercise Write down the form of a particular solution to the equation $y'' + 2y' + 2y = 5e^{-t}\sin t + 5t^3e^{-t}\cos t$.

Exercise Write down the form of a particular solution to the equation $y''' + 2y'' + y' = 5e^{-t}\sin t + 3 + 7te^{-t}$.

4.6 Variation of Parameters

The Method of Undetermined Coefficients is a procedure for determining a particular solution when the equation has constant coefficients and the nonhomogeneous term is of a special type.

Variation of Paramters is a more general method for finding a particular solution.

Consider a linear second-order equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

in the standard form

$$y'' + P(x)y' + Q(x)y = f(x)$$

Obtain the solution to the associated homogeneous equation

$$y = c_1y_1(x) + c_2y_2(x)$$

And replace the constants with functions

$$y = u_1y_1(x) + u_2y_2(x)$$

Substituting into the DE yields the system:

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ y_1'u_1' + y_2'u_2' &= f(x) \end{aligned}$$

By Cramer's Rule, the solution can be expressed in terms of determinants-

$$u_1' = \frac{W_1}{W} = \frac{-y_2 f(x)}{W} \qquad u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W}$$

The functions u_1 and u_2 are found by integrating.

A particular solution is $y_p = u_1 y_1 + u_2 y_2$.

Example

Find a general solution on $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y = \tan t$.

The standard form is $y'' + y = \tan t$.

The solutions to the homogeneous equation is $r = \pm i$, so $\alpha = 0$ and $\beta = 1$, so $y_1 = \cos t$ and $y_2 = \sin t$.

Using what was previously introduced, we end up with $y = c_1 \cos t + c_2 \sin t - \cos t \ln|\sec t + \tan t|$

Exercise Find a general solution on $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y = \tan t + 3t - 1$.

4.7 Variable-Coefficient Equations

We now consider equations with variable coefficients of the form

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$$

Typically, the equation is divided by the nonzero coefficient $a_2(t)$ and is expressed in standard form

$$y''(t) + p(t)y' + q(t)y(t) = g(t)$$

Theorem 4.6

Suppose $p(t), q(t)$, and $g(t)$ are continuous on an interval (a, b) that contains the point t_0. The, for any choice of the initial values Y_0 and Y_1, there exists a unique solution $y(t)$ on the same interval (a, b) to the initial value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \qquad y(t_0) = Y_0, \qquad y'(t_0) = Y_1$$

Example

Determine the largest interval for which the above theorem ensures the existence and uniqueness of a solution to the initial value problem

$$(t-3)y'' + y' + \sqrt{t}y = \ln t \qquad y(1) = 3, \qquad y'(1) = -5$$

First, let's put this into standard form. $y'' + \frac{1}{t-3}y' + \frac{\sqrt{t}}{t-3}y = \frac{\ln t}{t-3}$.

From this we can see that this is only continuous from $(0, 3)$.

Definition

A linear second-order equation that can be expressed in the form

$$at^2 y''(t) + bty'(t) + cy = f(t)$$

where a, b, and c are constants, is called a Cauchy-Euler, or equidimensional, equation.

To solve a Cauchy-Euler equation, for $t > 0$ look for solutions of the form

$$y = t^r$$

and substitute into the homogeneous form

$$at^2y''(t) + bty'(t) + cy = 0$$

the resulting equation $ar^2 + (b-a)r + c = 0$ is called the associated characteristic equation.

Example

Find two linearly independent solutions to the equation

$$3t^2y'' + 11ty' - 3y = 0 \qquad t > 0$$

The solutions are $y = t^r$ and plugging this into the equation results in $3r^2 + 8r - 3$, which factors to $r = \frac{1}{3}$ and $r = -3$.

The solutions are $y = t^{1/3}$ and $y = t^{-3}$.

If the roots of the associated characteristic equation r are equal, then independent solutions of the Cauchy-Euler equation on $(0, \infty)$ are given by

$$y_1 = t^r \qquad \text{and} \qquad y^2 = t^r \ln t$$

IF the roots are complex, $r = \alpha \pm \beta i$, then the independent solutions are given by

$$y_1 = t^a \cos(\beta \ln t) \qquad \text{and} \qquad y_2 = t^\alpha \sin(\beta \ln t)$$

Example

Find a pair of linearly independent solutions to the Cauchy-Euler equations for $t > 0$.

$t^2y'' + 5ty' + 5y = 0$

Answer: $y_1 = t^{-2}\cos(\ln t)$, $y_2 = t^{-2}\sin(\ln t)$

Exercise Do the same thing for $t^2y'' + ty' = 0$.

Lemma 4.7

If $y_1(t)$ and $y_2(t)$ are any two solutions to the homogeneous differential equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0$$

on an interval I where the functions $p(t)$ and $q(t)$ are continuous and if the Wronksian is zero at any point t of I, then y_1 and y_2 are linearly dependent on I.

Theorem 4.8

If $y_1(t)$ and $y_2(t)$ are any two solutions to the homogeneous differential equation that are linearly independent on an interval I containing t_0, then unique constants c_1 and c_2 can always be found so that $c_1y_1(t) + c_2y_2(t)$ satisfies the initial conditions $y(t_0) = Y_0$, $y'(t_0) = Y_1$ for any Y_0 and Y_1.

$y_h = c_1y_1 + c_2y_2$ is called a general solution to the homogeneous differential equation $y''(t) + p(t)y'(t) + q(t)y(t) = 0$ if y_1 and y_2 are linearly independent solutions on I.

For the nonhomogeneous equation $y''(t) + p(t)y'(t) + q(t)y(t) = g(t)$ a general solution on I is given by $y = y_p + y_h$ where $y_h = c_1y_1 + c_2y_2$ is a general solution to the corresponding homogeneous equation on I and y_p is a particular solution on I.

If linearly independent solutions to the homogeneous equation are known, y_p can be determined by the variation of parameters method.

Theorem 4.9

If y_1 and y_2 are two linearly independent solutions to the homogeneous equation on an interval I where $p(t)$, $q(t)$, and $g(t)$ are continuous, then a particular solution is given by $y_p = u_1 y_1 + u_2 y_2$, where u_1 and u_2 are determined up to a constant by the pair of equations, $y_1 v_1' + y_2 v_2' = 0$, $y_1' v_1' + y_2' v_2' = g$, which have the solutions

$$v_1(t) = \int \frac{-g(t) y_2(t)}{W(y_1, y_2)(t)} \mathrm{d}t$$

and

$$v_2(t) = \int \frac{g(t) y_1(t)}{W(y_1, y_2)(t)} \mathrm{d}t$$

Note that the formulation above presumes that the differential equation has been put into standard form (that is divided by $a_2(t)$).

Theorem 4.10

Let $y_1(t)$ be a solution, not identically zero, to the homogeneous equation in an interval I. Then

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t) \mathrm{d}t}}{y_1(t)^2} \mathrm{d}t$$

is a second, linearly independent solution.

Example

Given that $y_1(t) = t$ is a solution to

$$y'' - \frac{1}{t} y' + \frac{1}{t^2} y = 0$$

use the Reduction of Order formula to determine a second linearly independent solution for $t > 0$.

y_2 is equal to $t \int \frac{e^{\ln t}}{t^2} \mathrm{d}t = t \ln t$ from the formula, so the general solution is $y = c_1 t + c_2 t \ln t$.

Exercise The following equation arises in the mathematical modeling of reverse osmosis.

$$(\sin t) y'' - 2(\cos t) y' - (\sin t) y = 0, \qquad 0 < t < \pi$$

Find a general solution.

5 Introduction to Systems

Apologies for no examples since I have really bad allergies as of writing this.

5.1 Differential Operators and the Elimination Method for Systems

$y'(t) = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}y$ is used to emphasize that the derivative of a function y is the result of operation on the function y with the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}t}$. Similarly $y''(t) = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}y$. Commonly the symbol D is used instead of $\frac{d}{\mathrm{d}t}$.

$y'' + 4y' + 3y = 0$ is represented by $(D^2 + 4D + 3)[y] = 0$.

We use the convention that the operator "product" is a composition when it operates on functions. *Exercise* Show that the operator $(D + 1)(D + 3)$ is the same as $D^2 + 4D + 3$.

Exercise Show that $(D + 3t)D$ is not the same as $D(D + 3t)$.

Because the "algebra" of differential operators follows the same rules as the algebra of polynomials, we can adapt the elimination method used to solve algebraic operations to solve any system of linear differential equations with constant coefficients.

Example

Solve the system

$$x'(t) = 3x(t) - 4y(t) + 1$$
$$y'(t) = 4x(t) - 7y(t) + 10t$$

First we can solve this by writing this in differential operator form:

$$(D - 3)x + 4y = 1$$
$$-4x + (D + 7)y = 10t$$

By eliminating this using algebra, we get that $(D^2+4D-5)y = 14-30t$ (this is essentially $y''+4y-5y = 14 - 30t$).

The first step to solving this nonhomogeneous equation is to solve the homogeneous equation.

The homogeneous equation results in giving us that $y_h = c_1e^{-5t} + c_2e^t$.

We can guess the particular solution as $y_p = At + B$, so $y'_p = A$ and $y''_p = 0$. So we have that $0 + 4A - 5(At + B) = 14 - 30t$. From this, we have $4A - 5B = 14$ and $-5A = -30$, so $A = 6$ and $B = 2$.

So the general solution for y is $y = C_1e^{-5t} + C_2e^t + 6t + 2$.

Now we need to find the function x.

We can find x from elimination once again.

Using the same methods above, we have $x = k_1e^{-5t} + k_2e^t + 8t + 5$.

We should only end up with two constants, so we need to find the relationship between the constants of y and x.

So using the derivative of x we get $-5k_1e^{-5t} + k_2e^t + 8 = 3k_1e^{-5t} + 3k_2e^t + 24t + 15 - 4C_1e^{-5t} - 4C_2e^t - 24t - 8 + 1$.

Simplifying we get $-5k_1e^{-5t} - k_2e^t = (3k_1 - 4C_1)e^{-5t} + (3k_2 - 4C_2)e^t$. So we have that $-5k_1 = 3k_1 - 4C_1$ and $k_2 = 3k_2 - 4C_2$, so $C_1 = 2k_1$ and $C_2 = \frac{1}{2}k_2$.

To find a general solution for the system

$$L_1[x] + L_2[y] = f_1$$
$$L_3[x] + L_4[y] = f_2$$

where L_1, L_2, L_3, and L_4 are polynomials in $D = \mathrm{d}/\mathrm{d}t$.

- Make sure that the system is written in operator form.
- Eliminate one of the variables, say, y, and solve the resulting equation for $x(t)$. If the system is degenerate, stop! A separate analysis is required to determine whether or not there are solutions.
- (Shortcut) If possible, use the system to derive an equation that involves $y(t)$ but not its derivatives (Otherwise go to the next step). Substitute the found expression for $x(t)$ into this equation to get a formula for $y(t)$. The expressions for $x(t)$, $y(t)$ give the desired general solution.
- Eliminate x from the system and solve for $y(t)$. Solving for $y(t)$ gives more constants - in fact, twice as many as needed.
- Remove the extra constants by substituting the expressions for $x(t)$ and $y(t)$ into one or both of the equations in the system. Write the expressions for $x(t)$ and $y(t)$ in terms of the remaining constants.

Example

Find a general solution to

$$x''(t) + y'(t) - x(t) + y(t) = -1$$
$$x'(t) + y'(t) - x(t) = t^2$$

Subtracting the two, we get that $x'' - x' + y = -1 - t^2$.

We will now solve for x. In differential operator form we have $(D^2 - 1)x + (D + 1)y = -1$ and $(D - 1)x + Dy = t^2$.

Eliminating for x gives the $x_h = C_1e^t + C_2te^t + C_3e^{-t}$.

The particular solution will be in the form $At^2 + Bt + C$. The first derivative of this is $2At + B$ and the second derivative is $2A$.

Plugging this in gives us that $(D^2-2D+1)(D+1)x = -2t-t^2$. This is equal to $(D^3-D^2-D+1)x = -2t - t^2$.

This is $2-2A-2At-B+At^2+Bt+C = -2t-t^2$. From this we see that $A = -1, B = -4, C = -6$.

The particular solution $x_p = -t^2 - 4t - 6$.

Now since we have $x = c_1e^t + C_2te^t + C_3e^{-t} - t^2 - 4t - 6$, we can solve this for the y.

Plugging the general solution in gives us $y = -c_1e^t - c_2e^t - c_2e^t - c_2te^t - c_1e^{-t} + 2 + c_1e^t + c_2e^t + c_2te^t - c_3e^{-t} - 2t - 4 - 1 - t^2$.

Simplifying this gives $y = -c_2e^t - 2c_3e^{-t} - t^2 - 2t - 3$.

5.2 Solving Systems and Higher-Order Equations Numerically

If equations have variable coefficients, the solution process is limited. The solutions can be expressed as infinite series which can be very laborious (with the exception of the Cauchy-Euler equation). Fortunately all cases, constant and variable coefficient, nonlinear and higher order equations and systems can be addressed numerically.

We will express differential equations as a system in normal form and used the basic Euler method for computer solution that can be "vectorized" to apply to such systems.

A system of m differential equations in the m unknown functions $x_1(t), x_2(t), \ldots, x_m(t)$ expressed as

$$\begin{aligned} x_1'(t) &= f_1(t, x_1, x_2, \ldots, x_m) \\ x_2'(t) &= f_2(t, x_1, x_2, \ldots, x_m) \\ &\vdots \\ x_m'(t) &= f_m(t, x_1, x_2, \ldots, x_m) \end{aligned}$$

is said to be in normal form. It can be expressed in vector form as $x' = t(t, x)$.

A single higher-order equation can always be converted to an equivalent system of first-order equations. To convert an mth-order differential equation

$$y^{(m)}(t) = f(t, y, y', \ldots, y^{(m-1)})$$

into a first-order system, introduce additional unknowns, the sequence of derivatives of y:

$$x_1(t) = y(t), x_2(t) = y'(t), \ldots, x_m(t) = y^{(m-1)}(t)$$

We obtain this system

$$\begin{aligned} x_1'(t) &= y'(t) = x_2(t) \\ x_2'(t) &= y''(t) = x_3(t) \\ &\vdots \\ x_{m-1}'(t) &= y^{(m-1)}(t) = x_m(t) \\ x_m'(t) &= y^{(m)}(t) = f(t, x_1, x_2, \ldots, x_m) \end{aligned}$$

Example

Convert the initial value problem

$$y''(t) + 3ty'(t) + y(t)^2 = \sin t \qquad y(0) = 1, y'(0) = 5$$

into an initial value problem for a system in normal form.

We have $x_1 = y, x_1' = y', x_2 = y'$ and $x_2' = y''$.

In normal form, we would have $y'' = \sin t - 3ty' + y^2$ which is what x_2' and y'' are equal to. Note that this ia function of $f(t, y, y')$.

We also have $x_1' = y' = x_2$ and we also know that $\sin t - 3tx_2 + x_1^2 = x_2'$.

We now have a system where $x_1' = x_2, x_2' = \sin t - 3tx_2 + x_1^2$ where we know $x_1(0) = 1$ and $x_2(0) = 5$.

6 Theory of Higher-Order Linear Differential Equations

6.1 Basic Theory of Linear Differential Equations

A linear differential equation of order n is an equation that can be written in the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_0(x)y(x) = b(x)$$

where $a_0(x), a_1(x), \ldots, a_n(x)$ and $b(x)$ depend on on x, not y. When $a_0, a_1, \ldots, a_n$ are all constants, we say this equation has constant coefficients; otherwise it has variable coefficients. If $b(x) = 0$, this equation is called homogeneous; otherwwise it is nonhomogeneous.

We assume $a_0(x), a_1(x), \ldots, a_n(x)$ and $b(x)$ are all continuous on an interval I and $a_n(x) \neq 0$ on I.

We can rewrite the equation in standard form

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x)$$

where the functions $p_1(x), \ldots, p_n(x)$, and $g(x)$ are continuous on I.

Theorem 6.1

Suppose $p_1(x) \ldots p_n(x)$ and $g(x)$ are continuous on an interval (a, b) that contains the point x_0. Then, for any choice of the initial values, $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$, there exists a unique solution $y(x)$ on the whole interval (a, b) to the initial value problem

$$y^{(n)} + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x)$$

$$y(x_0) = \gamma_0, y'(x_0) = \gamma_1 \ldots y^{(n-1)}(x_0) = \gamma_{n-1}$$

Example

For the initial value problem

$$x(x-1)y''' - 3xy'' + 6x^2y' - (\cos x)y = \sqrt{x+5}$$
$$y(x_0) = 1, \qquad y'(x_0) = 0, \qquad y''(x_0) = 7$$

determine the values of x_0 and the intervals (a, b) containing x_0 for which the above theorem guarantees the existence of a unique solution on (a, b).

We know that $x \neq 0, 1$ from $x(x-1)$.

In standard form this becomes $y''' - \frac{3x}{x(x-1)}y'' + \frac{6x^2}{x(x-1)}y' - \frac{\cos x}{x(x-1)}y = \frac{\sqrt{x+5}}{x(x-1)}$.

These functions will be continuous when $x \neq 0$ and $x \neq 1$. We also know that $x \geq -5$ from the last term.

The intervals are $(-5, 0)$, $(0, 1)$ and $(1, \infty)$ in which all the x_0 can have an element from.

If we let the left-hand side of equation in the standard form define the differential operator L,

$$L[y] = \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_1\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + \cdots + p_n y = (D^n + p_1 D^{n-1} + \cdots + p_n)[y]$$

then the standard form equation can be expressed in the operator form

$$L[y](x) = g(x)$$

Keep in mind that L is a linear operator - that is, it satisfies

$$L[y_1 + y_2 + \cdots + y_m] = L[y_1] + L[y_2] + \cdots + L[y_m]$$
$$L[cy] = cL[y]$$

where c is any constant.

Definition: Wronksian

Let $f_1, \ldots, f_n$ be any n functions that are $(n-1)$ times differentiable.

The function

$$W[f_1, f_2, \ldots, f_n] = \begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronksian of $f_1, \ldots f_n$.

Theorem 6.2

Let $y_1, \ldots y_n$ be n solutions on (a, b) of

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0$$

where $p_1, \ldots, p_n$ are continuous on (a, b). If at some point x_0 in (a, b) these solutions satisfy

$$W[y_1, \ldots, y_n](x_0) \neq 0$$

then every solution of the above equation on (a, b) can be expressed in the form

$$y(x) = C_1 y_1(x) + \cdots + C_n y_n(x)$$

where $C_1, \ldots, C_n$ are constants.

Definition: Linear Dependence of Functions

The M functions $f_1, f_2, \ldots, f_m$ are said to be linearly dependent on an interval I if at least one of them can be expressed as a linear combination of the others on I; equivalently, they are linearly dependent if there exist constants $c_1, c_2, \ldots, c_m$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_m f_m(x) = 0$$

for all x in I. Otherwise, they are said to be linearly independent on I.

Example

Show that the functions $f_1(x) = e^x, f_2(x) = e^{-2x}$, and $f_3(x) = 3e^x - 2e^{-2x}$ are linearly dependent on $(-\infty, \infty)$.

We can see that $f_3 = 3f_1 - 2f_2$. We can see that f_3 is a linear combination of the other two functions.

We have a set of constants, not all zero that $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$ from $3f_1 - 2f_2 - 1f_3$, so the set of $\{f_1, f_2, f_3\}$ is linearly dependent.

If you do the Wronksian of the functions: $W[f_1, f_2, f_3]$, we get 0 which eans that it is linearly dependent. The process of writing the Wronksian takes a lot of paper, so it is easier likely to do the $c_1 f_1 + c_2 f_2 +$

$\cdots + c_n f_n = 0$ method.

To prove that functions $f_1, f_2, \ldots, f_m$ are linearly independent, a convenient approach is to assume the equation defined in the linear dependence definition holds and show that this forces $c_1 = c_2 = \cdots = c_m = 0$.

Example

Show that the functions $f_1(x) = x, f_2(x) = x^2$, and $f_3(x) = 1 - 2x^2$ are linearly independent on $(-\infty, \infty)$.

Assume $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$. If we can show this, then we can show its independence.

From this we will get $c_1 x + c_2 x^2 + c_3(1 - 2x^2) = 0$.

If we let $x = 0$, we get $c_3 = 0$.

If we let $x = 1$, we get $c_1 + c_2 - c_3 = 0$ and if we let $x = -1$, we get $-c_1 + c_2 - c_3 = 0$.

From this we see that $c_1 + c_2 = 0$ and $-c_1 + c_2 = 0$.

The functions are linearly independent when c_1, c_2 and c_3 are equal to 0, so x, x^2, and $1 - 2x^2$ are linearly independent.

There are other ways to do this as well.

Theorem 6.3

If $y_1, y_2, \ldots, y_n$ are n solutions to $y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0$ on the interval (a, b), with $p_1, p_2, \ldots, p_n$ continuous on (a, b), then the following statements are equivalent

1. $y_1, y_2, \ldots, y_n$ are linearly dependent on (a, b).
2. The Wronksian $W[y_1, y_2, \ldots, y_n](x_0)$ is zero at some point x_0 in (a, b).
3. The Wronksian $W[y_1, y_2, \ldots, y_n](x)$ is identically zero on (a, b).

The contrapositives of these statements are also equivalent:

1. $y_1, y_2, \ldots, y_n$ are linearly independent on (a, b).
2. The Wronksian $W[y_1, y_2, \ldots, y_n](x_0)$ is nonzero at some point x_0 in (a, b).
3. The Wronksian $W[y_1, y_2, \ldots, y_n](x)$ is never zero on (a, b)

Whenever the last 3 are met, $\{y_1, y_2, \ldots, y_n\}$ is called a fundamental solution set for linear independence theorem on (a, b).

It is useful to keep in mind the following sets consist of functions that are linearly independencet on every open interval (a, b):

$$\{1, x, x^2, \ldots, x^n\}$$

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx\}$$

$$\{e^{\alpha_1 x}, e^{\alpha_2 x}, \ldots, e^{\alpha_n x}\}$$

where α_i are distinct constants.

Theorem 6.4

Let $y_p(x)$ be a particular solution to the nonhomogeneous equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x)$$

on the interval (a, b) with $p_1, p_2, \ldots, p_n$ continuous on (a, b), and let $\{y_1, \ldots, y_n\}$ be a fundamental solution set for the corresponding homogeneous equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0$$

Then every solution of the original nonhomogeneous equation on the interval (a, b) can be expressed in the form

$$y(x) = y_p(x) + C_1 y_1(x) + \cdots + C_n y_n(x)$$

Example

Find a general solution on the interval $(-\infty, \infty)$ to

$$L[y] = y''' - 2y'' - y' + 2y = 2x^2 - 2x - 4 - 24e^{-2x}$$

given that $y_{p_1}(x) = x^2$ is a particular solution to $L[y] = 2x^2 - 2x - 4, y_{p_2}(x) = e^{-2x}$ is a particular solution to $L[y] = -12e^{-12x}$, and that $y_1(x) = e^{-x}, y_2(x) = e^x$, and $y_3(x) = e^{2x}$ are solutions to the corresponding homogeneous equation.

We know that $\{e^{-x}, e^x, e^{2x}\}$ is a fundamental solution set for homogeneous equations so we have $C_1 e^{-x} + C_2 e^x + C_3 e^{2x}$.

We know that $L[x^2] = 2x^2 - 2x - 4$ and $L[e^{-2x}] = -12e^{-2x}$. From the former, we have $L[2e^{-2x}] = -24e^{-2x}$.

We know that $Ly_p = 2x^2 - 2x - 4 - 24e^{-2x}$. We also know that $L[x^2 - 2e^{-2x}] = L[x^2] - 2L[e^{-2x}] = 2x^2 - 2x - 4 - 24e^{-2x}$.

The solution of the nonhomogeneous equation is $x^2 - 2e^{-2x}$.

The general solution is therefore $y(x) = x^2 - 2x^{-2x} + C_1 e^{-2x} + C_2 e^x + C_3 e^{2x}$.

6.2 Homogeneous Linear Equations with Constant Coefficients

Consider the homogeneous linear nth-order differential equation with constant coefficients

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = 0$$

e^{rx} is a solution to the equation, provided r is a root of the auxiliary (or characteristic equation)

$$P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 = 0$$

Distinct real roots: If the roots $r_1, r_2, \ldots, r_n$ of the auxiliary equation are real and distinct, then the n solutions to the first equation defined are

$$y_1(x) = e^{r_1 x}, \quad y_2(x) = e^{r_2 x} \quad , \ldots, \quad , y_n(x) = e^{r_n x}$$

Example

Find a general solution to

$$y''' - 2y'' - 5y' + 6y = 0$$

Using the auxiliary equation we get $r^3 - 2r^2 - 5r + 6 = 0$ from this.

From algebra, we know that the possible roots are $\pm 1, \pm 2, \pm 3, \pm 6$.

Let's assume $r = 1$ is a solution. From synthetic division, we see that $r = 1$ is a root. Now we can see that $(r-1)(r^2 - r - 6)$ is a solution.

Factoring this gives $(r-1)(r-3)(r+2)$.

The general solution is $y = C_1 e^x + C_2 e^{3x} + C_3 e^{-2x}$.

Looking at complex roots: If $\alpha + i\beta (\alpha, \beta$ real) is a complex root of the auxiliary equation, then so is its complex conjugate $\alpha - i\beta$. If we accept complex-valued functions as solutions, then both $e^{(\alpha + i\beta)x}$ and

$e^{(\alpha - i\beta)x}$ are solutions to the original homogeneous linear equation. The real-valued functions (which are linearly independent) corresponding to the complex roots $\alpha \pm i\beta$ are

$$e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)$$

Example

Find a general solution to

$$y''' + y'' + 3y' - 5y = 0$$

The auxiliary equation is $r^3 + r^2 + 3r - 5$.

The possible roots are $\pm 1, \pm 5$.

We know that 1 works from synthetic division and we get $(r-1)(r^2 + 2r + 5)$.

From $r^2 + 2r + 5$, we get that $-1 \pm 2i$ are the roots of this.

We get $c_1 e^x + c_2 e^{-x}\cos 2x + c_3 e^{-x}\sin 2x$.

If r_1 is a root of multiplicity m, then the m linearly independent solutions are

$$e^{r_1 x}, \quad xe^{r_1 x}, \quad x^2 e^{r_1 x}, \ldots, x^{m-1}e^{r_1}x$$

If $\alpha + i\beta$ is a repeated complex root of multiplicity m, then the $2m$ linearly independent real-valued solutions are

$$e^{\alpha x}\cos(\beta x), xe^{\alpha x}\cos(\beta x), \ldots, x^{m-1}e^{\alpha x}\cos(\beta x)$$
$$e^{\alpha x}\sin(\beta x), xe^{\alpha x}\sin(\beta x), \ldots, x^{m-1}e^{\alpha x}\sin(\beta x)$$

Example

Find a general solution to

$$y^{(4)} - y^{(3)} - 3y'' + 5y' - 2y = 0$$

The auxiliary equation is $r^4 - r^3 - 3r^2 + 5r - 2 = 0$.

The possible roots are $\pm 1, \pm 2$.

We know that $r = 1$ works from plugging in. Using synthetic division, we get $(r-1)(r^3 - 3r + 2)$.

From the $r^3 - 3r + 2$ term, we can factor this to $(r-1)(r^2 + r - 2)$.

The auxiliary equation ends up being $(r-1)^3(r+2)$.

The general solution ends up being $c_1 e^x + C_2 xe^x + C_3 x^2 e^x + C_4 e^{-2x}$.

Example

Find a general solution to

$$y^{(4)} - 8y^{(3)} + 26y'' - 40y' + 25y = 9$$

The auxiliary equation is $r^4 - 8r^3 + 26r^2 - 40r + 25 = 0$.

Let's assume we are told that $r_1 = 2 + i$ and $r_2 = 2 - i$.

This means that $(r - (2 + i))(r - (2 - i)) = r^2 - 4r + 5$ is a factor.

Dividing $r^4 - 8r^3 + 26r^2 - 40r + 25$ from this gives us $r^2 - 4r + 5$.

We know the roots are $2 + i, 2 - i, 2 + i, 2 - i$.

Since $2 + i$ and $2 - i$ have multiplicity of two, then the solution is $y = C_1e^{2x} \cos x + C_2e^{2x} \sin x + C_3xe^{2x} \cos x + C_4xe^{2x} \sin x$.

6.3 Undetermined Coefficients and the Annihilator Method

Previously we used the Method of Undetermined Coefficients to find a particular solution to a nonhomogeneous linear second-order constant coefficient equation

$$L[y] = (aD^2 + bD + c)[y] = f(x)$$

when $f(x)$ had a particular form (a product of a polynomial, an exponential, and a sinusoid) by observing a solution form y_p must resemble f. We also had to make accomodations when y_p was a solution to the homogeneous equation $L[y] = 0$.

The annihilator method uses the observation that suitable types of nonhomogeneities $f(x)$ are themselves solutions to homogeneous differential equations with constant coefficients.

1. Any nonhomogeneous term of the form $f(x) = e^{rx}$ satisfies $(D - r)[f] = 0$
2. Any nonhomogeneous term of the form $f(x) = x^ke^{rx}$ satisfies $(D-r)^m[f] = 0$ for $k = 0, 1, \ldots, m-1$.
3. Any nonhomogeneous term of the form $f(x) = \cos \beta x$ or $\sin \beta x$ satisfies $(D^2 + \beta^2)[f] = 0$
4. Any nonhomogeneous term of the form $f(x) = x^ke^{\alpha x} \cos \beta x$ or $x^ke^{\alpha x} \sin \beta x$ satisfies $[(D - \alpha)^2 + \beta^2]^m[f] = 0$ for $k = 0, 1, \ldots, m - 1$.

We have that D^n annihilates polynomial of degree $n - r$.

We have that $D - r$ annihilates e^{rx}.

We have that $(D - r)^k$ annihilates $x^{k-1}e^{rx}$

We have that $D^2 - 2\alpha D + (\alpha^2 + \beta^2)$ annihilates $e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x$.

If have a power of x^{k-1} to the above, then raise the above to the power of k to annihilate this. If we just have $\cos \beta x$ or $\sin \beta x$, then the operator becomes $D^2 + \beta^2$.

Definition

A linear differential operator A is said to annihilate a function f if

$$A[f](x) = 0$$

for all x. That is, A annihilates f if f is a solution to the homogeneous linear differential equation above on $(-\infty, \infty)$.

Example

Find a differential operator that annihilates

$$6xe^{-4x} + 5e^{x}\sin 2x$$

We know that $(D+4)^2$ will annihilate $6xe^{-4x}$.

We saw that the form that annihilates the other part of the equation is $D^2 - 2\alpha D + (\alpha^2 + \beta^2)$.

We know that $\alpha = 1$ and $\beta = 2$.

The operator that will annihilate that term is $D^2 - 2D + 5$, so this term annihilates $5e^x \sin 2x$.

The sum will be annihilated by multiplying $(D+4)^2$ and $(D^2 - 2D + 5)$.

Example

Find a general solution to

$$y'' - y = xe^x + \sin x$$

Method 1: Undetermined Coefficients

The homogeneous equation is $m^2 - 1$, so the solution to the homogeneous equation is $y_c = c_1e^x + c_2e^{-x}$

The form of the particular solution looks like $y_p = (Ax+B)e^x + C\sin x + D\cos x$.

Let's find the form of xe^x first.

We have that $y_p = (Ax+B)e^x$, then the derivative is $Ae^x + (Ax+B)e^x = Axe^x + (A+B)e^x$. The second derivative is $Ae^x + Axe^x + (A+B)e^x = Axe^x + (2A+B)e^x$.

Plugging this in gives $Axe^x + (2A+B)e^x - (Ax+B)e^x = xe^x$. We end up getting $2Ae^x = xe^x$.

Because of the overlap with the homogeneous equation, the particular solution is actually $y_p = x(Ax + B)e^x = (Ax^2 + Bx)e^x$.

The first derivative of this is $(2Ax+B)e^x + (Ax^2+Bx)e^x = [Ax^2 + (2A+B)x + B]e^x$. The second derivative is $(2Ax + 2A + B)e^x + [Ax^2 + (2A+B)x + B]e^x = [Ax^2 + (4A+B)x + (2A+2B)]e^x$.

Plugging this in gives $[Ax^2 + (4A+B)x + (2A+2B)]e^x - (Ax^2 + Bx)e^x = xe^x$.

Simplifying this gives $4A = 1$ and $2A + 2B = 0$. From this we get $A = 1/4$ and $B = -1/4$.

The solution for $y_p = (1/4x^2 - 1/4x)e^x = x(\frac{1}{4}x - \frac{1}{4})e^x$.

Now we need to solve the other part of y_p.

Doing derivatives and plugging in stuff we get $C = -1/2$ and $D = 0$, so $y_p = x(\frac{1}{4}x - \frac{1}{4})e^x - \frac{1}{2}\sin x$.

Therefore $y = c_1e^x + c_2e^{-x} + x(-\frac{1}{4}x - \frac{1}{4})e^x - \frac{1}{2}\sin x$

Method 2: Annihilator Method We know that $(D-1)^2$ annihilates xe^x.

We know that for $\sin x$ the form is $D^2 - 2\alpha D + (\alpha^2 + \beta^2)$.

So $D^2 + 1$ annihilates $\sin x$.

$(D-1)^2(D^2+1)$ annihilates $xe^x + \sin x$.

Rewrite the equation using differential operator notation. We end up getting $(D^2 - 1)y = xe^x + \sin x$.

This gives $(D+1)(D-1)y = xe^x + \sin x$. Applying $(D-1)^2(D^2+1)$ to both sides, we get $(D + 1)(D-1)^3(D^2+1)y = (D-1)^2(D+1)[xe^x + \sin x]$.

We get that $(D+1)(D-1)^3(D^2+1)y = 0$.

We would have $y = c_1e^{-x} + c_2e^x + c_3xe^x + c_4x^2e^x + c_5\sin x + c_6\cos x$ as the general solution to the homogeneous equation.

The particular solution is exactly what we got in the same form using the annihilator method.

Belpw for me later *Exercise* Find a general solution to $y''' - 3y'' + 4y = xe^2x$ pls later anastasia come back

6.4 Method Of Variation of Parameters

The method of undetermined coefficients and the annihilator method work only for linear equations with constant coefficients and when the nonhomogeneous term is a solution to some homogeneous linear equation with constant coefficients. The method of variation of parameters discussed in chapter 4 generalizes to higher-order linear equations with variable coefficients.

Our goal is to find a solution to the standard form equation

$$L[y](x) = g(x)$$

where $L[y] = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y$ and the coefficient functions $p_1, p_2, \ldots, p_n$ as well as g are continuous on (a, b).

A general solution to $L[y](x) = 0$ is $y_h(x) = C_1 y_1(x) + \cdots + C_n y_n(x)$.

In the method of variation of parameters, there exists a particular solution to the standard form equation of the form

$$y_p(x) = v_1(x)y_1(x) + \cdots + v_n(x)y_n(x)$$

The functions $v_1', v_2', \ldots, v_n'$ must satisfy the system

$$\begin{aligned} y_1 v_1' + \cdots + y_n v_n' &= 0 \\ \vdots \quad\ \vdots\ \vdots\ \vdots\ & \vdots \\ y_1^{(n-2)} v_1' + \cdots + y_n^{(n-2)} v_n' &= 0 \\ y_1^{(n-1)} v_1' + \cdots + y_n^{(n-1)} v_n' &= g \end{aligned}$$

Solving the system using Cramer's Rule, we find that $v_k'(x) = \frac{g(x)W_k(x)}{W[y_1,\ldots,y_n](x)}$ where $k = 1, \ldots, n$. and where $W_k(x)$ is the determinant of a matrix obtained from the Wronksian $W[y_1, \ldots, y_n](x)$ by replacing the kth column by Col$[0, \ldots, 0, 1]$.

Example

Find a general solution to the Cauchy-Euler equation

$$x^3 y''' + x^2 y'' - 2xy' + 2y = x^3 \sin x, \qquad x > 0$$

From Cauchy Euler, we see that $y = x^r, y' = rx^{r-1}, y'' = r(r-1)x^{r-2}, y''' = r(r-1)(r-2)x^{r-3}$.

Plugging this in the homogeneous equation gives $x^3 \cdot r(r-1)(r-2)x^{r-3} + x^2 r(r-1)x^{r-2} - 2xrx^{r-1} + 2x^r = 0$.

This gives $x^r[r^3 - 3r^2 + 2r] + x^r[r^2 - r] - 2x^r[r] + 2x^r = 0$. Factoring this gives $x^r[r^3 - 3r^2 + 2r + r^2 - r - 2r + 2] = 0$.

Assuming $x \neq 0$, we get $r^3 - 2r^2 - r + 2 = 0$. Factoring this gives $(r-2)(r-1)(r+1) = 0$.

The general solution to the homogeneous equation is $y = c_1 x^2 + c_2 x^{-1} + c_3 x$.

From above we see that $y_1 = x^2, y_2 x^{-1}, y_3 = x$.

The particular solution will be of the form $y_p = v_1 x^2 + v_2 x^{-1} + v_3 x$.

Starting with the Wronksian of x^2, x^{-1}, x.

Before, we get $g(x) = \frac{x^3 \sin x}{x^3} = \sin x$. This comes from dividing the $x^3 \sin x$ by the leading coefficient.

The next determinant for v_1 is the same as the original Wronksian above, but the first column has $0, 0, \sin x$ instead of $x^2, 2x, 2$.

The determinant for v_2 is the same as the original, but the second column is replaced by $0, 0, \sin x$ instead of $x^{-1}, -x^{-2}, 2x^{-3}$.

The determinant for v_3 is the same as the original, but the third column is replaced by $0, 0 \sin x$ instea dof $x, 1, 0$.

For the original Wronksian, we get $x(4x^{-2} + 2x^{-2}) - 1(2x^{-1} - 2x^{-1}) = 6x^{-1} = W$.

For the Wronksian of v_1, we get $\sin x(x^{-1} + x^{-1}) = 2x^{-1} \sin x$.

For the Wronksian of v_2, we get $-\sin x(x^2 - 2x^2) = x^2 \sin x$.

For the Wronksian of v_3, we get $\sin x(-1 - 2) = -3 \sin x$.

So we get $v_1' = \frac{W_1}{W} = \frac{2x^{-1} \sin x}{6x^{-1}} = \frac{1}{3} \sin x$

We get $v_2' = \frac{W^2}{W} = \frac{x^2 \sin x}{6x^{-1}} = \frac{1}{6} x^3 \sin x$

We get $v_3' = \frac{W^3}{W} = \frac{-3 \sin x}{6x^{-1}} = -\frac{1}{2} x \sin x$.

Integrating, we get $v_1 = -\frac{1}{3} \cos x$.

For v_3, we have $-\frac{1}{2} \int x \sin x \mathrm{d}x$. Let $u = x$ and $\mathrm{d}v = \sin x$. From this, $v = -\cos x$ and $\mathrm{d}u = 1$.

Integrating by parts should give $v_3 = \frac{1}{2} x \cos x - \frac{1}{2} \sin x$.

For v_2, we get $u = x^3$ then $3x^2, 6x, 6, 0$. and for $\mathrm{d}v$ we get $\sin x, -\cos x, \sin x, -\cos x, \sin x$.

This is tabular integration by parts. We get $v_2 = \frac{1}{6}[-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x]$.

Simplifying this gives $v_2 = -\frac{1}{6} x^3 \cos x + \frac{1}{2} x^2 \sin x + x \cos x - \sin x$.

We can now get y_p.

Yea, so y_p is simply $y_p = [-\frac{1}{3} \cos x]x^2 + [-\frac{1}{6} x^3 \cos x + \frac{1}{2} x^2 \sin x + x \cos x - \sin x]x^{-1} + [\frac{1}{2} x \cos x - \frac{1}{2} \sin x]x$.

Simplifying this gives $\cos x - x^{-1} \sin x$.

So the general solution is $y = \cos x - x^{-1} \sin x + c_1 x^2 + c_2 x^{-1} + c_3 x$.

7 Laplace Transforms

7.1 Definition of the Laplace Transform

Definition

Let $f(t)$ be a function on $[0, \infty)$. The Laplace transform of f is the function F defined by the integral

$$F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

The domain of $F(s)$ is all the values of s for which the integral above exists. The Laplace transform of f is denoted by both F and $\mathcal{L}\{f\}$.

Example

Determine the Laplace transform of the constant function $f(t) = 1, t \geq 0$.

Let $F(s) = \int_0^\infty e^{-st} 1\mathrm{d}t = \int_0^\infty e^{-st}\mathrm{d}t$. This is equal to $-\frac{1}{s}e^{-st}$ with bounds ∞ and 0.

Remember this is an improper integral where we have $\lim_{b\to\infty} -\frac{1}{s}e^{-st}$ from 0 to b.

This gives $-\frac{1}{s}e^{-sb} - \frac{1}{s}e^{0}$ on the inside of the limit, so we get $\lim_{b\to\infty} \left[-\frac{1}{s}e^{-sb} + \frac{1}{s}\right]$.

The above equals $\lim_{b\to\infty} \left[-\frac{1}{s} \cdot \frac{1}{e^{rb}} + \frac{1}{s}\right]$.

The restriction is $s > 0$ because $\frac{1}{e^{sb}}$ has to be greater than 0.

Our result ends up being $\frac{1}{s}$.

$\mathcal{L}\{1\} = \frac{1}{s}$.

Example

Determine the Laplace transform of $f(t) = t$.

We have $\mathcal{L}\{t\} = \int_0^\infty e^{-st} t\mathrm{d}t = \lim_{b\to\infty} \left[\int_0^b e^{-st} t\mathrm{d}t\right]$.

Integrating by parts gives the inside equal to $-\frac{1}{s} \cdot t \cdot \frac{1}{e^{st}} - \frac{1}{s^2}e^{-st}$ with bounds 0 to b.

Plugging this in gives $\lim_{b\to\infty} -\frac{1}{s} \cdot \frac{b}{e^{sb}} - \frac{1}{s} \cdot \frac{1}{e^{sb}} + \frac{1}{s^2}$.

We see that $\frac{b}{e^{sb}}$ is indeterminate, so using L'Hopital's Rule, the derivative is $\frac{1}{se^{sb}}$ and the limit as b approaches ∞ gives this as 0.

We are left with $\frac{1}{s^2}$.

$\mathcal{L}\{t\} = \frac{1}{s^2}$.

We will see that $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$.

Example

Determine the Laplace transform of $f(t) = e^{at}$, where a is a constant.

The integral is $\int_0^\infty e^{-st} \cdot e^{at}\mathrm{d}t = \int_0^\infty e^{-(s-a)t}\mathrm{d}t$.

Integrating this gives $-\frac{1}{s-a}e^{-(s-a)t}$ evaluated from 0 to ∞.

As t goes to infinity, we get 0 and then we get $0 - \frac{-1}{s-a}e^0 = \frac{1}{s-a}$.

So $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$.

If we were to find the Laplace of e^{5t}, from the above example it would be $\frac{1}{s-5}$.

Example

Find $\mathcal{L}\{\sin bt\}$, where b is a nonzero constant.

The integral this time is $\int_0^\infty e^{-st} \cdot \sin bt\mathrm{d}t$.

Integrating gives $-\frac{1}{s}\sin bte^{-st} + \frac{b}{s}\left[-\frac{1}{s}\cos bte^{-st} - \int -\frac{1}{s}e^{-st}(-b)\sin bt\mathrm{d}t\right]$.

(Do this example later)

Involves factoring Laplace stuff.

$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$.

Example

Determine the Laplace transform of

$$f(t) = \begin{cases} 2 & 0 < t < 5 \\ 0 & 5 < t < 10 \\ e^{4t} & t > 10 \end{cases}$$

To do this, you just do $\int_0^\infty e^{-st}f(t)\mathrm{d}t = \int_0^5 e^{-st} \cdot 2\mathrm{d}t + \int_5^{10} e^{-st} \cdot 0\mathrm{d}t + \int_1 0^\infty e^{-st} \cdot e^{4t}\mathrm{d}t$.

Evaluating this gives the laplace as $-\frac{2}{s}e^{-5s} + \frac{2}{s} + \frac{1}{s-4}e^{-(s-4)10}$

An important property of the Laplace transform is its linearity. That is, the Laplace transform $\mathcal{L}$ is a linear operator.

Theorem 7.1

Let f, f_1, and f_2 be functions whose Laplace transforms exist for $s > \alpha$ and let c be a constant. Then, for $s > \alpha$,

$$\mathcal{L}\{f_1 + f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}$$
$$\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$$

Exercise Determine $\mathcal{L}\{11 + 5e^{4t} - 6\sin 2t\}$.

A function $f(t)$ on $[a, b]$ is said to have a jump discontinuity at $t_0 \in (a, b)$ if $f(t)$ is discontinuous at t_0, but the one-sided limits

$$\lim_{t \to t_0^-} f(t) \qquad \text{and} \qquad \lim_{t \to t_0^+} f(t)$$

exist as finite numbers.

Definition

A function $f(t)$ is said to be piecewise continuous on a finite interval $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

A function $f(t)$ is said to be piecewise continuous on $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

In contrast, the function $f(t) = 1/t$ is not piecewise continuous on any interval containing the origin, since it has an "infinite jump" at the origin.

A function that is piecewise continuous on a finite interval is not necessarily integrable over that interval. However, piecewise continuity on $[0, \infty)$ is not enough to guarantee the existence (as a finite number) of the improper integral over $[0, \infty)$; we also need to consider the growth of the integrand for large t. The Laplace transform of a piecewise continuous function exists, provided the function does not grow "faster than an exponential".

Definition

A function $f(t)$ is said to be of exponential order α if there exist positive constants T and M such that

$$|f(T)| \leq Me^{\alpha t}$$

for all $t \geq T$.

Theorem 7.2

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then $\mathcal{L}\{f\}(s)$ exists for $s > a$.

Here are common Laplace transforms:

- $\mathcal{L}\{1\} = \frac{1}{s}$
- $\mathcal{L}\{t\} = \frac{1}{s^2}$
- $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
- $\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$
- $\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$

7.2 Properties of the Laplace Transform

Theorem 7.3

If the Laplace transform $\mathcal{L}\{f\}(s) = F(s)$ exists for $s > \alpha$, then

$$\mathcal{L}\{e^{\alpha t}f(t)\}(s) = F(s-a)$$

for $s > \alpha + a$

Example

Determine the Laplace transform of $e^{\alpha t}\sin bt$

We know the Laplace of $\sin bt$ is equal to $\frac{b}{s^2+b^2}$.

Multiplying by $e^{\alpha t}$ just shifts it $F(s-\alpha) = \frac{b}{(s-\alpha)^2+b^2}$

Theorem 7.4

Let $f(t)$ be continuous on $[0, \infty)$ and $f'(t)$ be piecewise continuous on $[0, \infty)$, with both of exponential order α. Then for $s > \alpha$,

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$

Theorem 7.5

Let $f(t), f'(t), \ldots, f^{(n-1)}(t)$ be continuous on $[0, \infty)$ and let $f^{(n)}(t)$ be piecewise continuous on $[0, \infty)$, with all these functions of exponential order α. Then, for $s > \alpha$,

$$\mathcal{L}\{f^{(n)}\}(s) = s^n \mathcal{L}\{f\}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

Example

Using the above theorems and the fact that $\mathcal{L}\{\sin bt\}(s) = \frac{b}{s^2+b^2}$, determine $\mathcal{L}\{\cos bt\}$

We know that $f'(t) = b\cos bt$ from this. So $\mathcal{L}\{b\cos bt\} = s\mathcal{L}\{\sin bt\} - f(0)$.

We know that $b\mathcal{L}\{\cos bt\} = s\mathcal{L}\{\sin bt\}$, since $f(0) = 0$.

So simplifying gives the Laplace transform as $\frac{s}{s^2+b^2}$

Example

Prove the following identity for continous functions $f(t)$ (assuming the transforms exist):

$$\mathcal{L}\left\{\int_0^t f(\tau)\mathrm{d}\tau\right\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(s)$$

We know $g(t) = \int_0^t f(\tau)\mathrm{d}\tau$. From this we know $g'(t) = f(t)$.

We get that $\mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0)$. and that $\mathcal{L}\{f(t)\} = s\mathcal{L}\{\int_0^t f(\tau)\mathrm{d}\tau\}$.

We also know $g(0) = 0$.

So the Laplace of the function is equal to $\frac{1}{s}\mathcal{L}\{f(t)\}$.

Theorem 7.6

Let $F(s) = \mathcal{L}\{f\}(s)$ and assume $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α. Then, for $s > \alpha$,

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{\mathrm{d}^n F}{\mathrm{d}s^n}(s)$$

Example

Determine $\mathcal{L}\{t\sin bt\}$.

We know $f(t) = \sin bt$ and that $n = 1$.

This is equal to $(-1)^1 \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\sin bt\}$.

We end up getting $-\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{b}{s^2+b^2}\right)$.

We end up getting $\frac{2bs}{(s^2+b^2)^2}$.

Here are some basic properties of Laplace Transforms

- $\mathcal{L}\{f + g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$.
- $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ for any constant c.
- $\mathcal{L}\{e^{at} f(t)\}(s) = \mathcal{L}\{f\}(s - a)$
- $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

- $\mathcal{L}\{f''(s)\} = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
- $\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
- $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}(\mathcal{L}\{f\}(s))$

7.3 Inverse Laplace Transform

7.4 Solving Initial Value Problems

7.5 Transforms of Discontinuous Functions

7.6 Transforms of Periodic and Power Functions

7.7 Convolution

7.8 Impulses and the Dirac Delta Function

7.9 Solving Linear Systems with Laplace Transforms

8 Series Solutions of Differential Equations

8.1 Introduction: The Taylor Polynomial Approximation

8.2 Power Series and Analytic Functions

8.3 Power Series Solutions to Linear Differential Equations

8.4 Equations with Analytic Coefficients

8.5 Method of Frobenius

9 Matrix Methods for Linear Systems

9.1 Introduction

9.2 Review 1: Linear Algebraic Equations

9.3 Review 2: Matrices and Vectors

9.4 Linear Systems in Normal Form

9.5 Homogeneous Linear Systems with Constant Coefficients

9.6 Complex Eigenvalues

9.7 Nonhomogeneous Linear Systems