Calculus III Notes

anastasia

Fall 2024 (Revised Summer 2025)

Contents

1 Three-Dimensional Coordinate Systems

1.1 Vectors in the Plane

The first goal for calculus 3 is to define the rectangular coordinate plane in 3-space.

In 2-space, this is the normal coordinate axis with coordinates (x, y).

In 3-space, you now have (x, y, z). We have an xy-plane, a yz-plane, and an xz-plane. Also we have octants in 3-space similar to quadrants in 2-space.

The z-axis is determined by the right-hand rule: if you curl the fingers on your right hand from the positive x-axis to the y-axis, your thumb points in the direction of the positive z-axis.

Exercise Graph $(2, 3, 4)$.

When graphing points in 3-space, it might be easier to draw a prism to see the three-dimensional components easier.

Exercise Graph $(-3, 2, -6)$.

In 3-space the distance formula is simliar to 2-space. It is the following:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Example

Find the distance between $(2, 3, 4)$ and $(-3, 2, -6)$.

Plug in the numbers into the formula and the answer gives $d = \sqrt{126}$

Similar to circles in 2-space, there are spheres in 3-space. The equation of a sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Example

Write the equation of a sphere with a diameter having endpoints $(-1, 2, 1)$ and $(0, 2, 3)$.

We can find the center with the midpoint formula, and the center is $(-\frac{1}{2}, 2, 2)$.

Using the $(-1, 2, 1)$ endpoint and the center calculated above, we can determine the radius to be $\sqrt{5/4}$.

Therefore the equation of the sphere is $(x + \frac{1}{2})^2 + (y - 2)^2 + (z - 2)^2 = \frac{5}{4}$.

Example

Find the center and radius of the sphere $x^2 + y^2 + z^2 - 2x - 4y + 8z + 17 = 0$.

When we complete the square for this, we end up getting $(x - 1)^2 + (y - 2)^2 + (z + 4)^2 = 4$.

Therefore, the center is $(1, 2, -4)$ and the radius is 2.

Theorem 1.1

An equation of the form $x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$ represents a sphere, a point, or has no graph.

We would get a point if the radius gives you 0, and if the radius is negative, then there is no graph.

Exercise Graph $y = 3$ in $\mathbb{R}^3$.

Exercise Describe and sketch the surface in $\mathbb{R}^3$ represented by $y = x$.

Exercise Graph $x^2 + z^2 = 1$.

Example

Describe the graph of $1 \leq x^2 + y^2 + z^2 \leq 4$. What if $z \leq 0$?

This will represent the region between the spheres and centered at the origin with radii of 1 and 2. (This is a sphere with center cut out).

The condition $z \leq 0$ will give us a hemisphere, it will only give the lower half of the figure.

1.2 Vectors

Definition: Vector

A vector indicates a quantity that has both magnitude and direction.

Examples include displacement, velocity, or force.

Some ways to notate vectors are **u**, $\overrightarrow{u}$, $\vec{u}$, $\overrightarrow{AB}$, $\vec{AB}$. For the last two of these, these are read as a vector starting at A, heading towards B.

We say that 2 vectors are equivalent (or equal) if they have the same length and direction.

0 vector (**0**) has a length of 0 and no direction. (Note that **0** is a vector because it is bolded.)

Definition: Vector Addition

If **u** and **v** are positioned so that the initial point of **v** is at the terminal point of **u**, then **u** + **v** is the vector from the initial point of **u** to the terminal point of **v**.

Note that $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

Definition: Scalar Multiplication

If c is a scalar and **v** is a vector, then c**v** is the vector whose length is $|c|$ times the length of **v** and whose direction is the same as **v** if $c > 0$ and is opposite if $c < 0$.

Using the two definitions above, we can find the difference **u** − **v**. From the above definitions, we can rewrite this as $\vec{u} + (-\vec{v})$, which is equivalent to $\vec{u} - \vec{v}$.

Exercise Sketch **a** − 2**b** and define **a** and **b** as you wish.

Coordinate Systems: Generally we place the initial point at the origin and then express a vector as **a** $= \langle a_1, a_2\rangle$ or **a** $= \langle a_1, a_2, a_3\rangle$ where (a_1, a_2) and (a_1, a_2, a_3) are terminal points.

Given points $A(x_1, y_1)$ and $B(x_2, y_2)$, vector **a** = **AB** is **a** $= \langle x_2 - x_1, y_2 - y_1\rangle$. (In 3-space the idea is similar.)

Example

Express vector $\overrightarrow{P_1P_2}$ in bracket notation if $P_1(1,3)$ and $P_2(4,-2)$.

Note that we start at the terminal point. Therefore we have $\overrightarrow{P_1P_2} = \langle 4-1, -2-3\rangle = \langle 3,-5\rangle$.

Note that the vector between the two points and the vector found have the same direction and same length.

Arithmetic Operations: If $\vec{v} = \langle v_1, v_2\rangle$ and $\vec{w} = \langle w_1, w_2\rangle$, then

- $\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2\rangle$
- $\vec{v} - \vec{w} = \langle v_1 - w_1, v_2 - w_2\rangle$
- $k\vec{v} = \langle kv_1, kv_2\rangle$

Example

If $\vec{a} = \langle -2, 0, 1\rangle$ and $\vec{b} = \langle 3, 5, -4\rangle$, find $\vec{a} + \vec{b}$ and $\vec{b} - 2\vec{a}$.

Finding $\vec{a} + \vec{b}$ is simple just add them together to get $\langle 1, 5, -3\rangle$.

To find $\vec{b} - 2\vec{a}$, multiply $\vec{a}$ by 2 to get $\langle -4, 0, 2\rangle$. Then subtracting gives you $\langle 7, 5, -6\rangle$.

Properties of Vectors:

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3. $\vec{a} + \mathbf{0} = \vec{a}$
4. $\vec{a} + (-\vec{a}) = \mathbf{0}$ (Additive Inverse)
5. $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
6. $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
7. $(cd)\vec{a} = c(d\vec{a})$
8. $1\vec{a} = \vec{a}$

Example

Prove property #2 from above.

Let $\vec{a} = \langle a_1, a_2\rangle$, $\vec{b} = \langle b_1, b_2\rangle$, and $\vec{c} = \langle c_1, c_2\rangle$.

From the left side of property 2, we have $\langle a_1, a_2\rangle + (\langle b_1, b_2\rangle + \langle c_1, c_2\rangle)$.

From this, we can simplify to get $\langle a_1 + a_2\rangle + \langle b_1 + c_1, b_2 + c_2\rangle$.

This gives us $\langle a_1 + (b_1 + c_1), a_2 + (b_2 + c_2)\rangle$.

From the associative law we can rewrite this as $\langle (a_1 + b_1) + c_1, (a_2 + b_2) + c_2\rangle$.

This is $\langle a_1 + b_1, a_2 + b_2\rangle + \langle c_1 + c_2\rangle$, which is equivalent to $(\vec{a} + \vec{b}) + \vec{c}$. $\square$

Unit Vectors: A unit vector is a vector with a length of 1.

In 2-space, define $\vec{i} = \mathbf{i} = \langle 1, 0\rangle$ and $\vec{j} = \mathbf{j} = \langle 0, 1\rangle$ and 3-space, $\vec{i} = \mathbf{i} = \langle 1, 0, 0\rangle$, $\vec{j} = \mathbf{j} = \langle 0, 1, 0\rangle$, and $\vec{k} = \mathbf{k} = \langle 0, 0, 1\rangle$.

$\vec{i}, \vec{j}$, and $\vec{k}$ are called unit or standard basis vectors. All have length 1 and point in the positive direction on the $x-$,$y-$, and $z-$ axes.

For example, $\vec{a} = \langle 1, 3, -4\rangle$ can be expressed as $\vec{a} = \vec{i} + 3\vec{j} - 4\vec{k}$.

Example

If $\vec{a} = \vec{i} + 2\vec{j} - 3\vec{k}$, and $\vec{b} = 4\vec{i} + 7\vec{k}$, find $2\vec{a} + 3\vec{b}$.

Adding them together gives $14\vec{i} + 4\vec{j} + 15\vec{k}$.

The norm (or magnitude or length) of a vector is defined as

$$|\vec{v}| = \|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$$

Example

Let $\vec{u} = \vec{i} - 3\vec{j} + 2\vec{k}$ and $\vec{v} = \vec{i} + \vec{j}$. Find:

- $\|\vec{u}\| + \|\vec{v}\|$ Use the above formula to get $\sqrt{14} + \sqrt{2}$.
- $\|\vec{u} + \vec{v}\|$. First add the two vectors to get $\langle 2, -2, 2\rangle$. The norm of this is $2\sqrt{3}$.
- $\frac{1}{\|\vec{v}\|}\vec{v}$ We previously found the magnitude of $\vec{v}$ to be $\sqrt{2}$. So we simply have $\frac{1}{\sqrt{2}}\langle 1, 1, 0\rangle$. This is $\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$.
- $\|\frac{1}{\|\vec{v}\|}\vec{v}\|$. The norm of $\langle\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\rangle$ is 1.

The vector found in the third part of the previous example is known as a unit vector because it has a norm of 1.

The process of obtaining a unit vector with the same direction is called normalizing $\vec{v}$.

Example

Find the unit vector in the direction of $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$.

The norm of a is $\|\vec{a}\| = \sqrt{4 + 1 + 4} = 3$. Then normalizing $\vec{a}$ gives $\langle\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\rangle$.

Vectors in Polar Form: Any vector can be written in the following form

$$\vec{v} = \|\vec{v}\|\langle\cos\theta, \sin\theta\rangle$$

We know that $\|\vec{v}\|$ is the magnitude of the vector and $\langle cos\theta, \sin\theta\rangle$ is the direction.

Example

Find the angle that $\vec{v} = \langle -\sqrt{3}, 1\rangle$ makes with the positive x-axis.

We know we can write this as $\langle -\sqrt{3}, 1\rangle = \|\vec{v}\|\langle\cos\theta, \sin\theta\rangle$.

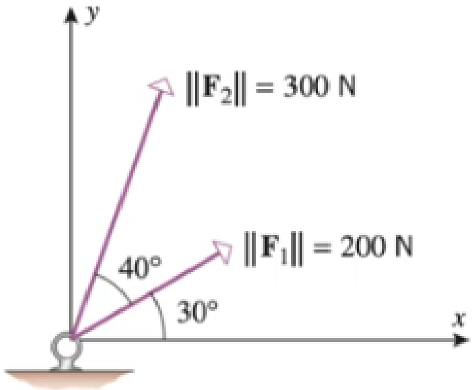
The magnitude of $\vec{v}$ is 2, so simplifying a little gives us $\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}\rangle = \langle\cos\theta, \sin\theta\rangle$.

From this, we can see that θ is the θ when $\cos\theta = -\frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$. So, $\theta = \frac{5\pi}{6}$.

Forces are often represented by vectors because they have a length and direction. If two forces are applied to the same point, they are concurrent. The two forces together form the resultant force, $\vec{F_1} + \vec{F_2}$.

Example

Suppose two forces are applied to an eye bracket. Find the magnitude of the resultant and the angle that it makes with the positive x-axis.



From the diagram, we can see that $\vec{F_1} = 200\langle\cos 30°, \sin 30°\rangle = \langle 100\sqrt{3}, 100\rangle$.

For $\vec{F_2}$, we get $\vec{F_2} = \langle 300\cos 70°, 300\sin 70°\rangle$.

Adding them gives $\vec{F} = \langle 100\sqrt{3} + 300\cos 70°, 100 + 300\sin 70°\rangle$.

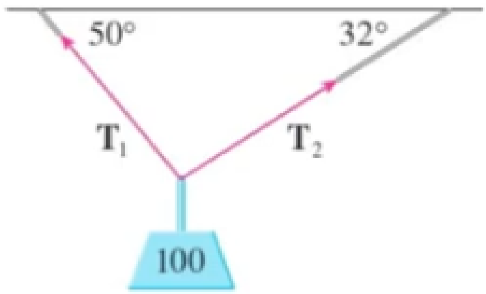
This approximates to $\langle 275.8, 381.9\rangle$. The magnitude of this approximates to $\|\vec{F}\| \approx 471$ N.

If we let $100\sqrt{3} + 300\cos 70°$ be $\|\vec{F}\|\cos\theta$, then we can determine the angle.

When we find θ in $\cos\theta = \frac{100\sqrt{3}+300\cos 70°}{471}$, we get $\theta \approx 54.2°$.

Example

A 100-lb weight hangs from two wires. Find the forces (tensions) $\vec{T_1}$ and $\vec{T_2}$ in both wires and the magnitude of those tensions.



Note that $\vec{T_1}$ does not have an angle of $50°$, rather it has a degree of $130°$ from the point.

Therefore, we have $\vec{T_1} = |\vec{T_1}|\langle\cos 130°, \sin 130°\rangle$.

Note we can rewrite this with to be in the first quadrant as $|\vec{T_1}|\langle -cos50°, \sin 50°\rangle$.

We also have $\vec{T_2} = |\vec{T_2}|\langle\cos 32°, \sin 32°\rangle$.

We also see that the weight of the block is $\vec{W} = 100\langle 0, -1\rangle = \langle 0, -100\rangle$, so we see to balance this out, both tension forces need to equal $\langle 0, 100\rangle$.

Therefore, we see that $-|\vec{T_1}|\cos 50° + |\vec{T_2}|\cos 32° = 0$ (addition of the $\vec{i}$ components).

We also have $|\vec{T_1}\sin 50° + |\vec{T_2}|\sin 32° = 100$ ($\vec{j}$ components).

Solving for $|\vec{T_1}|$ and $|\vec{T_2}|$ from this gives us 85.64 lbs and 64.91 lbs respectively.

Plugging these values back in gives us the tension vectors.

$\vec{T_1} \approx \langle -55.05, 65.60\rangle$

$\vec{T_2} \approx \langle 55.05, 34.40\rangle$

1.3 The Dot Product

Definition: The Dot Product

If $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$, then the dot product $\vec{a}\cdot\vec{b}$ is

$$\vec{a}\cdot\vec{b} = a_1b + 1 + a_2b_2 + a_3b_3$$

This is called the scalar or inner product. (Note: 2-space is a similar idea)

Example

$(\vec{i} + 2\vec{j} - 3\vec{k})\cdot(2\vec{j} - \vec{k})$

Using the dot product formula gives you 7.

Properties of dot products:

1. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$
2. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
3. $\vec{a}\cdot(\vec{b} + \vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$
5. $\vec{0} \cdot \vec{a} = 0$

Example

Prove the first property from above.

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$.

The dot product of $\vec{a}$ and $\vec{a}$ gives us $a_1^2 + a_2^2 + a_3^2$.

The magnitude of this is $\sqrt{a_1^2 + a_2^2 + a_3^2}^2$ which is equal to $|\vec{a}|^2$ $\square$

Example

Prove the third property from above.

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, likewise for $\vec{b}$ and $\vec{c}$.

Then when we do $\vec{a} \cdot (\vec{b} + \vec{c})$ we get

$$\begin{aligned}
&\langle a_1, a_2, a_3 \rangle \cdot (\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle) \\
&= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\
&= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\
&= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\
&= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\
&= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}
\end{aligned}$$

$\square$

Theorem 1.2

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

Corollary 1.3

$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

Example

Find the angle between $\vec{u} = \vec{i} - 2\vec{j} + 2\vec{k}$ and $\vec{v} = -3\vec{i} + 6\vec{j} + 2\vec{k}$.

The dot product of the two gives -11.

The magnitude of $\vec{u}$ is 3 and the magnitude of $\vec{v}$ is 7.

We see that $\cos\theta = -\frac{11}{3 \cdot 7}$, so $\theta = 2.12$ radians or $121.6°$.

Recall, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$. Since $|\vec{a}||\vec{b}|$ is always positive, the sign of the dot product is determined by $\cos\theta$.

- If $\vec{a} \cdot \vec{b} > 0$, then the angle is acute.
- If $\vec{a} \cdot \vec{b} < 0$, then the angle is obtuse.
- If $\vec{a} \cdot \vec{b} = 0$, then the vectors are orthogonal.

To determine if two vectors are parallel, the vectors have to be scalar multiples of each other.

Definition: Direction Angles

The direction angles α, β, and $\gamma([0,\pi])$ are the angles that $\vec{a}$ makes with the positive $x-$, $y-$, and $z-$ axes. Their cosines are called direction cosines.

$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{\Longrightarrow} \cos\alpha = \frac{a_1}{|\vec{a}|}$. Likewise, $\cos\beta = \frac{a_2}{|\vec{a}|}$ and $\cos\gamma = \frac{a_3}{|\vec{a}|}$.

Notice that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Also $\vec{a} = |\vec{a}|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$, which can be expressed as $\frac{\vec{a}}{|\vec{a}|} = \langle\cos\alpha, \cos\beta, \cos\gamma\rangle$.

So, the direction cosines form the unit vector in the direction of $\vec{a}$.

Example

Find the direction cosines of $\vec{a} = \langle 2, -4, 4\rangle$ and approximate the direction angles to the nearest degree.

The magnitude of $\vec{a}$ is 6.

From the formulas, we can find that $\cos\alpha = \frac{1}{3}$, $\cos\beta = -\frac{2}{3}$, and $\cos\gamma = \frac{2}{3}$.

Finding the angles gives us $\alpha \approx 71°$, $\beta \approx 132°$, and $\gamma \approx 48°$.

Example

Find the angle between a diagonal of a cube and one of its edges.

Let's call the vector from the diagonal to the edge as $\vec{d}$ and the length of the edge be a.

We get $\vec{d} = \langle a, a, a\rangle$ as a result. The magnitude of $\vec{a}$ ends up being $\sqrt{3a^2}$.

We get that $\cos\alpha = \frac{a}{\sqrt{3a^2}} = \frac{1}{\sqrt{3}}$.

This approximates $\alpha \approx 0.955$ radians or $54.7°$.

proj$_{\vec{a}}\vec{b}$ is the vector projection of $\vec{b}$ onto $\vec{a}$

comp$_{\vec{a}}\vec{b}$ is the scalar projection of $\vec{b}$ onto $\vec{a}$ (a signed magnitude of the vector projection).

If we look at the scalar projection, comp$_{\vec{a}}\vec{b} = |\vec{b}|\cos\theta$ and remember that $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$. Therefore comp$_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

The projection will just be the magnitude (which is the scalar projection) multiplied by the unit vector: $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\left(\frac{\vec{a}}{|\vec{a}|}\right)$.

This is equal to $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$.

Example

Find the scalar and vector projections of $\vec{b} = \langle 1, 1, 2\rangle$ onto $\vec{a} = \langle -2, 3, 1\rangle$.

The dot product of the vectors is 3, the magnitude of $\vec{a} = \sqrt{14}$.

From the formula above, the scalar projection is $\frac{3}{\sqrt{14}}$.

The vector projection gives $\frac{3}{(\sqrt{14})^2}\langle -2, 3, 1\rangle$, simplifying to $\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\rangle$.

Previously, you learnt work is defined as $W = F\cdot d$ (this only applies when the force is directed along the line of motion).

Now we can define work as $W = (|\vec{F}|\cos\theta)|\vec{d}|$. Moving stuff around gives $W = |\vec{F}||\vec{D}|\cos\theta$.

Now we see that $W = \vec{F}\cdot\vec{d}$. (Work is constant which means it should result in a scalar.)

Example

A wagon is pulled horizontally by exerting a constant force of 10 lb on the handle at an angle of $60°$ with the horizontal. How much work is done in moving the wagon 50 feet?

We can easily see that the displacement vector is $\vec{d} = \langle 50, 0 \rangle$.

The force vector we must divide into components, so we get $\vec{F} = 10\langle \cos 60°, \sin 60° \rangle = \langle 5, 5\sqrt{3} \rangle$.

So the dot product of both vectors gives us 250 ft-lb.

1.4 The Cross Product

To the interested reader that has gotten this far, review how to find the determinant of a matrix. *Exercise* $\begin{vmatrix} 4 & -2 \\ -5 & -1 \end{vmatrix}$

Exercise $\begin{vmatrix} 3 & -1 & 4 \\ 2 & -2 & 5 \\ 4 & -1 & 0 \end{vmatrix}$

Remember: when you see the straight vertical bars, you are trying to find the determinant.

Also remember

- If any two rows are the same, the determinant is zero.
- Interchanging two rows in a determinant multiplies the value by -1.

Definition: Cross Product

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product of $\vec{a}$ and $\vec{b}$ is:

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

OR

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Notice: This is a vector.

Example

$\vec{u} = \langle 1, 2, -2 \rangle$ and $\vec{v} = \langle 3, 0, 1 \rangle$. Find $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$.

First set up $\vec{u} \times \vec{v}$.

This is $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix}$.

This is $\vec{i}\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}$.

This gives you a vector $\vec{u} \times \vec{v} = 2\vec{i} - 7\vec{j} - 6\vec{k}$.

Now when we set up $\vec{v} \times \vec{u}$, notice that the two rows will be interchanged. Therefore $\vec{v} \times \vec{u} = -2\vec{i} + 7\vec{j} + 6\vec{k}$.

Important: $\vec{a} \times \vec{b}$ is orthogonal to BOTH $\vec{a}$ and $\vec{b}$. (If you are asked to find a vector orthogonal to both $\vec{a}$ and $\vec{b}$, this is useful.)

Use the Right-Hand Rule to find direction. If the fingers of your right hand curl in the direction of rotation (less than $180°$) from $\vec{a}$ to $\vec{b}$, then your thumb points in the direction of $\vec{a} \times \vec{b}$.

Properties of the Cross Product:

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

A few more important things:

- $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta$ (Similar to the dot product)
- $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$. This is because $\sin\pi = 0$.
- The length of $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

Example

Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

We need to find $\overrightarrow{PQ} \times \overrightarrow{PR}$ to find the vector perpendicular to the plane.

Finding $\overrightarrow{PQ}$ gives $\langle -3, 1, -7\rangle$ and finding $\overrightarrow{PR}$ gives $\langle 0, -5, -5\rangle$.

The cross product of this is $-40\vec{i} - 15\vec{j} + 15\vec{k}$. (All scalar multiples of this are also perpendicular.)

Example

Find the area of a triangle determined by $P_1(2, 2, 0), P_2(-1, 0, 2)$, and $P_3(0, 4, 3)$.

First you want to find the area of the parallelogram, then divide it by two.

Previously, the area of the parallelogram was determined by $\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\|$.

Do a similar process as the last example, the cross product is $\langle -10, 5, -10\rangle$.

The magnitude of this vector is 15, so the area of the triangle is $15/2$.

Now the scalar triple product: we have $\vec{a} \cdot (\vec{b} \times \vec{c})$. This gives a scalar.

Note this is $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

The absolute value of this scalar triple product gives the volume of a parallelpiped determined by $\vec{a}$, $\vec{b}$, and $\vec{c}$. (Note: absolute value, not magnitude.)

Example

Use the scalar triple product to show that $\vec{a} = \langle 1, 4, -7\rangle$, $\vec{b} = \langle 2, -1, 4\rangle$, and $\vec{c} = \langle 0, -9, 18\rangle$ are coplanar.

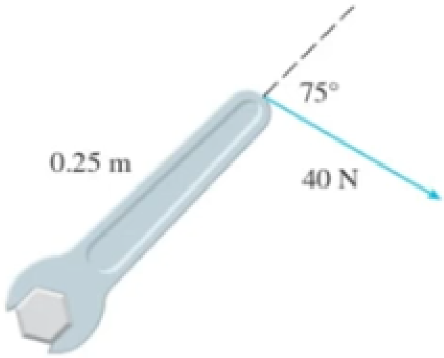
The scalar triple product of this will give a determinant of 0. The volume of the parallelpiped therefore is 0, which means that all three vectors are coplanar (meaning on the same plane).

Torque measures the tendency of a body to rotate about the origin. The direction of the torque vector indicates the axis of rotation.

$\vec{\tau} = \vec{r} \times \vec{F}$ where $\vec{r}$ represents position and $\vec{F}$ represents force.

Example

A bolt is tightened by applying a 40-N force to a 0.25 m as shown. Find the magnitude of the torque about the center of the bolt.



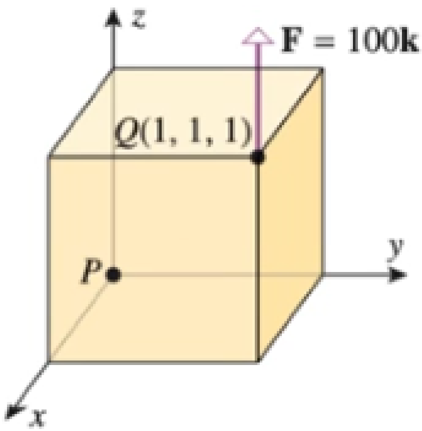
Remember: the $\vec{r}$ value must be in meters.

We know that $|\vec{tau}| = |\vec{r}||\vec{F}|\sin\theta$ (Cross Product)

So plug in numbers to get $|\vec{\tau}| \approx 9.66$ N-m.

Example

The figure shows a force of 100-N applied in the positive z-direction at the point $Q(1, 1, 1)$. Assuming the cube is free to rotate about $P(0, 0, 0)$, find the scalar moment (aka torque) of the force about P.



We know that $\vec{F} = \langle 0, 0, 100\rangle$ from the diagram.

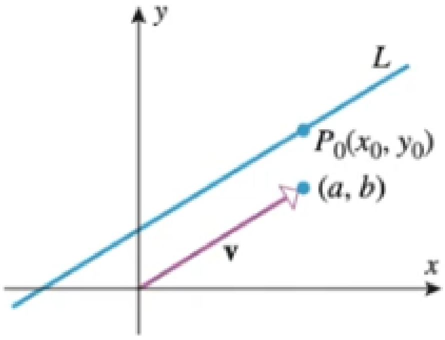
The vector $\vec{r}$ will be $\overrightarrow{PQ} = \langle 1, 1, 1\rangle$.

So the cross product of the two vectors gives the torque vector.

This vector is $\langle 100, -100, 0\rangle$.

1.5 Equations of Lines and Planes

To write the equation of a line, you need a point and the position.



In this figure, we know P_0 is a point on the line. The direction of the line is determined by a parallel vector $\vec{v}$. If $\vec{a} = \vec{P_0P}$, then $\vec{a} = t\vec{v}$.

Therefore, the vector equation of a line is $\vec{r} = \vec{r}_0 + t\vec{v}$. Each value of t gives a different point on the line. For $t > 0$, points are to the right.

Another expression is if we let $\vec{r} = \langle x, y, z\rangle$, $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$, and $\vec{v} = \langle a, b, c\rangle$, then we can write this as

$$\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$$

This leads us to the parametric equation of a line. The parametric equations of a line through (x_0, y_0, z_0) and parallel to the direction vector $\langle a, b, c\rangle$ are:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

Example

Find a vector and parametric equations for the line that passes through $(4, 2)$ and is parallel to $\vec{v} = \langle -1, 5\rangle$. Then find 2 other points on that line.

We have the vector $\vec{r} = \langle 4, 2\rangle + t\langle -1, 5\rangle$. This is the vector equation. This can also be written as $\vec{r} = (4\vec{i} + 2\vec{j}) + t(-\vec{i} + 5\vec{j})$. Also this can be written as $\vec{r} = (4 - t)\vec{i} + (2 + 5t)\vec{j}$.

For the parametric, we have $x = 4 - t$ and $y = 2 + 5t$.

Note: These equations are not unique. You can choose a different point or parallel vector.

To find 2 other points, just plug in t values.

For $t = 1$, we can get $x = 3$, and $y = 7$, so the point is $(3, 7)$.

Example

Find parametric equations of the line passing through $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$. Where does this line intersect the xy-plane?

We have $\overrightarrow{P_1P_2} = \langle 3, -4, 8\rangle$.

We can find the parametric equations as $x = 2 + 3t, y = 4 - 4t$, and $z = -1 + 8t$ using P_1. (Note any scalar multiple of $\overrightarrow{P_1P_2}$ will be parallel to the line).

This will intersect the xy-plane when $z = 0$. So if we let $-1 + 8t = 0$, $t = \frac{1}{8}$.

Plugging this in x and y gives the point $\left(\frac{19}{8}, \frac{7}{2}, 0\right)$ as the intersection point.

A third representation is symmetric equations. Setting the parametric equations and solving for t gives the following:

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example

For the last example, write the symmetric equations for the line.

This is just $\frac{x-2}{3} = \frac{y-4}{-4} = \frac{z+1}{8}$.

What happens if we limit t? You get a line segment.

Example

Find parametric equations for the line segment joining $P(2, 4, -1)$ and $Q(5, 0, 7)$.

First, $\overrightarrow{PQ} = \langle 3, -4, 8\rangle$.

The parametric equations are $x = 2 + 3t, y = -4 + 4t$, and $z = -1 + 8t$. We have to limit t though as $0 \leq t \leq 1$ for this to work.

A different representation for $\vec{r}(t) = (1 - t)\vec{r}_0 + t\vec{r}_1 = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)$.

So for this example, this would be $(1 - t)\langle 2, 4, -1\rangle + t\langle 5, 0, 7\rangle$.

Example

Consider the following lines. Are they parallel? Do they intersect?

$$L_1 \qquad x = 1 + 4t \quad y = 5 - 4t \quad z = -1 + 5t$$

$$L_2 \qquad x = 2 + 8t \quad y = 4 - 3t \quad z = 5 + t$$

Writing these in vector form gives L_1 as $\langle x, y, z\rangle = \langle 1, 5, -1\rangle + t\langle 4, -4, 5\rangle$, and L_2 as $\langle x, y, z\rangle = \langle 2, 4, 5\rangle + t\langle 8, -3, 1\rangle$.

We can see that the t terms are not scalar multiples of each other, so the lines are not parallel.

To find intersection points, we need to see if the points of the two lines can ever equal each other.

This means we have the equations $1 + 4t_1 = 2 + 8t_2$, $5 - 4t_1 = 4 - 3t_2$, and $-1 + 5t_1 = 5 + t_2$.

Solving for t_1 in this system gives us $t_1 = \frac{1}{4}$ and $t_2 = 0$.

This means when $t_1 = \frac{1}{4}$ and $t_2 = 0$, then L_1 and L_2 have the same x, y.

What we can see from this, is that for the last equation for z, they do not equal each other. That means both lines have a different z value. That means the line skews because they are not parallel or intersect.

Unlike a line, a vector parallel to a plane is not enough information to determine a plane. Instead we need a vector perpendicular to the plane.

The equation of a plane is $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$. (This is the vector equation of a plane.)

An alternate form is $\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$. This is also $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. (This is the scalar equation of a plane).

Note that "an equation" means that the equation is not unique.

Example

Find an equation of the plane passing through $(3, -1, 7)$ and perpendicular to the vector $\vec{n} = \langle 4, 2, -5\rangle$.

The equation is $\langle 4, 2, -5\rangle \cdot \langle x - 3, y + 1, z - 7\rangle = 0$.

This can be written as $4(x - 3) + 2(y + 1) - 5(z - 7) = 0$ or $4x + 2y - 5z + 25 = 0$. The second equation of this is known as the linear equation.

To graph this, we need to find 3 points (that are not co-linear). To find 3 points, you would need to come up with the intercepts, and then graph those and that gives you the plane.

Example

Find an equation of the plane that passes through $P(1, 3, 2)$, $Q(3, -1, 6)$ and $R(5, 2, 0)$.

If we find the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, this will find the vector perpendicular to the plane.

We can find that $\overrightarrow{PQ} = \langle 2, -4, 4\rangle$ and $\overrightarrow{PR} = \langle 4, -1, -2\rangle$ and that $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$.

The cross product is $\langle 12, 20, 14\rangle$. Dividing by 2 gives us $\langle 6, 10, 7\rangle$, which is also perpendicular to the plane.

So an equation of the plane is $6(x - 1) + 10(y - 3) + 7(z - 2) = 0$ or $6x + 10y + 7z = 50$. (There are many ways to represent this.)

Example

Consider the planes $x + y + z = 1$ and $x - 2y + 3z = 1$. Find the angle between the two planes and find symmetric equations for the line of intersection of the two planes.

We have $\vec{n}_1 = \langle 1, 1, 1\rangle$ and $\vec{n}_2 = \langle 1, -2, 3\rangle$.

We know that $\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|}$.

So plugging in numbers gives $\theta \approx 72°$.

The two planes of the two normal vectors will form a parallelogram. Remember $\vec{n}_1$ and $\vec{n}_2$ are normal vectors to the plane.

To find symmetric equations we need a point on the line and a parallel vector.

We will choose the point where the lines intersects the xy-plane ($z = 0$), so $x + y = 1$ and $x - 2y = 1$. Solving this system gives $x = 1$ and $y = 0$.

Both planes have the point $(1, 0, 0)$. If line L lies on both planes, then the line must be perpendicular to both $\vec{n}_1$ and $\vec{n}_2$.

The cross product of the two vectors is $\langle 5, -2, -3\rangle$.

So the symmetric equation is $\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}$. (There are other representations.)

Theorem 1.4

The distance between a point $P(x_1, y_1, z_1)$ and plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example

Find the distance between parallel planes $x + 2y - 2z = 3$ and $2x + 4y - 4z = 7$.

We have $\vec{n}_1 = \langle 1, 2, -2\rangle$ and $\vec{n}_2 = \langle 2, 4, -4\rangle$.

We can see that the point $(3, 0, 0)$ is on the first plane (letting $y = z = 0$).

For the second equation we have $2x + 4y - 4z - 7 = 0$.

From this, we have all the values needed now.

Plugging into the distance formula gives $\frac{1}{6}$.

Example

From a previous example, you found the lines are skew. Find the distance between them.

$$L_1 \qquad x = 1 + 4t \quad y = 5 - 4t \quad z = -1 + 5t$$

$$L_2 \qquad x = 2 + 8t \quad y = 4 - 3t \quad z = 5 + t$$

First find a point on L_1. We get a point $(1, 5, -1)$ (when $t = 0$).

We need to find a plane through L_2 and need a point and a perpendicular vector.

We know a point would be $(2, 4, 5)$. To find the perpendicular vector, we need to find a cross product of $\vec{L}_1 \times \vec{L}_2$.

The cross product is $\langle 11, -36, 20\rangle$. So the equation of the plane is $11(x-2)-36(y-4)+20(z-5) = 0$.

This can be written as $x - 36y + 20z + 22 = 0$.

Now we can find the distance since we have the plane and the point.

The distance is roughly $D \approx 3.918$ units.

1.6 Cylinders and Quadric Surfaces

Definition

A cylinder is a surface that consists of all lines parallel to a given line and passing through a given plane curve (i.e - in 3 space, the equation only has 2 variables).

For example, sketching $z = x^2$ in 3-space. This is essentialy a parabola on the xz-plane along a parallel line, and it will look like a piece of paper being in the process of being folded.

Exercise Sketch $y^2 + z^2 = 1$ in 3-space.

Definition

A quadric surface is the graph of a second-degree equation in 3 variables

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

Exercise Graph $z = x^2 + y^2$. (Hint $z \geq 0$.)

There are 6 different quadric surfaces.

First, the ellipsoid has the general equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. The intercepts of this are $(0, 0 \pm c)$, $(0 \pm b, 0)$. and $(\pm a, 0, 0)$. For the traces, we can see that if we let x, y, or z be a number, they will give ellipses. The shape of this looks like a football (American).

Next, the cone has the general equation $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. The axis is that $\frac{z^2}{c^2}$ term, but it can be y^2 or z^2. This has one intercept, $(0, 0, 0)$. The traces are z is some number, it gives an ellipse, if x or y is some number, you get a hyperbola.

Next, the elliptic paraboloid has the equation $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. Likewise with the cone, the $\frac{z}{c}$ term just tells you what axis (it can be x or y.) The intercept is the same as the cone, $(0, 0, 0)$. For the traces, z gives an ellipse, and x and y gives parabolas.

Next the hyperboloid of one sheet. The equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. (The term with a minus is the axis.) You have to find the intercept for this one. z gives an ellipse, and x and y gives hyperbolas.

Next, hyperboliods of 2 sheets. The equation is $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. (The term with the plus is the axis.) You have to find the intercept, and z gives an ellipse ($z > c$), and x and y gives ellipses.

Lastly, the hyperbolic paraboloid. The formula is $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$. You have to find the intercept. x and y gives parabolas and z gives a hyperbola. (This should kind of look like a saddle (on a horse).)

Example

Let's look at $y^2 = x^2 + \frac{z^2}{4}$.

The intercept is $(0, 0, 0)$.

We can assume that y is the axis we are drawing on.

When we look at the traces, they all give ellipses. For example, $y = 1$ gives $1 = x^2 + \frac{z^2}{4}$, $y = 2$ gives $4 = x^2 + \frac{z^2}{4}$ or $1 = \frac{x^2}{4} + \frac{z^2}{16}$.

When we let $x = 0$, then we can get $y^2 = \frac{z^2}{4}$, which gives $\pm y = \pm\frac{z}{2}$, which is two perpendicular lines essentially.

This is an elliptic cone.

Example

Let's look at $4x^2 + 4y^2 + z^2 + 8y - 4z = -4$.

Completing the square gives $x^2 + (y + 1)^2 + \frac{(z-2)^2}{4} = 1$.

The intercepts are $(0, 0, 2)$, $(0, -1, 0)$ as the two intercepts, (there are no x-intercepts.) From what we also see, we can see the center of the graph should be $(0, -1, 2)$.

Hopefully, you can see that all the traces are ellipses.

Cool! We get an ellipsoid.

Example

Lastly, graph $z = \frac{y^2}{4} - \frac{x^2}{9}$.

Starting with intercepts, we get $(0, 0, 0)$ as the only intercept.

For our traces, x and y give parabolas, and z gives a hyperbola.

This gives a hyperbolic paraboloid.

For a brief review on conic sections: the following.

Circles: A circle is the set of all points in the plane equidistant from a fixed point. The standard equation is $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center, and r is the radius.

Ellipses: An ellipse is the set of all points in the plane the sum of whose distances from two fixed points (the foci) is constant. The standard equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$. If $a>b$, we get the foci as $(h\pm c,k)$, and if $b>a$, we get the foci as $(h,k\pm c)$. For both of these, the center will be (h,k), and the vertices are the endpoints of the major axis. Use c to find the coordinates of each focus. The foci are located on the major axis and are each c units away from the center. If $a=b$, then the ellipse is just a circle, and a and b will equal r. The foci of a circle are located at the same point - the center. Eccentricity of an ellipse is $\frac{c}{a}$. (For a circle $e=0$.)

Hyperbolas: A hyperbola is the set of all points in the plane the difference of whose distances from two fixed points (the foci) is constant. The standard equation are the following:

- For hyperbolas that open left and right: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, the foci will be $(h\pm c,k)$
- For hypberolas that open up and down: $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$, the foci will be $(h,k\pm c)$

The center is (h,k) for both and the vertices are the turning points of the brances of the hyperbola. Use a and b to create the central rectangle around the center of the hyperbola. The diagonals of this rectangle form the asymptotes. The equation for the asymptotes are $y-k=\pm\frac{b}{a}(x-h)$. Use the value $c=\sqrt{a^2+b^2}$, to find the coordinates of each focus. The branches of a hyperbola will always bend towards the foci and away from the center.

Parabolas: A parabola is the set of all points in the plane equidistant from a fixed line (the directrix) and a fixed point (the focus). The graph of a parabola will always bend towards its focus and away from its directrix. The coordinates of the vertex are (h,k), the distance from the vertex to both the focus and directrix is given by $|p|$. For equation for a parabola that opens up and down is $(x-h)^2=4p(y-k)$, and for one that opens left or right it is $(y-k)^2=4p(x-h)$. If $p>0$, the parabola either opens up or right, and for $p<0$, the opposite happens.

2 Vector-Valued Functions

2.1 Vector Functions and Space Curves

Review: Parametric Curves

- $x = f(t)$
- $y = g(t)$
- $z = h(t)$

These represent a curve in 3-space (for 2-space, it is just x and y.)

The above represents a path in space that is traced in a specific direction as t increases (orientation). The domain is $(-\infty, \infty)$, unless specified otherwise.

Definition

$$\vec{r} = \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

At any given t value, $\vec{r}$ represents a vector whose initial point is at the origin and terminal point is $(f(t), g(t), h(t))$.

The domain is $(-\infty, \infty)$ and the range is the set of vectors.

Graphs of vector-valued functions: curve that is traced by connecting tips of "radius vectors".

Example

Graph $\vec{r}(t) = 2\cos t\vec{i} - 3\sin t\vec{j}$ for $0 \le t \le 2\pi$.

We could write this as $x = 2\cos t$ and $y = -3\sin t$ (parametric).

We could instead write a table.

t	x	y
0	2	0
$\pi/2$	0	-3
π	-2	0
$3\pi/2$	0	3
2π	2	0

As you draw this, you can see that this will be an ellipse.

Example

$$\vec{r}(t) = \langle 4\cos t, 4\sin t, t \rangle$$

We should know that since there are trig things in here, that we go from 0 to 2π, and if we put this on a table, we can see that x and y will give you a circle from the table. The z is moving up though, so basically the function will just be circling around a cylinder of radius 2.

Example

Find a vector and parametric equations for the line segment that joins $A(1,-3,4)$ to $B(-5,1,7)$.

We have $\vec{r} = \vec{AB} = \langle -6,4,3\rangle$. So $\vec{r}(t) = \langle 1-6t, -3+4t, 4+3t\rangle$, and we want to put the bound $0 \le t \le 1$

The parametrics are $x(t) = 1-6t, y(t) = -3+4t$, and $z = 4+3t$, with $0 \le t \le 1$.

Example

Find a vector function that represents the curve of intersection of $x^2+y^2=1$ and $y+z=2$.

$x^2+y^2=1$ is a cylinder and $y+z=2$ is a plane.

We can represent $x^2+y^2=1$ as $x=\cos t$ and $y=\sin t$, with bounds $0 \le t \le 2\pi$.

$y+z=2$ can be represented as $z=2-y$ or $z=2-\sin t$ with $0 \le t \le 2\pi$.

So $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + (2-\sin t)\vec{k} = \langle \cos t, \sin t, 2-\sin t\rangle$ with $0 \le t \le 2\pi$.

Example

Find the domain of $\vec{r}(t) = \langle \ln|t-1|, e^t, \sqrt{t}\rangle$.

The domain is all values of t for which $\vec{r}(t)$ is defined.

So we have $x = \ln|t-1|$, $y=e^t$ and $z=\sqrt{t}$.

For x, we have the domain as $(-\infty,1)\cup(1,\infty)$, for y we have the domain as $t \in \mathbb{R}$, and for z, we have $t \ge 0$, so combining them gives domain $[0,1)\cup(1,\infty)$.

Definition

If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, then $\lim\limits_{t\to a}\vec{r}(t) = \langle \lim\limits_{t\to a} f(t), \lim\limits_{t\to a} g(t), \lim\limits_{t\to a} h(t)\rangle$ (as long as all 3 limits exist).

Example

Let $\vec{r}(t) = t^2\vec{i} + e^t\vec{j} - (2\cos\pi t)\vec{k}$. Find $\lim\limits_{t\to 0}\vec{r}(t)$.

The limit of the $\vec{i}$ term is 0 as it goes to 0.

The limit of the $\vec{j}$ term is 1 as it approaches 0.

The limit of the $\vec{k}$ term is -2 as it approaches 0.

So the limit is $\lim\limits_{t\to 0}\vec{r}(t) = \vec{j} - 2\vec{k}$

Example

Let $\vec{r}(t) = \left(\frac{4t^3+5}{3t^3+1}\right)\vec{i} + \left(\frac{1-\cos t}{t}\right)\vec{j} + \left(\frac{\ln(t+1)}{t}\right)\vec{k}$. Find $\lim\limits_{t\to 0}\vec{r}(t)$.

For the first term, we get 5 as the limit.

For the other two, we will use L'Hopital's Rule.

Doing this and finding the limits should give that $\lim\limits_{t\to 0}\vec{r}(t) = \langle 5,0,1\rangle$.

Continuity: A vector function $\vec{r}(t)$ is continuous at a if: $\lim\limits_{t\to a}\vec{r}(t) = \vec{r}(a)$. (This is just AP Calculus BC)

2.2 Derivatives and Integrals of Vector Functions

Definition

If $\vec{r}(t)$ is a vector function, the derivative of $\vec{r}(t)$ with respect to t is

$$\vec{r'} = \vec{r}(t)' = \frac{d\vec{r}}{dt} = \frac{d}{dt}(\vec{r}(t)) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Geometrically, this would have $\vec{r}(t)$ as a vector tangent to the curve at the tip of $\vec{r}(t)$. It points in the direction of increasing parameter.

Theorem 2.1

If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, where f, g, and h are differentiable functions, then

$$\vec{r'}(t) = \langle f'(t), g'(t), h'(t)\rangle$$

Proof. Let $\vec{r}(t) = \langle x(t), y(t)\rangle$

By definition, $\vec{r'}(t) = \lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h}$.

This is equal to $\lim_{h\to 0} \frac{[x(t+h)\vec{i}+y(t+h)\vec{j}]-[x(t)\vec{i}+y(t)\vec{j}]}{h}$.

Which is equal to

$$\left(\lim_{h\to 0} \frac{x(t+h)\vec{i} - x(t)\vec{i}}{h}\right) + \left(\lim_{h\to 0} \frac{y(t+h)\vec{j} - y(t)}{h}\right)$$

Taking out the $\vec{i}$ and $\vec{j}$, allows us to see that this equals to $x'(t)\vec{i} + y'(t)\vec{j}$. □□

Example

$\vec{r}(t) = \frac{1}{t}\vec{i} + e^{2t}\vec{j} - 2\cos\pi t\vec{k}$. Find $\vec{r}(t)$.

The derivative of this is simply $\langle \frac{-1}{t^2}, 2e^{2t}, 2\pi\sin\pi t\rangle$.

$\vec{r'}(t)$ refers to the tangent vector. The tangent line is the line through P that is parallel to $\vec{r'}(t)$.

Unit Tangent Vector: $\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|}$.

Example

From the previous example, find the unit tangent vector at $t = 1$.

We know that $\vec{r'}(t) = \langle \frac{-1}{t^2}, 2e^{2t}, 2\pi\sin\pi t\rangle$.

From this, $\vec{r'}(1) = \langle -1, 2e^2, 0\rangle$, and the magnitude of this is $\sqrt{1+4e^4}$.

Therefore, $\vec{T}(1) = \langle \frac{-1}{\sqrt{1+4e^4}}, \frac{2e^2}{\sqrt{1+4e^4}}, 0\rangle$.

Exercise For the curve $\vec{r}(t) = \sqrt{t}\vec{i} + (2-t)\vec{j}$, find $\vec{r'}(t)$. Sketch $\vec{r}(1)$ and $\vec{r'}(1)$.

Example

Find parametric equations for the tangent line to the helix with equations $x = 2\cos t$, $y = \sin t$, and $z = t$ at the point $(0, 1, \pi/2)$.

We have $\vec{r}(t) = \langle 2\cos t, \sin t, t\rangle$, so $\vec{r'}(t) = \langle -2\sin t, \cos t, 1\rangle$.

We get $0 = 2\cos t$, $1 = \sin t$, and $\frac{\pi}{2} = t$, so we know that t is.

Plugging this in gives $\vec{r'}\left(\frac{\pi}{2}\right) = \langle -2, 0, 1\rangle$. This is the tangent vector.

So $\vec{r}(t) = \langle 0, 1, \frac{\pi}{2}\rangle + t\langle -2, 0, 1\rangle$.

Parametrically: $x = -2t$, $y = 1$, $z = \frac{\pi}{2} + t$.

Differentiation Rules:

1. $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$
2. $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$
3. $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
4. $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
5. $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$ (Order matters here)
6. $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\vec{u}'(f(t))$

Theorem 2.2

If $\vec{r}(t)$ is differentiable and $\|\vec{r}(t)\|$ is constant for all t, then $\vec{r}(t) \cdot \vec{r'}(t) = 0$.

This means they are orthogonal for all t.

Example

The graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ intersect at the origin. Find the degree measure of the acute angle between the tangent lines to the graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ at the origin.

We have $\vec{r}_1(t) = \langle \tan^{-1} t, \sin t, t^2\rangle$ and $\vec{r}_2(t) = \langle t^2 - t, 2t - 2, \ln t\rangle$.

$\vec{r}_1(t) = \langle 0, 0, 0\rangle$ at $t = 0$.

$\vec{r}_2(t) = \langle 0, 0, 0\rangle$ at $t = 1$.

We need the derivatives of the functions.

$\vec{r}_1'(t) = \langle \frac{1}{1+t^2}, \cos t, 2t\rangle$

$\vec{r}_2'(t) = \langle 2t - 1, 2, \frac{1}{t}\rangle$

$\vec{r}_1'(0) = \langle 1, 1, 0\rangle$ and $\vec{r}_2'(1) = \langle 1, 2, 1\rangle$.

If we want to find the angles between then we have to use the dot product.

We get $\cos\theta = \frac{1+2+0}{\sqrt{2}\cdot\sqrt{6}} = \frac{\sqrt{3}}{2}$.

So $\theta = \frac{\pi}{6}$.

Example

Calculate $\frac{d}{dt}\left[\vec{r}_1(t)\cdot\vec{r}_2(t)\right]$ and $\frac{d}{dt}\left[\vec{r}_1(t)\times\vec{r}_2(t)\right]$ by differentiating the product directly and using the formulas.

$$\vec{r}_1(t) = 2t\vec{i} + 3t^2\vec{j} + t^3\vec{k}$$
$$\vec{r}_2(t) = t^4\vec{k}$$

Directly:

The dot product $\vec{r}_1\cdot\vec{r}_2 = t^7$. The derivative of this is $7t^6$.

Formula: The formula is $\vec{r}_1'\cdot\vec{r}_2 + \vec{r}_1\cdot\vec{r}_2'$.

Using this formula gives you $3t^4t^6 = 7t^6$.

Now for the cross product.

Directly: The cross product gives $\langle 3t^6 - 0, -(2t^5 - 0), 0\rangle = \langle 3t^6, -2t^5, 0\rangle$.

The derivative of this is $\langle 18t^5, -10t^4, 0\rangle$.

Formula: The formula is $\vec{r}_1'\times\vec{r}_2 + \vec{r}_1'\times\vec{r}_2'$.

You should get the same answer.

$$\int_a^b \vec{r}(t) = \lim_{n\to\infty}\sum_{i=1}^{n}\vec{r}(t_i^*)\Delta t$$

Or, more helpfully

$$\int_a^b \vec{r}(t)dt = \left(\int_a^b f(t)dt\right)\vec{i} + \left(\int_a^b g(t)dt\right)\vec{j} + \left(\int_a^b h(t)dt\right)\vec{k}(t)$$

Example

Let $\vec{r}(t) = t^2\vec{i} + e^t\vec{j} - 2\cos\pi t\vec{k}$. Find $\int_0^1 \vec{r}(t)dt$.

Integrating each component and plugging in the limits of integration results in $\int_0^1 \vec{r}(t)dt = \frac{1}{3}\vec{i} + (e^t)\vec{j}$.

Example

Find $\int(2t\vec{i} + 3t^2\vec{j})dt$.

Remember in an indefinite integral to add a constant at the end.

The result is $t^2\vec{i} + t^3\vec{j} + \vec{c}$.

Example

Find $\vec{r}(t)$ given that $\vec{r}'(t) = \langle 3, 2t\rangle$ and $\vec{r}(1) = \langle 2, 5\rangle$.

If we start by integrating, then $\vec{r}(t) = \langle 3t, t^2\rangle + \vec{c}$.

We have $\langle 2, 5\rangle = \langle 3, 1\rangle + \langle c_1, c_2\rangle$.

We get $\vec{c} = \langle -1, 4\rangle$ from this.

So $\vec{r}(t) = \langle 3t - 1, t^2 + 4\rangle$.

2.3 Arc Length and Curvature

Consider a curve given by parametric equations $x = x(t)$ and $y = y(t), a \leq t \leq b$.

Then arc length

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The Arc Length of a Vector Valued Function is the exact same idea

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_a^b \|\vec{r'}(t)\| dt$$

Example

Find the arc length of the portion of the curve $x = 3\cos t,\ y = 3\sin t,\ z = 4t$ from $(3, 0, 0)$ to $(-3, 0, 4\pi)$.

If we use $z = 4t$ we get $t = 0$ and $t = \pi$ from both points.

The integral is $L = \int_0^\pi \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} dt$.

This is equal to $\int_0^\pi \sqrt{25} dt = 5\pi$.

A curve can be represented by more than one function.

Example

Given $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle, 1 \leq t \leq 2$.

If we use $t = e^u$ then $\vec{r}_1(u) = \langle e^u, e^{2u}, e^{3u} \rangle, 0 \leq u \leq \ln 2$.

Both represent the same curve. These are called parametrizations of the curve. Both can be use dto find arc length (because arc length does not depend on the parameter).

Example

Find the length of the curve above using both parametrizations.

$\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$.

$\vec{r}_1'(t) = \langle 1, 2t, 3t^2 \rangle$.

Then we integrate $L = \int_1^2 \sqrt{1 + 4t^2 + 9t^4} dt \approx 7.075$.

For $\vec{r}_2(u) = \langle e^4, e^{2u}, e^{3u} \rangle$.

The derivative of this is $\vec{r}_2'(u) = \langle e^u, 2e^{2u}, 3e^{3u} \rangle$.

The integral is $\int_0^{\ln 2} \sqrt{e^{2u} 4e^{4u} + 9e^{6u}} du \approx 7.075$.

As you can see, they are the same.

We want to parametrize a curve in terms of arc length, s, rather than an arbitrary value in a particular coordinate system.

We first must recognize that $s(t) = \int_a^t |\vec{r'}(u)| du$.

This of course is equal to

$$\int_a^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du$$

We can also see that $\frac{ds}{dt} = |\vec{r'}(t)|$.

Example

Find the arc length parametrization of $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$ with reference point $(1, 0, 0)$ and the same orientation as the helix.

We know that $\frac{ds}{dt} = |\vec{r'}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$.

$s = s(t) = \int_0^t \sqrt{2}du = \sqrt{2}t$.

We get that $t = \frac{s}{\sqrt{2}}$ as a result.

Therefore $\vec{r}(s) = \cos\left(\frac{s}{\sqrt{2}}\right)\vec{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\vec{j} + \left(\frac{s}{\sqrt{2}}\right)\vec{k}$.

Arc length formula guarantees same orientation.

This is useful because let's say we need to move along the curve for a certain amount of units, well we can just plug in that value and find the point at which we are.

For example, $\vec{r}(5) \approx (-0.923, -0.384, 3.5636)$.

Example

Find the arc length parametrization of the curve below measured from $(0, 0)$ in the direction of increasing t.

$$\vec{r}(t) = \langle 1/3t^2, 1/2t^2\rangle, t \geq 0$$

$\vec{r'}(t) = \langle t^2, t\rangle$ and the magnitude of this is $t\sqrt{t^2 + 1}$.

We are now integrating $s = \int_0^t u\sqrt{u^2 + 1}du$.

This gives you $\frac{1}{3}(u^2 + 1)^{3/2}$ from 0 to t.

Integrating this and solving for t gives you $t = \sqrt{(3s + 1)^{2/3} - 1}$.

Therefore the parametrization of this is $\vec{r}(s) = \langle\frac{1}{3}[(3s + 1)^{2/3} - 1]^{3/2}, \frac{1}{2}[(3s + 1)^{2/3} - 1]\rangle$.

Example

Let $\vec{r}(t) = \langle \ln t, 2t, t^2\rangle$. Find

(a) $\|\vec{r'}(t)\|$

$\vec{r'}(t) = \langle\frac{1}{t}, 2, 2t\rangle$, so the magnitude of this is $\sqrt{\frac{1}{t^2} + 4 + 4t^2} = 2t + \frac{1}{t}$.

(b) $\frac{ds}{dt}$

This is the exact same thing as $\|\vec{r'}(t)\| = 2t + \frac{1}{t}$

(c) $\int_1^3 \|\vec{r'}(t)\|dt$

We are integrating $\int_1^3 (2t + \frac{1}{t})dt = 9 + \ln 3 - 1 - 0 = 8 + \ln 3$.

A parametrization is called smoth on I if $\vec{r'}(t)$ is continuous and $\vec{r'}(t) \neq 0$ on I (a smooth curve has smooth parametrization). Smooth means no sharp corners or cusps.

Example

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

Is $\vec{r}(t)$ smooth?

The derivative of the vector is $\langle -\sin t, \cos t, 1 \rangle$. This is continuous on $(-\infty, \infty)$ and this is not equal to $\vec{0}$, so $\vec{r}(t)$ is smooth.

Recall: $\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|}$ (called unit tangent vector) indicated the direction of curve.

Curvature is as followed.

$$\kappa = \left| \frac{d\vec{T}}{ds} \right|$$

$\vec{T}$ has a constant length so κ is only affected by a change in direction.

Example

Show that the curvature of a circle with radius a is $1/a$.

$\vec{r}(t) = \langle a \cos t, a \sin t \rangle$.

The derivative $\vec{r'}(t) = \langle -a \sin t, a \cos t \rangle$.

$s(t) = \int_0^t \sqrt{a^2 \sin^2 u + a^2 \cos^2 u} du = \int_0^t a du$.

We get $s(t) = s = at$ so $t = \frac{a}{s}$.

The circle in terms of s is $\vec{r}(S) = \langle a \cos \frac{a}{s}, a \sin \frac{a}{s} \rangle$.

The derivative of this is $\langle -\sin \frac{a}{s}, \cos \frac{a}{s} \rangle$.

The magnitude of this is 1.

The unit tangent vector $\vec{T}(s) = \langle -\sin \frac{a}{s}, \cos \frac{a}{s} \rangle$.

The derivative of this vector is $\langle -\frac{1}{a} \cos \frac{a}{s}, -\frac{1}{a} \sin \frac{a}{s} \rangle$.

The magnitude of this vector is $\kappa = \frac{1}{a}$. A big radius means a small curvature.

The curvature of a straight line is $\kappa = 0$.

A circle has constant curvature.

Other formulas for κ are the following

$$\kappa = \left| \frac{d\vec{T}}{dS} \right| = \left| \frac{\frac{dT}{dt}}{\frac{dS}{dt}} \right|$$

$$\kappa = \frac{|\vec{T'}(t)|}{|\vec{r'}(t)|}$$

$$\kappa = \frac{|\vec{r'}(t) \times \vec{r''}(t)}{\vec{r'}(t)|^3}$$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}}$$

Exercise Use another formula to calculate κ for $\vec{r}(t) = \langle a \cos t, a \sin t \rangle$.

Example

Find κ for $\vec{r'}(t) = \langle 2t, t^2, -\frac{1}{3}t^3 \rangle$.

The derivative $\vec{r'}(t) = \langle 2, 2t, -t^2 \rangle$.

$|\vec{r'}(t)| = \sqrt{4 + 4t^2 + t^4} = t^2 + 2$

$\vec{T}(t) = \frac{\langle 2,2t,-t^2 \rangle}{t^2} + 2 = \langle \frac{2}{t^2+2}, \frac{2t}{t^2+2}, \frac{-t^2}{t^2+2} \rangle$.

$\vec{T'}(t) = \langle \frac{4t}{(t^2+2)^2}, \frac{-2t^2+4}{(t^2+2)^2}, \frac{-4t}{(t^2+2)^2} \rangle$.

$\|\vec{T'}(t)\| = \sqrt{\frac{16t^2+4t^4-16t^2+16+16t^2}{(t^2+2)^4}} = \frac{2}{t^2+2}$

$\kappa(t) = \frac{2/t^2+2}{t^2+2} = \frac{2}{(t^2+2)^2}$

We can also use the other formula using the cross product.

$\vec{r'}(t) = \langle 2, 2t, -t^2 \rangle$ and $\vec{r''}(t) = \langle 0, 2, -2t \rangle$.

The cross product of these two vectors will result in $\langle -4t^2 - -2t^2, -(-4t-0), 4-0 \rangle = \langle -2t^2, 4t, 4 \rangle$.

The magnitude of this is $2(t^2+2)$, so $\kappa(t) = \frac{2(t^2+2)}{(t^2+2)^3} = \frac{2}{(t^2+2)^2}$.

Both ways give an equivalent answer.

There is one more curvature formula in terms of x rather than t.

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

Example

Find the curvature of the parabola $y = x^2$ at the points $(0,0)$, $(1,1)$, and $(2,4)$.

So $f(x) = x^2$, $f'(x) = 2x$, and $f''(x) = 2$.

$\kappa(x) = \frac{|2|}{(1+(2x)^2)^{3/2}} = \frac{2}{(1+4x^2)^{7/2}}$

$\kappa(0) = 2$, $\kappa(1) \approx 0.18$, $\kappa(2) \approx 0.03$.

As $\kappa \to \infty$, $\kappa(x) \to 0$.

Radius of curvature: $\rho = \frac{1}{\kappa}$

We have also shown $\kappa = \frac{1}{a}$

Example

From the previous example, calculate the curvature at $(0,0)$. Then draw a circle of curvature.

$\kappa(0) = 2$ and $\rho(0,0) = \frac{1}{2}$.

At the point $(0,0)$, κ is same as circle with radius $\frac{1}{2}$.

Recall the unit tangent vector, $\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|}$ which points in the direction of increasing parameter.

The unit tangent vector is orthogonal to its derivative.

Unit normal vector $\vec{N}(t) = \frac{\vec{T'}(t)}{|\vec{T'}(t)|}$. This points inward towards the concave part of curve c.

Binormal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$.

$\|\vec{T} \times \vec{N}\| = \|\vec{T}\|\|\vec{N}\| \sin 90$. This is a also a unit vector.

Example

Find the unit tangent, unit normal, and binormal vectors for $\vec{r}(t) = \langle 3\sin t, 3\cos t, 4t\rangle$.

$\vec{r'}(t) = \langle 3\cos t, -3\sin t, 4\rangle$.

$\|\vec{r'}(t)\| = 5$

$\vec{T}(t) = \langle \frac{3}{5}\cos t, -\frac{3}{5}\sin t, \frac{4}{5}\rangle$.

$\vec{T'}(t) = \langle -\frac{3}{5}\sin t, -\frac{3}{5}\cos t, 0\rangle$

$\|\vec{T'}(t)\| = \frac{3}{5}$

$\vec{N}(t) = \langle -\sin t, -\cos t, 0\rangle$.

$\vec{B}(t) = \vec{T} \times \vec{N} = \langle \frac{4}{5}\cos t, -\frac{4}{5}\sin t, -\frac{3}{5}\rangle$

Another wayt to find $\vec{B}(t)$ is the following

$$\vec{B}(t) = \frac{\vec{r'}(t) \times \vec{r''}(t)}{\|\vec{r'}(t) \times \vec{r''}(t)\|}$$

Example

Consider $\vec{r}(t) = \langle t, \frac{\sqrt{2}}{2}t^2, \frac{1}{3}t^3\rangle$. Find $\vec{T}, \vec{N}$ at $t = 2$.

$\vec{r'}(t) = \langle 1, \sqrt{2}t, t^2\rangle$

$\|\vec{r'}(t)\| = \sqrt{1 + 2t^2 + t^4} = t^2 + 1$

$\vec{T}(t) = \langle \frac{1}{1+t^2}, \frac{\sqrt{2}t}{1+t^2}, \frac{t^2}{1+t^2}\rangle$

$\vec{T}(2) = \langle \frac{1}{5}, \frac{2\sqrt{2}}{5}, \frac{4}{5}\rangle$

Now to find $\vec{N}(2)$.

$\vec{T'}(t) = \langle \frac{-2t}{(1+t^2)2}, \frac{(1+t^2)\sqrt{2}-2t(\sqrt{2}t)}{(1+t^2)^2}, \frac{2t(1+t^2)-t^2(2t)}{(1+t^2)^2}\rangle = \langle \frac{-2t}{(1+t^2)^2}, \frac{-2t^2+2}{(1+t^2)^2}, \frac{2t}{(1+t^2)^2}\rangle$

$\vec{N}(t) = \frac{\vec{T'}(t)}{|\vec{T'}(t)|}$

We should instead of finding the magnitude, find $\vec{T'}(2) = \langle \frac{-4}{25}, \frac{-8+\sqrt{2}}{25}, \frac{4}{25}\rangle$

The magnitude of this is $\|\vec{T'}(2)\| = \sqrt{\frac{16}{625} + \frac{64-16\sqrt{2}+2}{625} + \frac{16}{625}} = \frac{\sqrt{98-16\sqrt{2}}}{25}$

So $\vec{N}(2) = \frac{\langle \frac{-4}{25}, \frac{-8+\sqrt{2}}{25}, \frac{4}{25}\rangle}{\sqrt{98-16\sqrt{2}}25}$

This is equal to $\langle \frac{-4}{\sqrt{98-16\sqrt{2}}}, \frac{-8+\sqrt{2}}{\sqrt{98-16\sqrt{2}}}, \frac{4}{\sqrt{98-16\sqrt{2}}}\rangle$.

A normal plane contains $\vec{N}$ and $\vec{B}$. It contains all lines perpendicular to $\vec{T}$.

The osculating plane contains $\vec{T}$ and $\vec{N}$. It is related to the circle of curvature or osculating circle.

The rectifying plane contains $\vec{T}$ and $\vec{B}$.

To find the equation of a plane you need a point and a perpendicular vector.

Example

Find the equations of the normal and osculating plantes at $(3, 0, 2\pi)$ for the following:

$$\vec{T}(t) = \langle \frac{3}{5}\cos t, -\frac{3}{5}\sin t, \frac{4}{5}\rangle$$

$$\vec{N}(t) = \langle -\sin t, -\cos t, 0\rangle$$

$$\vec{B}(t) = \langle \frac{4}{5}\cos t, -\frac{4}{5}\sin t, -\frac{3}{5}\rangle$$

The normal plane has point $(3, 0, 2\pi)$ and normalvector at $\frac{\pi}{2}$ is $\vec{T}\left(\frac{\pi}{2}\right) = \langle 0, -\frac{3}{5}, \frac{4}{5}\rangle$.

We have $0(x-3) + \frac{-3}{5}(y-0) + \frac{4}{5}(z-2\pi) = 0$ and this gives $\frac{-3}{5}y + \frac{4}{5}z = \frac{8}{5}\pi$.

The osculating plane we need the binormal vector. $\vec{B}\left(\frac{\pi}{2}\right) = \langle 0, -\frac{4}{5}, \frac{3}{5}\rangle$.

$0(x-3) + -\frac{4}{5}(y-0) + \frac{3}{5}(z-2\pi) = 0$ so we get $-\frac{4}{5}y - \frac{3}{5}z = -\frac{6}{5}\pi$

Example

Consider the ellipse given by

$$\vec{r}(t) = 2\cos t\vec{i} + 3\sin t\vec{j}, 0 \leq t \leq 2\pi$$

Note: $\kappa(t) = \frac{6}{[4\sin^2 t + 9\cos^2 t]^{3/2}}$

Find and draw the osculating circles at $(2, 0)$ and $(0, -3)$.

So we have $t = 0$ and $t = \frac{3\pi}{2}$.

For $(2, 0) \to \kappa(0) = \frac{2}{9}$. so circle with radius $\frac{9}{2}$ and diamater 9.

For $(0, -3)$, $\kappa\left(\frac{3\pi}{2}\right) = \frac{3}{4}$ so radius $r = \frac{3}{4}$ and diameter $\frac{8}{3}$.

For the point $(2, 0)$, we also have the point $(-7, 0)$, so the center is $\left(-\frac{5}{2}, 0\right)$.

So the equation for that is $\left(x + \frac{5}{2}\right)^2 + y^2 = \frac{81}{4}$.

2.4 Motion in Space - Velocity and Acceleration

1. Direction of motion time t is in the direction of $\vec{T}$.
2. speed $= \frac{ds}{dt}$ (instantaneous rate of change of the arc length traveled). This is a scalar
3. velocity vector $\vec{v}(t) = \frac{ds}{dt}\vec{T}(t)$

$\frac{ds}{dt}$ is the magnitude of $\vec{v}(t)$.

$\vec{T}(t)$ denotes direction.

$\vec{v}(t)$ points in direction of motion and has magnitude = speed

If $\vec{r}(t)$ is a position function, then $\vec{v}(t) = \frac{d\vec{r}}{dt}(t)$ and $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt}$.

Speed is $\|\vec{v}(t)\| = \frac{ds}{dt}$

Example

A particle moves along C: $\vec{r}(t) = \langle 2\sin\left(\frac{t}{2}\right), 2\cos\left(\frac{t}{2}\right)\rangle$.

(a) Find its velocity, acceleration, and speed at time t.

$\vec{v}(t) = \vec{r'}(t) = \langle \cos\frac{t}{2}, -\sin\frac{t}{2}\rangle = \vec{v}(t)$.

$\vec{a}(t) = \vec{v'}(t) = \langle -\frac{1}{2}\sin\frac{t}{2}, -\frac{1}{2}\cos\frac{t}{2}\rangle = \vec{a}(t)$

speed $= \|\vec{v}(t)\| = 1$

(b) Show that $\vec{a}(t)$ is orthogonal to $\vec{v}(t)$ for this path only.

$\vec{a}(t)\cdot\vec{v}(t) = -\frac{1}{2}\cos\frac{t}{2}\sin\frac{t}{2} + \frac{1}{2}\sin\frac{t}{2}\cos\frac{t}{2} = 0$.

This implies that $\vec{a}(t)$ is orthogonal to $\vec{v}(t)$.

Example

An object moves in 3-space so that $\vec{v}(t) = \langle 1, t, t^2\rangle$. Find the coordinates of the particle at time $t = 1$ given that at $t = 0$, the particle is at $(-1, 2, 4)$.

$\vec{r}(t) = \int \vec{v}(t)dt = \langle t, \frac{1}{2}t^2, \frac{1}{3}t^3\rangle + \vec{c}$

We know that $\vec{r}(0) = \langle -1, 2, 4\rangle$. This means that $\vec{c} = \langle -1, 2, 4\rangle$.

So, $\vec{r}(t) = \langle t - 1, \frac{1}{2}t^2, \frac{1}{3}t^3 + 4\rangle$.

$\vec{r}(1) = \langle 0, \frac{5}{2}, \frac{13}{3}\rangle$.

So this becomes the point $\left(0, \frac{5}{2}, \frac{13}{3}\right)$ at $t = 1$.

Example

An object with mass m that moves in a circular pattern with constant angular speed ω has position vector $\vec{r}(t) = a\cos\omega t\vec{i} + a\sin\omega t\vec{j}$. Find the force acting on the object and show that it is directed toward the origin.

We have a circle toward the origin with radius a and we have points on the circle P at an angle θ.

Newton's 2nd law states that $\vec{F}(t) = m\vec{a}(t)$.

We have the position vector.

$\vec{v}(t) = \langle -a\omega\sin\omega t, a\omega\cos\omega t\rangle$

$\vec{a}(t) = \langle -a\omega^2\cos\omega t, -a\omega^2\sin\omega t\rangle$

$\vec{F}(t) = m\vec{a}(t) = m\langle -a\omega^2\cos\omega t, -a\omega^2\sin\omega t\rangle$.

This can be simplified to $-m\omega^2\langle a\cos\omega t, a\sin\omega t\rangle$. As you can see the vector is just $\vec{r}(t)$.

So $\vec{F}(t) = -m\omega^2\vec{r}(t)$.

The force acts in direction opposite to radius vector $\vec{r}(t)$. It points towards the origin.

Newton's Second Law is $\vec{F} = m\vec{a}$ as we talked about earlier.

Assumptions:

- Mass is constant
- Only force acting on the object after launch is Earth's gravity
- Assume the force of gravity is constant because the object is sufficiently close to the earth

$\vec{F} = m\vec{a}$. m is mass, g is the acceleration due to gravity.

We can find $\vec{a}$ by letting $\vec{F} = -mg\vec{j}$, and we can rewrite as $m\vec{a} = -mg\vec{j}$.

This gives $\vec{a} = -g\vec{j}$.

$\vec{v}(t) = \int \vec{a}(t) = \int -g\vec{j}dt = -gt\vec{j} + c$ at $t = 0, v(0) = v_0$.

This leads us to $\vec{v}(t) = -gt\vec{j} + \vec{v_0}$.

To find position, we need to integrate once more.

$\vec{r}(t) = -\frac{1}{2}gt^2\vec{j} + \vec{v_0}t + \vec{c_2}$, we have initial conditions $\vec{r}(0) = s_0$ and $\vec{c_2} = s_0\vec{j}$ (up)

We can find that $\vec{r}(t) = -\frac{1}{2}gt^2\vec{j} + \vec{v_0}t + s_0\vec{j}$ or written as $\left(-\frac{1}{2}gt^2 + s_0\right)\vec{j} + t\vec{v_0}$

We can express $\vec{v_0}$ in two components, with the x component being $\vec{v_0}\cos\alpha$ and the y component being $\vec{v_0}\sin\alpha$.

So $\vec{v_0} = v_0\cos\alpha\vec{i} + v_0\sin\alpha\vec{j}$.

So $\vec{r}(t) = \left(-\frac{1}{2}gt^2 + s_0\right)\vec{j} + t(v_0\cos\alpha\vec{i} + v_0\sin\alpha\vec{j})$.

This simplifies to $\vec{r}(t) = (v_0\cos\alpha t)\vec{i} + (s_0 + v_0\sin\alpha t - \frac{1}{2}gt^2)\vec{j}$

So $x(t) = v_0\cos\alpha \cdot t$ and $y(t) = s_0 + v_0\sin\alpha \cdot t - \frac{1}{2}gt^2$.

Velocity in each direction is $v_x = v_0\cos\alpha$ and $v_y = v_0\sin\alpha - gt$

Example

A basketball is hit with an initial speed of 80 ft/sec at an angle of $30°$ and an initial height of 3 feet.

(a) Find parametric equations for the trajectory of the ball.

$x = 80\cos(30)t$ so $x(t) = 40\sqrt{3}t$

$y = 3 + 80\sin 30t - \frac{1}{2}(32)t^2$ so $y(t) = 3 + 40t - 16t^2$

(b) How high does the ball get?

We need to find the maximum of y so $\frac{dy}{dt} = 40 - 32t$. $0 = 40 - 32t$ and that gives $t = \frac{5}{4}$ seconds.

Substituting that back in gives $y\left(\frac{5}{4}\right) = 28$ ft.

Before $t = \frac{5}{4}$ the value is positive and after this time it is negative, so it is a maximum by the first derivative test.

(c) How far does it travel horizontally?

$0 = 3 + 40t - 16t^2$ and $t \approx 2.57$ sec. $x(2.57) \approx 178.25$ ft.

(d) What is the speed of the ball when it lands?

It lands at $t \approx 2.57$ sec and speed is $\|\vec{v}(t)\|$.

$\vec{v}(t) = \vec{r'}(t) = \langle 40\sqrt{3}, 40 - 32t\rangle$.

Speed is $\sqrt{(40\sqrt{3})^2 + (40 - 32(2.57))^2} \approx 81.19$ ft/sec

It if often useful to break acceleration into 2 components - one that is in the direction of the tangent vector and one in the direction of the normal vector.

We will define $\|\vec{v}(t)\| = v$. Then $\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\vec{v}(t)}{v}$

So, $\vec{v}(t) = v \cdot \vec{T}(t) = v \cdot \vec{T}$.

Differentiating this gives $\vec{v'} = v'\vec{T} + v\vec{T'}$.

To get $\vec{T'}$, use κ (curvature).

$\kappa = \frac{|\vec{T'}(t)|}{|\vec{r'}(t)|} = \frac{|\vec{T'}(t)|}{v} \implies |\vec{T'}(t)| = \kappa \cdot v$

Also $\vec{N} = \frac{\vec{T'}(t)}{|\vec{T'}(t)|} \implies \vec{T'}(t) = |\vec{T'}(t)|\vec{N}$

Substituting in gives $\vec{T'}(t) = \kappa v \cdot \vec{N}$.

$\vec{v'} = \vec{a}(t) = v'\vec{T} + v(\kappa v\vec{N}) = v'\vec{T} + \kappa v^2\vec{N}$

We can write $\vec{a}(t) = a_T\vec{T} + a_N\vec{N}$.

This tells us that the object always moves according to the direction of motion ($\vec{T}$) and direction the curve is turning ($\vec{N}$)

We can dot $\vec{a}(t)$ with v to get $\vec{v} \cdot \vec{a} = (v\vec{T}) \cdot (v'\vec{T} + \kappa v^2\vec{N})$

This gives us $\vec{v} \cdot \vec{a} = vv'\vec{T} \cdot \vec{T} + \kappa v^3\vec{T} \cdot \vec{N}$. Hence, $\vec{v} \cdot \vec{a} = vv'$.

We know that $v' = a_T$, so $a_T = \frac{\vec{v} \cdot \vec{a}}{v} = \frac{\vec{r'}(t) \cdot \vec{r''}(t)}{|\vec{r'}(t)|}$.

We also know that $a_N = \kappa v^2 = \frac{|\vec{r'}(t)| \times \vec{r''}(t)|}{|\vec{r'}(t)|^3} |r'\vec{(t)}^2$.

This gives $a_N = \frac{|\vec{r'}(t) \times \vec{r''}(t)|}{|\vec{r'}(t)|}$

In summary:

Scalar Tangential component of acceleration

$$a_T = \frac{\vec{r'}(t) \cdot \vec{r''}(t)}{\|\vec{r'}(t)\|}$$

Scalar Normal component of acceleration

$$a_N = \frac{\|\vec{r'}(t) \times \vec{r''}(t)\|}{\|\vec{r'}(t)\|}$$

Example

Suppose a particle moves along $C : \vec{r}(t) = \langle t, t^2, t^3 \rangle$.

(a) Find the scalar tangential and normal components of $\vec{a}$.

The first derivative is $\vec{r'}(t) = \langle 1, 2t, 3t^2 \rangle$ and $\vec{r''}(t) = \langle 0, 2, 6t \rangle$.

$\vec{r'}(t) \cdot \vec{r''}(t) = 4t + 18t^3$.

$|\vec{r'}(t)| = \sqrt{9t^4 + 4t^2 + 1}$.

So, $a_T = \frac{18t^3+4t}{\sqrt{9t^4+4t^2+1}}$.

The cross product of $\vec{r'}(t)$ and $\vec{r''}(t) = \langle 6t^2, -6t, 2 \rangle$.

The magnitude of this vector is $\sqrt{36t^4 + 36t^2 + 4}$.

The scalar normal component $a_N = \sqrt{\frac{36t^4+36t^2+4}{9t^4+4t^2+1}}$.

(b) Find the scalar tangential and normal components of $\vec{a}$ at $(1, 1, 1)$

Plug in to get $a_T = \frac{22}{\sqrt{14}}$ and $a_N = \sqrt{\frac{28}{7}}$.

(c) Find the vector tangential and normal components at $t = 1$.

$\vec{a} = a_T \vec{T} + a_n \vec{N}$.

$\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|}$. So $\vec{T}(1) = \frac{\langle 1,2,3 \rangle}{\sqrt{14}}$.

So $a_T \vec{T} = \frac{22}{\sqrt{14}} \frac{\langle 1,2,3 \rangle}{\sqrt{14}} = \langle \frac{11}{7}, \frac{22}{7}, \frac{33}{7} \rangle$.

Now to find the normal one, we can either find $\vec{N}$ or we can use that $\vec{a} = a_T \vec{T} + a_N \vec{N}$.

We know that $\vec{a}(1) = \langle 0, 2, 6 \rangle$ and we can substitute this to find $a_N \vec{N}$.

$\langle 0, 2, 6 \rangle - \langle \frac{11}{7}, \frac{22}{7}, \frac{33}{7} \rangle = a_N \vec{N} = \langle -\frac{11}{7}, -\frac{8}{7}, \frac{9}{7} \rangle$.

(d) Find the curvature of the path at the point $(1, 1, 1)$.

Remember $\kappa(t) = \frac{|\vec{r'} \times \vec{r''}|}{|\vec{r'}(t)|^3}$

Using what we previously found, $\vec{r'}(1) = \langle 1, 2, 3 \rangle$ and $\vec{r''}(1) = \langle 0, 2, 6 \rangle$.

The cross product of these gives $\langle 6, -6, 2 \rangle$.

$\kappa(1) = \frac{\sqrt{76}}{\sqrt{14}^3} = \frac{1}{14}\sqrt{\frac{38}{7}}$

Example

The position particle of a function is given by $\vec{r}(t) = \langle -5t^2, -t, t^2 + t\rangle$. At what time is the speed at a minimum?

speed is $\|\vec{v}(t)\|$.

$\vec{v}(t) = \langle -10t, -1, 2t + 1\rangle$

speed $= \sqrt{100t^2 + 1 + 4t^2 + 4t + 1} = \sqrt{104t^2 + 4t + 2}$

$\frac{d\text{speed}}{dt} = \frac{1}{2}(104t^2 + 4t + 2)^{-1/2}(208t + 4)$

$0 = \frac{1}{2}(104t^2 + 4t + 2)^{-1/2}(208t + 4)$

The first factor is never 0, the second factor is 0 when $t = -\frac{1}{52}$ sec

Now using the first derivative test, we see values before $-\frac{1}{52}$ are decreasing and after this point are positive, so $t = -\frac{1}{52}$ is a minimum.

3 Partial Derivatives

3.1 Functions of Several Variables

Before: $f(x)$ is a function in terms of x (one variable). An example of this is $y = 4x^2$.

Now: $z = f(x, y)$ (a function of 2 variables). An example is $A = \frac{1}{2}bh$. This is a function $f(b, h) = \frac{1}{2}bh$.

For $z = (x, y)$, z is the dependent variable and x and y are the independent variables.

In 3-space, this becomes $w = f(x, y, z)$.

Domain: The restrictions of the independent variables determine the domain of f.

Example

Find the domain of $f(x, y) = \ln(x, y)$.

$xy > 0$.

If both x and y are positive and x and y are negative, the product will be greater than zero.

This can be expressed as D : all ordered pairs in quadrants I and III. (not on axis)

Or written mathematically as $D : (x, y) : xy > 0$.

Example

Find the domain of $f(x, y, z) = \frac{x}{\sqrt{9-x^2-y^2-z^2}}$.

We know the quantity $9 - x^2 - y^2 - z^2 > 0$.

We can rewrite this as $x^2 + y^2 + z^2 < 9$.

The domain is D : all $(x, y, z) : x^2 + y^2 + z^2 < 9$.

This is also known as the set of all (x, y, z) inside sphere of $r = 3$ centered at the origin.

$z = f(x, y)$ is a surface in 3-space.

Example

$f(x, y) = \sqrt{4 - x^2 - y^2}$ graphed.

$f(x, y)$ is essentially z and $z \geq 0$.

Simplifying will give you $x^2 + y^2 + z^2 = 4$, which is a sphere $r = 2$ centered at the origin.

Exercise Graph $f(x, y) = 1 - x - \frac{1}{2}y$.

Definition

The level curves of a function f of two variables are the curves with equations $f(x, y) = k$ where k is a constant (in the range of f.)

Example

Describe the level curves of $z = x^2 + y^2$.

Remember from the first chapter, this will end up being a paraboloid.

Passing various planes through or making z a number gives us an idea that level curves are circles centered at the origin.

Contour plots contain sets of level curves for a function.

If we were to graph the set of level curves (the contour plot) of this equation, we would have circles of varying radii in the xy-plane.

If z was getting bigger, it would be going up, if z was getting smaller, it would be going downwards.

Exercise Sketch the contour plot of $f(x, y) = x^2 - 4y^2$.

Functions of 2 variables are surfaces we can project as level curves.

Functions of 3 variables are 4-D graphs that we can project as level surfaces.

Example

Describe the level surfaces of $f(s, y, z) = x^2 + y^2 + z^2$.

Let $w = x^2 + y^2 + z^2$.

We can write like $1 = x^2 + y^2 + z^2$, $4 = w$, $9 = w$, etc.

This is a graph in 3-space and the level surfaces are spheres centered at the origing. As distance from the origin increases, so will the value of f.

Exercise Graph the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Exercise Graph the hyperbola $y^2 - \frac{x^2}{4} = 1$

3.2 Limits and Continuity

In 2-space, $\lim\limits_{x \to x_0^+ f(x)} f(x) = \lim\limits_{x \to x_0^- f(x)} f(x)$.

In 3-space, we evaluate $\lim\limits_{(x,y) \to (x_0, y_0)} f(x, y)$.

Questions to consider:

1. Is there a point there or are we approaching a point?
2. How many directions/paths of approach do we have? Infinite.

Example

Consider $f(x, y) = \frac{-xy}{x^2+y^2}$. Find $\lim\limits_{(x,y) \to (2,1)} f(x, y)$.

$f(2, 1) = -\frac{2}{5}$.

This is the limit, since there is no funky behavior with this limit.

Example

Consider $f(x, y) = \frac{-xy}{x^2+y^2}$. Find $\lim\limits_{(x,y)\to(0,0)} f(x, y)$.

We can consider $\lim\limits_{x\to 0} = 0$ and this is along the x-axis.

0 is the limit from one direction.

Along the y-axis, we get the same limit $\lim\limits_{y\to 0} = 0$.

Along the line $y = x$, the limit is $\lim\limits_{(x,y)\to(0,0)} \frac{-x\cdot x}{x^2+x^2} = -\frac{1}{2}$.

It seems that the limit does not exist.

This process is inefficient and impractical.

Definition

Let f be a function of 2 variables and assume f is defined at all points of some open disk centered at (x_0, y_0) (except maybe at (x_0, y_0)). Then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if given $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x,y) - L| < \epsilon$ when the distance between (x,y) and (x_0,y_0) satisfies $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

Theorem 3.1

If $f(x,y) \to L$ as $f(x,y) \to (x_0,y_0)$, then $f(x,y) \to L$ as $(x,y) \to (x_0,y_0)$ along any smooth curve.

If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist along some smooth curve or if $f(x,y)$ has different values along different curves, the limit does not exist.

Options:

1. Plug in values.
2. Limit DNE $\to$ need to show that there is a path where DNE.
3. One other option with discontinuity "trick".

Example

Find $\lim\limits_{(x,y)\to(1,2)} \frac{5x^2y}{x^2+y^2}$.

Plug in numbers to get 2.

Example

Find $\lim\limits_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$.

The limit may not exist by plugging in $(0,0)$.

Start with the path along $x = 0$ and we get the limit $\lim\limits_{y\to 0} = 1$.

Along $y = x$, we get $\lim\limits_{x\to 0} = 0$.

Since these are two different limits, then the limit does not exist.

To prove that the limit does exist, you essentially have to be able to calculate it.

Definition

A function $f(x, y)$ is said to be continuous at (x_0, y_0) if $f(x_0, y_0)$ is defined and $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

1. If f is continuous on D, this means f is continuous at every point in an open set.
2. If f is continuous everywhere, this means f is continuous at every point in the xy-plane.

Theorem 3.2

$f(x, y) = g(x)h(y)$ is continuous at (x_0, y_0) if $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0.

Compositions are continuous $(f(x, y) = g(h(x, y)))$ if $h(x, y)$ is continuous at (x_0, y_0) and $g(u)$ is continuous at $u = h(x_0, y_0)$.

Example

(a) Determine continuity for $f(x, y) = 3x^2y^5$.

$g(x) = 3x^2$ and $h(y) = y^5$.

$g(x)$ and $h(y)$ are continuous everywhere, therefore $f(x, y)$ is continuous everywhere.

(b) Determine continuity for $f(x, y) = \sin(3x^2y^5)$.

We already know the function inside is continuous everywhere.

Therefore $\sin(u)$ is continuous everywhere, therefore $f(x, y)$ is continuous everywhere.

3.3 Partial Derivatives

$f(x, y)$ is a function of 2 variables x and y. What happens to x when we hold y constant?

Definition

If $z = f(x, y)$, then the first partial derivatives of f with respect to x and y are f_x and f_y, defined by $f_x(x, y) = \lim_{h\to 0} \frac{f(x+h), y - f(x,y)}{h}$ and $f_y(x, y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$ provided the limits exist.

Example

Find the first partial derivatives f_x and f_y for $f(x, y) = 3x - x^2y^2 + 2x^3y$.

For $f_x(x, y)$ we just treat y as a constant.

So we get $f_x(x, y) = 3 - 2xy^2 + 6x^2y$.

For $f_y(x, y)$ we treat x as a constant.

So $f_y(x, y) = -2x^2y + 2x^3$.

Different notations for partial derivatives are like $f_x = \frac{\partial f}{\partial x}$. If $z = f(x, y)$ then this is also equal to $\frac{\partial z}{\partial x} = z_x$.

Example

Consider $f(x, y) = xe^{x^2y}$. Find f_x and f_y and evaluate both at $(1, \ln 2)$.

$f_x = e^{x^2y} + 2x^2ye^{x^2y}$

$f_y = x^3 e^{x^2 y}$

Evaluating both of these at the point $(1, \ln 2)$:

$f_x(1, \ln 2) = e^{\ln 2} + 2(\ln 2)e^{\ln 2} = 2 + 4\ln 2$

$f_y(1, \ln 2) = 1 \cdot e^{\ln 2} = 2$

Geometrically $\frac{\partial f}{\partial x}$ is the slope in the x-direction. It tells us how f (or z) changes with respect to x when y is constant.

Example

Find slopes in x and y directions at $(1/2, 1)$ for $z = f(x, y) = -\frac{1}{2}x^2 - y^2 + \frac{25}{8}$. Then interpret these slopes.

$f_x = -x$

$f_y = -2y$

$f_x(1/2, 1) = -\frac{1}{2}$

$f_y(1/2, 1) = -2$

f_x is $\frac{\Delta z}{\Delta x}$ so this tells us z decreases 1 unit for every 2 unit increase in x at this point.

f_y is $\frac{\Delta z}{\Delta y}$ so this tells us that z decreases 2 units for every unit increase in y at this point.

Functions of 3 or more variables have a similar idea. You just hold the other variables constant.

Notation for higher-order partial derivatives. If we did the partial derivative of x and then the partial derivative of x once more, we would get $\frac{\partial^2 f}{\partial x^2} = f_{xx}$. If we did the partial derivative of y after the partial derivative of x we would get f_{xy}.

Example

Find f_{xx}, f_{yy}, f_{xy} and f_{yx} for $f(x, y) = 3xy^2 - 2y + 5x^2y^2$.

$f_x = 3y^2 + 10xy^2$, so $f_{xx} = 10y^2$ and $f_{xy} = 6y + 20xy$

$f_y = 6xy - 2 + 10x^2y$ so $f_{yy} = 6x + 10x^2$ and $f_{yx} = 6y + 20xy$

You will notice that $f_{xy} = f_{yx}$ in this case.

Theorem 3.3

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then for every (x, y) in r, $f_{xy}(x, y) = f_{yx}(x, y)$.

Example

Suppose $f(x, y, z) = ye^x + x\ln z$. Show that $f_{xzz} = f_{zxz} = f_{zzx}$.

$f_x = ye^x + \ln z$, so $f_{xz} = 1/z$ and $f_{xzz} = -\frac{1}{z^2}$.

$f_z = \frac{x}{z}$, so $f_{zx} = \frac{1}{z}$ and $f_{zxz} = -\frac{1}{z^2}$.

$f_z = \frac{x}{z}$ and $f_{zz} = -\frac{x}{z^2}$ and $f_{zzx} = -\frac{1}{z}^2$.

These three are equal to each other so this shows that order does not matter.

Example

Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the y-direction at $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

The goal is to find $\frac{\partial z}{\partial y} = z_y$ at the point given.

Let's start by differentiating by y.

We get $\frac{\partial}{\partial y} = \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(z^2) = \frac{\partial}{\partial y}(1)$.

This is $2y + 2z\frac{\partial z}{\partial y} = 0$.

So $\frac{\partial z}{\partial y} = -\frac{y}{z}$.

At the point given, this evaluates to $-\frac{1}{2}$.

3.4 Tangent Planes and Linear Approximations

Previously, if we consider the graph of 2-D differentiable function, when we zoom in on the graph a lot, the graph begins to look like its tangent line. We can approximate the value of the function at a specific point using this tangent line.

If we consider the graph of a 3-D differentiable function, when we zoom in on the graph a lot, the graph begins to look like its tangent plane. How do we find the equation of this tangent plane? If we find 2 tangent lines, then this tangent plane will contain both lines.

We choose 2 lines that result when they intersect $z = f(x, y)$ with the planes $y = y_0$ and $x = x_0$.

If we recall the equation of a plane: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ where (x_0, y_0, z_0) is a point on this plane, then $z - z_0 = -\frac{A}{C}(x - x_0) - \frac{B}{C}(y - y_0)$.

Let's call $-\frac{A}{C} = a$ and $-\frac{B}{C} = b$. So we get $z - z_0 = a(x - x_0) + b(y - y_0)$. This is the general form of a plane.

Now we use the 2 tangent lines that result from the planes $y = y_0$ and $x = x_0$.

The first plane results $T_1 : y = y_0$. We get $z - z_0 = a(x - x_0)$. We are looking at a line in point slope form with slope a. a is f_x.

Likewise, the second tangent line is $T_2 : x = x_0$. We get $z - z_0 = b(y - y_0)$. Similar idea, the slope b is f_y.

This leads us to the equation of the tangent plane as

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example

Find an equation of the tangent plane $z = 2x^2 + y^2$ at $(1, 1, 3)$.

Recall this equation makes an elliptic paraboloid.

$f_x = 4x$, so $f_x(1, 1) = 4$.

$f_y = 2y$ so $f_y(1, 1) = 2$.

What this gives us is $z - 3 = 4(x - 1) + 2(y - 1)$.

So $z = 4x + 2y - 3$.

In the above example, the linear function $L(x, y) = 4x + 2y - 3$ is a good approximation of $f(x, y)$ when (x, y) is near $(1, 1)$.

For example, $L(1.1, 0.95) = 3.3$. If we find $f(1.1, 0.95)$ this is equal to 3.3225.

L is called the linearization of f at $(1, 1)$.

The approximation is called the linear approximation or tangent line approximation of f at $(1, 1)$.

Recalling the equation of a tangent plane: $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

So the linearization of f at (a, b) is $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

And the linear approximation $f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$.

But only if $f(x, y)$ is differentiable at (a, b).

Definition

If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form $\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ where $\epsilon_1, \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Theorem 3.4

If f_x, f_y exist near (a, b) are are continuous at (a, b), then f is differentiable at (a, b). For functions of $3+$ variables it is similar.

Example

Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization. Then use it to approximate $f(1.1, -0.1)$.

$f_x(x, y) = e^{xy} + xye^{xy}$

$f_y(x, y) = x^2 e^{xy}$

These are both continuous at $(1, 0)$ and exist near $(1, 0)$. Therefore $f(x, y)$ is differentiable at $(1, 0)$.

$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$

$L(x, y) = 1 + 1(x - 1) + 1(y - 0) = x + y$.

$f(x, y) = xe^{xy} \approx x + y$.

$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$

In 2-space, $dy = f'(x)dx$.

Δy is the actual change in y from $x = a$ to $x = a + \Delta x$ and dy is the change in y when using tangent line.

When Δx is small, dy can be used to approximate Δy.

This is a similar idea in 3-space.

Total differential: $dz = f_x(x, y)dx + f_y(x, y)dy$ OR $dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$.

dx and dy are independent variables so these can be any numbers.

We take $dx = \Delta x = x - a$ so the linear approximation becomes $f(x, y) \approx f(a, b) + dz$

Example

Consider $z = xy^2$.

(a) Find the total differential.

$f_x = y^2$ and $f_y = 2xy$ so $dz = y^2 dx + 2xydy$.

(b) Approximate the change in z from $(0.5, 1.0)$ to $(0.503, 1.004)$.

$dz = (1)^2(0.503 - 0.5) + 2(0.5)(1)(1.004 - 1) = 0.007$

(c) Compare the approximation to the actual change.

$\Delta z = f(0.503, 1.004) - f(0.5, 1.0) = (0.503)(1.004)^2 - (0.5)(1)^2 = 0.007032\ldots$

Example

The base radius and height of a right circular cone are 10 cm and 25 cm with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

$V = \frac{1}{3}\pi r^2 h$.

So $V_r = \frac{2}{3}\pi rh$ and $V_h = \frac{1}{3}\pi r^2$

$dv = \frac{2}{3}\pi rhdr + \frac{1}{3}\pi r^2 dh$

The $|\Delta r| \leq 0.1$ and $|\Delta h| \leq 0.1$.

$dv = 20\pi \approx 63$ cm^3 from the formula.

This may over/underestimate the volume by as much as 63 cm^3.

Example

The dimensions of a rectangular box are 75 cm, 60 cm, and 40 cm. Each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated.

$V = xyz$, so $v_x = yz$, $v_y = xz$, and $v_z = xy$.

$dv = (yz)dx + (xz)dy + (xy)dz$, so $dv = 1980$ cm^3.

The volume of the box is 180000 cm^3, so the error is $\frac{1980}{180000} = 1.1\%$.

3.5 The Chain Rule

In previous the past, you were given $y(x)$ and $x(t)$. If you wanted to find $\frac{dy}{dx}$ for $y(x(t))$, you used the formula $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$.

Now we are looking at two variables, such as $z = f(x, y)$ where $x = x(t)$ and $y = y(t)$. Then the composition $z = f(x(t), y(t))$ expresses z as a single variable.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example

Consider $z = x^2 y - y^2$ where $x = \sin t$ and $y = e^t$. Find $\frac{dz}{dt}$ at $t = 0$.

$\frac{dz}{dt} = (2xy)(\cos t) + (x^2 - 2y)(e^t)$.

Replacing with what was defined gives $\frac{dz}{t} = 2\sin t \cos t e^t + (\sin^2 t - 2e^t)e^t$.

At $t = 0$, this is -2.

Without the chain rule, we would have to do $z(t) = (\sin^2 t)e^t - e^{2t}$. Then the derivative of this would be $2\sin t \cos t e^t + e^t \sin^2 t - 2e^{2t}$.

Suppose that u is a differentiable function of n variables, $x_1, x_2, \ldots, x_n$, and each x_i is a differentiable function of m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

If $z = f(x(t), y(t))$, then we can see that we can do partial derivatives to x and y, but non partial derivatives to get to t. So $\frac{dz}{dt} = \frac{\partial t}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$.

Example

Consider $z = f(x, y)$ where $x = x(u, v)$ and $y = y(u, v)$. Write the chain rule for $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

So we need partial derviatives to go from z to x to u and z to x to v, and likewise for y, we see we get that $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$.

Similarly, $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$.

Example

Consider $z = 2xy$ where $x = u^2 + v^2$ and $y = \frac{u}{v}$. Find $\frac{\partial z}{\partial u}$.

So $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$.

This is equal to $(2y)(2u) + (2x)\left(\frac{1}{v}\right)$.

We want this in terms of u.

Substitute to get $\frac{\partial z}{\partial u} = \left(2 \cdot \frac{u}{v}\right)(2u) + (2 \cdot (u^2 + v^2))\left(\frac{1}{v}\right) = \frac{4u^2}{v} + \frac{2u^2+2v^2}{v} = \frac{6u^2+2v^2}{v}$.

Example

Find a rule for $\frac{dw}{dx}$ if $w = xy + yz$, $y = \sin x$ and $z = e^x$.

$\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial w}{\partial t} \cdot \frac{\partial z}{\partial x}$.

This is $\frac{dw}{dx} = (y) + (x + z)(\cos x) + (y)(e^x)$.

Now everything needs to be in terms of x.

$\frac{dw}{dx} = \sin x + (x + e^x) \cos x + \sin x e^x$

Example

Scenario: Consider $y^3 + y^2 - 5y - x^2 + 4 = 0$.

Previously, we would use implicit differentiation.

$3y^2 \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$.

Get $\frac{dy}{dx} = \frac{2x}{3y^2+2y-5}$.

With partial derivatives: $f(x, y) = C$.

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$.

Isolate $\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$

$\frac{dy}{dx} = \frac{-(-2x)}{3y^2+2y-5}$.

Example

Consider $x^2 + y^2 + z^2 = 1$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$f(x, y, z) = x^2 + y^2 + z^2 = C = 1$

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$

$\frac{\partial z}{\partial x} = \frac{-\partial f/\partial x}{\partial f/\partial z}$

Likewise, $\frac{\partial z}{\partial y} = \frac{-\partial f/\partial y}{\partial f/\partial z}$

And we get $\frac{\partial z}{\partial x} = -\frac{x}{z}$.

And $\frac{\partial z}{\partial y} = -\frac{y}{z}$.

Implicitly we could use $2x\frac{\partial x}{\partial y} + 2y\frac{\partial y}{\partial y} + 2z\frac{\partial z}{\partial y} = 0$ and get $2y + 2z\frac{\partial z}{\partial y} = 0$ to get $\frac{\partial z}{\partial y} = -\frac{y}{z}$.

3.6 Directional Derivatives and the Gradient Vector

3.7 Maximum and Minimum Values

3.8 Lagrange Multipliers

4 Multiple Integrals

5 Vector Calculus

5.1 Vector Fields

5.2 Line Integrals

5.3 The Fundamental Theorem for Line Integrals

5.4 Green's Theorem

5.5 Curl and Divergence

5.6 Surface Integrals

5.7 Stokes' Theorem

5.8 The Divergence Theorem