STAT 88: Lecture 3

Contents

Last time

<u>**Sec 1.3**</u> Fundamental Rules

Three axioms of probability:

1. $P(A) \geq 0$ for all $A \subseteq \Omega$.
2. $P(\Omega) = 1$.
3. **Addition Rule.** If A and B are mutually exclusive, i.e. $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.

Based on the axiom, we have upper bound and lower bound as (why?)

$$\max\{P(A), P(B)\} \leq P(A \cup B) \leq P(A) + P(B). \tag{1}$$

Textbook notation : AB means $A \cap B$. In the lecture note, we will use $A \cap B$ to denote the intersection.

De Morgan's laws: $(A^c \cup B^c)^c = A \cap B$. Then

$$P(A \cap B) = P((A^c \cup B^c)^c) = 1 - P(A^c \cup B^c).$$

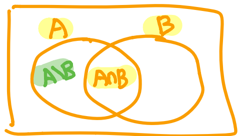
We know from (1) that $\max\{P(A^c), P(B^c)\} \leq P(A^c \cup B^c) \leq P(A^c) + P(B^c)$, therefore

$$\underbrace{1 - P(A^c) - P(B^c)}_{\substack{= \\ P(A)+P(B)-1 \\ \text{(using Complement rule)}}} \leq P(A \cap B) \leq \underbrace{1 - \max\{P(A^c), P(B^c)\}}_{\substack{= \\ \min\{1-P(A^c),\, 1-P(B^c)\} \\ = \min\{P(A),\, P(B)\}.}}.$$

Upper bound: $P(A\cap B) \le \min\{P(A), P(B)\}$. (followed by De Morgan's laws)

viz

Ω



$\underbrace{A = (A\cap B) \cup (A\setminus B)}_{\text{Mutually exclusive.}} \rightsquigarrow P(A) = P(A\cap B) + \underbrace{P(A\setminus B)}_{\geq 0} \geq P(A\cap B)$ (by addition rule)

Warm up: (Excercise 1.4.5) If a student applies to ten colleges with a 20% chance of being accepted to each, what are the chances that he will be accepted by at least one college? Be clear about any assumptions you are making.

Exercise

① Making no assumptions:

C_i = event get into ith college.

$$\begin{cases} P(C_1 \cup C_2 \cup \cdots \cup C_{10}) \geq P(C_1) = 20\% \\ P(C_1 \cup C_2 \cup \cdots \cup C_{10}) \leq \sum_{j=1}^{10} P(C_j) = 200\% \quad \text{(Bonferroni)} \end{cases}$$

$$\Rightarrow 20\% \leq P(C_1 \cup C_2 \cup \cdots \cup C_{10}) \leq 100\%$$

② Assume $C_1, C_2, \cdots, C_{10}$ independent.

$$\begin{aligned} P(C_1 \cup C_2 \cup \cdots \cup C_{10}) &= 1 - P((C_1 \cup \cdots \cup C_{10})^c) && \text{(Complement rule)} \\ &= 1 - P(C_1^c \cap C_2^c \cap \cdots \cap C_{10}^c) && \text{(De Morgan's laws)} \\ &= 1 - P(C_1^c)P(C_2^c) \cdots P(C_{10}^c) && \text{(Assuming Independence)} \\ &= 1 - (0.8)^{10} && (\because P(C_i^c) = 1 - P(C_i) = 0.8) \\ &\approx 0.89 \end{aligned}$$

2.1. The Chance of an Intersection

Pick two cards at random without replacement from a deck that contains one red, one blue, and one green card (R, B, G). Find

$$P(\text{1st Card B and 2nd Card R}).$$

The outcome space $\Omega = \{RB, RG, BG, BR, GR, GB\}$. The draws are at random, so all six pairs are equally likely. Then

$$P(\text{1st Card B and 2nd Card R}) = P(\{BR\}) = \frac{1}{6}. = \frac{\#\{BR\}}{\#\Omega}$$

Alternatively, we can use the multiplication rule: conditional prob.

$$P(\{BR\}) = \underbrace{P(\text{2nd Card R}|\text{1st Card B})}_{=1/2} \cdot \underbrace{P(\text{1st Card B})}_{=1/3}. \quad = 1/6$$

Multiplication rule For $A, B \subseteq \Omega$,

$$P(A \cap B) = P(A|B)P(B).$$

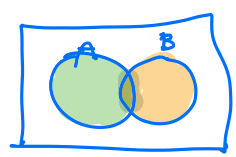
Since $P(A \cap B) = P(B \cap A)$, we also have $P(A \cap B) = P(B \cap A) = P(B|A)P(A)$.

Example: What is chance 1st card in a 52 card deck is queen "and" the last is queen?

$$P(\text{1st } Q \cap \text{last } Q) = \underbrace{P(\text{1st } Q)}_{=4/52} \underbrace{P(\text{last } Q \mid \text{1st } Q)}_{=3/51} = \frac{4}{52} * \frac{3}{51}$$

Inclusion-Exclusion For $A, B \subseteq \Omega$,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



Example: What is chance 1st card in a 52 card deck is queen "or" the last is queen?

$$P(\text{1st } Q \cup \text{last } Q) = P(\text{1st } Q) + P(\text{last } Q) - P(\text{1st } Q \cap \text{last } Q)$$

$$= \frac{4}{52} + \frac{4}{52} - \frac{4}{52} * \frac{3}{51}$$

Example: A deck of cards is shuffled. What is the chance that the top card is the king of spades or the bottom card is the king of spades?

Exercise

KS_i = event ith card is the king of spades. (Inclusion Exclusion)

$$P(KS_1 \cup KS_{52}) = P(KS_1) + P(KS_{52}) - P(KS_1 \cap KS_{52})$$

$$= \frac{1}{52} + \frac{1}{52} - 0$$ (KS_1 and KS_{52} are mutually exclusive)

$$= \frac{1}{26}$$

Intersection of Several Events Consider a poker hand (5 cards randomly drawn without replacement).

H_i: event that Card i is a heart.

What is the chance all 5 cards are hearts?

$P(H_1 \cap H_2 \cap \cdots \cap H_5) = P(H_1)P(H_2 | H_1)P(H_3 | H_1 \cap H_2) \cdots = \frac{13}{52} * \frac{12}{51} * \frac{11}{50} * \cdots$

What is the chance that all five cards are of $\boxed{\text{the same suit?}}$

P(all hearts or all diamonds or all spades or all clubs)

= 4 * answer from above.

2.2. Symmetry in Simple Random Sampling

Sampling individuals at random without replacement is one of the most natural ways to collect a random sample from a finite population. It is called simple random sampling and will be studied extensively in this course. We will examine simple random sampling in the context of dealing hands of cards from a deck (or population) of size 52.

Recall a deck of 52 cards has 4 suits.

How many possible pairs of cards are there? $52 * 51$

If you deal 2 cards, what is the chance the 2nd card is red?

B_i = black at card i

R_i = red at card i

Find $P(R_2)$. 2nd card is Red = (1st card is black "and" 2nd card Red) or (1st card Red "and" 2nd card Red)

Method 1

$$P(R_2) = P(B_1 \cap R_2) + P(R_1 \cap R_2)$$
$$= P(B_1)P(R_2 | B_1) + P(R_1)P(R_2 | R_1)$$
$$= \frac{26}{52} * \frac{26}{51} + \frac{26}{52} * \frac{25}{51}$$
$$= \frac{26}{52} * \underbrace{\left(\frac{26}{51} + \frac{25}{51}\right)}_{= 1} = \frac{26}{52}$$

(Note $= P(R_1)$)

Method2

Ω = pairs of cards. $\#\Omega = 52 * 51$

1st Card Black

2nd Card Red

2nd Card Red.

$R_2 = \{ (\text{Any card}, \text{Red card}) \}$ $\#R_2 = 26 * 26 + 26 * 25$

1st Card Red

$= 26 * (26 + 25)$

$= 26 * 51.$

$$p(R_2) = \frac{\#R_2}{\#\Omega} = \frac{26 * 51}{52 * 51} = \frac{26}{52}$$

Method3 Imagine someone deals 2 cards. Show you 2 cards backward.

You would not be able to tell that it was backwards.

i.e. nothing special about 1st card.

$P(R_2) = P(R_1) = \frac{26}{52}$. (Symmetry)