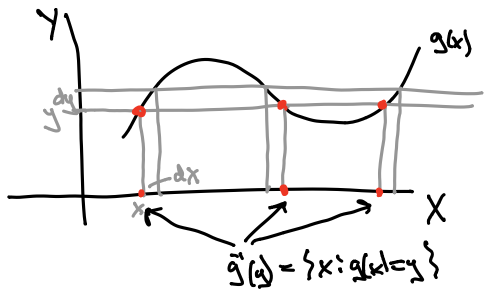
Stat 134 lec 25

Last 2 times

(1) sec 4.4 Change of Variable rule

many to one g:



$$f_Y(y)\,dy = f_X(x_1)dx_1 + f_X(x_2)dx_2 + f_X(x_3)dx_3$$

$$f_Y(y) = f_X(x_1)\frac{dx_1}{dy} + f_X(x_2)\frac{dx_2}{dy} + f_X(x_3)\frac{dx_3}{dy}$$

$$= \frac{f_X(x_1)}{g'(x_1)} + \frac{f_X(x_2)}{|g'(x_2)|} + \frac{f_X(x_3)}{g'(x_3)}$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{g'(x)}$$

$P(x \in dx_2) \geq 0$

(2) MGF (not in book)

$$M_X(t) = E(e^{tX})$$

Thm If a MGF exists in an interval around zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

ex $X \sim Pois(\mu)$

$$M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk}\frac{\mu^k e^{-\mu}}{k!} = e^{-\mu}\sum_{k=0}^{\infty}\frac{(\mu e^t)^k}{k!}$$

$$= e^{-\mu}e^{\mu e^t} = \boxed{e^{\mu(e^t-1)} \text{ for all } t}$$

then $E(X) = \frac{d}{dt}M_X(t)\Big|_{t=0} = e^{\mu(e^t-1)}\cdot\mu e^t\Big|_{t=0} = \mu$ ✓

Today

(1) Key properties of MGF

(2) Review Concept test responces from lec 23.

(3) Sec 3.5 CDF of a mixed distribution.

① Key Properties of MGF

ⓐ If an MGF exists in an interval containing zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

last time

ⓑ If X and Y are independent RVs,

$M_{X+Y}(t) = M_X(t)M_Y(t)$.

Proved in MGF HW.

ⓒ $M_X(t) = M_Y(t)$ for all t in an interval around 0 then $F_X(z) = F_Y(z)$ (i.e X and Y have the same distribution).

Skip proof — we can invert a MGF to get the CDF.

ex If $M_X(t) = \frac{1}{2}e^{1t} + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}$,

e^{xt} tells us the value of X and the associated coefficients tell us the probabilities

(i.e $X = 1, 2, 3$ w/ prob $1/2, 1/3, 1/6$.)

so MGF ⇒ distribution of X when X has finite # values,

Property ⓐ is useful to find $E(X)$, $Var(X)$,

Property ⓑ and ⓒ allow us to prove for example that sum of independent Poisson is Poisson.

$\underline{\underline{ex}}$ $\left.\begin{array}{l} X_1 \sim \text{Pois}(\mu_1) \\ X_2 \sim \text{Pois}(\mu_2) \end{array}\right\}$ independent.

show that $X_1 + X_2 \sim \text{Pois}(\mu_1 + \mu_2)$.

$M_{X_1}(t) = e^{\mu_1(e^t - 1)}$ for all t

$M_{X_2}(t) = e^{\mu_2(e^t - 1)}$ for all t

$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t) = \boxed{e^{(\mu_1+\mu_2)(e^t-1)}}$

MGF of Pois $(\mu_1 + \mu_2)$ for all t.

$\Rightarrow X_1 + X_2 \sim \text{Pois}(\mu_1 + \mu_2)$

$\underline{\underline{ex}}$ Let X be a RV and a a constant.

Show that $M_{aX}(t) = M_X(at)$

$\underline{\text{hint}}$ $M_{aX}(t) = E(e^{aXt})$

$= E(e^{Xat})$

$= M_X(at)$.

For $X \sim \text{Gamma}(r, \lambda)$

recall $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$ for $t < \lambda$

ex Let $X \sim \text{Exp}(\lambda)$ and $a > 0$.

Show that $Y = aX$ is also exponential, and specify the new parameter.

$$M_{aX}(t) = M_X(at) = \left(\frac{\lambda}{\lambda - at}\right)^1 \quad \text{for } at < \lambda$$

$$= \frac{\frac{\lambda}{a}}{\left(\frac{\lambda}{a} - t\right)} \quad \text{for } t < \frac{\lambda}{a}^{>0}$$

MGF for $\text{Exp}\left(\frac{\lambda}{a}\right)$

By uniqueness of MGF, $aX \sim \text{Exp}\left(\frac{\lambda}{a}\right)$.

2\. Let V be a standard normal RV. The distribution of $X = V^2$ is?

a Gamma

b Uniform

c Normal

d none of the above

$V \sim N(0,1)$

$f_V(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}$

$f_X(x) = \left.\frac{\frac{1}{\sqrt{2\pi}} e^{-v^2/2}}{2v}\right|_{v=\sqrt{x}} + \left.\frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}}{|2v|}\right|_{v=-\sqrt{x}}$

$\propto \frac{e^{-\frac{x}{2}}}{\sqrt{x}} = x^{-1/2} e^{-1/2 x}$

Gamma $(r = 1/2, \lambda = 1/2)$

$f_X(x) = \underbrace{\frac{1}{\Gamma(\frac{1}{2})} \left(\frac{1}{2}\right)^{1/2}}_{\text{constant part.}} \underbrace{x^{-1/2} e^{-1/2 x}}_{\text{variable part}}$

a	This is a change of variable from V=Norm(0,1) to X=V^2. Plugging in you see that the variable part consists of e^(-x/2)/sqrrootX. This matches gamma.

a	Gamma can talk on positive real numbers that aren't usually integers, so this has to be the only option.

d	It is like a gamma distribution and gaussian distribution mixed.

c	If we have a normal distribution and square our RV then we will see that it stays normal

② Sec 4.5 Cummulative Distribution Function (CDF)

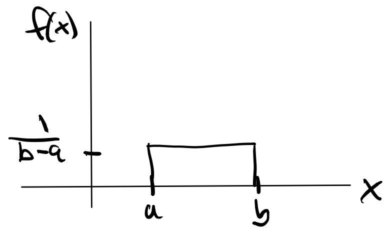
defn X RV

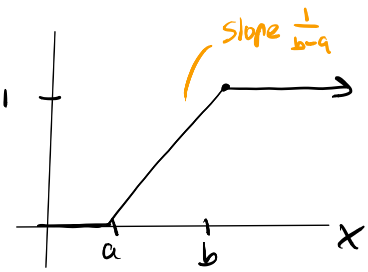
$$F_X(x) = P(X \le x)$$

Use Describes a distribution

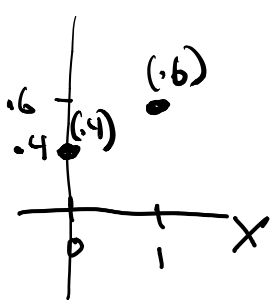
(equivalent to a density or probability mass function)

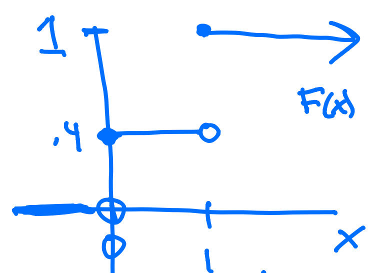
ex $X \sim \text{Unif}(a,b)$



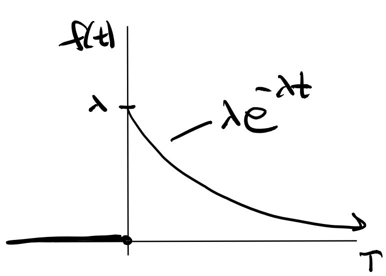


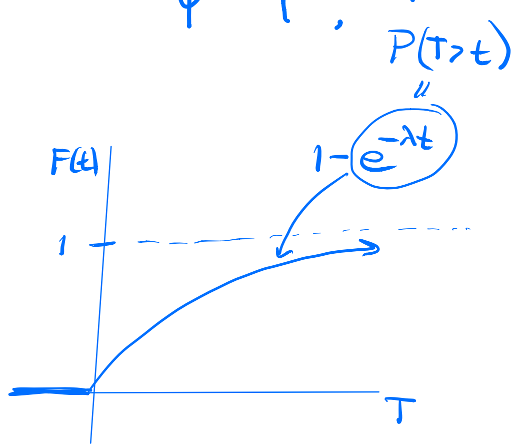
ex $X \sim \text{Bernoulli}(p = .6)$





ex $T \sim \text{Exp}(\lambda)$





Eg (Mixed distribution)

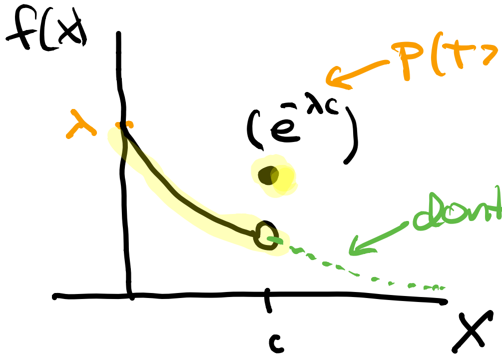
$T \sim \text{Exp}(\lambda)$

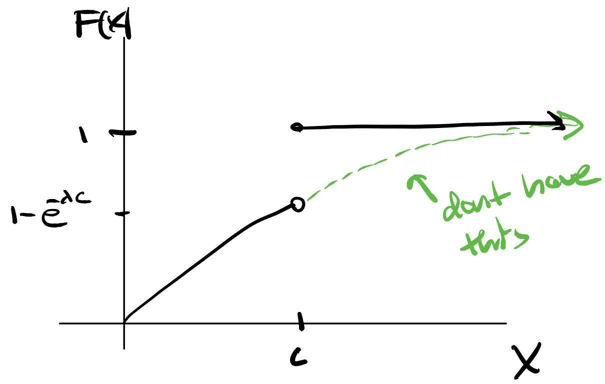
$c > 0$

$$X = \begin{cases} T & \text{if } x < c \\ c & \text{if } x = c \end{cases}$$

$X = \min(T, c)$

"T killed by c"





Stat 134

Friday March 22 2019

1. Suppose you are trying to discretly leave a party. Your time to leave is uniform from 0 to 2 minutes. However, if your walk to the exit takes more than 1 minute, you run into a friend at the door and and must spend the full 2 minutes to leave. Let T represent the time it takes you to leave. True or false, the graph of the cdf of T is: