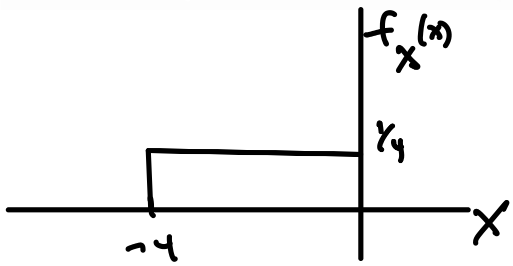
Stat 134 lec 25

Warmup 11:00-11:10

Suppose X has uniform $(-4,0)$ distribution. Find the density of X^2.



Change of variable formula:

$$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \frac{f_X(x)}{|g'(x)|} \quad \text{evaluated at } x = g^{-1}(y)$$

1) Find $g(x) = x^2$

2) Find $g'(x) = 2x$

3) Find $x = g^{-1}(y) = \pm\sqrt{y}$ — for neg x only $-\sqrt{y}$

4) Find $f_X(x) = \frac{1}{4} \cdot 1_{(x \in (-4,0))}$

5) Find $f_Y(y)$

$$\frac{\frac{1}{4} 1_{-\sqrt{y} \in (-4,0)}}{|-2\sqrt{y}|} = \frac{\frac{1}{4} 1_{y \in (0,16)}}{2\sqrt{y}}$$

$$= \boxed{\frac{1}{8\sqrt{y}} 1_{y \in (0,16)}}$$

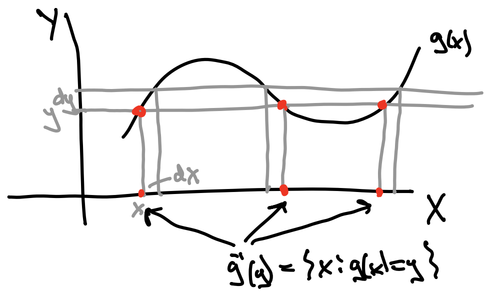
Feedback for EC due Sunday 10pm.

Announcement: Q3 next Wed covers Sec 4.1, 4.2, 4.4, MGF

Last 2 times

① sec 4.4 Change of Variable rule

many to one g:



$$f_Y(y)\,dy = f_X(x_1)dx_1 + f_X(x_2)dx_2 + f_X(x_3)dx_3$$

$$f_Y(y) = f_X(x_1)\frac{dx_1}{dy} + f_X(x_2)\frac{dx_2}{dy} + f_X(x_3)\frac{dx_3}{dy}$$

$$= \frac{f_X(x_1)}{g'(x_1)} + \frac{f_X(x_2)}{|g'(x_2)|} + \frac{f_X(x_3)}{g'(x_3)}$$

$P(x \in dx_2) \geq 0$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{g'(x)}$$

② MGF (not in book)

$$M_X(t) = E(e^{tX})$$

Thm If a MGF exists in an interval around zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$ Taylor for all $x \in \mathbb{R}$

ex $X \sim Pois(\mu)$

$$M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} \frac{\mu^k e^{-\mu}}{k!} = e^{-\mu} \sum_{k=0}^{\infty} \frac{(\mu e^t)^k}{k!} = e^{\mu e^t}$$ for all t

$$= e^{-\mu} e^{\mu e^t} = \boxed{e^{\mu(e^t - 1)} \text{ for all } t}$$

then $E(X) = \frac{d}{dt} M_X(t)\Big|_{t=0} = e^{\mu(e^t-1)} \cdot \mu e^t \Big|_{t=0} = \mu$ ✓

Today

① Key properties of MGF

② Recognizing a distribution from the variable part of its density.

1. Key Properties of MGF

(a) If an MGF exists in an interval containing zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

last time

(b) If X and Y are independent RVs,

$M_{X+Y}(t) = M_X(t) M_Y(t)$.

Proved in MGF HW.

(c) If $M_X(t) = M_Y(t)$ for all t in an interval around 0 then $F_X(z) = F_Y(z)$ (i.e. X and Y have the same distribution).

Skip proof — we can invert a MGF to get the CDF.

ex If $M_X(t) = \frac{1}{2}e^{1t} + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}$, $= E(e^{tX})$

e^{xt} tells us the value of X and the associated coefficients tell us the probabilities

(i.e. $X = 1, 2, 3$ w/ prob $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.)

so MGF ⇒ distribution of X when X has finite # values.

Property (a) is useful to find $E(X)$, $Var(X)$.

Property (b) and (c) allow us to prove for example that sum of independent Poisson is Poisson.

ex $X_1 \sim Pois(\mu_1)$, $X_2 \sim Pois(\mu_2)$ } independent.

Show that $X_1 + X_2 \sim Pois(\mu_1 + \mu_2)$.

$M_{X_1}(t) = e^{\mu_1(e^t - 1)}$ for all t

$M_{X_2}(t) = e^{\mu_2(e^t - 1)}$ for all t

$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t) = \boxed{e^{(\mu_1+\mu_2)(e^t - 1)}}$

MGF of Pois $(\mu_1 + \mu_2)$ for all t.

$\Rightarrow X_1 + X_2 \sim Pois(\mu_1 + \mu_2)$

ex Let X be a RV and a a constant.

Show that $M_{aX}(t) = M_X(at) \leftarrow E(e^{Xat})$

hint $M_{aX}(t) = E(e^{aXt})$

$= E(e^{Xat})$

$= M_X(at)$

For $X \sim \text{Gamma}(r, \lambda)$

recall $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$ for $t < \lambda$

ex Let $X \sim \text{Exp}(\lambda)$ and $a > 0$.

Show that $Y = aX$ is also exponential,

and specify the new parameter.

$M_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^1$ for $s < \lambda$, since $X \sim \text{Gamma}(1, \lambda)$

$M_Y(t) = M_{aX}(t) = M_X(at) = \left(\frac{\lambda}{\lambda - at}\right)^1$ for $at < \lambda$

$= \left(\frac{\frac{\lambda}{a}}{\frac{\lambda}{a} - t}\right)^1$ for $t < \frac{\lambda}{a}$ > 0

$\Rightarrow Y \sim \text{Exp}\left(\frac{\lambda}{a}\right)$

② Recognizing a distribution from the variable part of its density.

A density can be written as

$$f(t) = c\,h(t)$$

constant (c), variable part ($h(t)$).

$$1 = \int_{-\infty}^{\infty} f(t)\,dt = c\int_{-\infty}^{\infty} h(t)\,dt \implies c = \frac{1}{\int_{-\infty}^{\infty} h(t)\,dt}$$

So you can figure out the density from its variable part.

List of densities. Please circle their variable parts:

ex

$T \sim \text{Gamma}(r, \lambda) \quad f(t) = \frac{1}{\Gamma(r)} \lambda^r \, t^{r-1} e^{-\lambda t}, \; t > 0$

$T \sim \text{Normal}(\mu, \sigma^2) \quad f(t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$

$T \sim \text{Unif}(a, b) \quad f(t) = \frac{1}{b-a} \mathbb{1}_{(t \in (a,b))}$

$T_r \sim \text{Gamma}(r, \lambda)$, $r, \lambda > 0$ $\quad f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$

$T \sim \text{Exp}(\lambda)$, $\lambda > 0$ $\quad f(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$ ← variable part

ex Name the distribution with the following variable part **ex** Gamma$(r = 1/2, \lambda = 3)$

a) $h(t) = t^3 e^{-\frac{1}{2}t}$ Gamma$(r=4, \lambda = 1/2)$

b) $h(t) = e^{-\frac{1}{2}t^2}$ Normal$(\mu = 0, \sigma^2 = 1)$

c) $h(t) = e^{-3t}$ Exp$(\lambda = 3)$

d) $h(t) = t^{-1/2} e^{-t}$ Gamma$(r = 1/2, \lambda = 1)$

e) $h(t) = \mathbb{1}_{(t \in (0,1))}$ Unif$(0,1)$

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1. Let Z be a standard normal RV (with variable part $e^{-\frac{z^2}{2}}$). The variable part of the distribution $X = Z^2$ is?

   a Gamma $x^{-\frac{1}{2}}e^{-\frac{x}{2}}$

   b Gamma $x^{\frac{1}{2}}e^{-\frac{x}{2}}$

   c Exponential $e^{-\frac{x}{2}}$

   d Normal $e^{-\frac{x^2}{2}}$

   e none of the above

$$f_X(x) = \sum_{\{z:\, z=g(x)\}} \frac{f_Z(z)}{|g'(z)|}$$

$$f_X(x) \propto \frac{e^{-\frac{x}{2}}}{2\sqrt{x}} + \frac{e^{-\frac{x}{2}}}{2\sqrt{x}} = \frac{1}{\sqrt{x}}e^{-\frac{x}{2}} = x^{-\frac{1}{2}}e^{-\frac{x}{2}}$$

proportional

(don't worry about the constant)