$f(x)$ real-valued function s.t.

① $f(x) \geq 0$

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$

} then f is called a prob. density function (pdf), or "density"

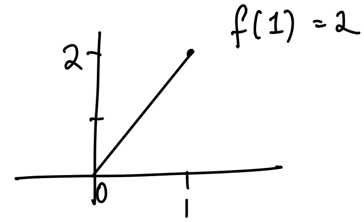
If X is a r.v. that takes any values in an interval, & $P(a < X < b) = \int_a^b f(x)\,dx$

then X is called a continuous r.v. & we say that X has density $f(x)$

$F(x) = \int_{-\infty}^{x} f(t)\,dt = P(X \leq x)$, $F(x)$ is the antiderivative of $f(x)$

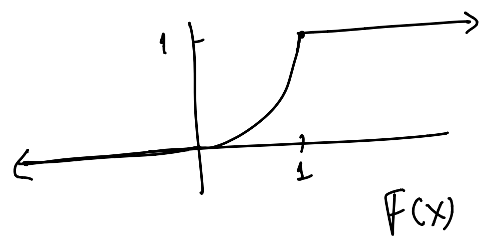
$F'(x) = \frac{d}{dx} F(x) = f(x)$

① $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{o/w ("other wise")} \end{cases}$



$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 2x\,dx = x^2\Big|_0^1 = 1$

$F(x) = \begin{cases} 0, & x \leq 0 \\ \int_0^x 2t\,dt, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$



$\int_0^x 2t\,dt = x^2, \quad 0 < x < 1$

$$F(5) = ? \quad \int_{-\infty}^{5} f(t)\,dt = \int_{-\infty}^{0} 0 \cdot dt + \int_{0}^{1} 2t\,dt + \int_{1}^{\infty} 0\,dt$$

$$= 0 + 1 + 0$$

If X is a r.v with density $f(x)$

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

discrete X.

$$\mathbb{E}(X) = \sum_{x} x \cdot P(X=x)$$

For X with density $f(x)$ defined above.

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot \underbrace{f(x)}_{\text{stand in for } P(X=x)}\,dx = \int_{0}^{1} x \cdot 2x\,dx = 2\int_{0}^{1} x^2\,dx = \underset{\mu}{\underset{\uparrow}{\frac{2}{3}}}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \mathbb{E}(X^2) - \mu^2$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

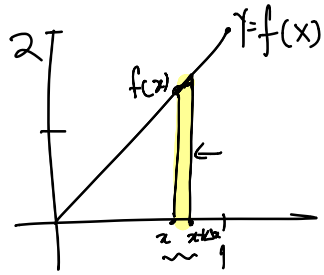
$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mu^2 = \mathbb{E}(X^2) - \left(\frac{2}{3}\right)^2$$

$$= \int_{0}^{1} x^2 \cdot 2x\,dx - \frac{4}{9} = 2\int_{0}^{1} x^3\,dx - \frac{4}{9}$$

$$= 2 \cdot \frac{x^4}{4}\Big|_{0}^{1} - \frac{4}{9} = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx$$

any function



$$P(x < X < x+\Delta x)$$

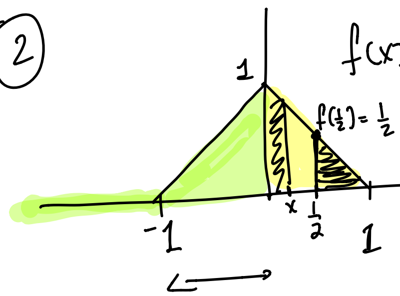
$$\approx f(x)\Delta x$$

$$\rightarrow f(x) \approx \frac{P(x<X<x+\Delta x)}{\Delta x}$$

data histograms from Data 8

$$\text{height of bin} = \frac{\%\text{ in bin}}{\text{width}}$$

(2)



$$f(x) = \begin{cases} 0, & x \le -1 \\ 1+x, & -1 < x < 0 \\ 1-x, & 0 \le x < 1 \\ 0, & x \geq 1 \end{cases}$$

Total area = area of both triangles

$$= \frac{1}{2} + \frac{1}{2} = 1 \qquad \left(\frac{1}{2} b \cdot h = \frac{1}{2} \cdot 1 \cdot 1\right)$$

$$P\left(\frac{1}{2} < X < 1\right) = \frac{1}{8} \qquad \left(\frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right)$$

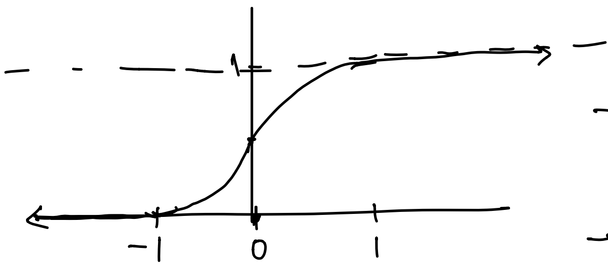
$$\int_{\frac{1}{2}}^{1} f(x)\,dx = \int_{\frac{1}{2}}^{1} (1-x)\,dx$$

$$F(x) = \begin{cases} 0, & x \le -1 \\ \int_{-1}^{x} (1+t)\,dt, & -1 < x < 0 \\ \frac{1}{2} + \int_{0}^{x} (1-t)\,dt, & 0 \le x < 1 \\ 1, & x \geq 1 \end{cases}$$

$$\longrightarrow \int_{-1}^{x} (1+t)\,dt = \frac{x^2+2x+1}{2}$$

$$\longrightarrow \frac{1}{2} + \frac{2x-x^2}{2}$$

$$= \frac{1+2x-x^2}{2}$$



$\rightarrow F(x)$ is increasing

$0 \le F(x) \le 1$

$\rightarrow F(x)$ is continuous

$\mathbb{E}(X)$, $Var(X)$ Exercise

Note : All the properties of $\mathbb{E}(X)$ & $Var(X)$ carry over.

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} f(x)\,dx, \quad Var(X) = \mathbb{E}(X^2) - \mu^2$$

$\mu = \mathbb{E}(X)$

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx.$$

① $\mathbb{E}(aX+b) = a\mathbb{E}(X)+b$

$Var(aX+b) = a^2\,Var(X)$

$SD(aX+b) = |a|\,SD(X)$

② $\mathbb{E}(aX+bY) = a\,\mathbb{E}(X) + b\,\mathbb{E}(Y)$

③ If for every interval A, B,

$P(X \in A \text{ and } Y \in B) = P(X \in A)P(Y \in B)$

say that X, Y are independent

(4) If X, Y are indep. $Var(X+Y) = Var(X) + Var(Y)$

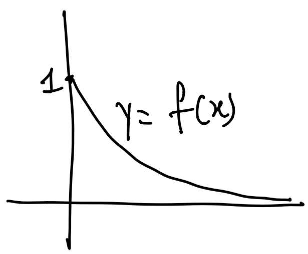
$$Var(X-Y) = Var(X) + Var(Y)$$

<u>Ex 3</u> $f(x) = e^{-x}, \; x > 0$, (0 everywhere else)

In this case X is called an exponential r.v. with rate 1. ($X \sim \exp(1)$)

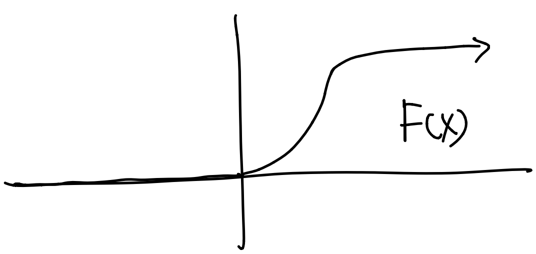
In general we say $X \sim \exp(\lambda)$

if $f(x) = \lambda e^{-\lambda x}, \; \lambda > 0, \; x > 0$ (λ is some constant)



$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x e^{-t}\,dt$$

$$= 1 - e^{-x}, \quad x > 0$$



$$\updownarrow$$

$$\lambda \int_0^x e^{-\lambda t} = \cancel{\lambda} \frac{e^{-\lambda t}}{-\cancel{\lambda}} \Big|_0^x$$

$\lambda > 0$

$f(x) = \lambda e^{-\lambda x}$

$$F(x) = \int_0^x \lambda e^{-\lambda t}\,dt = \underline{1 - e^{-\lambda x}}$$

We often denote the exponential r.v. by T

$$\int_{-\infty}^{\infty} f(t)\,dt = 1\,? = \int_0^{\infty} \lambda e^{-\lambda t}\,dt = \lambda \frac{e^{-\lambda t}}{-\lambda} \Big|_0^{\infty} = 1$$

$T \sim \exp(\lambda)$

$F(x) = 1 - e^{-\lambda x} = P(T \leq x)$

$P(T > x) = 1 - P(T \leq x) = 1 - (1 - e^{-\lambda x})$

$\boxed{P(T > x) = e^{-\lambda x}}$

Memoryless property of $T \sim \exp(\lambda)$

$P(T > s+t \mid T > t) = P(T > s)$

$\mathbb{E}(X) = \frac{1}{\lambda}$ ← use integration by parts

$Var(X) = \frac{1}{\lambda^2}$