STAT 88: Lecture 25

Contents

Warm up: (Exercise 7.4.11) Each Data 8 student is asked to draw a random sample and estimate a parameter using a method that has chance 95% of resulting in a good estimate.

Suppose there are 1300 students in Data 8. Let X be the number of students who get a good estimate. Assume that all the students' samples are independent of each other.

(a) Find the distribution of X.

(b) Find $E(X)$ and $\mathrm{SD}(X)$.

(c) Find the chance that more than 1250 students get a good estimate.

Last time

SD of sample sum:

Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim}$ with mean μ and SD σ. Let $S_n = X_1 + X_2 + \cdots + X_n$. Then

$$E(S_n) = n\mu, \qquad \mathrm{SD}(S_n) = \sqrt{n}\sigma.$$

SD of sample mean:

Let $\bar{X}_n = S_n/n$. Then

$$E(\bar{X}_n) = \mu, \qquad \mathrm{SD}(\bar{X}_n) = \frac{\sigma}{\sqrt{n}}.$$

The law of large numbers: For a fixed $c > 0$,

$$P(\mu - c < \bar{X}_n < \mu + c) = P(|\bar{X}_n - \mu| < c) \to 1 \text{ as } n \to \infty.$$

Today: How the shape of the distribution of S_n look like?

8.1. The Distribution of a Sample Sum

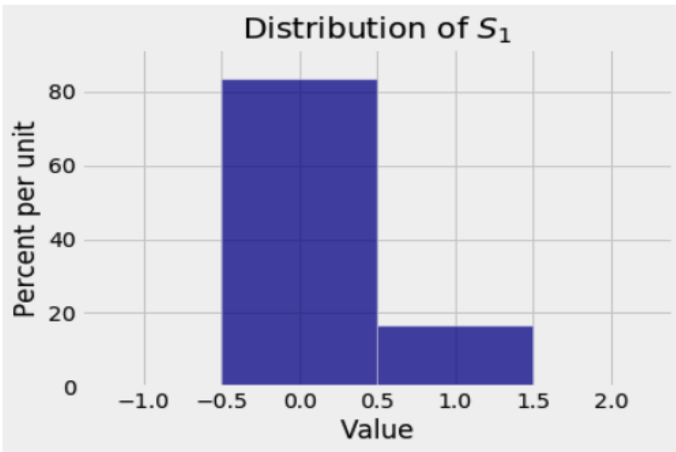
Sum of IID Indicators If $X_1, X_2, \dots, X_n \overset{\text{iid}}{\sim}$ Bernoulli(p), then $S_n = X_1 + X_2 + \cdots + X_n$ has the Binomial(n, p) distribution. What the distribution of S_n look like?

Lab 1 : $X_1, \cdots, X_n \sim$ Bernoulli(p)
Get $S_n = X_1 + X_2 + \cdots + X_n$

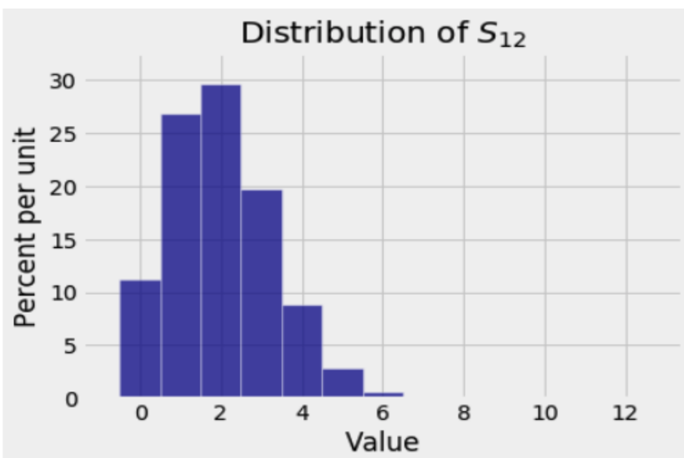
Lab 2 : $X_1, \cdots, X_n \sim$ Bernoulli(p)
Get $S_n = X_1 + X_2 + \cdots + X_n$

$\vdots$

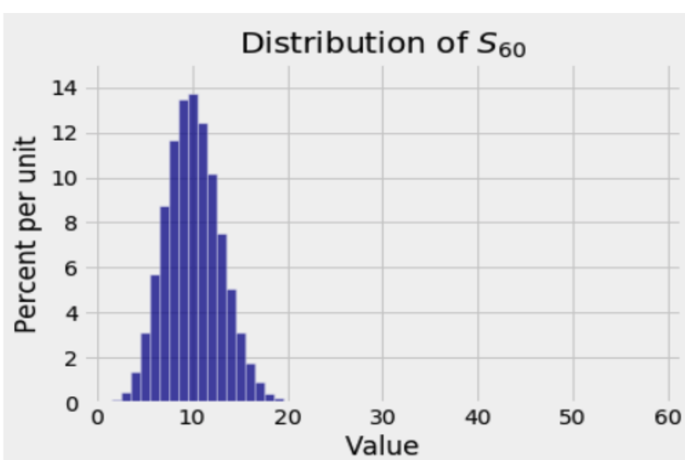
Lab 10,000 : $X_1, \cdots, X_n \sim$ Bernoulli(p)
Get $S_n = X_1 + X_2 + \cdots + X_n$
$\Downarrow$
Distribution of S_n



n=1

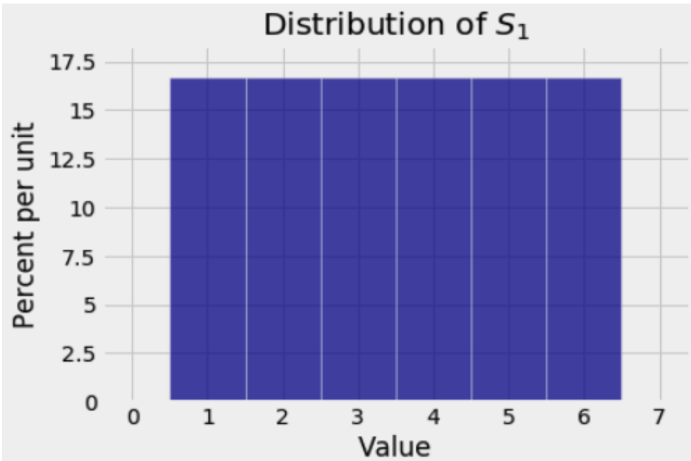


n=12

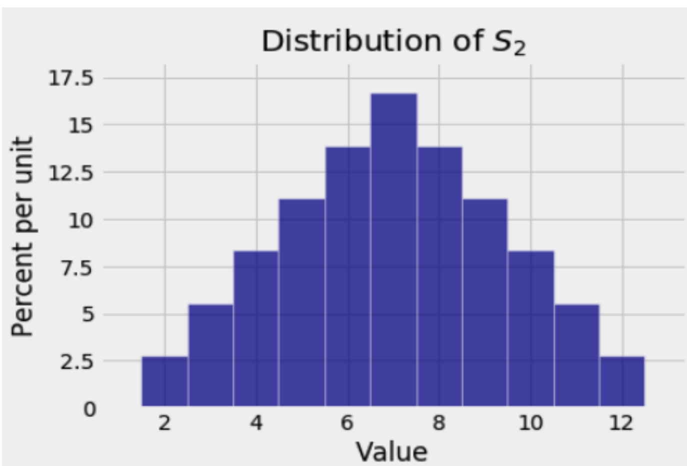


n=60

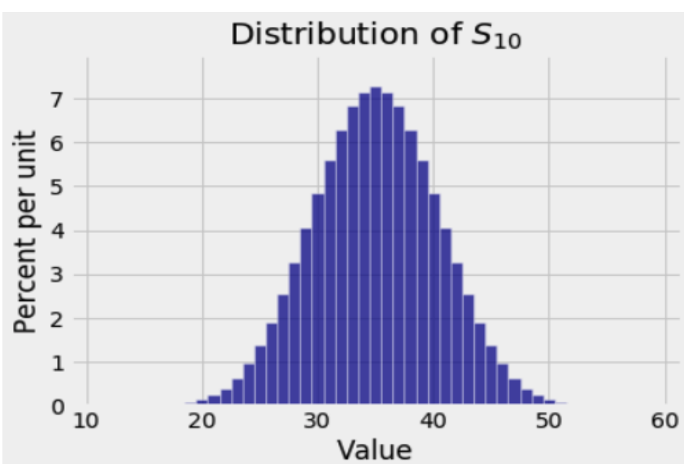
Sum of IID Uniform Random Variables Let $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} \text{Uniform}\{1, 2, 3, 4, 5, 6\}$ and $S_n = X_1 + X_2 + \cdots + X_n$. What the distribution of S_n look like?



n=1

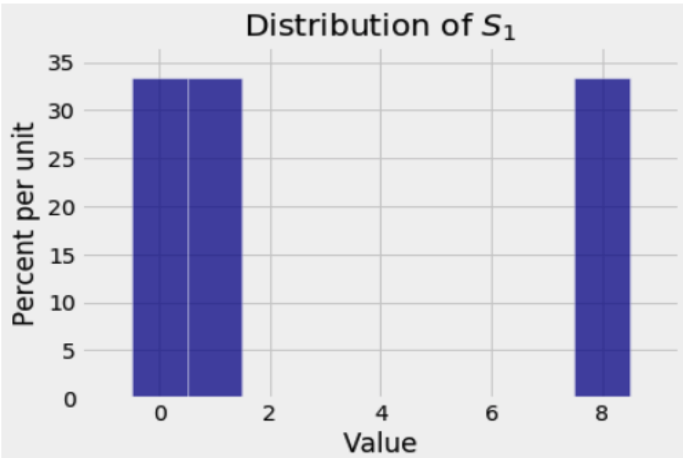


n=2

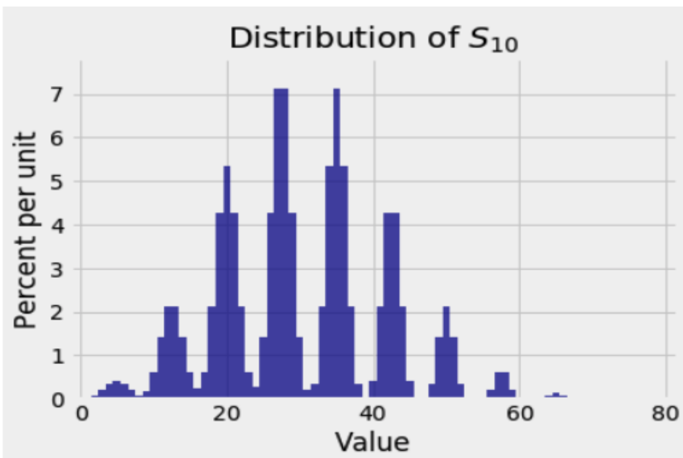


n=10

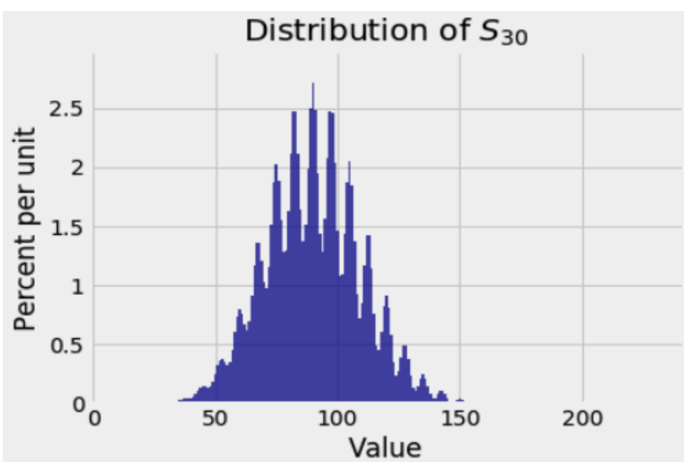
A Wild One Let $X_1, X_2, \dots, X_n \overset{\text{iid}}{\sim} \text{Uniform}\{0, 1, 8\}$ and $S_n = X_1 + X_2 + \cdots + X_n$. What the distribution of S_n look like?



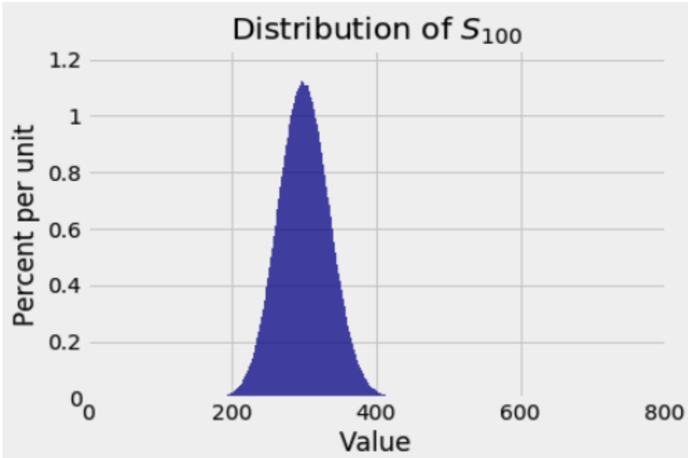
n=1



n=10

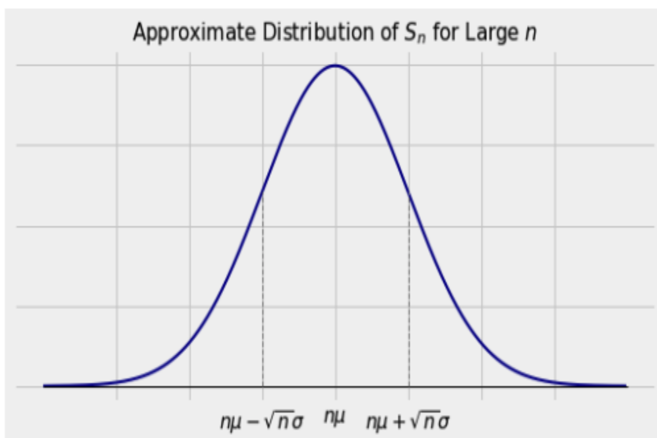


n=30



n=100

Central Limit Theorem Let $X_1, X_2, \ldots, X_n$ be i.i.d. with $E(X_1) = \mu$ and $\mathrm{SD}(X_1) = \sigma$. Let $S_n = X_1 + X_2 + \cdots + X_n$ be the sample sum. If n is large, the distribution of S_n is approximately normal (bell-shaped curve), regardless of the distribution of the X_i's.



Key idea: It is easier to approximate $P(X > 1250)$ using the fact that Binomial is almost Normal for large n.

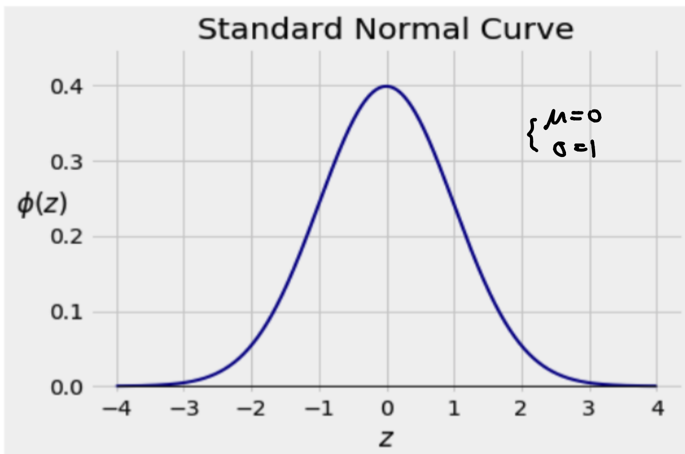
8.2. Standard Normal Curve

The normal or Gaussian curves are a family of bell-shaped curves named for the German mathematician and scientist Carl Friedrich Gauss.

The Standard Normal Curve

The standard normal curve is defined by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad -\infty < z < \infty.$$



Properties:

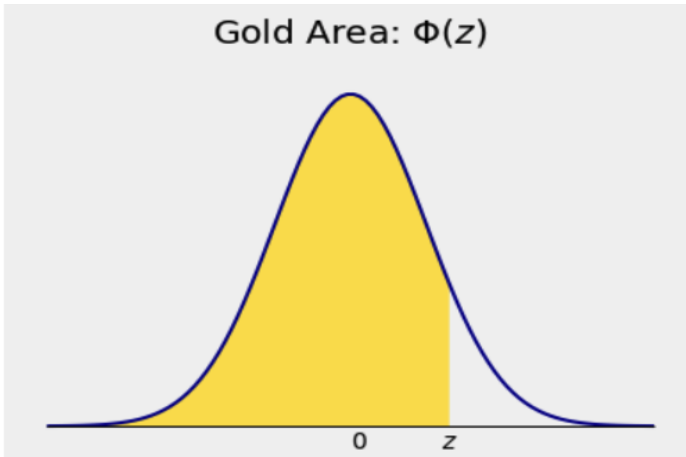
- The curve is bell-shaped and symmetric about 0.
- The points of inflection are at $z = -1$ and $z = 1$.
- For $|z| > 3$, the curve is pretty close to 0.
- The total area under the curve is 1.

The Standard Normal 'CDF'

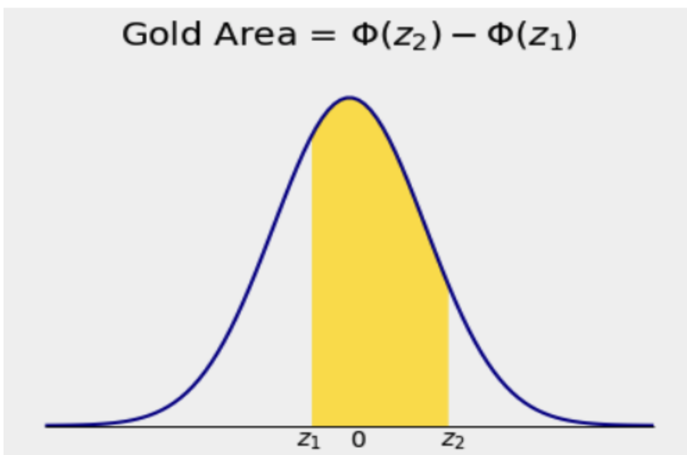
If you think of the standard normal curve as a probability histogram, then it is natural to think of areas under the curve as probabilities.

$$\Phi(z) = \int_{-\infty}^{z} \phi(x)dx.$$

Φ gives all the area under the curve to the left of z:



The area under the curve over any interval (z_1, z_2) is then $\Phi(z_2) - \Phi(z_1)$:



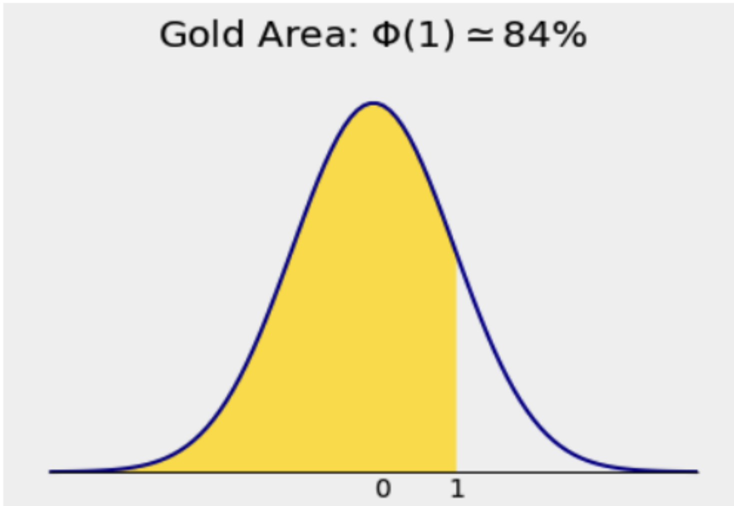
Numerical Values of the Areas

Calculating $\Phi(z)$ in Python:

```
stats.norm.cdf(1)
```

in scipy library

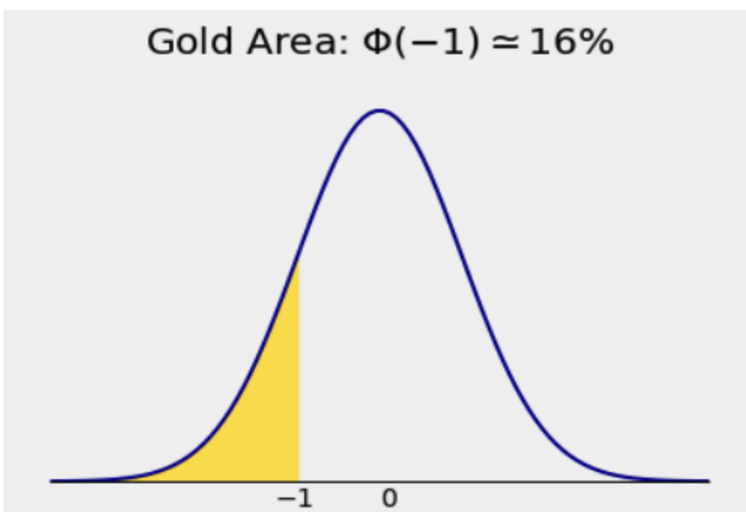
```
0.8413447460685429
```



By symmetry:

```
stats.norm.cdf(-1)
```

```
0.15865525393145707
```



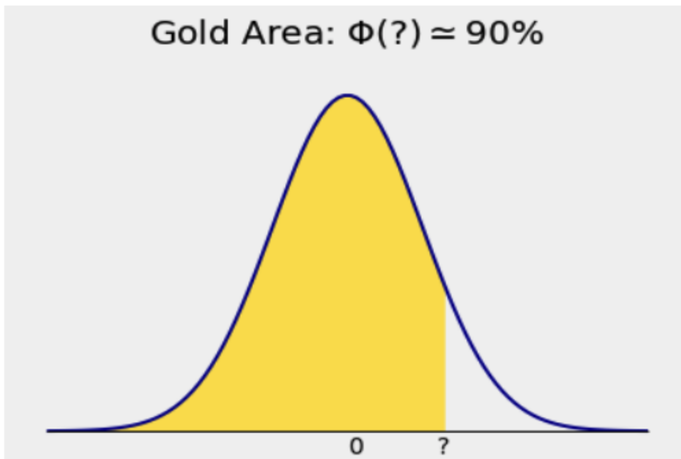
Percentiles

We saw the area under the curve to the left of 1 is about 84%

$$\Phi(1) \approx 84\%.$$

The point $z = 1$ is therefore called the 84th percentile of the curve. If you think of the curve as a probability histogram, then about 84% of the probability lies below $z = 1$.

The 90th percentile must be to the right of 1. But how far to the right?



We need to find the inverse of $\Phi(z)$. The 90th percentile is the point z such that $\Phi(z) = 0.9$, or

$$z = \Phi^{-1}(0.9).$$

Calculating $\Phi^{-1}(q)$ in Python:

Percent Point function

```
stats.norm.ppf(0.9)
```

```
1.2815515655446004
```

Example: Find the area

(a) to the right of 1.25.

(b) between -0.3 and 0.9.

(c) Outside -1.5 and 1.5.

Example: The standard normal curve is sketched below. Solve for z.