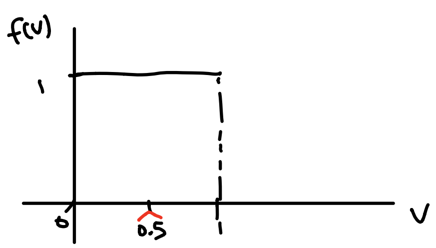
* Announcement
① Quiz 8 : ~ Saturday 9:00 AM
② Exam-prep section: 2~3 PM PT
③ Pre-final grade report (11/24)

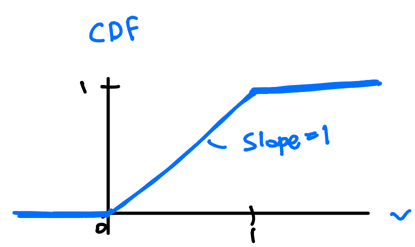
STAT 88: Lecture 31

Contents

Warm up: Let V have density

$$f(v) = \begin{cases} 1 & \text{if } 0 < v < 1 \\ 0 & \text{otherwise} \end{cases}$$





(a) Find the cdf of V.

(b) Find $E(V)$ and $\mathrm{Var}(V)$.

(a) $F(v) = P(V \le v)$

For $0<v<1$, $P(V \le v) = \int_{-\infty}^{v} f(u)\,du$

$= \int_0^v f(u)\,du$

$= \int_0^v 1\,du = v$

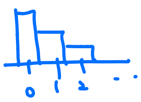
(b) $E(V) = 0.5 = \int v f(v)\,dv$

$E(V^2) = \int_0^1 v^2 f(v)\,dv = \int_0^1 v^2\,dv = \frac{1}{3} \rightsquigarrow \mathrm{Var}(V) = E(V^2) - (EV)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

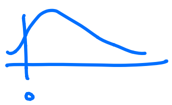
Last time

Density:

Pois
Geom
HG



For discrete random variables, such as $X \sim \text{Binom}(n, p)$, the probability mass function $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$, or the cdf $F(k) = P(X \leq k)$, describes the distribution.

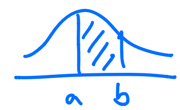


For continuous random variables, such as $Z \sim \mathcal{N}(0, 1)$, the density $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$, or the cdf $P(Z \leq z) = \Phi(z)$, describe the distribution.

A function f is called density if it is always nonnegative and integrates to 1.

If X is a continuous random variable, then f is a density of X if

$$P(a < X \leq b) = \int_a^b f(x)dx.$$



Expectation and Variance:

If a continuous random variable X has density f,

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx.$$

Discrete case

$E(X) = \sum_{x=-\infty}^{\infty} x \, P(X=x)$

For any function g, we have

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) dx.$$

$E g(x) = \sum_{x=-\infty}^{\infty} g(x) P(X=x)$

In particular, this shows

$\downarrow$ $g(t)=t^2$

$$\underbrace{E(X^2)}_{g(X)} = \int_{-\infty}^{\infty} \underbrace{x^2}_{g(x)} f(x) dx.$$

The variance and SD are then given by

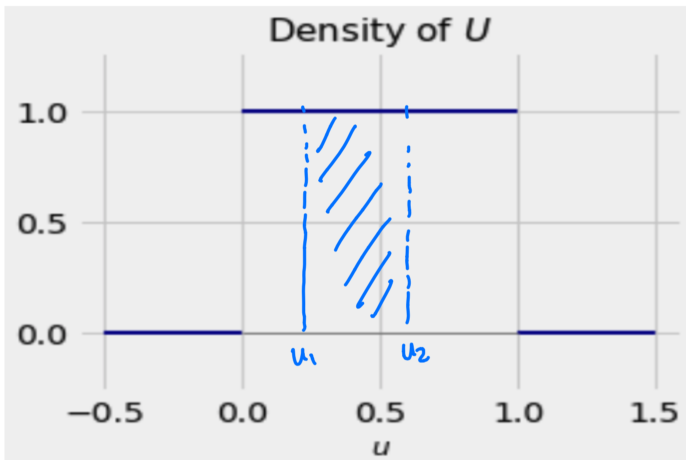
$$\text{Var}(X) = E(X^2) - (EX)^2 \text{ and } \text{SD}(X) = \sqrt{\text{Var}(X)}.$$

10.2. Expectation and Variance

Uniform$(0,1)$ **Distribution**

A random variable U has the uniform distribution on the unit interval $(0,1)$ if

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$



For $0 < u_1 < u_2 < 1$, what is $P(u_1 < U < u_2)$?

$$= \int_{u_1}^{u_2} f(u)\,du$$

$$= u_2 - u_1$$

Find and sketch the cdf of f, $F(x)$:

warm-up

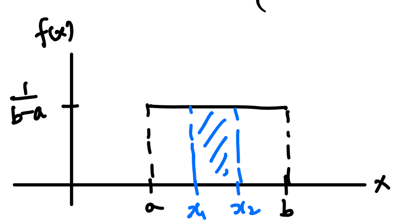
Find $E(U)$ and $\mathrm{Var}(U)$.

warm-up.

Uniform(a, b) **Distribution**

For $a < b$, the random variable X has the uniform distribution on the interval (a, b) if

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$



For $a < x_1 < x_2 < b$, what is $P(x_1 < X < x_2)$?

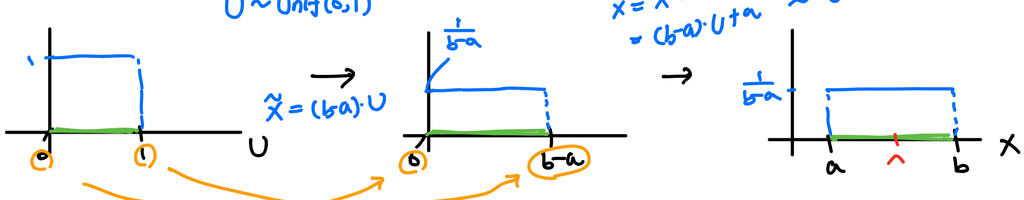
$$= \int_{x_1}^{x_2} f(x)\, dx$$

$$= \frac{x_2 - x_1}{b-a}$$

What function $X = g(U)$ stretches and shifts?

$U \sim \mathrm{Unif}(0,1)$

$X = \tilde{X} + a$
$= (b-a)\cdot U + a \sim \mathrm{Unif}(a,b)$



Find $E(X)$ and $\mathrm{Var}(X)$.

$E(X) = E((b-a)\cdot U + a) = (b-a)\underbrace{E(U)}_{=0.5} + a = 0.5(b-a) + a = 0.5(a+b)$

$\mathrm{Var}(X) = \mathrm{Var}((b-a)\cdot U + a) = (b-a)^2 \cdot \mathrm{Var}(U) = \frac{(b-a)^2}{12}$

Example: (Exercise 10.5.2) A class starts at 3:10 p.m. Seven students in the class arrive at random times $T_1, T_2, \ldots, T_7$ that are i.i.d. with the uniform distribution on the interval 3:07 to 3:12.

$\overset{\text{3:07}}{}= a$, $\overset{\text{3:12}}{}= b$

(a) Find $E(T_1)$. $0.5(a+b) = \frac{1}{2}(3{:}07 + 3{:}12)$

(b) What is the chance that all seven students arrive before 3:10?

(c) Let $X = \max(T_1, T_2, \ldots, T_7)$ be the time when the last of the seven students arrives. Find the cdf of X.

(b) $P(T_1 \le 3{:}10 \cap T_2 \le 3{:}10 \cap \cdots \cap T_7 \le 3{:}10)$

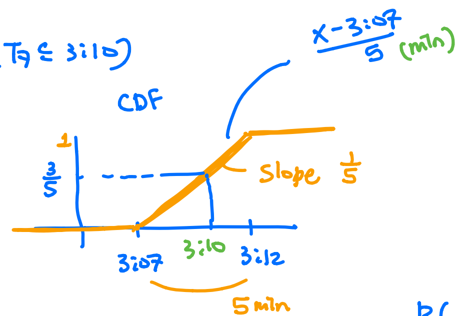
"And" $\leftrightarrow$ $\cap$

indep. $\Rightarrow$

$= P(T_1 \le 3{:}10)\, P(T_2 \le 3{:}10) \cdots P(T_7 \le 3{:}10)$

$= \left(P(T_1 \le 3{:}10)\right)^7$

$= \left(\frac{3}{5}\right)^7$



$P(\min(T_1, \ldots, T_7) \le x)$?

(c) $P(X \le x) = P(\max(T_1, \ldots, T_7) \le x)$

$\left(\max(a,b) \le m \iff a \le m \text{ and } b \le m\right)$

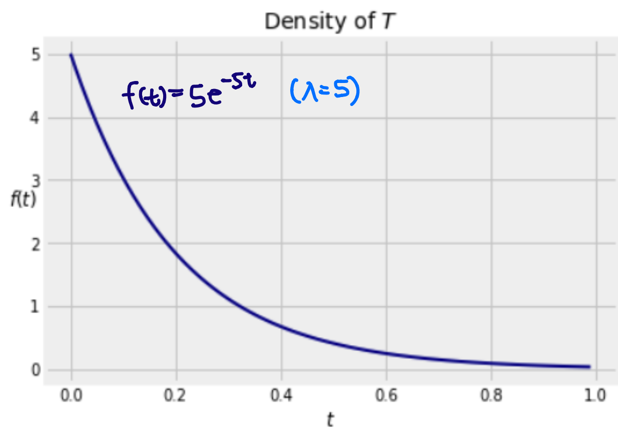
$= P(T_1 \le x, T_2 \le x, \ldots, T_7 \le x)$

$= \left(P(T_1 \le x)\right)^7$

$$= \begin{cases} 1 & \text{if } x > 3{:}12 \\ \left(\dfrac{x - 3{:}07}{5 \text{ (min)}}\right)^7 & \text{if } 3{:}07 \le x \le 3{:}12 \\ 0 & \text{if } x < 3{:}07 \end{cases}$$

10.3. Exponential Distribution

For $\lambda > 0$, a random variable T has the exponential distribution with rate λ (written $T \sim \text{Exp}(\lambda)$), if the density of T is $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$.



The exponential distribution is often used as a model for random lifetimes.

Example: Think of T as the lifetime of an object like a lightbulb and λ as a rate that a lightbulb burns out, e.g. $\lambda = 2$ bulbs/decade. Then the chance a bulb dies in 3 years is

time scale

$$
\begin{aligned}
\underset{\substack{\uparrow \\ \text{3 years}}}{P(T \leq 0.3)} = \int_0^{0.3} 2e^{-2t} dt &= -e^{-2t}|_{t=0}^{3} \\
&= -e^{-0.6} + 1 \\
&\approx 0.45.
\end{aligned}
$$

CDF and Survival Function The cdf $P(T \leq t)$ indicates the chance that the lightbulb dies before time t:

$$
\begin{aligned}
F(t) = P(T \leq t) &= \int_0^t \lambda e^{-\lambda s} ds \\
&= -e^{-\lambda s}|_{s=0}^{t} \\
&= 1 - e^{-\lambda t}.
\end{aligned}
$$

lifetime

$= (T > t)$

The survival function $S(t)$ is the chance that the lightbulb survives past time t:

$$S(t) = P(T > t) \underset{\text{complement rule}}{=} 1 - F(t) = e^{-\lambda t}.$$

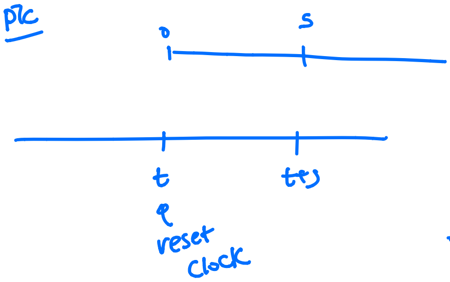
Memoryless Property Lets find the chance that T survives time $t+s$, given that it survives time t:

$$P(T > t+s \mid T > t) \overset{\text{Bayes'}}{=} \frac{P(T>t+s,\ T>t)}{P(T>t)}$$

If we know $T > t+s$ then $T > t$. $\Rightarrow$ $(T>t+s$ and $T>t)$ $= (T>t+s)$

$$= \frac{P(T>t+s)}{P(T>t)}$$

pic



$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}}$$

$$= e^{-\lambda s} = P(T>s)$$

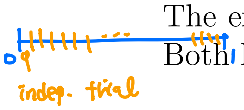
doesn't depend on t

$\Rightarrow$ So if the bulb lasts a decade, the chance it will last another decade is the same as that it lasts 1 decade (it forgets that it lived for 1 decade already)

Geom.

F F F F F S

S S S S S D Exp. dist'n



The exponential distribution is the continuous analog to the geometric distribution. Both have the *memoryless* property.

Mean and SD Let $T \sim \text{Exp}(\lambda)$. Then

$$E(T) = \int_0^\infty t\lambda e^{-\lambda t}dt =$$

The bigger λ is the sooner the bulb dies.

$$E(T^2) = \int_0^\infty t^2\lambda e^{-\lambda t}dt =$$

$$\text{Var}(T) = E(T^2) - (ET)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}, \quad \text{SD}(T) = \sqrt{\text{Var}(T)} = \frac{1}{\lambda}.$$

Example: Let $X \sim \mathrm{Exp}(\lambda)$ with $E(X) = 10$. Find $P(X > 25\,|X > 10)$.