① Review

② Final logistics

* Announcement :

① Midterm regrade request

~ Friday (10/23) 11:59 PM PT

② Quiz 6 : Ch 6

Ch6

How to compute Var(X), SD(X) given a dist. table?

If Y = aX+b, how to compute Var(X), SD(X)?

How to apply Markov's, Chebyshev's ineq?

After today's lecture

⇝ HW 8 Q5

STAT 88: Lecture 24

Contents

Warm up: Draw 6 cards and count the number of red cards you get. To increase your odds of getting 3 red cards should you draw with or without replacement?

Smaller SD ⇝ More chance of getting 3 red cards.

w/o replacement:

$X \sim HG(52, 26, 6)$

$P(X=3) = \frac{\binom{26}{3}\binom{26}{3}}{\binom{52}{6}} = 0.332$ ✓

w/ replacement:

$X \sim \text{Binomial}(6, \frac{1}{2})$.

$P(X=3) = \binom{6}{3}\left(\frac{1}{2}\right)^6 = 0.313$

Last time

If $X \sim \text{HG}(N, G, n)$, then

$$\text{SD}(X) = \sqrt{n \cdot \frac{G}{N} \cdot \frac{N-G}{N}} \cdot \underbrace{\sqrt{\frac{N-n}{N-1}}}_{\text{fpc}}.$$

fpc < 1 if n > 1

If $Y \sim \text{Binomial}(n, \frac{G}{N})$, we have

$$\text{SD}(X) = \text{SD}(Y) \cdot \text{fpc}.$$

This implies that

$$\text{SD}(X) < \text{SD}(Y).$$

When $n \ll N$, $SD(X) \approx SD(Y)$

7.3. The Law of Averages

Informally, the law of averages is the familiar statement that if you toss a coin many times you get about half heads and half tails.

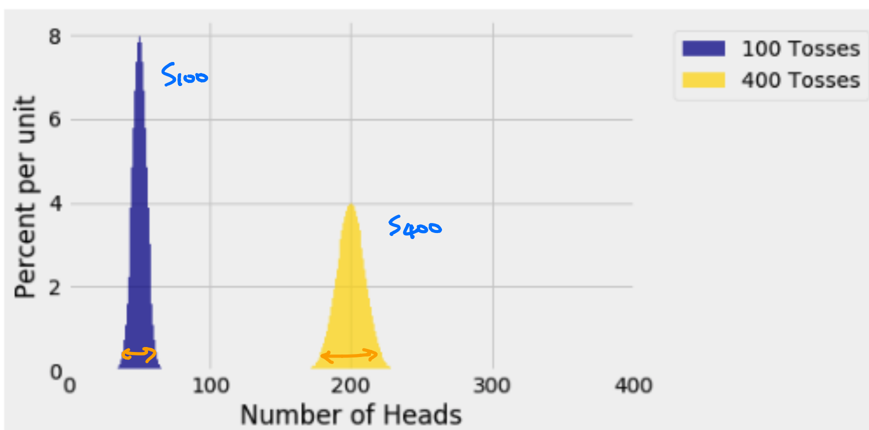
The Sample Sum

Let $X_1, \dots, X_n$ be i.i.d. samples from a population with mean μ and SD σ, and let $S_n = X_1 + X_2 + \cdots + X_n$. As a running example, let the population distribution be Bernoulli distribution, i.e. you toss a fair coin n times and the samples are

$$X_1, \dots, X_n \overset{\text{iid}}{\sim} \text{Bernoulli}\left(\frac{1}{2}\right).$$

$= \begin{cases} 1 & \text{if head} \\ 0 & \text{tail} \end{cases}$

Compare the distributions of S_{100} and S_{400}:



Observation: Both histograms have area 1. Yellow histogram is more spread out and hence $\underbrace{P(S_{400} = 200)}_{\text{Exactly half = heads}} < \underbrace{P(S_{100} = 50)}$.

We can see this mathematically:

$$\begin{aligned}
\text{Var}(S_n) &= \text{Var}(X_1 + \cdots + X_n) \\
&= \text{Var}(X_1) + \cdots + \text{Var}(X_n) \\
&= n\sigma^2,
\end{aligned}$$

and so

$$\text{SD}(S_n) = \sqrt{n}\sigma.$$

The Sample Average

As before, let $X_1, \ldots, X_n$ be i.i.d. samples from a population with mean μ and SD σ, and let $S_n = X_1 + X_2 + \cdots + X_n$. Let $\bar{X}_n = S_n/n$ be the sample average.

We can get

unbiased estimator.

$$E(\bar{X}_n) = \frac{1}{n}E(S_n) = \frac{1}{n}n\mu = \mu,$$
$$\mathrm{SD}(\bar{X}_n) = \frac{1}{n}\mathrm{SD}(S_n) = \frac{1}{n}\sqrt{n}\sigma = \frac{\sigma}{\sqrt{n}}.$$

① As n increases, $SD(\bar{X}_n) \to 0$
② $SD(\bar{X}_n)$ decreases in the rate of $\frac{1}{\sqrt{n}}$

The Square Root Law

S : unbiased estimator.
accuracy = SD(S)

In the language of estimation, the accuracy of an unbiased estimator can be measured by its SD: the smaller the SD, the more accurate the estimator.

For example,

$$\mathrm{SD}(\bar{X}_{400}) = \frac{\sigma}{\sqrt{400}} = \frac{\sigma}{\sqrt{4}\sqrt{100}} = \frac{1}{2}\mathrm{SD}(\bar{X}_{100}).$$

$= SD(\bar{X}_{100})$

So $\bar{X}_{400}$ is *twice* as accurate as $\bar{X}_{100}$. For double the accuracy, we have to multiply the sample size by a factor of $2^2 = 4$.

If you multiply the sample size by a factor, the accuracy only goes up by the square root of the factor. This is called the **square root law**.

$n \to 4 \cdot n$

$\frac{\sigma}{\sqrt{n}} \to \frac{\sigma}{2\sqrt{n}}$

Population dist'n. μ, σ

$n \to 9 \cdot n$

$\frac{\sigma}{\sqrt{n}} \to \frac{\sigma}{3\sqrt{n}}$

Concentration of Probabilities

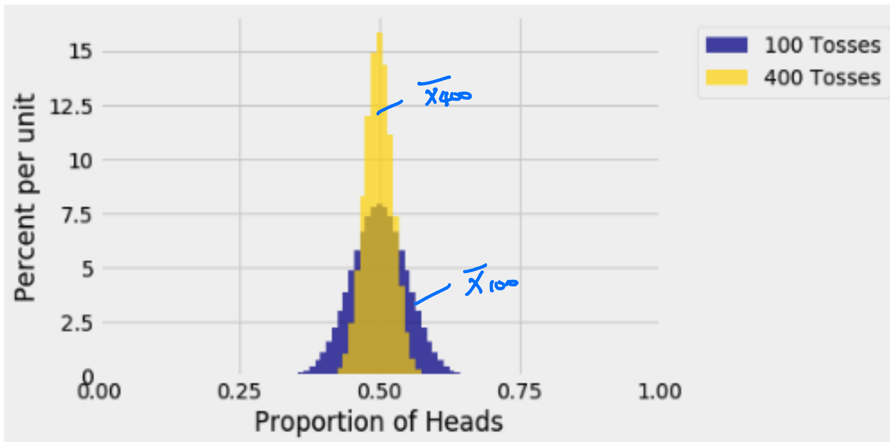
$\sigma = \sqrt{\frac{1}{2}\cdot\frac{1}{2}} = \frac{1}{2}$

You toss a fair coin n times. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Bernoulli}\left(\frac{1}{2}\right)$.

Compare the distributions of $\bar{X}_{100}$ and $\bar{X}_{400}$:

$E(\bar{X}_{100}) = E(\bar{X}_{400}) = \frac{1}{2}$

$SD(\bar{X}_{100}) = \frac{1}{2}\cdot\frac{1}{\sqrt{100}}$, $SD(\bar{X}_{400}) = \frac{1}{2}\cdot\frac{1}{\sqrt{400}}$



Observation: Both histograms balance at 0.5. Yellow histogram is more concentrated around 0.5.

In general, the larger the sample size n, the more likely it is that the sample average $\bar{X}_n$ will be close to the population average μ. = $SD(\bar{X}_n)$ is smaller.

Weak Law of Large Numbers: formally, for a fixed $c > 0$,

$$P(\mu - c < \bar{X}_n < \mu + c) = P(|\bar{X}_n - \mu| < c) \to 1 \text{ as } n \to \infty.$$

c = 0.001, 0.000001

Interpretation: no matter how small c is, the chance that the sample mean is in the interval $(\mu - c, \mu + c)$ increases to 1 as the sample size grows.

Pf $\quad \underbrace{P(|\bar{X}_n - \mu| < c) = 1 - \underbrace{P(|\bar{X}_n - \mu| \geq c)}_{\leq \frac{Var(\bar{X}_n)}{c^2} = \frac{\sigma^2}{n\cdot c^2} \to 0 \text{ as } n\to\infty}}_{\to 1 \text{ as } n \to \infty.}$ $\square$

[Strong law of large numbers:

$P(\lim_{n\to\infty} \bar{X}_n = \mu) = 1$]

The Law of Averages

Let $X_1, \dots, X_n \overset{\text{iid}}{\sim} \text{Bernoulli}(p)$. Let $\bar{X}_n$ be the sample mean or sample proportion of 1s in your sample. The law of averages (=the law of large numbers) is: for each fixed $c > 0$,

$$P(p - c < \bar{X}_n < p + c) = P(|\bar{X}_n - p| < c) \to 1 \text{ as } n \to \infty.$$

Interpretation: when the sample size is large, the sample proportion of 1's is hugely likely to be in a small interval around p.

In a large number of rolls, it is hugely likely that the observed proportion of times the face appears is close to 1/6.