Stat 88 lec 11

Warmup 2:00 – 2:10

HW 4 due Tuesday 10pm

a) True or False:

A Poisson distribution is an approximation of of the binomial distribution, Binomial(n, p), when n is large and p is small?

b) Exercise 4.5.11

11. A courtyard is paved with 100 identical tiles. In an instant of rain, the number of raindrops on each tile has the Poisson (10) distribution independently of all other tiles.

$P(X=k) = \frac{e^{-\mu}\mu^k}{k!}$ Poisson Formula $k = 0, 1, 2, \ldots$

Find the chance that more than 90 tiles have more than 5 raindrops on them.

X = # raindrops on a tile

$X \sim \text{Pois}(10)$

$p = P(X > 5) = 1 - P(X \le 5)$

$= \boxed{1 - \sum_{i=0}^{5} \frac{e^{-10}\,10^i}{i!}}$

use binomial $\text{Bin}(100, p)$

$$\boxed{\sum_{b=91}^{100} \binom{100}{b} p^b (1-p)^{100-b}}$$

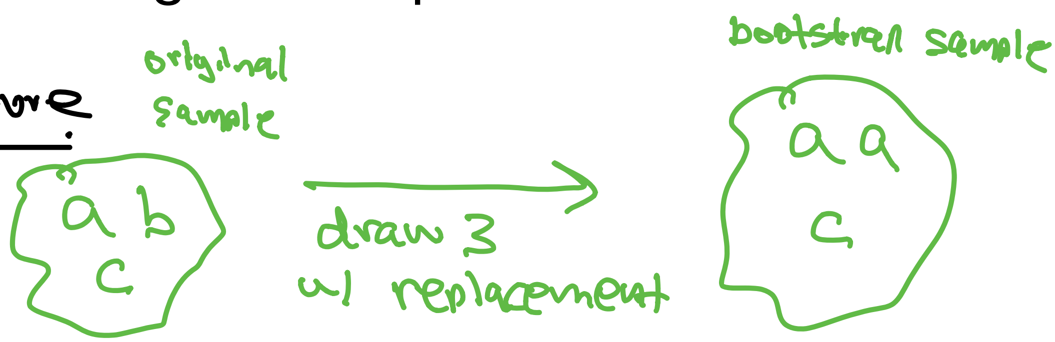
Last time

sec 4.3 <u>Exponential approximation</u>

ex (Bootstrap Sample)

A bootstrap sample is a sample of size n drawn at random with replacement from an original sample of n individuals.

Picture:



Our stat 88 class is a sample of Cal. Adam is in the 88 class of 200 People. Find the approximate chance a bootstrap sample of our class doesn't contain Adam?

$$x = \left(1 - \frac{1}{200}\right)^{200} \Rightarrow \log x = 200 \log\left(1 - \frac{1}{200}\right)$$

$$\approx 200\left(-\frac{1}{200}\right) = -1$$

$\log(1+\delta) \approx \delta$

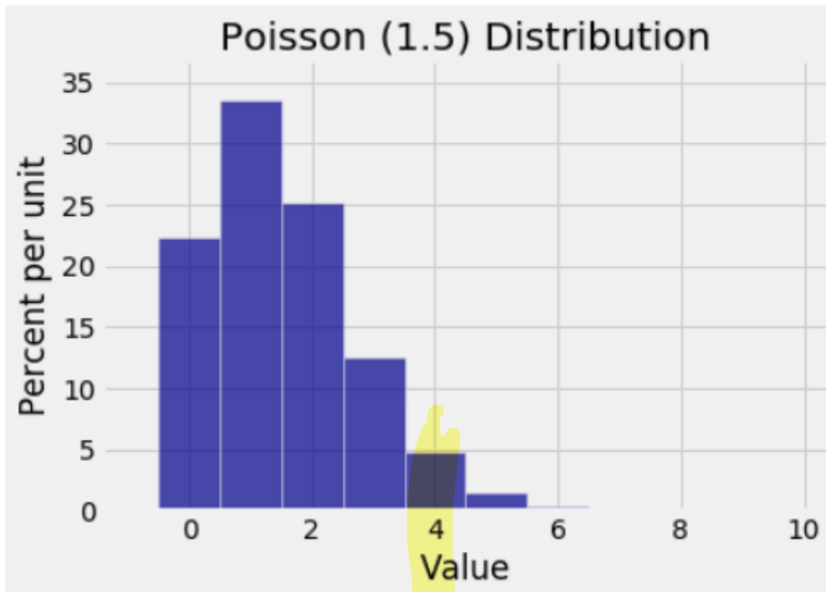
$$\Rightarrow \boxed{x \approx e^{-1}}$$

Sec 4.4 Poisson Distribution

X = # of times an event occurs

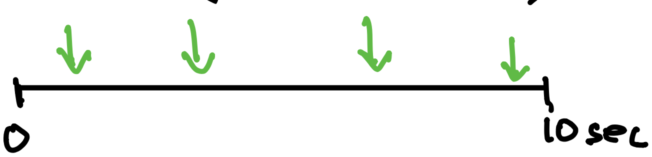
$X \sim \text{Pois}(\mu)$

$$P(X=k) = \frac{e^{-\mu}\mu^k}{k!} \qquad k = 0, 1, 2, 3, \ldots$$



ex # radioactive particles detected in 10 seconds follows a Poisson (1.5) distribution.

Picture



$$P(X=4) = \frac{e^{-1.5}\, 1.5^4}{4!} = .047$$

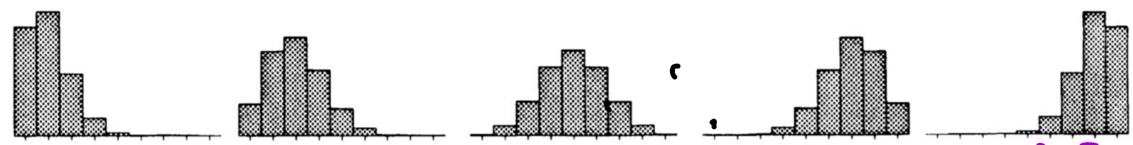
The exponential approximation is the key to showing that for large n and small p,

$$\text{Binomial}(n, p) \approx \text{Poisson}(\mu = np)$$

average number of successes

Binomial$(8, p)$

$n=8$



$p = 1/8$ $p = 1/4$ $p = 1/2$ $p = 3/4$ $p = 7/8$

this is approximately Poisson(1)

$\mu = np = 8\left(\frac{1}{8}\right) = 1$

Today

1. Sec 4.4 Poisson Distribution
2. Sec 5.1 Expectation

① Sums of independent poisson random variables

A useful property of the Poisson distribution is that if X and Y are random variables such that

- X and Y are independent,
- X has the Poisson (μ) distribution, and
- Y has the Poisson (λ) distribution,

then the sum $S = X + Y$ has the Poisson $(\mu + \lambda)$ distribution.

Lets use Poisson approx to Binomial to show this is true.

eg let X = # heads in 100 p coin tosses.
let Y = # heads in 200 p coin tosses.

$X+Y$ = # heads in 300 p coin toss

For p small,

$X \sim \text{Pois}(\underbrace{100p}_{=\mu})$

$Y \sim \text{Pois}(\underbrace{200p}_{=\lambda})$

$X+Y \sim \text{Pois}(\underbrace{300p}_{\mu+\lambda})$

ex

8. In the first hour that a bank opens, the customers who enter are of three kinds: those who only require teller service, those who only want to use the ATM, and those who only require special services (neither the tellers nor the ATM). Assume that the numbers of customers of the three kinds are independent of each other and also that:

- T = the number that only require teller service has the Poisson (6) distribution,
- A = the number that only want to use the ATM has the Poisson (2) distribution, and
- S = the number that only require special services has the Poisson (1) distribution.

Suppose you observe the bank in the first hour that it opens. In each part below, find the chance of the event described.

a) 12 customers enter the bank

$X = T + A + S$

$X \sim Pois(9)$

$P(X=12) = \boxed{\dfrac{e^{-9} 9^{12}}{12!}}$

b) customers do enter but none requires special services

and

$P(S=0 \text{ and } T+A>0)$

$= P(S=0)P(T+A>0)$

$= \boxed{e^{-1}(1-e^{-8})}$

c)

T = • the number that only require teller service has the Poisson (6) distribution,

A = • the number that only want to use the ATM has the Poisson (2) distribution, and

S = • the number that only require special services has the Poisson (1) distribution.

let $X = T + A + S$

Find $P(T=3 \mid X=10)$

$$P(A+S=7)$$

$$\frac{P(T=3 \text{ and } X=10)}{P(X=10)} = \frac{P(T=3)\,P(X=10 \mid T=3)}{P(X=10)}$$

$$\frac{\frac{\cancel{e^{-6}}\,6^3}{3!}\,\frac{\cancel{e^{-3}}\,3^7}{7!}}{\frac{\cancel{e^{-9}}\,9^{10}}{10!}} = \frac{10!}{3!\,7!}\,\frac{6^3\,3^7}{9^{10}}$$

$$\frac{10!}{3!\,7!} = \binom{10}{3}$$

$$= \binom{10}{3}\left(\frac{6}{9}\right)^3\left(\frac{3}{9}\right)^7$$

binomial $\left(10, \frac{6}{9}\right)$

avg # of teller service customers out of 9

Sec 5.1 Expectation

The *expectation* of a random variable X, denoted $E(X)$, is the average of the possible values of X weighted by their probabilities:

$$E(X) = \sum_{\text{all } x} x P(X = x)$$

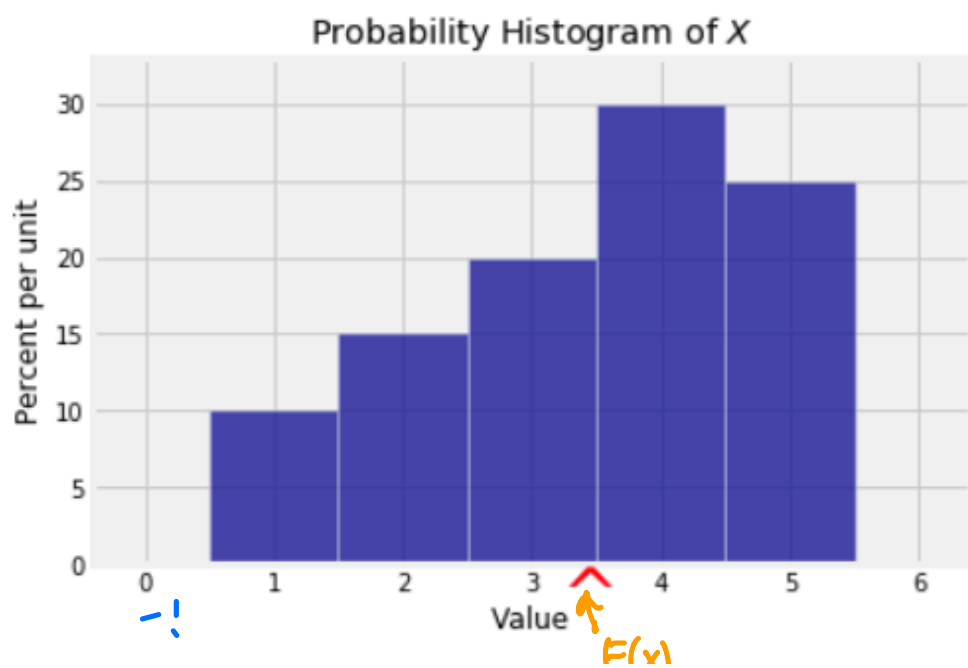
For example, suppose X has the following distribution table.

k	**1**	**2**	**3**	**4**	**5**
$P(X = k)$	0.1	0.15	0.2	0.3	0.25

Then

$$E(X) \;=\; 1(0.1) + 2(0.15) + 3(0.2) + 4(0.3) + 5(0.25) \;=\; 3.45$$

Here is the probability histogram of X with $E(X)$ marked in red on the horizontal axis.



$E(X)$ is the center of the distribution.

ex let X be the weight of a randomly selected person in Stat 88.

If there are n persons in the class we attach a probability of $\frac{1}{n}$ to each person's weight,

Then $E(X) = \sum_{i=1}^{n} \frac{1}{n} x_i = \frac{1}{n} \sum_{i=1}^{n} x_i$

is the average weight of the class.

Properties of Expectation.

1) $E(a) = a$ for constant a

2) $E(\underset{10}{\underset{\shortparallel}{a}}X + \underset{5}{\underset{\shortparallel}{b}}) = \underset{10}{\underset{\shortparallel}{a}}E(X) + \underset{5}{\underset{\shortparallel}{b}}$.

exercise 5.7.1

1. Let X have the distribution displayed in the table below.

x	-2	-1	0	1
$P(X = x)$	0.2	0.25	0.35	0.2

Find

a) $E(X)$ $= (.2)(-2) + (.25)(-1) + .35(0) + .2(1)$

$= -.45$

b) $E(X - 1)$ $= E(X) - 1 = -.45 - 1 = \boxed{-1.45}$

exercise 5.7.3

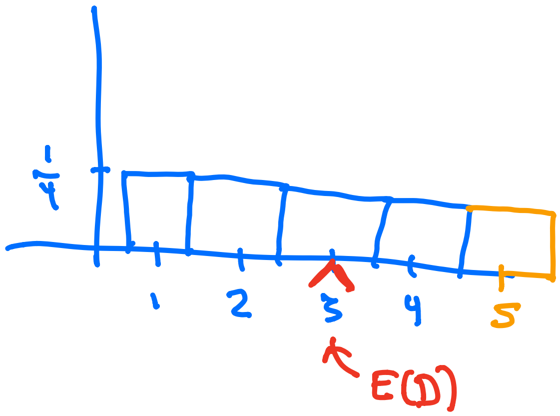
3. A box contains four blank index cards and one that has a gold star on it. Cards are drawn one by one at random without replacement until the gold star appears. Let D be the number of cards drawn.

(Misprint corrected)

a) Find the distribution of D.

1	2	3	4	5
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

b) Find $E(D)$.



$$E(D) = \frac{1}{5}(1+2+3+4+5) = \boxed{3}$$