AP Calculus BC Notes

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Contents

1 Infinite Sequences and Series

1.1 Defining Convergent and Divergent Infinite Series

Writing terms of a sequence.

$$a_n = \{1 + (-2)^n\}$$

ends up becoming $-1, 5, -7, 17, -31/$

A sequence is a collection of numbers that are in one-to-one correspondence with positive integers.

A monotonic sequence never decreases or never increases.

- $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$
- $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n$

Bounded Sequences

- $a_n \leq M$ (upper bound/above)
- $a_n \geq N$ (lower bound/below)
- $\{a_n\}$ bounded if both are true

An infinite series is

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n$$

Partial sums are

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

a_n is an expression that gives the nth term in a sequence.

s_n is an expression that gives the sum of the first n terms.

Example

Use the following sequence $2, 4, 6, 8, 10$ to find a_4 and S_4.

The fourth term is $a_4 = 8$.

$S_4 = 2 + 4 + 6 + 8 = 20$.

That leads us to $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$

Definition

For the infinite series $\sum_{n=1}^{\infty} a_n$, the nth partial sum is $S_n = a_1 + a_2 + a_3 + \cdots + a_n$.

If the sequence of the partial sum $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ converges. The limit S is called the sum of the series.

Likewise, if $\{S_n\}$ diverges, then the series diverges.

Example

Does the series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

The first term is $S_1 = \frac{1}{2}$. Then $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. $S_3 = \frac{7}{8}$ and $S_4 = \frac{15}{16}$.

This keeps on getting closer to 1, so the guess is that this converges.

A strategy is to find S_n. This ends up being $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$.

We get $\frac{1}{2}S_n = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^{n+1}}$.

We say that $S_n - \frac{1}{2}S_n$ by subtracting the two.

If we do this, we see that everything cancels all the way until you get $\frac{1}{2} + \frac{1}{2^{n+1}}$.

Simplifying for S_n gives $1 - \frac{1}{2^n}$.

Letting this appraoch infinity gives 1.

Example

Use a calculator to find the partial sum S_n of the series $\sum_{n=1}^{\infty} \frac{10}{n(n+2)}$ for $n = 200, 1000$.

Hint: Use a calculator

Example

Does the series converge or diverge?

$$\sum_{n=1}^{\infty} n$$

If you see this, it keeps getting bigger, so it diverges.

Exercise Given the infinite series $\sum_{n=1}^{\infty}(-1)^n$, find the sequence of partial sums S_1, S_2, S_3, S_4, and S_5. (Answer: $-1, 0, -1, 0, -1$)

Exercise Find the sequence of partial sums S_1, S_2, S_3, S_4, and S_5 for the infinite series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\ldots$. (Answer: $1, \frac{3}{2}, \frac{7}{4}, \frac{23}{12}, \frac{49}{24}$)

Exercise If the infinite series $\sum_{n=1}^{\infty} a_n$ has nth partial sum $S_n = (-1)^{n+1}$ for $n \geq 1$, what is the sum of the series. (Answer: $1, -1, 1, -1$ diverges)

Exercise The infinite series $\sum_{n=1}^{\infty} a_n$ has nth partial sum $S_n = \frac{n}{4n+1}$ for $n \geq 1$. What is the sum of the series? (The limit of the S_n to infinity gives $\frac{1}{4}$)

Exercise Use a calculator to find the partial sum S_n of the series $\sum_{n=1}^{\infty} \frac{6}{n(n+3)}$ for $n = 100, 500, 1000$. (Answer: Trivial)

Exercise Show that the sequence with the given nth term $a_n = 1 + 2n$ is monotonic. (Answer: $3, 5, 7, 9$ is monotic because it is strictly increasing)

Exercise What is the nth partial sum of the infinite series $\sum_{n=1}^{\infty} \frac{1}{2^{n+1}}$. (Answer: Do what was above you get $\frac{1}{2} - \frac{1}{2^{n+1}}$)

Special

Exercise Which of the following could be the nth partial sum for the infinite series $\sum_{n=1}^{\infty} \frac{1}{4^n}$? (Answer: DO same process as above, get $S_n = \frac{1}{3}(1 - \frac{1}{4^n})$) **(A)** $S_n = \frac{1}{3}(1 + \frac{1}{4^n})$ **(B)** $S_n = \frac{1}{3}(1 - \frac{1}{4^{n+1}})$ **(C)** $S_n = \frac{1}{3}(1 - \frac{1}{4^n})$ **(D)** $S_n = \frac{1}{4}(1 - \frac{1}{3^n})$

Exercise If the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent and has a sum of $\frac{7}{8}$, what could be the nth partial sum? (The limit as n goes to infinity for S_n is $\frac{7}{8}$) **(A)** $S_n = \frac{7n+1}{8n^2+1}$ **(B)** $S_n = \frac{7n^2+1}{8n+1}$ **(C)** $S_n =$

$2(\frac{7}{8} - \frac{1}{n+2} - \frac{1}{n+3})$ **(D)** $S_n = (\frac{7}{8} - \frac{1}{n+2} - \frac{1}{n+3})$

Exercise Which of the following sequences with the given nth term is bounded and monotonic? **(A)** $a_n = 2 + (-1)^n$ **(B)** $a_n = \frac{n^2}{n+1}$ **(C)** $a_n = \frac{3n}{n+2}$ **(D)** $a_n = \frac{\cos n}{n}$