Stat 88: Probability & Mathematical Statistics in Data Science

Lecture 12: 2/17/2021

Sums of Poisson rvs, Expectation, Functions of rvs

Sections 4.4, 5.1, 5.2

Agenda

- 4.4
    - Recall Poisson pmf, see connection to binomial
    - Ex. 4.5.7

    - Sums of Poisson random variables
    - Ex. 4.5.8

- 5.1 Expectation

- 5.2 Functions of random variable
    - Expectation
    - Joint distributions

The Poisson Distribution, and the Binomial Distribution

- Used to model rare events. X is the number of times a rare event occurs, X = 0, 1, 2, …
- We say that a random variable X has the Poisson distribution if

$$\boldsymbol{P(X = k) = e^{-\mu}\frac{\mu^k}{k!}}$$

- The parameter of the distribution is μ

Can think of μ as a rate: # of times an event happens in some time period.

- ***The Law of Small Numbers***: when $\boldsymbol{n}$ is large and $\boldsymbol{p}$ is small, the binomial (n,p) distribution is *well approximated* by the Poisson(μ) distribution where $\boldsymbol{\mu=np}$.

$X \sim \text{Pois}(\mu)$

$P(X=k) = e^{-\mu}\frac{\mu^k}{k!}$, $k=0,1,2.\ .\ .$

$Y = \text{Bin}(n,p)$

$P(Y=k) = \binom{n}{k} p^k (1-p)^{n-x}$

$k=0,\ldots\ n.$

Let n be large, p small, $\mu = np$.

$$P(X=0) = e^{-\mu}\frac{\mu^0}{0!} = e^{-\mu} \quad , P(Y=0) = \binom{n}{0} p^0 (1-p)^n = (1-p)^n \approx e^{-np} = e^{-\mu}$$

$$P(X=1) = e^{-\mu}\frac{\mu^1}{1!} = \mu e^{-\mu}$$

$$P(Y=1) = \binom{n}{1} p^1 (1-p)^{n-1}$$

$$= \underbrace{\overset{=\mu}{\overbrace{np}} \frac{\overbrace{(1-p)^n}^{\approx e^{-np} = e^{-\mu}}}{\underbrace{1-p}_{\approx 1}}}_{\approx \mu e^{-\mu}} = P(X=1)$$

$$P(X=2) = e^{-\mu}\frac{\mu^2}{2!} = \frac{e^{-\mu}\mu^2}{2}$$

$$P(Y=2) = \binom{n}{2} p^2 (1-p)^{n-2} = \frac{\overbrace{n(n-1)}^{n^2}}{2} p^2 \frac{(1-p)^n}{\underbrace{(1-p)^2}_{\approx 1}} = \frac{\overbrace{(np)^2}^{=\mu^2}\overbrace{(1-p)^n}^{\approx e^{-\mu}}}{2}$$

$$\approx \frac{\mu^2 e^{-\mu}}{2} = P(X=2)$$

If is n is large, p small then the binomial (n,p) probabilities are approximated very well by Poisson probabilities, $\mu = np$.

As $n \to \infty$, $p \to 0$ & $np = \mu$

$$\text{the } \underbrace{P(Y=k)}_{\text{Pois}(\mu)} \longrightarrow \underbrace{P(X=k)}_{\text{Bin}(n,p)}$$

Exercise 4.5.7

A book has 20 chapters. In each chapter the number of misprints has the Poisson distribution with parameter 2, independently of the misprints in other chapters.

a) Find the chance that Chapter 1 has more than two misprints.

b) Find the chance that the book has no misprints.

c) Find the chance that two of the chapters have three misprints each.

will be done in exam prep. on Friday

Sums of independent Poisson random variables

- If X and Y are random variables such that
- X and Y are independent,
- X has the Poisson(μ) distribution, and
- Y has the Poisson(λ)distribution,
- then the sum $S=X+Y$ has the Poisson $(\mu+\lambda)$ distribution.

$$S = X+Y \sim \text{Poiss}(\mu+\lambda)$$

Exercise 4.5.8

In the first hour that a bank opens, the customers who enter are of **three** kinds: those who only require teller service, those who only want to use the ATM, and those who only require special services (neither the tellers nor the ATM). Assume that the numbers of customers of the three kinds are independent of each other, and also that:

- the number that only require teller service has the Poisson (6) distribution, $X \sim \text{Pois}(6) \leftarrow$

- the number that only want to use the ATM has the Poisson (2) distribution, and $Y \sim \text{Pois}(2) \leftarrow$

- the number that only require special services has the Poisson (1) distribution. $Z \sim \text{Pois}(1) \leftarrow$

Suppose you observe the bank in the first hour that it opens. In each part below, find the chance of the event described.

$W = X+Y+Z \sim \text{Pois}(9)$

$$P(W=k) = e^{-\mu}\frac{\mu^k}{k!}$$

a) 12 customers enter the bank

b) more than 12 customers enter the bank

c) customers do enter but none requires special services

(a) $P(W=12)$ $\quad k=12,\ \mu=9$

$$= e^{-9}\cdot\frac{9^{12}}{12!}$$

(b) $P(W>12) = 1-P(W\leq 12) = 1-F(12)$

$\text{stats.poisson.cdf}(\overset{k}{\overset{\downarrow}{12}}, \overset{\mu}{\overset{\downarrow}{9}})$

$P(W \geq 12) \leftarrow$ at least 12 or 12 or more.

$$F(12) = \sum^{12} f(k)$$

$k=0$

$$\sum_{k=0}^{12} e^{-\mu} \frac{\mu^k}{k!}$$

$X+Y \sim \text{Pois}(8)$ $\quad P(X+Y=0) = e^{-8}$

$Z \sim \text{Pois}(1)$ $\quad P(Z=0) = e^{-1}$

(c) $$P(Z=0 \underset{\substack{\uparrow \\ \text{"and"}}}{,} X+Y>0) = \underbrace{P(Z=0)}_{e^{-1}} \underbrace{P(X+Y>0)}_{1-e^{-8}}$$

$$P(X+Y>0) = 1 - P(X+Y=0)$$

$$P(X+Y>0) = 1 - P(\underline{X+Y \leq 0})$$

But X & Y are nonnegative

$\Rightarrow X \geq 0, \ Y \geq 0 \Rightarrow \underline{X+Y \geq 0}$

$$P(X+Y \leq 0) = P(X+Y=0) + \underbrace{P(X+Y<0)}_{=0}$$

$$P(X=Y) = \underbrace{P(X=Y=0)} + P(X=Y=1) + - \quad -$$

5.1: Expected Value of a random variable

X, f, F

$E(X) \quad E(X)$

- The ***Expectation*** or ***Expected Value*** of a random variable X is defined to be the sum of all the products $(x \times f(x))$ over all possible values x of the random variable X: that is, the expectation of a random variable is a ***weighted average*** of all the possible values that the rv can take, weighted by the probability of each value.

$$\sum_x x f(x)$$

- $E(X) = \sum x \cdot P(X = x) = \sum x \cdot f(x)$

- For example, toss a coin 3 times, let X = # of heads. Write down $f(x)$, and then use the formula to compute the expectation of X.

x	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

$E(X) = 1.5$

$$E(X) = \sum_x x \cdot P(X=x)$$

$$= 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) + 3 \cdot P(X=3)$$

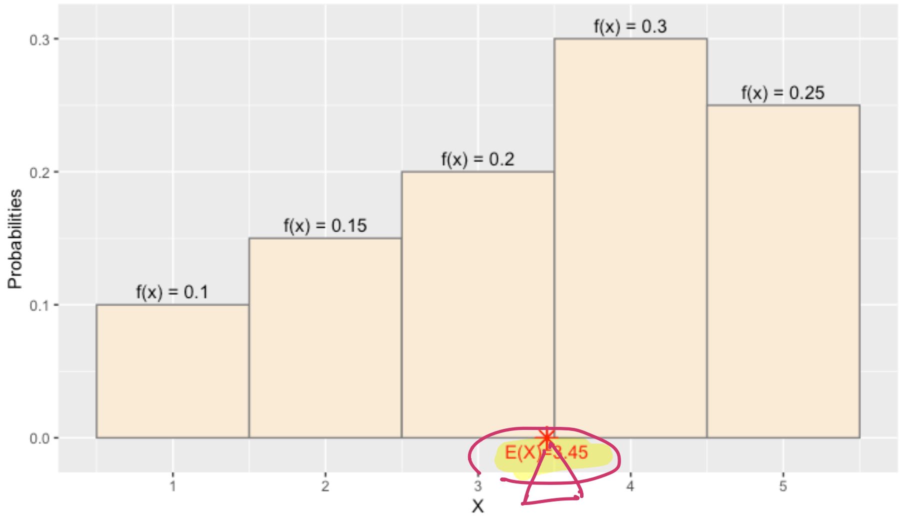
$$= 0 + 1(3/8) + 2 \cdot 3/8 + 3 \cdot 1/8 = 12/8$$

- Note: If X takes finitely many values, no problem. If it takes infinitely (countable, but not finite) many values (such as Poisson, or Geometric), then we have to be more careful when we take the sums.

Example (from text)

$$E(X) = \sum_{x} x \cdot P(X=x)$$

x	1	2	3	4	5
f(x)	0.1	0.15	0.2	0.3	0.25



Exp. value is the center of mass of the density

Notes about E(X)

- Same units as X
- Not necessarily an attainable value (for example, in In 2020, there was an average of 1.93 children under 18 per family in the United States)
- Expectation is a long-run average value of X

$$\mathbb{E}(X) = \sum_x x \cdot P(X=x)$$

- Center of mass or center of gravity (balancing point) for the distribution
- Expectation of a constant is that constant $E(c) = c$

If c is a const.
& r.v. X=c

$$P(X=c) = 1$$

$$\mathbb{E}(X) = \sum_x x \cdot P(X=x)$$

$$= c \cdot P(X=c)$$

$$= 1$$

- Linearity: $E(aX + b) = aE(X) + b$
- Example: for the X defined in the previous slide, find E(2X-1)

$$E(aX) = \sum_x ax \cdot P(X=x) = a \sum x P(X=x)$$

$$= a \cdot \mathbb{E}(X)$$

Special examples

$X \sim \text{Bernoulli}(p)$ if $X = \begin{cases} 0 & \text{w.p } 1-p \\ 1 & \text{w.p } p. \end{cases}$

- Bernoulli (Indicators)

$$\mathbb{E}(X) = 0 \cdot (1-p) + 1 \cdot p$$
$$= p = P(X=1)$$

We have special Bernoulli r.v. called 'Indicators'. Let A be an event $P(A) = p$.

Define $I_A = \begin{cases} 1, & \text{if } A \text{ occurs} \\ 0, & \text{o/w.} \end{cases}$

$$\mathbb{E}(I_A) = 1 \cdot P(I_A = 1) + 0 \cdot P(I_A = 0)$$
$$= 1 \cdot P(A)$$

- Uniform

- Poisson