Stat 134 lec 29

Quiz 5 Wednesday sec 4.4–4.6, MGF and 5.1, 5.2

Last time.

① Sec 4.6 Beta Distribution

Let $r, s > 0$

$P \sim \text{Beta}(r,s)$ if

$$f(p) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} p^{r-1}(1-p)^{s-1} \quad \text{for } 0 < p < 1$$

Applications

a) Beta(r, s) takes values between 0 and 1 and commonly models the prior distribution of a probability in Bayesian statistics.

b) generalization of uniform ordered statistic

If throw n darts at $[0,1]$

$U_{(k)} \sim \text{Beta}(k, n-k+1)$

Note $U(0,1) = \text{Beta}(1,1)$ — $n=1$, $k=1$

Sec 5.2 Competing exponentials.

Let $X \sim \text{Exp}(\lambda_1)$, $Y \sim \text{Exp}(\lambda_2)$ independent

Think of combined Poisson process — rate $\lambda_1 + \lambda_2$

we saw, $P(X<Y) = \frac{\lambda_1}{\lambda_1+\lambda_2}$

By Poisson thinning — is Poisson process rate $(\lambda_1+\lambda_2) \cdot \frac{\lambda_1}{\lambda_1+\lambda_2} = \lambda_1$

Today

① Sec 5.2 Competing exponentials

② Sec 5.2 Marginal density, expectation $E(g(X,Y))$

① sec 5.2 Competing exponentials

ex

(Exponential distributions) You are first in line to have your question answered by either of the 2 uGSIs, Brian and Yiming, who are busy with their current students. Your question will be answered as soon as the first uGSI finishes. The times spent with student by Brian and Yiming are independent and exponentially distributed with (positive) rates λ_B and λ_Y respectively, i.e. Brian's distribution is Exponential(λ_B), and Yiming's is Exponential(λ_Y).

(a) Find the probability that Yiming will be the one answering your questions.

B = waiting time for Brian $B \sim Exp(\lambda_B)$
Y = " " " Yiming $Y \sim Exp(\lambda_Y)$ } ind

$$\boxed{P(Y<B) = \frac{\lambda_Y}{\lambda_Y+\lambda_B}}$$

(b) What is the distribution of your wait time? Your answer should not include integrals.

$W = \min(Y, B)$ is shortest waiting time

$$P(W > w) = P(\min(Y,B) > w) = P(Y>w, B>w)$$
$$= e^{-w\lambda_Y} \cdot e^{-w\lambda_B} = e^{-w(\lambda_Y+\lambda_B)}$$

$$\Rightarrow W \sim Exp(\lambda_Y + \lambda_B)$$

Think of $\min(Y,B)$ as a combined Poisson process w/ rate $\lambda_Y + \lambda_B$

So $P(Y = \min(Y, B)) = \dfrac{\lambda_Y}{\lambda_Y + \lambda_B}$

If $X_1, \dots, X_n$ are indep exponentials with rates $\lambda_1, \dots, \lambda_n$

$P(X_i = \min(X_1, \dots, X_n)) = \ ?$ — $\dfrac{\lambda_i}{\lambda_1 + \lambda_2 + \dots + \lambda_n}$

Now have 3 GSI, Yiming, Bilan and Rowan. What is chance Yiming done first, then Bilan and then Rowan (indep exponentials with rates $\lambda_Y, \lambda_B, \lambda_R$)?

i.e. $P(Y < B < R)$

$Y = \min(Y, B, R)$, $B = \min(B, R)$ given $Y = \min(Y, B, R)$

Reset clock when $Y = \min(Y, B, R)$ so this event is independent of $B = \min(B, R)$

$\Rightarrow P(Y = \min(Y, B, R)) \cdot P(B = \min(B, R))$

$= \dfrac{\lambda_Y}{\lambda_Y + \lambda_B + \lambda_R} \cdot \dfrac{\lambda_B}{\lambda_B + \lambda_R}$

(2) Sec 5.2 Marginal densities

Recall marginal probability:

<u>discrete picture</u>

Y \ X	0	1	P(y)
2	0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
0	$\frac{1}{4}$	0	$\frac{1}{4}$
P(x)	$\frac{1}{2}$	$\frac{1}{2}$	

marginal probability of X

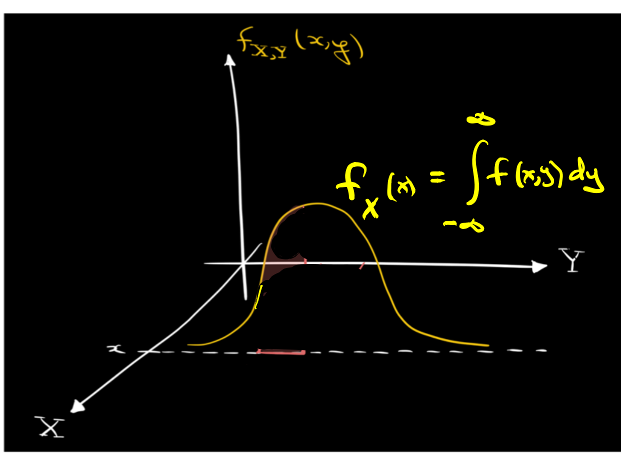
$P(x) = \sum_{y \in Y} P(x,y)$

Marginal prob of Y

$P(y) = \sum_{x \in X} P(x,y)$

2

Continuous Picture: marginal density



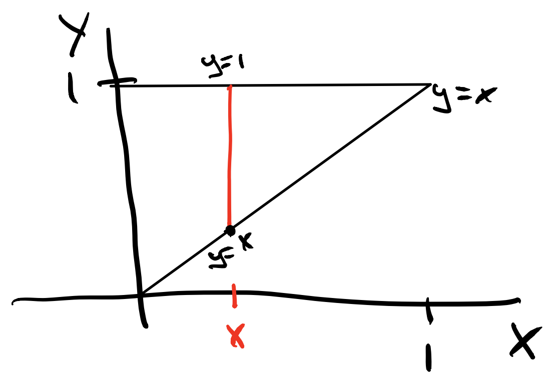
ex

joint density

$$f(x,y) = \begin{cases} 30(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$

$Y = U_{(6)}$



marginal density

$$f_X(x) = \int_{y=-\infty}^{y=\infty} f(x,y)\,dy$$

$$= \int_{y=x}^{y=1} 30(y-x)^4\,dy$$

$u = y-x$

$du = dy$

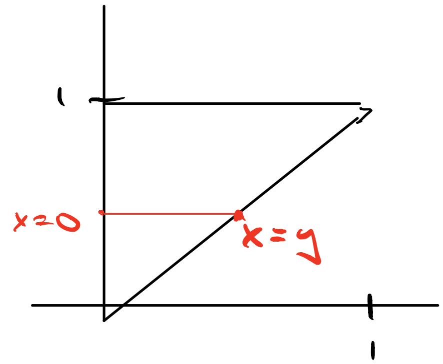
$$= \int_{u=0}^{u=1-x} 30u^4\,du = \frac{30u^5}{5}\Big|_0^{1-x} = \boxed{6(1-x)^5, \quad 0<x<1}$$

Find $f_Y(y)$

$$= \int_{x=0}^{x=y} 30\,(y-x)^4\,dx$$

$u = y-x$

$du = -dx$



$$= -\int_{u=y}^{u=0} 30u^4\,du = \frac{30u^5}{5}\Big|_0^y = \boxed{\begin{matrix} 6y^5 \\ 0<y<1 \end{matrix}}$$

Note

$$f(x,y) = 30\,(y-x)^4 \neq \underset{\underset{6(1-x)^5}{\shortparallel}}{f(x)}\;\underset{\underset{6y^5}{\shortparallel}}{f(y)}$$

so $X = U(1)$, $Y = U_{(6)}$ are dependent,

Expectation

$$E\big(g(x,y)\big) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} g(x,y) f(x,y)\,dx\,dy$$

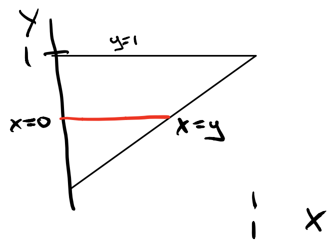
Ex

joint density

$$f(x,y) = \begin{cases} 30\,(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$

$Y = U_{(6)}$



Find

$$E(Y) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} y\, f(x,y)\,dx\,dy$$

$$= \int_{y=-\infty}^{y=\infty} y \underbrace{\int_{x=-\infty}^{x=\infty} f(x,y)\,dx}_{f_Y(y)}\,dy$$

$$= \int_{y=-\infty}^{y=\infty} y \underbrace{f_Y(y)}_{6y^5}\,dy = \int_{y=0}^{y=1} 6y^6\,dy$$

$$= \frac{6y^7}{7}\Big|_0^1 = 6/7$$

Note $Y \sim U_{(6)} = \text{Beta}(6,1) \Rightarrow E(Y) = \frac{6}{6+1}$ ✓

$\uparrow k \quad \uparrow n-k+1 = 6-6+1$

ex (5.2.9a)

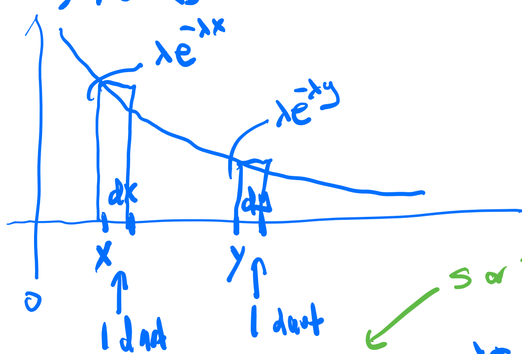
$S, T \overset{iid}{\sim} \text{Exp}(\lambda)$ $\left(f_S(s) = \lambda e^{-\lambda s}\right)$

$X = \min(S,T)$ $\leftarrow$ 1^{st} ordered statistic of Exp(λ)

$Y = \max(S,T)$ $\leftarrow$ 2^{nd} ordered statistic of Exp(λ)

Find the joint density of X and Y

$P(X \in dx, Y \in dy)$



S or T can go in dx

$$P(X \in dx, Y \in dy) = \binom{2}{1} \lambda e^{-\lambda x} dx \binom{1}{1} \lambda e^{-\lambda y} dy$$

$$= 2\lambda^2 e^{-\lambda(x+y)} dx\,dy$$

$$\Rightarrow \boxed{f(x,y) = 2\lambda^2 e^{-\lambda(x+y)}}$$

Next, lets find the marginal densities for X, Y.

Stat 134

Friday November 8 2019

1. S and T are i.i.d. Exp(λ). $X = \text{Min}(\text{S}, \text{T})$ and $Y = \text{Max}(\text{S}, \text{T})$. The joint density is $f(x, y) = 2\lambda^2 e^{-\lambda(x+y)}$. The marginal density of Y is:

   a $\lambda(1 - e^{-\lambda y})e^{-\lambda y}$ for $y > 0$

   b $2\lambda(1 - e^{-\lambda y})e^{-\lambda y}$ for $y > 0$

   c $2\lambda(1 - e^{-\lambda y})$ for $y > 0$

   d none of the above

$$f_Y(y) = 2\lambda^2 e^{-\lambda y} \int_0^y e^{-\lambda x}\,dx$$

$$= 2\lambda^2 e^{-\lambda y} \left(\frac{e^{-\lambda x}}{-\lambda}\right)\Bigg|_0^y$$

$$= \boxed{2\lambda(1 - e^{-\lambda y})e^{-\lambda y}} \quad y > 0$$