Stat 134 lec 26

Last two times:

sec 4.5 ① CDF, $F(t) = P(T \le t)$

useful to find distribution of min or max.

ex $X_i \overset{indep}{\sim} Exp(\lambda_i)$

let $M = \min(X_1, \dots, X_n)$.

What is distribution of M?

$$P(M > t) = P(X_1 > t, X_2 > t, \dots, X_n > t), \quad \text{for } t > 0$$

$$= P(X_1 > t) \cdot P(X_2 > t) \cdots P(X_n > t)$$

$$= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdots e^{-\lambda_n t}$$

$$= e^{-(\lambda_1 + \dots + \lambda_n)t}$$

$\Rightarrow M \sim \text{expon}(\lambda_1 + \dots + \lambda_n)$

class notes ② MGF

X RV, $t \in \mathbb{R}$.

$$M_X(t) = E(e^{tX})$$

ex if X is takes values $X = 1, 2, 3$ with prob $1/2, 1/3, 1/6$ respectively, $M_X(t) = \sum_{x=1}^{3} e^{tx} P(x) = e^t \cdot \frac{1}{2} + e^{2t} \cdot \frac{1}{3} + e^{3t} \cdot \frac{1}{6}$ for all t.

ex $X \sim Gamma(r, \lambda)$

$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$ for $t < \lambda$

Today

① Two RV with same MGF have same cdf.

② Review concept test responses from lec 24.

③ Finding E(X) using cdf.

1. Key Properties of MGF

(a) If an MGF exists in an interval containing zero 0, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

(b) If X and Y are independent RVs,
$M_{X+Y}(t) = M_X(t)M_Y(t)$.

Proved in MGF HW.

(c) $M_X(t) = M_Y(t)$ for all t in an interval around 0 then $F_X(z) = F_Y(z)$ (i.e X and Y have the same distribution).

Skip proof — we can invert a MGF to get the CDF.

ex If $M_X(t) = \frac{1}{2}e^{1t} + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}$,
e^{xt} tells us the value of X and the associated coefficients tell us the probabilities
(i.e $X = 1, 2, 3$ w/ prob $1/2, 1/3, 1/6$.)
so MGF ⇒ distribution of X when X has finite # values.

Property (a) is useful to find $E(X)$, $Var(X)$.
Property (b) and (c) allow us to prove for example that sum of independent Poisson is Poisson.

ex $\left.\begin{array}{l} X_1 \sim \text{Pois}(\mu_1) \\ X_2 \sim \text{Pois}(\mu_2) \end{array}\right\}$ independent.

$$P(X_1 = k) = \frac{e^{-\mu_1}\mu_1^k}{k!}$$

Show that $X_1 + X_2 \sim \text{Pois}(\mu_1 + \mu_2)$.

Step 1 Find $M_{X_1}(t) = E(e^{tX_1})$

$$E(e^{tX_1}) = \sum_{k=0}^{\infty} e^{kt} \frac{e^{-\mu_1}\mu_1^k}{k!} = e^{-\mu} \sum_{k=0}^{\infty} \frac{e^{kt}\mu_1^k}{k!}$$

$$= e^{-\mu_1} \underbrace{\sum_{k=0}^{\infty} \frac{(e^t\mu_1)^k}{k!}}_{= e^{e^t \mu_1}} = e^{-\mu_1} e^{e^t \mu_1}$$

$$= e^{e^t\mu_1 - \mu_1} \quad \text{for all } t$$

$\Rightarrow$

$$M_{X_1}(t) = e^{\mu_1(e^t - 1)} \quad \text{for all } t$$

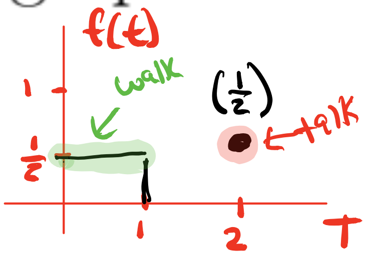
$$M_{X_2}(t) = e^{\mu_2(e^t - 1)} \quad \text{for all } t$$

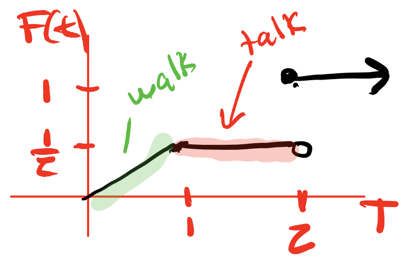
$$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t) = \boxed{e^{(\mu_1+\mu_2)(e^t - 1)}}$$

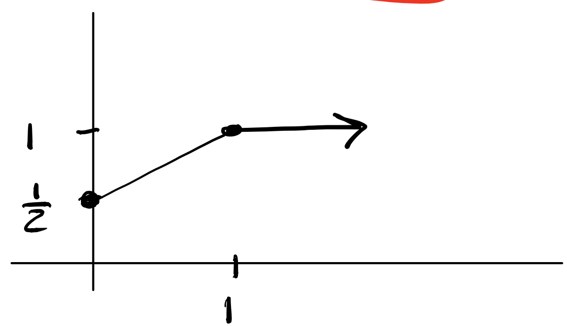
MGF of Pois $(\mu_1 + \mu_2)$

(2) Review concept test responses from lec 24,

Suppose you are trying to discretly leave a party. Your time to leave is uniform from 0 to 2 minutes. However, if your walk to the exit takes more than 1 minute, you run into a friend at the door and and must spend the full 2 minutes to leave. Let T represent the time it takes you to leave. True or false, the graph of the cdf of T is:

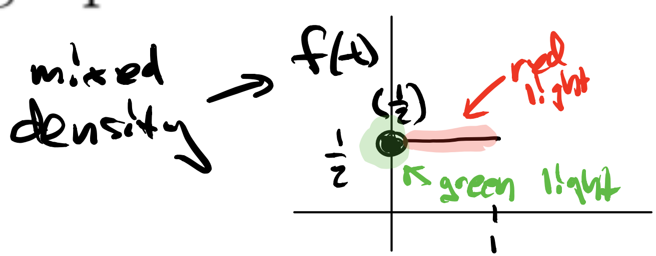


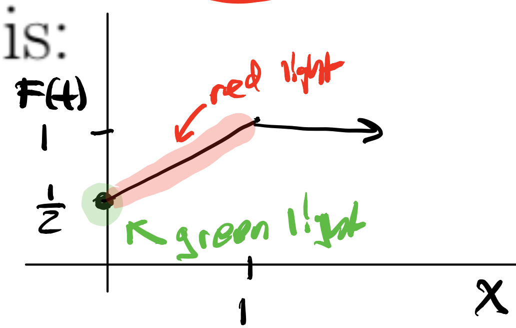




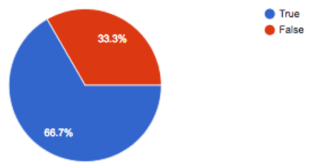
10/17/2018 9:50:3	FALSE	FALSE	should be steeper
10/17/201	FALSE	FALSE	The graph shouldn't start at 1/2 that would imply that half the probability is assigned to leaving at 0. It should start at 0 and increase uniformly to t=1. Then there should be a constant line from t=1 to t<2 then a jump to 1 at t=2
10/17/2018 9:47:0	FALSE	FALSE	Increase from 0 to 1/2 between 1 and 0 and then flat from 1-2 with jump up to 1 at 2
10/17/2018 9:47:2	FALSE	FALSE	
10/17/2018 9:47:3	FALSE	FALSE	Should be constant with a jump up to one after half the time
10/17/2018 9:51:4	TRUE	FALSE	the values should start at 0 for uniform
10/17/2018 9:47:4	FALSE	FALSE	
10/17/2018 9:47:4	FALSE	FALSE	If it's from uniform from 0-2 minutes it should start at 0 and go to a half, and then jump up to one?
10/17/201	FALSE	FALSE	The probability it takes you 0 minutes to leave is not 1/2 because it should reflect uniform from 0 to 2 minutes— the probability should accumulate over time starting at 0
10/17/2018 9:48:1	TRUE	FALSE	the CDF should be a flat line at 1 at 1 minute in, as at that point the time taken is guaranteed to be 2 min total. up until that point. Since the probability graph is uniform, the line before and up to 1 min in should be a flat line with a constant slope. However, the graph should start at the origin, as there is a 0% chance you can make it to the door in less than or equal to 0 sec.
10/17/2018 9:48:1	FALSE	FALSE	should start from 0
10/17/2018 9:48:2	FALSE	FALSE	should be a jump once it hits 2 minutes because if you stay any point past 1 minute, you have to take 2 minutes
10/17/2018 9:48:2	FALSE	FALSE	You shouldn't start at 1/2 and it should not be 1 until 2 min
10/17/2018 9:48:3	FALSE	FALSE	Must spend 2mins to leave so in (0, 2) p=0
10/17/2018 9:48:3	FALSE	FALSE	
10/17/2018 9:48:3	FALSE	FALSE	You aren't necessarily gone after two minutes. Also you don't have a .5 probability of already being gone by Time = 0
10/17/201	FALSE	FALSE	0—>1/2——1/2—>1
10/17/201	FALSE	FALSE	The sloped part of the function should be shifted down by 1/2, and there should be a jump discontinuity at x=1.
10/17/2018 9:50:3	FALSE	FALSE	
10/17/201	FALSE	FALSE	The probability of leaving < 1.5 minutes is not 1.
10/17/2018 9:54:0	FALSE	FALSE	
10/17/2018 9:54:3	FALSE	FALSE	
10/17/201	FALSE	FALSE	The cdf cannot start at 1/2 because that means there is a 0.5 probability that you're already gone by the time you decide to leave
10/17/2018 9:54:3	FALSE	FALSE	Start the discountunity from 2

Suppose stop lights at an intersection alternately show green for one minute, and red for one minute (no yellow). Suppose a car arrives at the lights at a time distributed uniformly from 0 to 2 minutes. Let X be the delay of the car at the lights (assuming there is only one car on the road). True or false, the graph of the cdf of X is:





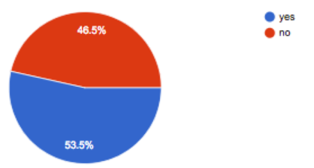
45 responses



feedback

Did student feedback from part 1 help you?

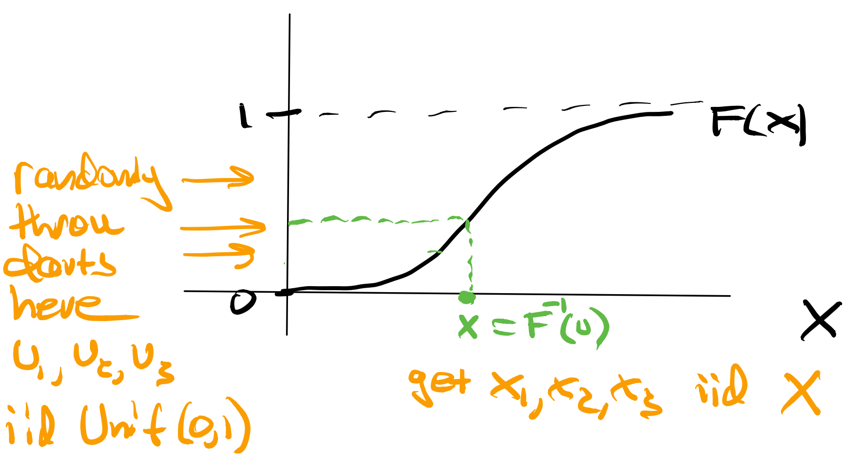
43 responses



③ Sec 4.5 Using CDF to find $E(X)$. sec 4.5.9

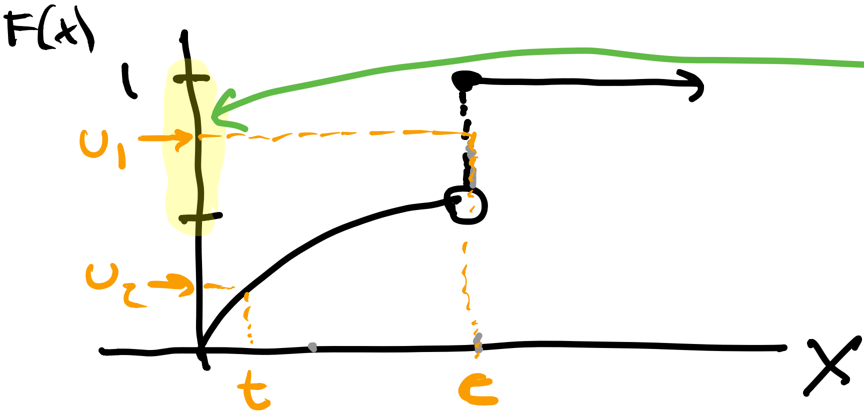
Inverse distribution function, $F^{-1}(u)$

Let X have CDF $F(x)$.



\ doesn't have to be continuous

eg $X = \min(T, c)$, $T \sim \exp(\lambda)$



This yellow region gets assigned the single value c.

<u>Thm</u> (p322) — Proof at end of lecture.

Let X have CDF F.

Then the RV $F^{-1}(U) = X$

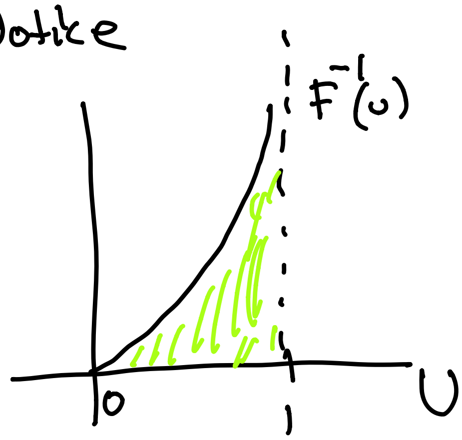
How is this useful to us finding $E(X)$?

It is sometimes easier to calculate $E(X)$ using the cdf (avoid doing integration by parts):

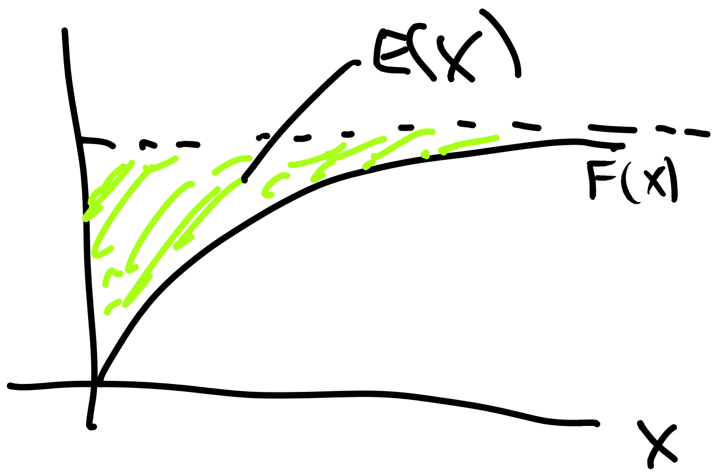
1 since $U \sim Unif(0,1)$

$$E(X) = E\left(F^{-1}(U)\right) = \int_0^1 F^{-1}(u) f_U(u)\,du$$

Notice



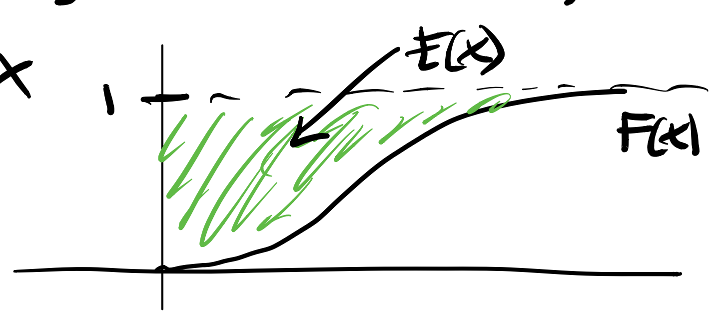
Now reflect the above graph about the diagonal $y=x$



We can find the shaded region by integrating $1-F(x)$ with respect to x:

Thm Let X be a pos. random variable, with CDF F. (continuous, discrete, mixed),

$$E(X) = \int_0^\infty (1-F(x))\,dx$$



ex

$T \sim \text{expon}(\lambda)$

$F_T(t) = 1 - e^{-\lambda t}$

Calculate $E(T)$.

$$E(T) = \int_0^\infty (1 - F(t))\,dt = \int_0^\infty e^{-\lambda t}\,dt$$

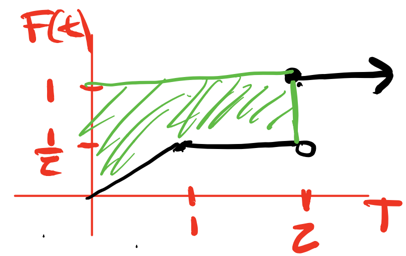
$$= \frac{1}{\lambda} e^{-\lambda t}\Big|_0^\infty = \frac{1}{\lambda}$$

wow that was easy!

ex

Suppose you are trying to discretly leave a party. Your time to leave is uniform from 0 to 2 minutes. However, if your walk to the exit takes more than 1 minute, you run into a friend at the door and and must spend the full 2 minutes to leave. Let T represent the time it takes you to leave. True or false, the graph of the cdf of T is:

$E(X) = \frac{5}{4}$



Concept test:

Stat 134

Monday October 21 2018

1. Let X have density $f(x) = xe^{-x}$ for $x > 0$.
   The MGF is?

   a $M_X(t) = \frac{1}{1-t}$ for $t < 1$

   b $M_X(t) = \frac{1}{(1-t)^2}$ for $t < 1$

   c $M_X(t) = \frac{1}{(1+t)^2}$ for $t > -1$

   d none of the above

variable part of Gamma $x^{r-1} e^{-\lambda x}$

$\Rightarrow X \sim \text{Gamma}(2,1)$

know MGF of Gamma(r,λ) is $\left(\frac{\lambda}{\lambda - t}\right)^r$, $t < \lambda$

so $\boxed{M_X(t) = \frac{1}{(1-t)^2} \text{ for } t < 1}$

2. The MGF of X is $M_X(t) = \frac{1}{\sqrt{1-t}}$ for $t < 1$. The distribution of X is:

a Gamma($\mathrm{r} = 1/2, \lambda = 1$) and possibly another distribution.

b Gamma($\mathrm{r} = 2, \lambda = 1$)

c Gamma($\mathrm{r} = -1/2, \lambda = 1$)

d none of the above

$X \sim \text{Gamma}(r = 1/2, \lambda = 1)$

uniquely determined from

$M_X(t) = \frac{1}{\sqrt{1-t}}$ for $t < 1$.

Appendix

See p 322 in book

Claim for any CDF F

$X = F^{-1}(U)$ is a RV with cdf F.

Unif$(0,1)$

Proof/ let $X = F^{-1}(U)$

$$F_X(x) = P(X \le x)$$

we will show $F_X = F$

$$= P(F^{-1}(U) \le x)$$

$$= P(F \cdot F^{-1}(U) \le F(x))$$

since F is increasing

$$= P(U \le F(x))$$

$$= F(x)$$

since $P(U \le u) = u$

$\square$