* Announcement

① HW5 due Mon 9/28

② Midterm logistics early next week
(Friday 10/9)

③ Exam-prep section
today 2-3pm

STAT 88: Lecture 13

After today's lecture
$\rightsquigarrow$ HW5 Q3

Contents

Section 5.3: Method of Indicators
Section 5.4: Unbiased Estimators

Warm up:

(a) (X_1, X_2) has joint distribution:

		$X_2=0$ ($\frac{1}{6}$)	$X_2=1$ ($\frac{5}{6}$)
$\frac{30}{36} = \frac{5}{6}$	$X_1=0$	$\frac{5}{36}$	$\frac{25}{36}$
$\frac{6}{36} = \frac{1}{6}$	$X_1=1$	$\frac{1}{36}$	$\frac{5}{36}$

Are X_1 and X_2 independent?

$P(X_1=a, X_2=b) = P(X_1=a)\,P(X_2=b)$ for all (a,b)

$\frac{5}{36} = P(X_1=0, X_2=0) = P(X_1=0)\,P(X_2=0) = \frac{5}{36}$ $\rightsquigarrow$ X_1 and X_2 indep.

(b) A die is rolled 10 times. Find the expectation of the number of times an odd number of spots appears.

$\sim \text{Binomial}(10, \frac{1}{2})$

$\rightarrow 10 \cdot \frac{1}{2} = 5$

Last time

The expectation of a random variable X, denoted $E(X)$, is the average of the possible values of X weighted by their probabilities:

$$E(X) = \sum_{\text{all } x} x P(X = x).$$

Recall (Bernoulli (indicator) random variable)

success

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

failure

Then

$$E(X) = 1 \cdot p + 0 \cdot (1 - p) = p.$$

Additivity of Expectation $E(X_1 + X_2) = E(X_1) + E(X_2)$.

Method of indicator to find $E(X)$

Counting the number of successful trials is the same as adding zeros and ones.

Example: A success is blue and failure non blue.

B R R G B R B B

1 0 0 0 1 0 1 1

$$\#\text{blue} = 1 + 0 + 0 + 0 + 1 + 0 + 1 + 1 = 4.$$

random count = sum of indicators

Suppose a trial is blue with probability p. Find the expected # blue in n trials.

Step 1: Write down what X is.

$X = \#$ trials out of n that are blue. ~ Binomial (n,p)

Step 2: Find I_j (jth trial).

p

$$I_j = \begin{cases} 1 & \text{if } j\text{th trial is blue} \\ 0 & \text{otherwise} \end{cases}$$

Step 3: Find p.

Step 4: Write X as a sum of indicators:

$$X = I_1 + I_2 + \cdots + I_n.$$

Step 5: Find $E(X)$.

$$E(X) = E(I_1 + I_2 + \cdots + I_n) = E(I_1) + E(I_2) + \cdots + E(I_n) = nE(I_1) = np.$$

Additivity

Conclusion: If $X \sim \text{Binomial}(n, p)$, $E(X) = np$.

n large, p small $\quad$ Binomial$(n,p) \approx$ pois(μ)

$\mu = np$

$E(X) = np \quad E(X) = \mu$

5.3. Method of Indicators (Continued)

Example: Let X be the number of spades in 7 cards dealt with replacement from a well shuffled deck of 52 cards containing 13 spades. Find $E(X)$.

Step 1: Write down what X is.

X = # trials out of 7 that is a spade

Step 2: Find I_j (jth trial).

$$I_j = \begin{cases} 1 & \text{if } j\text{th trial is spade} \\ 0 & \text{else} \end{cases}, \quad j = 1, \dots, 7$$

Step 3: Find p.

$$p = \frac{13}{52} = \frac{1}{4}$$

Step 4: Write X as a sum of indicators.

$$X = I_1 + I_2 + \cdots + I_7$$

Step 5: $E(X) = E(I_1 + I_2 + \cdots + I_7)$

$$= E(I_1) + \cdots + E(I_7)$$

$$= 7 \cdot \left(\frac{1}{4}\right)$$

$X \sim \text{Binomial}(7, \frac{1}{4})$

Example: Let X be the number of spades in 7 cards dealt without replacement from a well shuffled deck of 52 cards containing 13 spades. Find $E(X)$.

Step 1 $X = \#$ cards out of 7 that are spade

Step 2 $I_j = \begin{cases} 1 & \text{if } j\text{th trial is spade} \\ 0 & \text{else} \end{cases}$

Step 3 $p = \frac{1}{4}$ (by symmetry)

$p = \frac{G}{N}$ by symmetry

Step 4 $X = I_1 + I_2 + \cdots + I_7$

Step 5 $E(X) = 7 \cdot \left(\frac{1}{4}\right)$

$X \sim HG(52, 13, 7)$ $\rightsquigarrow$ $X \sim HG(N, G, n)$

$E(X) = n \cdot \frac{G}{N}$

If X is not binomial or hypergeometric be thoughtful how define your indicator. You want each indicator to have same p.

Example: (Exercise 5.7.6) A die is rolled 12 times. Find the expectation of

(a) the number of times the face with five spots appears.

(c) the number of faces that don't appear.

(a) $X \sim \text{Binomial}(12, \frac{1}{6}) \rightsquigarrow E(X) = 2$

(c) Step 1 $X = \#$ faces out of 6 that don't appear

Step 2 $I_j = \begin{cases} 1 & \text{if } j\text{th face doesn't appear} \\ 0 & \text{else} \end{cases}$ $j = 1, \cdots, 6$

Step 3 Find $p = P(j\text{th face doesn't appear})$

$= \left(\frac{5}{6}\right)^{12}$

$j=1$, $2 \sim 6$th faces show up in 12 trials

$\parallel$

$\left(\frac{5}{6}\right)^{12}$

Step 4 $X = I_1 + I_2 + \cdots + I_6$

Step 5 $E(X) = 6 \cdot \left(\frac{5}{6}\right)^{12}$

$Y = \#$ jth face out of 12 trials

$\sim \text{Binomial}(12, \frac{1}{6})$

$p = P(Y=0) = \binom{12}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{12}$

$= \left(\frac{5}{6}\right)^{12}$

(d) "the number of faces that do appear"

$\tilde{X} = \#$ faces out of 6 that appear.

$X + \tilde{X} = 6$

$\rightsquigarrow E(X) + E(\tilde{X}) = 6$

$\rightsquigarrow E(\tilde{X}) = 6 - E(X)$

$= 6 - 6 \cdot \left(\frac{5}{6}\right)^{12}$

Example: n people with hats have had a bit too much to drink at a party. As they leave the party, each person randomly grabs a hat. A match occurs if a person gets his or her own hat.

(a) The expected number of matches depends n.

(b) The expected number of matches is 1

(c) The number of matches is hypergeometric. a wrong.

(d) More than one of the above.

indicators dependent but $X \not\sim HG$.
For HG, you must be able to tell if element in population is good before you draw. Here you can only tell after you draw.

Step1 $X = \#$ ppl out of n who get a match

Step2 $I_j = \begin{cases} 1 & \text{if } j\text{th person gets a match} \\ 0 & \text{else} \end{cases}$

" gets his/her own hat.

Step3 $p = \frac{1}{n}$

Step4 $X = I_1 + \cdots + I_n$

Step5 $E(X) = E(I_1) + \cdots + E(I_n)$

$= 1$

Example: A building has 10 floors above the basement. If 12 people get into an elevator at the basement, and each chooses a floor at random to get out, independently of the others, at how many floors do you expect the elevator to make a stop to let out one or more of these 12 people?

Step 1 $X =$ # floors out of 10 that's chosen

Step 2 $I_j = \begin{cases} 1 & \text{if } j\text{th floor is chosen} \\ 0 & \text{else} \end{cases}$

Step 3 $p = P(\text{1st floor is chosen})$

by symmetry

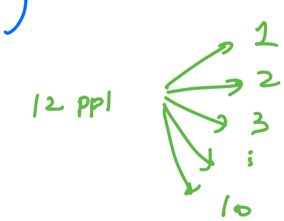
$= 1 - P(\text{1st floor is not chosen})$

Complement rule

$= 1 - \left(\frac{9}{10}\right)^{12}$

Step 4 $X = I_1 + I_2 + \cdots + I_{10}$

Step 5 $E(X) = 10\left(1 - \left(\frac{9}{10}\right)^{12}\right)$



$p = P(\text{1st floor chosen})$

$= P(Y \geq 1)$

$= 1 - P(Y < 1)$

$= 1 - P(Y = 0)$

$= 1 - \binom{12}{0}\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^{12}$

$= 1 - \left(\frac{9}{10}\right)^{12}$

$Y =$ # pple who choose 1st floor

$\sim \text{Binomial}(12, \frac{1}{10})$