Stat 134 lec 23

Last time sec 4.2 Gamma Distribution

$T \sim Exp(\lambda), \lambda > 0$

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$$

variable part

$T \sim Gamma(r, \lambda), \; r, \lambda > 0$

$$f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$$

where $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$ Gamma function

Notes about $T \sim Gamma(r, \lambda)$:

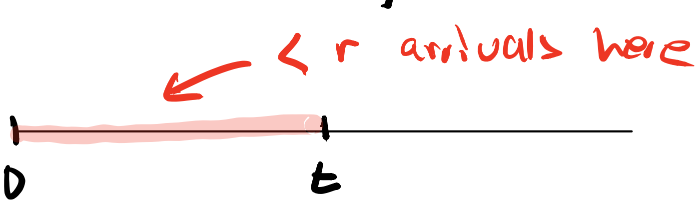
$\Gamma(r) = (r-1)!$ for $r \in \mathbb{Z}^+$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

If $r \in \mathbb{Z}^+$, T = time to r^{th} arrival

$Exp(\lambda) = Gamma(r=1, \lambda)$

$P(T > t) = P(N_t < r)$ where $N_t \sim Pois(\lambda t)$



Today sec 4.4 (skip 4.3)

1. Recognizing a distribution from the variable part of its density.
2. Change of variable formula for densities.

Pre Sec 4.4

Recognizing a distribution from the variable part of its density.

A density can be written as

$$f(t) = c\, h(t)$$

c ← constant; $h(t)$ ← variable part.

$$1 = \int_{-\infty}^{\infty} f(t)\,dt = c\int_{-\infty}^{\infty} h(t)\,dt \Rightarrow c = \frac{1}{\int_{-\infty}^{\infty} h(t)\,dt}$$

So the constant is a function of the variable part $h(t)$.

Often it is advantageous to ignore the constant part of a density and just work with the variable part.

List of densities and their variable parts:

ex $T \sim \text{Gamma}(r,\lambda)$ $\quad f(t) = \frac{1}{\Gamma(r)}\lambda^r t^{r-1} e^{-\lambda t}, \; t > 0$

$T \sim \text{Normal}(\mu,\sigma^2)$ $\quad f(t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$

$T \sim \text{Unif}(a,b)$ $\quad f(t) = \frac{1}{b-a} \mathbb{1}_{(t\in(a,b))}$

↑ indicator

ex Name the distribution with the following variable part ex Gamma($r=\frac{1}{2}, \lambda=3$).

a) $h(t) = t^3 e^{-\frac{1}{2}t}$ Gamma($r=4, \lambda=\frac{1}{2}$)

b) $h(t) = e^{-\frac{1}{2}t^2}$ Normal($\mu=0, \sigma=1$)

c) $h(t) = e^{-3t}$ Gamma($r=1, \lambda=3$)

d) $h(t) = t^{-\frac{1}{2}} e^{-t}$ Gamma($r=\frac{1}{2}, \lambda=1$)

e) $h(t) = \mathbb{1}_{(t\in(0,1))}$ Unif$[0,1)$

Sec 4.4 Change of variable formula for densities

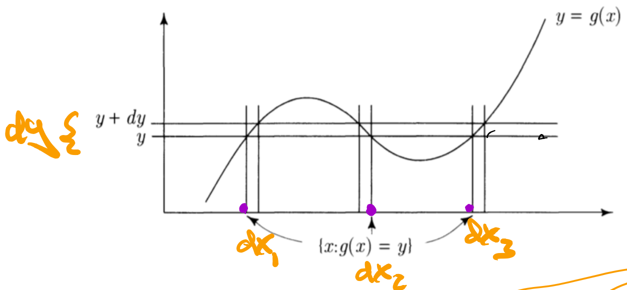
Thm (P 307)

Let X be a continuous RV with density $f_X(x)$.

Let $Y = g(X)$ have a derivative that is zero at only finitely many pts.

then
$$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \frac{f_X(x)}{|g'(x)|}$$
evaluated at $x = g^{-1}(y)$.

Picture $\qquad g^{-1}(y) = \{x_1, x_2, x_3\}$



mutually exclusive events.

$Y \in dy$ iff $X \in dx_1$ or $X \in dx_2$ or $X \in dx_3$

$$P(Y \in dy) = P(X \in dx_1) + P(X \in dx_2) + P(X \in dx_3)$$

$$f_Y(y)\,dy = f_X(x_1)\,dx_1 + f_X(x_2)\,dx_2 + f_X(x_3)\,dx_3$$

$$f_Y(y) = f_X(x_1)\frac{dx_1}{dy} + f_X(x_2)\frac{dx_2}{dy} + f_X(x_3)\frac{dx_3}{dy}$$

$$= \frac{f_X(x_1)}{g'(x_1)} + \frac{f_X(x_2)}{|g'(x_2)|} + \frac{f_X(x_3)}{g'(x_3)}$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{g'(x)}$$

$P(X \in dx_2) \geq 0$

ex let $X \sim N(0,1)$ $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

let $Y = \sigma X + \mu$, $\sigma > 0$ be a change of scale,

Find $f_Y(y)$. $x = \frac{y-\mu}{\sigma}$

Note $y = \sigma x + \mu$ is increasing since $\sigma > 0$

$\Rightarrow$ only one x for every y.

Use variable part of $f_X(x)$,

$$f_Y(y) \propto \left.\frac{e^{-\left(\frac{x^2}{2}\right)}}{\sigma}\right|_{x = \frac{y-\mu}{\sigma}} = \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$$

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}}$$

$$E(Y) = E(\sigma X + \mu) = \sigma \underbrace{E(X)}_{=0} + \mu = \mu$$

$$Var(Y) = Var(\sigma X + \mu) = \sigma^2 \underbrace{Var(X)}_{=1} = \sigma^2$$

We will see later in the semester that Y is normal.

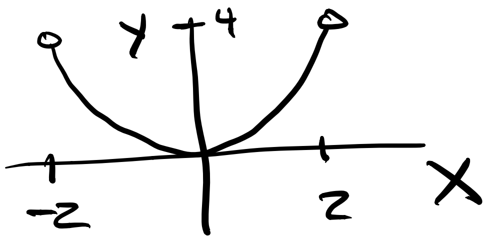
$\Rightarrow$ $\boxed{Y \sim N(\mu, \sigma^2)}$

ex

let $X \sim \text{Unif}(-2,2)$ $\quad f_X(x) = \frac{1}{2+2} \mathbb{1}_{(x \in (-2,2))}$

Find density of $Y = X^2$

$X = \pm\sqrt{y}$



Note that Y takes values $(0,4)$

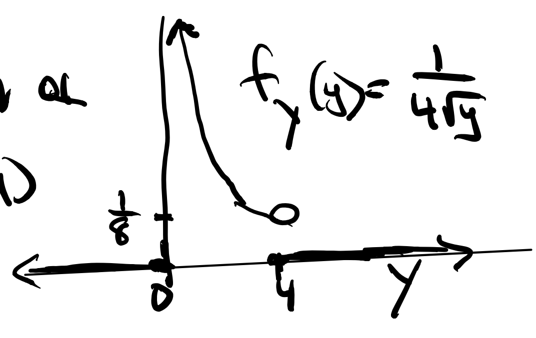
$$f_Y(y) \propto \left.\frac{\mathbb{1}_{(x \in (-2,2))}}{2x}\right|_{x=\sqrt{y}} + \left.\frac{\mathbb{1}_{(x\in(-2,2))}}{|2x|}\right|_{x=-\sqrt{y}}$$

$$= \frac{\mathbb{1}_{(y \in (0,4))}}{2\sqrt{y}} + \frac{\mathbb{1}_{(y\in(0,4))}}{2\sqrt{y}}$$

$$= \frac{\cancel{2}\,\mathbb{1}_{(y\in(0,4))}}{\cancel{2}\sqrt{y}}$$

$$f_Y(y) = \frac{1}{4}\frac{\mathbb{1}_{(y\in(0,4))}}{\sqrt{y}} \iff \boxed{f_Y(y) = \begin{cases} \frac{1}{4\sqrt{y}} & 0<y<4 \\ 0 & \text{else} \end{cases}}$$

graph of $f_Y(y)$



Stat 134

Monday October 15 2018

1. Let V be a standard normal RV. The distribution of $X = V^2$ is?

   a Gamma

   b Uniform

   c Normal

   d none of the above

$$f_V(v) \propto e^{-\frac{v^2}{2}} \qquad x = v^2, \quad v = \pm\sqrt{x}$$

$$\frac{dx}{dv} = 2v$$

$$f_X(x) \propto \left.\frac{e^{-\frac{v^2}{2}}}{|2v|}\right|_{v=\pm\sqrt{x}}$$

$$= \frac{\cancel{2}\cdot e^{-\frac{1}{2}x}}{\cancel{2}\cdot x^{1/2}} = x^{\overset{1-r}{-1/2}} e^{\overset{\lambda}{-\frac{1}{2}}x} \Rightarrow X \sim \text{Gamma}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$\Rightarrow f_X(x) = \underbrace{\frac{1}{\Gamma(\frac{1}{2})}\left(\frac{1}{2}\right)^{1/2}}_{\text{Constant.}} \cdot x^{-1/2} e^{-1/2 x}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

2. Suppose a particle's velocity V has a standard normal distribution. The distribution of the particle's kinetic energy, $K = \frac{1}{2}mV^2$, is:

$m > 0$

a Gamma(r = 1/m, λ = 1/2)

b Gamma(r = 1/2, λ = 1/m)

c Gamma(r = 1/2, λ = 1/2)

d none of the above

By Part 1, $X \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

$K = \frac{m}{2} X$

$f_X(x) \propto x^{-1/2} e^{-\frac{1}{2}x}$ $\qquad K' = \frac{m}{2}, \quad x = \frac{2K}{m}$

$$f_K(k) \propto \left. \frac{x^{-1/2} e^{-1/2 x}}{\frac{m}{2}} \right|_{x = \frac{2k}{m}} = \left(\frac{2}{m}\right)^{-1/2} K^{-1/2} e^{-\frac{1}{m} K}$$

$\Rightarrow K \sim \text{Gamma}(r = \frac{1}{2}, \lambda = 1/m)$