Stat 134 lec 34

warmup 1:00-1:10

8 transistors (type 1) are distributed $Exp(\frac{1}{100})$ and 4 transistors (type 2) are $Exp(\frac{1}{200})$.
Let T be the lifetime of a randomly picked transistor.

a) Find $E(T \mid \text{transistor is type } 1)$

b) Find $E(T)$

a) $T \mid \text{type } 1 \sim Exp(\frac{1}{100})$

$E(T \mid \text{type } 1) = 100$

b)

Let X = type of transistor

$$E(T) = E(E(T \mid X))$$

$$= E(T \mid X=1) \cdot P(X=1) + E(T \mid X=2) P(X=2)$$

$$= 100 \cdot \frac{8}{12} + 200 \cdot \frac{4}{12} = 133.3$$

Practice midterm 2 and review materials will be posted soon.

Please ask questions on b-courses/discussion for in class wednesday review.

Last time (Sec 6.1, 6.2)

Rule of average conditional probabilities (discrete case)

marginal prob of X

$$P(X=x) = \sum_{s} P(X=x, S=s)$$

$$= \sum_{s} P(X=x \mid S=s) P(S=s)$$

Conditional expectation

$$E(T \mid S=s) = \sum_{t \in T} t \cdot P(T \mid S=s)$$

$E(T \mid S)$ is a RV (function of S)

Law of Iterated Expectation

$$E(T) = E(E(T \mid S))$$

$$E(T) = \sum_{\text{all } s} E(T \mid S=s) \cdot P(S=s)$$

recall

$N \sim \text{Geom}(p)$ on $n = 1, 2, 3, \ldots$

$$P(N=n) = pq^{n-1}$$

Useful identities

$$\sum_{n=0}^{\infty} q^n = 1+q+q^2+\cdots = \frac{1}{1-q} \quad \text{geometric sum}$$

$$\frac{d}{dq}: \quad \sum_{n=1}^{\infty} n q^{n-1} = 1+2q+3q^2+\cdots = \frac{1}{(1-q)^2}$$

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

ex

Let $N \sim \text{Geom}(p)$ on $1,2,3,\ldots$

Suppose $X|N=n \sim \text{Unif}(0,1,2,\ldots,n)$

Find $E(X)$

$X|N=n \sim \text{Unif}(0,1,2,\ldots,n)$

$$E(X|N=n) = \frac{0+1+\cdots+n}{n+1} = \frac{n(n+1)}{2(n+1)} = \frac{n}{2}$$

$$E(X) = \sum_{n=1}^{\infty} \underbrace{E(X|N=n)}_{=\frac{n}{2}} \underbrace{P(N=n)}_{=q^{n-1}p} = \frac{p}{2}\sum_{n=1}^{\infty} n q^{n-1} = \frac{p}{2}\frac{1}{p^2} = \boxed{\frac{1}{2p}}$$

or $\quad E(X|N) = \frac{N}{2}$

$$E(X) = E(E(X|N)) = E\left(\frac{N}{2}\right) = \frac{1}{2p}$$

Today

1. Sec 5.4 General convolution formula
2. Sec 6.2 Properties of conditional expectation.
3. Sec 6.3

   Conditional densities.

① sec 5.4 General convolution formula (not on midterm)

we have different convolution formulas for sums and quotients.

we can write a general convolution formula for any operation.

1 dimensional change of variables

RV	transformed RV
Y	$z(Y)$ a 1-1 differentiable function
f_Y	$f_z = \left\lvert \frac{\partial y}{\partial z} \right\rvert f_Y$

$\frac{1}{\left\lvert \frac{\partial z}{\partial y} \right\rvert}$

ex $z = y^3$

2 dimensional change of variables

RV	transformed RV
(X, Y)	$(X, z(x,y))$ a 1-1 differentiable function
$f_{X,Y}$	$f_{X,Z}$

ex $Z = X+Y$

ex $Z = \frac{X}{Y}$

$$f_{X,Z} = \left\lvert \det \frac{\partial(x,y)}{\partial(x,z)} \right\rvert f_{X,Y}$$

$$= \left\lvert \det \begin{bmatrix} \underset{=1}{\frac{\partial x}{\partial x}} & \frac{\partial x}{\partial z} \\ \underset{=0}{\frac{\partial y}{\partial x}} & \frac{\partial y}{\partial z} \end{bmatrix} \right\rvert f_{X,Y}$$

$$= f_{X,Y} \left\lvert \frac{dy}{dz} \right\rvert$$

Convolution formula

let $z(x,y)$ be a 1-1 differentiable function of x, y

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Z}(x,z)\, dx = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y) \left\lvert \frac{\partial y}{\partial z} \right\rvert dx$$

Ex Let $Z = \frac{Y}{X}$. Find the convolution formula for Z.

$\Rightarrow Y = XZ \Rightarrow \frac{\partial y}{\partial z} = x$

$$\Rightarrow f_Z(z) = \int_{x=-\infty}^{x=\infty} f_{X,Y}(x, xz)\,|x|\,dx$$

Convolution formula for quotient.

Ex Let $Z = \frac{X}{X+Y}$. Find the convolution formula for Z.

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\left|\frac{\partial y}{\partial z}\right| dx$$

Step 1 Solve for Y

$Zx + ZY = X \Rightarrow ZY = X - ZX$

$\Rightarrow Y = \frac{X(1-Z)}{Z}$

Step 2 Find $\frac{\partial y}{\partial z}$

$$\Rightarrow \frac{\partial Y}{\partial Z} = x\left[\frac{(1-Z)'Z - (1-Z)(Z)'}{Z^2}\right]$$

$$= x\left[\frac{-Z - 1 + Z}{Z^2}\right] = \frac{-x}{Z^2}$$

Step 3 Substitute $y, \frac{dy}{dz}$ in $f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\left|\frac{\partial y}{\partial z}\right| dx$

$$\Rightarrow f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}\left(x, \frac{x(1-Z)}{Z}\right)\frac{|x|}{Z^2}\,dx$$

② Sec 6.2 Properties of Conditional expectation

$(Y+Z)|X=x \;=\; Y|X=x \;+\; Z|X=x$ so

$$E(Y+Z|X=x) = E(Y|X=x) + E(Z|X=x)$$

What is $E(X+Z|X=5) = ?$

$5 + E(Z|X=5)$

$$E(X+Y|X) = \underbrace{E(X|X)}_{= X} + E(Y|X)$$

Properties

1. $E(X) = E(E(X|Y))$ equality of numbers
2. $E(aY+b|X) = aE(Y|X)+b$
3. $E(Y+Z|X) = E(Y|X)+E(Z|X)$
4. $E(g(X)|X) = g(X)$
5. $E(g(X)Y|X) = g(X)E(Y|X)$

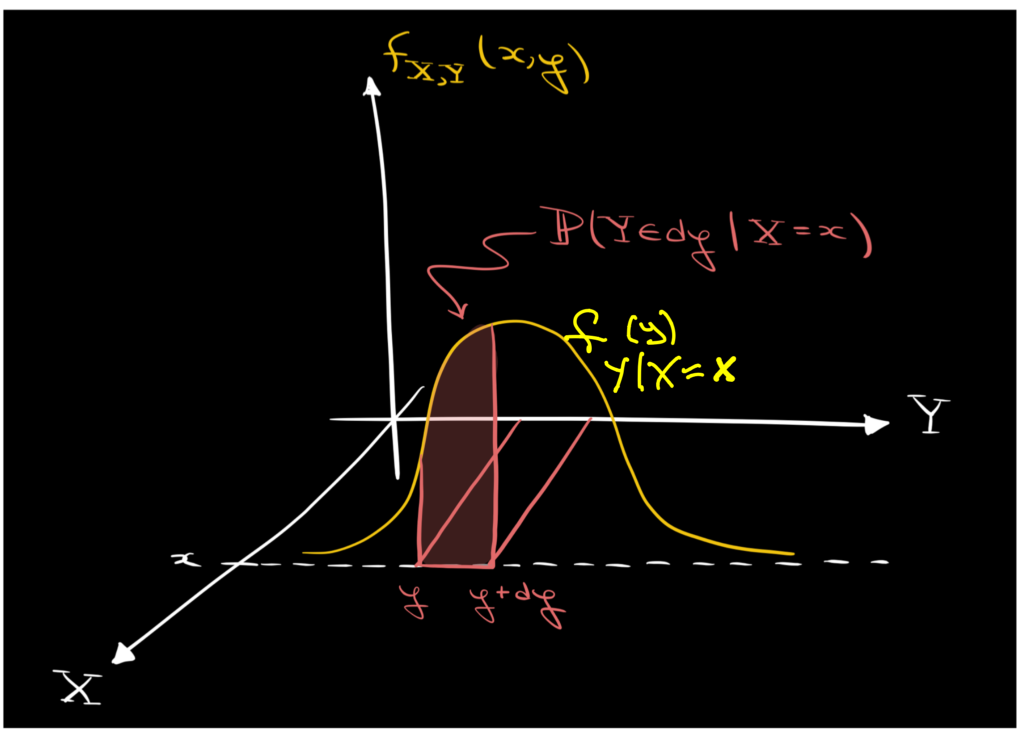
equality of RV

③ sec 6.3 Conditional Density:

ⓐ Let X, Y be continuous RVs with joint density $f_{X,Y}(x,y)$

Let $f_{Y|X=x}(y)$ be a slice of $f_{X,Y}(x,y)$ through $X=x$.

Define $P(Y \in dy \mid X=x)$ as the area under $f_{Y|X=x}(y)$ for $Y \in dy$



By Baye's rule,

$$\underbrace{P(Y\in dy \mid X=x)}_{= f_{Y|X=x}(y)\,dy} = \lim_{dx\to 0} \frac{\overbrace{P(Y\in dy, X\in dx)}^{= f(x,y)\,dx\,dy}}{\underbrace{P(X\in dx)}_{\approx f_X(x)\,dx}}$$

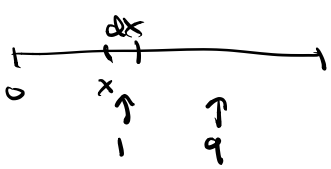
$$\boxed{f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)}}$$

Conditional density of Y given $X=x$

ex Let $U_1,\dots,U_{10} \overset{iid}{\sim} U(0,1)$

$X=U_{(1)}$, $Y=U_{(10)}$

$f(x,y)=90\,(y-x)^8$



$$f_X(x) = \binom{10}{1,9}(1-x)^9 = 10(1-x)^9$$

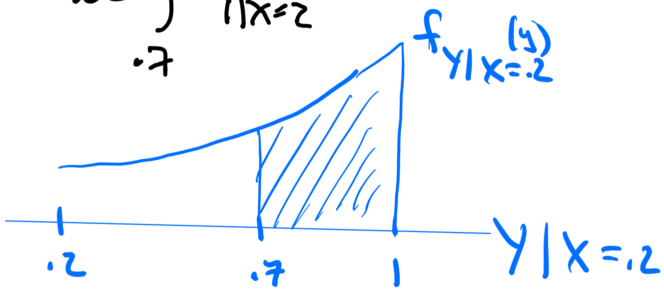
Find $P(Y>.7 \mid X=.2) = \int_{.7}^{1} f_{Y|X=2}(y)\,dy$

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)}$$

a) Find $f_{Y|X=.2}(y)$

$$f_{Y|X=.2}(y) = \frac{f(.2,y)}{f_X(.2)} = \boxed{\frac{90(y-.2)^8}{10(.8)^9}}$$

b) Find $\int_{.7}^{1} f_{Y|X=2}(y)\,dy$



$$\frac{9}{(.8)^9}\int_{.7}^{1}(y-.2)^8\,dy$$

let $u = y-.2$

$$= \frac{9}{(.8)^9}\int_{u=.5}^{u=.8} u^8\,du$$

$$= \frac{\cancel{9}}{(.8)^9}\frac{u^9}{\cancel{9}}\Bigg|_{.5}^{.8} = \boxed{1-\left(\frac{.5}{.8}\right)^9}$$

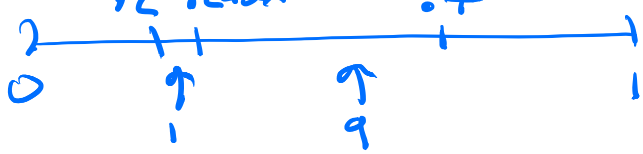
Method 2 (use fact that $X = U_{(1)}$ and $Y = U_{(10)}$)

$P(Y > .7 \mid X = .2) = 1 - P(Y < .7 \mid X = 0.2)$

By Bayes' rule,

$$P(Y < .7 \mid X = 0.2) = \lim_{dx \to 0} \frac{P(Y < .7,\ X \in .2 + dx)}{P(X \in .2 + dx)}$$

Picture



$P(Y < .7, X \in .2 + dx)$ is the chance that the $U_{(1)}$ is in dx and the remaining 9 darts land between .2 and .7

this is $\binom{10}{1,9} 1\, dx (.7 - .2)^9 = 10\, dx (.5)^9$.

$P(X \in .2 + dx)$ is the chance that the $U_{(1)}$ is in dx and the remaining 9 darts land between .2 and 1.

this is $\binom{10}{1,9} 1\, dx (1 - .2)^9 = 10 \cdot dx (.8)^9$.

Hence $$P(Y > .7 \mid X = .2) = \lim_{dx \to 0} 1 - \frac{\cancel{10\, dx} (.5)^9}{\cancel{10\, dx} (.8)^9}$$

$$= \boxed{1 - \left(\frac{.5}{.8}\right)^9}$$