STAT 88: Lecture 22

Contents

Section 6.3: Markov's Inequality

Section 6.4: Chebyshev's Inequality

Section 7.1: Sums of Independent Random Variables

Warm up:

- (a) State Markov's inequality.
- (b) Is it possible that half of US flights have delay times at least 3 times the national average?

Last time

Upper bounds for tail probability:

Markov's inequality For a non-negative random variable X and a positive constant c > 0,

$$P(X \ge c) \le \frac{E(X)}{c}.$$

Chebyshev's inequality For a random variable X with mean μ and SD σ and a positive constant c > 0,

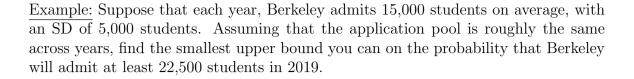
$$P(|X - \mu| \ge c) \le \frac{\sigma^2}{c^2} = \frac{\operatorname{Var}(X)}{c^2}.$$

One tail bound:

$$P(X - \mu \ge c) \le \frac{\sigma^2}{c^2} = \frac{\operatorname{Var}(X)}{c^2}.$$

Alternative formula for Var(X):

- $Var(X) = E((X \mu)^2).$
- $\operatorname{Var}(X) = E(X^2) E(X)^2$. This implies $E(X^2) = E(X)^2 + \operatorname{Var}(X)$.



Example: Suppose a list of numbers $x = \{x_1, \ldots, x_n\}$ has mean μ and standard deviation σ . Let k be the smallest number of standard deviations away from μ we must go to ensure the range $(\mu - k\sigma, \mu + k\sigma)$ contains at least 50% of the data in x. What is k?

Example: A list of non negative numbers has an average of 1 and an SD of 2. Let p be the proportion of numbers over 4. To get an upper bound for p, you should:

- (a) Assume a binomial distribution.
- (b) Use Markov's inequality.
- (c) Use Chebyshev's inequality.
- (d) None of the above.

Example: Let X be a non negative random variable such that E(X) = 100 = Var(X).

- (a) Can you find $E(X^2)$ exactly? If not what can you say?
- (b) Can you find P(70 < X < 130) exactly? If not what can you say?

7.1. Sums of Independent Random Variables

We know that E(X+Y) = E(X) + E(Y) but does Var(X+Y) = Var(X) + Var(Y)?

Let X be the number of hours a student is awake a day and let Y be the number of hours a student is asleep a day. Then X + Y = 24, so trivially

$$Var(X + Y) = Var(24) = 0 \neq Var(X) + Var(Y).$$

So when X and Y are dependent, it is possible that $Var(X+Y) \neq Var(X) + Var(Y)$.

Theorem If X and Y are independent,

$$Var(X + Y) = Var(X) + Var(Y).$$

Ex: Let X_1, X_2, \ldots, X_n be a i.i.d. random sample with mean μ and SD σ . Let $S_n = \sum_{i=1}^n X_i$. Then

$$\operatorname{Var}(S_n) = \operatorname{Var}(X_1) + \operatorname{Var}(X_2) + \dots + \operatorname{Var}(X_n) = n\sigma^2.$$

So, $SD(S_n) = \sigma \sqrt{n}$.

SD of Binomial Let $X \sim \text{Bernoulli}(p)$. Then

$$Var(X) = E(X^2) - E(X)^2 = p - p^2 = p(1 - p).$$

Now let $X \sim \text{Binomial}(n, p)$. Write X as a sum of n Bernoulli random variables and find SD(X).

SD of Poisson Recall that Binomial(n, p) can be approximated by Poisson(np) for large n and small p.

Binomial (n,p)
$$\frac{n \to \infty}{p \to 0}$$
 Poisson ($\mu = np$)

So we can find SD of Poisson(μ) from limit of SD of Binomial(n, p) as $n \to \infty, p \to 0, np \to \mu$.

$$\sqrt{npc-p} \qquad \frac{n\rightarrow\infty}{p\rightarrow0}$$

$$\Rightarrow \text{ If } X \sim pois(\mu),$$

$$SD(X) = \sqrt{\mu}$$

SD of Geometric Fact: If $X \sim \text{Geom}(p)$,

$$SD(X) = \frac{\sqrt{1-p}}{p}.$$

Ex: (Waiting till the 10th success) Suppose you roll a die until the 10th success. Let R be the number of rolls required. Find SD(R).