* Announcement

① Quiz 3: Thu (9/24) 9:00 AM PT
~ Fri (9/25) 9:00 AM PT

Ch4
- Waiting time until rth success
  $P(T_r=k)$, $P(T_r \le k)$, $P(T_r > k)$
- Exp. Approximation
  No success in n trials, At least one success

Poisson dist'n.
ss
Binomial (n,p) for n large, p small
Poisson formula, independence

# STAT 88: Lecture 12

After today's lecture
⇝ HW5 Q1(b)(c)(d), Q4, Q3 try

**Contents**



**Last time**

**Sec 5.1** The expectation of a random variable $X$, denoted $E(X)$, is the average of the possible values of $X$ weighted by their probabilities:

$$E(X) = \sum_{\text{all } x} xP(X = x).$$

(Bernoulli RV)

$$X = \begin{cases} 1 & \text{with probability } p \quad \text{– success} \\ 0 & \text{with probability } 1 - p \quad \text{– failure.} \end{cases}$$

Then

$$E(X) = 1 \cdot p + 0 \cdot (1 - p) = p.$$

(Uniform distribution) $X$ has a uniform distribution on $\{1, 2, \ldots, n\}$ if

$$P(X = k) = \frac{1}{n} \text{ for } k = 1, \ldots, n.$$

Then

$$E(X) = \sum_{k=1}^{n} k \cdot \frac{1}{n} = \frac{n+1}{2}.$$

(Poisson distribution) $X$ has a Poisson distribution if

$$P(X = k) = \frac{e^{-\mu}\mu^k}{k!} \text{ for } k = 0, 1, \ldots.$$

Then

$$E(X) = \sum_{k=0}^{\infty} k \cdot \frac{e^{-\mu}\mu^k}{k!} = \mu.$$

average of counts

**Sec 5.2** Let $Y = g(X)$ be a function of the random variable $X$. Then

$$E(Y) = E(g(X)) = \sum_{\text{all } x} g(x)P(X = x).$$

Warm up: (Exercise 5.7.1) Let $X$ have the distribution displayed in the table below.

| $x$ | $-2$ | $-1$ | $0$ | $1$ |
|---|---|---|---|---|
| $P(X = x)$ | 0.2 | 0.25 | 0.35 | 0.2 |
| $g(x) = \lvert x-1 \rvert$ | 3 | 2 | 1 | 0 |
| $g(x) = (x-1)^2$ | 9 | 4 | 1 | 0 |

(c) Find $E\underbrace{|X-1|}_{Y}$.

(d) Find $E((X-1)^2)$.

(c) $E|X-1| = 3*(0.2) + 2*(0.25) + 1*(0.35) + 0*(0.2)$

$= 1.45$

(d) $E((X-1)^2) = 9*(0.2) + 4*(0.25) + 1*(0.35) + 0*(0.2)$

$= 3.15$

## 5.2. Functions of Random Variables (Continued)

Recall that a random variable is a function on the outcome space $\Omega$, $X : \Omega \to \mathbb{R}$.

Example: Flip a fair coin twice. $\Omega = \{HH, HT, TH, TT\}$. Let $X = \#$ heads in 2 coin tosses. Then $X(HH) = 2$.

$X(HT) = 1 = X(TH)$
$Y(HT) = 1^2 = Y(TH)$

Now if $Y = g(X) = X^2$, this is also a function of the outcome space

$$Y(HH) = (\underset{\overset{\shortparallel}{2}}{X(HH)})^2 = 4.$$

So a function of a random variable is itself a random variable.

$$\begin{tikzcd} \Omega \arrow[r, "X"] \arrow[rr, bend right, "Y = g \circ X \ : \ RV"'] & \mathbb{R} \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

**Joint Distribution** Suppose two draws are made at random without replacement from a population that has five elements labeled 1,2,**2**,3,3. Define the following random variables:

- $X_1$ is the number on the first draw   1, 2, 3
- $X_2$ is the number on the second draw   1, 2, 3

The pair $(X_1, X_2)$ is a random variable (vector) and we describe its distribution in a table

$$P(X_1 = 1 \underset{\text{"and"}}{,} X_2 = 1) \underset{\text{Multiplication rule}}{=} P(X_1 = 1)P(X_2 = 1 | X_1 = 1) = \frac{1}{5} \cdot 0 = 0.$$

$P(X_1 = 1, X_2 = 2)$?

$P(X_1=1)\,P(X_2=2 | X_1=1) = \frac{1}{5} * \frac{2}{4} = \frac{2}{20} = \frac{1}{10}$

This fills out the table:

| | $X_2 = 1$ | $X_2 = 2$ | $X_2 = 3$ |
|---|---|---|---|
| $X_1 = 1$ | 0 | $\frac{2}{20}$ | $\frac{2}{20}$ |
| $X_1 = 2$ | $\frac{2}{20}$ | $\frac{2}{20}$ | $\frac{4}{20}$ |
| $X_1 = 3$ | $\frac{2}{20}$ | $\frac{4}{20}$ | $\frac{2}{20}$ |

Sum to 1

Find $P(X_1 + X_2 = 4)$?   $P(X_1=1, X_2=3) + P(X_1=2, X_2=2) + P(X_1=3, X_2=1)$

$= \frac{2}{20} + \frac{2}{20} + \frac{2}{20} = \frac{6}{20}$

$P(X_1 = X_2) = P(X_1=1, X_2=1) + P(X_1=2, X_2=2) + P(X_1=3, X_2=3)$

$= \frac{4}{20}$

0

$+ P(X_1=1, X_2=4)$

**Marginal Distribution** Note that

$$P(X_1 = 1) = P(X_1 = 1, X_2 = 1) + P(X_1 = 1, X_2 = 2) + P(X_1 = 1, X_2 = 3).$$

↑ Addition rule

So we can recover the distribution of $X_1$ from the joint distribution.

| | $X_2 = 1$ | $X_2 = 2$ | $X_2 = 3$ | **Dist of** $X_1$ |
|---|---|---|---|---|
| $X_1 = 1$ | 0 | $\frac{2}{20}$ | $\frac{2}{20}$ | $\frac{4}{20} = \frac{1}{5}$ |
| $X_1 = 2$ | $\frac{2}{20}$ | $\frac{2}{20}$ | $\frac{4}{20}$ | $\frac{8}{20} = \frac{2}{5}$ |
| $X_1 = 3$ | $\frac{2}{20}$ | $\frac{4}{20}$ | $\frac{2}{20}$ | $\frac{8}{20} = \frac{2}{5}$ |
| Dist of $X_2$ | $\frac{4}{20}$ | $\frac{8}{20}$ | $\frac{8}{20}$ | |

You can get the distribution of $X_2$ by summing along columns.

Are $X_1$ and $X_2$ independent? $P(X_1=2, X_2=2) \neq P(X_1=2)P(X_2=2) \to$ No

$P(X_1=2, X_2=2) = \frac{2}{20}$; $P(X_1=2)P(X_2=2) = \frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$

$X_1$ and $X_2$ are independent if

$$P(X_1 = a, X_2 = b) = P(X_1 = a)P(X_2 = b)$$

for all cells in the joint table.

**Expectations of Functions** We can find the expectation of any function $g$ of two random variables $X$ and $Y$ by extending our method for finding the expectation of a function of $X$.

- Take a cell of the joint distribution table of $X$ and $Y$. This corresponds to one possible value $(x, y)$ of the pair $(X, Y)$.
- Apply the function $g$ to get $g(x, y)$.
- Weight this by the probability in the cell, to get the product $g(x, y)P(X = x, Y = y)$.
- Add these products over all the cells of the table.

Example: Find $E|X_1 - X_2|$. $g(X_1, X_2) = |X_1 - X_2|$

| | $X_2 = 1$ | $X_2 = 2$ | $X_2 = 3$ |
|---|---|---|---|
| $X_1 = 1$ | $0$ $g(1,1)=0$ | $\frac{2}{20}$ $g(1,2)=1$ | $\frac{2}{20}$ $g(1,3)=2$ |
| $X_1 = 2$ | $\frac{2}{20}$ $g(2,1)=1$ | $\frac{2}{20}$ $g(2,2)=0$ | $\frac{4}{20}$ $g(2,3)=1$ |
| $X_1 = 3$ | $\frac{2}{20}$ $g(3,1)=2$ | $\frac{4}{20}$ $g(3,2)=1$ | $\frac{2}{20}$ $g(3,3)=0$ |

$$\sum_{\text{all cells}} g(x_1, x_2)\, P(X_1 = x_1, X_2 = x_2) = 0 * 0 + 1 * \frac{2}{20} + 2 * \frac{2}{20} + 1 * \frac{2}{20} + \cdots$$

$$= 1$$

Example: A joint distribution for two random variables $M$ and $S$ is given below.

| | $M = 2$ | $M = 3$ |
|---|---|---|
| $S = 2$ | $0$ | $\frac{1}{3}$ |
| $S = 1$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| Dist of M | $\frac{1}{3}$ | $\frac{2}{3}$ |

Find $E(M)$.

Exercise

$$E(M) = 2 * \left(\frac{1}{3}\right) + 3 * \left(\frac{2}{3}\right)$$

$$= \frac{8}{3}$$

## 5.3. Method of Indicators

**Preliminary: Additivity of Expectation** No matter what the joint distribution of $X$ and $Y$ is, we have

$$E(X+Y) = E(X) + E(Y).$$

This additivity is one of the most important properties of expectation, because it is true whether the random variables are dependent or independent.

Furthermore, for any constants $a$ and $b$, the linearity also holds:

$$E(aX + bY) = aE(X) + bE(Y). \qquad a=1, b=1$$

**Method of Indicators to find $E(X)$:**

Key idea Counting the number of successful trials is the same as adding zeros and ones.

Example: A success is blue, and failure non blue

B R R G B R B B

1 0 0 0 1 0 1 1

$$\#\text{blue} = 1+0+0+0+1+0+1+1 = 4.$$

Recall (Bernoulli (indicator) random variable)

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

**Binomial** $X$ is the number of successes in $n$ independent trials all with the same probability $p$ of success. To find $E(X)$, write $X$ as a sum of $n$ indicators.

$$X = I_1 + I_2 + \cdots + I_n,$$

where

prob p

$$I_j = \begin{cases} 1 & \text{if } j\text{th trial is success} \\ 0 & \text{otherwise} \end{cases}$$

Then

$E(I_j) = p$

$$E(X) = E(I_1 + I_2 + \cdots + I_n) = E(I_1) + E(I_2) + \cdots + E(I_n) = np.$$

Additivity  p  p  p

pop size $N$

$G$ $\quad N-G$

$n$

**Hypergeometric** $X$ is the number of good elements in a simple random sample of size $n$ drawn from a population $N$ elements of which $G$ are good. To find $E(X)$, write $X$ as a sum of $n$ indicators.

$$X = I_1 + I_2 + \cdots I_n,$$

where

$$I_j = \begin{cases} 1 & \text{if } j\text{th draw yields a good element} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$E(X) = E(I_1 + I_2 + \cdots + I_n) = E(I_1) + E(I_2) + \cdots + E(I_n) = n\frac{G}{N}.$$

Additivity

$$\underbrace{E(I_j) = \frac{G}{N}}_{\text{by Symmetry}}$$