* Announcement

① HW11 due today (11:59 PM PT)

② HW12 ~ 11/23

③ Quiz 10 : Ch 10.3 ~ Ch 11.1

After today's lecture (HW12)

→ Q1, Q2, Q3 (Sec 10.4)

Q4, Q5 (Sec 11.1 ~ 11.2)

STAT 88: Lecture 34

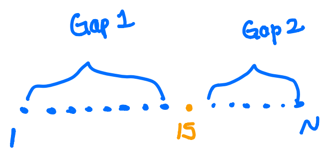
Contents

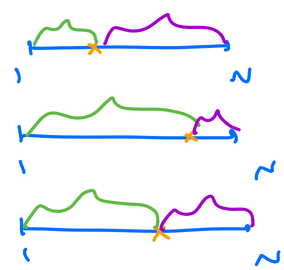
Warm up:

parameter

$E(N) = N$

N consecutive positive integers, $1, 2, \ldots, N$ are in a hat, with N unknown. You randomly sample one number from the hat and get 15. Estimate N.





→ Gap 1 + Gap 2 + 1 = N

→ $\underbrace{E(\text{Gap }1)}_{= G} + \underbrace{E(\text{Gap }2)}_{= G} + 1 = N$ (by symmetry, $E(\text{Gap }1) = E(\text{Gap }2)$)

→ $2G + 1 = N$

G ← 14

⇒ Estimate G by 14

⇒ Estimate N by $2 \cdot 14 + 1 = 29$

Last time

Suppose you have two independent samples, $X_1, X_2, \ldots, X_n$ i.i.d. with mean μ_X and SD σ_X, and $Y_1, Y_2, \ldots, Y_m$ are i.i.d. with mean μ_Y and SD σ_Y. By CLT, $\bar{X} - \bar{Y}$ is approximately distributed as

$$\mathcal{N}\left(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right).$$

An approximate 95% CI for $\mu_X - \mu_Y$ is given by

$$\bar{X} - \bar{Y} \pm 2\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}.$$

If we wish to test $H_0 : \mu_X = \mu_Y$ vs $H_A : \mu_X > \mu_Y$, $T = \bar{X} - \bar{Y}$ is our test statistic. Under H_0, we have

$$T \sim \mathcal{N}\left(0, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right).$$

If t is an observed value of our test statistic, p-value is given by

$$\text{p-val} = P(T \geq t) = P\left(Z \geq \frac{t}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}\right) = 1 - \Phi\left(\frac{t}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}\right).$$

Next, you have two independent populations of 0's and 1's, so $X_1, X_2, \ldots, X_n$ are i.i.d. from Bernoulli(p_X), and $Y_1, Y_2, \ldots, Y_m$ are i.i.d. from Bernoulli(p_Y). By CLT, $\bar{X} - \bar{Y}$ is approximately distributed as

$$\bar{X} - \bar{Y} \sim \mathcal{N}\left(p_X - p_Y, \frac{p_X(1-p_X)}{n} + \frac{p_Y(1-p_Y)}{m}\right).$$

An approximate 95% CI for $p_X - p_Y$ is given by

Estimate $p_X \leftarrow \bar{X}$
$p_Y \leftarrow \bar{Y}$

$$\bar{X} - \bar{Y} \pm 2\sqrt{\frac{p_X(1-p_X)}{n} + \frac{p_Y(1-p_Y)}{m}}.$$

If we wish to test $H_0 : p_X = p_Y = p$ vs $H_A : p_X > p_Y$, $T = \bar{X} - \bar{Y}$ is our test statistic. Under H_0, we have

Estimate p by $\frac{n}{n+m}\bar{X} + \frac{m}{n+m}\bar{Y}$

$$T \sim \mathcal{N}\left(0, \frac{p(1-p)}{n} + \frac{p(1-p)}{m}\right).$$

If t is an observed value of our test statistic, p-value is given by

$$\text{p-val} = P(T \geq t) = P\left(Z \geq \frac{t}{\sqrt{\frac{p(1-p)}{n} + \frac{p(1-p)}{m}}}\right) = 1 - \Phi\left(\frac{t}{\sqrt{\frac{p(1-p)}{n} + \frac{p(1-p)}{m}}}\right).$$

11.1. Bias and Variance

Bias-Variance Decomposition

Some estimators for a parameter θ are better than others. A good estimator is one with a small mean square error.

estimator

$$\mathrm{MSE}_\theta(T) = E_\theta((T-\theta)^2) = B_\theta^2(T) + \mathrm{Var}_\theta(T),$$

parameter

where

$$B_\theta(T) = E_\theta(T) - \theta \text{ and } \mathrm{Var}_\theta(T) = E_\theta((T - E_\theta(T))^2).$$

Bias   Variance

11.2. The German Tank Problem

The Allies during WWII needed to estimate how many Tanks N the Germans had produced. The idea was to model the observed serial numbers as random draws from $1, 2, \dots, N$ and then estimate N.

So we will now assume, as the Allies did, that the serial numbers of the observed tanks are random variables $X_1, \dots, X_n$ drawn uniformly at random without replacement from $\{1, 2, \dots, N\}$. That is, we have a simple random sample of size n from the population $\{1, 2, \dots, N\}$ and we have to estimate N. $X_1, \dots, X_n \sim \mathrm{Unif}\{1, 2, \dots, N\}$

Now we will compare several estimators.

T_1: By symmetry, for each i,

Expectation of Unif$\{1, \dots, N\}$ = pop. mean

$$E(X_i) = \frac{N+1}{2}.$$

Since $\bar{X}$ is an unbiased estimator for the pop. mean,

$\frac{X_1 + \dots + X_n}{n}$

$$E(\bar{X}) = \frac{N+1}{2}.$$

$\rightarrow 2E(\bar{X}) - 1 = N$

$\rightarrow E(2\cdot\bar{X} - 1) = N$

This is a linear function of N so we can find an unbiased estimator for N:

$$E(\underbrace{2\bar{X} - 1}_{=T_1}) = N.$$

If we define

$$T_1 = 2\bar{X} - 1, \quad \rightsquigarrow \quad E(T_1) = N$$

it is an unbiased estimator of N.

$X_1, \dots, X_n \sim \text{Unif}\{1, \dots, N\}$

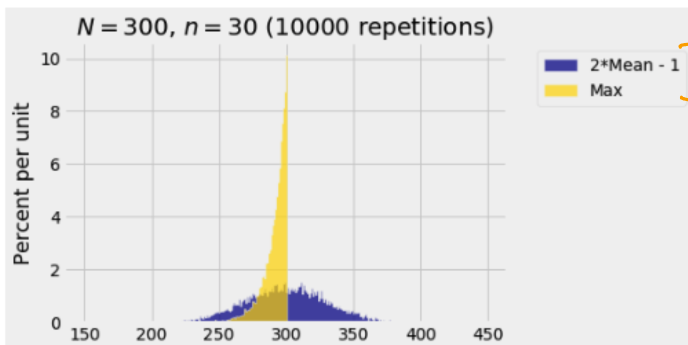
T_2: Another natural estimator is

$$T_2 = \max\{X_1, X_2, \dots, X_n\},$$

the maximum of the observed numbers.

$X_1, \dots, X_n \sim \text{Unif}\{1, \dots, N\}$

$\begin{cases} T_1 = 2\bar{X} - 1 \\ T_2 = \max\{X_1, \dots, X_n\} \end{cases}$



Unbiased and large var

biased and small var

$E(T_2) < N$

$\rightsquigarrow$ with some chance, $T_2 = N$

w/ other chance $T_2 < N$

$\rightsquigarrow E(T_2) < N$

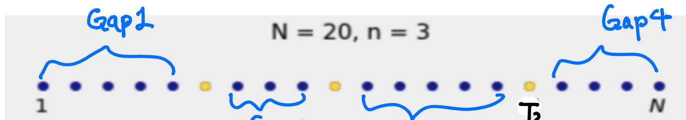
$300 = N$

T_2 is clearly biased. We compute the bias of T_2.

The Bias of the Sample Maximum The bias of T_2 is

$$\underbrace{B(T_2)}_{\text{Bias of } T_2} = E(T_2) - N.$$

Imagine a row of N spots for the serial numbers 1 through N, with marks at the spots corresponding to the observed serial numbers.



Gap1, Gap2, Gap3, Gap4, T_2

$$\rightarrow \boxed{\begin{aligned} \text{Gap4} &= N - T_2 \\ E(\text{Gap4}) &= E(N - T_2) \\ &= N - E(T_2) \\ &= -B(T_2) \end{aligned}}$$

The $n = 3$ sampled spots create $n + 1 = 4$ blue "gaps" between sampled values: one before the leftmost gold spot, two between successive gold spots, and one after the rightmost gold spot that is at position T_2.

By symmetry, the lengths of all four gaps have the same distribution. Therefore all four gaps have the same expected length.

- The gaps are made up of $N - n = 17$ blue spots;

$\text{Gap 1} + \text{Gap 2} + \text{Gap 3} + \text{Gap 4} = 17$

$\rightsquigarrow E(\text{Gap 1}) + E(\text{Gap 2}) + \cdots + E(\text{Gap 4}) = 17$

$\rightsquigarrow$ By symmetry, $E(\text{Gap 1}) = \cdots = E(\text{Gap 4}) = G,$

so $4 \cdot G = 17$

$\rightsquigarrow G = \frac{17}{4} \quad \underset{\text{Generalize}}{\rightsquigarrow} \quad \frac{N-n}{n+1}$

- Since each of the four gaps has the same expected length, the expected length of a single gap is $\frac{17}{4}$.

More generally,

$$\text{expected length of gap} = \frac{N-n}{n+1}.$$

Note that $N - E(T_2)$ is the expected length of the last gap. So,

$$\underset{E(\text{Gap}4)}{\overset{\shortparallel}{N - E(T_2)}} = \frac{N-n}{n+1},$$

and therefore

$$B(T_2) = E(T_2) - N = \frac{-(N-n)}{n+1}. \quad < 0 \quad \Rightarrow \quad E(T_2) < N$$

T_3: (the "augmented maximum")

What is $E(T_2)$?

$$E(T_2) = N + B(T_2) = N + \frac{-(N-n)}{n+1}$$
$$= \frac{n}{n+1}(N+1)$$

Since $E(T_2)$ is a linear function of N, we can make a new unbiased estimator by solving for N.

$$E(T_2) = \frac{n}{n+1}(N+1) \quad \Rightarrow \quad \frac{n+1}{n}E(T_2) - 1 = N$$
$$\Rightarrow E\Bigg(\underbrace{\frac{n+1}{n} \cdot T_2 - 1}_{= T_3}\Bigg) = N.$$

If we define

$$T_3 = \frac{n+1}{n} \cdot T_2 - 1, \qquad E(T_3) = N$$

it is an unbiased estimator of N.

- How does $\text{Var}(T_3)$ and $\text{Var}(T_2)$ compare? $\text{Var}(T_3) = \text{Var}\left(\frac{n+1}{n} \cdot T_2\right) = \left(\frac{n+1}{n}\right)^2 \text{Var}(T_2)$
  $\rightsquigarrow$ If n is large, $\text{Var}(T_3) \approx \text{Var}(T_2)$
- How does $B(T_3)$ and $B(T_2)$ compare?
  $B(T_3) = 0$, $B(T_2) = -\frac{N-n}{n+1}$

So for large n, T_3 is better than T_2.

$\text{MSE}(T_3) = \text{bias}^2 + \text{var}$
$= 0 + \text{Var}(T_3)$

$\Rightarrow$ T_3 has lower MSE than T_2

$\text{MSE}(T_2) = \left(\frac{N-n}{n+1}\right)^2 + \text{Var}(T_2)$

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Average of Augmented Maxes: 300.18587333333335
SD of of Augmented Maxes: 8.947086216787126
Average of Maxes: 291.4702
SD of Maxes: 8.65847053237464
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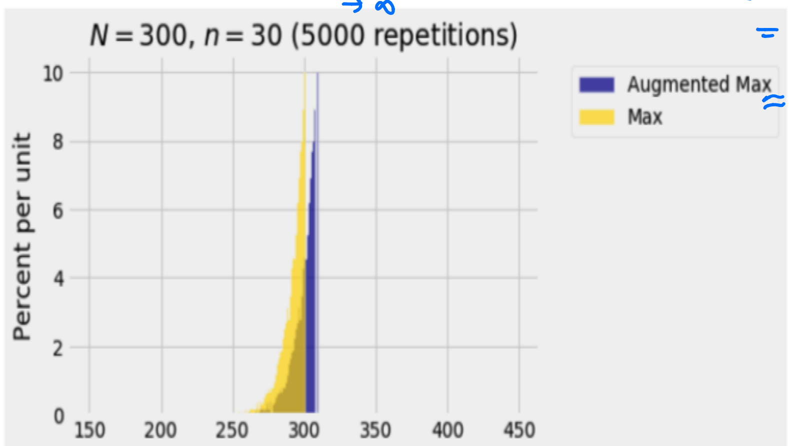
$= E(T_3)$, $= SD(T_3)$, $= E(T_2)$, $= SD(T_2)$

"T_3" Bias^2 + Var $= (300.186 - 300)^2 + 8.95^2 \approx 79.956$ ($300 = N$)

"T_2" Bias^2 + Var $= (291.47 - 300)^2 + 8.66^2 \approx 147.58$



$\max\{X_1, \dots, X_n\}$

In summary, we can have many different estimators for a parameter. In this lecture, ($2\bar{X} - 1 =$) T_1 was unbiased but had a large variance, T_2 was biased but had a smaller variance. ($\frac{n+1}{n} T_2 - 1 =$) T_3 was unbiased and had a bigger variance but for large n $\mathrm{Var}(T_3) \approx \mathrm{Var}(T_2)$. The estimator with the smallest MSE $=$ Bias^2 + Var is the best.

Another way to think of the "augmented maximum" If we could see the gap to the right of T_2, we would see N. But we can't. So we can try to do the next best thing, which is to augment T_2 by the estimated size of that gap.

What is the estimated gap length?

Hence we can try to improve upon T_2 using the estimator

$$T_3 =$$