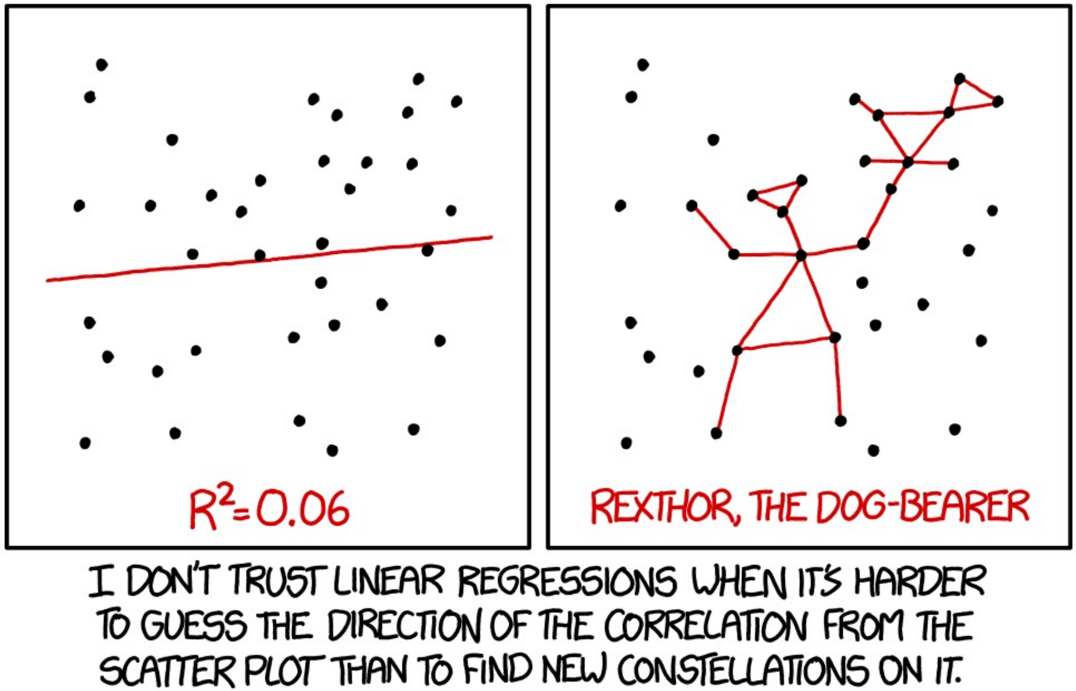
Stat 88: Probability & Mathematical Statistics in Data Science



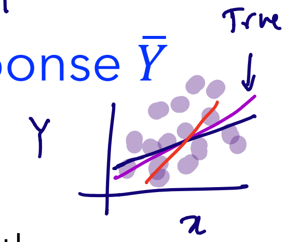
Lecture 40 : 4/30/2021

Chapter 12

Finishing up regression

https://xkcd.com/1725/

$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ (True reg. line) : $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$



The individual response Y_i and the average response $\bar{Y}$

$\varepsilon_i \sim N(0, \sigma^2)$ $Y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$

- Y_i are normal with expectation $\beta_0 + \beta_1 x_i$ and variance σ^2
- Note that the individual responses are independent of each other.

- Let $\bar{Y}$ be the average response. $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$
- $E(\bar{Y}) = \beta_0 + \beta_1 \bar{x}$ (the expected average response is the *signal* at the average value of the predictor variable)

- $Var(\bar{Y}) = \frac{\sigma^2}{n}$ (only involves the error variance since the randomness in the Y_i's comes only from the errors or noise)

- Since $\bar{Y}$ is a linear combination of independent normally distributed random variables, it is also normal.

$$\bar{Y} \sim N\left(\beta_0 + \beta_1 \bar{x}, \frac{\sigma^2}{n}\right)$$

from Ch 11, $\hat{a} = \frac{E(D_X D_Y)}{\sigma_X^2}$

The estimated slope $\hat{\beta}_1$ (estimates the true slope β_1, which is an unobservable parameter)

- The least squares estimate of the true slope β_1 is $\hat{\beta}_1 = \frac{\cancel{\frac{1}{n}}\sum_{i=1}^{n}(x_i - \bar{x})(Y_i - \bar{Y})}{\cancel{\frac{1}{n}}\sum_{i=1}^{n}(x_i - \bar{x})^2}$ } sample cor of x, Y } sample variance

Let's define $a_i = (x_i - \bar{x})$

$\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} a_i^2 \leftrightarrow A$

$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} a_i (Y_i - \bar{Y})}{A} = \sum_{i=1}^{n} \frac{a_i}{A}(Y_i - \bar{Y})$

- Notice that $\hat{\beta}_1$ is random (because of the Y_i).
- Also, since Y_i is normal, and $\bar{Y}$ is normal, so is $Y_i - \bar{Y}$, therefore $\hat{\beta}_1$ ***is also normally distributed***

$$E(\hat{\beta}_1) = \beta_1$$

- $E(Y_i - \bar{Y}) = \beta_1(x_i - \bar{x})$
- $E(\hat{\beta}_1) = \beta_1$, so $\hat{\beta}_1$ is an ***unbiased*** estimator of β_1
- $Var(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$ (to be taken as fact, proof beyond the scope of this class)

SD of the estimated slope $\hat{\beta}_1$

- $SD(\hat{\beta}_1) = \frac{\sigma}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}}$

  constant unknown

  we know these

- Need to estimate σ, which we will do by using the SD of the residuals. The larger the n, the better our estimate of σ

$$\hat{\sigma} = SD(D_1, D_2, \dots, D_n) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(D_i - \bar{D})^2},$$

- $D_i = Y_i - \widehat{Y_i} = Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i$ (The D_i are the residuals and estimate the errors)

  residuals

- Since we are estimating the SD from the data, we will call it the *standard error* of the estimator.

- That is, we will denote this estimated $SD(\hat{\beta}_1)$ by $\boldsymbol{SE(\hat{\beta}_1)}$.

x	Y
1	3
0	2
-1	2

: $\hat{\beta}_1$ & compute $\hat{Y}_i$

: $\hat{\beta}_0$

$Y_i - \hat{Y}_i$

$$SE(\hat{\beta}_1) = \frac{\hat{\sigma}}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}}$$

estimate of σ which estimated by the sample SD of residuals

D_i: $Y_i - \hat{Y}_i$ = residual

Confidence intervals for β_1

- $SE(\hat{\beta}_1) = \frac{\hat{\sigma}}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}}$, for large n, $SE(\hat{\beta}_1) \to SD(\hat{\beta}_1)$

Therefore, for large n, the distribution of $\hat{\beta}_1$, standardized, is approximately standard normal.

$$T = \frac{\hat{\beta}_1 - \beta_1}{SE(\hat{\beta}_1)} \sim N(0,1)$$

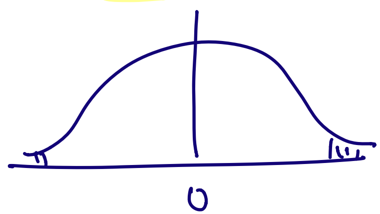
$\leftarrow$ if n not large, then $T \nsim N(0,1)$

- A 95% CI for β_1 is given by $\hat{\beta}_1 \pm 2SE(\hat{\beta}_1)$

(___ , ___)

If 95% C.I. does not contain 0, then can reject $H_0: \beta_1 = 0$ (vs. $\beta_1 \neq 0$) at 5% significance level.

- Note that if the sample size is not large enough, the distribution of T is not necessarily normal, since the assumption that $SE(\hat{\beta}_1) \approx SD(\hat{\beta}_1)$ may not hold.
- In this situation, we model the distribution of T using a family of bell-shaped distributions, called the *t-distributions*.



Hypothesis tests to test $\beta_1 = 0$

- $\beta_1 = 0$ is a very important question: is there any linear relationship at all?

- A 95% CI for β_1 is given by $\hat{\beta}_1 \pm 2SE(\hat{\beta}_1)$: we can use this CI. If 0 is not in this interval, then we reject the null hypothesis of the slope being 0 at the 5% significance level.
- We can set up a test: $\boxed{H_0: \beta_1 = 0 \text{ vs } H_1:\ \beta_1 \neq 0}$ and use the fact that under the null hypothesis,

$$T = \frac{\hat{\beta}_1 - 0}{SE(\hat{\beta}_1)} \sim N(0,1)$$

$H_0: \beta_1 = 0$

$H_1: \beta_1 \neq 0$

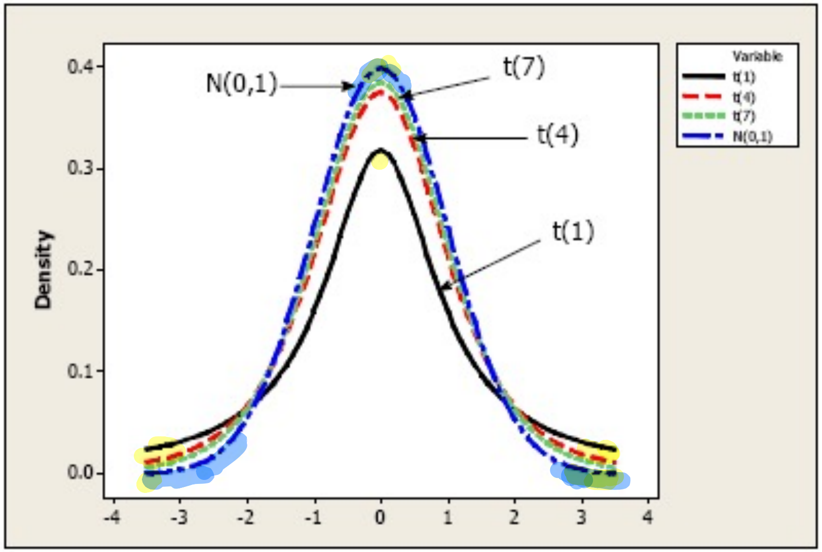
- Let's look at the example from the text on pulse rates after looking at the t-distribution

The t-distribution

t-dsn has parameter "degrees of freedom"

Rather than a normal curve, a t-curve is used. For regression, "degrees of freedom" for T equals $n-2$. For large enough n, use the normal curve.

(When the sample size n is large, so is $n-2$, so we might as well use the normal curve. When the sample is size is small, using the appropriate t curve gives more accurate answers.)



$$T = \frac{\hat{\beta}_1 - \beta_1}{SE(\hat{\beta}_1)}$$

From onlinecourses.science.psu.edu

Example (12.4.3)

slope, intercept, r, p, se_slope=

$\hat{\beta}_1$	(1.142879681904831,
$\hat{\beta}_0$	13.182572776013345,
r	0.6041870881060092,
p-value	1.7861044071652305e-24,
$SE(\hat{\beta}_1)$	0.09938884436389145)

$(H_0 : \beta_1 = 0 \text{ vs } H_1 : \beta_1 \neq 0)$

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mean_active, sd_active
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(91.29741379310344, 18.779629284683832)
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$\bar{Y}$, $SD(Y_1, \ldots Y_n)$

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mean_resting, sd_resting
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(68.34913793103448, 9.927912546587986)
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$\bar{x}$ $SD(x_1, \ldots x_n)$

$SD(x_i)$

c) Find the SD of the residuals.

$\hat{\sigma}$??

$SD(\text{residuals}) \approx \sqrt{1-r^2}\, SD(y)$

≈ 14.96

$$SE(\hat{\beta}_1) = \frac{\hat{\sigma}}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}}$$

$$\hat{\sigma} = SE(\hat{\beta}_1) \cdot \underbrace{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}}_{n \cdot (SD(\text{resting}))^2}$$

$232 \cdot (\hat{\sigma}_x)^2$

Quick look back