Stat 155 Lec 5

Warmup 11:00-11:10

Consider a single item auction where the second highest bidder gets the item at the price of third highest bid.

Assuming players bid truthfully does this auction maximize social welfare? — No If $v_1 > v_2$ then social welfare is v_2 since second highest bidder wins.

Is the allocation rule monotone? (i.e if you bid more do you get more?)

No suppose other bids are $50, $100

if you bid

$30 — lose
$90 — win
$120 — lose

Myerson's Lemma will show that for a SPE an allocation rule is implementable iff it is monotone.

Hence this auction isn't DSIC (in fact no payment rule can make it DSIC).

Last time

In lecture 4 we asked if we can design a sponsored search auction to be ideal (i.e ① DSIC, ② maximize social welfare and be ③ computationally efficient). By allocating the best slots to those who pay the most and assuming that people bid truthfully, we can satisfy ② and ③. We need to know if we can choose a payment rule which coupled with the above allocation rule makes the auction DSIC.

Myerson's Lemma will show that this is possible. Myerson's Lemma holds for all single parameter environments.

Sec 3.1 Single Parameter Environment (SPE).

- n bidders each with a single private valuation v_i
- A feasible set X of allowable allocations. For example for a single item auction X is the set of all 0-1 n vectors with at most a single 1.

Sec 3.2 Allocation and payment rules.

$b = (b_1, \ldots, b_n)$ bid profile what everyone bids

feasible allocation $x(b) = (x_1(b), \ldots, x_n(b))$ who gets what

Payment $P(b) = (P_1(b), \ldots, P_n(b))$ who pays what

We will focus on payment rules $P_i(b) \in [0, b_i x(b)]$

Each bidder has a quasilinear utility

$$U_i(b) = v_i \cdot x_i(b) - P_i(b)$$

The pair $(x(b), p(b))$ for a DSIC is called a direct revelation mechanism since it directly reveals the bidder's valuation

Sec 3.3 Myerson's Lemma

Def^n An allocation rule x, for a SPE is implementable if there is a payment rule p such that (x, p) is DSIC.

Today

Sec 3.3 ① Statement of Myerson's Lemma

Sec 3.4 ② Proof of Myerson's Lemma.

1) Sec 3.3 Statement of Myerson's Lemma

Defn (Monotone Allocation Rule)

An allocation rule for a single parameter environment is monotone if for every bidder i and bids b_{-i} by other bidders, the allocation $\underbrace{x_i(z, b_{-i})}_{b = (b_i, b_{-i})}$ to i is non decreasing in z.

i.e in a monotone allocation rule, if you bid more you get more.

ex Is a single item auction that awards the highest bidder monotone?

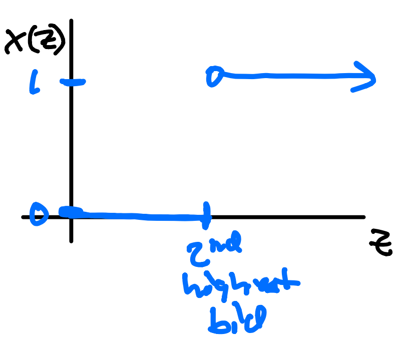
Fix i, and b_{-i}

player i's bid.

Lets see if $x_i(z, b_{-i})$ is monotone as a function of z.

Shorthand:
write $x(z)$ for $x_i(z, b_{-i})$
$P(z)$ for $p_i(z, b_{-i})$

For 2nd price auction what does $x(z)$ look like?

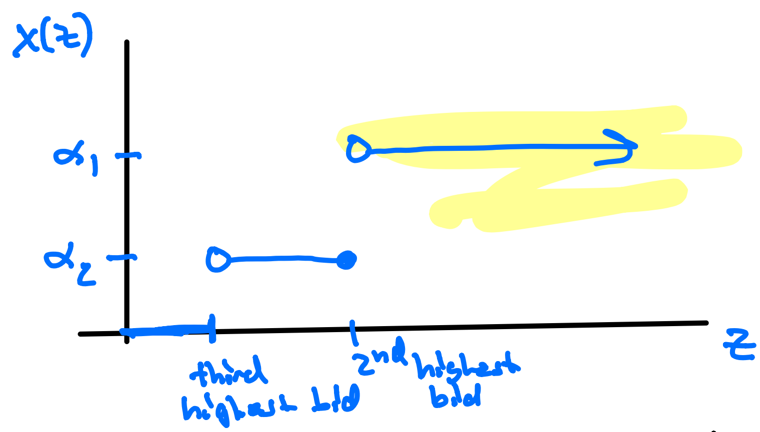


ex Is a single item auction that awards the second highest bidder monotone?

No (see warmup question)

ex Is our allocation in sponsored search monotone?

For sponsored search auction ($k=2$) what does $x(z)$ look like?



$n \geq 2$

$k=2$

Suppose $v_1 > v_2$

$x = (\alpha_1, \alpha_2)$

So these auctions have monotone allocations

foundation for most of mechanism theory
Got Nobel Prize for Econ in 2007.

Thm (Myerson's Lemma)

Fix a single-parameter environment.

a) An allocation rule x is implementable iff it is monotone.

b) If x is monotone, then there is a unique payment rule for which (x, p) is DSIC (and $p_i(b) = 0$ when $b_i = 0$)

c) The payment rule in (b) is given by an explicit formula in the proof.

ex In single item auction is the allocation rule that awards the second highest bidder implementable? — No since not monotone

ex Is our allocation rule in Sponsored Search auction implementable? — yes since monotone

Checking whether an allocation is monotone isn't hard. It also tells us what the unique payment rule is.

(2) sec 3.4 Proof of Myerson's Lemma

Fix a SPE and consider an allocation rule x. Focus on part (a) and parts (b), (c) will follow naturally.

x implementable $\Rightarrow$ x monotone.

x implementable means $\exists\, p$ such that (x, p) is DSIC.

Fix i, b_{-i}

To simplify notation

$$x(z) = x_i(z, b_{-i})$$
$$p(z) = p_i(z, b_{-i})$$

To be DSIC means to bid truthfully is always giving $\geq$ utility then over or underbidding.

Let $0 \leq z < y$ be two numbers.

It could be the case that $v_i = z$ and bidder i overbids y

By DSIC,

$$u_i(\underset{=v_i}{z}, b_{-i}) \geq u_i(\underset{\text{overbid}}{y}, b_{-i}) \qquad (\text{overbidding})$$

In terms of x, p

$$x(z)\cdot z - p(z) \geq x(y) z - p(y)$$

algebra $\rightsquigarrow$ $$z(x(y) - x(z)) \leq p(y) - p(z)$$

Simultaneously we must also consider the possibility that $y = v_i$ and z is underbid,

$$\overset{v_i}{U_i(y, b_{-i})} \geq U_i(\overset{\text{underbid}}{z}, b_{-i})$$ (under bidding)

$$x(y)y - p(y) \geq x(z)y - p(z)$$

algebra
$\rightsquigarrow$ $$p(y) - p(z) \leq y(x(y) - x(z))$$

Both inequalities are true at the same time because a DSIC auction has to guard simultaneously against a bidder overbidding and underbidding.

This gives a payment difference sandwich.

$$z(x(y) - x(z)) \leq p(y) - p(z) \leq y(x(y) - x(z))$$

In exercise 3.1 of HW #1 you will conclude that x is monotone.