Stat 134 lec 16

last time

sec 3.4 Geometric distribution $(Geom(p))$

$X = \#$ p-coin tosses until the first heads

takes values $1, 2, 3, \ldots$

$P(X=k) = q^{k-1}p$

$E(X) = \frac{1}{p}$

$Var(X) = \frac{q}{p^2}$

Negative Binomial distribution $(NegBin(r,p))$

$T_r = W_1 + \cdots + W_r$ where $W_1, W_2, \ldots W_r \overset{iid}{\sim} Geom(p)$

$P(T_r = k) = \binom{k-1}{r-1} p^r q^{k-r}$

$E(T_r) = \frac{r}{p}$

$Var(T_r) = \frac{rq}{p^2}$

Today

1. Finish sec 3.4 Minimum of independent geometrics
2. Poisson distribution
3. Poisson random scatter (PRS) AKA Poisson Process
4. Poisson thinning

① Sec 3.4 Minimum of independent geometrics

Adam, Beth and John independently flip a p_1, p_2, p_3 coin respectively, let X = # trials until Adam, Beth or John get a heads.

eg

A	TTT	$X_1 \sim \text{Geom}(p_1)$
B	TTT	$X_2 \sim \text{Geom}(p_2)$
J	$\underbrace{\text{TTH}}_{X=3}$	$X_3 \sim \text{Geom}(p_3)$

a) what is probability Adam, Beth or John get a head?

p = Prob(A or B or J get heads)

= 1 − Prob(A, B, J don't get heads)

$= \boxed{1 - q_1 q_2 q_3}$

b) what distribution is X?

$X = \min(X_1, X_2, X_3) \sim \text{Geom}(1 - q_1 q_2 q_3)$

② Sec 3.5 Poisson distribution $(\text{Pois}(\mu))$

$X \sim \text{Pois}(\mu)$

$$P(X=k) = \frac{e^{-\mu}\mu^k}{k!} \quad k=0,1,2,\ldots$$

Find $E(X)$, $Var(X)$.

Recall $e^{\mu} = 1 + \mu + \frac{\mu^2}{2!} + \cdots$ Taylor series.

$$E(X) = \sum_{k=0}^{\infty} k \cdot P(X=k) = \sum_{k=0}^{\infty} k e^{-\mu} \frac{\mu^k}{k!}$$

$$= \sum_{k=1}^{\infty} \cancel{k} e^{-\mu} \frac{\mu^{k-1}\mu}{(k-1)!\cancel{k}} \qquad \left(\text{note } 0 \cdot e^{-\mu}\frac{\mu^0}{0!} = 0\right)$$

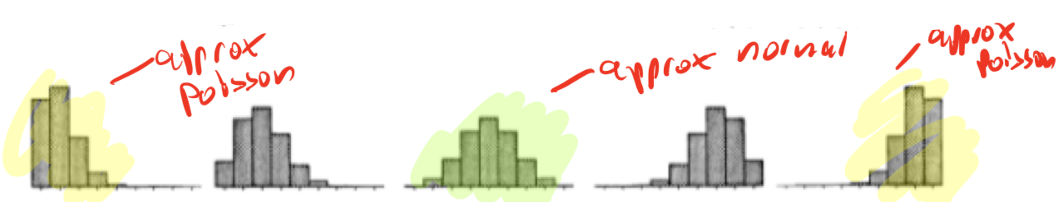
$$= \mu e^{-\mu} \sum_{k=1}^{\infty} \frac{\mu^{k-1}}{(k-1)!}$$

$$= \mu e^{-\mu} \underbrace{\left(1 + \mu + \frac{\mu^2}{2!} + \cdots\right)}_{e^{\mu}} = \boxed{\mu}$$

This makes sense since

$\text{Bin}(n,p) \to \text{Pois}(\mu)$ when $n \to \infty$, $p \to 0$, $np \to \mu$.

n = 8



Also we expect

$$npq \longrightarrow \mu q \simeq \mu \quad \text{since } q \to 1$$

So $\mathrm{Var}(X)$ should be μ,

let's check:

$$\mathrm{Var}(X) = E(X^2) - E(X)^2$$
$$= E(X^2) - E(X) + E(X) - E(X)^2$$
$$= E(X(X-1)) + E(X) - E(X)^2$$

$$E(X(X-1)) = \sum_{k=0}^{\infty} \cancel{k}(\cancel{k-1}) \underbrace{P(X=k)}_{= \frac{e^{-\mu}\mu^k}{\cancel{k}(\cancel{k-1})(k-2)!}}$$

$$= e^{-\mu} \sum_{k=2}^{\infty} \frac{\mu^k}{(k-2)!}$$

$$= e^{-\mu} \mu^2 \underbrace{\sum_{k=2}^{\infty} \frac{\mu^{k-2}}{(k-2)!}}_{= e^{\mu}} = \mu^2$$

$$\Rightarrow \mathrm{Var}(X) = \mu^2 + \mu - \mu^2 = \boxed{\mu}$$

ex Let $X \sim \text{Pois}(\mu)$

Find $E(X(X+1))$

$$= E(X^2+X) = E(X^2) + E(X)$$

$$= \text{Var}(X) + (E(X))^2 + E(X)$$

$$= \mu + \mu^2 + \mu$$

$$= \boxed{2\mu + \mu^2}$$

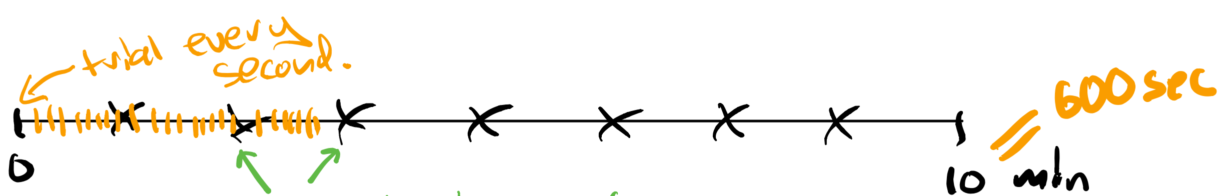
(3) Poisson Random Scatter (PRS)

$\text{Pois}(np) \approx \text{Bin}(n,p)$ with large n and small p

models situations with low probability, independent trials.

ex X = # number of calls coming into a hotel reservation center in 10 minutes.

Say $\mu = 5$ call in 10 min



The distribution of calls should look random not clustered since we have independent trials w/ same p.

PRS assumptions

1) No time interval gets more than one call

2) Have n iid Bernoulli p trials with $\mu = np$ large n, small p.
(i.e all calls are independent of each other with the same probability)

μ = mean number of calls in t min

let $\lambda = \frac{\mu}{t}$ be the intensity (rate) of calls/min for our PRS.

ex $\lambda = 5/10 = 1/2$ calls/min in above example.

Stat 134

Friday September 28 2018

1. Which of the following can be modeled as a Poisson Random Scatter with intensity $\lambda > 0$?

a The number of blueberries in a 3 cubic inch blueberry muffin

b The number of patients entering a doctor's office in a 24 hour period.

— no patients outside regular hours

c The number of times a day a person feels hungry

— not indep hits

d The number of air pulses counted every second from cars driving over an empty rubber hose lying across a highway between noon and 1pm.

— not indep hits
also can have multiple hits each trial.

e more than one of the above

ex (number of bb in a bb muffin)

A tub of bb muffin batter has
$\lambda = 2 \text{ bb}/\text{in}^3$.

A muffin is 3 in^3.

On average how many bb are there per muffin?

$\mu = \lambda \cdot 3 \text{ in}^3$
$= 2 \text{ bb}/\text{in}^3 \cdot 3 \text{ in}^3 = \boxed{6 \text{ bb}}$

Let X_1 = # bb in 1st muffin

$X_1 \sim \text{Pois}(6)$

Another muffin is 4 in^3 (from the same batter)

let X_2 = # bb in 2nd muffin.

a) Find P(5 bb in each muffin)

$X_1 \sim \text{Pois}(6)$
$X_2 \sim \text{Pois}(8)$

$$P(X_1 = 5, X_2 = 5) = \boxed{\frac{e^{-6} 6^5}{5!} \cdot \frac{e^{-8} 8^5}{5!}}$$

b) Find P (10 bb total in both muffin together)

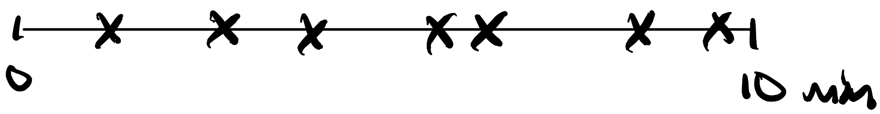
$X_1 + X_2 \sim \text{Pois}(14)$

$$P(X_1 + X_2 = 10) = \boxed{\frac{e^{-14} 14^{10}}{10!}}$$

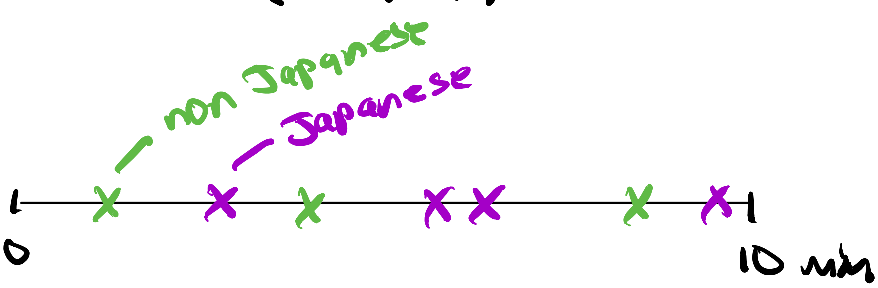
④ **Sec 3.5** **Poisson thinning**

Cars arrive at a toll booth according to a Poisson process at a rate $\lambda = 3$ arrivals/min

X = # cars arriving at a toll booth in 10 min. $X \sim \text{Pois}(\underbrace{\lambda \cdot 10}_{30})$



Of cars arriving, it is known, over the long term, that 60% are Japanese imports.



Call Japanese cars a success and non Japanese a failure.

Each hit is a success with $P = .6$, independent of all other hits.

Then the process of "success" hits in your PRS is a PRS with intensity λp and the process "failure" hits in your PRS is a PRS with intensity λq