# 1 Multiple Integrals

## 1.1 Double Integrals over Rectangles

Recall that $\int_a^b f(x)dx$ gives the area under the curve from $x = a$ to $x = b$.

Also, $\int_a^b f(x)dx = \lim\limits_{x\to\infty} \sum\limits_{k=1}^{n} f(x_k^i)\Delta x_k$ (Riemann sums)

Now we have volume.

Volume Problem: Given a function of $2$ variables that is continuous and non-negative on a region $R$ in the $xy$-plane, find the volume of the solid enclosed between the surface $z = f(x, y)$ and the $xy$-plane.

Plan:

1. Partition $R$ in rectangles.
2. Choose a point $(x_k^*, y_k^*)$ in each rectangle.
3. Map onto $z$.
4. Form parallelpiped.

We can then approximate the volume using rectangular parallelpipeds.

Volume $\approx$ area of rectangle $\times$ height $\to$ Volume $\approx \Delta A_k f(x_k^*, y_k^*)$.

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Volume $= \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$ (this is the formal definition for the volume problem using Riemann sums)

If $f$ has both positive and negative values, then the volume is the difference in volumes between $R$ and the surface above and below the $xy$-plane.

**Definition**

If $f(x, y) \geq 0$, then the volume of the solid that lies above the rectangle $R$ and below the surface $z = f(x, y)$ is:

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$
$$= \iint_R f(x, y)dA$$

**Example**

Estimate the volume of the solid that lies above the square $R = [0, 2]\times[0, 2]$ and below $z = 16-x^2-2y^2$. Divide $R$ into $4$ equal squares and choose the sample point to be the upper right corner of each square.

So $V = \iint_R (16 - x^2 - 2y^2)dA$.

This is $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$. We will use approximately $4$ parallelpipeds.

We get $V \approx 1 \cdot f(1, 1) + 1 \cdot f(2, 1) + 1 \cdot f(1, 2) + 1 \cdot f(2, 2)$.

This is equal to $V \approx 34$ u$^3$. This is an estimate.

**Example**

If $R = \{(x, y)| - 1 \leq x \leq 1, -2 \leq y \leq 2\}$, evaluate $\iint_R \sqrt{1 - x^2} dA$.

The graph will look like half of a cylinder.

The volume of a cylinder is $\pi r^2 h$.

$V = \frac{1}{2}\pi r^2 h = \frac{1}{2}\pi(1)^2(4)$, so $\iint_R dA = 2\pi$.

Midpoint Rule - This tells us to evaluate $\iint_R f(x, y)dA$ using Riemann sums and midpoints.

Evaluating Double Integrals $\rightarrow$ uses iterated integration.

$$\int_c^d \int_a^b f(x, y)dxdy = \int_c^d \left[\int_a^b f(x, y)dx\right] dy$$

This integral can be $dxdy$ or $dydx$.

**Example**

Integrate $\int_0^3 \int_1^2 x^2 y dy dx$.

First integrate $\int_1^2 x^2 y dy$.

This is $x^2 \cdot \frac{1}{2}y^2$ from $y = 1$ to $y = 2$.

And then we have $\int_0^3 \left(\frac{1}{2}x^2 \cdot 2^2 - \frac{1}{2}x^2 \cdot 1^2\right) dx$.

This is $\int_0^3 \frac{3}{2}x^2 dx = \frac{1}{2}x^3$ from $x = 0$ to $x = 3$, and the answer is $\frac{27}{2}$.

**Example**

Evaluate $\int_1^2 \int_0^3 x^2 y dx dy$.

First evaluate $\int_0^3 x^2 y dx$ to get $y \cdot \frac{1}{3}x^3$ from $x = 0$ to $x = 3$.

Then we have $\int_1^2 9ydy$ and this is $\frac{9}{2}y^2$ from $y = 1$ to $y = 2$, so the result of the integral is $\frac{27}{2}$.

$\frac{27}{2}$ in both cases is the area underneath $f(x, y) = x^2 y$ and above $R : [0, 3] \times [1, 2]$. Note that $dx$ and $dy$ are not always interchangable.

**Theorem 1.1: Fubini's Theorem**

Let $R$ be a rectangular region defined by $a \leq x \leq b$, $c \leq y \leq d$. If $f(x, y)$ is continuous on this rectangle, then:

$$\iint_R f(x, y)dA = \int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx$$

**Example**

Evaluate $\iint_R (x - 3y^2)dA$ with $R = [0, 2] \times [1, 2]$.

The integral is $\int_0^2 \int_1^2 (x - 3y^2)dydx$.

We start with the inner integral and get $xy - y^3$ from $y = 1$ to $y = 2$.

Then we evaluate $\int_0^2 (2x - 8) - (1x - 1)dx$ to get $\int_0^2 (x - 7)dx$.

The result of this integral is $-12$.

A few properties:

1. $\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$
2. $\iint_R [f(x,y) \pm g(x,y)]dA = \iint_R f(x,y)dA \pm \iint_R g(x,y)dA$
3. $\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$
4. $\iint_R g(x)h(y)dA = \int_a^b g(x)\int_c^d h(y)dy$ where $R = [a,b]\times[c,d]$.

**Example**

Evaluate $\iint_R \sin x\cos y dA$ where $R = \left[0, \frac{\pi}{2}\right]\times[0,\pi]$.

We can use the fourth property above to get $\int_0^{\pi/2}\sin x dx\cdot\int_0^{\pi}\cos y dy$.

We get $\cos x$ from $0$ to $\pi/2$ and subtract $\sin y$ from $0$ to $\pi$ from this.

The answer is $0$.

**Example**

Find the volume of the solid that is bounded above by $f(x,y) = y\sin(xy)$ and below by $R = [1,2]\times[0,\pi]$.

$V = \int_0^{\pi}\int_1^2 y\sin(xy)dxdy = \int_1^2\int_0^{\pi} y\sin(y)dydx$.

The first option is better.

So we start with $y\cdot -\frac{1}{y}\cos(xy)$ from $x = 1$ to $x = 2$

Then, $\int_0^{\pi}(-\cos 2y + \cos y)dy = 0$.

Average Value: $f_{avg} = \frac{1}{A(R)}\iint_R f(x,y)dA$

**Example**

Find the average value of $f(x,y) = x^2y$ over $R$ with vertices $(-1,0)$, $(-1,5)$, $(1,5)$, and $(1,0)$.

The area of the rectangle is $A_R = 10$.

Set up the integral to get $f_{avg} = \frac{1}{10}\int_{-1}^{1}\int_0^5 x^2ydydx/$

## 1.2 Double Integrals over General Regions

There are 2 Types of regions:

The biggest question is finding the limits of integration.

So if we have $y$ in terms of $x$, then we use $\iint_D f(x,y)dA = \iint_{g_1(x)}^{g_2(x)} f(x,y)dydx$.

If we have $x$ in terms of $y$, then $\iint_D f(x,y)dA = \iint_{h_1(y)}^{h_2(y)} f(x,y)dxdy$

**Example**

Evaluate $\iint_D (x + 2y) dA$ where $D$ is the region bounded by $y = 2x^2$ and $y = 1 + x^2$.

We have both equations being $y =$ something, so we are integrating with respect to $y$ first in this case. Though the reason for this is mostly because we are integrating vertically.

So the integration is $\int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y) dy dx$.

We then get $xy + y^2$ with limits of integration $y = 2x^2$ to $y = 1 + x^2$.

So this results in $\int_{-1}^{1}[x(1+x^2)+(1+x^2)^2]-[x(2x^2)+(2x^2)^2]dx = \int_{-1}^{1}(-3x^4-x^3+2x^2+x+1)dx = \frac{32}{15}$.

**Example**

Set up only! Evaluate $\iint_R xydA$ where $R$ is the region bounded by $y=-x+1, y=x+1$, and $y=3$.

Draw to see the region. We can see from the graph that integrating by $x$ first is probably the better idea, because $y$ would require 2 double integrals.

So setting up the integral gives $\int_1^3\int_{1-y}^{y-1} xydxdy$. (Setting the equations in terms of $x=$)

**Example**

Find the volume of the solid that lies under $z=xy$ and above $D$ where $D$ is the region bounded by $y=x-1$ and $y^2=2x+6$.

By drawing this, we see we will go by $dx$ first.

Setting up the integral is $\int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xydxdy$.

Starting the integration, we get $\frac{1}{2}x^2y$ with the limits of the first integral.

This gives $\frac{1}{2}\int_{-2}^{4}(y+1)^2\cdot y-\left(\frac{1}{2}y^2-3\right)^2\cdot ydy$.

Simplifying some more gives $\frac{1}{2}\int_{-2}^{4}-\frac{1}{4}y^5+4y^3+2y^2-8ydy$ and this gives 36 as an answer.

**Example**

$\int_0^1\int_x^1\sin(y^2)dydx$

If we draw the region, we have $y=1$ and $y=x$.

We go from $x=0$ to $x=y$ and for the $y$ limits, we go from $0$ to $1$.

Therefore the integral is $\int_0^1\int_0^y\sin(y^2)dxdy$.

Solving this integral gives $-\frac{1}{2}\cos(1)+\frac{1}{2}$.

**Example**

Use a double integral to find the area of the region enclosed between $y=x^3$ and $y=2x$ in the first quadrant.

Set up the integral $A=\int_0^{\sqrt{2}}\int_{x^3}^{2} x$ to get the area.

*Exercise* Evaluate by reversing the order of integration: $\int_0^2\int_{y/2}^{1} e^{x^2}dxdy$.

## 1.3 Double Integrals in Polar Coordinates

Recall that polar coordinates are in form $(r,\theta)$ and rectangular coordinates are in $(x,y)$.

In polar, for the unit circle, we can write $0\le r\le 1$ or $0\le\theta\le 2\pi$.

We have 3 equations for converting:

- $r^2=x^2+y^2$
- $x=r\cos\theta$
- $y=r\sin\theta$

To find the volume, the process is similar to the Riemann sum process for double integrals.

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k = \iint f(r,\theta)dA$$

In the above, $\Delta A_k$ represents the area of the polar rectangle.

Now we need to find the area of a polar rectangle.

First we know that the area of a sector is $\frac{1}{2}r^2\theta$ and that $\Delta A_k$ is the large sector minus the little sector.

Therefore we have $\frac{1}{2}\left(r_k^* + \frac{1}{2}\Delta r_k\right)^2 \Delta\theta_k - \frac{1}{2}\left(r_k^* - \frac{1}{2}\Delta r_k\right)^2 \Delta\theta k$.

And this simplifies to $r_k^*\Delta r_k\Delta\theta_k$, so $\Delta A_k = r_k^*\Delta r_k\Delta\theta_k$.

So, $V = \iint_R f(r,\theta)dA = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(r_k^*, \theta_k^*)r_k^*\Delta r_k\Delta\theta_k$.

So we have

$$V = \iint_R f(r,\theta)rdrd\theta$$

**Example**

Find $\iint_R \sin\theta dA$ where $R$ is the region outside the circle $r = 2$ and inside $r = 2 + 2\cos\theta$ in the 1st quadrant.

We see that the circle will be hit first, then the other polar curve.

So the integral is $\int_0^{\pi/2}\int_2^{2+2\cos\theta} \sin\theta \cdot rdrd\theta$.

Note the outer integral goes to $\frac{\pi}{2}$ because that is the first quadrant.

So the inner integral becomes $\sin\theta \cdot \frac{1}{2}r^2$ from the bounds $r = 2$ to $r = 2 + 2\cos\theta$.

We then integrate $\frac{1}{2}\int_0^{\pi/2} \sin\theta[(2 + 2\cos\theta)^2 - 2^2]d\theta$.

Solving this gives $8/3$.

**Example**

Find the volume of the solid bounded by $z = 0$ and $z = 1 - x^2 - y^2$.

The graph of $z = 1 - x^2 - y^2$ will be a paraboloid.

So $R$ is a circle with radius $1$ when we draw this paraboloid.

We could integrate as $V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1 - x^2 - y^2)dydx$, or we could convert to polar.

In polar, we know $R$ is a circle and then we can convert to polar to get $\int_0^{2\pi}\int_0^1(1 - r^2) \cdot rdrd\theta$, which is equal to $V = \frac{\pi}{2}$.

Area is the same as before, recall this was $A = \iint_R 1 \cdot dA$, and now it is $\iint_R rdrd\theta$.

**Example**

Use a double integral to find the area enclosed by one loop of the four-leaf rose of $r = \cos 2\theta$.

If you know how to draw this, then we can find the area $A = \int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} rdrd\theta$, and this integral is simple to solve, the answer is $\pi/8$.

**Example**

Find the volume of the solid that lies under $z = x^2 + y^2$, above the $xy$-plane, and inside $x^2 + y^2 = 2x$.

The graph $x^2 + y^2 = 2x$ can be rearranged to complete the square. We have then $(x-1)^2 + y^2 = 1$.

So if we were to convert $x^2 + y^2$ to polar, we get $r^2$.

We have $r^2 = 2r\cos\theta$ and $r = 2\cos\theta$.

The integral then we get $\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} (r^2) \cdot r dr d\theta$.

The integral is $V = \frac{3\pi}{2}$

## 1.4 Surface Area

The formula for surface area is

$$A(S) = \lim_{n\to 0} \sum_{k=1}^{n} \sqrt{(z_x)^2 + (z_y)^2 + 1}\Delta A$$

which becomes

$$A(S) = \iint_R \sqrt{(z_x)^2 + (z_y)^2 + 1} dA$$

**Example**

Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region with vertices $(0,0), (1,0)$, and $(1,1)$.

$z = x^2 + 2y$, $z_x = 2x$ and $z_y = 2$.

So $A(S) = \int_0^1 \int_0^x \sqrt{(2x)^2 + (2)^2 + 1} dy dx$.

This integral gievs $\frac{1}{12}(27 - 5\sqrt{5})$.

**Example**

Find the surface area of the portion of $z = x^2 + y^2$ below the plane $z = 9$.

Paraboloid!

The integral is $A(S) = \iint_R \sqrt{(2x)^2 + (2y)^2 + 1} dA$.

We might be able to see that simplifying this gives $4x^2 + 4y^2 + 1$ inside the square root, and we have an $x^2 + y^2 = 9$ in the paraboloid.

We should use polar.

So we now have $\int_0^{2\pi} \int_0^3 \sqrt{4r^2 + 1} \cdot r dr d\theta$.

This gives you $\frac{\pi}{6}(37^{3/2} - 1)$.

**Example**

Find the surface area of $z = \sqrt{4 - x^2}$ above $R : [0,1] \times [0,4]$.

Setting up the integral gives $\iint_R \sqrt{\left(-\frac{x}{\sqrt{4-x^2}}\right)^2 + 0^2 + 1}$.

It doesn't really matter the way we integrate, so we get $\int_0^1 \int_0^4 \sqrt{\frac{x^2}{4-x^2} + \frac{4-x^2}{4-x^2}} dy dx$.

This simplifies to $\int_0^1 \frac{8}{\sqrt{4-x^2}} dx$.

We notice that this becomes $8\sin^{-1}\left(\frac{x}{2}\right)$ with bounds $0$ to $1$, and this gives $\frac{4\pi}{3}$.

## 1.5 Triple Integrals

## 1.6 Triple Integrals in Cylindrical Coordinates

## 1.7 Triple Integrals in Spherical Coordinates

## 1.8 Change of Variables in Multiple Integrals