Stat 134 lec 23

Warm up 11:00–11:10

A random variable X has non negative values and density cx^4e^{-3x} for $0 \le x < \infty$.

Find $Var(X)$ as a simple fraction without involving the constant c.

$X \sim Gamma(r, \lambda)$

$f(x) = \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}$

$\int_0^\infty f(x)\,dx = 1$

$\frac{1}{\Gamma(r)} \lambda^r \int x^{r-1} e^{-\lambda x}\,dx$

$\frac{1}{\Gamma(r)} \lambda^r = \frac{1}{\int x^{r-1} e^{-\lambda x}\,dx}$

$X \sim Gamma(r=5, \lambda=3)$

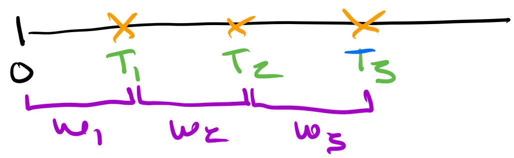
$X = W_1 + W_2 + W_3 + W_4 + W_5$

$E(X) = 5E(W_1) = \frac{5}{\lambda} = \boxed{\frac{5}{3}}$

$Var(X) = 5 \cdot Var(W_1) = \frac{5}{\lambda^2} = \boxed{\frac{5}{9}}$

Wednesday lecture on Moment Generating Functions (not in book)

<u>Last time</u> <u>sec 4.2</u> <u>Gamma Distribution</u>



$T_1 \sim Exp(\lambda)$, $\lambda > 0$ $\qquad f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$ ← variable part

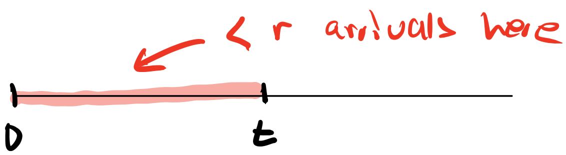
$T_r \sim Gamma(r, \lambda)$, $\lambda > 0$, $r > 0$ $\qquad f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$ where $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$

$T_r = W_1 + W_2 + \dots + W_r$, $W_i \overset{iid}{\sim} Exp(\lambda)$

$E(W_1) = \frac{1}{\lambda} \Rightarrow E(T_r) = \frac{r}{\lambda}$

$Var(W_1) = \frac{1}{\lambda^2} \Rightarrow Var(T_r) = \frac{r}{\lambda^2}$

$P(T_r > t) = P(N_t < r)$ where $N_t \sim Pois(\lambda t)$



<u>Today</u> Sec 4.4 (skip 4.3)

1. Gamma example
2. Change of Variable formula for densities.

① Gamma example

Suppose customers are arriving at a ticket booth at rate of five per minute, according to a Poisson arrival process. Find the probability that:

(a) Starting from time 0, the $9th$ customer doesn't arrive within 5 minutes;

(b) At least one customer arrives within 40 seconds after the arrival of the $13th$ customer.

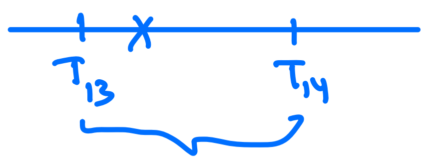
a) $P(T_9 > 5) = P(N_5 < 9)$ ≤ 8 $\qquad N_5 \sim Pois(\lambda t)$, $\lambda t = 25$

$$= \boxed{\sum_{k=0}^{8} \frac{e^{-25} 25^k}{k!}}$$

b)

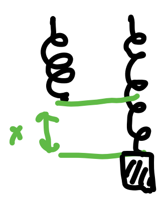
$$P\left(W_{14} < \tfrac{2}{3}\right) = P\left(T_1 < \tfrac{2}{3}\right)$$

$$= \boxed{1 - e^{-5 \cdot 2/3}}$$



$W_{14} = T_1 \sim Exp(5)$

Hooke's Law $\quad F = kX$



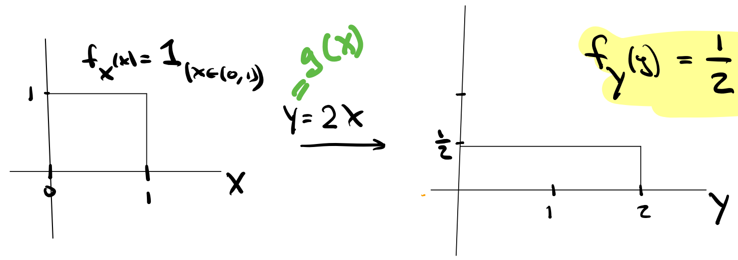
displacement of a spring

② Sec 4.4 Change of Variable formula for densities

ex let X = displacement of a spring

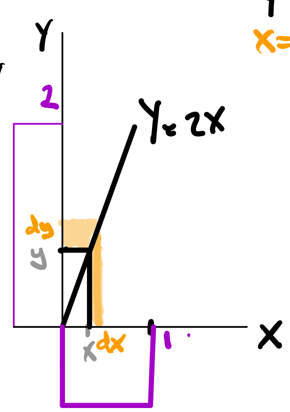
The density of $X \sim U(0,1)$ is $f_X(x) = 1_{(x \in (0,1))}$

What is density of $Y = 2X$, the force on the spring.



$f_Y(y) = \frac{1}{2} 1_{(y \in (0,2))}$

Another view:



$y = g(x) = 2x$

$x = g^{-1}(y) = \frac{y}{2}$

$P(Y \in dy) = P(X \in dx)$

$f_Y(y)\,dy = f_X(x)\,dx$

$\Rightarrow f_Y(y) = f_X(x) \frac{dx}{dy}$

$= \frac{1}{\frac{dy}{dx}} f_X(x) = \frac{1}{2} \cdot 1_{(x \in (0,1))}$ evaluated at $x = g^{-1}(y)$

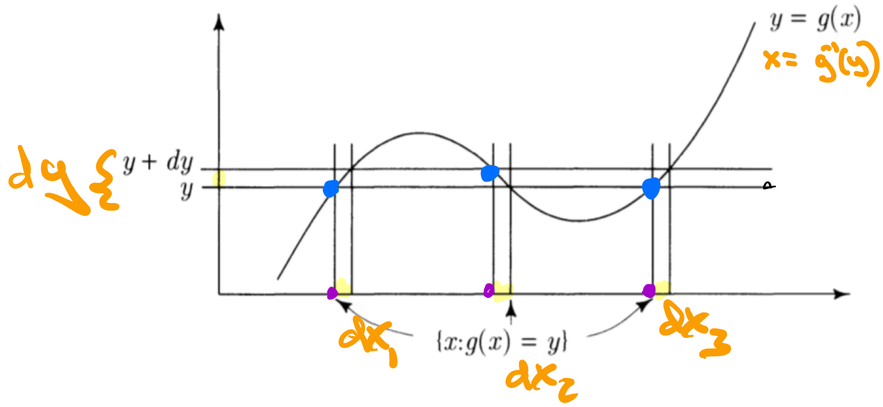
$\frac{dy}{dx} = g'(x) = 2$

$= \frac{1}{2} 1_{(\frac{y}{2} \in (0,1))}$

$= \frac{1}{2} 1_{(y \in (0,2))}$

Here the transformation of X to $Y = g(X)$ is linear and one to one.

What if $Y = g(X)$ isn't linear and one-one?



mutually exclusive events.

$Y \in dy$ iff $X \in dx_1$ or $X \in dx_2$ or $X \in dx_3$

$$P(Y \in dy) = P(X \in dx_1) + P(X \in dx_2) + P(X \in dx_3)$$

$$f_Y(y)\,dy = f_X(x_1)\,dx_1 + f_X(x_2)\,dx_2 + f_X(x_3)\,dx_3$$

$$f_Y(y) = f_X(x_1)\frac{dx_1}{dy} + f_X(x_2)\frac{dx_2}{dy} + f_X(x_3)\frac{dx_3}{dy}$$

$$= \frac{f_X(x_1)}{g'(x_1)} + \frac{f_X(x_2)}{|g'(x_2)|} + \frac{f_X(x_3)}{g'(x_3)}$$

evaluated at $x = g^{-1}(y)$

$P(X \in dx_2) \geq 0$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{g'(x)}$$

Thm (P 307) **Change of Variable Formula for densities**

Let X be a continuous RV with density $f_X(x)$.

Let $Y=g(X)$ have a derivative that is zero at only finitely many pts,

then $$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \frac{f_X(x)}{|g'(x)|}\Bigg|_{x=g^{-1}(y)}$$

evaluated at $x=g^{-1}(y)$

ex

let $X=N(0,1)$, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ * — we will show later in the semester that this is a density

Find the density of $Y=\sigma X+\mu$ where $\sigma>0$, $\mu\in\mathbb{R}$

Steps

1) Find $g(x)=\sigma x+\mu$
2) Find $g'(x)=\sigma$
3) Find $x=g^{-1}(y) = \frac{y-\mu}{\sigma}$
4) Find $f_Y(y) = \frac{f_X(x)}{|g'(x)|}\Bigg|_{x=\frac{y-\mu}{\sigma}}$

$$= \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}{\sigma}\Bigg|_{x=\frac{y-\mu}{\sigma}}$$

$$\boxed{= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}}$$

Note:

$E(Y) = E(\sigma X+\mu) = \sigma \underset{0}{E(X)} + \mu = \mu$

$Var(Y) = Var(\sigma X+\mu) = \sigma^2 \underset{1}{Var(X)} = \sigma^2$

$\Rightarrow \boxed{Y \sim N(\mu,\sigma^2)}$

Change of variable formula:

$$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \frac{f_X(x)}{|g'(x)|}\Bigg|_{x=g^{-1}(y)}$$

ex $X \sim \text{Unif}(-2,2)$, $f_X(x) = \frac{1}{4}\mathbb{1}_{(x\in(-2,2))}$

Find density of $Y = X^2$

note: $g(x) = x^2$
$x = g^{-1}(y) = \pm\sqrt{y}$

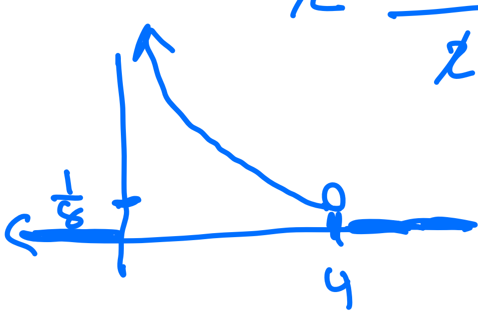
$$f_Y(y) = \sum_{x\in g^{-1}(y) = \{\pm\sqrt{y}\}} \frac{f_X(x)}{|g'(x)|}\Bigg|_{x=\pm\sqrt{y}}$$

$$= \frac{\frac{1}{4}\mathbb{1}_{x\in(-2,2)}}{|2x|}\Bigg|_{x=\sqrt{y}} + \frac{\frac{1}{4}\mathbb{1}_{x\in(-2,2)}}{|2x|}\Bigg|_{x=-\sqrt{y}}$$

$$= \frac{\frac{1}{4}\mathbb{1}_{\sqrt{y}\in(0,2)}}{2\sqrt{y}} + \frac{\frac{1}{4}\mathbb{1}_{-\sqrt{y}\in(-2,0)}}{2\sqrt{y}}$$

$= \sqrt{y}\in(0,2)$

$$= \cancel{2}\cdot\frac{\frac{1}{4}\mathbb{1}_{\sqrt{y}\in(0,2)}}{\cancel{2}\sqrt{y}} = \frac{\frac{1}{4}\mathbb{1}_{y\in(0,4)}}{\sqrt{y}}$$



$$= \boxed{\begin{cases} \frac{1}{4\sqrt{y}} & 0<y<4 \\ 0 & \text{else} \end{cases}}$$

ex (extra problem)

(3 pts) Suppose the random variable X, which measures the magnitude of an earthquake (on the Richter scale) in the Bay Area, follows the Exponential (λ) distribution. Since the Richter scale is logarithmic, we want to study the distribution of the total energy of earthquakes. Find the distribution of $Y = e^X$.

$\leftarrow g(x)$

Change of variable formula:

$$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \frac{f_X(x)}{|g'(x)|} \quad \text{evaluated at } x = g^{-1}(y).$$

Find $g(x) = e^x$

$g'(x) = e^x$

$g^{-1}(x) = \ln y$

$$f_Y(y) = \frac{\lambda e^{-\lambda x}}{e^x}\bigg|_{x=\ln y} = \lambda e^{(-\lambda-1)x}\bigg|_{x=\ln y}$$

$$= \lambda e^{(-\lambda-1)\ln y} = \lambda \big(\underbrace{e^{\ln y}}_{y}\big)^{(-\lambda-1)} = \boxed{\lambda y^{(-\lambda-1)}, \quad y > 1}$$