第二章

最小二乘估计（二）

Outline

渐近理论基础

Various Modes of Convergence

定义 (Convergence in Probability)

A sequence of random scalars $\{z_n\}$ converges in probability to a constant α if, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathrm{Prob}(|z_n - \alpha| > \varepsilon) = 0.$$

The constant α is called the probability limit of z_n and is written as $\operatorname*{plim}_{n\to\infty} z_n = \alpha$ or $z_n \underset{p}{\to} \alpha$.

A sequence of K-dimensional random vectors $\{\mathbf{z}_n\}$ converges in probability to a K-dimensional vector of constants $\boldsymbol{\alpha}$ if, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathrm{Prob}(|z_{nk} - \alpha_k| > \varepsilon) = 0 \quad \text{for all } k = 1, 2, \ldots, K,$$

where z_{nk} is the k-th element of $\mathbf{z}_n$ and α_k the k-th element of $\boldsymbol{\alpha}$.

定义 (Almost Sure Convergence)

A sequence of random scalars $\{z_n\}$ converges almost surely to a constant α if

$$\text{Prob}\left(\lim_{n\to\infty} z_n = \alpha\right) = 1.$$

We write this as $z_n \underset{a.s.}{\to} \alpha$. The extension to random vectors is analogous to that for convergence in probability.

定义 (Convergence in Mean Square)

A sequence of random scalars $\{z_n\}$ converges in mean square (or in quadratic mean) to a constant α (written as $z_n \underset{m.s.}{\to} \alpha$) if

$$\lim_{n\to\infty} \mathbb{E}\left[(z_n - \alpha)^2\right] = 0.$$

Convergence to a Random Variable

We say that a sequence of K-dimensional random variables $\{\mathbf{z}_n\}$ converges in probability to a K-dimensional random variable $\mathbf{z}$ and write $\mathbf{z}_n \underset{p}{\to} \mathbf{z}$ if $\{\mathbf{z}_n - \mathbf{z}\}$ converges in probability to $\mathbf{0}$:

$$\text{“}\mathbf{z}_n \underset{p}{\to} \mathbf{z}\text{”} \quad \text{is the same as} \quad \text{“}\mathbf{z}_n - \mathbf{z} \underset{p}{\to} \mathbf{0}.\text{”}$$

Similarly,

$$\text{“}\mathbf{z}_n \underset{a.s.}{\to} \mathbf{z}\text{”} \quad \text{is the same as} \quad \text{“}\mathbf{z}_n - \mathbf{z} \underset{a.s.}{\to} \mathbf{0}.\text{”}$$

$$\text{“}\mathbf{z}_n \underset{m.s.}{\to} \mathbf{z}\text{”} \quad \text{is the same as} \quad \text{“}\mathbf{z}_n - \mathbf{z} \underset{m.s.}{\to} \mathbf{0}.\text{”}$$

定义 (Convergence in Distribution)

Let $\{z_n\}$ be a sequence of random scalars and F_n be the cumulative distribution function (c.d.f) of z_n. We say that $\{z_n\}$ converges in distribution to a random scalar z if the c.d.f. F_n of z_n converges to the c.d.f. F of z at every continuity point of F. We write $z_n \underset{d}{\rightarrow} z$ or $z_n \underset{L}{\rightarrow} z$ and call F the asymptotic (or limit or limiting) distribution of z_n.

定理 (Multivariate Convergence in Distribution Theorem)

Let $\{\mathbf{z}_n\}$ be a sequence of K-dimensional random vectors. Then:

$$\mathbf{z}_n \underset{d}{\rightarrow} \mathbf{z} \;\Leftrightarrow\; \boldsymbol{\lambda}'\mathbf{z}_n \underset{d}{\rightarrow} \boldsymbol{\lambda}'\mathbf{z} \text{ for any } K\text{-dimensional vector of real numbers } \boldsymbol{\lambda}.$$

Remark: For convergence in distribution, element-by-element convergence does not necessarily mean convergence for the vector sequence.

定理 (Relationship among the four modes of convergence)

(a) $\mathbf{z}_n \underset{m.s.}{\rightarrow} \boldsymbol{\alpha} \;\Rightarrow\; \mathbf{z}_n \underset{p}{\rightarrow} \boldsymbol{\alpha}, \quad \mathbf{z}_n \underset{m.s.}{\rightarrow} \mathbf{z} \;\Rightarrow\; \mathbf{z}_n \underset{p}{\rightarrow} \mathbf{z}.$

(b) $\mathbf{z}_n \underset{a.s.}{\rightarrow} \boldsymbol{\alpha} \;\Rightarrow\; \mathbf{z}_n \underset{p}{\rightarrow} \boldsymbol{\alpha}, \quad \mathbf{z}_n \underset{a.s.}{\rightarrow} \mathbf{z} \;\Rightarrow\; \mathbf{z}_n \underset{p}{\rightarrow} \mathbf{z}.$

(c) $\mathbf{z}_n \underset{p}{\rightarrow} \boldsymbol{\alpha} \;\Leftrightarrow\; \mathbf{z}_n \underset{d}{\rightarrow} \boldsymbol{\alpha}.$ *(That is, if the limiting random variable is a constant [a trivial random variable], convergence in distribution is the same as convergence in probability.)*

Viewing Estimators as Sequences of Random Variables

Let $\hat{\boldsymbol{\theta}}_n$ be an estimator of a parameter vector $\boldsymbol{\theta}$ based on a sample of size n. We say that an estimator $\hat{\boldsymbol{\theta}}_n$ is consistent for $\boldsymbol{\theta}$ if $\hat{\boldsymbol{\theta}}_n \underset{p}{\rightarrow} \boldsymbol{\theta}$.

Continuous Mapping Theorem

定理 (Continuous Mapping Theorem)

Let $\mathbf{a}(\cdot)$ *be a vector-valued function that does not depend on* n.

(a) *Suppose* $\mathbf{a}(\cdot)$ *is continuous at* $\boldsymbol{\alpha}$. *Then,* $\mathbf{z}_n \underset{p}{\to} \boldsymbol{\alpha} \;\Rightarrow\; \mathbf{a}(\mathbf{z}_n) \underset{p}{\to} \mathbf{a}(\boldsymbol{\alpha})$.

Stated differently,

$$\operatorname*{plim}_{n\to\infty} \mathbf{a}(\mathbf{z}_n) = \mathbf{a}(\operatorname*{plim}_{n\to\infty} \mathbf{z}_n).$$

(b) *Suppose* $\mathbf{a}(\cdot)$ *is continuous almost everywhere. Then,*

$$\mathbf{z}_n \underset{d}{\to} \mathbf{z} \;\Rightarrow\; \mathbf{a}(\mathbf{z}_n) \underset{d}{\to} \mathbf{a}(\mathbf{z}).$$

Slutsky's Theorem

定理 (Slutsky's Theorem)

(a) "$\mathbf{x}_n \underset{d}{\rightarrow} \mathbf{x},\ \mathbf{y}_n \underset{p}{\rightarrow} \boldsymbol{\alpha}$" $\Rightarrow$ "$\mathbf{x}_n + \mathbf{y}_n \underset{d}{\rightarrow} \mathbf{x} + \boldsymbol{\alpha}$."

(b) "$\mathbf{x}_n \underset{d}{\rightarrow} \mathbf{x},\ \mathbf{y}_n \underset{p}{\rightarrow} \mathbf{0}$" $\Rightarrow$ "$\mathbf{y}_n' \mathbf{x}_n \underset{p}{\rightarrow} 0$."

(c) "$\mathbf{x}_n \underset{d}{\rightarrow} \mathbf{x},\ \mathbf{A}_n \underset{p}{\rightarrow} \mathbf{A}$" $\Rightarrow$ "$\mathbf{A}_n \mathbf{x}_n \underset{d}{\rightarrow} \mathbf{A}\mathbf{x}$." *In particular, if* $\mathbf{x} \sim N(\mathbf{0}, \boldsymbol{\Sigma})$, *then* $\mathbf{A}_n \mathbf{x}_n \underset{d}{\rightarrow} N(\mathbf{0}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$.

(d) "$\mathbf{x}_n \underset{d}{\rightarrow} \mathbf{x},\ \mathbf{A}_n \underset{p}{\rightarrow} \mathbf{A}$" $\Rightarrow$ "$\mathbf{x}_n' \mathbf{A}_n^{-1} \mathbf{x}_n \underset{d}{\rightarrow} \mathbf{x}' \mathbf{A}^{-1} \mathbf{x}$," *provided* $\mathbf{A}$ *is nonsingular.*

Delta Method

定理 (Delta Method)

Suppose $\{\mathbf{x}_n\}$ *is a sequence of* K*-dimensional random vectors such that* $\mathbf{x}_n \underset{p}{\rightarrow} \boldsymbol{\beta}$ *and*

$$\sqrt{n}(\mathbf{x}_n - \boldsymbol{\beta}) \underset{d}{\rightarrow} \mathbf{z},$$

and suppose $\mathbf{a}(\cdot) : \mathbb{R}^K \rightarrow \mathbb{R}^r$ *has continuous first derivatives with* $\mathbf{A}(\boldsymbol{\beta})$ *denoting the* $r \times K$ *matrix of first derivatives evaluated at* $\boldsymbol{\beta}$*:*

$$\mathbf{A}(\boldsymbol{\beta}) \equiv \frac{\partial \mathbf{a}(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}'}.$$

Then

$$\sqrt{n}[\mathbf{a}(\mathbf{x}_n) - \mathbf{a}(\boldsymbol{\beta})] \underset{d}{\rightarrow} \mathbf{A}(\boldsymbol{\beta})\mathbf{z}.$$

In particular:

$$\sqrt{n}(\mathbf{x}_n - \boldsymbol{\beta}) \underset{d}{\rightarrow} N(\mathbf{0}, \boldsymbol{\Sigma}) \;\Rightarrow\; \sqrt{n}[\mathbf{a}(\mathbf{x}_n) - \mathbf{a}(\boldsymbol{\beta})] \underset{d}{\rightarrow} N(\mathbf{0}, \mathbf{A}(\boldsymbol{\beta})\boldsymbol{\Sigma}\mathbf{A}(\boldsymbol{\beta})').$$

证明.
由中值定理，存在 $\mathbf{y}_n$ between $\mathbf{x}_n$ and $\boldsymbol{\beta}$ 使得:

$$\mathbf{a}(\mathbf{x}_n) - \mathbf{a}(\boldsymbol{\beta}) = \mathbf{A}(\mathbf{y}_n)(\mathbf{x}_n - \boldsymbol{\beta}).$$

等式两边同时乘以 $\sqrt{n}$,

$$\sqrt{n}[\mathbf{a}(\mathbf{x}_n) - \mathbf{a}(\boldsymbol{\beta})] = \mathbf{A}(\mathbf{y}_n)\sqrt{n}(\mathbf{x}_n - \boldsymbol{\beta}).$$

因为 $\mathbf{x}_n \underset{p}{\rightarrow} \boldsymbol{\beta}$，所以: $\mathbf{y}_n \underset{p}{\rightarrow} \boldsymbol{\beta}$. 再由连续映射定理可以得到:

$$\mathbf{A}(\mathbf{y}_n) \underset{p}{\rightarrow} \mathbf{A}(\boldsymbol{\beta}).$$

根据 Slutsky's Theorem 和 $\sqrt{n}(\mathbf{x}_n - \boldsymbol{\beta}) \underset{d}{\rightarrow} \mathbf{z}$ 易知定理结论成立。 □

大数定律和中心极限定理

定理 (Law of Large Numbers, LLN)

- *Khintchine's weak LLN: Let $\{z_i\}$ be i.i.d. with a finite mean μ. Then*

$$\bar{z}_n \equiv \frac{1}{n}\sum_{i=1}^{n} z_i \underset{p}{\to} \mu.$$

- *Kolmogorov's strong LLN: Let $\{z_i\}$ be i.i.d. random variables. Then*

$$\bar{z}_n \underset{a.s.}{\to} \mu \quad \text{iff} \quad \mathbb{E}|z_i| < \infty \text{ and } \mathbb{E}(z_i) = \mu.$$

定理 (Lindeberg-Levy Central Limit Theorem, CLT)

Let $\{\mathbf{z}_i\}$ be i.i.d. with $\mathbb{E}(\mathbf{z}_i) = \boldsymbol{\mu}$ and $\mathrm{Var}(\mathbf{z}_i) = \boldsymbol{\Sigma}$. Then

$$\sqrt{n}(\bar{\mathbf{z}}_n - \boldsymbol{\mu}) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\mathbf{z}_i - \boldsymbol{\mu}) \underset{d}{\to} N(\mathbf{0}, \boldsymbol{\Sigma}).$$

时间序列分析的基本概念

(Strictly) Stationarity

The process $\{\mathbf{z}_i\}$ $(i = 1, 2, \ldots)$ is said to be (strictly) stationary if $(\mathbf{z}_{i_1}, \ldots, \mathbf{z}_{i_r})$ and $(\mathbf{z}_{i_1+k}, \ldots, \mathbf{z}_{i_r+k})$ have the same joint distribution, for any $r \in \mathbb{N}$ and any $k \in \mathbb{Z}$.

Weakly Stationarity

The process $\{\mathbf{z}_i\}$ is said to be weakly stationary (or covariance stationary or second-order stationary) if:

(i) $\mathbb{E}(\mathbf{z}_i)$ does not depend on i, and

(ii) $\mathrm{Cov}(\mathbf{z}_i, \mathbf{z}_{i-j})$ exists, is finite, and depends only on j but not on i.

Ergodicity

A stationary process $\{z_i\}$ is said to be ergodic if, for any two bounded functions $f : \mathbb{R}^{k+1} \to \mathbb{R}$ and $g : \mathbb{R}^{l+1} \to \mathbb{R}$,

$$
\begin{aligned}
&\lim_{n\to\infty} \Big| \mathbb{E}[f(z_i, \ldots, z_{i+k}) g(z_{i+n}, \ldots, z_{i+n+l})] \Big| \\
&= \Big| \mathbb{E}[f(z_i, \ldots, z_{i+k})] \Big| \Big| \mathbb{E}[g(z_i, \ldots, z_{i+l})] \Big| .
\end{aligned}
$$

A stationary process that is ergodic will be called ergodic stationary.

Remark: Heuristically, a stationary process is ergodic if it is asymptotically independent.

定理 (Ergodic Theorem)

Let $\{\mathbf{z}_i\}$ *be a stationary and ergodic process with* $\mathbb{E}(\mathbf{z}_i) = \boldsymbol{\mu}$. *Then*

$$
\bar{\mathbf{z}}_n \equiv \frac{1}{n} \sum_{i=1}^{n} \mathbf{z}_i \underset{a.s.}{\rightarrow} \boldsymbol{\mu}.
$$

Martingale

A vector process $\{\mathbf{z}_i\}$ is called a martingale if

$$\mathbb{E}(\mathbf{z}_i \mid \mathbf{z}_{i-1}, \ldots, \mathbf{z}_1) = \mathbf{z}_{i-1} \quad \text{for } i \geq 2.$$

Martingale Difference Sequence

A vector process $\{\mathbf{g}_i\}$ with $\mathbb{E}(\mathbf{g}_i) = \mathbf{0}$ is called a martingale difference sequence (m.d.s.) if the expectation conditional on its past values, too is zero:

$$\mathbb{E}(\mathbf{g}_i \mid \mathbf{g}_{i-1}, \ldots, \mathbf{g}_1) = \mathbf{0} \quad \text{for } i \geq 2.$$

A martingale difference sequence has no serial correlation.

定理 (Ergodic Stationary Martingale Differences CLT)

Let $\{\mathbf{g}_i\}$ be a vector martingale difference sequence that is stationary and ergodic with $\mathbb{E}(\mathbf{g}_i\mathbf{g}_i') = \mathbf{\Sigma}$, and let $\bar{\mathbf{g}} \equiv \frac{1}{n}\sum_{i=1}^{n}\mathbf{g}_i$. Then

$$\sqrt{n}\bar{\mathbf{g}} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{g}_i \xrightarrow[d]{} N(\mathbf{0}, \mathbf{\Sigma}).$$

Remark:

- This CLT, being applicable not just to i.i.d. sequences but also to stationary martingale differences such as ARCH(1) processes, is more general than Lindeberg-Levy.
- Ergodic Theorem is an LLN for serially correlated processes. A central limit theorem for serially correlated processes can be found in Section 6.5 of [Hayashi, 2000].

最小二乘估计的大样本性质

大样本性质的推导基于以下关于数据生成过程（Data Generating Process, DGP）的假设：

1. Assumption 2.1 (linearity):

$$y_i = \mathbf{x}_i'\boldsymbol{\beta} + \varepsilon_i \quad (i = 1, 2, \ldots, n)$$

   where $\mathbf{x}_i$ is a K-dimensional vector of regressors, $\boldsymbol{\beta}$ is a K-dimensional coefficient vector, and ε_i is the unobservable error term.

2. Assumption 2.2 (ergodic stationarity): The $(K+1)$-dimensional vector stochastic process $\{y_i, \mathbf{x}_i\}$ is jointly stationary and ergodic.

3. Assumption 2.3 (predetermined regressors): All the regressors are predetermined in the sense that they are orthogonal to the contemporaneous error term: $\mathbb{E}(x_{ik}\varepsilon_i) = 0$ for all i and k $(= 1, 2, \ldots, K)$. This can be written as $\mathbb{E}[\mathbf{x}_i \cdot (y_i - \mathbf{x}_i'\boldsymbol{\beta})] = \mathbf{0}$ or $\mathbb{E}(\mathbf{g}_i) = \mathbf{0}$ where $\mathbf{g}_i \equiv \mathbf{x}_i \cdot \varepsilon_i$.

4. Assumption 2.4 (rank condition): The $K \times K$ matrix $\mathbb{E}(\mathbf{x}_i \mathbf{x}_i')$ is nonsingular (and hence finite). We denote this matrix by $\mathbf{\Sigma}_{\mathbf{xx}}$.
5. Assumption 2.5 ($\mathbf{g}_i$ is a martingale difference sequence with finite second moments): $\{\mathbf{g}_i\}$ is a martingale difference sequence. The $K \times K$ matrix of cross moments, $\mathbb{E}(\mathbf{g}_i \mathbf{g}_i')$, is nonsingular.

Remark

- Let $\mathbf{S} \equiv \mathrm{Avar}(\bar{\mathbf{g}})$, the variance of the asymptotic distribution of $\sqrt{n}\bar{\mathbf{g}}$. Then, $\mathbf{S} = \mathbb{E}(\mathbf{g}_i \mathbf{g}_i')$ under Assumption 2.5.
- This DGP accommodates conditional heteroskedasticity (条件异方差)。
- Assumption 2.5 is stronger than Assumption 2.3.

最小二乘估计的大样本性质

定理 (asymptotic distribution of the OLS Estimator)

(a) *(Consistency of* $\mathbf{b}$ *for* $\boldsymbol{\beta}$*) Under Assumptions 2.1-2.4,* $\operatorname*{plim}_{n\to\infty} \boldsymbol{b} = \boldsymbol{\beta}$.

(b) *(Asymptotic Normality of* $\mathbf{b}$*) If Assumption 2.3 is strengthened as Assumption 2.5, then*

$$\sqrt{n}(\boldsymbol{b}-\boldsymbol{\beta}) \underset{d}{\rightarrow} N(\mathbf{0}, \operatorname{Avar}(\boldsymbol{b})) \quad \text{as} \quad n \to \infty, \tag{1}$$

where

$$\operatorname{Avar}(\boldsymbol{b}) = \boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}\mathbf{S}\boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}.$$

(c) *(Consistent Estimate of* $\operatorname{Avar}(\mathbf{b})$*) Suppose there is available a consistent estimator,* $\hat{\mathbf{S}}$*, of* $\mathbf{S}$*. Then, under Assumption 2.2,* $\operatorname{Avar}(\mathbf{b})$ *is consistently estimated by*

$$\widehat{\operatorname{Avar}(\boldsymbol{b})} = \mathbf{S}_{\mathbf{xx}}^{-1}\hat{\mathbf{S}}\mathbf{S}_{\mathbf{xx}}^{-1},$$

where

$$\mathbf{S}_{\mathbf{xx}} \equiv \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i' = \frac{1}{n}\mathbf{X}'\mathbf{X}.$$

证明.

(a) We first write $\boldsymbol{b}-\boldsymbol{\beta}$ in terms of sample means.

$$\begin{aligned}\boldsymbol{b}-\boldsymbol{\beta} &= \left(\frac{1}{n}\mathbf{X}'\mathbf{X}\right)^{-1}\left(\frac{1}{n}\mathbf{X}'\boldsymbol{\varepsilon}\right)=\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\cdot\varepsilon_i\right)\\ &\equiv \mathbf{S}_{\mathbf{xx}}^{-1}\bar{\mathbf{g}}.\end{aligned}$$

Since by Assumption 2.2 $\{\mathbf{x}_i\mathbf{x}_i'\}$ is ergodic stationary, $\mathbf{S}_{\mathbf{xx}}\underset{p}{\to}\boldsymbol{\Sigma}_{\mathbf{xx}}$. Since $\boldsymbol{\Sigma}_{\mathbf{xx}}$ is invertible by Assumption 2.4, $\mathbf{S}_{\mathbf{xx}}^{-1}\underset{p}{\to}\boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}$ by Continuous Mapping Theorem. Similarly, $\bar{\mathbf{g}}\underset{p}{\to}\mathbb{E}(\mathbf{g}_i)=\mathbf{0}$. So by Continuous Mapping Theorem again, $\mathbf{S}_{\mathbf{xx}}^{-1}\bar{\mathbf{g}}\underset{p}{\to}\boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}\mathbf{0}=\mathbf{0}$. Therefore, $\operatorname*{plim}_{n\to\infty}\boldsymbol{b}=\boldsymbol{\beta}$.

(b) Write

$$\sqrt{n}(\boldsymbol{b}-\boldsymbol{\beta})=\mathbf{S}_{\mathbf{xx}}^{-1}(\sqrt{n}\bar{\mathbf{g}}).$$

By Assumption 2.5, $\sqrt{n}\bar{\mathbf{g}}\underset{d}{\to}N(\mathbf{0},\mathbf{S})$. So, by Slutsky's Theorem, $\sqrt{n}(\boldsymbol{b}-\boldsymbol{\beta})\underset{d}{\to}N(\mathbf{0},\mathrm{Avar}(\boldsymbol{b}))$.

(c) Since $\mathbf{S}_{\mathbf{xx}}\underset{p}{\to}\boldsymbol{\Sigma}_{\mathbf{xx}}$ and $\hat{\mathbf{S}}\underset{p}{\to}\mathbf{S}$, $\widehat{\mathrm{Avar}(\boldsymbol{b})}\underset{p}{\to}\mathrm{Avar}(\boldsymbol{b})$ by Continuous Mapping Theorem. □

s^2 is consistent

定理 (consistent estimation of error variance)

Let $e_i \equiv y_i - \mathbf{x}_i'\boldsymbol{b}$ *be the OLS residual for observation* i. *Under Assumptions 2.1-2.4,*

$$s^2 \equiv \frac{1}{n-K}\sum_{i=1}^{n} e_i^2 \underset{p}{\to} \mathbb{E}(\varepsilon_i^2),$$

provided $\mathbb{E}(\varepsilon_i^2)$ *exists and is finite.*

证明.

Since

$$s^2 = \frac{n}{n-K}\left(\frac{1}{n}\sum_{i=1}^{n} e_i^2\right),$$

证明.

it suffices to prove that the sample mean of e_i^2 converges in probability to $\mathbb{E}(\varepsilon_i^2)$. The relationship between e_i and ε_i is given by

$$e_i \equiv y_i - \mathbf{x}_i'\boldsymbol{b} = y_i - \mathbf{x}_i'\boldsymbol{\beta} - \mathbf{x}_i'(\boldsymbol{b}-\boldsymbol{\beta}) = \varepsilon_i - \mathbf{x}_i'(\boldsymbol{b}-\boldsymbol{\beta}),$$

so that

$$e_i^2 = \varepsilon_i^2 - 2(\boldsymbol{b}-\boldsymbol{\beta})'\mathbf{x}_i \cdot \varepsilon_i + (\boldsymbol{b}-\boldsymbol{\beta})'\mathbf{x}_i\mathbf{x}_i'(\boldsymbol{b}-\boldsymbol{\beta}).$$

Summing over i, we obtain

$$\frac{1}{n}\sum_{i=1}^{n} e_i^2 = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i^2 - 2(\boldsymbol{b}-\boldsymbol{\beta})'\bar{\mathbf{g}} + (\boldsymbol{b}-\boldsymbol{\beta})'\mathbf{S}_{\mathbf{xx}}(\boldsymbol{b}-\boldsymbol{\beta}).$$

Therefore,

$$\operatorname{plim}(\frac{1}{n}\sum_{i=1}^{n} e_i^2) = \operatorname{plim}(\frac{1}{n}\sum_{i=1}^{n} \varepsilon_i^2) = \mathbb{E}(\varepsilon_i^2).$$

□

Estimating S consistently

A consistent estimator of $\mathbf{S} = \mathbb{E}(\mathbf{g}_i \mathbf{g}_i') = \mathbb{E}(\varepsilon_i^2 \mathbf{x}_i \mathbf{x}_i')$ is required to calculate the estimated asymptotic variance, $\widehat{\mathrm{Avar}(\boldsymbol{b})}$. Let

$$\hat{\mathbf{S}} \equiv \frac{1}{n} \sum_{i=1}^{n} \hat{\varepsilon}_i^2 \mathbf{x}_i \mathbf{x}_i', \tag{2}$$

where $\hat{\varepsilon}_i \equiv y_i - \mathbf{x}_i' \hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\beta}}$ is some consistent estimator of $\boldsymbol{\beta}$. For this estimator to be consistent for $\mathbf{S}$, we need to make a fourth-moment assumption about the regressors.

6. Assumption 2.6 (finite fourth moments for regressors): $\mathbb{E}[(x_{ik} x_{ij})^2]$ exists and is finite for all $k, j = 1, 2, \ldots, K$.

定理 (consistent estimation of $\mathbf{S}$)

Suppose the coefficient estimate $\hat{\boldsymbol{\beta}}$ *used for calculating the residual* $\hat{\varepsilon}_i$ *for* $\hat{\mathbf{S}}$ *in (2) is consistent, and suppose* $\mathbf{S} = \mathbb{E}(\mathbf{g}_i\mathbf{g}_i')$ *exists and is finite. Then, under Assumptions 2.1, 2.2, and 2.6,* $\hat{\mathbf{S}}$ *given in (2) is consistent for* $\mathbf{S}$*.*

证明.

Following the same steps as in the proof of the previous theorem, we have

$$\hat{\varepsilon}_i \equiv y_i - \mathbf{x}_i'\hat{\boldsymbol{\beta}} = y_i - \mathbf{x}_i'\boldsymbol{\beta} - \mathbf{x}_i'(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \varepsilon_i - \mathbf{x}_i'(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}),$$

so that

$$\hat{\varepsilon}_i^2 = \varepsilon_i^2 - 2(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\mathbf{x}_i \cdot \varepsilon_i + (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\mathbf{x}_i\mathbf{x}_i'(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}).$$

证明.

Therefore, for the (l,m)-th element of $\hat{\mathbf{S}}$,

$$\begin{aligned}\hat{\mathbf{S}}_{lm} &\equiv \frac{1}{n}\sum_{i=1}^{n}\hat{\varepsilon}_i^2 x_{il}x_{im} \quad (l,m=1,2,\ldots,K)\\ &= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2 x_{il}x_{im} - 2(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\cdot x_{il}x_{im}\varepsilon_i\right)\\ &\quad + (\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i' x_{il}x_{im}\right)(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}).\end{aligned}$$

By Cauchy-Schwarz inequality,[a] and ergodic stationarity, it is easy to show that $\hat{\mathbf{S}}_{lm} \underset{p}{\to} \mathbf{S}_{lm}$. □

[a] $\mathbb{E}(|f\cdot h|) \le \sqrt{\mathbb{E}(f^2)\mathbb{E}(h^2)}$.

大样本统计推断

Testing Linear Hypotheses

定理 (robust t-ratio and Wald statistic)

Suppose Assumptions 2.1-2.5 hold, and suppose there is available a consistent estimate $\hat{\mathbf{S}}$ *of* $\mathbf{S}$. *Then*

(a) *Under the null hypothesis* $\mathbb{H}_0 : \beta_k = \bar{\beta}_k$,

$$t_k \equiv \frac{\sqrt{n}(b_k - \bar{\beta}_k)}{\sqrt{\widehat{\text{Avar}(b_k)}}} = \frac{b_k - \bar{\beta}_k}{SE^*(b_k)} \underset{d}{\rightarrow} N(0,1),$$

where b_k *is the* k*-th element of* $\boldsymbol{b}$, $\widehat{\text{Avar}(b_k)}$ *is the* (k,k) *element of* $\widehat{\text{Avar}(\boldsymbol{b})}$, *and*

$$SE^*(b_k) \equiv \sqrt{\frac{1}{n} \cdot \widehat{\text{Avar}(b_k)}} = \sqrt{\frac{1}{n} \cdot \left(\mathbf{S}_{\mathbf{xx}}^{-1} \hat{\mathbf{S}} \mathbf{S}_{\mathbf{xx}}^{-1}\right)_{kk}}.$$

$SE^*(b_k)$ *is called the heteroskedasticity-consistent standard error, (heteroskedasticity-)robust standard error, or White's standard error.*

(b) *Under the null hypothesis* $\mathbb{H}_0 : \mathbf{R}\boldsymbol{\beta} = \mathbf{r}$, *where* $\mathbf{R}$ *is a* $J \times K$ *matrix of full row rank,*

$$W \equiv n \cdot (\mathbf{R}\boldsymbol{b} - \mathbf{r})' \left\{ \mathbf{R} [\widehat{\text{Avar}(\boldsymbol{b})}] \mathbf{R}' \right\}^{-1} (\mathbf{R}\boldsymbol{b} - \mathbf{r}) \rightarrow \chi^2(J)$$

证明.

(a) It is easy to establish this result using (1) and Slutsky's Theorem.
(b) Write W as

$$W = \mathbf{c}_n' \mathbf{Q}_n^{-1} \mathbf{c}_n \quad \text{where} \quad \mathbf{c}_n \equiv \sqrt{n}(\mathbf{R}\boldsymbol{b} - \mathbf{r}) \text{ and } \mathbf{Q}_n \equiv \mathbf{R}\widehat{\mathrm{Avar}(\boldsymbol{b})}\mathbf{R}'.$$

Under $\mathbb{H}_0$, $\mathbf{c}_n = \mathbf{R}\sqrt{n}(\boldsymbol{b} - \boldsymbol{\beta})$. So by (1),

$$\mathbf{c}_n \underset{d}{\to} \mathbf{c} \quad \text{where} \quad \mathbf{c} \sim N(\mathbf{0}, \mathbf{R}\mathrm{Avar}(\boldsymbol{b})\mathbf{R}').$$

Also note that

$$\mathbf{Q}_n \underset{p}{\to} \mathbf{Q} \quad \text{where} \quad \mathbf{Q} \equiv \mathbf{R}\mathrm{Avar}(\boldsymbol{b})\mathbf{R}'.$$

Because $\mathbf{R}$ is of full row rank and $\mathrm{Avar}(\boldsymbol{b})$ is positive definite, $\mathbf{Q}$ is invertible. Therefore,

$$W \underset{d}{\to} \mathbf{c}'\mathbf{Q}^{-1}\mathbf{c}.$$

Since the J-dimensional random vector $\mathbf{c}$ is normally distributed with mean $\mathbf{0}$ and since $\mathbf{Q}$ equals $\mathrm{Var}(\mathbf{c})$,

$$\mathbf{c}'\mathbf{Q}^{-1}\mathbf{c} \sim \chi^2(J).$$

Testing Nonlinear Hypotheses

定理 (continued)

(c) *Under the null hypothesis with J restrictions $\mathbb{H}_0 : \mathbf{a}(\boldsymbol{\beta}) = \mathbf{0}$ such that $\mathbf{A}(\boldsymbol{\beta})$, the $J \times K$ matrix of continuous first derivatives of $\mathbf{a}(\boldsymbol{\beta})$, is of full row rank, we have*

$$W \equiv n \cdot \mathbf{a}(\boldsymbol{b})' \left\{ \mathbf{A}(\boldsymbol{b}) \widehat{\mathrm{Avar}(\boldsymbol{b})} \mathbf{A}(\boldsymbol{b})' \right\}^{-1} \mathbf{a}(\boldsymbol{b}) \underset{d}{\rightarrow} \chi^2(J).$$

证明.

(c) The asymptotic result given in (1) and Delta Method imply that

$$\sqrt{n}[\mathbf{a}(\boldsymbol{b}) - \mathbf{a}(\boldsymbol{\beta})] \underset{d}{\rightarrow} \mathbf{c}, \quad \mathbf{c} \sim N(\mathbf{0}, \mathbf{A}(\boldsymbol{\beta}) \mathrm{Avar}(\boldsymbol{b}) \mathbf{A}(\boldsymbol{\beta})').$$

Under $\mathbb{H}_0$, $\mathbf{a}(\boldsymbol{\beta}) = \mathbf{0}$, therefore

$$\sqrt{n}\mathbf{a}(\boldsymbol{b}) \underset{d}{\rightarrow} \mathbf{c}, \quad \mathbf{c} \sim N(\mathbf{0}, \mathbf{A}(\boldsymbol{\beta}) \mathrm{Avar}(\boldsymbol{b}) \mathbf{A}(\boldsymbol{\beta})').$$

证明.

Since $\boldsymbol{b} \underset{p}{\rightarrow} \boldsymbol{\beta}$, by Continuous Mapping Theorem, $\mathbf{A}(\boldsymbol{b}) \underset{p}{\rightarrow} \mathbf{A}(\boldsymbol{\beta})$. Hence,

$$\mathbf{A}(\boldsymbol{b})\widehat{\mathrm{Avar}(\boldsymbol{b})}\mathbf{A}(\boldsymbol{b})' \underset{p}{\rightarrow} \mathbf{A}(\boldsymbol{\beta})\mathrm{Avar}(\boldsymbol{b})\mathbf{A}(\boldsymbol{\beta})' = \mathrm{Var}(\mathbf{c}).$$

Because $\mathbf{A}(\boldsymbol{\beta})$ is of full row rank and $\mathrm{Avar}(\boldsymbol{b})$ is positive definite, $\mathrm{Var}(\mathbf{c})$ is invertible. Then

$$\sqrt{n}\mathbf{a}(\boldsymbol{b})' \left\{\mathbf{A}(\boldsymbol{b})\widehat{\mathrm{Avar}(\boldsymbol{b})}\mathbf{A}(\boldsymbol{b})'\right\}^{-1} \sqrt{n}\mathbf{a}(\boldsymbol{b}) \underset{d}{\rightarrow} \mathbf{c}'\mathrm{Var}(\mathbf{c})^{-1}\mathbf{c} \sim \chi^2(J).$$

$\square$

Implications of Conditional Homoskedasticity

7. Assumption 2.7 (conditional homoskedasticity): $\mathbb{E}(\varepsilon_i^2|\mathbf{x}_i) = \sigma^2 > 0$.

定理 (large-sample properties of $\boldsymbol{b}$, t, and F under Conditional Homoskedasticity)

Suppose Assumptions 2.1-2.5 and 2.7 are satisfied. Then

(a) *(Asymptotic distribution of* $\mathbf{b}$*) The OLS estimate* $\mathbf{b}$ *of* β *is consistent and asymptotically normal with* $\mathrm{Avar}(\boldsymbol{b}) = \sigma^2 \boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}$.

(b) *(Consistent estimation of asymptotic variance) Under the same set of assumptions,* $\mathrm{Avar}(\boldsymbol{b})$ *is consistently estimated by* $\widehat{\mathrm{Avar}(\boldsymbol{b})} = s^2 \mathbf{S}_{\mathbf{xx}}^{-1} = n \cdot s^2 \cdot (\mathbf{X}'\mathbf{X})^{-1}$.

(c) *(Asymptotic distribution of the* t *and* F *statistics of the finite-sample theory) Under* $\mathbb{H}_0 : \beta_k = \bar{\beta}_k$, *the usual* t*-ratio is asymptotically distributed as* $N(0,1)$. *Under* $\mathbb{H}_0 : \mathbf{R}\boldsymbol{\beta} = \mathbf{r}$, $J \cdot F$ *is asymptotically* $\chi^2(J)$, *where* F *is the* F *statistic and* J *is the number of restrictions in* $\mathbb{H}_0$.

证明.

(a) Under Assumption 2.7, we have

$$\mathbf{S} = \mathbb{E}(\mathbf{x}_i \mathbf{x}_i' \varepsilon_i^2) = \mathbb{E}[\mathbf{x}_i \mathbf{x}_i' \mathbb{E}(\varepsilon_i^2 | \mathbf{x}_i)] = \sigma^2 \mathbf{\Sigma}_{\mathbf{xx}}.$$

Therefore, $\mathrm{Avar}(\boldsymbol{b}) = \mathbf{\Sigma}_{\mathbf{xx}}^{-1} \mathbf{S} \mathbf{\Sigma}_{\mathbf{xx}}^{-1} = \sigma^2 \mathbf{\Sigma}_{\mathbf{xx}}^{-1}$.
(b) By ergodic stationary, $\mathbf{S}_{\mathbf{xx}} \underset{p}{\to} \mathbf{\Sigma}_{\mathbf{xx}}$. Therefore, $\mathbf{S}_{\mathbf{xx}}^{-1} \underset{p}{\to} \mathbf{\Sigma}_{\mathbf{xx}}^{-1}$. Note that s^2 is consistent for σ^2 under Assumptions 2.1-2.4. So, $s^2 \mathbf{S}_{\mathbf{xx}}^{-1} \underset{p}{\to} \sigma^2 \mathbf{\Sigma}_{\mathbf{xx}}^{-1}$.[a]

(c) Recall that the usual t-ratio is given by

$$t_k \equiv \frac{b_k - \bar{\beta}_k}{SE(b_k)} \equiv \frac{b_k - \bar{\beta}_k}{\sqrt{s^2 \cdot \left((\mathbf{X}'\mathbf{X})^{-1}\right)_{kk}}}.$$

And, the robust t-ratio is

$$t_k = \frac{b_k - \bar{\beta}_k}{SE^*(b_k)} \quad \text{with} \quad SE^*(b_k) = \sqrt{\frac{1}{n} \cdot \widehat{\mathrm{Avar}(b_k)}} = \sqrt{s^2 \cdot \left((\mathbf{X}'\mathbf{X})^{-1}\right)_{kk}}.$$

So the usual finite-sample t-ratio is numerically identical to the robust t-ratio, and hence asymptotically distributed as $N(0,1)$.

[a] Under conditional homoskedasticity, we do not need Assumption 2.6 for consistency.

证明.

(c) Similarly,

$$\begin{aligned} W &= n \cdot (\mathbf{R}\boldsymbol{b} - \mathbf{r})' \left\{ \mathbf{R}[\widehat{\mathrm{Avar}(\boldsymbol{b})}]\mathbf{R}' \right\}^{-1} (\mathbf{R}\boldsymbol{b} - \mathbf{r}) \\ &= n \cdot (\mathbf{R}\boldsymbol{b} - \mathbf{r})' \left\{ \mathbf{R}[n \cdot s^2 \cdot (\mathbf{X}'\mathbf{X})^{-1}]\mathbf{R}' \right\}^{-1} (\mathbf{R}\boldsymbol{b} - \mathbf{r}) \\ &= (\mathbf{R}\boldsymbol{b} - \mathbf{r})' \left\{ \mathbf{R}[(\mathbf{X}'\mathbf{X})^{-1}]\mathbf{R}' \right\}^{-1} (\mathbf{R}\boldsymbol{b} - \mathbf{r})/s^2 \\ &= J \cdot F \\ &= (SSR_R - SSR_U)/s^2. \end{aligned}$$

Thus, the Wald statistic W is numerically identical to $J \cdot F$. □

特例：解释变量联合显著性检验

For a regression with a constant, consider the null hypothesis that the coefficients of the $K-1$ nonconstant regressors are all zero. We have shown that, under conditional homoskedasticity,

$$F = \frac{R^2/(K-1)}{(1-R^2)/(n-K)}.$$

From this equation, we can derive:

$$nR^2 = \frac{1}{\dfrac{n-K}{n} + \dfrac{1}{n}(K-1)F}(K-1)F.$$

Since $(K-1)F$ converges in distribution to a random variable, $\dfrac{1}{n}(K-1)F \underset{p}{\to} 0$. Then, the asymptotic distribution of the RHS is the same as that of $(K-1)F$, which is $\chi^2(K-1)$.

Resampling Methods: Jackknife and Bootstrap

Jackknife

- The jackknife estimates moments of estimators using the distribution of the leave-one-out estimators.
- Let $\hat{\boldsymbol{\theta}}$ be any estimator of a vector-valued parameter $\boldsymbol{\theta}$ which is a function of a random sample of size n. Let $\boldsymbol{V}_{\hat{\boldsymbol{\theta}}} = \operatorname{Var}(\hat{\boldsymbol{\theta}})$ be the variance of $\hat{\boldsymbol{\theta}}$.
- Define the leave-one-out estimators $\hat{\boldsymbol{\theta}}_{(-i)}$ which are computed using the formula for $\hat{\boldsymbol{\theta}}$ except that observation i is deleted.
- The jackknife estimator for $\boldsymbol{V}_{\hat{\boldsymbol{\theta}}}$ is defined as

$$\hat{\boldsymbol{V}}_{\hat{\boldsymbol{\theta}}}^{\text{jack}} = \frac{n-1}{n}\sum_{i=1}^{n}\left(\hat{\boldsymbol{\theta}}_{(-i)} - \bar{\boldsymbol{\theta}}\right)\left(\hat{\boldsymbol{\theta}}_{(-i)} - \bar{\boldsymbol{\theta}}\right)', \tag{3}$$

where $\bar{\boldsymbol{\theta}}$ is the sample mean of the leave-one-out estimators

$$\bar{\boldsymbol{\theta}} = \frac{1}{n}\sum_{i=1}^{n}\hat{\boldsymbol{\theta}}_{(-i)}.$$

To motivate (3), consider the case where $\hat{\theta}$ is the mean of $(x_1, \ldots, x_n)$. We have

$$
\begin{aligned}
\hat{\theta}_{(-i)} &= \frac{1}{n-1}\sum_{j\neq i} x_j = \frac{n}{n-1}\bar{x} - \frac{1}{n-1}x_i, \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \\
\bar{\theta} &= \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_{(-i)} = \bar{x}, \\
\hat{\theta}_{(-i)} - \bar{\theta} &= \frac{1}{n-1}(\bar{x} - x_i).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\hat{\boldsymbol{V}}_{\hat{\theta}}^{\text{jack}} &= \frac{n-1}{n}\sum_{i=1}^{n}\left(\frac{1}{n-1}\right)^2(\bar{x} - x_i)^2 \\
&= \frac{1}{n}\left(\frac{1}{n-1}\right)\sum_{i=1}^{n}(\bar{x} - x_i)^2.
\end{aligned}
$$

This is the conventional estimator for the variance of $\bar{x}$.

- Formula (3) is quite general and does not require any technical calculations. However, it requires n separate estimations, which in some cases can be computationally costly.
- In most cases $\hat{\boldsymbol{V}}_{\hat{\boldsymbol{\theta}}}^{\text{jack}}$ will be similar to a robust asymptotic variance matrix estimator.
- The main attractions of the jackknife estimator are that it can be used when an explicit asymptotic variance formula is not available.

Bootstrap

- The bootstrap distribution is obtained by estimation on independent samples created by i.i.d. sampling (sampling with replacement) from the original dataset.
- Let B be the number of bootstrap draws. The bootstrap estimator of variance of an estimator $\hat{\boldsymbol{\theta}}$ is the sample variance across the bootstrap draws:

$$\begin{aligned}\hat{\boldsymbol{V}}_{\hat{\boldsymbol{\theta}}}^{\text{boot}} &= \frac{1}{B-1}\sum_{b=1}^{B}\left(\hat{\boldsymbol{\theta}}^*(b)-\bar{\boldsymbol{\theta}}^*\right)\left(\hat{\boldsymbol{\theta}}^*(b)-\bar{\boldsymbol{\theta}}^*\right)', \\ \bar{\boldsymbol{\theta}}^* &= \frac{1}{B}\sum_{b=1}^{B}\hat{\boldsymbol{\theta}}^*(b),\end{aligned}$$

where $\hat{\boldsymbol{\theta}}^*(b)$ denotes the bootstrap estimate of $\boldsymbol{\theta}$ based on the b-th bootstrap sample.

- Let $\left\{\hat{\theta}_1^*, \hat{\theta}_2^*, \ldots, \hat{\theta}_B^*\right\}$ be an i.i.d. sample of bootstrap estimates of a parameter θ.
- For any $0 < \tau < 1$, one can calculate the empirical quantile q_τ^* of these bootstrap estimates.
- The percentile bootstrap $100(1-\tau)\%$ confidence interval is

$$C^{\mathrm{pc}} = \left[q_{\tau/2}^*, \quad q_{1-\tau/2}^*\right].$$

For example, if $B = 1000$, $\tau = 0.05$, and the empirical quantile estimator is used, then $C^{\mathrm{pc}} = \left[\hat{\theta}_{(25)}^*, \quad \hat{\theta}_{(975)}^*\right]$.

Bootstrap Hypothesis Tests

To test $\mathbb{H}_0 : \theta = \theta_0$ against $\mathbb{H}_1 : \theta \neq \theta_0$, the bootstrap t-statistic is

$$t^* = \frac{\hat{\theta}^* - \hat{\theta}}{s(\hat{\theta}^*)},$$

where $\hat{\theta}^*$ is the bootstrap estimator of θ and $s(\hat{\theta}^*)$ is the bootstrap standard error. The bootstrap p-value is given by

$$p^* = \frac{1}{B}\sum_{b=1}^{B} \mathbf{1}(|t^*(b)| > |t|),$$

that is, the percentage of bootstrap t-statistics that are larger than the observed t-statistic.

When standard errors are not available or are not reliable, we can use the non-studentized statistic $t = \hat{\theta} - \theta_0$. The bootstrap version is $t^* = \hat{\theta}^* - \hat{\theta}$. The bootstrap p-value is

$$p^* = \frac{1}{B} \sum_{b=1}^{B} \mathbf{1} \left(\left| \hat{\theta}^*(b) - \hat{\theta} \right| > \left| \hat{\theta} - \theta_0 \right| \right).$$

Similarly, the bootstrap Wald statistic for testing $\mathbb{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ is

$$W^* = \left(\hat{\boldsymbol{\theta}}^* - \hat{\boldsymbol{\theta}} \right)' \hat{\boldsymbol{V}}_{\hat{\boldsymbol{\theta}}}^{*-1} \left(\hat{\boldsymbol{\theta}}^* - \hat{\boldsymbol{\theta}} \right).$$

The bootstrap p-value is given by

$$p^* = \frac{1}{B} \sum_{b=1}^{B} \mathbf{1} \left(W^*(b) > W \right).$$

If a reliable covariance matrix estimator $\hat{\boldsymbol{V}}_{\hat{\boldsymbol{\theta}}}$ is not available, a Wald-type test can be implemented with any positive-definite weight matrix instead of $\hat{\boldsymbol{V}}_{\hat{\boldsymbol{\theta}}}$.

Bootstrap for Regression

Consider the linear regression model

$$y_i = \mathbf{x}_i' \boldsymbol{\beta} + \varepsilon_i, \quad \mathbb{E}(\varepsilon_i | \mathbf{x}_i) = 0, \quad \mathbb{E}(\varepsilon_i \varepsilon_j) = 0, \forall i \neq j$$

where there are n observations.

1. The Residual Bootstrap
   - Obtain the OLS estimates $\boldsymbol{b}$ and residuals e_i.
   - Obtain the rescaled residuals using $u_i \equiv \left(\frac{n}{n-K}\right)^{1/2} e_i$.
   - Generate the bootstrap sample by the equation

$$y_i^* = \mathbf{x}_i' \boldsymbol{b} + u_i^*, \quad u_i^* \sim \text{EDF}(u_i).$$

Remark: The residual bootstrap is not valid if the error terms are not independently and identically distributed.

2 The Wild Bootstrap

- Obtain the OLS estimates $\boldsymbol{b}$ and residuals e_i.
- Generate the bootstrap sample by the equation

$$y_i^* = \mathbf{x}_i'\boldsymbol{b} + e_i v_i^*,$$

where v_i^* is a random variable with mean 0 and variance 1. In practice, the following two-point distribution is most commonly used to generate v_i^*:

$$v_i^* = \begin{cases} \frac{-(\sqrt{5}-1)}{2} & \text{with probability } \frac{(\sqrt{5}+1)}{2\sqrt{5}} \\ \frac{(\sqrt{5}+1)}{2} & \text{with probability } \frac{(\sqrt{5}-1)}{2\sqrt{5}}. \end{cases}$$

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