Differential Equations Notes

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1 Introduction to Differential Equations

1.1 Background

In a variety of subject areas, mathematical models are developed to aid in understanding. These models often yield an equation that contains derivatives of an unknown function. Such an equation is called a differential equation.

One example is free fall of a body. An object is released from a certain height above the ground and falls under the force of gravity. Newton's second law states that an object's mass times its acceleration equals the total force acting on it.

$$m\frac{\mathrm{d}^2h}{dt^2} = -mg$$

We have $h(t)$ as position, $\frac{\mathrm{d}h}{\mathrm{d}t}$ as velocity and $\frac{\mathrm{d}^2h}{\mathrm{d}t}$ as acceleration. The independent variable is t and the dependent variable is h.

$m\frac{\mathrm{d}^2h}{dt^2} = -mg$ is a differential equation and $h(t)$ is the unknown function that we are trying to find.

From this we have $\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = -g$ and the integral of this is $\frac{\mathrm{d}h}{\mathrm{d}t} = -gt + C_1$. To find h we integrate again and we get $h(t) = -\frac{gt^2}{2} + C_1t + C_2$.

Another example is the decay of a radioactive substance. The rate of decay is proportional to the amount of radioactive substance present.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -kA, \qquad k > 0$$

where A is the unknown amount of radioactive substance present at time t and k Is the proportionality constant.

We are looking for $A(t)$ that satisfies this equation. We can solve this from $\frac{1}{A}\mathrm{d}A = k\mathrm{d}t$ and integrating both sides we get that $\ln|A| + C_1 = -kt + C_2$. We can rewrite this as $\ln|A| = -kt + C$. So, $e^{-kt+C} = A$.

So $A(t) = e^{-kt} + e^C$, so $A(t) = Ce^{-kt}$. Remember A is the dependent variable and t is the independent variable.

Notice that the solution of a differential equation is a function, not merely a number.

When a mathematical model involves the rate of change of one variable with respect to another, a differential equation is apt to appear.

Terminology

If an equation involves the derivative of one variable with respect to another, then the former is called a dependent variable and the latter an independent variable.

In $\frac{\mathrm{d}h}{\mathrm{d}t}$, h is dependent and t is independent.

A differential equation involving only ordinary derivatives with respect to a single independent variable is called an ordinary differential equation. A differential equation involving partial derivatives with respect to more than one independent variable is a partial differential equation.

For example we have $z = f(x,y) = 4x^2 + 5xy$, so $\frac{\partial z}{\partial x} = 8x + 5y$ and that is partial differentiation.

The order of a differential equation is the order of the highest-order derivatives present in the equation.

For example, $\frac{\mathrm{d}^2h}{dt^2} = -g$ has a order of 2.

A linear differential equation is one in which the dependent variable y and its derivatives appear in additive combinations of their first powers. A differential equation is linear if it has the format.

$$a_n(x)\frac{\mathrm{d}^n y}{dx^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = F(x)$$

$2x + 3y = 7$ is linear, $2x^2 + 5xy + 7y + 8y = 1$ is second-degree. Nothing can have a second degree for this to be linear.

You are just looking at the dependent variable and the derivatives and adding their powers.

If an ordinary differential equation is not linear, we call it nonlinear.

Example

For each differential equation, classify as ODE or PDE, linear or nonlinear, and indicate the dependent/independent variables and order.

(a) $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + a\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0$

Dependent is x, independent is t and the order is 2. This is an ODE and linear.

(b) $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x - 2y$

The dependent variable is u and the independent variables are x, y, so this is a PDE. The order is 1.

(c) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^3 = 0$

The dependent variable is y, the independent variable is x, the order is 2 and this is an ODE and this is nonlinear.

(d) $t^3\frac{\mathrm{d}x}{\mathrm{d}t} = t^3 + x$

Dependent is x, independent is t, order is 1, this is an ODE. We can rewrite this as $t^3\frac{\mathrm{d}x}{\mathrm{d}t} - 1x = t^3$, and this matches the form of the linear equation so this is linear.

(e) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - y\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x$

The dependent is y, the independent is x, the order is 2 and this is an ODE and this is nonlinear because of $y\frac{\mathrm{d}y}{\mathrm{d}x}$.

1.2 Solutions and Initial Value Problems

An nth-order ordinary differential equation is an equality relating the independent variable to the nth derivative (and usually lower-order derivatives as well) of the dependent variable.

Example

Identify the order, independent and dependent variable.

(a) $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3$. Independent: x, dependent: y, order: 2

(b) $\sqrt{1 - \left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right)} - y = 0$. Independent: t, dependent: y, order: 2

(c) $\frac{\mathrm{d}^4x}{\mathrm{d}t^4} = xt$. Independent: t, dependent: x, order: 4. (This is also linear.)

A general form for an nth-order equation with x independent, y dependent can be expressed as

$$F(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}) = 0$$

where F is a function that depends on x, y, and the derivatives of y up to order n. We assume the equations holds for all x in an open interval I. In many cases, we can isolate the highest-order term and write the previous equation as

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}\right)$$

This is called the normal form.

A function $\phi(x)$ that when substituted for y in either the previous two equations satisfies the equation for all x in the interval I is called an explicit solution to the equation on I.

Example

Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to the linear equation $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - frac2x^2 y = 0$ but $\psi(x) = x^3$ is not.

So we have $y = x^2 - x^{-1}$. The first derivative of this is $2x + 1x^{-2}$. The second derivative is $y'' = 2 - 2x^{-3}$. If we plug in the values we end up getting from the derivatives, we get that $2 - 2x^{-3} - 2x + 2x^{-3} = 0$, so this is satisfied.

For the second part, the first derivative is $3x^2$ and the second derivative is $6x$. Plugging this in, we get $4x$ which is not 0, so $\psi(x)$ is not a solution.

Example

Show that for any choice of the constants c_1 and c_2, the function $\phi(x) = c_1 e^{-x} + c_2 e^{2x}$ is an explicit solution to the linear equation $y'' - y' - 2y = 0$.

We have that the first derivative is $-c_1 e^{-x} + 2c_2 e^{2x}$ and the second derivative is $c_1 e^{-x} + 4c_2 e^{2x}$. When we plug this in, we find that this does satisfy the solution for the differential equation.

Methods for solving differential equations do not always yield an explicit solution for the equation. A solution may be defined implicitly.

Example

Show that the relation $y^2 - x^3 + 8 = 0$ implicitly defines a solution to the nonlinear equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{2y}$ on the interval $(2, \infty)$.

We have from the given that $y = \pm\sqrt{x^3 - 8}$. The derivative (of the positive version) of this is $\frac{3x^2}{2\sqrt{x^3-8}}$. This is the same and defined on the interval.

A relation $G(x, y) = 0$ is said to be an implicit solution to the previous equation on the interval I if it defines one or more explicit solutions on I.

Example

Show that $x + y + e^{xy} = 0$ is an implicit solution to the nonlinear equation $(1 + xe^{xy})\frac{\mathrm{d}y}{\mathrm{d}x} + 1 + ye^{xy} = 0$.

Taking the derivative of both sides gets us that $1 + \frac{\mathrm{d}y}{\mathrm{d}x} + e^{xy}\frac{\mathrm{d}}{\mathrm{d}x}(xy) = 0$. This does simplify to what was given in the problem.

Example

Verify that for every constant C the relation $4x^2 - y^2 = C$ is an implicit solution to $y\frac{\mathrm{d}y}{\mathrm{d}x} - 4x = 0$. Graph the solution curves for $C = 0, \pm 1, \pm 4$.

The derivative of what is given is $8x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. This simplifies to what is given, so it is clearly an implicit solution.

For $C = 0$, the solution curves for this is $2x = y$ and $-2x = y$.

For $C = \pm 4$, the solution curves is given by a hyperbola $\frac{x^2}{-}\frac{y^2}{4} = 1$.

The collection of all solutions in the previous example is called a one-parameter family of solutions.

In general, the methods for solving nth-order differential equations evoke n arbitrary constants. We often can evalute these constants if we are given n initial values $y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)$.

Definition

By an initial value problem for an nth-order differential equation

$$F(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^n y}{\mathrm{d}x^2}) = 0$$

we mean: Find a solution so the differential equation on an interval I that satisfies at x_0 the n initial conditions

$$y(x_0) = y_0, \quad \frac{\mathrm{d}y}{\mathrm{d}x}(x_0) = y_1 \cdots \frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}(x_0) = y_{n-1}$$

where $x_0 \in I$ and $y_0, y_1, \ldots, y_{n-1}$ are constants.

Example

Show that $\phi(x) = \sin x - \cos x$ is a solution to the initial value problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0; \qquad y(0) = -1 \qquad \frac{\mathrm{d}y}{\mathrm{d}x}(0) = 1$$

We have $y = \sin x - \cos x$, $y' = \cos x + \sin x$, and $y'' = -\sin x + \cos x$. These satisfy the conditions.

Theorem 1.1: Existence and Uniqueness of Solution

Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \qquad y(x_0) = y_0$$

If f and $\partial f / \partial y$ are continuous functions in some rectangle

$$R = \{(x, y) : a < x < b, c < y < d\}$$

that contains the point (x_0, y_0), then the initial value problem has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 \delta$, where δ is a positive number.

Example

Does the theorem above imply the existence for this problem.

$3\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - xy^3, \qquad y(1) = 6$

The derivative exists for all (x, y) and is continuous in all intervals containing $x = 1$ and all rectangular regions containing $(1, 6)$.

When we consider the partial derivatives, $\partial f / \partial y = -xy^2$, and this exists and is continuous for all rectangular regions in the xy plane.

1.3 Direction Fields

One technique useful in visualizing (graphing) the solutions to a first-order differential equation is to sketch the direction field for the equation. A first-order equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

specifies a slope at each point in the xy-plane where f is defined.

Definition

A plot of short line segments drawn at various points in the xy-plane showing the slope of the solution curve htere is called a direction field for the differential equation.

For example, consider the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - y$. When we plug in the point $(1, 0)$, the slope is 1. When the point is $(0, 1)$ the slope is -1. Notice the further right we get, the steeper the graph goes. When we look at this function, $f(x, y) = x^2 - y$, so taking the partial of y results in -1 which is continuous so there is a solution curve at any point.

Note that we are basically just drawing slope fields from AP Calculus BC.

When we consider an equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}$, we have a unique solution when $x \neq 0$ because $f(x, y)$ is continuous if $x \neq 0$ and $\frac{\partial f}{\partial y} = -\frac{1}{x}$, so if $x \neq 0$, then there is a unique solution.

Example

Consider the direction field for $\frac{\mathrm{d}y}{\mathrm{d}x} = 3y^{2/3}$. Is there a unique solution passing through $(2, 0)$?

$\frac{\partial f}{\partial y} = \frac{2}{\sqrt[3]{y}}$ is not continuous when $y = 0$.

Example

The logistic equation of the population p (in thousands) at time t of a certain species is given by $\frac{\mathrm{d}p}{\mathrm{d}t} = p(2 - p)$. Use its direction field to answer the following questions.

(a) If the initial population is $3000[p(0) = 3]$, what can you say about the limiting population $\lim_{t\to\infty} p(t)$?

We start at the point $(0, 3)$ and as the field approaches $p = 2$, the rate of change becomes 0 so the limit is equal to 2.

(b) Can a population of 1000 ever decline to 500?

No

(c) Can a population of 1000 ever increase to 3000.

No

A differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$ is autonomous if the independent variable t does not appear explicitly: $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$. An autonomous equation has the following properties

- The slopes in the direction field are all identical among horizontal lines
- New solutions can be generated from old ones by time shifting [i.e., replacing $y(t)$ with $y(t - t_0)$]

The constant, or equilibrium, solutions $y(t) = c$ for autonomous equations are of particular interest. The equilibrium $y = c$ is called a stable equilibrium, or sink, if neighboring solutions are attracted to it as $t \to \infty$. Equilibria that repel neighboring solutions, are known as sources; all other equilibria are called nodes. Sources and nodes are unstable equilibria. (Nodes are sometimes called semi stable).

A phase line indicates the zeros and signs of $f(y)$ to describe the nature of the equilibrium solutions for an autonomous equation.

Example

SKetch the phase line for $y' = -(y-1)(y-3)(y-5)^2$ and state the nature of its equilibria.

We have $y' = 0$ and $y = 1, 3, 5$. When $y = 0$, then $y' < 0$ that means that it is decreasing for values below 1. When we let $y = 2$ then $y' > 0$, so any value between 1 and 3 is increasing. When we put $y = 4$, then $y' < 0$ so any values between 3 and 5 are decreasing. When $y = 6$ then $y' < 0$ so it is decreasing.

At $y = 5$, above it is is attracting but repeling below, so it is semi-stable or a node. At $y = 3$, it is stable or an attractor because it is approaching on both sides. At $y = 1$, it is a repeller.

Hand sketching the direction field for a differential equation is often tedious. Fortunately, several software programs are available for this task. When hand sketching is necessary, the method of isoclines can be helpful reducing the work.

A isocline for the differential equation

$$y' = f(x, y)$$

is a set of points in the xy-plane where all the solutions have the same slope $\frac{\mathrm{d}y}{\mathrm{d}x}$; thus, it is a level curve for the function $f(x, y)$.

Example

Find isocline curves of $y' = f(x, y) = x + y$ for a few select values of c. Use the isoclines to draw hash marks with slope c along the isocline $f(x, y) = c$.

When $c = 0, y' = 0, x + y = 0$ and $y = -x$. When $c = 1, y' = 1, x + y = 1$ and $y = -x + 1$. When $c = 2, y' = 2, x + y = 2$ and $y = -x + 2$.

1.4 The Approximation Method of Euler

Euler's method (or the tangent-line method) is a procedure for constructing approximate solutions to an initial value problem for a first-order differential equation

$$y' = f(x, y), \qquad y(x_0) = y_0$$

Euler's method can be summarized by the recursive formulas $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n, y_n)$, where $n = 0, 1, 2, \ldots$.

h is the step size, y' is the m of the tangent line. Remember that $y - y_0 = m(x - x_0)$ and that $y - y_0$ is just $f(x_0, y_0)(x - x_0)$ so $y = y_0 + f(x_0, y_0) \cdot h$.

Example

Use Euler's method with step size $h = 0.1$ to approximate the solution to the initial value problem

$$y' = x\sqrt{y}, \qquad y(1) = 4$$

at the points $x = 1.1, 1.2, 1.3, 1.4$ and 1.5.

We know the x points so we can find the y values from $y_n = y_{n-1} + f(x_{n-1}, y_{n-1})(0.1)$.

We have $(1, 4)$ then $(1.1, 4.2)$ and continuing the calculations, we get $(1.2, 4.43)$, $(1.3, 4.68)$, $(1.4, 4.96)$ and $(1.5, 5.27)$.

Example

Use Euler's method to find approximations to the solution of the initial value problem

$$y' = y, \qquad y(0) = 1$$

at $x = 1$, taking $1, 2, 4, 8$ and 16 steps.

Let $y = e^x$ and a point $(0, 1)$. The recursion formula is $y_n = y_{n-1} + f(x_{n-1} + y_{n-1})h$.

If we use a step size of 1, then $y(1) = 2$.

If we use a step size of 0.5 then $y(1) = 2.25$

If we use a step size of 0.25 then $y(1) = 2.44$

Using technology we can see with a step size of 0.125 that $y(1) = 2.57$ and with a step size of 0.0625, $y(1) = 2.64$.

2 First-Order Differential Equations

2.1 Separable Equations

Definition

If the right-hand side of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

can be expressed as a function $g(x)$ that depends only on x times a function $p(y)$ that depends only on y, then the differential equation is called separable.

To solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)p(y)$$

multiply by $\mathrm{d}x$ and by $h(y) = 1/p(y)$ to obtain

$$h(y)\mathrm{d}y = g(x)\mathrm{d}x$$

Then integrate both sides and you end up getting $H(y) = G(x)+C$, where we have merged the two constants of integration into a single symbol C. The last equation gives an implicit solution to the differential equation.

Example

Solve the nonlinear equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-5}{y^2}$$

This can be rewritten as $y^2\mathrm{d}y = (x-5)\mathrm{d}x$. Integrating both sides results in $\frac{y^3}{3} = \frac{x^2}{2} - 5x + C$.

To get the explicit form just solve for y, which is trivial.

Example

Solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-1}{x+3} \qquad y(-1) = 0$$

Doing Calc BC stuff gives us $y = 1 - \frac{1}{2}(x+3)$.

Be careful because you can be losing solutions. Ok bye!

2.2 Linear Equations

Remember a linear first-order equation is an equation that can be expressed in the form

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0y = b(x)$$

where $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on the independent variable x, not on y.

Method for solving linear equation:

- Write the equation in the standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

- Calculate the integrating factor $\mu(x)$ by the formula

$$\mu(x) = \exp\left[\int P(x)\mathrm{d}x\right]$$

- Multiply the equation in standard form by $\mu(x)$ and, recalling that the left-hand side is just $\frac{\mathrm{d}}{\mathrm{d}x}[\mu(x)y]$, obtain

$$\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\mu(x)y = \mu(x)Q(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\mu(x)y] = \mu(x)Q(x)$$

- Integrate the last equation and solve for y by dividing by $\mu(x)$ to obtain.

Example

Find the general solution to

$$\frac{1}{x}\mathrm{d}y\mathrm{d}x - \frac{2y}{x^2} = x\cos x \qquad x > 0$$

We have $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x}y = x^2\cos x$.

The integrating factor $\mu(x) = e^{\int P(x)\mathrm{d}x}$ which in this case is $e^{-2\int \frac{1}{x}\mathrm{d}x}$ and this is equivalent to $\frac{1}{x^2}$.

Using this we can multiply through in standard form then we have $\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x^3}y = \cos x$.

The left side is just $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^2}\cdot\right) = \cos x$.

Integrating and solving for y we get that $y = x^2\sin x + Cx^2$.

Example

For the initial value problem

$$y' + y = \sqrt{1+\cos^2 x} \qquad y(1) = 4$$

find the value of $y(2)$.

Our $P(x)$ is 1 here, so $\mu = e^x$.

So the equation after multiplying through by it gives us that $\mu y' + \mu y = \mu\sqrt{1+\cos^2 x}$, or $e^x y' + e^x y = e^x\sqrt{1+\cos^2 x}$.

This is equivalent to basically $\frac{\mathrm{d}}{\mathrm{d}x}(e^x y) = e^x\sqrt{1+\cos^2 x}$.

This is $e^x y = \int e^x\sqrt{1+\cos^2 x}\mathrm{d}x$.

Using a calculator $y(2) = 2.127$.

Theorem 2.1: Existence and Uniqueness of Solution

Suppose $P(x)$ and $Q(x)$ are continuous on an interval (a, b) that contains the point x_0. Then for any choice of initial value y_0, there exists a unique solution $y(x)$ on (a, b) to the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Qx \qquad y(x_0) = y_0$$

In fact the solution is given for a suitable value of C.

2.3 Exact Equations

Definition: Exact Differential Form

The differential form $M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y$ is said to be exact in a rectangle R is there is a function $F(x,y)$ such that
$$\frac{\partial F}{\partial x}(x,y) = M(x,y) \qquad \text{and} \qquad \frac{\partial F}{\partial y}(x,y) = N(x,y)$$
for all (x,y) in R. That is, the total differential of $F(x,y)$ satisfies
$$\mathrm{d}F(x,y) = M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y$$
If $M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y$ is an exact differential form, then the equation
$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0$$
is called an exact equation.

Theorem 2.2: Test for Exactness

Suppose the first partial derivatives of $M(x,y)$ and $N(x,y)$ are continuous in a rectangle R. Then
$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0$$
is an exact equation in R if and only if the compatibility condition
$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$$
holds for all (x,y) in R.

Example

Solve the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy^2+1}{2x^2y}$$
Ok so this is not separable or linear, so we use exactness.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2xy^2+1}{2x^2y} &= 0 \\
\mathrm{d}y + \frac{2xy^2+1}{2x^2y}\mathrm{d}x &= 0 \\
\frac{2xy^2+1}{2x^2y}\mathrm{d}x + 1\mathrm{d}y &= 0
\end{aligned}$$

This is the same form we want.

Another form we can get is $(2xy^2+1)\mathrm{d}x + 2x^2y\mathrm{d}y = 0$.

Another form we can get is $1\mathrm{d}x + \frac{2x^2y}{2xy^2+1}\mathrm{d}y = 0$.

We are now looking for a $F(x,y) = c$ and we know this is true when $\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$.

So the second one of these is probably the best, so we now have $m = 2xy^2+1$ and $n = 2x^2y$.

Doing the partial of m with respect to y we get $4xy$ and the partial of n with respect to x is $4xy$ and these are the same.

Let $F(x, y) = x^2y^2 + x = C$. The partial of this function with respect to x is $2xy^2 + 1$ and the partial of this function with respect to y is $2x^2y$ and this is the same as previous.

Method for Solving Exact Equations:

- If $M\mathrm{d}x + N\mathrm{d}y = 0$ is exact, then $\partial F/\partial x = M$. Integrate this last equation with respect to x to get

$$F(x, y) = \int M(x, y)\mathrm{d}x + g(y)$$

- To determine $g(y)$, take the partial derivative with respect to y of both sides of the aboev equation and substitute N for $\partial F/\partial y$. We can now solve for $g'(y)$.
- Integrate $g'(y)$ to obtain $g(y)$ up to a numerical constant. Substituting $g(y)$ into the equation from step 1 gives $F(x, y)$
- The solution to $M\mathrm{d}x + N\mathrm{d}y = 0$ is given implicitly by

$$F(x, y) = C$$

(Alternatively, starting with $\partial F/\partial y = N$, the implicit solution can be foudn by first integrating with respect to y.)

Example

Solve

$$(2xy - \sec^2 x)\mathrm{d}x + (x^2 + 2y)\mathrm{d}y = 0$$

Let m be the first term and n be the second term, and the partial derivatives of these are the same, so they are exact.

Let $F(x, y) = \int 2xy - \sec^2 x\mathrm{d}x$. When we integrate this, we get $yx^2 - \tan x$. In this case, the constant is anything with y, so the integral is equivalent to $yx^2 - \tan x + g(y)$.

Now we take the $\frac{\partial F}{\partial y} = x^2 - 0 + g'(y)$. These two are n so $x^2 + 2y = x^2 + g'(y)$, so solving for $g(y)$ we get that this is equal to $y^2 + C$.

So $F(x, y) = xy^2 - \tan x + y^2 = C$.

Exercise Solve $(1 + e^x y + xe^x y)\mathrm{d}x + (xe^x + 2)\mathrm{d}y = 0$.

Solution: $x + xye^x + 2y = C$.

Example

Solve

$$(x + 3x^3 \sin y)\mathrm{d}x + (x^4 \cos y)\mathrm{d}y = 0$$

Doing the partials originally makes them not equal to each other.

We can get this to exact form by multiplying through by x^{-1}. When we do this we get $(1+3x^2 \sin y)\mathrm{d}x + x^3 \cos y\mathrm{d}y = 0$ and the partials of these are the same.

x^{-1} is called an integrating factor.

Integrating m with respect to x, we get that $F(x, y) = \int 1 + 3x^2 \sin y\mathrm{d}x = x + \sin y \cdot x^3 + g(y)$.

Doing the partial of F with respect to y, we get $\frac{\partial F}{\partial y} = x^3 \cos y = 0 + x^3 \cos y + g'(y)$, and this gets that $g(y) = C$.

So the answer is $x + x^3 \sin y = C$.

2.4 Special Integrating Factors

Definition

If the equation
$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0$$
is not exact, but the equation
$$\mu(x,y)M(x,y)\mathrm{d}x + \mu(x,y)N(x,y)\mathrm{d}y = 0$$
which results from multiplying the first equation by the function $\mu(x,y)$, is exact, then $\mu(x,y)$ is called an integrating factor of the first equation.

Theorem 2.3: Special Integrating Factors

If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then
$$\mu(x) = \exp\left[\int\left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right)\mathrm{d}x\right]$$
is an integrating factor for an equation. If $(\partial N/\partial x - \partial M/\partial y)/M$ is continuous and depends only on y, then
$$\mu(y) = \exp\left[\int\left(\frac{\partial M/\partial x - \partial N/\partial y}{M}\right)\mathrm{d}y\right]$$
is an integrating factor for the same equation.

Method for Finding Special Integrating Factors:

If $M\mathrm{d}x + N\mathrm{d}y = 0$ is neither separable nor linear, compute $\partial M/\partial y$ and $\partial N/\partial x$. If $\partial M/\partial y = \partial N/\partial x$, then the equation is exact. If it is not exact, consider
$$\frac{\partial M/\partial y - \partial N/\partial x}{N}$$
If this is a function of just x, then an integrating factor is given by the formula above of $\mu(x)$. If not consider
$$\frac{\partial N/\partial x - \partial M/\partial y}{M}$$
If this is a function of just y¡ then an integrating factor is given by above of $\mu(y)$.

Example

Solve $(2x^2 + y)\mathrm{d}x + (x^2y - x)\mathrm{d}y = 0$

When we do the partials, we get that $1 \neq 2xy - 1$.

So lets look at $\frac{\partial m/\partial y - \partial n/\partial x}{N}$, which is $\frac{1-(2xy-1)}{x^2y-x} = \frac{-2}{x}$ which is just a function of x. So we have that $\mu = e^{\int -\frac{2}{x}\mathrm{d}x}$, so we don't have to look at the one in terms of y.

Doing the integral of all this gives us that $e^{-2\ln x} = x^{-2}$. So when we multiply through by x^{-2}, we get that $(2 + x^{-2}y)\mathrm{d}x + (y - x^{-1})\mathrm{d}y = 0$.

The partials are equal to each other, so this equation is now exact.

Now we find $F(x,y)$ by integrating m, so $\int(2 + x^{-2}y)\mathrm{d}x = 2x + -x^{-1}y + g(y) = F(x)$

Now we differentiation with respect to y so $\frac{\partial F}{\partial y} = y - x^{-1} = -x^{-1} + g'(y)$, so $g(y) = \frac{y^2}{2}$

The solution is therefore $2x - x^{-1}y + \frac{y^2}{2} = C$.

2.5 Substitutions and Transformations

Substitution Procedure:

- Identify the type of equation and determine the appropriate substitution or transformation
- Rewrite the original equation in terms of new variables
- Solve the transformed equation
- Express the solution in terms of the original variables

Definition: Homogeneous Equation

If the right-hand side of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

can be expressed as a function of the ratio y/x alone, then we say the equation is homogeneous.

To solve a homogeneous equation, use the substitution $v = \frac{y}{x}$; $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ to transform the equation into a separable equation.

Example

Solve $(xy + y^2 + x^2)\mathrm{d}x - x^2\mathrm{d}y = 0$.

Solving for $\frac{\mathrm{d}y}{\mathrm{d}x}$ we get that this is equal to $\frac{-x^2-y^2-xy}{-x^2}$ and this simplifies to $1 + \left(\frac{y}{x}\right)^2 + \frac{y}{x}$.

This is equivalent to $v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v^2 + v$. We end up getting that $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v^2+1}{x}$ and this can be done by separation. The solution is $y = x\tan(\ln|x| + C)$ after solving.

To solve an equation of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = G(ax+by)$, use the substitution $z = ax+by$ to transform the equation into a separable equation.

Example

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = y - x - 1 + (x - y + 2)^{-1}$

First we have $\frac{\mathrm{d}y}{\mathrm{d}x} = -(x - y) - 1 + (x - y + 2)^{-1}$

So substituting with $z = x - y$, we have that $\frac{\mathrm{d}z}{=}1 - \frac{\mathrm{d}z}{\mathrm{d}x}$. Knowing this, the equation is equal to $1 - \frac{\mathrm{d}z}{\mathrm{d}x} = -z - 1 + (z + 2)^{-1}$. From this this simplifies to $\frac{\mathrm{d}z}{\mathrm{d}x} = z + 2 - (z + 2)^{-1}$.

So now we write this into a separable equation with $\frac{(z+2)\mathrm{d}z}{(z+2)^2-1} = \mathrm{d}x$.

Separating by parts and substituting gives $(x - y + 2)^2 = ce^{2x} + 1$

Definition: Bernoulli Equation

A first-order equation that can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

where $P(x)$ and $Q(x)$ are continuous on the interval (a, b) and n is a real number, is called a Bernoulli equation.

To solve a Bernoulli equation use the substitution $v = y^{1-n}$ to transform the equation into a linear equation.

Example

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = -\frac{5}{2}xy^3$.

From above, we have $v = y^{-2}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = -2y^{-3}\frac{\mathrm{d}u}{\mathrm{d}x}$ and $-\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}x} = y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}$.

So multiplying through by y^{-3} and substituting, we get that $-\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}x} - 5v = -\frac{5}{2}x$.

This is equal to $\frac{\mathrm{d}v}{\mathrm{d}x} + 10v = 5x$. The integrating factor here is $\mu = e^{\int P(x)\mathrm{d}x}$, which is e^{10x} in this case.

Multiplying through by μ, we get that $e^{10x}\frac{\mathrm{d}v}{\mathrm{d}x} + 10ve^{10x} = 5xe^{10}x$ and the LHS should be equal to $\frac{\mathrm{d}}{\mathrm{d}x}(e^{10x}v) = 5xe^{10x}$.

Using elementary integration techniques the answer is $y^{-2} = \frac{x}{2} - \frac{1}{20} + Ce^{-10x}$.

3 Mathematical Models and Numerical Methods Involving First-Order Equations

3.1 Compartmental Analysis

We assume that the growth rate is proportional to the population present. A mathematical model called the Malthusian, or exponential law of population growth, model is given by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = kp, \qquad p(0) = p_0$$

where $k > 0$ is the proportionality constant of the growth rate and p_0 is the population at time $t = 0$.

Example

In 1790, the population of the United States was 3.93 million, and in 1890 it was 62.98 million. Assuming the Mathusian model, estimate the U.S. population as a function of time.

Let $t = 0$ be year 1790. Using separation of variables and integration from $\frac{\mathrm{d}p}{\mathrm{d}t} = kp$, we get that $\ln P = kt + C$. Solving for P we get that $e^{\ln P} = P = e^{kt+C}$, so this is equal to $P = Ce^{kt}$.

At $P(0) = 3.93$, and using this we can find k. $3.93 = Ce^0$, so $C = 3.93$. So we have $P(t) = P_0 e^{kt}$. We can now find k by plugging in $P(100) = 62.98$ from this, and solving for k, we get that $k \approx 0.027742$.

So $P(t) = 3.93e^{.027742t}$

The Malthusian model considered only death by natural casues. Other factors such as premature deaths due to malnutrition, inadequate medical supplies, communicable diseases, violent crimes, etc involve a competition within the population. The logistic model is given by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -Ap(p - p_1), \qquad p(0) = p_0$$

Note that this equilibrium has two equilibrium solutions $p(t) = p_1$ and $p(t) = 0$.

Example

Taking the population of 3.93 million as the initial population and given the 1840 and 1890 populations of 17.07 and 62.98 million respectively, use the logistic model to estimate the population at time t.

We have $P(50) = 17.07$ and $P(100) = 62.98$. Using a previously derived formula, $p(t) = \frac{p_0 p_1}{p_0 + (p_1 - p_0)e^{-Ap_1 t}}$, we get $17.07 = \frac{3.93p_1}{3.93 + (p_1 - 3.93)e^{-50Ap_1}}$ and $62.98 = \frac{3.93p_1}{3.93 + (p_1 - 3.93)e^{-100Ap_1}}$.

The answers are that $p_1 \approx 251.78$ and $A \approx 0.0001210$, so the logistic equation ended up being $p(t) = \frac{989.5}{3.93 + 247.85e^{-0.030463t}}$.

Example

Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days, $x(4) = 50$.

We are solving the initial value problem $\frac{\mathrm{d}x}{\mathrm{d}t} = kx(1000-x), x(0) = 1$. Doing some stuff we get that $\ln\frac{x}{1000-x} = 1000kt + C$.

After doing some more things we get that $x = \frac{1000Ce^{1000kt}}{1+Ce^{1000kt}}$ and from this we can get that $C = \frac{1}{999}$.

So the function is now $x(t) = \frac{\frac{1000}{999}e^{1000kt}}{1+\frac{1}{999}e^{1000kt}}$, so simplifying we get that $x(t) = \frac{1000}{999e^{-1000kt}+1}$.

Doing some plugging in stuff we get that $k \approx -.9906$, so after 6 days approximately 276 students are infected.

3.2 Numerical Methods: A Closer Look At Euler's Algorithm

The numerical method defined by the formula

$$y_{n+1} = y_n + h\frac{f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})}{2}$$

where

$$y^*_{n+1} = y_n + hf(x_n, y_n)$$

is known as the improved Euler's method.

The improved Euler's method is an example of a predictor-corrector method.

It is recommended you use technology to find the answer.

3.3 Higher-Order Numerical Methods: Taylor and Runge-Kutta

We wish to obtain a numerical approximation of the solution $\phi(x)$ to the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0$$

To derive the Tayler methods, let $\phi_n(x)$ be the exact solution to the initial value problem

$$\phi_n'(x) = f(x, \phi_n), \qquad \phi_n(\text{problem}x_n) = y_n$$

The Taylor series for $\phi_n(x)$ about the point x_n is

$$\phi_n(x) = \phi_n(x_n) + h\phi_n'(x_n) + \frac{h}{2!}\phi_n''(x_n) + \ldots$$

where $h = x - x_n$. Since ϕ_n satisfies the initial value, we can write this series in the form

$$\phi_n(x) = y_n + hf(x_n, y_n) + \frac{h^2}{2!}\phi_n''(x_n) + \ldots$$

Example

Determine the recursive formula for the Taylor method of order 2 for the initial value

$$y' = \sin(xy), y(0) = \pi$$

We know that $\phi_n''(x_n) = \frac{\partial f}{\partial x}(x_n, y_n) + \frac{\partial f}{\partial y}(x_n, y_n) \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$.

So $\frac{\partial f}{\partial x} = \cos(xy) \cdot y$ and $\frac{\partial f}{\partial y} = \cos(xy) \cdot x$.

Putting this all together we get that $\phi_n(x) = y_{n+1} = y_n + h\sin(x_ny_n) + \frac{h^2}{2}[y\cos(xy) + x\cos(xy) \cdot \sin(xy)]$.

Doing some magic, $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h\sin(x_ny_n) + \frac{h^2}{2}[y_n\cos(x_ny_n) + \frac{x_n}{2}\sin(2x_ny_n)]$.

4 Linear Second-Order Equations

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