* Announcement

① HW2 due today 11:59 pm PT   OH ~ 7pm.

② Quiz 1 tomorrow ~ Chapter 1 & 2.

③ HW3 due next Tuesday (9/15)

STAT 88: Lecture 6

Contents

Last time

Sec 2.5 A and B are independent iff

$$P(B|A) = P(B).$$

Then $P(A \cap B) = P(A)P(B)$.

$= P(B|A)P(A)$

If A and B are independent, non-empty sets, then they must overlap, i.e.

$$P(A \cap B) = P(A)P(B) > 0.$$

$A \cap B \neq \phi$

In other words, A and B are not mutually exclusive.

Indep $\Rightarrow$ not mutually exclusive.

Warm up:

(a) You flip a coin 8 times. What is the chance that you get all heads?

(b) Everyone in a class of 100 people flip a coin 8 times. What is the chance that at least one person gets all heads?

(a) $P(\text{all heads}) = P(\text{1st head} \cap \text{2nd head} \cap \cdots \text{8th head})$

$\overset{\text{indep}}{=} P(\text{1st head}) \cdot P(\text{2nd head}) \cdots P(\text{8th head})$

$= \left(\frac{1}{2}\right)^8$

$= \frac{1}{256}$

8 flips.

O O O O O O O O

"trial" = flip. "independent" trials.

"sucess" = head.

(b) "at least" → Complement rule.

$P(\text{At least one all heads}) = 1 - P(\underbrace{\text{no one gets all heads}}_{= \text{ everyone gets at least one tail.}})$

$P(\text{you get "at least" one tail})$

$= 1 - P(\text{you get all heads})$

$= 1 - \frac{1}{256}$ (from part (a))

100 people.

flip coin 8 times O O O --- O

O O --- O

"trial" = flip 8 coins

"success" = gets all heads.

$= 1 - P(\text{everyone gets at least one tail})$

$= 1 - \left(1 - \frac{1}{256}\right)^{100}$

3.1. Success and Failure

Read Section 3.1 of textbook, Paul the Octopus.

3.2. Random Variables

Random Variables (RV) help reduce the amount of writing involved in phrases like “the chance that there are no more than 1 head in three tosses of a coin”.

You can instead write:

> Let X be the number of heads in three coin tosses. Find $P(X \leq 1)$.

Formally a random variable X is a function from the outcome space to the real numbers, i.e. $X : \Omega \to \mathbb{R}$.

Ω = outcome	X(outcome)	Probability
HHH	3	1/8
HHT	2	1/8
HTH	2	1/8
THH	2	1/8
HTT	1	1/8
THT	1	1/8
TTH	1	1/8
TTT	0	1/8

$$P(X \leq 1) = \underbrace{P(X=0)}_{=\frac{1}{8}} + \underbrace{P(X=1)}_{=\frac{3}{8}} = \boxed{\frac{1}{2}}$$

$$\frac{1}{8} + \frac{3}{8} = \frac{4}{8}.$$

$$P(X=3) = \frac{1}{8}$$

$$P(X=2) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}.$$

$$P(X=1) = \frac{3}{8}$$

$$P(X=0) = \frac{1}{8}.$$

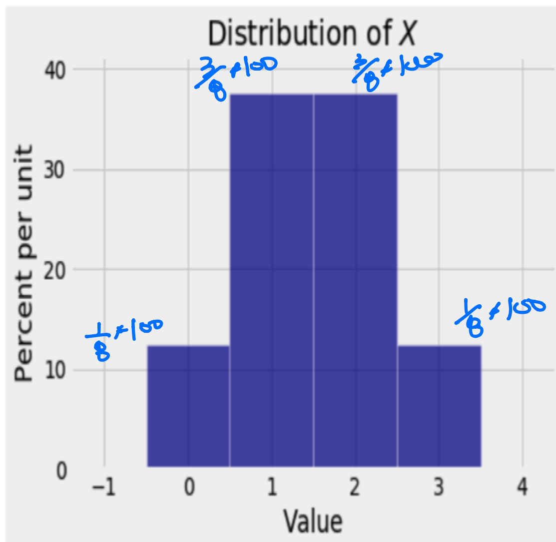
Probability distribution table for X, known for short as a distribution table.

Possible value x	0	1	2	3
$P(X = x)$	1/8	3/8	3/8	1/8

Distribution (pmf) The probability distribution of a random variable, or distribution for short, is the set of all possible values of the random variables along with all of the corresponding probabilities.

The probabilities in a distribution must add up to 1. The distribution of a random variable is sometimes called a probability mass function, abbreviated to pmf.

Probability Histogram The distribution or probability mass function (Pmf) allows us to visualize the probability for each value of X.



Equality Two RVs can have the same distribution but not be equal. Let X_1 be the number of heads and X_2 be the number of tails in three tosses. If the outcome of three tosses is HTH, then $X_1(HTH) = 2$ and $X_2(HTH) = 1$ so as functions on the outcome space, $X_1 \neq X_2$. But both RVs have the same distribution.

X_1	0	1	2	3
	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$.

X_2	0	1	2	3
	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$.

$X_1 \neq X_2$ but dist'n of X_1, X_2 can be same.

$X_1 = X_2 \rightsquigarrow$ dist'n of X_1, X_2 are always same.

3.3. The Binomial Distribution

A binomial distribution Binomial(n, p) has n independent trials, each with probability p for success.

Example: $X = \#$ heads out of 5 coin tosses of a $p = 1/4$ coin (chance of landing head is $1/4$). Let's find $P(X = 2)$.

Here $n = 5$ independent coin tosses, $p = 1/4$ is chance for heads, and $k = 2$.

First what is the chance that you get HHTTT? HTHTT?

$(\frac{1}{4})^2(\frac{3}{4})^3$ $(\frac{1}{4})^2(\frac{3}{4})^3$

How many permutations of 5 letters abcde? In case of HHTTT we must divide by 2!3!, giving

5!

Notation. "5 choose 2"

$$\binom{5}{2} = \frac{5!}{2!3!}.$$

total number of permutation for 2Hs, 3Ts

This shows that

$$P(X = 2) = \binom{5}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3.$$

"Binomial formula"

What values does X take?

k represents possible values for X

$k = 0, 1, 2, \dots, 5$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$(\underbrace{HH \cdots H}_{k}\ \underbrace{TT \cdots T}_{n-k})$

More generally, $X \sim$ Binomial(n,p)

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \dots, n$$

X = # sucess in n independent trials each having probability p for success

Roll a die. 10 times. # 1 out of 10 rolls.

$\underbrace{\text{Roll a die}}_{\text{trial}}$ $\underbrace{\text{10 times}}_{n=10}$ $\underbrace{\text{\# 1 out of 10 rolls}}_{p=\frac{1}{6}}$ $\Rightarrow$ Binomial$(10, \frac{1}{6})$

Example: (Exercise 3.6.3) Yi likes to bet on "red" at roulette. Each time she bets, her chance of winning is 18/38 independently of all other times. Suppose she bets repeatedly on red. Find the chance that:

X = # wins in n trials $\sim$ Binomial$\left(n, \frac{18}{38}\right)$

(a) she wins four of the first 10 bets

(b) she wins at most four of the first 10 bets

$\begin{cases} n\ ? \\ p\ ? \\ k\ ? \end{cases}$

(c) the third time she wins is on the 10th bet

(d) she needs more than 10 bets to win five times = up to 10 bets, she has at most 4 wins

(a) $n = 10$, $p = 18/38$, $k = 4$

$$P(X=4) = \binom{10}{4}\cdot\left(\frac{18}{38}\right)^4\left(\frac{20}{38}\right)^{10-4}$$

(b) At most four.

$n = 10$, $p = 18/38$

$$P(X\le 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$$= \sum_{k=0}^{4}\binom{10}{k}\cdot\left(\frac{18}{38}\right)^k\left(\frac{20}{38}\right)^{10-k}$$

$1 - p = \frac{20}{38}$

(c)

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}_{P(\text{2 wins for the first 9 bets})}\ \underset{\text{10th bet}}{\overset{W - \frac{18}{38}.}{\_}}$$

10th bet ← 3rd time win

P(2 wins for the first 9 bets)

$n = 9$, $k = 2$

$\sim$ Binomial$\left(9, \frac{18}{38}\right)$ $= \binom{9}{2}\left(\frac{18}{38}\right)^2\left(\frac{20}{38}\right)^{9-2}$

→ answer: $\binom{9}{2}\left(\frac{18}{38}\right)^2\left(\frac{20}{38}\right)^7 * \underbrace{\frac{18}{38}}_{\text{win on 10th bet}}$

$$= \binom{9}{2}\left(\frac{18}{38}\right)^3\left(\frac{20}{38}\right)^7$$

(d)

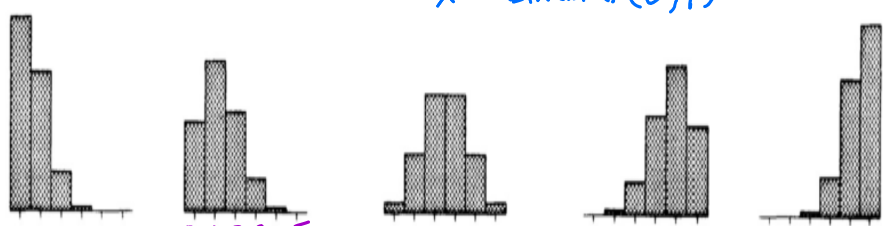
$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}_{\text{wins} \le 4 \text{ in 10 bets.}}$$

$X \sim$ Binomial$\left(10, \frac{18}{38}\right)$

$$P(X\le 4) = \sum_{k=0}^{4}\binom{10}{k}\left(\frac{18}{38}\right)^k\left(\frac{20}{38}\right)^{10-k}$$

Binomial Probabilities in Python SciPy is a compendium of Python software that is enormously useful in data science. In particular, its stats module contains numerous functions and methods used by data scientists.

$X \sim \text{Binomial}(5, p)$



0 1 2 3 4 5

$p = 1/8$ $\quad$ $p = 1/4$ $\quad$ $p = 1/2$ $\quad$ $p = 3/4$ $\quad$ $p = 7/8$

In Python:

Binomial(n,p)
$P(X=k) = \cdots$

```
from scipy import stats
import numpy as np
```

$k \quad n \quad p$

```
stats.binom.pmf(2, 5, 1/4)
```

```
0.26367187499999994
```

$= \binom{5}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3$

```
stats.binom.pmf(np.arange(6), 5, 1/4)
```

"$(0,1,\ldots,5)$"

```
array([0.23730469, 0.39550781, 0.26367187, 0.08789062, 0.01464844,
       0.00097656])
```

$\binom{5}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{5-x}$ for $x = 0, 1, 2, \ldots, 5$

Why does $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$? (so Binomial(n,p) is a distribution)

"Binomial theorem"

$$\underbrace{(p + (1-p))^n}_{=1} = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

Example: Ten cards are dealt off the top of a well shuffled deck. The binominal formula doesn't apply to find the chance of getting exactly three diamonds because:

1. The probability of a trial being successful changes
2. The trials aren't independent
3. There isn't a fixed number of trials
4. More than one of the above

The ith trial is whether the ith card is a diamond. By symmetry, this always has probability $\frac{13}{52} = \frac{1}{4}$. Hence the probability of a trial being successful doesn't change (i.e. choose a diamond)

However, the trials are "dependent" b/c the cards are dealt w/o replacement. For example, the chance the second card is diamond depends on what the first card is.

This is an example where we have 10 dependent trials all prob $1/4$.