Linear Algebra Notes

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1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

A linear equation in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ where b and the coefficients $a_1, a_2, \ldots, a_n$ are real or complex numbers.

Example: $4x_1 - 5x_2 + 2 = x_1, x_2 = 2(6^{1/2} - x_1) + x_3$ are both linear. Not linear examples are $4x_1 - 5x_2 = x_1x_2, x_2 = 2(x_1^{1/2}) - 6$, and $2x_1^{-1} + \sin x_2 = 0$.

Systems of Linear Equations: an $(m \times n)$ system of linear equations is a system of m linear equations with n unknowns.

Example: the 2×3 system of equations below has a solution $x_1 = 5, x_2 = 6.5$, and $x_3 = 3$:

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 - 4x_3 &= -7 \end{aligned}$$

Two linear systems are called equivalent if they have the same solution set.

A system of linear equations can have: infinitely many solutions, no solution, or a unique solution. Coincident lines have infinitely many solutions, parallel lines have no solution, and intersecting lines have a unique solution.

Matrix Notation:

Let's say we have $x_1 - 2x_2 + x_3 = 0, 2x_2 - 8x_3 = 8$, and $5x_1 - 5x_3 = 10$.

The coefficient matrix is $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$ and the augmented matrix is $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$. The augmented matrix is 3×4 (3 rows, 4 columns).

To solve a linear system: if one of the following elementary operations is applied to a system of linear equations, the resulting system is equivalent, that is the resulting system has the same set of solutions as the original:

1. interchange two equations
2. multiply an equation by a non-zero scalar
3. add a constant multiple of one equation to another

Let's use the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

So using row operations and rearranging the rows, we can do $R_1 \leftrightarrow R_2$ to swap the first and second row. Now we can multiply the second line by $1/2$ so $1/2R_2$. The next step is to do $R_2 + R_1$.

The matrix that results from this is $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$

Doing the operations $R_3 - 5R_1$, $R_3 - 5R_2$, and $1/2R_2$ and $1/30R_3$, we end up getting $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, so from this we have the equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned}$$

, so the solution set is $(1, 0, -1)$. Since a solution exists, the system is consistent.

Let's see if this one is consistent.

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned}$$

Doing the row operations $R_1 \leftrightarrow R_2$, $R_3 - 2R_1$, $R_3 + 2R_2$ we get that $0 = 15$, so this is inconsistent.

This last example has infinitely many solutions:

$$\begin{aligned} x_1 - 2x_2 - x_3 &= -2 \\ 2x_1 + x_2 + 3x_3 &= 1 \\ -3x_1 + x_2 - 2x_3 &= 1 \end{aligned}$$

Doing the row operations $R_2 - 2R_1$ and $R + 3 + 3R_1$, then $R_3 + R_2$ and then $1/5R_2$ results in $\begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solving this, we have x_3 with no restrictions, it is a "free parameter" and $x_2 = 1 - x_3$ and $x_1 = -x_3$ so x_1 and x_2 are parameterized by x_3.

1.2 Row Reduction and Echelon Forms

Leading entry of a row: the first (counting from left to right) non-zero entry (in a nonzero row)

Echelon Form: upper right-hand stair-case, triangle

1. all rows that consist entirely of zeros are grouped together at the bottom of the matrix
2. the first (counting from left to right) non-zero entry in the $(i+1)$st row must appear in a column to the right of the first non-zero entry in the ith row.
3. all entries in a column below a leading entry are zeros

Reduced Echelon Form: an echelon Form matrix that also has the following properties:

1. the leading entry in each nonzero row is 1
2. each leading one is the only nonzero entry in its column

A pivot point: a location in a matrix A that corresponds to a leading 1 in a reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

Steps to solving a system of linear equations:

1. begin with the leftmost nonzero column. This is the pivot column, the pivot position is at the top.
2. select a non zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
3. Use row replacement operations to create zeros in all positions below the pivot.
4. Cover (or ignore) the row containing the pivot position and cover all rows (if any) above it. Apply previous steps to the sub matrix that remains. Repeat until there are no more nonzero rows to modify.
5. Beginning with the rightomst pivot, create zeros above each pivot. Make each pivot equal to 1 by scaling.

Example

Determine the existence and uniqueness of the solution to

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

We can find that $x_5 = 4$, and x_3, x_4 are of infinite number of solutions.

Example

Find the general solution of the linear system whose augmented matrix has been reduced to $\left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$

We have that $x_5 = 7$, $x_3 = 1/2(3 + 8x_4 + 7)$ and $x_1 = -4 - 6x_2 - 2(1/2)(3 + 8x_4 + 7) + 14$

Theorem 1.1: Existence and Uniqueness

A linear system is consistent if and only if the right most column of the augmented matrix is not a pivot column, that is if and only if an echelon form of the augmented matrix has no row of the form $[0, \ldots, 0b]$ with b nonzero. If the system if consistent in the solution contains either a unique solution when there are no free variables or infinitely many solutions when there is at least one free variable.

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