Chapter Fourteen

Asset Returns and Stylized Facts of Financial Data

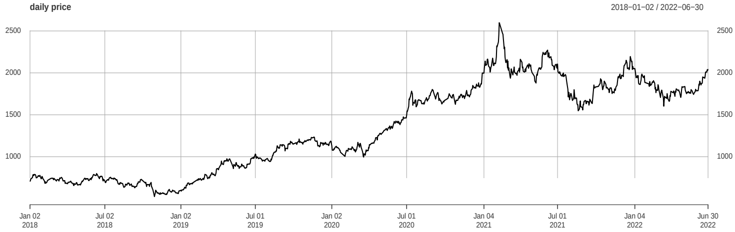
Outline

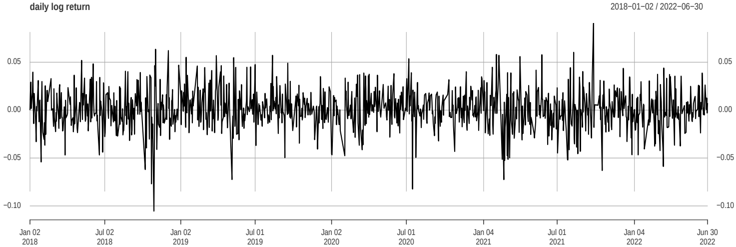
1. Asset Returns
2. Behavior and Stylized Features of Financial Return Data

Section One

Section One: Asset Returns

KWEICHOW MOUTAI: January 2018 - June 2022





Reasons for using returns

- complete summary of the investment opportunity
- unit-free
- more attractive statistical properties

Simple Returns: One-period

- Let P_t be the price of an asset at time t. Assume that the asset pays no *dividends*.
- One-period Simple Return:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1$$

  which is the profit rate of holding the asset from time $t-1$ to t. Often, we write $R_t = 100R_t\%$.
- One-period Gross Return is defined as

$$\frac{P_t}{P_{t-1}} = R_t + 1.$$

Simple Returns: Multiperiod

- k-period Simple Return:

$$R_t[k] = \frac{P_t - P_{t-k}}{P_{t-k}} = \frac{P_t}{P_{t-k}} - 1.$$

- k-period Gross Return is defined as

$$1 + R_t[k] = \frac{P_t}{P_{t-k}}.$$

- Compound Return:

$$\begin{aligned} 1 + R_t[k] &= \frac{P_t}{P_{t-1}} \times \frac{P_{t-1}}{P_{t-2}} \times \cdots \times \frac{P_{t-k+1}}{P_{t-k}} \\ &= (1+R_t)(1+R_{t-1})\cdots(1+R_{t-k+1}) \\ &= \prod_{j=0}^{k-1}(1+R_{t-j}). \end{aligned}$$

Simple Returns: Multiperiod (Cont'd)

- That is, k-period gross return is the product of the k one-period gross returns involved.
- When the one-period returns are small,

$$R_t[k] \approx R_t + R_{t-1} + \cdots + R_{t-k+1}.$$

Annualized Returns

- The actual holding period is important in comparing returns.
- If the time interval is not given, it is implicitly assumed to be one year.
- If the asset was held for k years, the annualized return is defined as

$$\begin{aligned}\text{Annualized}\{R_t[k]\} &= \left[\prod_{j=0}^{k-1}(1+R_{t-j})\right]^{1/k} - 1 \\ &= \exp\left[\frac{1}{k}\sum_{j=0}^{k-1}\ln(1+R_{t-j})\right] - 1.\end{aligned}$$

Annualized Returns (Cont'd)

- If the one-period returns are small, using first-order Taylor expansion, we have

$$\text{Annualized}\{R_t[k]\} \approx \frac{1}{k}\sum_{j=0}^{k-1} R_{t-j}.$$

Accuracy of the approximation may not be sufficient in some application.

Log Returns

- Log Return:

$$r_t = \ln(P_t) - \ln(P_{t-1}) = \ln(1 + R_t)$$

  which is also called continuously compounded return.

- Advantage:
  - comparable (frequency of compounding of the return does not matter)
  - time-additive

$$\begin{aligned} r_t[k] &= \ln(1 + R_t[k]) = \ln\left[\prod_{j=0}^{k-1}(1 + R_{t-j})\right] \\ &= \sum_{j=0}^{k-1}\ln(1 + R_{t-j}) = \sum_{j=0}^{k-1} r_{t-j}. \end{aligned}$$

Log Returns (Cont'd)

- Disadvantage: not additive across a portfolio
  - The simple return of an N-asset portfolio is the weighted average of the individual simple returns of the N assets, denoted R_{it}, for $i = 1, \cdots, N$.
  - Let p denote the portfolio with investment weight ω_i on asset i and $\sum_{i=1}^{N} \omega_i = 1$. The portfolio return is

$$R_{p,t} = \sum_{i=1}^{N} \omega_i R_{it}.$$

  - The portfolio log return $r_{p,t}$ satisfies

$$e^{r_{p,t}} = \sum_{i=1}^{N} \omega_i e^{r_{it}}.$$

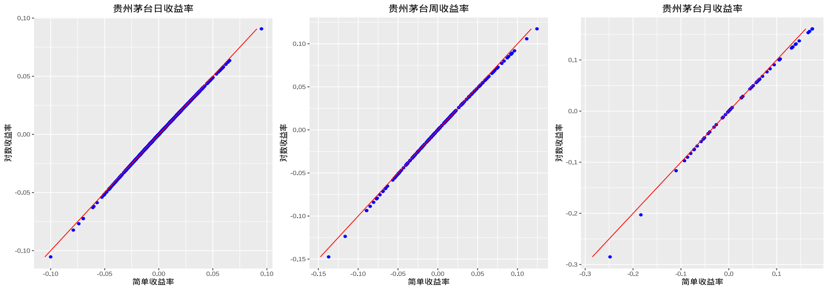
  - However, $r_{p,t} \neq \sum_{i=1}^{N} \omega_i r_{it}$.

Log Returns (Cont'd)

- If the simple returns are all small in magnitude, then

$$\begin{aligned} r_t &= \ln(1+R_t) \approx R_t, \\ r_{p,t} &\approx \sum_{i=1}^{N} \omega_i r_{it}. \end{aligned}$$

Log Returns (Cont'd)



- When the holding period increases, the discrepancy is more apparent.
- Simple returns are always greater than the corresponding log returns.

Continuous Compounding

- The log return r_t is also called *continuously compounded return* due to its close link with the concept of compound rates or interest rates.
- Suppose that the quoted simple interest rate per annum is r and is unchanged, and the earnings are paid m times per annum.
- Let the initial deposit be $\$1.00$. One year later, the value of the deposit becomes

$$\$1(1+\frac{r}{m})^m$$

- Let $m \to \infty$, $\lim_{m\to\infty}(1+\frac{r}{m})^m = e^r$.

Continuous Compounding (Cont'd)

Type	Number of Payments	Interest Rate per Period	Net Value
Annual	1	0.1	$\$1.10000$
Semiannual	2	0.05	$\$1.10250$
Quarterly	4	0.025	$\$1.10381$
Monthly	12	0.0083	$\$1.10471$
Weekly	52	$0.1/52$	$\$1.10506$
Daily	365	$0.1/365$	$\$1.10516$
Continuously	∞		$\$1.10517$

Table: Illustration of effects of compounding: $r = 10\%$.

$$e^{0.1} = 1.105170918075648\cdots.$$

Continuous Compounding (Cont'd)

- r has two interpretations:
  - the simple annual return if the interest is only paid once at the end of the year
  - the annual log return if the interest is compounded continuously

Adjustment for Dividends

- Many assets pay dividends to their shareholders.
- A dividend is typically allocated as a fixed amount of cash per share.
- Assume that all dividends are cashed out and are not re-invested in the asset, then

$$R_t = \frac{P_t + D_t}{P_{t-1}} - 1, \quad r_t = \ln(P_t + D_t) - \ln P_{t-1},$$

where D_t is the dividend payment of an asset between dates $t-1$ and t.

Excess Returns

- Excess Return: the difference between an asset's return and the return on some reference asset (risk-free asset*):

$$Z_t = R_t - R_{0t}, \quad z_t = r_t - r_{0t},$$

  where R_{0t} and r_{0t} are the simple and log returns of the reference asset, respectively.

- Commonly used risk-free rates:
  - 10-month Treasury bonds
  - 10-year CDB Bond

*Risk-free means there is no default or credit risk.

Section Two

Section Two: Behavior and Stylized Features of Financial Return Data

Statistical Distributions and Their Moments

- Moments of a Random Variable: the l-th moment of a continuous random variable X is

$$m'_l = E(X^l) = \int_{-\infty}^{\infty} x^l f(x)dx,$$

where $E(\cdot)$ stands for expectation and $f(x)$ is the pdf of X. If $l = 1$, $E(X)$ is the mean or expectation of X, denoted as μ_X.

- The l-th central moment of X is

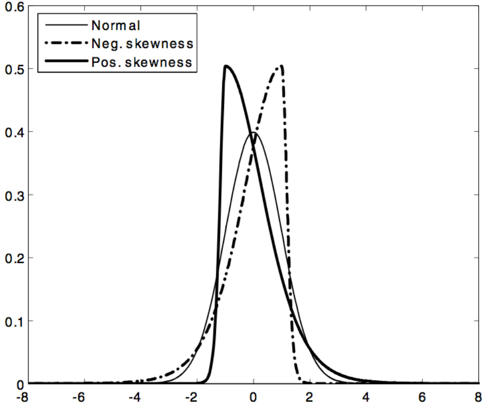
$$m_l = E[(X - \mu_X)^l] = \int_{-\infty}^{\infty} (x - \mu_X)^l f(x)dx,$$

- When $l = 2$, $E[(X - \mu_X)^2]$ is called the variance of X, denoted as σ_X^2. The square root, σ_X, of variance is called standard deviation.

Statistical Distributions and Their Moments (Cont'd)

Two useful normalized central moments:

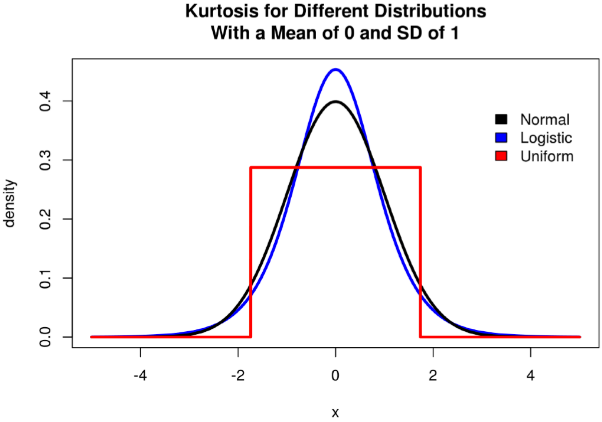
- skewness: $S(X) = E[\frac{(X-\mu_X)^3}{\sigma_X^3}]$, which measures the symmetry of X with respect to its mean.



Statistical Distributions and Their Moments (Cont'd)

Two useful normalized central moments:

- kurtosis: $K(X) = E[\frac{(X-\mu_X)^4}{\sigma_X^4}]$, which measures the tail behavior of X, and $K(X) - 3$ is called excess kurtosis.



Statistical Distributions and Their Moments (Cont'd)

- In practice, moments can be estimated by their sample counterparts.
- Let $\{x_1, \ldots, x_T\}$ be a random sample of X with T observations.
- Sample mean: $\hat{\mu}_X = \frac{1}{T}\sum_{t=1}^{T} x_t$
- Sample variance: $\hat{\sigma}_X^2 = \frac{1}{T-1}\sum_{t=1}^{T}(x_t - \hat{\mu}_X)^2$
- Sample skewness: $\hat{S}(X) = \frac{1}{(T-1)\hat{\sigma}_X^3}\sum_{t=1}^{T}(x_t - \hat{\mu}_X)^3$
- Sample kurtosis: $\hat{K}(X) = \frac{1}{(T-1)\hat{\sigma}_X^4}\sum_{t=1}^{T}(x_t - \hat{\mu}_X)^4$

Statistical Distributions and Their Moments (Cont'd)

Some useful test:

- For an asset, an interesting question is whether the mean of its return is zero. We can consider the hypothesis testing $H_0 : \mu_X = 0$ vs. $H_a : \mu_X \neq 0$. The test statistic is

$$t = \frac{\sqrt{T}\hat{\mu}_X}{\hat{\sigma}_X},$$

which $\xrightarrow{d} N(0, 1)$.

Statistical Distributions and Their Moments (Cont'd)

- If X is normal, then $\hat{S}(X)$ and $\hat{K}(X) - 3$ are distributed asymptotically as normal with zero mean and variances $6/T$ and $24/T$, respectively.
- Given an asset return series $\{r_1, \ldots, r_T\}$, to test the skewness of the returns, we consider $H_0 : S(r) = 0$ vs. $H_a : S(r) \neq 0$. The test statistic is

$$t = \frac{\hat{S}(r)}{\sqrt{6/T}},$$

which $\xrightarrow{d} N(0, 1)$.

Statistical Distributions and Their Moments (Cont'd)

- One can also test the excess kurtosis of the return series using $H_0 : K(r) - 3 = 0$ vs. $H_a : K(r) - 3 \neq 0$. The test statistic is

$$t = \frac{\hat{K}(r) - 3}{\sqrt{24/T}},$$

which $\xrightarrow{d} N(0, 1)$.

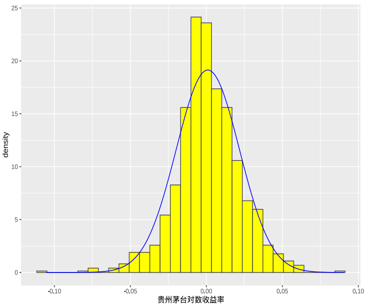
- Jarque & Bera (1987)'s normality test

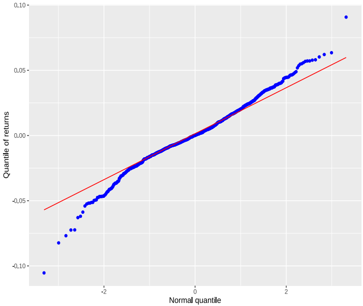
$$\mathrm{JB} = \frac{\hat{S}^2(r)}{6/T} + \frac{(\hat{K}(r) - 3)^2}{24/T},$$

which $\xrightarrow{d} \chi_2^2$.

Stylized Features of Financial Return Data

- **Heavy tails**: the probability distribution of return r_t ofter exhibits heavier tails than those of a normal distribution.
- **Asymmetry**: the distribution of return r_t is often negatively skewed, suggesting that extreme negative returns are more frequent than extreme positive returns.

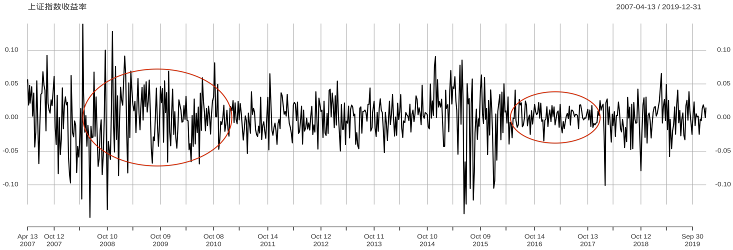


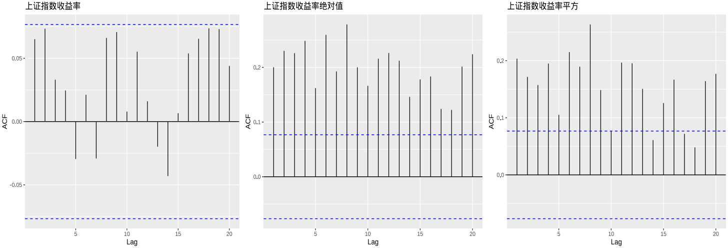


Stylized Features of Financial Return Data (Cont'd)

- Stationarity: most return sequences can be modeled as a stochastic processes with at least time-invariant first two moments.
- Volatility clustering: large price changes occur in clusters.
- Long range dependence: daily squared and absolute returns exhibit small and significant autocorrelations.

Stylized Features of Financial Return Data (Cont’d)





Distribution of Returns

- The most general model for $\{r_{it}; i = 1, \ldots, N; t = 1, \ldots, T\}$ is its joint distribution function

$$F_r(r_{11}, \ldots, r_{N1}; \ldots; r_{1T}, \ldots, r_{NT}; \boldsymbol{Y}; \boldsymbol{\theta}),$$

  where $\boldsymbol{Y}$ is a state vector and $\boldsymbol{\theta}$ is a vector of parameters that uniquely determines the distribution function $F_r(\cdot)$.

- Cross-sectional aspect of financial econometrics: the distribution of $\{r_{1t}, \ldots, r_{Nt}\}$ for a given time index t.

Distribution of Returns (Cont'd)

- Time series aspect of financial econometrics: the distribution of $\{r_{i1}, \ldots, r_{iT}\}$ for a given asset i,

$$\begin{aligned} F(r_{i1}, \ldots, r_{iT}; \boldsymbol{\theta}) &= F(r_{i1})F(r_{i2}|r_{i1}) \cdots F(r_{iT}|r_{i,T-1}, \ldots, r_{i1}) \\ &= F(r_{i1}) \prod_{t=2}^{T} F(r_{it}|r_{i,t-1}, \ldots, r_{i1}). \end{aligned}$$

and, using probability density functions,

$$f(r_{i1}, \ldots, r_{iT}; \boldsymbol{\theta}) = f(r_{i1}; \boldsymbol{\theta}) \prod_{t=2}^{T} f(r_{it}|r_{i,t-1}, \ldots, r_{i1}; \boldsymbol{\theta}).$$

Distribution of Returns (Cont'd)

Some useful statistical distributions:

- Normal Distribution: the simple returns $\{R_{it}|t = 1, \ldots, T\}$ are traditionally assumed to be $i.i.d.$Normal.
  - Advantage: tractable statistical properties
  - Disadvantage:
    (1) simple return has lower bound -1;
    (2) $R_{it}[k]$ is not normally distributed;
    (3) inconsistent with the stylized facts of asset returns.

Distribution of Returns (Cont'd)

- Lognormal Distribution: the log returns r_t of an asset are
$$r_t \sim i.i.d. N(\mu, \sigma^2).$$
The simple return are then $i.i.d.$ lognormal random variables with mean and variance given by
$$E(R_t) = \exp(\mu + \frac{\sigma^2}{2}) - 1, \mathrm{Var}(R_t) = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1].$$
Let m_1 and m_2 be the mean and variance of the simple return R_t, which is lognormally distributed. Then the mean and variance of the corresponding log return r_t are
$$E(r_t) = \ln\left[\frac{m_1 + 1}{\sqrt{1 + m_2/(1 + m_1)^2}}\right], \mathrm{Var}(r_t) = \ln\left[1 + \frac{m_2}{(1 + m_1)^2}\right].$$
    - Advantage: (1) $r_t[k]$ is normally distributed; (2) no lower bound for r_t.
    - Disadvantage: inconsistent with all the properties of historical stock returns, for example, many stock returns exhibit a positive excess kurtosis.

Distribution of Returns (Cont'd)

- Scale Mixture of Normal Distribution: Recent studies of stock returns tend to use scale mixture or finite mixture of normal distributions. An example of finite mixture of normal distributions is

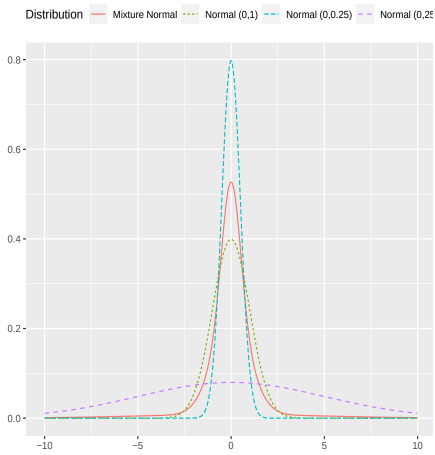
$$r_t \sim (1 - X)N(\mu, \sigma_1^2) + XN(\mu, \sigma_2^2), \quad \sigma_1^2 < \sigma_2^2,$$

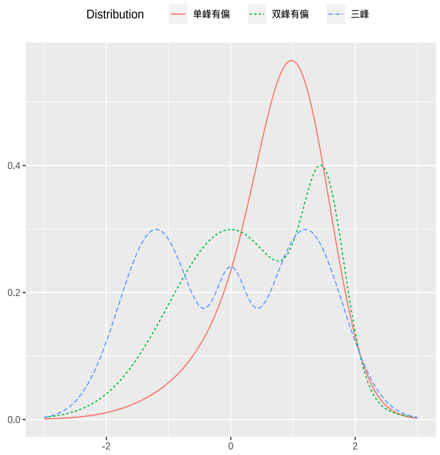
where X is a Bernoulli random variable such that $P(X = 1) = \alpha$ and $P(X = 0) = 1 - \alpha$ with $0 < \alpha < 1$.

For instance, $\alpha = 0.05$:

$$\begin{aligned} &N(\mu, \sigma_1^2), \quad \text{with the probability } 95\%, \\ &N(\mu, \sigma_2^2), \quad \text{with the probability } 5\%. \quad \text{Heavy Tails!} \end{aligned}$$

  - Advantage: maintain the tractability of normal; have finite higher order moments; capture the excess kurtosis.
  - Disadvantage: hard to estimate the mixture parameters.


Distribution
Mixture Normal
Normal (0,1)
Normal (0,0.25)
Normal (0,25
0.8
0.6
0.4
0.2
0.0
−10
−5
0
5
10


Distribution
单峰有偏
双峰有偏
三峰
0.4
0.2
0.0
-2
0
2