A Basic Guide to High School Mathematics

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Contents

1 Proof

Introduction to Proof

Introduction to Proof

In this section we will working with these topics:

- Consequence and Equivalence
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Introducing Consequence and Equivalence

When we look at consequence, we essentially say that "a implies b", or:

$$a \rightarrow b$$

If the arrow points the other way, we say that "b implies a", or:

$$a \leftarrow b$$

Let's say that statement a states that p is a prime number > 2.

Let's say that statement b states that p is an odd number.

For these statements, we see that a does imply b, so we can write that

$$a \rightarrow b$$

The other way however does not work, since because p is an odd number, it does not imply that p is a prime number.

However, if this was true, we can write that a implies b and b implies a, or:

$$a \leftrightarrow b$$

which is sometimes written as "a if and only b" or "a iff b".

Let's show a logical equivalence. Let a be the statement n^2 is odd and b be the statement n is odd.

We know that when n^2 is odd, that n is odd when we list out the odd squared numbers. We can see the converse is true as well in this statement since every time a number n is squared, we are given an odd number, therefore:

$$a \leftrightarrow b$$

Consequence and Equivalence Examples

Let's give some examples where we determine whether one of the statements implies the other statement.

Given that an object is a cube and an object has six faces. If an object is a cube, it definitely has six faces. Therefore, The object is a cube $\implies$ The the object has six faces. The opposite is not true, because it can be a cuboid, for example.

Given $x = 29$ and $x > 10$, then $x = 29 \implies x > 10$. The opposite is not true, since there are many more values where $x > 10$.

Given $x^3 = x$ and $x = -1$. We need to find the solutions of $x^3 = x$ first. By subtracting and obtaining $x^3 - x = 0$, we can factor this to $x(x^2 - 1) = 0$. Then we have $x(x-1)(x+1) = 0$, and the solution of this equation are $0, 1$, and -1. Therefore $x^3 = x$ does not imply $x = -1$. However, going the other way, $x = -1 \implies x^3 = x$.

Given n is a positive integer greater than 1, we are given the statements that n is a prime number and n has exactly two factors. n always has two factors if it is prime, then n is a prime number $\implies$ n has exactly two factors. If n has exactly two factors, then it must be prime, so we can see that n has exactly two factors $\implies$ n is a prime number, so n is a prime number $\leftrightarrow$ n has exactly two factors.

Proof by Exhaustion

Introducing Proof by Exhaustion

Proof by Exhaustion is trying all possible variations to prove a statement is true.

We are going to prove a conjecture, which is a statement that we believe to be correct but needs to be proved.

The conjecture is "97 is a prime number". To show this, we need to show that 97 has two factors, 1 and itself.

Let's try some numbers.

$97 \div 2$ is 48.5, clearly 2 is not a factor of 97. $97 \div 3$ is $32.\overline{3}$. Therefore, 3 is not a factor either. We wouldn't need to try 4 since 2 already isn't a factor. Let's try 5. $97 \div 5$ is 19.4, so 5 is also not a factor of 97. We don't need to try 6 since 3 and 2 are both not factors of 97. Now we try 7. $97 \div 7 = 13.85\ldots$, so 7 is not a factor either. It's clear we are just working through all the prime numbers now.

We don't need to go further than this because when we square root 97, we will get a number a little less than 10. Because the square root of 97 is a little less than 10, when we go beyond 10, if we are to find any factor above 10, then there would have to have been a factor less than 10 to multiply with to make 97.

In other words, because there were no factors below the square root of 97, this implies there are no factors larger than the square root of 97, indicating that 97 is a prime number.

Proof by Exhaustion Examples

Let's do three examples.

- No square number ends in an 8

  This problem looks at squaring each unit digit. If a number ends in a 1, the square one gets will end in a 1 as well. If the number ends in a 2, and I square it, then this number will end with a 4. If the number ends with a 3, the number will end with a 9. If the number ends with a 4, the squared number will end with a 6. If the number ends with a 5, the squared number will end with a 5. If the number ends with a 6, the squared number will end with a 6. If the number ends with a 7, the squared number will end with a 9. If the number ends with a 8, the squared number will end with a 4. If the number ends with a 9, the squared number will end with a 1. If the number ends with a 0, the squared number will end with up with a 0.

  As we can see, there are no numbers that can have a unit digit of 8.

- If n is an integer and $2 \leq n \leq 7$, then $A = n^2 + 2$ is not divisible by 4.

  To show this, lets consider all values of n.

n	n^2+2	divisible by 4?
2	6	no
3	11	no
4	18	no
5	27	no
6	38	no
7	51	no

so in none of these cases, none of these values of A are divisible by 4 and we have gone through every single part of this and show that this is never divisible by 4.

- Every integer that is a perfect cube is either a multiple of 9, is 1 more than a multiple of 9, or is 1 less than a multiple of 9.

  The first statement says that $n = 3k$, that the number is a multiple of 3, or $n = 3k - 1$, one less than a multiple of three, or $n = 3k - 2$, a number is two less than a multiple of 3.

  Let's start by cubing. $n^3 = 27k^3$. Because 27 is a multiple of 9, k is an integer and n^3 is a multiple of 9.

  Let's look at $n = 3k - 1$. $n^3 = 27k^3 - 27k^2 + 9k - 1$. If we factor a 9 out, we get $9(3k^3 - 3k^2 + k) - 1$. This is clearly 1 less than a multiple of 9.

  Now let's look at $n = 3k - 2$. $n^3 = 27k^3 - 54k^2 + 36k - 8$. If I write the 8 as a $-9 + 1$, we can factor out the 9 and get $9(3k^3 - 6k^2 + 4k - 1) + 1$, or one more than a multiple of 9.

Proof by Deduction

Introduction Proof by Deduction

Proof by deduction is all about going through a logical sequence of arguments where you will start with something you know to be true, and subsequently, the next thing is true, etc, until the conjecture is true.

Conjecture: "The sum of any two consecutive odd numbers is a multiple of 4."

We can start with an odd number $2n + 1$, since $2n$ is always an even number, so adding 1 will make it odd. If we are looking for the next consecutive odd number, then we can see this as $2n + 3$. The conjecture talks about the sum of the consecutive odd numbers. Adding them together, we get $4n + 4$, which factors to $4(n + 1)$, which is always a multiple of 4.

Proof by Deduction Example

Example

For any four consecutive integers, the difference between the prodcut of the last two and the product of the first two of these numbers is equal to their sum.

Let's first label four consecutive integers as $n, n + 1, n + 2, n + 3$. We have to find the product of the last two and the product of the last two and to find the difference between the two things.

Therefore, we are finding $(n+2)(n+3)-n(n+1)$. Expanding this, we get $n^2+5n+6-n^2-n$. Simplifying, we get $4n + 6$.

Adding the consecutive integers, we have $n + n + 1 + n + 2 + n + 3 = 4n + 6$. We have shown that the difference between the products of the last two and the first two is the same as the sum of the four numbers.

Example

$k^3 - k$ is divisible by 6 for all integers $k > 1$.

First we can factor $k^3 - k$ to $k(k^2 - 1)$. We can factor this further as $k(k-1)(k+1)$. Now if we write this in a slightly different order, as $(k-1)(k)(k+1)$. What we have here is the product of three consecutive integers. At least one of these integers therefore will be an even integer, so $k^3 - k$ is divisible by 2.

Now because we have three consecutive integers, precisely one of them will be a multiple of 3 because since $k > 1$, there will always be a number that is divisible by 3 when consecutively counting. Therefore $k^3 - k$ is also divisible by 3.

Because $k^3 - k$ is divisible by 2 and 3, then it is divisible by 6.

Disprove by Counter-Example

Introducing Disproof by Counter Example

Sometimes we are asked to find a single example where a conjecture fails.

Let's start with the conjecture "The value of $n^2 + n + 11$ is prime for all integers $n > 0$"

When $n = 11$, we can see that $11^2 + 11 + 11$ which is equal to $11(13)$ which is evidently not prime.

Disproof by Counter Example Examples

Example

If $x^2 > x$, then $x > 1$.

When we plug in $x = -2$, we can see that $4 > -2$, but -2 is not greater than 1.

Example

If n is prime, then $n^2 + n + 1$ is prime.

When we plug in $n = 7$, we get $n^2 + n + 1 = 57$, which is not prime, so this conjecture fails.

Example

The sum of n consecutive integers is divisible by n (where n is a positive integer).

We can easily disprove this in one example. $1 + 2 + 3 + 4 = 10$, which is not divisible by 4.

2 Algebra & Functions

2.1 Indices

Subsets of Real Numbers

Introducing Subsets of Real Numbers

Natural numbers are represented by $\mathbb{N}$. They are just the counting numbers - like $1, 2, 3, 4, 5, 6, \ldots$. This does not include 0 or negative numbers.

Integers are represented by $\mathbb{Z}$. This includes all the natural numbers and also includes $0, -1, -2, -3, \ldots$. It is twice the size of natural numbers plus a zero.

Rational numbers are represented by $\mathbb{Q}$. This would include $\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, -\frac{5}{7}, -\frac{7}{2}$ along with the natural numbers and integers.

The real numbers are represented by $\mathbb{R}$. This includes everything above, along with things such as $\sqrt{2}, \sqrt{3}, \pi, e$.

The complex numbers are based on if we allowed to square root -1. We define this as i. The complex numbers will include things such as $2i, 3 + i$.

The Laws of Indices

The Laws of Indices

We should know that $x^2 = x \times x$, and $x^3 = x \times x \times x$. The index tells us how many times we are multiplying x by itself.

When we put the x as x^2, we can see that $x^2 \times x^2 = x \times x \times x \times x = x^4$ or $(x^2)^2$.

As we can see, when multiplying $x^p \times x^q = x^{p+q}$.

Also when we have $(x^p)^q = x^{pq}$. Of course we know that $pq = qp$, and ad can also see that $(x^q)^p = (x^p)^q$.

Now let's imagine what we have $x^5 \div x^3 = \frac{x \times x \times x \times x \times x}{x \times x \times x} = x \times x = x^2$.

When we are dividing, then $x^p \div x^q = x^{p-q}$.

Let's say we have $x^{3.5}$. As long as the power is a rational number (in this case $3.5 = \frac{7}{2}$), then we can have an idea on what it is. We can write $x^{\frac{7}{2}}$ as $x^{\frac{1}{2} \times 7}$. This is the same now as $(x^{\frac{1}{2}})^7$.

This shows us our next rule - $x^{\frac{1}{p}} = \sqrt[p]{x}$.

So the above equation can be written as $(\sqrt{x})^7$.

Now let's also consider x^0. If you think about writing this as x^{2-2}, this equals $\frac{x^2}{x^2} = 1$.

Therefore, $x^0 = 1$.

Now we can look at $x^{-1} = x^{4-5} = \frac{x^4}{x^5}$. So from this we get $\frac{x \times x \times x \times x}{x \times x \times x \times x \times x} = \frac{1}{x}$.

This means that $x^{-1} = \frac{1}{x}$.

We have the rule then that $x^{-p} = \frac{1}{x^p}$.

Examples of Negative Indices

Exercise $2^{-3} =$

Exercise $3^{-4} =$

Exercise $5^{-2} =$

Exercise $\left(\frac{1}{4}\right)^{-2} =$

Exercise $\left(\frac{2}{3}\right)^{-3} =$

Examples of Positive Rational Indices

Exercise $36^{\frac{1}{2}} =$

Exercise $81^{\frac{1}{4}} =$

Exercise $\left(\frac{1}{8}\right)^{\frac{1}{3}} =$

Exercise $25^{\frac{3}{2}} =$

Exercise $\left(\frac{8}{27}\right)^{\frac{2}{3}} =$

Examples of Negative Rational Indices

Exercise $8^{-\frac{1}{3}} =$

Exercise $16^{-\frac{3}{4}} =$

Exercise $4^{-\frac{5}{2}} =$

Exercise $\left(\frac{36}{49}\right)^{-\frac{1}{2}} =$

Exercise $\left(\frac{10000}{16}\right)^{-\frac{5}{4}} =$

More Complicated Examples

Example

$2^3 \times 8^{-\frac{5}{3}} \times \frac{1}{\sqrt{2}} = 2^k$. Find k.

For this problem, you want to write everything in terms of 2 to the power of something. We can rewrite this equation as

$$2^3 \times (2^3)^{-\frac{5}{3}} \times 2^{-\frac{1}{2}}$$

So this can be rewritten as $2^3 \times 2^{-5} \times 2^{-\frac{1}{2}}$, and using laws of indices, we can see that this is equivalent to $2^{-\frac{5}{2}}$. So $k = -\frac{5}{2}$.

Exercise Write $\frac{x^2y^5}{\sqrt{x}} \div \frac{x^{\frac{3}{2}}}{y^7}$ as a product of powers of x and y.

Examples of Simplifying Expressions

Exercise $5a^3b^2c \times 6a^8bc^{-3} =$

Exercise $(60a^4b^2c) \div (12a^8b^5c^{-4}) =$

Exercise $\frac{(3x)^3 \times (2x^3)^4}{(6x^8)^2} =$

Write in the form of 2^k

Example

Write $\frac{\sqrt{2}}{4^3}$ in the form 2^k.

We can rewrite $\sqrt{2} = 2^{\frac{1}{2}}$ and $4^3 = (2^2)^3 = 2^6$. So now we have $\frac{2^{\frac{1}{2}}}{2^6} = 2^{-\frac{11}{2}}$.

Exercise Write $8^4 \times \frac{2}{\sqrt[3]{16}}$ in terms of 2^k.

Write in the form of 3^k

Example

Write $\sqrt[3]{3} \times \sqrt[3]{9}$ in terms of 3^k.

We can rewrite this as $3^{\frac{1}{3}} \times (3^2)^{\frac{1}{3}} = 3^{\frac{1}{3}} \times 3^{\frac{2}{3}} = 3^1$.

Exercise Write $\frac{\sqrt[5]{27}}{\sqrt{3}} \times 81$ in terms of 3^k.

Write in the form of 4^k

Example

Write $\frac{16}{\sqrt[4]{5}}$ in terms of 4^k.

This can be rewritten as $\frac{4^2}{4^{\frac{1}{5}}}$, so this is equivalent to $4^{\frac{9}{5}}$.

Exercise Rewrite $2 \times \sqrt[3]{16} \times \sqrt[5]{64}$ in terms of 4^k.

Write in the form 5^k

Example

Rewrite $\frac{125}{\sqrt[3]{25}} \times \sqrt{5}$ in terms of 5^k.

We first start off with $\frac{5^3}{(5^2)^{\frac{1}{3}}} \times 5^{\frac{1}{2}}$.

This is equal to $5^{\frac{7}{3}} \times 5^{\frac{1}{2}} = 5^{\frac{17}{6}}$.

Exercise Rewrite $\frac{\sqrt[3]{50}}{\sqrt{625}} \times \sqrt[3]{12.5}$ in terms of 5^k.

2.2 Surds

Simplifying Surds

Introducing Surds and Simplifying Surds

You can quite easily build up a list of what you believe are surds - $\sqrt{1} = 1$, $\sqrt{2}$ and $\sqrt{3}$ are surds, $\sqrt{4} = 2$. If the number under the square root if a square number, then obviously it will not be a surd.

Let's get an example with $\sqrt{8}$. Using our indices knowledge, we can write this as $8^{\frac{1}{2}} = (4 \times 2)^{\frac{1}{2}} = 4^{\frac{1}{2}} \times 2^{\frac{1}{2}}$.

This can be written as $\sqrt{4} \times \sqrt{2} = 2\sqrt{2}$.

Now let's look at $\sqrt{12}$. Now we can write this as $\sqrt{6} \times \sqrt{2}$, but there is no real point in doing this, since none of these can be simplified. We are looking for any square numbers that can go in 12. So we can write this as $\sqrt{4} \times \sqrt{3}$ which is equivalent to $2\sqrt{3}$.

This leads us to the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.

Simplifying Surds Examples

Exercise $\sqrt{18}$

Exercise $\sqrt{200}$

Exercise $\sqrt{48}$

Exercise $\frac{\sqrt{12}}{\sqrt{300}}$

Exercise $\sqrt{24} \times \sqrt{150}$

Adding/Subtracting Surds

Let's start with an example.

If we are given $\sqrt{20} + \sqrt{180}$, we cannot add the two together. In most cases $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$.

We have $\sqrt{4} \times \sqrt{5} + \sqrt{36} \times \sqrt{5}$ when we simplify this. Now we can simplify this as $2\sqrt{5} + 6\sqrt{5}$. Surds can be combined like 'like' terms in algebra. So the answer so this expression is $8\sqrt{5}$.

Here's two more examples to try with the answer given.

Example 1: $\sqrt{63} - \sqrt{28} = \sqrt{7}$

Example 2: $\sqrt{108} + \sqrt{72} = \sqrt{2} + \sqrt{3} + \sqrt{5}$

Example

$\sqrt{3}(\sqrt{2} + 5)$.

Expanding this, we get that $\sqrt{6} + 5\sqrt{3}$. Note that if you can split up a surd, you can also do the reverse and multiply them back together.

Exercise $6(\sqrt{3} + \sqrt{6})$

Exercise $\sqrt{5}(8 - \sqrt{7})$

Exercise $\sqrt{6}(\sqrt{15} - 2\sqrt{2})$

Exercise $\sqrt{12}(\sqrt{50} + 3\sqrt{10})$

Example

$(2 + \sqrt{2})(3 - \sqrt{5})$. This example is simliar to above, but with double brackets instead.

So using a technique of your choice, you should end up with $6 - 2\sqrt{5} + 3\sqrt{2} - \sqrt{10}$.

Exercise $(2 - \sqrt{5})(2 + \sqrt{5})$

Exercise $(3 + \sqrt{2})(2 + \sqrt{3})$

Exercise $(\sqrt{2} + 1)(\sqrt{3} - \sqrt{5})$

Exercise $(2\sqrt{3} + 3\sqrt{5})(2\sqrt{2} - 5\sqrt{3})$

Rationalising the Denominator

In general, we do not want a surd in the denominator of a fraction such as $\frac{1}{\sqrt{2}}$. We can use a technique known as rationalising the denominator. In order to do this, we want to multiply by $1 = \frac{\sqrt{2}}{\sqrt{2}}$. So when we multiply $\frac{1}{\sqrt{2}}$ by this, we get that the answer is $\sqrt{2}2$.

Exercise $\frac{2}{\sqrt{3}}$

Exercise $\frac{10}{\sqrt{5}}$

Exercise $\frac{9}{2\sqrt{3}}$

Example

Let's try rationalising $\frac{1}{1+\sqrt{2}}$

For this, we must use the identity $x^2 - y^2 = (x-y)(x+y)$. If we multiply $(1+\sqrt{2})$ by $(1-\sqrt{2})$, we are able to rationalise the denominator in this case.

So multiplying by $\frac{1-\sqrt{2}}{1-\sqrt{2}}$ all the way simplifies to $-1+\sqrt{2}$, which is the rationalised form of what was given.

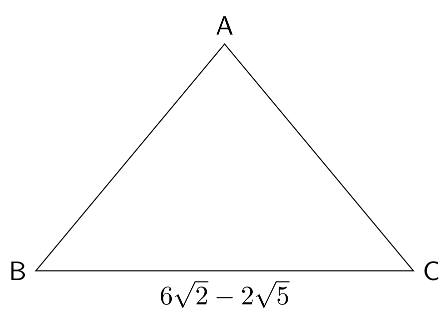
Exercise $\frac{2}{\sqrt{2}+2}$

Exercise $\frac{3}{4-\sqrt{5}}$

Exercise $\frac{1+\sqrt{2}}{3-\sqrt{2}}$

Exercise $\frac{4+2\sqrt{3}}{3+3\sqrt{2}}$

Challenge Problems

1. 

   $\triangle ABC$ has area 5. Find the exact perpendicular height of the triangle.

2. Rationalise the denominator of $\frac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$

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SOHCAHTOA

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