AP Physics C: Mechanics Notes

anastasia

Spring 2025

Contents

1 Kinematics

1.1 Scalars and Vectors

Scalars are quantities described by magnitude only, vectors are quantities described by both magnitude and direction.

Vectors can be visually modeled as arrows with appropriate direction and lengths proportional to their magnitudes.

Vectors can be expressed in unit vector notation or as a magnitude and a direction.

- Unit vector notation can be used to represent vectors as the sum of their constituent components in the x, y, and z directions, denoted by $\hat{i}, \hat{j}$ and $\hat{k}$.

$$\vec{r} = A\hat{i} + B\hat{j} + C\hat{k}$$

- The position vector of a point is given by $\vec{r}$ and the unit vector in the direction of the position vector is denoted $\hat{r}$.
- A resultant vector is the vector sum of the addend vectors' components.

$$\vec{C} = \vec{A} + \vec{B} = (A_x + B_y)\hat{i} + (A_y + B_y)\hat{j}$$

In a given one-dimensional coordinate system, opposite directions are denoted by opposite signs.

Example

If $\vec{a} = 3\hat{i} + 4\hat{j} - \hat{k}$ and $\vec{b} = -2\hat{i} + \hat{y} + 2\hat{k}$, what is (a) $\vec{c} = \vec{a} + \vec{b}$

$\vec{c} = (3 + -2)\hat{i} + (4 + 1)\hat{j} + (-1 + 2)\hat{k} = \hat{i} + 5\hat{j} + \hat{k}$

(b) $\vec{c} = \vec{a} - \vec{b}$

$\vec{c} = (3 + 2)\hat{i} + (4 - 1)\hat{j} + (-1 - 2)\hat{k} = 5\hat{i} + 3\hat{j} - 3\hat{k}$

(c) $\vec{c} = \vec{b} - \vec{a}$

$\vec{c} = (-2 - 3)\hat{i} + (1 - 4)\hat{j} + (2 + 1)\hat{k} = -5\hat{i} - 3\hat{j} + 3\hat{k}$

Exercise An object moves in the xy-plane a distance A at an angle θ measured counterclockwise from the positive x-direction, where $0 < \theta < 90°$. The object then moves a distance B in the positive x-direction. The change in the x-component of the object's position is equal to the change in the y-component of its positions. What is B in terms of A and θ?

Exercise An object is moving with an initial velocity $\vec{v}_1 = (3.00\hat{i} + 4.00\hat{j})$ m/s. After a certain time interval, its velocity is $\vec{v}_2 = (-8.00\hat{i} + 15.0\hat{j})$ m/s. What is the magnitude of the change in the velocity of the object over this time interval?

1.2 Displacement, Velocity, and Acceleration

When using the object model, the size, shape and internal configuration are ignored.

- The object may be treated as a single point with extensive properties such as mass and charge.

Displacement is the change in an object's position: $\Delta x = x - x_0$

Averages of velocity and acceleration are calculated considering the initial and final states of an object over an interval of time.

Average velocity is the displacement of an object divided by the interval of time in which that displacement occurs:

$$\vec{v}_{avg} = \frac{\Delta \vec{x}}{\Delta t}$$

Average acceleration is the change in velocity divided by the interval of time in which that change in velocity occurs.

$$\vec{a}_{avg} = \frac{\Delta \vec{v}}{\Delta t}$$

As the time interval used to calculate the average value of a quantity approaches zero, the average value of that quantity approaches the value of the quantity that is instant, called the instantaneous value.

- $\vec{v} = \frac{\mathrm{d}x}{\mathrm{d}t}$
- $\vec{a} = \frac{\mathrm{d}v}{\mathrm{d}t}$

Time dependent functions and instantaneous values of position, velocity and acceleration can be determined using differentiation and integration.

Example

A particle moves along the x-axis with an acceleration of $a = 18t$, where a has units of m/s^2. If the particle at $t = 0$ is at the origin with a velocity of -12 m/s, what is its position at $t = 4.0$ s?

The velocity is the integral of acceleration: $v = \int 18t\mathrm{d}t = 9t^2 + C$. Substituting the initial conditions gives $v(0) = -12$.

Integrating the velocity function: $\int 9t^2 - 12\mathrm{d}t = 3t^3 - 12t + C$.

Since we know $x(0) = 0$, plug this in and we find that $C = 0$. Therefore, $x(4) = 144$ m.

Exercise An object moves in one dimension along the x-axis. At time $t = 0$, the object is located at position $x = 1$ m and has a velocity of $v = 1$ m/s. The object's acceleration varies as $a = 3t$, where a is in m/s^2 and t is time in seconds. A student incorrectly derives the equation for the object's position as $x = \frac{1}{2}t^3 + 1$ where x is in meters. What is a possible error that could have resulted in the incorrect equation?

Exercise Two objects, Object 1 and Object 2, have velocities $\vec{v}_1 = (3t^2\hat{\imath} + 5t\hat{\jmath})$ m/s and $\vec{v}_2 = (5t^2\hat{\imath} - 3t\hat{\jmath})$ m/s, respectively, where t is time in seconds. What is the relationship between the magnitudes of the acceleration a_1 of Object 1 at $t = 1$ s and the acceleration a_2 of Object 2 at $t = 1$ s?

1.3 Representing Motion

Motion can be represented by motion diagrams, figures, graphs, equations and narrative descriptions.

For constant acceleration, three kinematics equations can used to describe the instantaneous linear motion in one dimension:

- $v = v_0 + at$
- $x = x_0 + v_0 t + \frac{1}{2}at^2$
- $v^2 = v_0^2 + 2a\Delta x$

Near the surface of the Earth, the vertical acceleration caused by the force of gravity is downward, constant and has a measured value of $g = 9.8$ m/s^2 or $g = 10$ m/s^2.

Graphs of position, velocity and acceleration as functions of time can be used to find the relationships between those quantities.

Example

A large cat, running at a constant velocity of 5.0 m/s in the positive x direction, runs past a small dog that is initially at rest. Just as the cat passes the dog, the dog begins accelerating at 0.5 m/s^2 in the positive x direction. (a) How much time passes before the dog catches up to the cat?

We know from the formula $x = v_0 t + \frac{1}{2}at^2$ that the cat will have position $5t$ and the dog $0.25t^2$.

Setting these equal to each other gives $t = 20$ s.

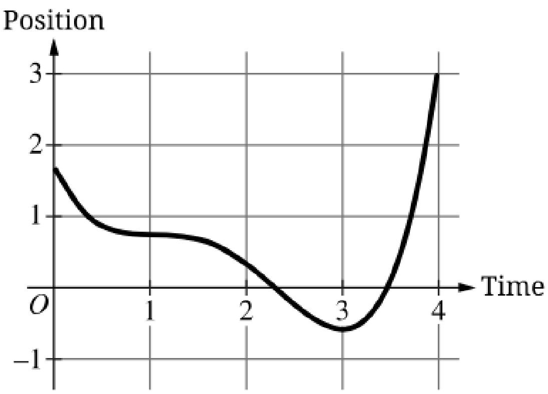
(b) How far has the dog traveled at this point?

Plug in $t = 20$ s to get 100 m.

(c) How fast is the dog traveling at this point?

The formula $v = v_0 + at$ can be used to get 10 m/s.

Exercise



An object moves in one dimension along the x-axis as described by the position versus time graph shown in the figure. During the time interval of the graph, how many times does the object change direction and what feature or features of the graph justifies this response?

Exercise On a distant planet where the acceleration due to gravity is g_P, an object takes a time t_P to reach the ground when dropped from a height h_0. On a small moon of the planet, the acceleration due to gravity is $\frac{g_P}{16}$. How long does it take an object to reach the ground when it is dropped from the same height on the moon?

1.4 Reference Frames and Relative Motion

The choice of reference frame will determine the direction and magnitude of quantities measured by an observer in that reference frame.

Measurements from a given reference frame may be converted to measurements from another reference frame.

The observed velocity of an object results from the combination of the object's velocity and the velocity of the observer's reference frame.

- Combining the motion of an object and the motion of an observer in a given reference frame involves the addition or subtraction of vectors.
- The acceleration of any object is the same as measured from all inertial reference frames.

Example

A cat passes a dog, traveling in the positive x-direction at 5.0 m/s. As the cat passes, the dog begins accelerating at 0.5 m/s^2 in the positive y-direction.

(a) What is the cat's acceleration relative to the dog?

The acceleration of the cat relative to the ground is $0\hat{i}$ and the acceleration of the dog relative to the ground is $0.5\hat{j}$ so the acceleration of the cat relative to the dog is $0\hat{i} - 0.5\hat{j}$.

(b) What is the cat's velocity relative to the dog at time $t = 5.0$ seconds after the dog begins running?

We have $v = v_{cat} + a_{CD}t = (5\hat{i} - 2.5\hat{j})$ m/s

(c) What is the cat's position relative to the dog at $t = 5.0$ seconds after the dog begins running?

From $\Delta r = v_{CD}t + \frac{1}{2}a_{CD}t^2$, we get that $\Delta r = 25\hat{i} - 6.25\hat{j}$.

Exercise A toy plane which maintains an airspeed v_p flies between points A and B in a time t_0 when there is negligible wind. When the air is moving at a velocity of $\frac{1}{2}v_p$ from point B to point A, the toy plane can make the trip from A to B in t_{AB}, and the return trip from B to A in t_{BA}. How do the three travel times compare?

Exercise Car A is traveling east with a speed of 30 m/s. Car B is traveling north with speed of 40 m/s. What is the direction of the velocity of Car B relative to Car A?

1.5 Motion in Two or Three Dimensions

Motion in two or three dimensions can be analyzed using one-dimensional kinematic relationships if the motion is separated into componenets.

Velocity and acceleration may be different in each dimension and be nonuniform.

Motion in one dimension may be changed without causing a change in the perpendicular dimension.

Projectile motion is a special case of two-dimensional motion that has zero acceleration in one dimension and constant, nonzero acceleration in the second dimension.

Example

The motin of an object can be described by the equations

- $x(t) = 4t^2 - 3t$
- $y(t) = 3t^3 - 2t^2 - 9t$

(a) What is the objects displacement after 2.5 s?

Plug in $t = 2.5$ for both equations to get $\Delta\vec{r} = (17.5\hat{i} + 11.9\hat{j})$ m.

(b) Find the two equations that describe the object's velocity in the x and y directions.

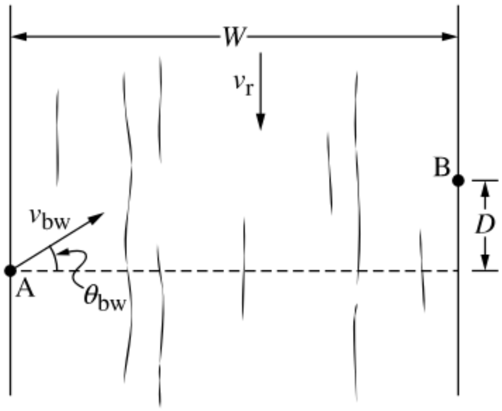
Take the derivative of both equations to get $v_x = 8t - 3$ and $v_y = 9t^2 - 4t - 9$.

(c) What is the object's velocity after 2.5 s?

Plug in 2.5 to both equations from part (b) to get $v = (17\hat{i} + 37.25\hat{j})$ m/s.

Exercise Object 1 is launched at an initial speed v_0 at an angle θ above the horizontal and reaches a maximum height of y_1. Object 2 is launched at an initial speed $2v_0$ at the same angle θ, reaching a maximum height of y_2. What is the relationship between y_1 and y_2?

Exercise



A river has width W and a current v_r. A boat maintains a constant velocity as it travels from Point A to Point B. Point B is located at a distance D upstream from point A. The boat's water speed is v_{bw} with a heading of angle θ_{bw}, as shown in the figure. What is a correct expression for D?

2 Force and Translational Dynamics

2.1 Systems and Center of Mass

- System properties are determined by the interactions between objects within the system.
- If the properties or interactions of the constituent objects within a system are not important in modeling the behavior of a macroscopic system, the system can itself be treated as a single object.
- Systems may allow interactions between constituent parts of the system and the environment, which may result in the transfer of energy or mass.
- For objects with symmetrical mass distributions, the center of mass is located on lines of symmetry.
- The location of a system's center of mass along a given axis can be calculated using the equation:

$$x_{cm} = \frac{\sum m_i x_i}{M}$$

- For a nonuniform solid that can be considered as a collection of differential masses, $\mathrm{d}m$, the solid's center of mass can be calculated using

$$x_{cm} = \int x\mathrm{d}m/M$$

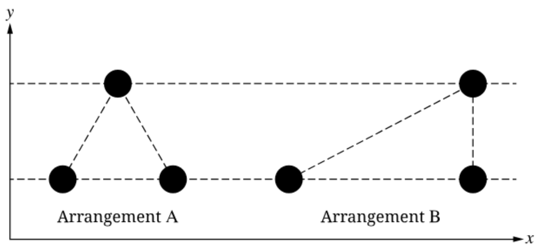
- A system can be modeled as a singular object that is located at the system's center of mass.

Example

A triangular rod of length L and mass M has a nonuniform linear mass density given by the equation $\lambda = \gamma x^2$ where $\gamma = \frac{3M}{L^2}$ and x is the distance from the left end of the rod. Determine the horizontal location of the center of the mass relative to point P. Express your answer in terms of L.

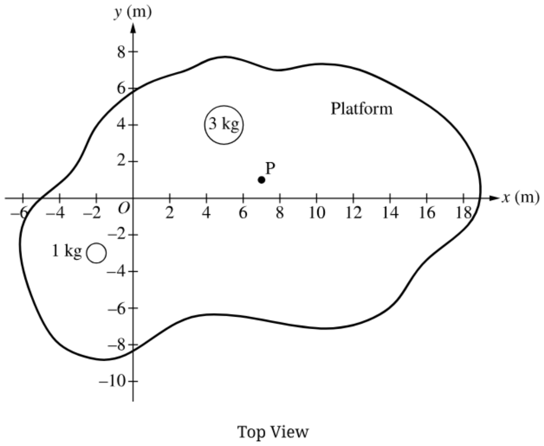
From $\lambda = \frac{\mathrm{d}m}{\mathrm{d}l}$ we know that $\mathrm{d}m = \lambda \mathrm{d}l$. Plugging this into the center of mass formula $x_{cm} = \frac{\int \mathrm{d}m}{M}$ gives us $x_{cm} = \frac{\gamma \int_0^L x^3 \mathrm{d}x}{M} = \frac{3}{4}L$.

Exercise



Six identical uniform spheres are arranged on a set of coordinate axes in two different triangular arrangements, A and B, as shown. How does the y-coordinate of the center of mass of the three spheres in arrangement A, $y_{cm,A}$ compare to the y-coordinate of the center of mass of the three spheres in arrangement B, $y_{cm,B}$?

Exercise

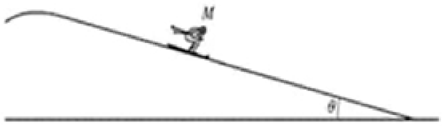


The center of mass of an irregularly shaped platform is balanced on a pivot Point P with coordinates (7.0 m, 1.0 m). Two rocks are then placed on top of the platform, as shown in the top view. One rock has a mass of 1.0 kg and is located at (-2.0 m, -3.0 m), and the second rock has a mass of 3.0 kg and is located at (5.0 m, 4.0 m). At what coordinates should a third rock of mass 4.0 kg be placed such that the three rock-platform system is balanced.

2.2 Forces and Free-Body Diagrams

- Forces are vector quantitites that describe interactions between objects or systems.
- Contact forces describe the interaction of an object or system touching another object or system.
- Free-body diagrams (FBDs) are useful tools for visualizing forces exerted on a single object or system and for determining the equations that represent a physical situation.
- The FBD of an object or system shows each of the forces exerted on the object or system by the environment.
- Forces exerted on an object or system are represented as vector originating from the center of mass, such as a dot.
- Choose a coordinate system such that one axis is parallel to the acceleration of the object or system.

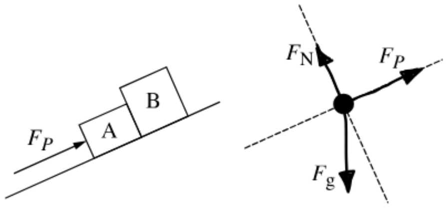
Example



A skier of mass M is skiing down a frictionless hill that makes an angle θ with the horizontal, as shown in the diagram. The skier starts from rest at time $t = 0$ and is subject to a velocity-dependent drag force due to air resistance of the form $F = -bv$, where v is the velocity of the skier and b is a positive constant. Express all algebraic answer in terms of M, b, θ, and fundamental constants. Draw a dot that represents the skier, and draw a free-body diagram indicating and labeling all of the forces that act on the skier while the skier descends the hill.

The correct answer will be $F_g = mg$ pointing downwards, a normal force an angle and the force $-bv$ perpendicular to this force.

Exercise



Two different blocks, A and B, are next to each other on an inclined smooth surface which has negligible friction. An applies force, F_P, pushes Block A as shown and the blocks move up the ramp. A student sketch of the free-body diagram representing the forces is given. What changes should be made to this free-body diagram?

Exercise A block is at rest on a desk's horizontal surface. A student correctly identifies the force exerted on the block as the force of Earth on the block and the force of the desk on the block. A book then is placed between the block and the desk. Which objects exert forces of equal magnitude on the block after the book has been introduced?

2.3 Newton's Third Law

Newton's third law describes the interaction of two objects or systems in terms of the paired forces that exerts on the other.

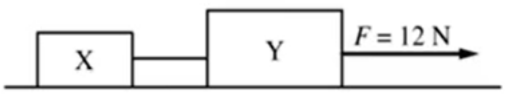
$$\vec{F}_{\text{A on B}} = -\vec{F}_{\text{B on A}}$$

Interactions between objects within a system do not influence the motion of a system's center of mass.

Tension is the macroscopic net results of forces that infinitesimal segments of a string, cable, chain or similar systme exert on each other in response to an external force.

- An ideal string has negligible mass and does not stretch when under tension.
- The tension in an ideal string is the same at all points within the string.
- In a string with nonneglibible mass, tension may not be the same at all points within the string.
- An ideal pulley that has negligible mass and rotates about an axle through its center of mass with negligible friction.

Example



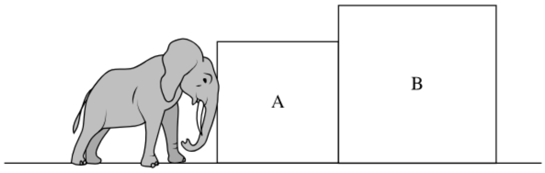
Blocks X and Y of masses 3.0 kg and 5.0 kg, respectively, are connected by a light string and are both on a level horizontal surface of negligible friction. A force $F = 12$ N is exerted on Block Y, as shown in the figure above. What is the tension in the string connecting the two blocks?

After drawing a free body diagram, we see that the $\sum F_x = ma_x$ and we can find that $a_x = 1.5$ m/s^2.

We alsk now that $F_T = ma_x$, so $F_T = 4.5$ N.

Exercise A cart moving to the right collides with a stationary block, resulting in the two objects sliding together along the horizontal surface until coming to a stop. During the collision, the cart exerts a force F_1 on the block, the surface exerts a force of friction F_2 on the block, and the block exerts a force F_3 on the cart. Which two forces are equal during the collision?

Exercise



An elephant pushes two heavy boxes across a rough surface. The force that Box A exerts on Box B is F_{AB} and the force that Box B exerts on Box A is F_{BA}. What must be true of the two boxes to support that $|F_{AB}| = |F_{BA}|$?

2.4 Newton's First Law

The net force on a system is the vector sum of all forces exerted on the system.

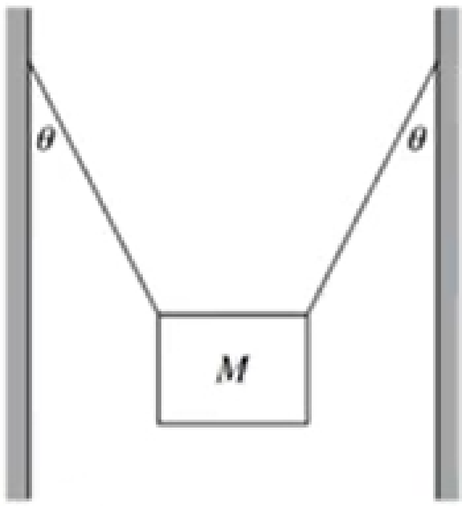
Translational equilibrium is the configuration of forces that the net force exerted on a system is zero.

$$\sum F = 0$$

Newton's first law states that if the net force exerted on a system is zero, the velocity of that system will remain constant.

Forces may be balanced in one dimension but unbalanced in another.

Example



A heavy sign of mass M is held at rest by two supporting wires between two buildings, with each wire making an angle θ with the vertical, as shown in the figure. What is the tension in each wire?

Drawing the free body diagram of the system results in the following:

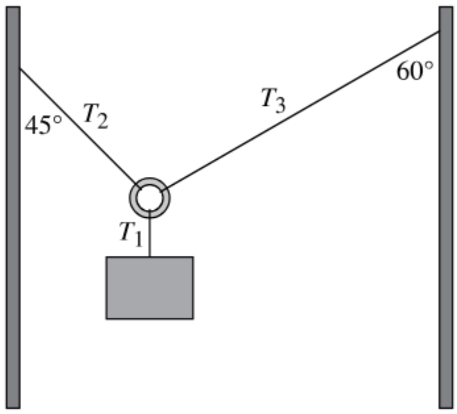
In the x-direction, we get $T = T$.

In the y-direction we get $2T\cos\theta = Mg$.

Solving this for T gives $T = \frac{Mg}{2\cos\theta}$

Exercise An object is moving while a constant force is exerted on it. Could the addition of a force of the same magnitude cause the object to move with a constant velocity? Why or why not?

Exercise



A heavy block is suspended by a string which is attached to a plastic ring. The ring is attached to two other strings which are tied to vertical supports at the angles shown. The masses of the ring and strings are negligible. Compare the magnitudes of the tensions in the strings T_1, T_2, and T_3.

2.5 Newton's Second Law

Unbalanced forces are a configuration of forces such that the net force exerted on a system is not equal to zero.

Newton's second law of motion states that the acceleration of a system's center of mass has a magnitude proportional to the magnitude of the net force exerted on the system and is in the same direction of the force.

$$\sum F = ma = 0$$

The velocity of a system's center of mass will only change if a nonzero net external force is exerted on that system/

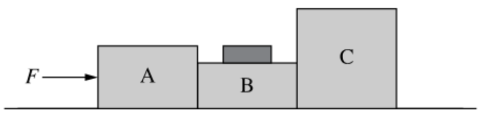
Example

An object of mass 10 kg starts from rest at time $t = 0$ and moves in a straight line. For time $t > 0$, the object's velocity as a function of time t is given by $v = 2t + 3t^2$, where v is in m/s and t is in seconds. What is the instantaneous net force that acts on the object at $t = 2$ s?

The acceleration function is given by $2 + 6t$, so $a(2) = 14$ m/s^2.

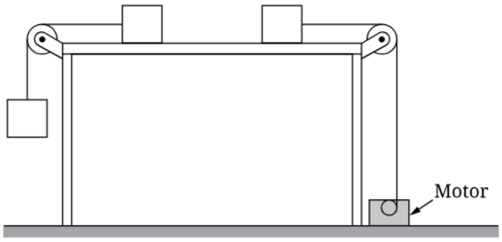
$F = ma$, so plugging in numbers gives 140 N.

Exercise



Three large blocks, A, B, and C, and a small block attached to Block B slide across a horizontal surface as a constant force F is exerted on Block A, as shown in the figure. There is negligible friction between the blocks and the horizontal surface. Block A pushes Block B with a force F_{AB}. The small block is then removed from Block B and attached to Block C and the same force F is exerted on Block A. How does F_{AB} compare in the second situation to the first situation and why?

Exercise



Two identical blocks are placed on a table as shown in the figure. The block on the left is attached to another identical block hanging over the edge of the table. The block on the right is attached to a motor pulling downward with a constant tension equal to the weight of one block. The mass of the strings and friction between the blocks and table are negligible and the pulleys are ideal. How do the magnitudes of the acceleration of the blocks compare and why?

2.6 Gravitational Force

Newton's law of universal gravitation describes the gravitational force between two objects as directly proportional to each of their masses and inversely proportional to the square of the distance between their centers.

$$F_G = \frac{Gm_1m_2}{d^2}$$

A field models the effects of a noncontact force exerted on an object at various positions in space.

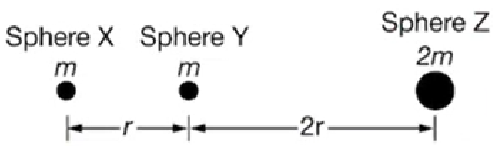
The magnitude of the gravitational field created by a system of mass M at a point in space is equal to the ratio of the gravitational force exerted by the system on a test object of mass m to the mass of the test object.

$$\vec{g} = \frac{\vec{F}_g}{m}$$

If a system is accelerating, the apparent weight of the system is not equal to the magnitude of the gravitational force exerted on the system.

Newton's shell law theorem describes the net gravitational force exerted on an object by a uniform spherical shell of mass.

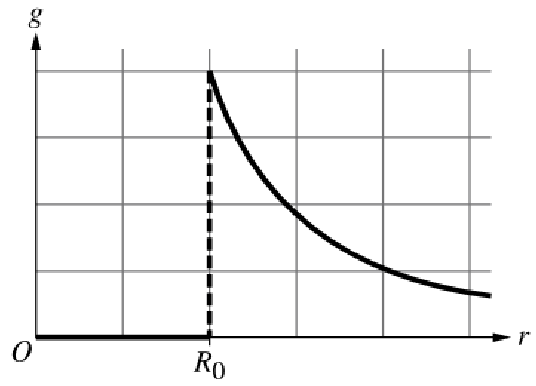
Example



Spheres X, Y, and Z have the masses and locations indicated in the figure above. What is the magnitude of the net gravitational force on sphere X due to the other two spheres?

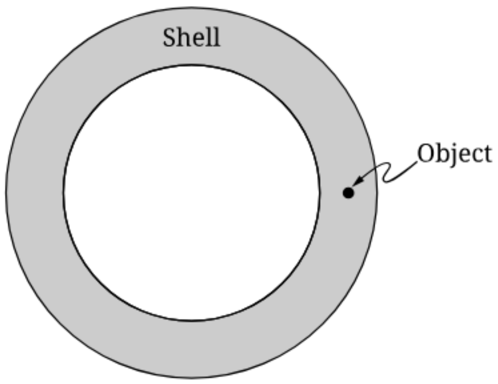
We have that $F_y = \frac{Gm^2}{r^2}$ and $F_z = \frac{1}{2}\frac{Gm^2}{r^2}$ so adding these two together gives $\frac{3}{2}\frac{Gm^2}{r^2}$.

Exercise



The gravitational field g of a spherically symmetric object of radius R_0 as a function of distance r from the object's center is shown in the graph. What best describes the object?

Exercise



A large spherical shell with a uniform mass distribution contains a small object within the thickness of the shell, as shown in the figure. At which locations could the object be moved to increase the magnitude of the gravitational force exerted on the object by the shell?

2.7 Kinetic and Static Friction

Kinetic friction occurs when two surfaces in contact move relative to each other.

- It opposes the direction of motion.
- The surface area of contact is not a factor.

The magnitude of the kinetic friction force exerted on an object is the product of the normal force the surface exerts on the object and the coefficient of kinetic friction.

$$f_k = \mu_k F_N$$

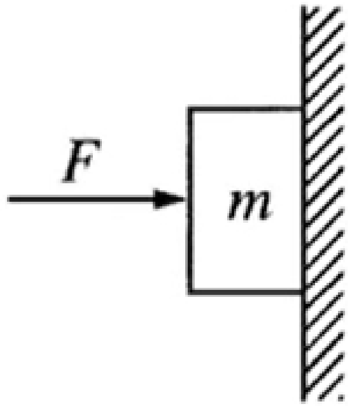
Static friction may occur between the contacting surfaces of two objects that are not moving relative to each other.

Static friction adopts the value and direction required to prevent an object from slipping or sliding on a surface.

$$f_s \leq \mu_s F_N$$

The coefficient of static friction is typically greater than the coefficient of kinetic friction for a given pair of surfaces.

Example

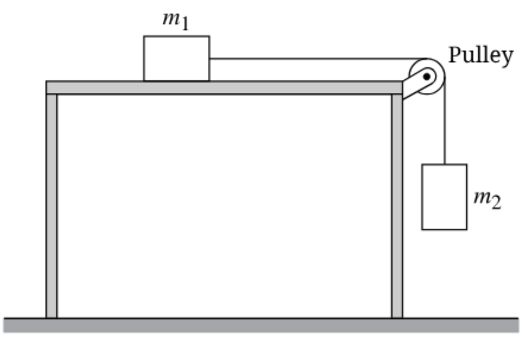


A horizontal force F pushes a block of mass m against a vertical wall. The coefficient of friction between the block and the wall is μ. What value of F is necessary to keep the block from slipping down the wall?

In the x direction the forces result in $F_N = F$.

In the y direction the forces end up with $f = F_g$ or $\mu F_N = mg = \mu F = mg$. The force is therefore $F = \frac{mg}{\mu}$.

Exercise



A block of mass m_1 rests on a rough horizontal tabletop, as shown in the figure. The block is connected by a string to a second block of mass m_2, which hangs below a pulley at the edge of the table. The coefficient of static friction between the tabletop and the first block is μ_s. The masses of the string and the pulley are negligible, and the pulley can rotate with negligible friction on its axle. What is the minimum mass m_2 that will cause the blocks to start moving?

Exercise A rectangular block is pushed by a constant force and accelerates along a rough horizontal surface. The block can be oriented to slide along any of three different sides, A, B, and C. Sides A, B, and C have surface areas S_A, S_B, and S_C, respectively where $S_A < S_B < S_C$. On which side should the block be placed to have the greatest magnitude of acceleration?

2.8 Spring Forces

An ideal spring has negligible mass and exerts a force that is proportional to the change in its length as measured from its relaxed length.

The magnitude of the force exerted by an ideal spring on an object is given by Hooke's Law:

$$F_{sp} = -k\Delta x$$

The force exerted on an object by a spring is always directed toward the equilibrium position of the object-spring system.

A collection of springs that exert forces on an object may behave as though they were a single spring with an equivalent spring constant.

- Springs in series: $\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots$
- Springs in parallel: $k_{eff} = k_1 + k_2 + \ldots$

Example

To illustrate a human soft tissue deformation, a science teacher uses two ideal springs and a small sphere. The sphere of mass m_s is attached to the free ends of the two springs. Then, the system is suspended vertically. The upper string has an equilibrium L_u and a spring constant k_u. The lower spring has an equilibrium length L_l and a spring constant k_l. The teacher fixes an additional small block of mass m_b to the free end of the lower spring. Find the expression of the system's total length.

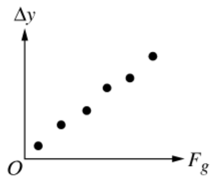
The upper string is given as $F_{sp} = k_u \Delta x_u$. Plugging in total mass and gravity we get $(m_b + m_s)g = k_u \Delta x_u$. Solving for Δx_u gives $\Delta x_u = \frac{(m_b+m_s)g}{k_u}$

The lower string is given by a similar approach and gives us $\Delta x_l = \frac{m_b g}{k_l}$.

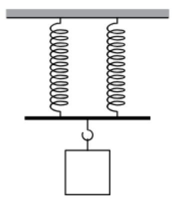
The total length is therefore $L_T = L_l + L_u + \frac{(m_b+m_s)g}{k_u} + \frac{m_b g}{k_l}$.

Exercise When a block of mass M is hung vertically from a spring, the spring is stretched by a distance D compared to its unstretched length. If a second identical spring is connected in series with the first spring and a larger block of mass $2M$ is then hung vertically from the two-spring combination, by how much is the combination stretched compared to its unstretched length?

Exercise



Data for Single Spring



Two-Spring Arrangement

Some students attach a single spring to a clamp and let the spring hang vertically. Objects of different mass are attached to the free end of the spring and allowed to hang at rest. The students measure the distance Δy the spring stretches from its equilibrium length for each object. The students produce the graph of Δy as a function of the weight F_g of the objects shown in the figure, and the slope of the best-fit line to the data is determined to be S_1. Next, the students take a second spring that is identical to the first and arrange the two springs as shown in the two-spring arrangement next to the graph. Once again, the objects of different mass are attached to the two-spring arrangement, Δy is measured, and the data is plotted on another graph showing Δy as a function of F_g. What best describes the slope of the best-fit line to the data collected for the two-spring arrangement?

2.9 Resistive Forces

A resistive force is defined as a velocity-dependent force in the opposite direction of an object's velocity.

$$F_R = -kv[F_R = -bv^2]$$

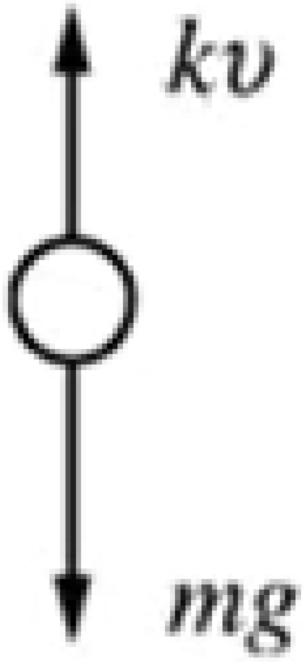
Applying Newton's second law to an object upon which a resistive force is exerted results in a differential equation for velocity.

- The differential portion of a=the equation comes from substituting in $a = \frac{\mathrm{d}v}{\mathrm{d}t}$

Terminal velocity is defined as the maximum speed achieved by an object moving under the influence of a constant force and a resistive force that are exerted on the object in opposite directions.

- For a falling object, this occurs when the air resistance equals the weight of the object.

Example



The object of mass m shown above is dropped from rest near Earth's surface and experiences a resistive force of magnitude kv, where v is the speed of the object and k is a constant. Derive an expression for the velocity of the object at any point in time. (Assume that the direction of the gravitational force is positive.)

We have that $\Delta F = ma$ so we have $mg - kv = ma$. We also have $mg - kv = m\frac{\mathrm{d}v}{\mathrm{d}t}$ as well as $mg - kv_T = 0$, so $v_T = \frac{mg}{k}$.

From $mg - kv = m\frac{\mathrm{d}v}{\mathrm{d}t}$ we can simplify this to $\int_0^t \mathrm{d}t = \int_0^{v(t)} \frac{\mathrm{d}v}{g - \frac{kv}{m}}$.

Solving this gives $t = -\frac{m}{k}\ln(1 - \frac{kv}{mg})$.

Simplifying for $v(t)$ gives $v(t) = \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right)$.

Exercise An object is released from rest and falls to the ground near Earth's surface. The resistive force exerted on the object is directly proportional to the speed of the object which results in a velocity function which includes the term $e^{-\frac{t}{\beta}}$, where β is a positive constant. What best describes the motion of the object if it falls for a time equal to β?

Exercise Two spheres, A and B, of identical size and surface material, but different masses, are dropped from rest near the surface of Earth. While falling, each sphere experiences a resistive force which is proportional to the sphere's velocity. What are the relationships of the magnitude of the initial acceleration a_0 of each sphere and of the terminal speed v_T of each sphere if $m_A < m_B$?

2.10 Circular Motion

Centripetal acceleration is the component of an object's acceleration directed toward the center of the object's circular path.

- The magnitude of the acceleration for an object moving in a circular path is the ratio of the object's tangential speed squared to the radius of the circular path.

$$a_c = v^2/r$$

Centripetal acceleration can result from a single force, more than one force, or components of forces that are exerted on an object in circular motion.

Tangential acceleration is the rate at which an object's speed changes and is directed tangent to the object's circular path.

$$a = \sqrt{a_c^2 + a_T^2}$$

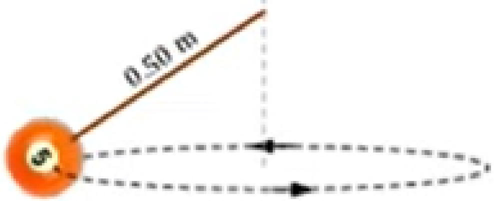
The net acceleration of an object moving in a circle is the vector sum of the centripetal acceleration and tangential acceleration.

The revolution of an object traveling in a circular path at a constant speed (UCM) can be described using period and frequency.

$$v = \frac{2\pi r}{T} = 2\pi r f \qquad T = \frac{1}{f}$$

Example

A billiard ball (mass $m = 0.150$ kg) is attached to a light string that is 0.50 meters long and swung so that it travels in a horizontal, circular path of radius 0.40 m, as shown.



a. On the diagram, draw a free-body diagram of the forces acting on the billiard ball.

There will be a force T in the direction of the string, a_c pointing right from the billiard ball and F_g pointing downwards.

b. Calculate the force of tension in the string as the ball swings in a horizontal circle.

We know that $T \sin\theta = F_g$. From this we can determine that $T = 2.5$ N.

c. Determine the magnitude of the centripetal acceleration of the ball as it travels in the horizontal circle.

We know that $T \cos\theta = ma_c$, so solving for a_c gives us 13.3 m/s^2.

d. Calculate the period T (time for one revolution) of the ball's motion.

We know that $a_c = \frac{v^2}{r}$ so we can find that that $v =$ 2.30 m/s. We also know that $v = \frac{2\pi r}{T}$, so solving for T gives 1.15.

Exercise An object of mass m is attached to the end of a spring. The string is spun around in a vertical circle of radius r. When the object is at the top of its path, the speed of the object is v and the string has a tension F_T. Write an expression for v at the top of the circular path.

Exercise Two small blocks, P and Q rotate without slipping on a horizontal disk with Block P being twice as far from the rotational axis of the disk as Block Q. The blocks are made of the same material and Block P is half the mass of Block Q. As the disk increases in speed, which block will be the first to begin to slide on the disk's surface?

3 Work, Energy and Power

3.1 Translational Kinetic Energy

An object's translational kinetic energy is given by the equation

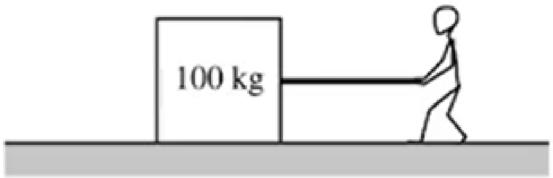
$$K = \frac{1}{2}mv^2$$

Translational kinetic energy is a scalar quantity.

Different observers may measure different values of the translational kinetic energy of an object, depending on the observer's frame of reference.

Example

A 100 kg box shown is being pulled along the x-axis by a student. The box slides across a rough surface, and its position x varies with time according to the equation $x = 0.5t^3 + 2t$, where x is in meters and t is in seconds.



(a) Determine the speed of the box at time $t = 0$. The derivative of the position function is $1.5t^2 + 2$, so $v(0) = 2$ m/s

(b) Determine the kinetic energy of the box as a function of time.

We know that $K = \frac{1}{2}mv^2$. Plugging in m and v, we get $K(t) = 50(1.5t^2 + 2)^2$.

Exercise Two identical blocks, Block A and Block B, slide across a horizontal surface. Block A has a speed v, and kinetic energy K_A. Block B has a speed $2v$ and a kinetic energy K_B. What is the ratio $K_A : K_B$ with a correct justification?

Exercise

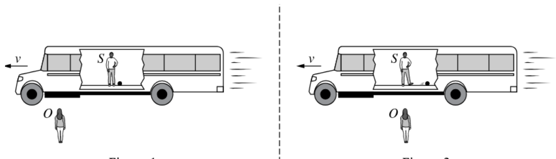


Figure 1 Figure 2

A student stands on a bus moving with a constant speed v to the left as shown in Figure 1. The student S is at rest relative to the bus and a ball sits at rest on the floor of the bus next to the student. Outside the bus standing at rest relative to the ground is an observer O. The kinetic energies of the ball as measured by the student and the observer at this moment are K_{S1} and K_{O1} respectively. As the bus passes the observer, the student kicks the ball toward the back of the bus with a constant speed less than v as shown in Figure 2. The kinetic energies of the ball as measured by the student and the observer after the ball is kicked are K_{S2} and K_{O2}, respectively. How do the kinetic energies measured by the student and observer compare before and after the ball is kicked?

3.2 Work

Work is the amount of energy transferred into or out of a system by a force exerted on that system over a distance.

- The work done by a conservative force is path-independent and only depends on the initial and final configurations of that system.
- The work done by a nonconservative force is path-dependent.

Work is scalar quantity that may be positive, negative or zero.

The work done on an object by a variable force is calculated as

$$W = \int_a^b \vec{F}(r) \cdot \mathrm{d}\vec{r} \qquad [W = Fd\cos\theta]$$

The work-energy theorem states that the change in an object's kinetic energy is equal to the sum of the work being done by all forces exerted on the object.

$$\delta K = W$$

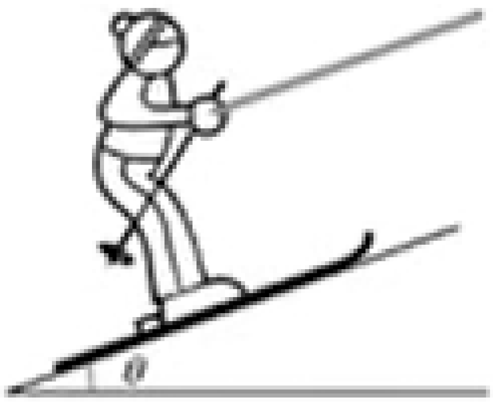
Work is equal to the area under the curve of a graph of F as a function of displacement.

Example

A skier of mass m will be pulled up by a hill by a rope, as shown. The magnitude of the acceleration as a function of time t can be modeled by

- $a = a_{max} \sin \frac{\pi t}{T} (0 < t < T)$
- $a = 0 (t \geq T)$

Where a_{max} and T are constants. The hill is inclined at an angle θ above the horizontal, and friction between the skis and the snow is negligible. Express your answers in terms of given quantities and fundamental constants.



(a) Derive an expression for the velocity of the skier as a function of time during the acceleration. Assume the skier starts from rest.

We have $v = \int a(t)\mathrm{d}t = \int_0^t a_{max} \sin \frac{\pi t}{T} \mathrm{d}t$,

Integrating this and applying the limits of integration give $-\frac{a_{max}T}{\pi}\left(1 - \cos\frac{\pi t}{T}\right)$ for v.

(b) Derive an expression for work done by the net force from the skier from rest until terminal speed is reached.

We know $W = \Delta k$ and that $\Delta k = \frac{1}{2}mv^2$ in this case.

Plugging in the v just derived gives $W = -2a_{max}m\frac{T^2}{\pi^2}$.

Exercise A block of mass m slides with an initial velocity v_0 along a rough surface where the coefficient of kinetic friction between the block and the surface is μ. The box comes to rest after sliding a distance d_0. A new block of unknown mass slides with an initial velocity of $2v_0$ across a surface where the coefficient of kinetic friction between the new block and the surface is $\frac{\mu}{2}$. Write an expression that represents the distance the new block slides before coming to rest in terms of d_0.

Exercise A force F is exerted on an obejct which is initially at rest. The force varies with positions x and can be described by the equation $\vec{F} = (Ax - B)\hat{\imath}$, where A and B are constants with appropriate units. After moving a distance D_0, the block again comes to rest. An identical object, also initially at rest, experiences a force $2F$. The second object comes to rest again after moving a distance D_1. Describe the relationship between D_0 and D_1.

3.3 Potential Energy

A system composed of two or more objects has potential energy if the objects within that system only interact with each other through conservative forces.

Potential energy is a scalar quantity associated with the position of objects within a system.

The definition of zero potential energy for a given system is a decision made by the observer considering the situation to simplify or otherwise assist in analysis.

The relationship between conservative forces exerted on a system and the system's potential energy is

$$\delta U = -\int \vec{F}(r) \cdot \mathrm{d}\vec{r}$$

The conservative forces exerted on a single dimension can be determined using the slope of a system's potential energy with respect to position in that dimension, these forces point in the direction of decreasing potential energy.

$$F_x = -\mathrm{d}u(x)/\mathrm{d}x$$

The potential energy of common physical systems can be described using the physical properties of that system.

3.4 Conservation of Energy

A system that contains objects that interact via conservative forces or that can change its shape reversibly may have both kinetic and potential energies.

Mechanical energy is the sum of a system's kinetic and potential energy.

A system may be selected so that the total energy of the system is constant.

If the total energy of a system changes, that change will be equivalent to the energy transferred into or out of the system.

Energy is conserved in all interactions.

If the work done on a selected system is zero and there are no nonconservative interactions within the system, the total mechanical energy of the system is constant.

If the work done on a selected system is nonzero, the energy is transferred between the system and the environment.

3.5 Power

Power is the rate at which energy changes with respect to time, either by transfer into or out of a system or by converstion from one type to another within the system.

Average power is the amount of energy being transferred or converted, divided by the time it took for that transfer to happen.

The instantaneous power delivered to an object by a force is given by the equation:

$$p_{ins} = \frac{\mathrm{d}E}{\mathrm{d}t}$$

The instantaneous power delivered to an object by the component of a constant force parallel to the object's velocity can be described with the derived equation:

$$p_{ins} = Fv\cos\theta$$

4 Linear Momentum

4.1 Linear Momentum

4.2 Change in Momentum and Impulse

4.3 Conservation of Linear Momentum

4.4 Elastic and Inelastic Collisions

5 Torque and Rotational Dynamics

5.1 Rotational Kinematics

5.2 Connecting Linear and Rotational Motion

5.3 Torque

5.4 Rotational Inertia

5.5 Rotational Equilibrium and Newton's First Law in Rotational Form

5.6 Newton's Second Law in Rotational Form

6 Energy and Momentum of Rotating Systems

6.1 Rotational Kinetic Energy

6.2 Torque and Work

6.3 Angular Momentum and Angular Impulse

6.4 Conservation of Angular Momentum

6.5 Rolling

6.6 Motion of Orbiting Satellites

7 Oscillations

7.1 Defining Simple Harmonic Motion (SHM)

7.2 Frequency and Period of SHM

7.3 Representing and Analyzing SHM

7.4 Energy of Simple Harmonic Oscillators

7.5 Simple and Physical Pendulums