* Announcement

① HW 12 due today (11:59 PM PT)

② HW 13 Part I ~ 12/7

③ No quiz this week

HW 13 Part I

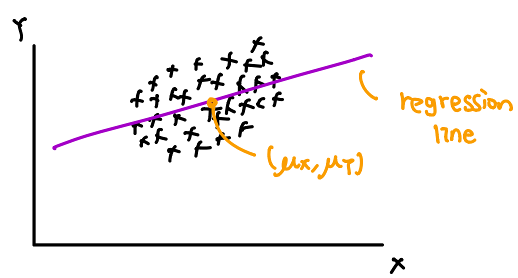
Q1, Q2 ← Ch 11.3 ~ Ch 11.5

STAT 88: Lecture 37

Contents

Warm up:

Let (X, Y) be a random pair. The average of this pair is (μ_X, μ_Y), called the point of averages. Show that the point of averages lies on the regression line.



$\hat{y} = \hat{a}x + \hat{b}$

Plug-in $x = \mu_X$

$\Rightarrow \hat{a}\mu_X + \hat{b} = \hat{a}\mu_X + (\mu_Y - \hat{a}\mu_X)$

$= \mu_Y$

(μ_X, μ_Y) is on the regression line $\hat{y} = \hat{a}x + \hat{b}$

Last time

Least squares regression

Let (X, Y) be a random pair. We write

- $E(X) = \mu_X$, $\mathrm{SD}(X) = \sigma_X$.
- $E(Y) = \mu_Y$, $\mathrm{SD}(Y) = \sigma_Y$.
- Correlation

cov.

$$r = \frac{E((X-\mu_X)(Y-\mu_Y))}{\sigma_X \sigma_Y}.$$

The regression line $\widehat{Y} = \widehat{a}X + \widehat{b}$ is the best fitting line, in the sense that it minimizes the mean squared error

$$\mathrm{MSE}(a, b) = E((Y - (aX + b))^2).$$

We showed that

$$\widehat{a} = r\frac{\sigma_Y}{\sigma_X} \text{ and } \widehat{b} = \mu_Y - \widehat{a} \cdot \mu_X.$$

Correlation

Let $X^\star$ be X in standard units and $Y^\star$ be Y in standard units:

$$X^\star = \frac{X - \mu_X}{\sigma_X} \text{ and } Y^\star = \frac{Y - \mu_Y}{\sigma_Y}.$$

Then

$$r = r(X, Y) = \frac{E((X-\mu_X)(Y-\mu_Y))}{\sigma_X \sigma_Y} = E\left(\frac{X-\mu_X}{\sigma_X} \cdot \frac{Y-\mu_Y}{\sigma_Y}\right) = E(X^\star Y^\star).$$

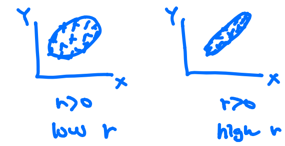
The correlation satisfies the following properties:

- $-1 \le r(X, Y) \le 1$.
- $r(X, Y) = r(Y, X)$.
- $r(aX + b, cY + d) = \begin{cases} r(X, Y) & \text{if } ac > 0 \\ -r(X, Y) & \text{if } ac < 0 \end{cases}$

11.5. The Error in Regression

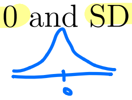
r As a Measure of Linear Association The error in the regression estimate is called the residual and is defined as

$$D = Y - \widehat{Y}.$$



We showed

$$E(D) = 0 \text{ and } \mathrm{SD}(D) = \sqrt{1 - r^2}\,\sigma_Y.$$



$\hat{a}X + \hat{b}$

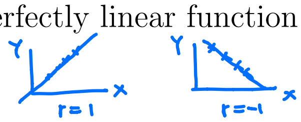
$\|$

So if r is close to ± 1, $\mathrm{SD}(D)$ is close to 0, which implies that Y is close to $\widehat{Y}$. In other words, Y is close to being a linear function of X. ← r measures strength of linear relationship between X and Y

In the extreme case $r = \pm 1$, $\mathrm{SD}(D) = 0$ and Y is a perfectly linear function of X.

$E(D)=0 \Rightarrow D = Y - \hat{Y} = 0$

$\Rightarrow Y = \hat{Y}$



The Residual is Uncorrelated with X We will show that the correlation between X and residual D is zero. Note that

$$r(D, X) = \frac{E((D - \overset{=0}{\mu_D})\overbrace{(X - \mu_X)}^{D_X})}{\sigma_D \sigma_X} = \frac{1}{\sigma_D \sigma_X} E(D D_X),$$

because $\mu_D = 0$. We thus show $E(DD_X) = 0$:

$$\begin{aligned} E(DD_X) &= E((D_Y - \hat{a}D_X)D_X) \\ &= E(D_X D_Y) - \hat{a}E(D_X^2) \\ &= r\sigma_X\sigma_Y - r\frac{\sigma_Y}{\sigma_X}\sigma_X^2 \\ &= 0. \end{aligned}$$

additivity & linearity

$$\begin{aligned} D &= Y - \hat{Y} \\ &= Y - (\hat{a}X + \hat{b}) \\ &= Y - (\hat{a}X + \mu_Y - \hat{a}\mu_X) \\ &= \underbrace{(Y - \mu_Y)}_{D_Y} - \hat{a}\underbrace{(X - \mu_X)}_{D_X} \\ &= D_Y - \hat{a}D_X \end{aligned}$$

$$\begin{aligned} E(D_X^2) &= E((X - \mu_X)^2) \\ &= Var(X) \\ &= \sigma_X^2 \end{aligned}$$

After you subtract $\hat{Y}$ from Y,

it should contain no information about linear function of X.

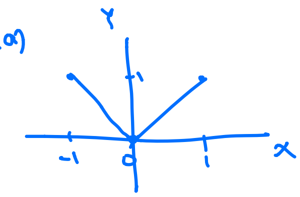
$\Rightarrow r(\underbrace{Y - \hat{Y}}_{= D}, X) = 0$

Example: (Exercise 11.6.11) Let X have the uniform distribution on the three points -1, 0, and 1. Let $Y = X^2$.

(a) Show that X and Y are uncorrelated.

(b) Are X and Y independent?

(a)



$X \sim \text{Unif}\{-1, 0, 1\}$

$E(X) = 0$

Want to show: $r(X,Y) = 0$

$$r(X,Y) = \frac{E(D_X D_Y)}{\sigma_X \sigma_Y}$$

Show $E(D_X D_Y) = 0$:

$$\begin{aligned} E(D_X D_Y) &= E((X-\mu_X)(Y-\mu_Y)) \\ &= E(X(Y-\mu_Y)) \quad (\because \mu_X = 0) \\ &= E(XY) - E(\mu_Y X) \\ &= E(XY) - \mu_Y \underbrace{E(X)}_{=0} \\ &= E(X^3) \quad (\because Y = X^2) \\ &= E(X) = 0 \end{aligned}$$

(b) $Y = X^2$ so X and Y are dependent.

$$\begin{cases} P(X=1, Y=1) = \frac{1}{3} \\ P(X=1) = \frac{1}{3}, \; P(Y=1) = \frac{2}{3}. \end{cases} \qquad P(X=1)P(Y=1) \neq P(X=1, Y=1)$$

Conclusion

$$\begin{cases} \text{Uncorrelated} \not\Rightarrow \text{independent} \\ \text{independent} \Rightarrow \text{Uncorrelated} \end{cases}$$

Ch11: linear regression in pop. version
Ch12: " in finite sample version.

12.1. The Simple Linear Regression Model

The model involves a variable called the response and another called a predictor variable or feature.

$$Y = \text{response} \quad \text{and} \quad x = \text{predictor variable/covariate/feature}$$

big Y (random) Small x (fixed / not random)

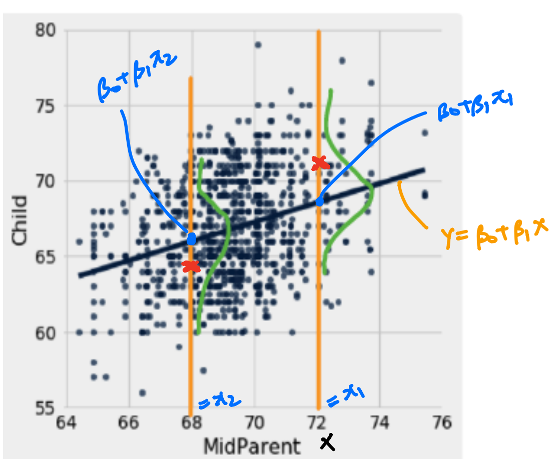
We observe the response and the predictor variable of n individuals, i.e. $(x_1, Y_1), \ldots, (x_n, Y_n)$. Our assumption is

$$Y_i = \underbrace{\beta_0 + \beta_1 x_i}_{\text{signal}} + \underbrace{\epsilon_i}_{\text{noise}},$$

where

- β_0 and β_1 are unobservable constant parameters.
- x_i is the value of the predictor variable for individual i and is assumed to be constant (that is, not random).
- The errors $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are i.i.d. normal $\mathcal{N}(0, \sigma^2)$ random variables.
- The error variance σ^2 is an unobservable constant parameter, and is assumed to be the same for all individuals i.

Ch11: (X,Y) random pair



Data generation process

Pop of X (temperature) → Generate $Y = \beta_0 + \beta_1 X + \varepsilon$ (thermometer) where $\varepsilon \sim N(0, \sigma^2)$ → Pop distn of (X,Y)

↓ Draw n samples

$X_1, X_2, \ldots, X_n$

↳ $x_1, x_2, \ldots, x_n$ → Generate $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ where $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$

Fix (Fix temperature on certain days)

↓ (x_1, Y_1), (x_2, Y_2), $\vdots$, (x_n, Y_n)

Individual Responses Fix x_i, then

only source of randomness
$\sim N(0, \sigma^2)$

$$Y_i = \underbrace{\beta_0 + \beta_1 x_i}_{\text{fixed/constant}} + \epsilon_i \sim \mathcal{N}(\ \beta_0 + \beta_1 x_i\ ,\ \sigma^2\ ).$$

$Y_1, Y_2, \ldots, Y_n$ are independent because $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are independent.

Average Response Let $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ denote the average response of the n individuals.

What distribution does $\bar{Y}$ follow? Becaus Y_i's are indep. and from Normal dist'n, $\bar{Y}$ follows normal distribution. (it is exact, not approximate)

Find $E(\bar{Y})$.

$$\begin{aligned} E(\bar{Y}) &= E\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} E(Y_i) \\ &= \frac{1}{n}\sum_{i=1}^{n} (\beta_0 + \beta_1 x_i) \\ &= \beta_0 + \beta_1 \bar{x} \quad \left(\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i\right) \end{aligned}$$

Find $\mathrm{Var}(\bar{Y})$.

$$\begin{aligned} Var(\bar{Y}) &= Var\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) \\ &= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} Y_i\right) \\ &= \frac{1}{n^2}\sum_{i=1}^{n} Var(Y_i) \\ &= \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n} \end{aligned}$$

$$\Rightarrow \bar{Y} \sim N\left(\beta_0 + \beta_1 \bar{x}, \frac{\sigma^2}{n}\right)$$