Calculus III Notes

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Contents

1 Vectors and the Geometry of Space

1.1 Vectors in the Plane

Vectors are quantities with both magnitude and direction. The vector who's tail is at point P and head at point Q is denoted as $\overrightarrow{PQ}$.

Two vectors, **u** and **v** are equal if they have equal length and point in the same direction. They do not necessarily have to be in the same location.

Scalar multiplication happens when a c is multiplied to vector **v**. If $c < 0$, then vector cv and v will point in opposite directions, otherwise they will point in the same direction. Two vectors are parallel if they are scalar multiples of each other.

If you place the tail of a vector **v** at the head of another vector **u**, the sum **u+v** is the vector that extends from the tail of **u** to the head of **v**.

The vector difference **u-v** is defined as **u+(-v)**.

In order to do calculations with vectors, we must introduce a cartesian plane. Angle brackets $\langle a, b\rangle$ show the components of a vector.

The magnitude of a vector is simply its length. Given points $P(x_1, y_1)$ and $Q(x_1, y_1)$, the magnitude of the vector $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1\rangle$ is denoted as $|\vec{PQ}|$, is equal to:

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We can also now do vector addition with components. Given two vectors **u** and **v**, the vector sum is:

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2\rangle$$

For a scalar c and a vector **u**, the scalar multiple is c**u**. i.e $|c\mathbf{u}| = |c||\mathbf{u}|$

A unit vector is any vector with length 1. **i** is a unit vector in the x-direction and **j** is a unit vector in the y-direction.

Example

Determine the necessary air speed and heading that a pilot must maintain in order to fly her commercial jet north at a speed of 480 mi/hr relative to the ground in a crosswind that is blowing $60\deg$ south of east at 20 mi/hr.

Let $\vec{p}$ be the velocity vector we are trying to find.

We have ground vector $\langle 0, 480\rangle$ and a crosswind vector $\langle 10, -10\sqrt{3}\rangle$. Note we got the x-component and the y-component of the crosswind vector from the formula: $\vec{v} = \langle a\cos\theta, b\cos\theta\rangle$

We can find the vector $\vec{p}$ from adding this vector to the crosswind vector resulting in: $\vec{p} = \vec{g} - \vec{c} = \langle -10, 480 + 10\sqrt{3}\rangle$.

The magnitude of this vector is the speed and is equal to 497.2 mi/hr roughly.

1.2 Vectors in Three Dimensions

We can create a z-axis to create a three dimensional system.

The xyz-plane is divided into octants and has 3 planes, the xy-plane, the xz-plane, and the yz-plane.

We can also extend the distance formula to 3 dimensions. It is similar to the distance formula in two dimensions, with the z-component added, essentially:

$$|PQ| = \sqrt{|PR|^2 + |RQ|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The midpoint formula works the same way:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2} + \frac{y_1 + y_2}{2} + \frac{z_1 + z_2}{2}\right)$$

The normal form of the circle equation is:

$$(x - h)^2 + (y - k)^2 = r^2$$

For a disk we have

$$(x - h)^2 + (y - k)^2 \leq r^2$$

We can generalize this to a sphere. A sphere centered at (a, b, c) with radius r is the set of points satisfying:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

A ball centered at (a, b, c) with radius r is the set of points satisfying:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2$$

Example

Find an equation of the sphere passing through $P(-4, 2, 3)$ and $Q(0, 2, 7)$ with its center at the midpoint of PQ.

We can find the midpoint from the midpoint formula and it is equal to $(-2, 2, 5)$.

The radius can be found through the distance formula and is equal to $\sqrt{8}$.

The equation is $(x + 2)^2 + (y - 2)^2 + (z - 5)^2 = 8$

All the vector operations from two-dimensions work in three-dimensions.

Example

A model airplane is flying horizontally due east at 10 mi/hr when it encounters a horizontal crosswind blowing south at 5 mi/hr and an updraft blowing vertically upward at 5 mi/hr.

- Find the position vector that represents the velocity of the plane relative to the ground.
- Find the speed of the plane relative to the ground.

The velocity vector of the model plane $\vec{p}$ is equal to $\langle 10, 0, 0\rangle$

The velocity vector of the horizontal crosswind $\vec{w}$ is equal to $\langle 0, -5, 0\rangle$

The velocity vector of the updraft $\vec{u}$ is $\langle 0, 0, 5\rangle$

Adding the three vectors results in the speed: $\langle 10, -5, 5\rangle$

The magnitude of this is roughly 12.25.

1.3 Dot Products

Definition

Given two nonzero vectors **u** and **v** in two or three dimensions, the dot product is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$$

The dot product is 0 when $\theta = \frac{\pi}{2}$, negative when $\theta > \frac{\pi}{2}$ and positive when $\theta < \frac{\pi}{2}$.

Two vectors are parallel if and only if $\mathbf{u}{\cdot}\mathbf{v}{=}{\pm}|\mathbf{u}||\mathbf{v}|$

When the dot product is zero, we call $\vec{u}$ and $\vec{v}$ orthogonal.

Theorem 1.1: Dot Product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

We now apply the dot product to vector projections.

Definition

The orthogonal projection of **u** on **v**, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$ is:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right)$$

The orthogonal projections can also be computed with the formulas:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{scal}_{\mathbf{v}}\mathbf{u} \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v}$$

where the scalar component of **u** in the direction of **v** is

$$\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$$

1.4 Cross Products

Definition

Given two vectors **u** and **v** in $\mathbb{R}^3$, the cross product $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$$

Note that $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

There are some useful properties of the cross product.

- The cross product $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
- The cross product is zero when $\sin(\theta) = 0$.
- Two vectors are parallel if the cross product between them is zero.

We define the determinant of a 2×2 array $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ as $ad - bc$.

For a matrix $\begin{vmatrix} a & b & c \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

the determinant is $a\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - b\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + c\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$ or:

$$a(u_2v_3 - u_3v_2) - b(u_1v_3 - u_3v_1) + c(u_1v_2 - u_2v_1)$$

1.5 Lines and Planes in Space

Recall that in two dimensions, we needed a point and a slope to write an equation for a line.

We can write an equation in three dimensions as well.

A vector equation of a line passing through the point $P_0(x_0, y_0, z_0)$ in the direction of vector $\mathbf{v} = \langle a, b, c\rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ or:

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

The corresponding parametric equations of the line also are:

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct$$

The general equation of a plane in R^3 with the plane passing through $P_0(x_0, y_0, z_0)$ with vector $\mathbf{v} = \langle a, b, c\rangle$ is described by:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

1.6 Cylinders and Quadric Surfaces

A cylinder is a surface that is parallel to a line.

A trace of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the xy-trace, the yz-trace, and the xz-trace.

To sketch quadric surfaces:

- Determine the points where the surface intersects the coordinate axes.
- Finding traces of the surface helps visualize the surface
- Sketch at least two traces in parallel planes

Example

For:

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 1$$

We set certain variables to zero to find the x-intercept to be ± 3, the y-intercept to be ± 4 and the z-intercept to be ± 5.

We can also find the traces:

- xy: $\frac{x^2}{9} + \frac{y^2}{16} = 1$
- xz: $\frac{x^2}{9} + \frac{z^2}{25} = 1$
- yz: $\frac{y^2}{16} + \frac{z^2}{25} = 1$

The resulting shape is a ellipsoid.

In general the equation of an ellipsoid is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

In this all traces are ellipses.

For an elliptic cone the equation is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.

For an elliptic paraboloid the equation is:

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.

For a hyperbolic paraboloid:

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ or $y = y_0$ are parabolas.

For a hyperboloid of one sheet:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Traces with $z = z_0$ are ellipses for all z_0. Traces with $x = x_0$ or $y = y_0$ are hyperbolas.

For a hyperboloid of two sheets:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Traces with $z = z_0$ with $|z_0| > |c|$ are ellipses. Traces with $x = x_0$ and $y = y_0$ are hyperbolas.

2 Vector-Valued Functions

2.1 Vector-Valued Functions

A vector valued function has the form $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

A set of parametric equations $x = x(t), y = y(t), z = z(t)$ can describe a curve in space.

It can also be viewed as a vector function where each variable varies with respect to an independent variable t.

A point $(x(t), y(t), z(t))$ on the curve is the head of the vector $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

We can consider the vector-valued function of the form:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

or

$$f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

where f, g, h are defined on some interval a to b.

The positive orientation of a curve is the direction the curve is generated as the parameter increases.

Example

Find the domain of $\mathbf{r}(t) = \frac{4}{\sqrt{1-t}}\mathbf{i} + \frac{2}{t+3}\mathbf{j}$

The domain is the largest set of values of t on which both f and g are defined.

The first component has $1 - t > 0$, therefore the domain is $(-\infty, 1)$.

The second component's domain is $(-\infty, -3) \cup (-3, \infty)$.

We now find the intersection which is $(-\infty, -3) \cup (-3, 1)$

Definition

A vector valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$$

, provided

$$\lim_{t \to a} |\mathbf{r}(t) - \mathbf{L}| = 0$$

A function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is continuous at $\mathbf{a}$ provided $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$.

2.2 Calculus of Vector-Valued Functions

We can define the derivative of a vector-valued function as:

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

The result of the derivative is a vector-valued function.

Geometrically, as $\Delta t \to 0$, $\frac{\Delta \mathbf{r}}{\Delta t} \to \mathbf{r}'(t)$, which is a tangent vector at point P.

Much like single variable calculus, we can simply use the power rule as we know, rather than the limit definition.

The unit tangent vector for a particular value t is:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

A vector-valued function is smooth if it is differentiable and the derivative is not equal to $\langle 0, 0, 0\rangle$.

We can also integrate vector-valued functions. The rules remain the same as in single variable calculus, just breaking it into three vector components.

2.3 Motion in Space

The derivative of the position vector function is the velocity vector function.

The speed of a scalar function is

$$|v(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

Acceleration is the derivative of the velocity vector function.

Straight-line motion has a uniform velocity. Given:

$$\mathbf{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle$$

the velocity $\mathbf{v}(t) = \langle a, b, c\rangle$

Circular motion has constant $|r(t)|$. We let $\mathbf{r}(t) = \langle A\cos t, A\sin t\rangle$.

$\mathbf{r}(t)$ describes a circular trajectory counter-clockwise around a circle with radius A and center at the origin.

We have:

$$|\mathbf{r}(t)| = \sqrt{(A\cos t)^2 + (A\sin t)^2} = A$$

and

$$\mathbf{v}(t) = \langle -A\sin t, A\cos t\rangle$$

as well as

$$\mathbf{a}(t) = \langle -A\cos t, -A\sin t\rangle = -\mathbf{r}(t)$$

Also there are some important properties - the position and acceleration vectors are both orthogonal to the velocity vector as seen:

- $\mathbf{r}(t) \cdot \mathbf{v}(t) = -A^2\cos t\sin t + A^2\sin t\cos t = 0$
- $\mathbf{a}(t) \cdot \mathbf{v}(t) = A^2\sin t\cos t - A^2\cos t\sin t = 0$

Let $\mathbf{r}$ describe a path on which $|\mathbf{r}|$ is constant.

We can show that $\mathbf{r} \cdot \mathbf{v} = 0$, showing that the position and velocity vectors are always orthogonal.

For two-dimensional motion in a gravitational field:

The gravitational force is $\mathbf{F} = \langle 0, -mg\rangle$.

Therefore: $\mathbf{F} = m\mathbf{a}(t) = \langle 0, -mg\rangle$.

This shows that $\mathbf{a}(t) = \langle 0, -g\rangle$.

We can summarize this:

The velocity of the object is

$$\mathbf{v}(t) = \langle x'(t), y'(t)\rangle = \langle u_0, -gt + v_0\rangle$$

where $\mathbf{v}(0) = \langle u_0, v_0\rangle$ and $\mathbf{r}(0) = \langle x_0, y_0\rangle$.

The position is:

$$\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle u_0 t + x_0, -\frac{1}{2}gt^2 + v_0 t + y_0\rangle$$

2.4 Length of Curves

We know the arc length of a parametric equation is:

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\mathrm{d}t$$

Now we can consider this equation in three dimensions.

The arc length of a parametrized curve is:

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\mathrm{d}t = \int_a^b |\mathbf{r}'(t)|\mathrm{d}t$$

To find the arc length of a curve given $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ for $t \geq a$, we have:

$$s(t) = \int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2}\mathrm{d}u = \int_a^t |\mathbf{v}(u)|\mathrm{d}u$$

Suppose $|\mathbf{v}(t)| = 1$, then we have $t - a$. This shows the parameter t corresponds to arc length.

2.5 Curvature and Normal Vectors

Curvature will be a measure of how fast a curve $\mathbf{r}(t)$ turns at a point.

Recall the unit tangent vector:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The curvature is

$$\kappa(s) = \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right|$$

where s denotes arc length, and $\mathbf{T}$ denotes the tangent vector.

Lines have zero curvature.

We can write the curvature in terms of arclength:

$$\kappa(t) = \frac{1}{|\mathbf{v}|}\left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right| \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

Circles have constant curvature of $\frac{1}{R}$.

An alternative curvature formula is used for trajectories of moving objects in three-space:

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

where $\mathbf{v} = \mathbf{r'}$ is the velocity and $\mathbf{a} = \mathbf{v'}$ is the acceleration.

The principal unit normal vector will determine the direction in which the curve turns.

The principal unit normal vector at point P on the curve at which $kappa \neq 0$ is:

$$\mathbf{N}(s) = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}s}{|\mathrm{d}\mathbf{T}/\mathrm{d}s|} = \frac{1}{\kappa}\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$$

For other parameters t, we use the equivalent formula:

$$\mathbf{N}(s) = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}t}{|\mathrm{d}\mathbf{T}/\mathrm{d}t|}$$

There are two ways to change the velocity of an object or accelerate - to change its speed or its direction of motion.

The acceleration vector of an object moving in space along a smooth curve has the following representation of its tangential component a_T (in the direction of $\mathbf{T}$) and its normal component a_N (in the direction of $\mathbf{N}$):

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$$

where $a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}}{|\mathbf{v}|}$ and $a_T = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$.

Alternatively $\mathbf{a}\cdot\mathbf{N} = a_N$ and $\mathbf{a}\cdot\mathbf{T} = a_T$.

If we have the unit tangent and principal unit vectors $\mathbf{T}$ and $\mathbf{N}$. The unit binormal vector at each point in the curve is:

$$\mathbf{B} = \mathbf{T}\times\mathbf{N}$$

and the torsion is:

$$\tau = -\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\cdot\mathbf{N}$$

The binormal vector is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$.

$|\tau| = \left|\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\right|$ and the torsion gives the rate at which the curve moves out of the osculating plane formed by $\mathbf{T}$ and $\mathbf{N}$.

3 Functions of Several Variables

3.1 Graphs and Level Curves

Definition

A function $z = f(x, y)$ assigns to each point (x, y) in a set D in $\mathbb{R}^2$ a unique real number z in a subset of $\mathbb{R}$. The set D is the domain of f. The range of f is the set of all real numbers z that are assumed as the points (x, y) vary over the domain.

Example

Find the domain of

$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2 - 25}}$$

Note that the denominator cannot be zero or is negative.

Therefore, the domain is $D = (x, y) : x^2 + y^2 > 25$.

Definition

For a function of two variables $f(x, y)$ a level curve is the set of points (x, y) in the xy-plane where $f(x, y)$ is equal to a constant z_0.

We can extend our defintions to functions of more than two variables.

Definition

For a function of three variables $f(x, y, z)$ a level surface is the set of points (x, y, z) in xyz-space where $f(x, y, z)$ is equal to a constant w_0.

We can describe level surfaces as well.

3.2 Limits and Continuity

We can write the limit of two variables as:

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L$$

There are some laws of limits.

Theorem 3.1

Let a, b, and c be real numbers.

- Constant function $f(x, y) = c$:

$$\lim_{(x,y)\to(a,b)} c = c$$

- Linear function $f(x, y) = x$:

$$\lim_{(x,y)\to(a,b)} x = a$$

- Linear function $f(x, y) = y$:

$$\lim_{(x,y)\to(a,b)} y = b$$

If we let the limit of a function $f(x, y) = L$ and the limit of a function $g(x, y) = M$, we can see:

- The sum of the limits is $L + M$.
- The difference of the limits is $L - M$.
- If we multiply a function by a constant c, the limit is cL.
- The product of the limits is LM.
- The difference of the limits is $\frac{L}{M}$.
- Putting a limit to a power n is L^n.

Definition

Let R be a region in $\mathbb{R}^2$. An interior point P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R.

A boundary point Q of R lies on the edge of R in the sense that every disk centered at Q contains at least one point in R and at least one point not in R.

Definition

A region is open if it consists entirely of interior points. A region is closed if it contains all its boundary points.

The two-path test for nonexistence of limits:

If $f(x, y)$ approaches the two different values as (x, y) approaches (a, b) along two different paths in the domain of f, then the limit does not exist.

Definition

The function f is continuous at the point (a, b) provided:

- f is defined at (a, b).
- The limit for f exists.
- The two quantities must be equal.

3.3 Partial Derivatives

The idea of a partial derivative is to differentiate f with respect to one variable, treating the other variable as a constant.

Definition

The partial derivative of f with respect to x at the point (a, b) is:

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

The partial derivative of f with respect to y at the point (a, b) is:

$$f_y(a, b) = \lim_{h\to 0} \frac{f(a + h) - f(a, b)}{h}$$

provided these limits exist.

The notation for these are:

$$f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

We can first partially differentiate f with respect to x and then with respect to x denoted as:

$$\frac{\partial^2 f}{\partial x^2}$$

We can do it first with respect to x and then with respect to y denoted as:

$$\frac{\partial^2 f}{\partial y \partial x}$$

We can also differentiate with respect to y and then with respect to x:

$$\frac{\partial^2 f}{\partial x \partial y}$$

Lastly, we can differentiate with respect to y twice:

$$\frac{\partial^2 f}{\partial y^2}$$

Theorem 3.2: Equality of Mixed Partial Derivatives

Assume f is defined on an open set D of $\mathbb{R}^2$, and that f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

3.4 The Chain Rule

Theorem 3.3

Let z be a differentiable function on x and y on its domain, where x and y are differentiable functions of t on an interval I. Then

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Let's suppose that we have a function $w = f(x, y, z)$ where x, y, z depend on t.

Then we have:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial w}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

Theorem 3.4

Let z be a differentiable function of x and y, where x and y are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Theorem 3.5

Let F be differentiable on its domain and suppose $F(x, y) = 0$ defines y as a differentiable function of x. Provided $F_y \neq 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$$

3.5 Directional Derivatives and the Gradient

Suppose we have the function $f(x, y)$.

$f_x(x, y)$ measures the rate of change in the direction of positive x-axis and $f_y(x, y)$ measures the rate of change in the direction of positive y-axis.

Definition

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The directional derivative of f at (a, b) in the direction of u is

$$D_{\mathbf{u}}f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

provided the limit exists.

The directional derivative $D_{\mathbf{u}}f(a, b)$ measures the rate of change of f at the point (a, b) in the direction of $\mathbf{u}$.

Theorem 3.6

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The directional derivative of f at (a, b) in the direction of u is

$$D_{\mathbf{u}}f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle$$

Definition

Let f be differentiable at the point (x, y). The gradient of f at (x, y) is the vector-valued function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

The applications of this are:

Directional derivatives: $D_{\mathbf{u}}f(a, b) = \nabla f(a, b) \cdot \mathbf{u}$

The gradient gives the direction of maximum increase of a function.

Gradient is normal to level curves.

Theorem 3.7

Let f be differentiable at (a, b) with $\nabla(a, b) \neq 0$.

f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $|\nabla f(a, b)|$. (This will be maximized if $\theta = 0$.)

f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-|\nabla f(a, b)|$. (This will be minimized if $\theta = \pi$.)

The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

Theorem 3.8

Given a function f differentiable at (a,b), the line tangent to the level curve of f at (a,b) is orthogonal to the gradient $\nabla f(a,b)$, provided $\nabla f(a,b) \neq 0$.

We can now consider gradients of functions of three variables.

Theorem 3.9

Let f be differentiable at (a,b,c) and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector. The directional derivative of f at (a,b,c) in the direction of $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}} f(a,b,c) &= \nabla(a,b,c) \cdot \mathbf{u} \\ &= \langle f_x(a,b,c), f_y(a,b,c), f_z(a,b,c) \rangle \cdot \langle u_1, u_2, u_3 \rangle \end{aligned}$$

Assuming $\nabla f(a,b,c) \neq 0$, the gradient in three dimensions has the following properties.

f has its maximum rate of increase at (a,b,c) in the direction of the gradient $\nabla f(a,b,c)$, and the rate of change in this direction is $|\nabla f(a,b,c)|$.

f has its maximum rate of decrease at (a,b,c) in the direction of $-\nabla f(a,b,c)$, and the rate of change in this direction is $-|\nabla f(a,b,c)|$.

The directional derivative is zero in any direction orthogonal to $\nabla f(a,b,c)$.

3.6 Tangent Planes and Linear Approximation

Definition

Let F be differentiable at the point $P_0(a,b,c)$ with $\nabla F(a,b,c) \neq 0$. The plane tangent to the surface $F(x,y,z) = 0$ at P_0, called the tangent plane, is the plane passing through P_0 orthogonal to $\nabla F(a,b,c)$. An equation of the tangent plane is

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$$

Definition

Let f be differentiable at the point (a,b). An equation of the plane tangent to the surface $z = f(x,y)$ at the point $(a,b,f(a,b))$ is

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

One of the main applications of tangent planes is linear approximation for a function near a point.

Definition

Let f be differentiable at (a,b). The linear approximation to the surface $z = f(x,y)$ at the point $(a,b,f(a,b))$ is the tangent plane at that point, given by the equation

$$L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

For a function of three variables, the linear approximation to $w = f(x,y,z)$ at the point $(a,b,c,f(a,b,c))$ is given by

$$L(x,y,z) = f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) + f(a,b,c)$$

The idea for linear approximations of functions gives the defintion of the differential.

Definition

Let f be differentiable at the point (x, y). The change in $z = f(x, y)$ as the independent variables change from (x, y) to $(x + \mathrm{d}x, y + \mathrm{d}y)$ is denoted Δz and is approximated by the differential $\mathrm{d}z$.

$$\Delta z \approx \mathrm{d}z = f_x(x, y)\mathrm{d}x + f_y(x, y)\mathrm{d}y$$

Note that Δz is the change in f on the surface and is equal to $f(x, y) - f(a, b)$.

Note that $\mathrm{d}z$ is the change in l on the tangent plane and is equal to $L(x, y) - f(a, b)$.

3.7 Maximum/Minimum Problems

Definition

Suppose (a, b) is a point in a region R on which $f(x, y)$ is defined. If $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a local maximum value of f. If $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a local minimum value of f. Local maximum and local minimum values are also cllaed local extreme values or local extrema.

Theorem 3.10

If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.

Definition

An interior point (a, b) in the domain of f is a critical point of f is either:

- $f_x(a, b) = f_y(a, b) = 0$ or
- at least one of the partial derivatives f_x and f_y does not exist at (a, b).

Theorem 3.11

Suppose the second partial derivatives of f are continuous throughout an open disk centered at the point (a, b), where $f_x(a, b) = f_y(a, b) = 0$. Let $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$.

- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b).
- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b).
- If $D(a, b) < 0$, then f has a saddle point at (a, b).
- If $D(a, b) = 0$, then the test is inconclusive.

Definition

Let f be defined on a set R in $\mathbb{R}^2$ containing the point (a, b). If $f(a, b) \geq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an absolute maximum value of f on R. If $f(a, b) \leq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an absolute minimum value of f on R.

We can describe the procedure for finding the absolute maximum/minimum values on closed bounded sets:

Let f be continuous on a closed bounded set R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R:

- Determine the values of f at all critical points in R.
- Find the maximum and minimum values of f on the boudnary of R.
- The greatest function value found in the previous steps is the absolute maximum value of f on R, and the least function value found is the absolute minimum value of f on R.

3.8 Lagrange Multipliers

The goal is to maximize (or minimize) a function $f(x, y)$ subject to the constraint $g(x, y) = 0$.

To find find absolute extrema on closed and bounded constraint curves using Lagrange Multipliers:

Let the objective function f and the constraint function g be differentiable on a region of $\mathbb{R}^2$ with $\nabla g(x, y) \neq 0$ on the curve $g(x, y) = 0$. To locate the absolute maximum and minimum values of f subject to the constraint $g(x, y) = 0$, carry out the following steps:

1. Find the values of x, y, and λ (if they exist) that satisfy the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y) \qquad \text{and} \qquad g(x, y) = 0$$

2. Evalute f at the values (x, y) found in Step 1 and at the endpoints of the constraint curve (if they exist). Select the largest and smallest corresponding function values. These values are the absolute maximum and minimum values of f subject to the constraint.

(This section is kinda annoying, I don't like the amount of steps each question has) Basically, I hate optimization.

4 Multiple Integration

4.1 Double Integrals over Rectangular Regions

Definition

A function f defined on a rectangular region R in the xy-plane is integrable on R if

$$\lim_{\Delta\to 0}\sum_{k=1}^{n} f\left(x_k^*, y_k^*\right)\Delta A_k$$

exists for all partitions of R and for all choices of (x_k^*, y_k^*) within those partitions. The limit is the double integral of f over R, which we write

$$\iint_R f(x,y)\mathrm{d}A = \lim_{\Delta\to 0}\sum_{k=1}^{n} f\left(x_k^*, y_k^*\right)\Delta A_k$$

We usually use Fubini's theorem.

Theorem 4.1

Let f be continuous on the rectangular region $R = (x,y) : a \leq x \leq b, c \leq y \leq d$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint_R f(x,y)\mathrm{d}A = \int_c^d \int_a^b f(x,y)\mathrm{d}x\mathrm{d}y = \int_a^b \int_c^d f(x,y)\mathrm{d}y\mathrm{d}x$$

Definition

The average value of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R}\iint_R f(x,y)\mathrm{d}A$$

The average height of $f(x,y)$ is $\frac{1}{Area(R)}$(Volume under $f(x,y)$).

4.2 Double Integrals over General Regions

In order to do a double integral over a general integral:

- Divide the plane into rectangles.
- In each rectangle R_k inside R, choose a point (x_k^*, y_k^*).
- Let $\Delta A_k = \text{area}(R_k)$, calculate $\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$
- Take the limit as $\Delta \to 0$, where Δ is the maximum length of diagonal of R_k.

We get

$$\iint_R f(x,y)\mathrm{d}A = \lim_{\Delta\to 0}\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

Given the graph of a surface $z = f(x,y)$ for (x,y) in a planar region R where $f(x,y) \geq 0$ for all (x,y) in R, the volume of the solid bounded by the surface $z = f(x, y)$ and the set R in the xy-plane is given by

$$\text{Volume} = \iint_R f(x, y)\mathrm{d}A$$

If R can be described as $a \leq x \leq b$ and $g(x) \leq y \leq h(x)$ then

$$\iint_R f(x, y)\mathrm{d}A = \int_b^a \int_{g(x)}^{h(x)} f(x, y)\mathrm{d}y\mathrm{d}x$$

If R can be described as $g(y) \leq x \leq h(y)$ and $c \leq y \leq d$ then

$$\iint_R f(x, y)\mathrm{d}A = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\mathrm{d}x\mathrm{d}y$$

To evaluate a $\mathrm{d}y\mathrm{d}x$ double integral of the form

$$\int_a^b \int_{g(x)}^{f(x)} f(x, y)\mathrm{d}y\mathrm{d}$$

1. Integrate $f(x, y)$ with respect to y.
2. Substitute $y = h(x)$, $y = g(x)$ and subtract, resulting in a function of x (call it $A(x)$).
3. Evaluate the integral of the resulting function

$$\int_a^b A(x)\mathrm{d}x$$

To evaluate a $\mathrm{d}x\mathrm{d}y$ double integral of the form

$$\int_c^d \int_{g(y)}^{h(y)} f(x, y)\mathrm{d}x\mathrm{d}$$

1. Integrate $f(x, y)$ with respect to x.
2. Substitute $x = h(y)$, $x = g(y)$ and subtract, resulting in a function of y (call it $A(y)$).
3. Evaluate the integral of the resulting function

$$\int_c^d A(y)\mathrm{d}y$$

Definition

Let R be a region in the xy-plane. Then

$$\text{area of } R = \iint_R \mathrm{d}A$$

4.3 Double Integrals in Polar Coordinates

A cartesian rectangle can be described as:

$$R = (x, y) : a \leq x \leq b, c \leq y \leq d$$

A polar rectangle is:

$$R = (r, \theta) : a \leq r \leq b, \alpha \leq \theta \leq \beta$$

Approximation to polar double integral: Given $f(x, y) = f(r\cos\theta, r\sin\theta)$

If we let ΔA_k be the area of the kth polar rectangle, then the approximation is

$$\sum_{k=1}^{n} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*)\Delta A_k$$

Which can be written as:

$$\iint_R f(x, y)\mathrm{d}A = \lim_{\Delta\to 0}\sum_{k=1}^{n} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*)\Delta A_k$$

Over polar rectangles we have:

$$\iint_R f(x, y)\mathrm{d}A = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)r\mathrm{d}r\mathrm{d}\theta$$

$$R = (r, \theta) : a \le r \le b, \alpha \le \theta \le \beta$$

Theorem 4.2

Let f be continuous on the region R in the xy-plane expressed in polar coordinates as

$$R = (r, \theta) : 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta$$

where $0 < \beta - \alpha \le 2\pi$. Then

$$\iint_R f(x, y)\mathrm{d}A = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)r\mathrm{d}r\mathrm{d}\theta$$

4.4 Triple Integrals

Consider $f(x, y, z)$ defined on D.

- Divide region containing D into rectangular boxes, numbered $k = 1, 2, \ldots, n$.
- Let ΔV_k be the volume of the kth box.
- For each k, choose a point (x_k^*, y_k^*, z_k^*) in the kth box.

  Approximation $= \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$.
- Set Δ = maximum length of a diagonal of a box.

$$\iiint_D f(x, y, z)\mathrm{d}V = \lim_{\Delta\to 0}\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

Two applications of triple integrals:

1.
$$\iiint_D 1\mathrm{d}V = \text{Volume}(D)$$

2. If $\rho(x, y, z)$ represents the density of a solid at any point (x, y, z) of a solid then

$$\iiint_D \rho(x, y, z)\mathrm{d}V = \text{mass}(D)$$

Possible orders for integration:

1. $\mathrm{d}x\mathrm{d}y\mathrm{d}z$
2. $\mathrm{d}x\mathrm{d}z\mathrm{d}y$
3. $\mathrm{d}y\mathrm{d}x\mathrm{d}z$
4. $\mathrm{d}y\mathrm{d}z\mathrm{d}x$
5. $\mathrm{d}z\mathrm{d}x\mathrm{d}y$
6. $\mathrm{d}z\mathrm{d}y\mathrm{d}x$

Let's consider the $\mathrm{d}z\mathrm{d}y\mathrm{d}x$ order.

Theorem 4.3

Let f be continuous over the region

$$D = (x, y, z) : a \leq x \leq b, g(x) \leq y \leq h(x), G(x, y) \leq z \leq H(x, y)$$

, where g, h, G, and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral

$$\iiint_D f(x, y, z)\mathrm{d}V = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x, y, z)\mathrm{d}z\mathrm{d}y\mathrm{d}x$$

4.5 Triple Integrals in Cylindrical and Spherical Coordinates

A point P in three-dimensional space can be described in cylindrical coordinates $P(r, \theta, z)$.

- P^* is the projection of P into the xy-plane.
- (r, θ) is the polar coordinates of P^*.
- (r, θ, z) is the cylindrical coordinates of P.

We can transform from Rectangular to cylindrical:

$$r^2 = x^2 + y^2 \qquad tan\theta = y/x \qquad z = z$$

To convert from cylindrical to rectangular:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Approximate volume given a cylindrical point: $(r_k^*, \theta_k^*, z_k^*)$ and rectangular point: (x_k^*, y_k^*, z_k^*), the approximate volume is $\Delta V_k = r_k^*\Delta r\Delta\theta\Delta z$.

$$\iiint_D f(x, y, z)\mathrm{d}V = \lim_{\Delta\to 0} \sum_{k+1}^{n} f(r_k^*\cos\theta_k^*, r_k^*\sin\theta_k^*, z_k^*)r_k^*\Delta r\Delta\theta\Delta z$$

Theorem 4.4

Let f be continuous over the region D, expressed in cylindrical coordinates as

$$D = (r, \theta, z) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta, G(x, y) \leq z \leq H(x, y)$$

Then f is integrable over D, and the triple integral of f over D is

$$\iiint_D f(x, y, z)\mathrm{d}V = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\mathrm{d}zr\mathrm{d}r\mathrm{d}\theta$$

In a triple integral in spherical coordinates, the coordinate is described as $P(\rho, \phi, \theta)$.

- ρ is the distance from the origin to P.
- ϕ is the angle between the positive z-axis and the line from the origin to P.
- θ is the same angle as in cylindrical coordinates; measures rotation around the z-axis relative to x-axis.

Some relations:

- $x^2 + y^2 = r^2$.
- $\tan\theta = \frac{y}{x}$.
- $x = r\cos\theta$.
- $y = r\sin\theta$.
- $x^2 + y^2 + z^2 = \rho^2$.
- $r = \rho\sin\phi$.
- $z = \rho\cos\phi$.
- $\tan\phi = \frac{r}{z}$.
- $x = \rho\sin\phi\cos\theta$.
- $y = \rho\sin\phi\sin\theta$.

Given the spherical coordinate $(\rho_k^*, \phi_k^*, \theta_k^*)$ and the rectangular coordinate (x_k^*, y_k^*, z_k^*).

The approximate volume of would be $\Delta V_k = \rho_p^{*2}\sin\phi_k^*\Delta\rho\Delta\phi\Delta\theta$.

$$\iiint_D f(x,y,z)\mathrm{d}V = \lim_{\Delta\to 0}\sum_{k=1}^{n} f(\rho_k^*\sin\phi_k^*\cos\theta_k^*, \rho_k^*\sin\phi_k^*\sin\theta_k^*, \rho_k^*\cos\phi_k^*)\rho_k^{*2}\sin\phi_k^*\Delta\rho\Delta\phi\Delta\theta$$

Theorem 4.5

Let f be continuous over the region D, expressed in spherical coordinates as

$$D = (\rho,\phi,\theta) : 0 \leq g(\phi,\theta) \leq \rho \leq h(\phi,\theta), a \leq \phi \leq b, \alpha \leq \theta \leq \beta$$

Then f is integrable over D and the triple integral of f over D is

$$\iiint_D f(x,y,z)\mathrm{d}V = \int_\alpha^\beta\int_a^b\int_{g(\phi,\theta)}^{h(\phi,\theta)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta$$

4.6 Integrals for Mass Calculations

If masses $m_1, m_2, \ldots, m_n$ are arranged on the x-axis at coordinates $x_1, x_2, \ldots, x_n$ respectively the masses will be balanced about the point $\bar{x}$ if

$$\sum_{k=1}^{n} m_k(x_k - \bar{x}) = 0 \implies \bar{x} = \frac{\sum_{k=1}^n m_k x_k}{\sum_{k=1}^n m_k}$$

We can obtain the center of mass in 1 dimension as

$$\bar{x} = \frac{\int_a^b x\rho(x)\mathrm{d}x}{\int_a^b \rho(x)\mathrm{d}x}$$

Definition

Let ρ be a integrable density function on the interval $[a, b]$ (which represents a thin rod or wire). The center of mass is located on the point $\bar{x} = \frac{M}{m}$, where the total moment M and mass m are

$$M = \int_a^b x\rho(x)\mathrm{d}x \qquad m = \int_a^b \rho(x)\mathrm{d}x$$

Definition

Let ρ be an integrable area density function defined over a closed bounded region R in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by R are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\rho(x,y)\mathrm{d}A \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\rho(x,y)\mathrm{d}A$$

where $m = \iint_R \rho(x,y)\mathrm{d}A$ is the mass, and M_y and M_x are the moments with respect to the y-axis and x-axis, respectively. If ρ is constant, the center of mass is called the centroid and is independent of the density.

$$M_y = \iint_R x\rho(x,y)\mathrm{d}A \qquad M_x = \iint_R y\rho(x,y)\mathrm{d}A$$

Definition

Let ρ be an integrable density function on a closed bounded region D in $\mathbb{R}^3$. The coordinates of the center of mass of the region are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint_D x\rho(x,y,z)\mathrm{d}V \quad \bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint_D y\rho(x,y,z)\mathrm{d}V$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint_D z\rho(x,y,z)\mathrm{d}V$$

$$M_{yz} = \iiint_D x\rho(x,y,z)\mathrm{d}V \quad M_{xz} = \iiint_D y\rho(x,y,z)\mathrm{d}V \quad M_{xy} = \iiint z\rho(x,y,z)\mathrm{d}V$$

4.7 Change of Variables in Multiple Integrals

We have substitution in single integrals as:

$$\int_a^b f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int_{u(a)}^{u(b)} f(u)\mathrm{d}u$$

For double integrals, we could use polar coordiantes using the following we know:

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r^2 = x^2 + y^2$
- $\tan\theta = \frac{y}{x}$.
- $\mathrm{d}A \to r\mathrm{d}r\mathrm{d}\theta$

Definition

A transformation T from a region S to a region R is one-to-one on S if $T(P) = T(Q)$ only when $P = Q$, where P and Q are points in S.

Definition

Given a transformation $T : x = g(u, v), y = h(u, v)$, where g and h are differentiable on a region of the uv-plane, the Jacobian determinant (or Jacobian) of T is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial y} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Theorem 4.6

Let $T : x = g(u, v), y = h(u, v)$ be a transformation that maps a closed bounded region S in the uv-plane to a region R in the xy-plane. Assume T is one-to-one on the interior of S and g and h have continuous first partial derivatives there. If f is continuous on R, then

$$\iint_R f(x,y)\mathrm{d}A = \iint_S f(g(u,v), h(u,v)) \mid J(u,v) \mid \mathrm{d}A$$

Definition

Given a transformation $T : x = g(u, v, w), y = h(u, v, w)$, and $z = p(u, v, w)$, where g, h, and p are differentiable on a region of uvw-space, the Jacobian determinant (or Jacobian) of T is

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

We can express $J(u,v,w) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\frac{\partial z}{\partial w} + \frac{\partial x}{\partial v}\frac{\partial y}{\partial w}\frac{\partial z}{\partial u} + \frac{\partial x}{\partial w}\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial w}\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial y}{\partial w}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\frac{\partial z}{\partial w} - \frac{\partial x}{\partial w}\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}$

Theorem 4.7

Let $T : x = g(u, v, w), y = h(u, v, w)$, and $z = p(u, v, w)$ be a transformation that maps a closed bounded region S in uvw-space to a region $D = T(S)$ in xyz-space. Assume T is one-to-one on the interior of S and g, h, and p have continuous first partial derivatives there. If f is continuous on D, then

$$\iiint_D f(x,y,z)\mathrm{d}V = \iiint_S f(g(u,v,w), h(u,v,w), p(u,v,w,)) \mid J(u,v,w) \mid \mathrm{d}V$$

5 Vector Calculus

5.1 Vector Fields

Definition

Let f and g be defined on a region R of $\mathbb{R}^2$. A vector field in $\mathbb{R}^2$ is a function **F** that assigns to each point (x, y) in R a vector $\mathbf{F}(x, y)$ where

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$$

or

$$F(x, y) = \langle f(x, y), g(x, y)\rangle$$

The vector field **R** is continuous or differentiable on R is f and g are continuous or differentiable on R.

Note: A vector field is both a vector valued function and a function of several variables.

Definition

Let f, g, and h be defined on a region D of $\mathbb{R}^3$. A vector field in $\mathbb{R}^3$ is a function **F** that assigns to each point (x, y, z) in D a vector $\mathbf{F}(x, y, z)$ where

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

or

$$\mathbf{F}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z)\rangle$$

The vector field **F** is continuous or differentiable on D is f, g, and h are continuous or differentiable on D.

Definition: Radial Vector Field in $\mathbb{R}^2$

Let $\mathbf{r} = \langle x, y\rangle$ and p is any real number, then

$$\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y\rangle}{|\mathbf{r}|^p}$$

is a radial vector field.

Definition

Let φ be a differentiable region of $\mathbb{R}^2$ or $\mathbb{R}^3$. The vector field **F**$= \nabla\varphi$ is a gradient field and the function φ is a potential function for **F**.

Recall $\nabla\varphi = \langle \varphi_x, \varphi_y \rangle$ or $\langle \varphi_x, \varphi_y, \varphi_z \rangle$ and that the vector field **F** $= \nabla_\varphi$ is orthogonal to the level curves of φ at (x, y).

In $\mathbb{R}^3$ the gradient field will be orthogonal to level surfaces of φ.

Definition

Let φ be a potential function for a vector field in **F** in $\mathbb{R}^2$. That is, **F**$= \nabla_\varphi$.

The level curves of a potential function are called equipotential curves.

Also, the vector field may be visualized by drawing continuous flow curves or streamlines that are everywhere orthogonal to the equipotential curves.

These ideas can be extended to $\mathbb{R}^3$ in which case we will have equipotential surfaces.

5.2 Line Integrals

Definition

Suppose the scalar-valued function f is defined on the region containing the smooth curve C given by **r**$(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$.

The line integral of f over C is

$$\int_C f(x(t), y(t))\mathrm{d} = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*))\Delta s_k$$

provided this limit exists over all partitions of $[a, b]$.

If $f > 0$ then the line integral computes the area of the "curtain" under f and over C.

Scalar line integrals are independent of the orientation and parameterization of the curve C.

Evaluting Scalar Line integrals in $\mathbb{R}^2$.

$$\int_C f\mathrm{d}s$$

What is $\mathrm{d}s$?

Let C be given by **r**$(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$.

Recall: The length of the curve C over $[a, t]$ is given by $s(t) = \int_a^t |\mathbf{r}'(u)|\mathrm{d}u$.

By differentiating both sides, $s'(t) = |\mathbf{r}'(t)|$.
Thus, $\mathrm{d}s = s'(t)\mathrm{d}t = |\mathbf{r}'(t)|$.

Theorem 5.1

Let f be continuous on a region containing a smooth curve C: $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$. Then

$$\int_C f \mathrm{d}s = \int_a^b f(x(t), y(t))|\mathbf{r}'(t)|\mathrm{d}t = \int_a^b f(x(t), y(t))\sqrt{((x'(t))^2 + (y'(t))^2)}\mathrm{d}t$$

Theorem 5.2

Let f be continuous on a region containing a smooth curve C: $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $a \le t \le b$. Then

$$\int_C f \mathrm{d}s = \int_a^b f(x(t), y(t), z(t))|\mathbf{r}'(t)|\mathrm{d}t$$
$$= \int_a^b f(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\mathrm{d}t$$

Definition: Line Integral of a Vector Field

Let **F** be a vector field that is continuous on a region containing a smooth oriented curve C parametrized by arc length. Let **T** be the unit tangent vector at each point of C consistent with the orientation. The line integral of **F** over C is

$$\int_C \mathbf{F}\cdot\mathbf{T}\mathrm{d}s$$

Observations

- $\mathbf{F}\cdot\mathbf{T}$=$|\mathbf{f}||\mathbf{T}|\cos\theta = |\mathbf{F}|\cos\theta$
- The line integral adds up these components
- The orientation of the curve matters! $\int_{-C}\mathbf{F}\cdot\mathbf{T}\mathrm{d}s = -\int_C \mathbf{f}\cdot\mathbf{T}\mathrm{d}s$

Evaluating The Line Integral of a Vector Field

$$\int_C \mathbf{F}\cdot\mathbf{T}\mathrm{d}s = \int_a^b \mathbf{F}\cdot\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}|\mathbf{r}'(t)|\mathrm{d}t = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\mathrm{d}t$$

Different Forms of Line Integrals of Vector Fields: Given $\mathbf{F} = \langle f, g, h\rangle$ and C with parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \le t \le b$:

$$\int_C \mathbf{F}\cdot\mathbf{T}\mathrm{d}s = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\mathrm{d}t = \int_a^b (f(t)x'(t) + g(t)y'(t) + h(t)z'(t))\mathrm{d}t$$
$$= \int_C (f\mathrm{d}x + g\mathrm{d}y + h\mathrm{d}z)$$
$$= \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$$

This works similarly for vector fields in $\mathbb{R}^2$.

Definition

Let **F** be a continuous force field in a region D of $\mathbb{R}^3$. Let

$$C : \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle \text{for} a \le t \le b$$

be a smooth curve in D with a unit tangent vector **T** consistent with the orientation.

The work done (by the force field) in moving an object along C in the positive direction is

$$W = \int_C \mathbf{F} \cdot \mathbf{T} \mathrm{d}s = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t) \mathrm{d}t$$

- $\mathbf{F} \cdot \mathbf{T} = |\mathbf{F}| \cos\theta$ is the tangential component of $\mathbf{F}$ along C (in direction of the motion).
- The vector line integral sums the work done at each point along C.

Definition

Definition: Circulation

Let $\mathbf{F}$ be a continuous vector field on a region R of $\mathbb{R}^2$, and let C be a closed smooth oriented curve in R. The circulation of $\mathbf{F}$ on C is $\int_C \mathbf{F} \cdot \mathbf{T} \mathrm{d}s$, where $\mathbf{T}$ is the unit vector tangent to C consistent with the orientation.

A curve C in $\mathbb{R}^2$ is closed if its initial and terminal points are the same.

Circulation is a measure of how much of the vector field points in the direction of C.

Definition

Definition: Flux

Let $\mathbf{F}$ be a continuous vector field on a region R of $\mathbb{R}^2$, and let C be a closed smooth oriented curve in R. The flux of the vector field $\mathbf{F}$ across C is $\int_C \mathbf{F} \cdot \mathbf{n} \mathrm{d}s$, where $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ is the unit normal vector and $\mathbf{T}$ is the vector tangent to C consistent with the orientation.

Flux is a measure of how much the vector field points orthogonally to C.

In practice, use $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ and $\mathrm{d}s = |\mathbf{r}'(t)| \mathrm{d}t$.

Given $\mathbf{F} = \langle f, g \rangle$ and $C : \mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$, then

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{n} \mathrm{d}s &= \int_a^b (f(t)y'(t) - g(t)x'(t)) \mathrm{d}t \\ &= \int_C f \mathrm{d}y - g \mathrm{d}x \end{aligned}$$

5.3 Conservative Vector Fields

Definition: Simple and Closed Curves

Suppose a curve C (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \leq t \leq b$.

- Then C is a simple curve if $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for all t_1 and t_2, with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints.
- The curve C is closed if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same.

Definition: Connected and Simply Connected Regions

- An open region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is connected if it is possible to connect any two points in R by a continuous curve lying in R. (Think: R is in one piece)
- An open region R is simply connected if every closed simple curve in R can be deformed and contracted to a point in R. (Think: R has no holes.)

Recall that all points of an open set are interior points. An open set does not contain any of its boundary points.

Definition: Conservative Vector Fields

A vector field $\mathbf{F}$ is said to be conservative on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if there exists a scalar function φ such that $\mathbf{F} = \nabla\varphi$ on that region.

Recall that when $\mathbf{F} = \nabla\varphi$, the function φ is a potential function for $\mathbf{F}$.

Note: any function of the form $\varphi(x, y) = xy + C$ would be a potential function for $\mathbf{F}$.

Definition: Test for Conservative Vector Fields

Suppose $\mathbf{F} = \langle f, g\rangle$ has continuous first partial derivatives on a connected and simply connected region D in $\mathbb{R}^2$.

If $\mathbf{F}$ is conservative $\implies$ there is a function φ such that $\mathbf{F} = \nabla\varphi$.

$$\implies (1) f = \varphi_x \qquad \text{and} \qquad (2)\ g = \varphi_y.$$

Now by taking partial derivatives,

$$f_y = \varphi_{xy} \qquad \text{and} \qquad g_x = \varphi_{yx}.$$

By equality of mixed partial derivatives,

$$\varphi_{xy} = \varphi_y x.$$

Thus we can conclude, $f_y = g_x$.

The other direction is also true. That is, if $f_y = g_x$ then $\mathbf{F}$ is conservative.

This provides us with a test for a conservative vector field in two dimensions, which can be extended to the following test for vector fields in three dimensions.

Theorem 5.3: Test for Conservative Vector Fields

Let $\mathbf{F} = \langle f, g, h\rangle$ be a vector field defined on a connected and simply connected region D in $\mathbb{R}^3$, where f, g, and h have continuous first partial derivatives on D.

Then $\mathbf{F}$ is a conservative vector field on D if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x} \qquad \text{and} \qquad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Theorem 5.4: Fundamental Theorem for Line Integrals

Let R be a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let φ be a differentiable potential function defined on R. If $\mathbf{F} = \nabla\varphi$ (which means that $\mathbf{F}$ is conservative), then

$$\int_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \varphi(B) - \varphi(A)$$

for all points A and B in R and all piecewise-smooth oriented curves C in R from A to B.

Why? Let $\mathbf{r}(t)$ be any parameterization of C for $a \leq t \leq b$ and one can use the chain rule to show that $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \mathbf{F} \cdot \mathbf{r}'(t)$. This,

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\mathrm{d}t = \int_a^b \frac{\mathrm{d}\varphi}{\mathrm{d}t}\mathrm{d}t = \varphi(B) - \varphi(A)$$

Interpretation: If $\mathbf{F}$ is a conservative vector field, then the value of a line integral of $\mathbf{F}$ depends only on the endpoints of the path!

Definition: Path Independence

Let $\mathbf{F}$ be a continuous vector field with domain R. If

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F}\mathrm{d}\mathbf{r}$$

for all piecewise-smooth curves C_1 and C_2 in R with the same initial and terminal points, then the line integral is independent of path.

Theorem 5.5

Let $\mathbf{F}$ be a continuous vector field on an open connected region R in $\mathbb{R}^2$. If

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

is independent of path, then $\mathbf{F}$ is conservative; that is, there exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ on R.

Line Integrals on Closed Curves Notation: We will ues $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ to denote a line integral over a closed curve C.

Theorem 5.6

Let R be an open connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\mathbf{F}$ is a conservative vector field on R if an only if $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ on all simple closed piecewise-smooth oriented curves C in R.

Why?
$\mathbf{F}$ is a conservative $\implies \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \varphi(B) - \varphi(A) = \varphi(A) - \varphi(A) = 0$
$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0 \implies 0 = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$
$\implies \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{-C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$

5.4 Green's Theorem

Theorem 5.7: Green's Theorem - Circulation Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g\rangle$ where f and g have continuous first partial derivatives in R. Then

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_C f\mathrm{d}x + g\mathrm{d}y = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}A$$

Green's Theorem relates the circulation on C to a double integral over the region R.

If needed: $\oint_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$

Definition: Two-Dimensional Curl

The two-dimensional curl of the vector field $\mathbf{F} = \langle f, g \rangle$ is

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$$

If the curl is zero throughout a regio, the vector field is irrotational throughout that region.

Recall: If $\mathbf{F} = \langle f, g \rangle$ is conservative then $f_y = g_x$

Thus, $g_x - f_y = 0$ and the curl of $\mathbf{F}$ is zero.

Under the conditions of Green's Theorem: $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}A = 0.$

Circulation integrals of conservative vector fields are always zero!

Theorem 5.8: Area of a Plane Region by Line Integrals

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane.

Then the area of R is given by:

$$\oint_C x\mathrm{d}y = -\oint_C y\mathrm{d}x = \frac{1}{2}\oint_C (x\mathrm{d}y - y\mathrm{d}x)$$

Theorem 5.9: Green's Theorem - Flux Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$ where f and g have continuous first partial derivatives in R. Then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\mathrm{d}s = \oint_C f\mathrm{d}y - g\mathrm{d}x = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) \mathrm{d}A$$

Interpretation:

- Green's theorem says that the net divergence throughout the region R equals the flux across the boundary of R.

Definition: Two-Dimensional Divergence

The two-dimensional divergence of the vector field $\mathbf{F} = \langle f, g \rangle$ is

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

If the divergence is zero throughout a region, the vector field is source free throughout that region.

The outward flux of a source free vector field is always zero!

5.5 Divergence and Curl

Definition: Divergence

The divergence of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\text{div}\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

If div $\mathbf{F} = 0$, the vector field is source free.

Divergence measures the expansion or contraction of the vector field at each point.

Del operator: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

Alternation notation: $\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = \text{div}\mathbf{F}$

Theorem 5.10: Divergence of Radial Vector Fields

For a real number, p, the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{p/2}} \qquad \text{is} \qquad \nabla \cdot \mathbf{F} = \frac{3 - p}{|\mathbf{r}|^p}$$

Definition: Curl

The curl of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable in a region of $\mathbb{R}^3$ is

$$\text{curl}\mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathbf{k}$$

Curl is a measure of rotation within a vector field at each point.

We can express the curl as a cross product:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

If curl $\mathbf{F} = \mathbf{0}$, the vector field is irrotational.

The **k**-component of the curl (or the two-dimensional curl) gives the rotation of the vector field in the xy-plane at a point.

The other components of the curl give similar information about the rotation of the vector field.

Theorem 5.11: Curl of a Conservative Vector Field

Suppose $\mathbf{F}$ is a conservative vector field on an open region D of $\mathbb{R}^3$. Let $\mathbf{F} = \nabla\varphi$, where φ is a potential function with continuous second partial derivatives on D.

Then curl **F=0** and F is irrotational.

Theorem 5.12: Divergence of the Curl

Suppose $\mathbf{F} = \langle f, g, h \rangle$, where f, g, and h have continuous second partial derivatives, then

$$\text{div curl } \mathbf{F} = 0.$$

That is, the divergence of the curl is zero.

Note: If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$ then div $\mathbf{F}$ is a scalar valued function and not a vector field. Hence, curl div $\mathbf{F}$ is not defined.

General Rotation Field:

$\mathbf{F} = \mathbf{a} \times \mathbf{r} = \langle a_1, a_2, a_3 \rangle \times \langle x, y, z \rangle$.

- The vector $\mathbf{a}$ is the axis of rotation for the vector field $\mathbf{F}$.
- The length of the curl of $\mathbf{F}$, $|\nabla \times \mathbf{F}| = 2|\mathbf{a}|$.
- The divergence of the vector field $\mathbf{F}$ is zero, or $\mathbf{F}$ is source free.

5.6 Surface Integrals

Recall a curve in $\mathbb{R}^2$ is defined parametrically by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$.

For a surface in $\mathbb{R}^3$ we'll need two parameters and three dependent variables:

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

A cylinder with radius $a > 0$ and height $h > 0$ can be described parametrically as $\mathbf{r}(u, v) = \langle a \cos u, a \sin u, v \rangle$ where $0 \leq u \leq 2\pi$ and $0 \leq v \leq h$.

A cone with radius $a > 0$ and height $h > 0$ can be described parametrically as $\mathbf{r}(u, v) \left\langle \frac{av}{h} \cos u, \frac{av}{h} \sin u, v \right\rangle$ where $0 \leq u \leq 2\pi$ and $0 \leq v \leq h$.

A sphere with radius $a > 0$ can be described parametrically as $\mathbf{r}(u, v) = \langle a \sin u \cos v, a \sin u \sin v, a \cos u \rangle$, where $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$.

For an explicitly defined surface:

$$z = g(x, y) \text{ on } R = (x, y) : a \leq x \leq b, c \leq y \leq d$$

can be parametrically described:

$$\mathbf{u{,}v} = \langle u, v, g(u, v) \rangle$$

where $a \leq u \leq b$ and $c \leq v \leq d$.

Now we will develop the surface integral of a scalar-valued function f defined on a smooth parametrized surface S.

$$\iint_S f(x, y, z) \mathrm{d}S$$

Applications:

- Compute the surface area of S.
- Compute the mass of a thin sheet described by the surface S with mass density function f.
- Compute the average value of f over the surface S.

Definition: Surface Integrals of Scalar-Valued Functions

Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, where u and v vary over the rectangle $R = (u, v) : a \leq u \leq b, c \leq v \leq d$. Assume also that the tangent vectors

$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \qquad \text{and} \qquad \mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

are continuous on R and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R.

The surface integral of f over S is

$$\iint_S f(x, y, z) \mathrm{d}S = \iint_R f(x(u, v), y(u, v), z(u, v)) |\mathbf{t}_u \times \mathbf{t}_v| \mathrm{d}A$$

If $f(x, y, z) = 1$, this integral equals the surface area of S.

Theorem 5.13: Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $z = g(x, y)$, for (x, y) in a region R. The surface integral of f over S is

$$\iint_S f(x,y,z)\mathrm{d}S = \iint_R f(x,y,g(x,y))\sqrt{z_x^2 + z_y^2 + 1}\mathrm{d}A$$

If $f(x, y, z) = 1$, the surface integral equals the area of the surface.

Orientable Surfaces: To be orientable, a surface must have a choice of normal vectors that varies continuously over the surface. (The surface is two-sided.)

If a surface encloses a region then we will choose normal vectors to point in the outward direction. For other surfaces, we must specify the direction of the normal vector.

Definition

Flux Integrals: Consider a continuous vector field $\mathbf{F} = \langle f, g, h\rangle$. Let S be a smooth oriented surface with unit normal vector $\mathbf{n}$.

The flux integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\mathrm{d}S$$

computes the net flux of the vector field across the surface.

$\mathbf{F} \cdot \mathbf{n} = |\mathbf{F}||\mathbf{n}| \cos\theta = |\mathbf{F}| \cos\theta$.

The flux integral adds up the components of the vector field $\mathbf{F}$ normal to the surface.

Definition: Surface Integral of a Vector Field

Suppose $\mathbf{F} = \langle f, g, h\rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{u,v} = \langle x(u, v), y(u, v), z(u, v)\rangle$, for (u, v) in a region R, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\mathrm{d}S = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v)\mathrm{d}A$$

where $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$ and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$ are continuous on R, and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R, and the direction of the normal vector is consistent with the orientation of S.

Definition: Surface Integral of a Vector Field

If S is defined in the form $z = w(x, y)$ for (x, y) in a region R, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\mathrm{d}S = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v)\mathrm{d}A = \iint_R (-f z_x - g z_y + h)\mathrm{d}A$$

5.7 Stokes' Theorem

Theorem 5.14: Stokes' Theorem

Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume $\mathbf{F} = \langle f, g, h\rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \mathrm{d}S$$

where $\mathbf{n}$ is the unit normal vector to S determined by the orientation of S.

Stokes' Theorem says that the line integral around the boundary C of the tangential component of $\mathbf{F}$ is equal to the surface integral over S of the normal component of the curl of $\mathbf{F}$.

The right hand rule relates the orientations of S and C and determines the choice of normal vectors.

Stokes' Theorem is the three-dimensional version of the circulation form of Green's Theorem.

Note: If a closed curve C is the boundary of two different smooth oriented surfaces S_1 and S_2 which both have orientation consistent with that of C, then the integrals of $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ on the two surfaces are equal.

5.8 Divergence Theorem

Theorem 5.15: Divergence Theorem

Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D in $\mathbb{R}^3$ enclosed by an oriented surface S. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \mathrm{d}S = \iiint_D \nabla \cdot \mathbf{F} \mathrm{d}V$$

where $\mathbf{n}$ is the outward unit normal vector to S.

The Divergence Theorem says that the flux of $\mathbf{F}$ across the boundary surface of D is equal to the triple integral over the divergence of $\mathbf{F}$ over D.

Theorem 5.16: Divergence Theorem for Hollow Regions

Suppose the vector field $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on a region D in $\mathbb{R}^3$ bounded by two oriented surfaces S_1 and S_2 where S_1 lies within S_2. Let S be the entire boundary of $D (S = S_1 \cup S_2)$ and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the outward unit normal vectors for S_1 and S_2 respectively. Then

$$\iiint_D \nabla \cdot \mathbf{F} \mathrm{d}V = \iint_S \mathbf{F} \cdot \mathbf{n} \mathrm{d}S = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 \mathrm{d}S - \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1 \mathrm{d}S$$

This form ot the Divergence Theorem is applicable to vector fields that are not differentiable at the origin, as is the case with some important radial vector fields of the form:

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p}$$