Stat 134 lec 28

Last time

Sec 4.6 uniform order statistic

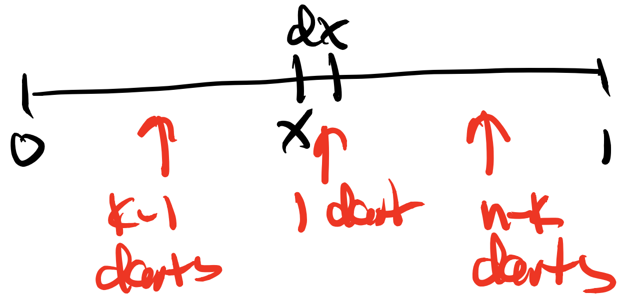
$U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$

$U_{(1)}, \dots, U_{(n)}$ order statistics

ex $U_1 = .2 \quad U_2 = .7 \quad U_3 = .1$

$\Rightarrow U_{(1)} = .1, \; U_{(2)} = .2, \; U_{(3)} = .7$

$P(U_{(k)} \in dx) = f(x)dx$



Note

$$\binom{n}{a,b,c} = \frac{n!}{a!\,b!\,c!} = \binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}$$

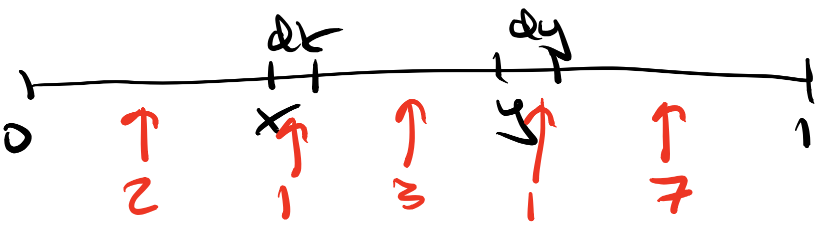
$$f_{U_{(k)}}(x) = \binom{n}{k-1,\,1,\,n-k} x^{k-1}(1-x)^{(n-k+1)-1} \quad \text{on } 0<x<1$$

Sec 5.1, 5.2 Continuous Joint Distribution

$P(X \in dx, Y \in dy) = f(x,y)\,dx\,dy.$

ex What joint density, $f(x,y)$ has variable part $x^2(y-x)^3(1-y)^7$ on $0<x<y<1$.

expect X and Y are unif ordered statistics.



$\Rightarrow X = U_{(3)}$
$Y = U_{(7)}$
(throw 14 darts).

today ① sec 4.6 Beta distribution

② sec 5.1, 5.2 Calculate probabilities with $f(x,y)$.

① Sec 4.6 Beta Distribution

Let $U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$

$$f_{U_{(k)}}(x) = \binom{n}{k-1, 1, n-k} x^{k-1} (1-x)^{(n-k+1)-1} \quad \text{on } 0<x<1$$

Let r, s be pos. integers.

$\boxed{\text{Def}^n \quad X \sim Beta(r,s) \text{ if } f(x) = \dfrac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} x^{r-1}(1-x)^{s-1} \text{ for } 0<x<1.}$

Notice that $f_{U_{(k)}}(x)$ and $f_{Beta(r,s)}(x)$ have the same variable part of their density when $r = k$, $s = n-k+1$

then $\Gamma(s+r) = \Gamma(n-k+1+k) = \Gamma(n+1) = n!$

$\Gamma(r) = (k-1)!$

$\Gamma(s) = (n-k)!$

$$\Rightarrow \frac{\Gamma(s+r)}{\Gamma(r)\Gamma(s)} = \binom{n}{k-1, 1, n-k}$$

$\Rightarrow$ uniform ordered statistics are beta!

see appendix of notes

Thm $X \sim \mathrm{Beta}(r, s)$

$$E(X) = \frac{r}{r+s}$$

Hence if $X \sim U_{(k)}$

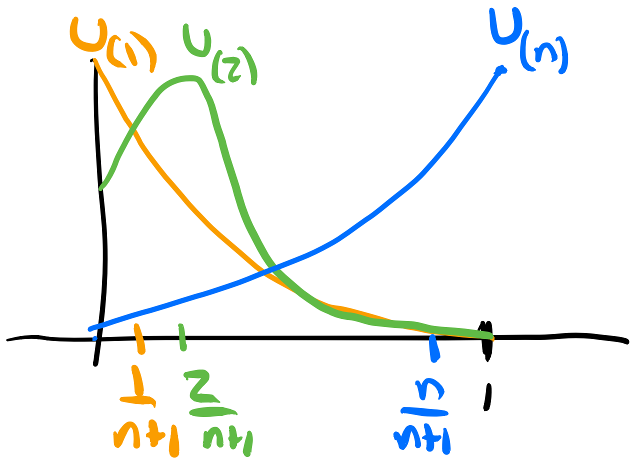
$$E(X) = \frac{k}{n-k+1+k} = \boxed{\frac{k}{n+1}}$$

$$E(U_{(1)}) = \frac{1}{n+1}$$

$$E(U_{(2)}) = \frac{2}{n+1}$$

$$\vdots$$

$$E(U_{(n)}) = \frac{n}{n+1}$$



Stat 134

Wednesday October 24 2018

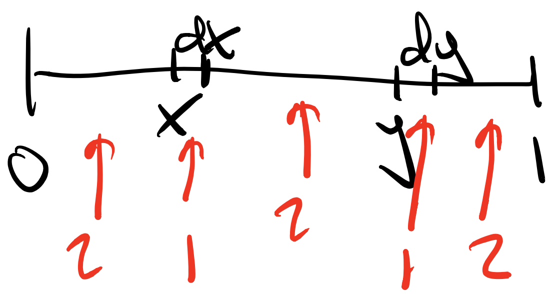
1. I throw down 8 darts on $(0,1)$. The variable part of the joint density of $X = U_{(3)}$ and $Y = U_{(6)}$ is:

   a $x(y-x)^5(1-y)^2$

   b $x^2(y-x)^2(1-y)^2$

   c $x^4(y-x)^2(1-y)^2$

   d none of the above



2. Is $f(x,y) = \binom{6}{1,4,1}(y-x)^4$ on $0 < x < y < 1$ a joint density function?

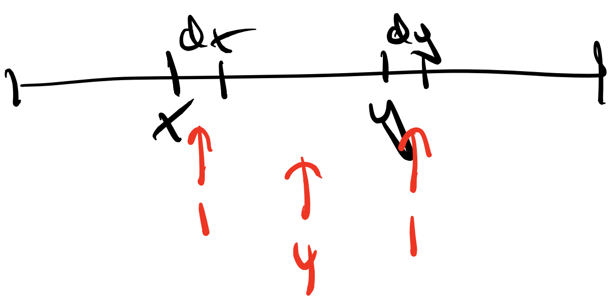
a yes

b no

c not enough info to decide

$X = U_{(1)}$

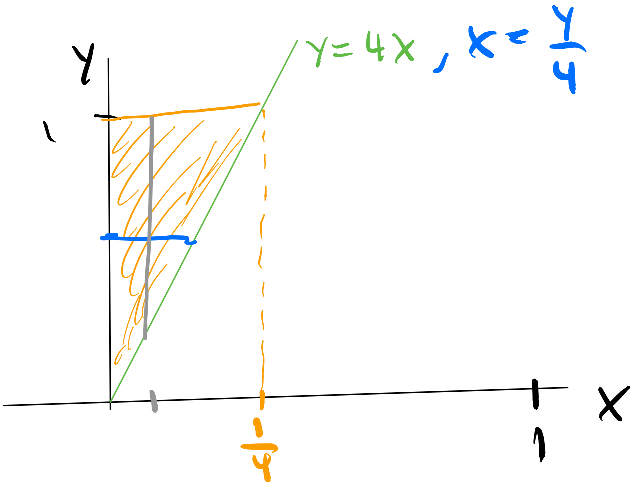
$Y = U_{(6)}$



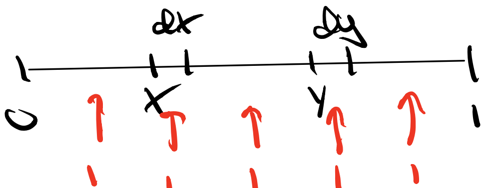
Throw down 5 darts on $(0,1)$.

ex $X = U_{(2)}$ $Y = U_{(4)}$

Find $P(Y > 4X)$



Find density



$$f(x,y) = \binom{5}{1,1,1,1,1} x(y-x)(1-y)$$

$$= \boxed{5!\, x(y-x)(1-y)}$$

$$P(Y > 4X) = \int_{y=0}^{y=1} \int_{x=0}^{x=\frac{y}{4}} 5!\, x(y-x)(1-y)\, dx\, dy$$

(or)

$$P(Y > 4X) = \int_{x=0}^{x=1/4} \int_{y=4x}^{y=1} 5!\, x(y-x)(1-y)\, dy\, dx$$

Which is better? — first one starts at $x=0$ and $y=0$ which is easier.

details:

$$P(Y > 4X) = \int_{y=0}^{y=1} \int_{x=0}^{y/4} 120\, x\,(y-x)(1-y)\,dx\,dy$$

$$= 120 \int_{y=0}^{y=1} (1-y) \int_{x=0}^{x=y/4} (xy - x^2)\,dx\,dy$$

$$= \int_0^1 120\,(1-y)\left(\frac{x^2 y}{2} - \frac{x^3}{3}\right)\Bigg|_0^{y/4}$$

$$= \int_0^1 120\,(1-y)\left(\frac{y^3}{32} - \frac{y^3}{3 \cdot 64}\right)dy$$

$$= \frac{5}{192} \cdot 120 \int_0^1 y^3 - y^4\,dy = \frac{5 \cdot 120}{192}\left(\frac{y^4}{4} - \frac{y^5}{5}\right)\Bigg|_0^1$$

$$= \frac{5 \cdot 120}{192}\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{30}{192} = .156$$

independent RVs

defn X and Y are independent if

$$P(X \in dx, Y \in dy) = P(X \in dx)\, P(Y \in dy)$$

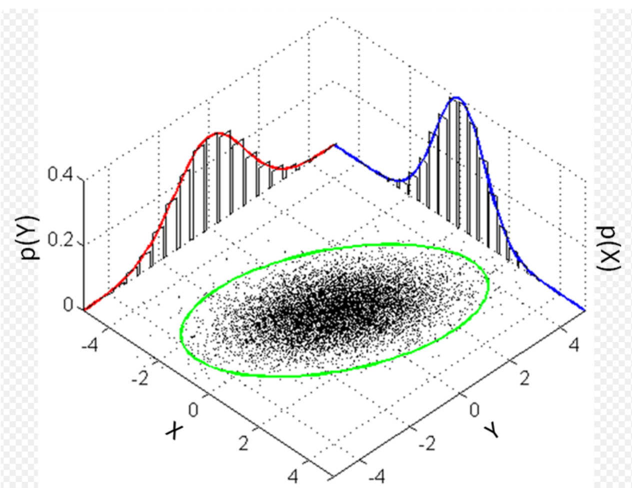
$$\overset{\parallel}{f(x,y)\,dx\,dy} \qquad \overset{\parallel}{f(x)\,dx} \qquad \overset{\parallel}{f(y)\,dy}$$

$$\Leftrightarrow \boxed{f(x,y) = f(x)f(y)}$$

ex $X, Y \overset{iid}{\sim} N(0,1)$

$$f(x,y) = \phi(x)\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$



Not a great picture because the oval in green should be a circle. This is the picture of a correlated bivariate normal from chapter 6 instead of an uncorrelated bivariate normal.

ex

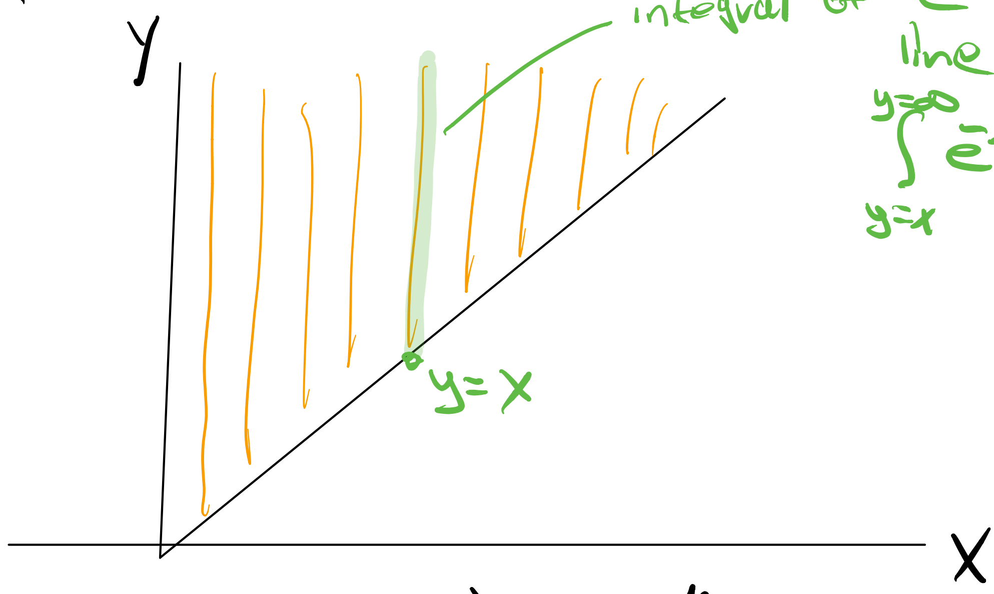
Let $X \sim \text{Exp}(\lambda)$, $Y \sim \text{Exp}(\mu)$ be independent lifetimes of two bulbs.

Find $P(Y > X)$.

Soln

integral of $e^{-\mu y}$ along this line is easy!

$$\int_{y=x}^{y=\infty} e^{-\mu y}\, dy = \frac{e^{-\mu x}}{\mu}$$



$$f(x,y) = \lambda e^{-\lambda x} \mu e^{-\mu y}$$

$$P(Y>X) = \lambda\mu \int_{x=0}^{x=\infty} e^{-\lambda x} \underbrace{\int_{y=x}^{y=\infty} e^{-\mu y}\, dy}_{-\frac{e^{-\mu y}}{\mu}\Big|_{y=x}^{y=\infty} = \frac{e^{-\mu x}}{\mu}}\, dx$$

$$= \lambda \int_{x=0}^{\infty} e^{-(\lambda+\mu)x}\, dx = \boxed{\frac{\lambda}{\lambda+\mu}}$$

ex

If $X, Y \overset{iid}{\sim} U(0,1)$

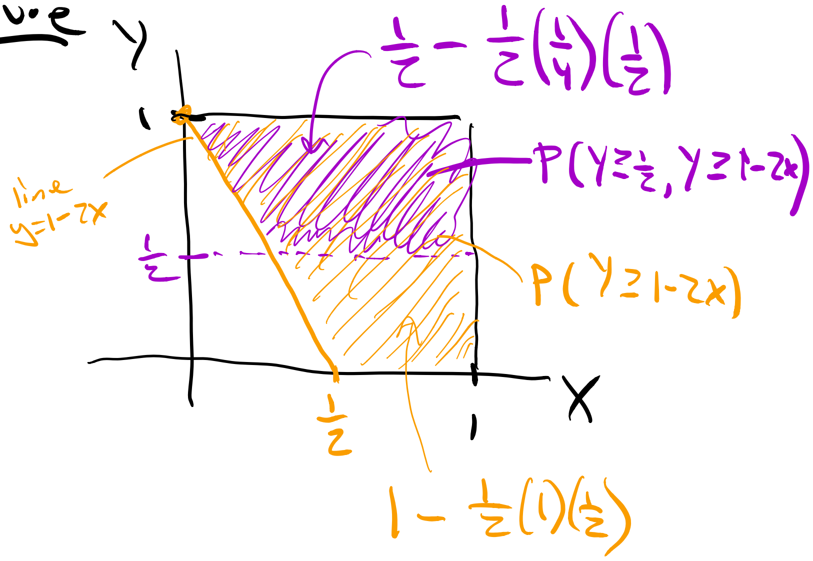
Find $P(Y \geq \frac{1}{2} \mid Y \geq 1-2X)$

soln

$f(x,y) = \underset{1}{\underset{\parallel}{f(x)}}\,\underset{1}{\underset{\parallel}{f(y)}} = 1$ for $0 < x, y < 1$

$$P(Y \geq \tfrac{1}{2} \mid Y \geq 1-2X) = \frac{P(Y \geq \frac{1}{2}, Y \geq 1-2X)}{P(Y \geq 1-2X)}$$

Picture



So,

$$\frac{P(Y \geq \frac{1}{2}, Y \geq 1-2X)}{P(Y \geq 1-2X)} = \frac{\frac{1}{2} - \frac{1}{2}(\frac{1}{4})(\frac{1}{2})}{1 - \frac{1}{2}(1)(\frac{1}{2})} = \frac{7/16}{3/4} = 7/12$$

Appendix

Let $X \sim \text{Beta}(r, s)$

then $E(X) = \frac{r}{r+s}$.

Pf/ Note that $\int_0^1 f(x)\,dx = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = 1$

$$\Rightarrow \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}.$$

$$E(X) = \int_0^1 x f(x)\,dx = \frac{\Gamma(r+s)}{\Gamma(r)\cancel{\Gamma(s)}} \underbrace{\int_0^1 x\, x^{r-1}(1-x)^{1-s}\,dx}_{= \frac{\cancel{\Gamma(s)}\Gamma(r+1)}{\Gamma(s+r+1)}}$$

x^{r+1-1}

$$= \frac{(r+s-1)!\; r!}{(s+r)!} = \boxed{\frac{r}{r+s}}$$

$\square$