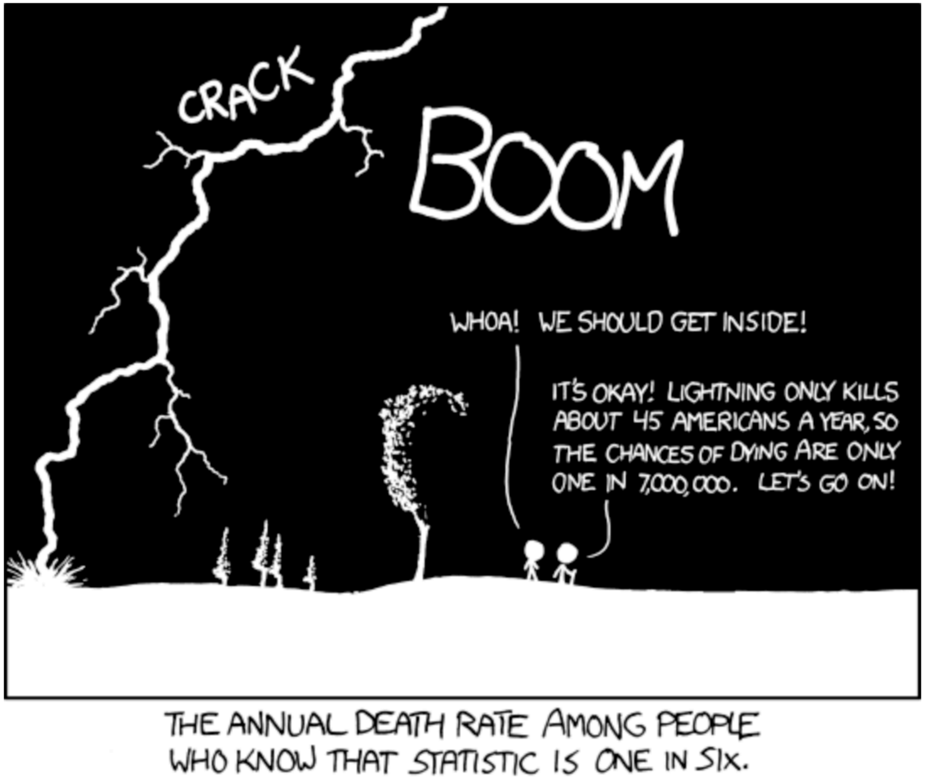
Stat 88: Probability & Math. Statistics in Data Science



https://xkcd.com/795/

Lecture 16: 2/26/2021

Conditional expectation and expectation by conditioning

Sections 5.5, 5.6

Agenda

- Example 5.7.11
- Example from 5.5
- 5.6: Expectation by conditioning

Example: (5.7.11)

A data scientist believes that a randomly picked student at his school is twice as likely not to own a car as to own one car. He knows that no student has three cars, though some students do have two cars. He therefore models the probability distribution for the number of cars owned by a random student as follows. The model involves an unknown positive parameter θ.

Each X_k has the dsn :

# of cars	0	1	2
Probability	2θ	θ	$1-3\theta$

(a) Find $E(X_k)$ = $\mathbb{E}(X_k) = \sum_x x P(X_k = x) = 0 \cdot 2\theta + 1 \cdot \theta + 2(1-3\theta)$
$= 2 - 5\theta$

(b) Let $X_1, X_2, \ldots X_n$ be the numbers of cars owned by n random students picked independently of each other. Assuming that the data scientist's model is good, use the entire sample to construct an unbiased estimator of θ.

$\mathbb{E}(\bar{X}) = 2 - 5\theta$ $\qquad$ $\star = ?$ $\qquad$ $5\theta = 2 - \mathbb{E}(\bar{X})$

$\theta = \dfrac{2 - \mathbb{E}(\bar{X})}{5}$

$\mathbb{E}(\star) = \theta$, $\mathbb{E}\left(\dfrac{2-\bar{X}}{5}\right) = \dfrac{2 - \mathbb{E}(\bar{X})}{5} = \theta$

$\star = \dfrac{2-\bar{X}}{5}$

Example: (5.7.11)

We know that

$$\theta = \frac{2 - E(\bar{X})}{5} = E\left(\frac{2-\bar{X}}{5}\right)$$

$$E\left(\frac{2-\bar{X}}{5}\right) = \theta$$

$\frac{2-\bar{X}}{5}$ is an unbiased estimator of θ.

UNBIASED

"On average, the estimator S hits its target θ"

means $E(S) = \theta$

i.i.d = "independent & identically distributed"

Conditional Expectation: An example

- Let X and Y be iid rvs with the distribution described below, and let $S = X + Y$:

x	1	2	3
P(X=x)	1/4	1/2	1/4

$S = X + Y$

$S = X + Y,\ X = 2$

$2 = 2 + Y$

- Let's write down the joint distribution of X and S:

$S = X + Y$

S \ X	1	2	3	Marg. dsn of S $f_S(z)$
2	$\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}$	0	0	1/16 $= P(S=2)$
3	$\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$	1/8	0	1/4 = 4/16 $= P(S=3)$
4	$\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}$	$\frac{1}{4}\left(=\frac{1}{2}\cdot\frac{1}{2}\right)$	$\frac{1}{16}$ $P(X=3,Y=1)$	6/16 $= P(S=4)$
5	0	$X=2, Y=3$ $\frac{1}{2}\cdot\frac{1}{4}=1/8$	1/8	4/16
6	0	0	1/16 $X=Y=3$	1/16
marginal dsn of X $f_X(x)$	4/16	8/16	4/16	

$P(X=1, S=2)$

Note that S, X

are NOT indep.

Conditional Expectation: An example

Given $S=3$, what is $P(X=1)$

	$P(X=1)$	$P(X=2)$	$P(X=3)$	$f_S(3)$
$S=3$	$1/8$ $P(X=1, S=3)$	$1/8$	0	$4/16$

Need $P(X=1 \mid S=3) = \dfrac{P(X=1, S=3)}{P(S=3)} = \dfrac{1/8}{4/16} = \dfrac{1/8}{2/8} = \dfrac{1}{2}$

Conditional distribution of X given $S=3$

X:	1	2	3
	$1/2$	$1/2$	0

} conditioned on $S=3$

Do this for each value of S. (get the conditional dsn of X)

Given	$P(X=1)$	$P(X=2)$	$P(X=3)$	Conditional Expectation $E(X \mid S=s)$
$S=2$	1	0	0	1
$S=3$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}=1.5$
$S=4$	$= P(X=1,S=4)/P(S=4)$ $1/6$	$4/6$	$1/6$	$1\cdot\frac{1}{6}+2\cdot\frac{4}{6}+3\cdot\frac{1}{6}=$ $12/6=2$
$S=5$	0	$\frac{P(X=2,S=5)}{P(S=5)}$ $\frac{1}{2}$	$\frac{1}{2}$	$5/2=2\cdot\frac{1}{2}+\frac{3}{2}$
$S=6$	0	0	1	$3\cdot 1=3$

function of s $\longrightarrow E(X \mid S=1) = 1 \qquad E(X \mid S=3) = 3 \quad \ldots$

$$P(X=1 \mid S=4) = \frac{P(X=1, S=4)}{P(S=4)} = \frac{1/16}{6/16} = \frac{1}{6}$$

In general, V, W are 2 random variables on the same out come space.

If we fix the value of W to be $W=w$

conditional distribution of $V \mid W=w$

is the probabilities $P(V=v \mid W=w)$ over all the ossible. values of V.

Given V, W. construct a joint distribution

$V \backslash W$	w_1	w_2	$\cdots$	w_m	$f_V(v)$
v_1					$f_V(v_1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
v_n					
$f_W(w)$				$f(w,v)$	

From the joint distribution we can get the marginal dsns and the conditional dsns.

$$f(v,w) = P(V=v, W=w)$$

$$f_V(v) = \sum_w f(v,w)$$

$$f_W(w) = \sum_v f(v,w)$$

A, B

$P(AB) = P(A)P(B)$

$f(x,y) = P(X=x, Y=y)$

$P(X=x, Y=y) = P(X=x)P(Y=y)$

$f(x,y) = f_X(x) \cdot f_Y(y)$

$f(v,w) = f_V(v) \cdot f_W(w)$ for every choice of v, w then V, W are independent.

Further, can compute the conditional distribution

Expectation by Conditioning

- In the example we just worked out, once we fix a value *s* for S, then we have a distribution for X, and can compute its expectation using that distribution that depends on *s*: $E(X \mid S = s) = \sum_x xP(X = x \mid S = s)$, with the sum over all values of X.

- Note that $E(X \mid S = s)$ is a *function of s*. We can think of $E(X \mid S)$ as a rv. changes with diff values of s. $E(X \mid S)$

- This means that if we want to compute $E(X)$, we can just take a weighted average of these conditional expecations $E(X \mid S = s)$:

$$\boldsymbol{E(X) = \sum_s E(X \mid S = s)\, P(S = s)}$$

- This is the *law of iterated expectation*

$$E(E(X \mid S)) = E(X)$$

Law of iterated expectation

- Note that $E(X \mid S = s)$ is a function of s. That is, if we change the value of s we get a different value. (It is not a function of x, though.)

- Therefore we can define the function $g(s) = E(X \mid S = s)$, and the random variable $g(S) = E(X \mid S)$.

- In general, recall that $E\big(g(S)\big) = \sum_s g(s)f(s) = \sum_s g(s)P(S = s)$.
- How can we use this to find the expected value of the rv $g(S)$?