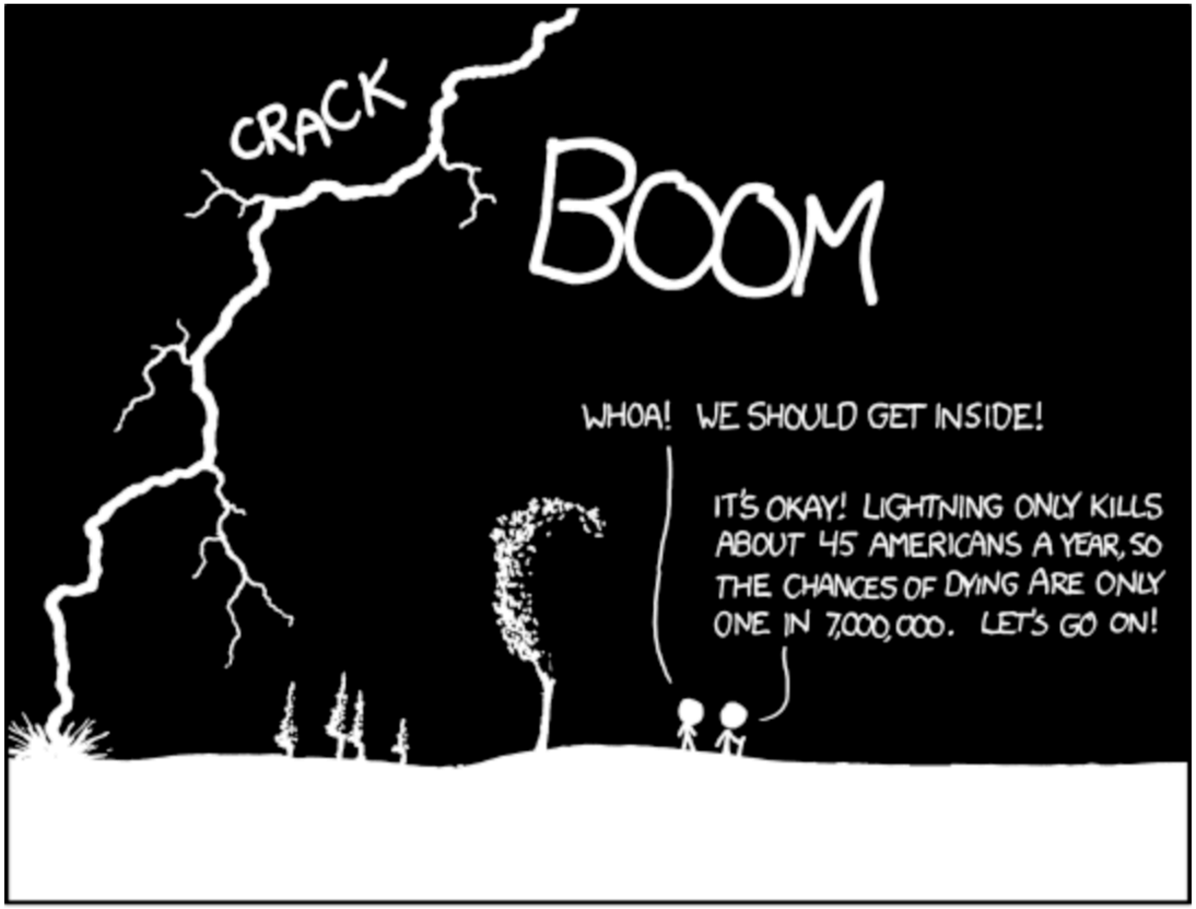
Stat 88: Probability & Math. Statistics in Data Science



THE ANNUAL DEATH RATE AMONG PEOPLE WHO KNOW THAT STATISTIC IS ONE IN SIX.

https://xkcd.com/795/

Lecture 16: 2/26/2021

Conditional expectation and expectation by conditioning

Sections 5.5, 5.6

Agenda

- Example 5.7.11
- Example from 5.5
- 5.6: Expectation by conditioning

Example: (5.7.11)

A data scientist believes that a randomly picked student at his school is twice as likely not to own a car as to own one car. He knows that no student has three cars, though some students do have two cars. He therefore models the probability distribution for the number of cars owned by a random student as follows. The model involves an unknown positive parameter θ.

# of cars	0	1	2
Probability	2θ	θ	$1-3\theta$

(a) Find $E(X_k)$

(b) Let $X_1, X_2, \dots X_n$ be the numbers of cars owned by n random students picked independently of each other. Assuming that the data scientist's model is good, use the entire sample to construct an unbiased estimator of θ.

Example: (5.7.11)

Conditional Expectation: An example

- Let X and Y be iid rvs with the distribution described below, and let $S = X + Y$:

x	1	2	3
P(X=x)	1/4	1/2	1/4

- Let's write down the joint distribution of X and S:

Conditional Expectation: An example

Expectation by Conditioning

- In the example we just worked out, once we fix a value s for S, then we have a distribution for X, and can compute its expectation using that distribution that depends on s: $E(X \mid S = s) = \sum xP(X = x \mid S = s)$, with the sum over all values of X.

- Note that $E(X \mid S = s)$ is a *function of* s. We can think of $E(X \mid S)$ as a rv.

- This means that if we want to compute $E(X)$, we can just take a weighted average of these conditional expecations $E(X \mid S = s)$:

$$\boldsymbol{E(X) = \sum_{s} E(X \mid S = s)\, P(S = s)}$$

- This is the *law of iterated expectation*

Law of iterated expectation

- Note that $E(X \mid S = s)$ is a function of s. That is, if we change the value of s we get a different value. (It is not a function of x, though.)

- Therefore we can define the function $g(s) = E(X \mid S = s)$, and the random variable $g(S) = E(X \mid S)$.

- In general, recall that $E\big(g(S)\big) = \sum_s g(s)f(s) = \sum_s g(s)P(S = s)$.
- How can we use this to find the expected value of the rv $g(S)$?

Examples from the text: Time to reach campus

- 2 routes to campus, student prefers route A (expected time =15 minutes) and uses it 90% of the time. 10% of the time, forced to take route B which has an expected time of 20 minutes. What is the expected duration of her trip on a randomly selected day?

Catching misprints

- The number of misprints is a rv N ~ Pois(5) dsn. Each misprint is caught before printing with chance 0.95 independently of all other misprints. What is the expected number of misprints that are caught before printing?

Exercise 5.7.13

- A survey organization in a town is studying families with children. Among the families that have children, the distribution of the number of children is shown below. Suppose each child has chance 0.51 of being male, independently of all other children. What is the expected number of male children in a family picked at random from those that have children?

n	1	2	3	4	5
prop. with n children	0.2	0.4	0.2	0.15	0.05

Expectation of a Geometric waiting time

- $X \sim Geom(p)$: X is the number of trials until the first success
- $P(X = k) = (1-p)^{k-1}\, p, \quad k = 1, 2, 3, \dots$
- Let $x = E(X)$
- Recall that $P(X > 1) = P(first\ trial\ is\ F) = 1 - p$
- We can split the possible situations into when the first trial is a success and the first trial is a failure, and condition on this and compute the *conditional expectation:*

$$E(X) = E(\,X \mid X = 1\,)P(X = 1) + E(\,X \mid X > 1\,)P(X > 1)$$

Expected waiting time until k sixes have been rolled

- Let T_k be the waiting time until the kth six is rolled.
- For example, consider $k = 1$ (geometric) , expected waiting time=6 rolls
- What about if $k = 2$? $k = 3$? $k = 10$?

- General formula for expected waiting time until kth success where P(S) = p

Example

- A fair die is rolled repeatedly. Find the expected waiting time (number of rolls) until two *different* faces are rolled.