Stat 134 lec 30

sec 5.3

warm up 1:

$X \sim N(0,1)$

$Y = g(X) = -X$

find f_Y

$$f_Y(y) = \frac{1}{|g'(x)|} f_X(x) \quad \text{where } x = g^{-1}(y)$$

$$= f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad x = -y$$

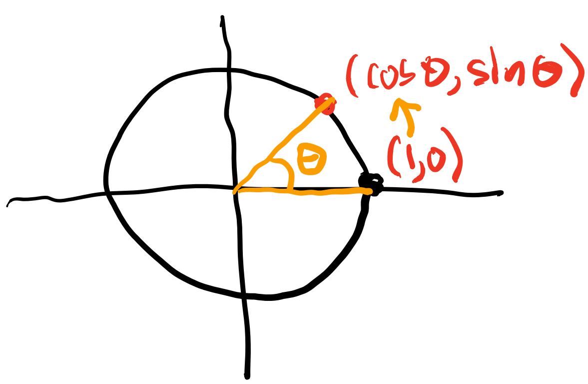
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}.$$

Note if g doesn't stretch the line (i.e. g is a rigid motion) then $\boxed{f_Y(y) = f_X(x)}$

warm up 2:

rotation matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$



Find the rotation matrix for rotation by $-\theta$

$\cos -\theta = \cos\theta$
$\sin -\theta = -\sin\theta$

$$\rightsquigarrow \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Linear combinations and rotations

If X, Y are <u>independent</u> normal RV is $aX + bY$ normal?

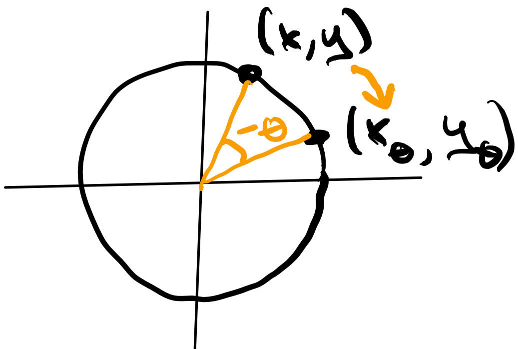
First we will show that if $X, Y \overset{iid}{\sim} N(0,1)$ then $\cos\theta X + \sin\theta Y \sim N(0,1)$.

let $X, Y \overset{iid}{\sim} N(0,1)$

and (x,y) a pt in the plane.

Know $f_{(X,Y)}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$

let (x_θ, y_θ) be rotation of (x,y) clockwise by $-\theta$



$$x^2 + y^2 = x_\theta^2 + y_\theta^2$$

What is joint density of (X_θ, Y_θ)?

No stretching of the plane so

$$f_{(X_\theta, Y_\theta)}(x_\theta, y_\theta) = f_{(X,Y)}(x,y)$$

The change of variables rule for joint densities is beyond the scope of this class but for rigid motions it is super simple.

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(\underbrace{x^2+y^2}_{= x_\theta^2 + y_\theta^2})}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(x_\theta^2 + y_\theta^2)}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_\theta^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_\theta^2}$$

$\Rightarrow$ compute marginal density

$$\boxed{f_{X_\theta}(x_\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_\theta^2}}$$

$$\Rightarrow \boxed{X_\theta \sim N(0,1)}$$

Next, write X_θ as a function of X and Y using the rotation matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

i.e Find

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta X + \sin\theta Y \\ -\sin\theta X + \cos\theta Y \end{bmatrix}$$

$$\Rightarrow \boxed{X_\theta = \cos\theta X + \sin\theta Y}$$

It follows that for any α, B with $\alpha^2 + B^2 = 1$ and X, Y iid $N(0,1)$

$$\alpha X + BY \sim N(0,1)$$

Hence, $Z = \frac{1}{\sqrt{2}}(X+Y) \sim N(0,1)$

For $Z \sim N(0,1)$ using change of variable rule you can show

$$X = \mu + \sigma Z \sim N(\mu, \sigma^2)$$

Normal (μ, σ^2) Distribution

If Z has standard normal distribution and μ and σ are constants with $\sigma \geq 0$, then

$$X = \mu + \sigma Z$$

has mean μ, standard deviation σ, and variance σ^2. The distribution of X is called the *normal distribution with mean μ and variance σ^2*, abbreviated normal (μ, σ^2). So X has normal (μ, σ^2) distribution if and only if the standardized variable

$$Z = (X - \mu)/\sigma$$

has normal $(0, 1)$ or *standard normal* distribution. To find $P(c < X < d)$, change to standard units and use the standard normal table

$$P(c < X < d) = P(a < Z < b) = \Phi(b) - \Phi(a)$$

where $\quad a = (c - \mu)/\sigma \qquad Z = (X - \mu)/\sigma \qquad b = (d - \mu)/\sigma$

Formula for the normal (μ, σ^2) density. For $\sigma > 0$, the formula is

$$f_X(x) = \frac{1}{\sigma}\phi((x - \mu)/\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} \qquad (-\infty < x < \infty).$$

It follows that for $X, Y \overset{iid}{\sim} N(0,1)$,

$$\boxed{X + Y = \sqrt{2}\, Z \sim N(0,2)}$$

<u>Summary</u>

- If you add two indep std normals you get a normal.

- If add two independent normals you get a normal.

Why? Please read the proof on the next page carefully!

- Linear combinations of independent normals is normal (follows from first two bullet points)

P 363

Sums of Independent Normal Variables

If X and Y are independent with normal (λ, σ^2) and normal (μ, τ^2) distributions, then $X + Y$ has normal $(\lambda + \mu, \sigma^2 + \tau^2)$ distribution.

Proof. Recall that X has normal (λ, σ^2) distribution if and only if $(X - \lambda)/\sigma$ has normal $(0, 1)$ distribution. Transform all the variables to standard units by letting

$$U = (X - \lambda)/\sigma \quad \text{and} \quad V = (Y - \mu)/\tau \quad \text{and} \quad W = \frac{X + Y - (\lambda + \mu)}{\sqrt{\sigma^2 + \tau^2}}$$

Then U and V are independent normal $(0, 1)$ random variables. By algebra,

$$W = \alpha U + \beta V \quad \text{where} \quad \alpha^2 = \frac{\sigma^2}{\sigma^2 + \tau^2} \quad \text{and} \quad \beta^2 = \frac{\tau^2}{\sigma^2 + \tau^2} \quad \text{so} \quad \alpha^2 + \beta^2 = 1$$

Apply (d) above with (U, V) instead of (X, Y) to deduce that W has normal $(0, 1)$ distribution. So $X + Y = (\lambda + \mu) + \sqrt{\sigma^2 + \tau^2} W$ has normal $(\lambda + \mu, \sigma^2 + \tau^2)$ distribution. $\square$

Stat 134

Wednesday April 4 2018

1. Let $X \sim N(68, 3^2)$ and $Y \sim N(66, 2^2)$ be independent. $P(X > Y)$ equals

   a $1 - \phi(\frac{0-2}{\sqrt{3^2+2^2}})$

   b $1/2$

   c $1 - \phi(\frac{66-68}{\sqrt{3^2-2^2}})$

   d none of the above

$\boxed{1 - \Phi\left(\frac{0-2}{\sqrt{13}}\right)}$

X, Y indep normals

$P(X > Y) = P(X - Y > 0)$

Normal(2, 13)

$X = N(68, 3^2)$

$Y = N(66, 2^2)$

$E(X-Y) = E(X) - E(Y) = 68 - 66$
$= 2$

$\text{var}(X-Y) = \text{var}(X) + \text{var}(Y)$
$= 13$

Chi square distribution – based iid $N(0,1)$

$\lambda > 0$, pos integer r,
gamma (r, λ) density

$$f(t) = \frac{\lambda^r}{\Gamma(r)} t^{r-1} e^{-\lambda t}, \quad t > 0$$

density of T_r = the time until the r^{th} arrival of a Poisson Process with rate λ.

ex let Z = std normal

$X = Z^2$ change of variable rule.

Find $$f_X(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-\frac{1}{2}x}, \quad x > 0$$

$$= \frac{\left(\frac{1}{2}\right)^{1/2}}{\sqrt{\pi}} x^{\frac{1}{2}-1} e^{-\frac{1}{2}x}$$

$\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right)$

so $Z^2 \sim \text{gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$ – called $\chi^2(1)$

1 deg freedom

ex let $Z_1, Z_2 \overset{iid}{\sim} N(0,1)$

$X = Z_1^2 + Z_2^2 \sim \exp\left(\frac{1}{2}\right)$ (we saw this last class)

$= \text{gamma}\left(\frac{2}{2}, \frac{1}{2}\right)$

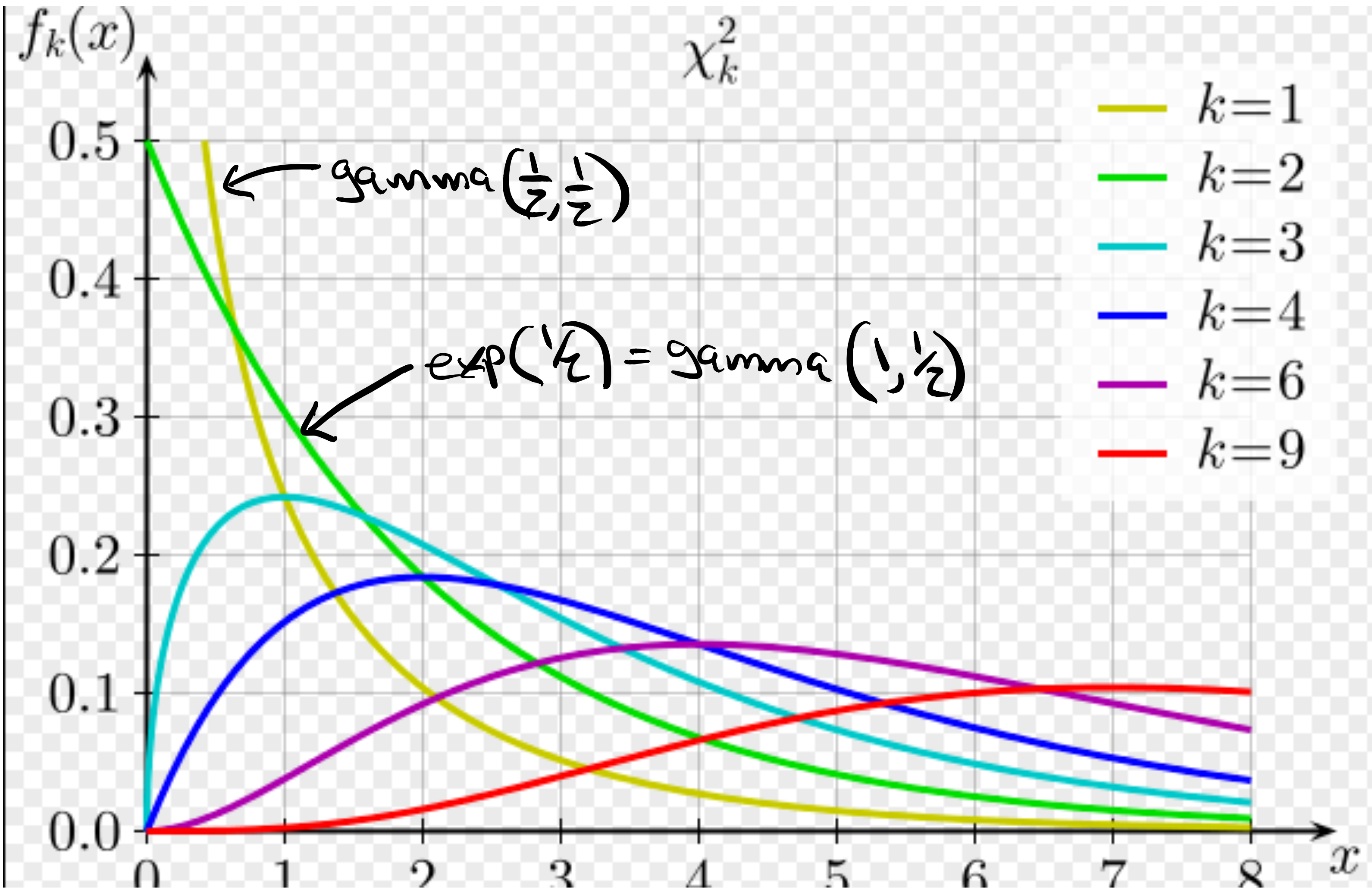
$= \chi^2(2)$ ← 2 degrees of freedom

Chi Square

Let $Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$

$X = Z_1^2 + \dots + Z_n^2 \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right) = \chi^2(n)$

n degrees of freedom



Recall, for $X \sim \text{Gamma}(r, \lambda)$

$$E(X) = \frac{r}{\lambda}, \quad SD = \frac{\sqrt{r}}{\lambda}$$

so if $X \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right) = \chi^2(n)$

$$E(X) = \frac{\frac{n}{2}}{\frac{1}{2}} = n, \quad SD = \sqrt{2n}$$