

Stat 88: Probability & Mathematical Statistics in Data Science



Lecture 13: 2/19/2021

Functions of rvs, Method of indicators

Sections 5.2, 5.3

Joint distributions

- Draw two tickets without replacement from a box that has 5 tickets marked 1, 2, 2, 3, 3. Let X_1 and X_2 represent the values of the tickets drawn on the first and second draws respectively.
- Create a table of all possible outcomes for the pair (X_1, X_2) (which is also a random variable), and write down the probabilities using the multiplication rule.

	$X_2 = 1$	$X_2 = 2$	$X_2 = 3$
$X_1 = 1$			
$X_1 = 2$			
$X_1 = 3$			

Marginal distributions

- What is $P(X_1 = 1)$? Write down the pmf for X_1 and X_2
- Are they independent?
- Use the table to compute $P(X_1 + X_2 = 5)$
- Use the table to compute $E(g(X_1, X_2))$, where $g(X_1, X_2) = |X_1 - X_2|$ (the expected distance between the two draws)
- If $S = X_1 + X_2$, find $E(S)$.

Method of indicators

- Additivity of Expectation: This is a very useful property – no matter what the joint distribution of X and Y may be, we have:

$$E(X + Y) = E(X) + E(Y)$$

- Whether X and Y are dependent or independent, this holds, making it enormously useful.
- We also have linearity: $E(aX + bY) = aE(X) + bE(Y)$
- Recall that we talked about “classifying and counting” – so, we divide up the outcomes into those that we are interested in (successes), and everything else (failures), and then count the number of successes.
- We can represent these outcomes as 0 and 1, where 1 marks a success and 0 and failure, so if we model the trials as draws from a box, we can count the number of success by counting up the number of times we drew a 1.
- We can represent each draw as a Bernoulli trial, where $p = P(S)$

Using indicators and additivity

- For example, say we roll a die 10 times, and success is rolling a 1.
- Then $p=1/6$, and we can define a Bernoulli rv as $X = \begin{cases} 0, & \text{w.p. } 5/6 \\ 1, & \text{w.p. } 1/6 \end{cases}$
- We can also define an event A: let A be the event of rolling a 1 and define a rv I_A that takes the value 1 if A occurs and 0 otherwise.
- This is a Bernoulli rv, what is its expectation?
- Now let $X \sim \text{Bin}(10, \frac{1}{6})$, so X counts the number of successes in 10 rolls. Let's find $E(X)$ using additivity and indicators:

Using indicators

- Binomial
- Hypergeometric: Did we use the independence of the trials for the binomial? If not, we can use the same method to compute the expected value of a hypergeometric rv:

Exercise 5.7.6: A die is rolled 12 times. Find the expectation of:

- a) the number of times the face with five spots appears
- b) the number of times an odd number of spots appears
- c) the number of faces that don't appear
- d) the number of faces that do appear

Example

- Let X be the number of spades in 7 cards dealt **with replacement** from a well shuffled deck of 52 cards containing 13 spades. Find $E(X)$.
 1. Write down what X is
 2. Define an indicator for the k th trial: I_k
 3. Find $p = P(I_k = 1)$
 4. Write X as a sum of indicators
 5. Now compute $E(X)$ using additivity
- Do the same thing if we deal 7 cards **without replacement**.

Missing classes

- We can use indicators to compute the chance that something *doesn't* occur.
- For example, say we have a box with balls that are red, white, or blue, with 35% being red, 30% being white, and 35% blue. If we draw n times with replacement from this box, what is the expected number of colors that *don't* appear in the sample?

Examples

1. An instructor is trying to set up office hours during RRR week. On one day there are 8 available slots: 10-11, 11-noon, noon-1, 1-2, 2-3, 3-4, 4-5, and 5-6. There are 6 GSIs, each of whom picks one slot. Suppose the GSIs pick the slots at random, independently of each other. Find the expected number of slots that no GSI picks.

2. A building has 10 floors above the basement. If 12 people get into an elevator at the basement, and each chooses a floor at random to get out, independently of the others, at how many floors do you expect the elevator to make a stop to let out one or more of these 12 people?