Math Formula Sheet

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Shapes

Area of a Triangle: $\frac{1}{2}\times$ base $\times$ height

Area of a Parallelogram: base $\times$ height

Area of a Rectangle: length $\times$ width

Area of a Trapezoid: $\frac{1}{2}$(sum of parallel sides)$\times$ height

Circumference & Area: Circle: $c = 2\pi r, A = \pi r^2$

Cuboid Surface Area: $SA = 2xy + 2xz + 2yz$, where x, y, and z are side lengths

Cuboid Volume: $V = xyz$, where x, y, and z are side lengths

Cylinder Surface Area: $SA = 2\pi rh + 2\pi r^2$. Note: Curved Part: $2\pi rh$

Cylinder Volume: $V = \pi r^2 h$

Cone Surface Area: $SA = \pi rl + \pi r^2$. Note Curved part: πrl, where l is slant length

Cone Volume: $V = \frac{1}{3}\pi r^2 h$

Sphere Surface Area: $SA = 4\pi r^2$. Note: Hemisphere $= 2\pi r^2 + \pi r^2 = 3\pi r^2$

Sphere Volume: $V = \frac{4}{3}\pi r^3$. Note: Hemisphere $= \frac{2}{3}\pi r^3$

Prism Volume: $V=$ Area of cross section $\times$ height

Pyramid Volume: $V = \frac{1}{3}\times$ base area $\times h$

Indices

Multiplication:

- $x^a \times x^b = x^{a+b}$
- $(x^a)^b = x^{ab}$
- $(cx^a y^b)^d = c^d x^{ad} y^{bd}$

Division: $x^a \div x^b = \frac{x^a}{x^b} = x^{a-b}$

Negative Powers: $x^{-n} = \frac{1}{x^n}$

Fractions:

- $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$
- $\left(\frac{x}{y}\right)^{-n} = \frac{y^n}{x^n}$

Rational Powers: $a^{\frac{n}{m}} = (a^{\frac{1}{m}})^n = (\sqrt[m]{a})^n = (a^n)^{\frac{1}{m}} = \sqrt[m]{a^n}$

Series

Arithmetic sequence: nth term - $u_n = a + (n-1)d$ where a is the first term and d is the common difference.

Arithmetic sequence: sum of n terms - $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$ where a is the first term, d is the common difference and l is the last term.

Geometric sequence: nth term - $u_n = ar^{n-1}$ where a is the first term and r is the common ratio.

Geometric sequence: sum of n terms - $S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, r \neq 1$ where a is the first term and r is the common ratio

Geometric sequence: sum to infinity - $S_\infty = \frac{a}{1-r}, |r| < 1$, where a is the first term and r is the common ratio

Compound interest: $FV = PV\left(1+\frac{r}{100k}\right)^{kt}$ where FV is the future value, PV is the present value, t is the no. of years, r is the nominal annual interest rate, and k is the no. of compounding periods per year

Binomial Theorem: integer powers - $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n$

Binomial Theorem: Fractional & Negative powers $= (a+b)^n = a^n\left(1 + n(\frac{b}{a}) + \frac{n(n-1)}{2!}\frac{b}{a}^2 + \cdots\right)$

Binomial Coefficient: $\binom{n}{r} = nc_r = \frac{n!}{(n-r)!r!}$

Geometry

Straight Line: Equation (gradient means slope), Parallel $\implies$ same slope. Perpendicular $\implies$ "flip fraction and change the sign" (slopes multiply to make -1)

- Slope intercept form: $y = mx + c$
- General form: $ax + by + d = 0$
- Point slope form: $y - y_1 = m(x - x_1)$

Straight Line: Gradient - $m = \frac{y_2 - y_1}{x_2 - x_1}$

Distance between 2 points $(x_1, y_1), (x_2, y_2)$: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Coordinates of midpoint of $(x_1, y_1), (x_2, y_2)$: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Circles: $(x-a)^2 + (y-b)^2 = r^2$, where the centre is (a, b) the radius is r.

Quadratics

Quadratic Function: Solutions to $ax^2 + bx + c = 0$ - $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$

Quadratic Function: Axis of Symmetry - $f(x) = x^2 + bx + c \implies x = -\frac{b}{2a}$

Quadratic Function: Discriminant - $\Delta = b^2 - 4ac$

- > 0 (2 real distinct roots)
- $= 0$ (2 real repeated/double roots)
- < 0 (no real roots)

Completing The Square $ax^2 \pm bx + c = 0$ - $a\left(x \pm \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$

Max/Min Value: $c - \frac{b^2}{4a}$

Exponential and Logarithmic Functions

- $a^x = e^{x \ln a}$
- $\log_a a^x = x = a^{\log_a x}$

where, $a, x > 0, a \neq 1$

Exponential and Logarithm Rules

- $c \log_a b = \log_a b^c$
- $\log_a b = c \implies a^c = b, a, b > 0, a \neq 1$
- $\log_a b + \log_a c = \log_a bc$
- $\log_a b - \log_a c = \log_a \frac{b}{c}$
- $\log_a b = \frac{\log_c b}{\log_c a}$
- Solving a power of x: log both sides if 2 terms or use substitution if 3 terms
- Solving an exponential: $\ln$ both sides

- Solving a logarithm: raise e both side or write as $\log_e$ as proceed as usual for $\log$

Trigonometry

Sine Rule:

- Finding a side: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- Finding an angle: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Cosine Rule:

- Finding a side: $a^2 = b^2 + c^2 - 2bc \cos A$
- Finding an angle: $A = \cos^{-1}\left(\frac{b^2+c^2-a^2}{2bc}\right)$

Area of Triangle: $\frac{1}{2}ab \sin C$

Degrees $\rightleftarrows$ radians

- Degrees to radians: $\times \frac{\pi}{180}$
- Radians to degrees: $\times \frac{180}{\pi}$

Length of an arc: $\frac{\theta}{360} \times 2\pi r$ (degrees) or $r\theta$ (radians)

Area of a sector: $\frac{\theta}{360} \times \pi r^2$ (degrees) or $\frac{1}{2}r^2\theta$ (radians)

Small Angle Approximations

- $\sin\theta \approx \theta$
- $\cos\theta \approx 1 - \frac{\theta^2}{2}$
- $\tan\theta \approx \theta$

Pythagorean identity 1: $\sin^2 x + \cos^2 x = 1$

Pythagorean identity 2: $1 + \tan^2 x = \sec^2 x$

Pythagorean identity 3: $1 + \cot^2 x = \csc^2 x$

Confunction

- $\cos x = \sin(90 - x)$
- $\sin x = \cos(90 - x)$

Identity of $\tan x$: $\tan x = \frac{\sin x}{\cos x}$

Reciprocal:

- $\sec x = \frac{1}{\cos x}$
- $\csc x = \frac{1}{\sin x}$
- $\cot x = \frac{1}{\tan x}$

Double Angle

- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 \implies \cos^2 x = \frac{\cos 2x+1}{2} = 1 - 2\sin^2\theta \implies \sin^2 x = \frac{1-\cos 2x}{2}$
- $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

Half Angle

- $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$
- $\cos\frac{x}{2} = \pm\sqrt{\frac{\cos x+1}{2}}$

- $\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$

Compound Angle

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Factor Formula: sum to product (Note: For product t to sum rearrange andd let $\frac{A+B}{2}$ and $\frac{A-B}{2}$ equal your given angles and solve for A and B simultaneously)

- $\sin A + \sin B \equiv 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
- $\sin A - \sin B \equiv 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$
- $\cos A + \cos B \equiv 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
- $\cos A - \cos B \equiv -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

If $t = \tan\frac{1}{2}x \implies \sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

Vectors

Notations: vector = **a**, $\underline{a}$, $\overrightarrow{OA}$. Distance = OA

Vector Form: $a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \equiv \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Properties:

- $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \pm \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a \pm d \\ b \pm e \\ c \pm f \end{pmatrix}$
- $\lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \end{pmatrix}$
- $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = ad + be + cf$

Magnitude of a vector: $\left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| = \sqrt{a^2 + b^2 + c^2}$

Unit Vector: Unit vector of $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{\sqrt{a^2+b^2+c^2}}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Parallel means vectors are a multiple of each other.

Perpendicular means scalar product equals zero.

Midpoint of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$: $\left(\frac{a+d}{2}, \frac{b+e}{2}, \frac{c+f}{2}\right)$

Scalar Product: $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right|\left|\begin{pmatrix} d \\ e \\ f \end{pmatrix}\right|\cos\theta$, where θ is the angle between $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$

Vector Product: Note: θ is the angle between $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$:

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} bf - ec \\ -(af - cd) \\ ae - bd \end{pmatrix}$ or $\left| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right| = \left| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right| \left| \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right| \sin\theta$

Angle Between 2 vectors (This is just a re-arrangement of above): $\theta = \cos^{-1}\left(\frac{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix}}{\left| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right| \left| \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right|} \right)$

Vector Equation of a line: To find this we need: point and direction (if given 2 points find the directions and use either point) $r = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} d \\ e \\ f \end{pmatrix}$, $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is position, and $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$ is direction (parallel to)

Cartesian Equation of a line: $\frac{x-a}{d} = \frac{y-b}{e} = \frac{z-c}{f}$

Parametric form of a line: $x = a\lambda d$, $y = b + \lambda e$, $z = c + \lambda f$

Equation of a plane: $\mathbf{r} \cdot \mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \mathbf{n}$, where n is the normal vector

Vector Equation of a plane: To find this we need: a point in plane and perp direction. If not given perp direction take the cross product of 2 direction vectors. Remember to find a direction we subtract 2 position vectors; $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} d \\ e \\ f \end{pmatrix} + \mu \begin{pmatrix} r \\ s \\ t \end{pmatrix}$, where $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is position, $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$ and $\begin{pmatrix} r \\ s \\ t \end{pmatrix}$ are directions (parallel to)

Cartesian Equation of a plane: $ax + by + cz = d$, where d is the distance form origin to plane and $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the direction vector (perpendicular to)

Area of a Parallelogram: $A = \left| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right|$ where $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$ form 2 adjacent sides of a parallelogram

Perp Distance between point and plane from (α, β, γ) to $ax + by + cz = d$: $\frac{|a(\alpha)+b(\beta)+c(\gamma)+d}{\sqrt{a^2+b^2+c^2}}$

Scalar Product Properties:

- $0 \cdot \mathbf{a} = \mathbf{a}$
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $(-\mathbf{a}) \cdot \mathbf{b} = -(\mathbf{a} \cdot \mathbf{b})$
- $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b})$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- If a and b are parallel: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$, moreover $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

Cross Product Properties:

- $\mathbf{a} \times \mathbf{a} = 0$
- $\mathbf{a} \times 0 = 0 \times \mathbf{a} = 0$
- $\lambda(\mathbf{a} \times \mathbf{b}) = (\lambda\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\lambda\mathbf{b})$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$

- $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$
- $\mathbf{b}(\mathbf{c} \times \mathbf{a}) = \mathbf{c}(\mathbf{a} \times \mathbf{b})$

Probability

Mean:

- If no frequency: $\overline{x} = \frac{\sum x}{n}$
- If frequency, $\overline{x} = \frac{\sum fx}{\sum f}$

Variance:

- If no frequency: $\sigma^2 = \frac{\sum x^2}{n} - \overline{x}^2 = \frac{\sum (x-\mu)^2}{n}$
- If frequency: $\sigma^2 \frac{\sum fx^2}{\sum f} - \overline{x}^2 = \frac{\sum f(x-\mu)^2}{\sum f}$ Note: can also use the formula $\frac{s_{xx}}{n}$

Standard Deviation: $\sigma = \sqrt{\text{variance}}$

s_{xx}: $\sum (x_i - \overline{x})^2 = \sum x_i^2 - \frac{(\sum x_i)^2}{n}$

Probability of event A: $P(A) = \frac{n(A)}{n(U)} = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$

Complementary Events: $P(A') = 1 - P(A)$ i.e. probabilities add up to 1

Combined Events (Addition Rule): $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Mutually Exclusive Events: $P(A \cap B) = 0$ Addition rule becomes: $P(A \cup B) = P(A) + P(B)$

Independent Events: $P(A \cap B) = P(A)P(B)$, Addition rule becomes: $P(A \cup B) = P(A) + P(B) - P(A)P(B)$. To find whether independent: Find $P(A), P(B)$ and $P(A \cap B)$ and see whether the former 2 multiply to make the latter or show that $P(A|B) = P(A)$

Conditional "A given B": $P(A|B) = \frac{P(A \cap B)}{P(B)}$. If independent, $P(A|B) = P(A)$

Bayes Theorem: $P(A|B) = \frac{P(B|A)P(A)}{PB|AP(A) + P(B|A')P(A')}$

Binomial Distribution (binompd ($=$), binomcd ($\leq$))

- $x \sim B(n,p)$
- $E(X) =$ Mean $= np$, Var$(X) = np(1-p)$
- $P(X = x) = \binom{n}{x} p^x (1-p)^x$

Normal Distribution (normcd (given x, want prob), invnorm (given prob, want x))

- $x \sim N(\mu, \sigma^2)$
- Standardised variable $z = \frac{x-\mu}{\sigma}$

Interquartile Range: IQR $= Q_3 - Q_1$

Outliers: Ant=y values $> UQ + 1.5(IQR)$ or $< LQ - 1.5(IQR)$

Mechanics

SUVAT (5 formulas)

- $v = u + at$
- $s = vt - \frac{1}{2}at^2$
- $s = \left(\frac{u+v}{2}\right) t$
- $s = ut + \frac{1}{2}at^2$
- $v^2 = u^2 + 2as$

Centres of Mass for Uniform Bodies

- Triangular Lamina: $\frac{2}{3}$ along median from vertex
- Circular arc: radius r, angle at centre $2\alpha = \frac{r \sin\alpha}{\alpha}$ from centre
- Sector of circle, radius r, angle at centre 2α : $\frac{2r\sin\alpha}{3\alpha}$ from centre
- Solid hemisphere, radius r : $\frac{3}{8}r$ from centre
- Hemispherical Shell, radius r : $\frac{1}{2}r$ from centre
- Solid cone or pyramid of height h : $\frac{1}{4}h$ above the base on the line from centre to base of vertex
- Solid cone or pyramid of height h : $\frac{3}{4}h$ from the vertex
- Conical shell of height h : $\frac{1}{4}h$ above the base on the line from centre to base of vertex

Motion in a circle

- Transverse velocity: $v = rth\dot{e}ta$
- Transverse acceleration: $\dot{v} = r\ddot{\theta}$
- Radial acceleration: $-r\dot{\theta}^2 = -\frac{v^2}{r}$ Note: Mag $= r\dot{\theta}^2$ or $\frac{v^2}{r}$

Motion of a projectile: Equation of a trajectory: $y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta}$

Elastic Strings and Springs: F is force needed to extend or compress, T is tension, x is length of extension/compression, k is the stiffness constant (spring constant measured in N/m), λ is the modulus of elasticity (spring modulus) measured in Newtons, and l is the natural length of the spring.

- Hooke's Law: $F = -kx$
- Tension in elastic spring/string: $T = \frac{\lambda}{l}x = \frac{\lambda x}{l}$

Energy: Note if answer is negative then means a loss

- Kinetic Energy: $\frac{1}{2}mv^2$
- Gravitational Potential Energy: mgh
- Elastic Potential Energy: $\frac{\lambda x^2}{2l}$
- Change in Kinetic Energy: $\frac{1}{2}m_1v_1^2 - \frac{1}{2}m_2v_2^2$
- Change in potential energy: $m_1gh_1 - m_2gh_2$

Work Done: W is the work done, F is the magnitude of the force, d is the distance moved in the direction of the force, θ is the angle between the force and the displacement, and total energy is kinetic + potential + elastic

- If work done against an opposing force: Final total energy = Initial total energy - work done against force where work is done against opposing force is W
- $W = Fd\cos\theta$ total energy lost (aka work energy principle) and potential energy (if talking about against gravity)
- Note: Total energy lost = change in total energy = change in kinetic + change in potential
- If no work done against an opposing force, final total energy = initial total energy

Induction Template

Let P_n be the proposition...:

- Let $n = 1$, Plug in $n = 1$ to both the LHS and RHS. Show that LHS = RHS $\implies P_1$ true
- Assume $n = k$ true i.e P_k true: replace n with k. There is nothing to prove here, we just assume this to be true

- Let $n = k + 1$: Replace n with $k + 1$. Usually only need work on the LHS side by simplifying and using assumed P_k step to show that what we get for LHS is equal to RHS (sometimes we may need to work on the RHS also) $\implies P_{k+1}$ is true
- So P_k true $\implies P_{k+1}$ true and P_1 true then $P_2, P_3, P_4, \dots$ true $\therefore$ true for all $n \in \dots$.

Calculus

Turning/Stationary Points (Max/Min): Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

Proving whether Max/Min: If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ min and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ max. Or can do sign change test for $\frac{\mathrm{d}y}{\mathrm{d}x}$ using number line

Points of Inflection: solve $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$

Increasing/Decreasing (use number line to solve)

- To find where increasing: solve $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$
- To find where decreasing: solve $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$

Convex/Concave (use number line to solve)

- To find where concave up/convex: solve $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$
- To find where concave down/concave: solve $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$

Tangents and Normals: $y - y_1 = m(x - x_1)$. Differentiate to get m (tangent means $\parallel$, Normal means $\perp$)

Implicit: "every time we differentiate a y we write $\frac{\mathrm{d}y}{\mathrm{d}x}$"

Area Between

- curve & x-axis: $\int_{x=a}^{x=b} y\mathrm{d}x$
- curve & y-axis: $\int_{y=a}^{y=b} x\mathrm{d}y$
- Between 2 curves: $\int_{x=a}^{x=b}$(top curve - bottom curve)$\mathrm{d}x$

Remember to split up if separate areas.

Kinematics:

- Distance $= \int_{t_1}^{t_2} |v(t)|\mathrm{d}t$
- Displacement $= \int_{t_1}^{t_2} v(t)\mathrm{d}t$
- Velocity: $\int_{t_1}^{t_2} a(t)\mathrm{d}t$ or $\frac{\mathrm{d}s}{\mathrm{d}t}$
- Acceleration $= \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2s}{\mathrm{d}t^2}$

Differentiation 1st Principles: $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Chain Rule: $y = g(u), u = f(x) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$

Product Rule: $y = uv \implies \frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

Quotient Rule: $y = \frac{u}{v} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Derivatives:

- $x^n \implies nx^{n-1}$
- $(f(x))^n \implies n(f(x))^{n-1}f'(x)$
- $\ln(f(x)) \implies \frac{f'(x)}{f(x)}$
- $\sin f(x) \implies f'(x)\cos f(x)$

- $\cos f(x) \implies -f'(x)\sin f(x)$
- $e^{f(x)} \implies f'(x)e^{f(x)}$
- $a^{f(x)} \implies f'(x)a^{f(x)}\ln a$
- $\tan f(x) \implies f'(x)\sec^2 f(x)$
- $\sec f(x) \implies f'(x)\sec f(x)\tan f(x)$
- $\csc f(x) \implies -f'(x)\csc f(x)\cot f(x)$
- $\cot f(x) \implies -f'(x)\csc^2 f(x)$
- $\sin^{-1} f(x) \implies \frac{f'(x)}{\sqrt{1-(f(x))^2}}$
- $\cos^{-1} f(x) \implies -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$
- $\tan^{-1} f(x) \implies \frac{f'(x)}{1+(f(x))^2}$
- $\sec^{-1} f(x) \implies \frac{f'(x)}{f(x)\sqrt{(f(x))^2-1}}$
- $\csc^{-1} f(x) \implies -\frac{f'(x)}{f(x)\sqrt{(f(x))^2-1}}$
- $\cot^{-1} f(x) \implies -\frac{f'(x)}{1+(f(x))^2}$

Integrals:

- $\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + c, n \neq -1$
- $\int \frac{1}{kx}\mathrm{d}x = \frac{1}{k}\ln|X| + c$
- $\int \sin kx\mathrm{d}x = -\frac{1}{k}\cos kx + c$
- $\int \cos kx\mathrm{d}x = \frac{1}{k}\sin kx + c$
- $\int e^{kx}\mathrm{d}x = \frac{1}{k}e^{kx} + c$
- $\int a^{kx}\mathrm{d}x = \frac{1}{k\ln a}a^{kx} + c$
- $\int \sec^2 kx\mathrm{d}x = \frac{1}{k}\tan kx + c$
- $\int \sec kx\tan kx\mathrm{d}x = \frac{1}{k}\sec kx + c$
- $\int \csc kx\cot kx\mathrm{d}x = -\frac{1}{k}\csc kx + c$
- $\int \csc^2 kx\mathrm{d}x = -\frac{1}{k}\cot kx + c$
- $\int \sec kx\mathrm{d}x = \frac{1}{k}\ln|\sec kx + \tan kx| + c$
- $\int \csc kx\mathrm{d}x = -\frac{1}{k}\ln|\csc kx + \cot kx| + c$
- $\int \frac{1}{\sqrt{a^2-(bx)^2}}\mathrm{d}x = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + c$
- $\int -\frac{1}{\sqrt{a^2-(bx)^2}}\mathrm{d}x = \frac{1}{b}\cos^{-1}\left(\frac{bx}{a}\right) + c$
- $\int \frac{1}{a^2+(bx)^2}\mathrm{d}x = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + c$

Integration by parts: $\int u\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$

Trapezium Rule: $h = \frac{b-a}{\text{number of strips}}$: $\frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4 + \cdots) + y_n]$. Simply put, $\frac{1}{2}h$[1st$y$ + 2(middle y's + last y)]

Newton Raphson: For solving $f(x) = 0 : x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Functions

Inverse: Replace $f(x)$ with y, swap x & y, solve for y

Composite: $fg(x)$ means plug $g(x)$ into $f(x)$

Odd and Even Functions

- Even: $f(-x) = f(x)$
- Odd: $f(-x) = -f(x)$

Transformations: $af(bx + c) + d$ "anything in a bracket affects x and does the opposite".

- a is vertical stretch sf a
- b is horizontal stretch sf $\frac{1}{b}$
- c is translation c units x direction
- d is translation d units y direction
- $f(-x)$ reflection in y axis
- $-f(x)$ reflection in x axis

Linear: $y = mx + b$

- Domain: $x \in \mathbb{R}$
- Range: $y \in \mathbb{R}$

Quadratic: $y = \pm a(bx + c)^2 + d$

- Domain: $x \in \mathbb{R}$
- Range: $y \geq d$ if min, $y \leq d$ if max

Exponential: $y = ae^{bx+c} + d$

- Domain: $x \in \mathbb{R}$ (Hint: power of exp can be anything, so no restriction)
- Range: $y > d$ if $a > 0$, $y < d$ if $a < 0$ (Hint: exp can't be zero)
- Asymptote: $y = d$

Logarithm: $y = a\ln(bx + c) + d$

- Domain: $x > -\frac{c}{d}$ (Hint: $\ln$ can't take a neg number so $bx + c > 0$)
- Range: $y \in \mathbb{R}$
- Asymptote: $x = -\frac{c}{b}$

Root: $y = a\sqrt{bx + c} + d$

- Domain: $x \geq -\frac{c}{b}$ (Hint: underneath root must be positive so $bx + c \geq 0$)
- Range: $y \geq d$ if $a > 0$ and $y \leq d$ if $a < 0$

Modulus: $y = a|bx + c| + d$

- Domain: $x \in \mathbb{R}$
- Range: $y \geq d$ if $a > 0$ and $y \leq d$ if $a < 0$

Note: Definition of $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

Rational: $y = \frac{ax+b}{cx+d} + e$

- Domain: $x \in \mathbb{R}, x \neq -\frac{d}{e}$ (hint: denom $\neq$ 0)
- Range: $y \in \mathbb{R}, y \neq \frac{a}{c} + e$

- Asymptotes: $x = -\frac{d}{c}, y = \frac{a}{c} + e$ Note: often a and or e are zero

Trigonometry: $y = a\sin(bx+c)+d,\ y = a\cos(bx+c)+d$

- Domain: $x \in \mathbb{R}$
- Range: $-a+d \leq y \leq a+d$ Note: If asked to find values of a, b, c, d:
- $a =$ amplitude $= \frac{\text{max}y - \text{min}y}{2}$
- $b = \frac{2\pi}{\text{period}}$ or $\frac{360}{\text{period}}$
- $d =$ principal axis $= \frac{\text{max}y + \text{min}y}{2}$
- $c =$ phase shift (plug in point to find)

Inverse trig: $y = \sin^{-1} x$

- Domain: $-1 \leq x \leq 1$
- Range: $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Inverse trig: $y = \cos^{-1} x$

- Domain: $-1 \leq x \leq 1$
- Range: $0 \leq x \leq \pi$

Inverse trig: $y = \tan^{-1} x$

- Domain: $-\infty \leq x \leq \infty$
- Range: $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Summation Results

Properties:

- Can take constants out: $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$
- Can split up: $\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$
- Inclusive-Exclusive principle: $\sum_{k=m}^{n}(a_k) = \sum_{k=1}^{n}(a_k) - \sum_{k=1}^{m-1}(a_k)$

Results:

- $\sum_{i=1}^{n} 1 = n$
- $\sum_{i=1}^{n} c = cn$
- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Matrix Transformations

Reflection in the line $y = (\tan\theta)x$: $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

Horizontal stretch by scale factor k: $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$

Vertical stretch by scale factor k: $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$

Enlargement by scale factor k centre $(0,0)$: $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

Anti-clockwise rotation of angle θ about origin: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \theta > 0$

Clockwise rotation of angle θ bout origin: $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta > 0$

Matrices

Determinant:

- $2 \times 2 : \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = ad - bc$
- $3 \times 3 : \left|\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$

Inverse: $\frac{1}{\text{determinant}} \times$ adjugate

- $2 \times 2 : \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$
- $3 \times 3 : \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1}$

To find adjugate:

1. Find matrix of minors (cross of corresponding row and column for each element and the determinant of each remaining part forms the new element)
2. Find matrix of cofactors (get the correct signs)
3. Transpose all the elements. In other words swap their positions over the diagonal (the diagonal stays the same)

3 types of solutions for systems of linear equations

- Consistent/unique - one solution (a point). To find unknowns - use basic elimination or solve $\det \neq 0$ (if unknowns on LHS)
- Consistent/non-unique - infinite solutions ($\det = 0$). To find unknowns - Use basic elimination with 2 pairs of the same variable eliminated and look to get $0 = 0$
- Inconsistent/non unique - no sol ($\det = 0$). To find unknowns - use basic elimination with 2 pairs of the same variable eliminated and look to get an inconsistency

Roots

Quadratic:

- form: $(x-\alpha)(x-\beta)$, α and β are the roots
- form: $x^2 + bx + c$. Sum roots $= -b = \alpha + \beta$. product roots $= c = \alpha\beta$
- form: $ax^2 + bx + c$. Sum: $-\frac{b}{a} = \alpha$, product: $\frac{c}{a} = c = \alpha\beta$

Cubic:

- form: $(x-\alpha)(x-\beta)(x-\gamma)$, α, β, γ are the roots
- form: $ax^3 + bx^2 + cx + d = 0$, sum: $\alpha + \beta + \gamma = -\frac{b}{a}$, product: $\alpha\beta\gamma = -\frac{d}{a}$, sum of all possible products of pairs of roots: $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$

Quartic:

- form: $(x-\alpha)(x-\beta)(x-\gamma)(x-\sigma)$, α, β, γ, σ are the roots

- form: $ax^4 + bx^3 + cx^2 + dx + e = 0$, sum: $\alpha + \beta + \gamma + \sigma = -\frac{b}{a}$, product: $\alpha\beta\gamma\sigma = \frac{e}{a}$, sum of all possible products of pairs of roots: $\alpha\beta + \alpha\gamma + \alpha\sigma + \beta\gamma + \beta\sigma + \gamma\sigma = \frac{c}{a}$, sum of all possible products of triples of roots: $\alpha\beta\gamma + \beta\gamma\sigma + \gamma\alpha\sigma + \sigma\alpha\beta = -\frac{d}{a}$

Form: $\sum_{i=0}^{n} a_i x^i = 0$

- Sum $= \frac{-a_{n-1}}{a_n}$
- Product $= \frac{(-1)^n a_0}{a_n}$

Polar Coordinates

Area of Sector: $\frac{1}{2}\int r^2 \mathrm{d}\theta$

Complex Numbers

Definition: $\sqrt{-1} = i, i^2 = -1$

Cartesian Form: $z = a + bi$

Modulus Argument Form

- $z = r(\cos\theta + i\sin\theta) = r \operatorname{cis}\theta$
- $r = \sqrt{a^2 + b^2} \arg = \theta = \tan^{-1}\left(\left|\frac{b}{a}\right|\right)$

Then draw the angle θ in the quadrant where the complex number $a + bi$ lies. Read off θ by starting on the positive x axis (like when you solve for trig using the CAST diagram). Remember that $-\pi \leq \theta < \pi$. Or you can use the following to get θ:

- If Quadrant 1 or 4, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$
- If Quadrant 2, $\theta = \tan^{-1}\left(\frac{b}{a}\right) + \pi$
- If Quadrant 3, $\theta = \tan^{-1}\left(\frac{b}{a}\right) - \pi$

Eulers Form: $z = re^{i\theta}$

De Moivre's Theorem: $z^n = r^n(\cos n\theta + i\sin n\theta) = r^n \operatorname{cis} n\theta$

Roots of $z^n = 1$: $z = e^{\frac{2\pi k i}{n}}$, for $k = 0, 1, 2, \ldots, n-1$.

Linear Algebra

Eigenvalues: Set characteristic polynomial which is $\det(A - \lambda I)$ or $\det(\lambda - AI)$ equal to zero, where $A =$ matrix and $I =$ identity matrix. Solve this for eigenvalue.

Eigenvectors: $\mathbf{Av} = \lambda\mathbf{v}$, where $\lambda =$ eigenvector & v = eigenvector. $Av - \lambda\mathbf{v} = 0$ i.e. $A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$. Solve with value of λ above. Quick way: Put λ into $(A - \lambda I)$ and you'll obviously have a matrix. Multiply this matrix by $\begin{pmatrix} x \\ y \end{pmatrix}$ and set equal to 0. Solve for x in terms of y.

Conics

Ellipse:

- Standard form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- Parametric Form: $(a\cos\theta, b\sin\theta)$
- Eccentricity: $e < 1$, $b^2 = a^2(1 - e^2)$
- Foci: $(\pm ae, 0)$

- Directrices: $x = \pm\frac{a}{e}$
- Asymptotes: none

Parabola:

- Standard Form: $y^2 = 4ax$
- Parametric Form: $(at^2, 2at)$
- Eccentricity: $e = 1$
- Foci: $(a, 0)$
- Directrices: $x = -a$
- Asymptotes: none

Hyperbola:

- Standard Form: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- Parametric Form: $(a\sec\theta, b\tan\theta)$, $(\pm a\cosh\theta, b\sinh\theta)$
- Eccentricity: $e > 1$, $b^2 = a^2(e^2 - 1)$
- Foci: $(\pm ae, 0)$
- Directrices: $x = \pm\frac{a}{e}$
- Asymptotes: $\frac{x}{a} = \pm\frac{y}{b}$

Rectangular Hyperbola:

- Standard Form: $xy = c^2$
- Parametric Form: $\left(ct, \frac{c}{t}\right)$
- Eccentricity: $e = \sqrt{2}$
- Foci: $(\pm\sqrt{2}c, \pm\sqrt{2}c)$
- Directrices: $x + y = \pm\sqrt{2}c$
- Asymptotes: $x = 0, y = 0$

Groups

Order of a group: Number of elements in the group

Order of an element: Least positive integer n such that $g^n = e$ (how many times before you get e in modular arithmetic). If $g \in G$ then $g \ldots g = e$ i.e. $g^n = e$. If no n such that $x^n = e$ then we say element has infinite order.

Definition: A set G with a binary operation * on G such that

1. G is associative
2. G is closed under *
3. G has an identity element (usually denoted e)
4. Each element of G has an inverse

Hyperbolics

Definitions:

- $\sinh x = \frac{e^x - e^{-x}}{2}$
- $\cosh x = \frac{e^x + e^{-x}}{2}$

- $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
- $\mathsf{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$
- $\mathsf{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$
- $\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Identities:

- $\cosh^2 - \sinh^2 x = 1$
- $\tanh^2 x + \mathsf{sech}^2 x = 1$
- $\coth^2 x - \mathsf{csch}^2 x = 1$
- $\tanh x = \frac{\sinh x}{\cosh x}$
- $\sinh 2x = 2 \sinh x \cosh x$
- $\cosh 2x = \cosh^2 x + \sinh^2 x$

Inverse:

- $\mathsf{sech} x = \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), x \geq 1$
- $\mathsf{csch} x = \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$
- $\coth x = \tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), |x| < 1$

Number Theory

Fermat's Theorem: $a^p \equiv a \pmod{p}$ is p is prime and a is any integer