STAT 88: Lecture 40

Contents

Warm up: You get the following readout for the simple linear regression model:

	coef	std err	t	P>|t|	[0.025	0.975]
const	13.1826	6.864	1.920	0.056		
Rest	1.1429	0.099	11.499	0.000		

What can you conclude from this table about β_1?

If I don't give you t in this table, can you figure it out from the rest of the table?

If I don't give you $P > |t|$ in this table, can you figure it out from the rest of the table?

Can you find the 95% CI for β_1 from the table above?

Last time

We have n samples $(x_1, Y_1), (x_2, Y_2), \ldots, (x_n, Y_n)$ generated from the following simple linear regression model:

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \text{ where } \epsilon_i \sim \mathcal{N}(0, \sigma^2).$$

The least-squares estimates of β_0 and β_1 are given by

$$\widehat{\beta}_0 = \bar{Y} - \widehat{\beta}_1 \bar{x} \text{ and } \widehat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(Y_i - \bar{Y})}{\sum_{i=1}^n (x_i - \bar{x})^2}.$$

We can show

$$\widehat{\beta}_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{x})^2}\right),$$

and hence

standardize

$$\frac{\widehat{\beta}_1 - \beta_1}{\mathrm{SD}(\widehat{\beta}_1)} \sim \mathcal{N}(0, 1).$$

Since σ is an unknown parameter, we approximate it with the SD of the residuals, denoted as $\widehat{\sigma}$. A resulting statistic is $T = \frac{\widehat{\beta}_1 - \beta_1}{\mathrm{SE}(\widehat{\beta}_1)}$ where

$$\mathrm{SE}(\widehat{\beta}_1) = \frac{\widehat{\sigma}}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}},$$

and a known fact is that

$$T = \frac{\widehat{\beta}_1 - \beta_1}{\mathrm{SE}(\widehat{\beta}_1)} \sim t(n-2).$$

When n is large, the t-distribution with degree of freedom $n-2$ is close to the standard normal distribution, so

$$T = \frac{\widehat{\beta}_1 - \beta_1}{\mathrm{SE}(\widehat{\beta}_1)} \sim \mathcal{N}(0, 1).$$

We can use the distribution of T to construct 95% CI for β_1 or conduct hypothesis testing $H_0 : \beta_1 = 0$ vs $H_A : \beta_1 \neq 0$.

12.3. Towards Multiple Regression

Below is data on a random sample of Hodgkin cancer patients.

Simple Regression

We predict difference from base:

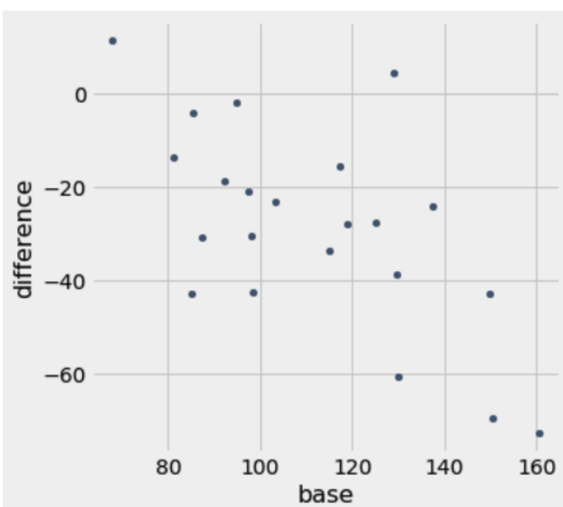
```
hodgkins
```

Health before chemo (bigger means more healthy)

height	rad	chemo	base	month15	difference
164	679	180	160.57	87.77	-72.8
168	311	180	98.24	67.62	-30.62
173	388	239	129.04	133.33	4.29
157	370	168	85.41	81.28	-4.13
160	468	151	67.94	79.26	11.32
170	341	96	150.51	80.97	-69.54
163	453	134	129.88	69.24	-60.64
175	529	264	87.45	56.48	-30.97
185	392	240	149.84	106.99	-42.85
178	479	216	92.24	73.43	-18.81

... (12 rows omitted)

```
hodgkins.scatter('base', 'difference')
```



OLS Regression Results

Dep. Variable:	difference	R-squared:	0.397

	coef	std err	t	P>|t|	[0.025	0.975]
const	32.1721	17.151	1.876	0.075	-3.604	67.949
base	-0.5447	0.150	-3.630	0.002	-0.858	-0.232

What difference do you predict if you have base health 100?

Multiple Regression

What if we want to regress on both base and chemo? Here chemo is very uncorrelated with base.

```
h_data.corr()
```

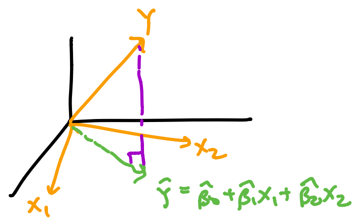
	height	rad	chemo	base	month1
height	1.000000	-0.305206	0.576825	0.354229	0.39052
rad	-0.305206	1.000000	-0.003739	0.096432	0.04061
chemo	0.576825	-0.003739	1.000000	0.062187	0.44578
base	0.354229	0.096432	0.062187	1.000000	0.56137
month15	0.390527	0.040616	0.445788	0.561371	1.00000
difference	-0.043394	-0.073453	0.346310	-0.630183	0.28879

Conceptual picture:

Model: $Y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i$, $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$

Estimate $\beta_0, \beta_1, \beta_2$ by minimizing $\frac{1}{n}\sum_{i=1}^{n} (Y_i - a - b x_{1i} - c x_{2i})^2$ w.r.t. a, b, c

write $Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$, $X_1 = \begin{pmatrix} x_{11} \\ \vdots \\ x_{1n} \end{pmatrix}$, $X_2 = \begin{pmatrix} x_{21} \\ \vdots \\ x_{2n} \end{pmatrix}$:



OLS Regression Results

Dep. Variable:	difference	R-squared:	0.546

	coef	std err	t	P>|t|	[0.025	0.975]
const	-0.9992	20.227	-0.049	0.961	-43.335	41.336
base	-0.5655	0.134	-4.226	0.000	-0.846	-0.285
chemo	0.1898	0.076	2.500	0.022	0.031	0.349

What can you conclude here about the fit and $\beta_0, \beta_1, \beta_2$?

What if we include all features?

```
h_data.corr()
```

	height	rad	chemo	base	month1
height	1.000000	-0.305206	0.576825	0.354229	0.39052
rad	-0.305206	1.000000	-0.003739	0.096432	0.04061
chemo	0.576825	-0.003739	1.000000	0.062187	0.44578
base	0.354229	0.096432	0.062187	1.000000	0.56137
month15	0.390527	0.040616	0.445788	0.561371	1.00000
difference	-0.043394	-0.073453	0.346310	-0.630183	0.28879

Note that we have multi-collinearity (i.e. some features are highly correlated with each other).

a very minor improvement

OLS Regression Results

Dep. Variable:	difference	**R-squared:**	0.550

	coef	std err	t	P>|t|	[0.025	0.975]
const	33.5226	101.061	0.332	0.744	-179.698	246.743
base	-0.5393	0.160	-3.378	0.004	-0.876	-0.202
chemo	0.2124	0.103	2.053	0.056	-0.006	0.431
rad	-0.0062	0.031	-0.203	0.841	-0.071	0.059
height	-0.2274	0.658	-0.346	0.734	-1.615	1.160

Overview of Class

Ch 6: Measuring Variability

- You learned how the variance and SD is the average spread of your data from the mean.
- You should be able to compute $\mathrm{Var}(X)$ and $\mathrm{SD}(X)$ given a distribution table / a density function.
- If there are two random variables X and Y, and Y is a linear function of X, i.e. $Y = aX + b$, how to compute $\mathrm{Var}(Y)$ and $\mathrm{SD}(Y)$ from $\mathrm{Var}(X)$ and $\mathrm{SD}(X)$?
- If we don't assume anything about the population distribution except the mean and SD, you can use Chebyshev's inequality to get an upper bound on the tail probability.

Ch 7: The Variance of a Sum

- If two random variables X and Y are independent, $\mathrm{Var}(X + Y)$ is given by the sum of $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.
- If $X_1, X_2, \ldots, X_n$ are i.i.d. samples from a population distribution and $S = X_1 + \cdots + X_n$ is the sample sum, $\mathrm{Var}(S) = n\sigma^2$ and $\mathrm{SD}(S) = \sqrt{n}\sigma$, where $\sigma = \mathrm{SD}(X)$.
- The law of large number says the sample mean $\bar{X}$ converges to $\mu = E(X)$ as the sample size n grows. In particular, we proved the weak law of large numbers using Chebyshev's inequality.

Ch 8: The Central Limit Theorem

- The central limit theorem (CLT) says the distribution of sample mean $\bar{X}$ is always approximately normal, i.e. $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$ when n is large enough.
- For any random variable $X \sim \mathcal{N}(\mu, \sigma^2)$, we can define a new random variable $X^\star$, called X in standard units, as $X^\star = \frac{X-\mu}{\sigma}$. $X^\star$ follows a standard normal distribution $\mathcal{N}(0, 1)$. The CDF of the standard normal distribution is written as $\Phi(x) = P(X^\star \leq x)$.

Ch 9: Inference

- Given n i.i.d. samples from a population distribution (say Bernoulli, normal, etc), you learned how to estimate the population parameter such as the population mean or population proportion.
- From our samples, we can make hypotheses about the value of the parameter of the population distribution. Assuming the null hypothesis is true, we compute a test statistic and compute the p-value. If the p-value is less that 0.05 at level 5%, we reject the null.
- A 95% CI for an unknown parameter tells you the rough uncertainty of the parameter.
- You should be able to conduct hypothesis testing for the population mean (both for one-sided and two-sided alternative hypotheses) and also construct 95% confidence interval.
- Interpretation of CI is important and you should be able to tell what is the right/wrong interpretation.

Ch 10: Probability Density

- For a continuous random variable, we compute the probability, expectation, and variance using the probability density function.
- The exponential distribution is used to model the random life time of an object.
- You should be able to perform hypothesis testing and construct confidence interval for the difference between two groups.

Ch 11: Bias, Variance, and Least Squares

- The mean squared error (MSE) can be decomposed into the squared bias + variance.
- The German Tank Problem discusses the parameter estimation for the discrete uniform distribution case, Unif$\{1, 2, \ldots, N\}$. In class, we also discussed a similar problem for the continuous uniform distribution case, Unif$(0, \theta)$.
- In regression, we aim to predict Y from a linear function of X, i.e. $\widehat{Y} = \widehat{a}X + \widehat{b}$. It is important to understand the properties of correlation and the residuals and its connection to regression.

Ch 12: Inference in Regression

- The simple linear regression model assumes $Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ with $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$. Looking at the scatter plot, can you determine whether the linear regression model is satisfied?
- You learned how to use the regression line $\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_i$ to estimate the true linear curve $\beta_0 + \beta_1 x_i$.
- You should be able to compute statistic/quantities and conduct hypothesis testing from the python output of the linear regression.