Lesson 2

There are some special sets of numbers.

There is the set of positive integers

$$\{1, 2, 3, 4, \dots\}$$

However, we cannot ignore 0 or the negative numbers, so we have $\mathbb{Z}$, the entire set of integers.

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

Addition in $\mathbb{Z}$ gives some nice axioms

- **commutativity**: for any $a, b \in \mathbb{Z}$, $a + b = b + a$
- **associativity**: for any $a, b, c \in \mathbb{Z}$, $(a + b) + c = a + (b + c)$
- $\mathbb{Z}$ has an additive identity element, namely 0, for any $a \in \mathbb{Z}, a + 0 = 0 + a = a$
- $\mathbb{Z}$ has additive inverses: for any $a \in \mathbb{Z}$, there exists $b \in \mathbb{Z}$ such that $a + b = b + a = 0$. (Of course, $b = -a$)

We know that $\mathbb{Z}$ has multiplication that is also commutative, associative, and has an identity element, 1.

We also have that

$$\text{for any } a, b, c \in \mathbb{Z},\ a(b + c) = (ab) + (ac)$$

We move to the rational numbers due to not having multiplicative inverses.

$$\mathbb{Q} = \{\tfrac{m}{n} | m, b \in \mathbb{Z} \text{ and } n \neq 0\}$$

The set $\mathbb{Q}$ together with the operations of addition and multiplication is what is called a field, and these operation satisfy all the nice properties we want them to have. However there is something missing from $\mathbb{Q}$.

For example there is no number $x \in \mathbb{Q}$ such that $x^2 = 2$, since, yes in $\mathbb{R}$ we have a solution of $\sqrt{2}$, but in $\mathbb{Q}$ this does not exist.

Definition

Let S be a subset of $\mathbb{R}$.

- A number $x \in \mathbb{R}$ is called an upper bound for S if for all $y \in S$, we have $y \leq x$. If an upper bound for S exists, we say that S is bounded above.
- A number $x \in \mathbb{R}$ is called a lower bound for S if for all $y \in S$, we have $y \geq x$. If a lower bound for S exists, we say that S is bounded below.
- If S has an upper bound and a lower bound, we say that S is bounded.

There are some special bounds.

Definition

Let S be a subset of $\mathbb{R}$.

- A number $x \in \mathbb{R}$ is called a least upper bound of S if x is an upper bound for S and if for every $z \in \mathbb{R}$ such that z is an upper bound of S, we have $x \leq z$.
- A number $x \in \mathbb{R}$ is called a greatest lower bound of S if x is a lower bound for S and if for every $z \in \mathbb{R}$ such that z is a lower bound of S, we have $x \geq z$.

In other words, an upper bound of a subset $S \subseteq \mathbb{R}$ is any number greater than or equal to all of the numbers in S. If we can find a smallest possible upper bound for S, then this number is the least upper bound of S. Similarly, a lower bound of S is a number that is less than or equal to all of the numbers in S. If we can find a greatest possible lower bound for S, then this number is the greatest lower bound of S.

The most important subsets of $\mathbb{R}$ that we will be regularly using are intervals:

Definition

A subset I of $\mathbb{R}$ is called an interval if, for any $a, b \in I$ and $x \in \mathbb{R}$ such that $a \leq x \leq b$, we have $x \in I$.

Informally, an interval I is a subset of $\mathbb{R}$ without any "holes". If a and b are in I, then all the numbers between a and b are also in I. There are two types of intervals commonly used.

Definition

Let $a \leq b$ be any two real numbers. The open interval (a, b) is defined as the set

$$(a, b) = \{x \in \mathbb{R} | a < x < b\}$$

The closed interval $[a, b]$ is defined as the set

$$[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$$

There are also half-open intervals.

We have one more definition

Definition

Let S be a subset of $\mathbb{R}$. If sup$S \in S$, we say that sup$S$ is the maximal element of S (or simply the maximum of S). If inf $\in S$, we say that inf$S$ is the minimal element of S (or simply the minimum of S).

Homework

1. Show that there is no number $x \in \mathbb{Q}$ such that $x^2 = 2$.
2.
   - Show that the intersection of any two intervals is an interval.
   - Show that the union of any two intervals need not be an interval.
3. If A, B are bounded intervals with $A \cap B \neq \emptyset$, then show that sup$(A \cap B) = \min\{$sup$A$, sup$B\}$.