Stat 134 Lec 10

Last time Sec 3.1

- Random variables are great!
  (1) RVs provide an efficient way describe an event.

ex Flip a coin 3 times. $X = \#$ heads.
$X=1$ means $\{HTT, THT, TTH\}$.

(2) We can describe a RV by its distribution
ex $X \sim Bin(n,p)$ or $X \sim Hyper(N,G,n)$ or
$X \sim N(\mu, \sigma^2)$

- We may combine 3 RVs to make a joint distribution (X,Y,Z).
  Independence means
  $$P(X=x, Y=y, Z=z) = P(X=x)P(Y=y)P(Z=z)$$
  for all $x \in X$, $y \in Y$, $z \in Z$.

- The sum of independent binomials with the same p is binomial.

$$\left.\begin{array}{l} X_1 \sim Bin(n_1, p) \\ X_2 \sim Bin(n_2, p) \\ X_3 \sim Bin(n_3, p) \end{array}\right\} \text{independent}$$

$E(X_1) = n_1 p$
$E(X_2) = n_2 p$
$E(X_3) = n_3 p$

$X_1 + X_2 + X_3 \sim Bin(n_1 + n_2 + n_3, p)$ $\quad E(X_1+X_2+X_3) = (n_1+n_2+n_3)p$

Note $E(X_1+X_2+X_3) = E(X_1) + E(X_2) + E(X_3)$

<u>Today</u> – <u>Sec 3.1</u> Sum of independent Poissons is Poisson.

– <u>Sec 3.2</u> expectation of RVs.

<u>Sec 3.1</u> <u>Sum of independent Poisson</u>

<u>Recall</u> binomial theorem

$$(a+b)^3 = \binom{3}{3}a^3b^0 + \binom{3}{2}a^2b^1 + \binom{3}{1}a^1b^2 + \binom{3}{0}a^0b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

<u>Recall</u> $X \sim \text{Pois}(\mu)$

$$P(X=k) = \frac{e^{-\mu}\mu^k}{k!} \leftarrow$$

Claim If $X \sim \text{Pois}(\mu)$ and $Y \sim \text{Pois}(\lambda)$ are independent then

$$S = X + Y \sim \text{Pois}(\mu + \lambda).$$

Pf/

$$P(S=s) = P(X=0, Y=s) + P(X=1, Y=s-1) + \dots P(X=s, Y=0)$$

addition rule

$$= \sum_{k=0}^{s} P(X=k, Y=s-k)$$

independence of X, Y

$$= \sum_{k=0}^{s} P(X=k)P(Y=s-k)$$

$$= \sum_{k=0}^{s} \frac{e^{-\lambda}\mu^{k}}{k!} \cdot \frac{e^{-\mu}\mu^{s-k}}{(s-k)!}$$

$\frac{s!}{s!} = 1$

$$= e^{-(\lambda+\mu)} \frac{1}{s!} \sum_{k=0}^{s} \frac{s!}{k!(k-s)!} \mu^{k} \lambda^{s-k}$$

$= \binom{s}{k}$

binomial theorem

$$= e^{-(\lambda+\mu)} \frac{1}{s!} (\mu+\lambda)^{s}$$

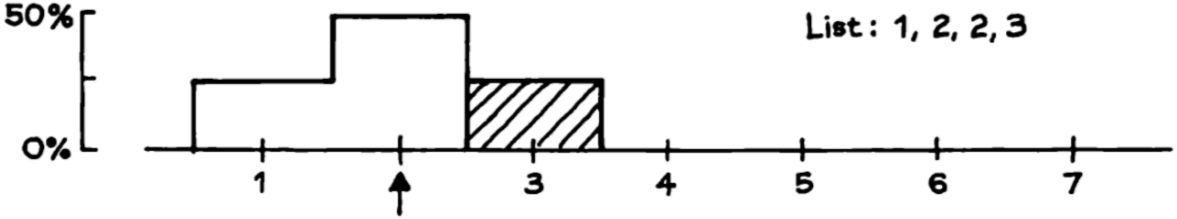
$$\Rightarrow S \sim \text{Pois}(\mu+\lambda).$$

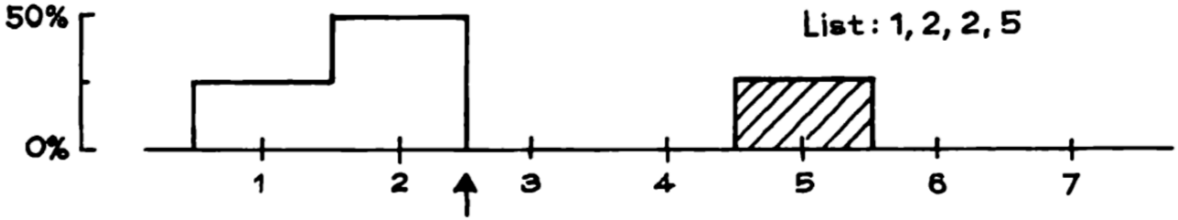
$\square$

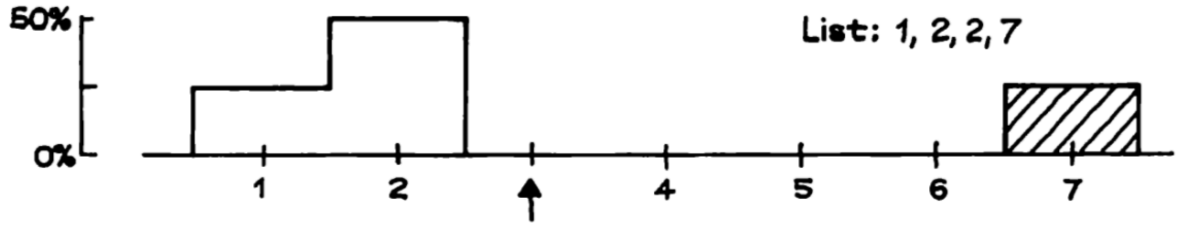
Sec 3.2 Expectation

$$E(X) = \sum_{x \in X} x \cdot P(X=x)$$

$$E(X) = \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 3 = 2$$







Properties of Expectation

① $E(c) = c$

② $E(X+Y) = E(X) + E(Y)$ (X, Y <u>don't</u> need to be independent)

③ $E(aX+b) = aE(X) + b$

Indicators

An indicator is a RV that has only 2 values 1 (w/ prob p) and 0 (with prob 1-p).

RV that are counts can often be written as a sum of indicators.

$$I = \begin{cases} 1 & \text{prob } p \\ 0 & \text{prob } 1-p \end{cases}$$

$$E(I) = 1 \cdot p + 0 \cdot (1-p) = p$$

ex $X \sim Bin(n, p)$

← # successes in n Bernoulli p trials,

ex X = # heads in n flips of p coin

$$X = I_1 + I_2 + \cdots + I_n$$

where $I_j = \begin{cases} 1 & \text{if } j^{th} \text{ trial success} \\ 0 & \text{else} \end{cases}$

p

$$E(X) = \underset{\overset{\shortparallel}{p}}{E(I_1)} + \cdots + \underset{\overset{\shortparallel}{p}}{E(I_n)} = \boxed{np}$$

indicators are independent since trials are indep.

Ex $X \sim \text{Hyper}(N, G, n)$

ex $X = \#$ aces in a poker hand from a deck of cards

0, 1, 2, 3, 4

$N = 52$
$G = 4$
$n = 5$

Find $E(X)$.

$X = I_1 + I_2 + \cdots + I_n$

$$\text{where } I_j = \begin{cases} 1 & \text{if } j^{th} \text{ card is ace} \xleftarrow{} \frac{G}{N} \\ 0 & \text{else} \end{cases}$$

$$\boxed{E(X) = n \cdot \frac{G}{N}}$$

These indicators are dependent since the trials are dependent.

ex Suppose a fair die is rolled 10 times. Find the expected number of different faces that appear in 10 rolls.

ex if roll 2,3,4,2,3,5,2,3,3,2 then $X=4$

Steps

Step 1 Write what X is and find its range of values.

Step 2 Write X as a sum of indicators

Step 3 Find p.

$X = $ # of different faces in 10 rolls

1, 2, 3, 4, 5, 6

ex roll 2,3,4,2,3,5,2,3,3,2

$$X = I_1 + I_2 + I_3 + I_4 + I_5 + I_6$$

$$1 - \left(\frac{5}{6}\right)^{10}$$

where $I_j = \begin{cases} 1 & \text{if } j^{th} \text{ face appears} \\ 0 & \text{else.} \end{cases}$

i.e

$I_2 = \begin{cases} 1 & \text{if the face 2 appears} \\ 0 & \text{else} \end{cases}$

$$E(X) = \boxed{6\left(1-\left(\frac{5}{6}\right)^{10}\right)}$$

Stat 134

Chapter 3 **Friday September 14 2018**

1. A forgetful valet is attempting to return n cars to their n rightful owners. For each driver, the valet remembers the car correctly 5% of the time; otherwise the valet retrieves a car at random (possibly the correct car). Let N be the number of drivers who retrieve their own car. $E(N)$ is:

   a $.05n$

   b $.05n + .95$

   c $.05n + 1$

   d none of the above

N = # drivers who retrieve their own car

0, 1, 2, ... n.

P

$N = I_1 + I_2 + \cdots + I_n$, $I_j = \begin{cases} 1 & \text{if } j^{th} \text{ driver retrieves their own car.} \\ 0 & \text{else.} \end{cases}$

Note first that I_1 and I_2 are <u>dependent</u> since if the 1st person gets their own car it increases the chance the second person will get their own car.

Note also that P is an <u>unconditional</u> probability. So the chance the 2^{nd} person gets their own car is the same the chance the 1^{st} person get their own car.

add^n rule

$$P = P(\text{valet remembers}) + P(\text{valet forgets and chooses correctly})$$

$$P(\text{valet remembers}) = .05$$

$$P(\text{valet forgets and chooses correctly}) = P(\text{chooses correct} \mid \text{forgets}) \cdot P(\text{forgets}) \quad \text{— mult rule}$$

$$P(\text{chooses correct} \mid \text{forgets}) = \frac{1}{n}, \qquad P(\text{forgets}) = .95$$

$$\Rightarrow E(N) = n\left(.05 + \frac{.95}{n}\right) = \boxed{.05n + .95}$$