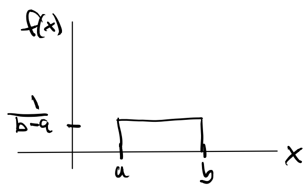
Stat 134 lec 26

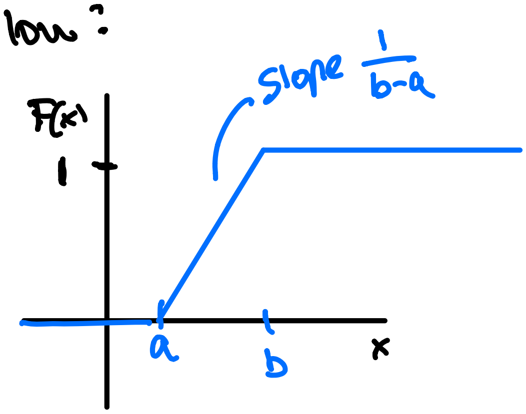
Warmup: 1:00–1:10

Recall that the cummulative distribution function (CDF) for a RV X is $F(x) = P(X \le x)$.

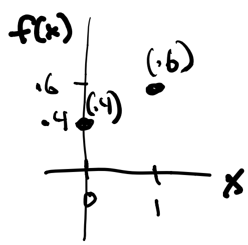
Draw the CDF for each of the distributions below:

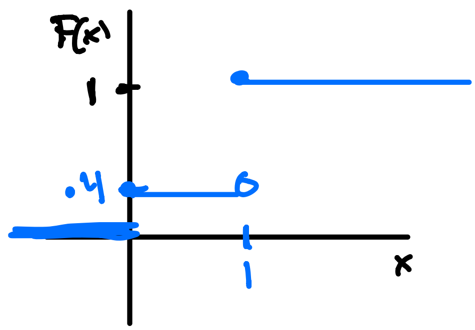
ex $X \sim \text{Unif}(a,b)$



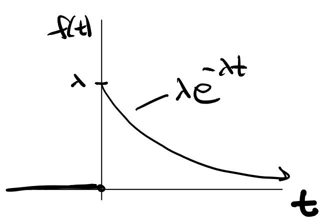


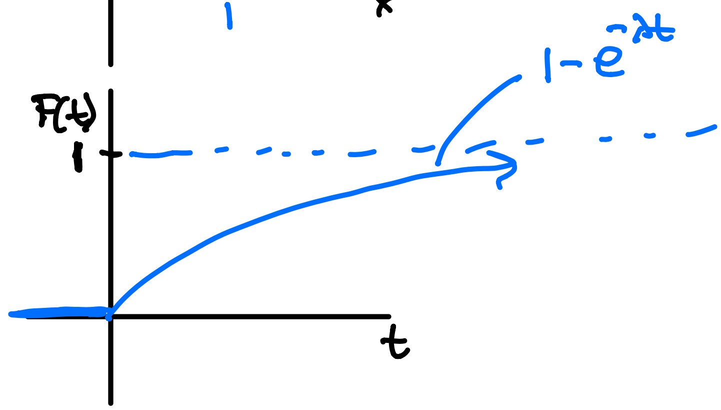
ex $X \sim \text{Bernoulli}(p=.6)$





ex $T \sim \text{Exp}(\lambda)$





Today

1. review MGF
2. Sec 4.5 Find CDF of a mixed distribution
3. sec 4.5 Using CDF to find $E(X)$

① Review MGF

X RV, $t \in \mathbb{R}$

$$M_X(t) = E(e^{tX})$$

ex $X \sim \text{Gamma}(r, \lambda)$ ← variable part

$$f_X(x) = \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}, \quad x > 0$$

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r, \quad t < \lambda$$

Property of MGF:

$M_X(t) = M_Y(t)$ for all t in a neighborhood of zero, iff $X \overset{d}{=} Y$ (i.e. X and Y have the same distribution)

ex If $M_X(t) = \dfrac{1}{\sqrt{1-t}}$ for $t < 1$

what distribution is X? $X \sim \text{Gamma}(r = 1/2, \lambda = 1)$

Knowing the distribution uniquely specifies the MGF

And

Knowing the MGF uniquely specifies the distribution tinyurl.com/oct26-pt1

Stat 134

1. Let X have density $f(x) = xe^{-x}$ for $x > 0$.
The MGF is?

a $M_X(t) = \frac{1}{1-t}$ for $t < 1$

b $M_X(t) = \frac{1}{(1-t)^2}$ for $t < 1$

c $M_X(t) = \frac{1}{(1+t)^2}$ for $t > -1$

d none of the above

$X \sim \text{Gamma}(2,1)$

Know MGF of $\text{Gamma}(r,\lambda)$ is $\left(\frac{\lambda}{\lambda-t}\right)^r$, $t < \lambda$

$M_X(t) = \frac{1}{(1-t)^2}$ for $t < 1$

(2) sec 4.5 CDF of mixed distributions

ex (Mixed distribution)

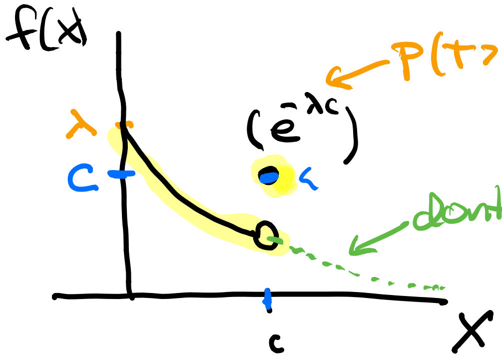
$T \sim Exp(\lambda)$

$c > 0$

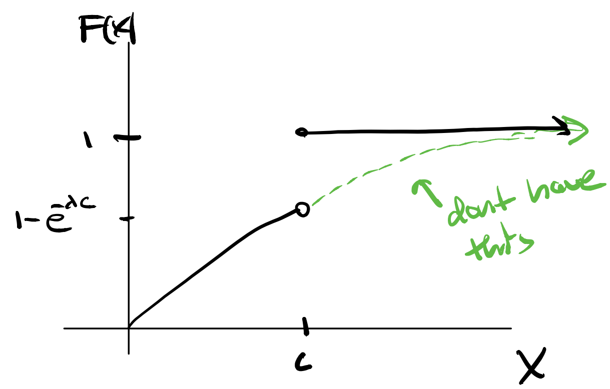
$$X = \begin{cases} T & \text{if } x < c \\ c & \text{if } x = c \end{cases}$$

$X = \min(T, c)$

"T killed by c"

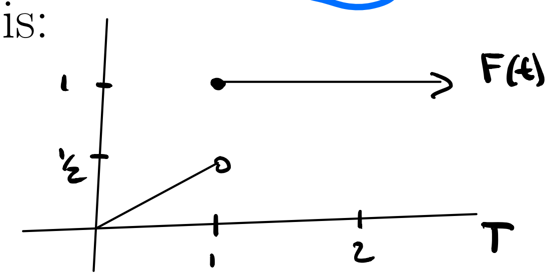


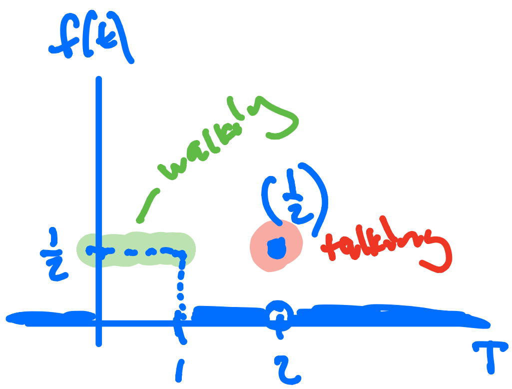
$P(T < c) = 1 - e^{-\lambda c}$

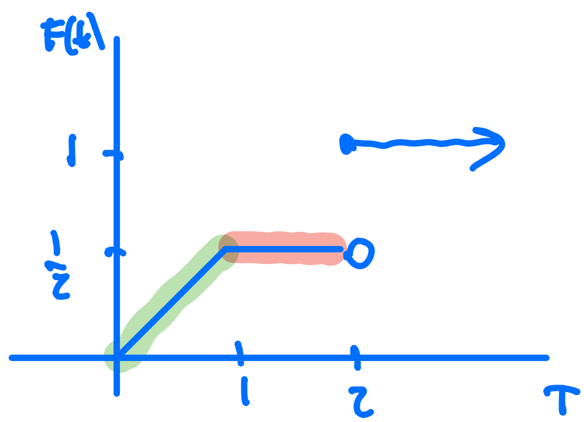


ex

Suppose you are trying to discretly leave a party. Your time to leave is uniform from 0 to 2 minutes. However, if your walk to the exit takes more than 1 minute, you run into a friend at the door and and must spend the full 2 minutes to leave. Let T represent the time it takes you to leave. True or false, the graph of the cdf of T is:





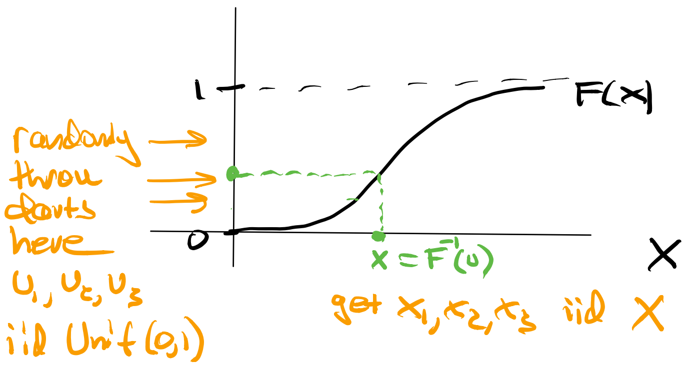


③ Sec 4.5 Using CDF to find $E(X)$ for $X \geq 0$

see #9 p324

Inverse distribution function, $F^{-1}(u)$

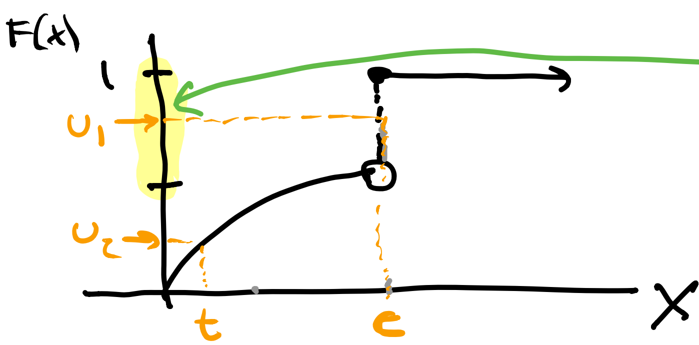
Let X have CDF $F(x)$.



$$\boxed{F^{-1}(u) = x}$$

Note: doesn't have to be continuous RV.

ex $X = \min(T, c)$, $T \sim \text{Exp}(\lambda)$



This yellow region gets assigned the single value c.

Thm (P322) — Proof at end of lecture.

Let X have CDF F.

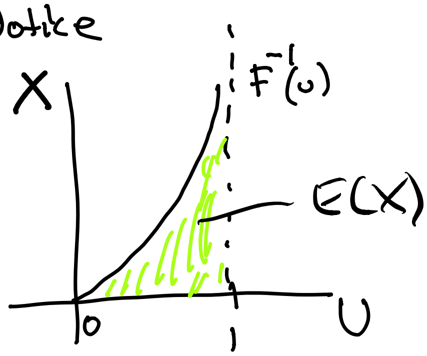
Then the RV $F^{-1}(U) = X$

How is this useful to us finding $E(X)$?

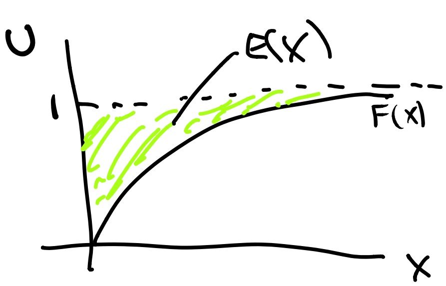
1 since $U \sim \text{Unif}(0,1)$

$$E(X) = E(F^{-1}(U)) = \int_0^1 F^{-1}(u) f_U(u)\, du$$

Notice



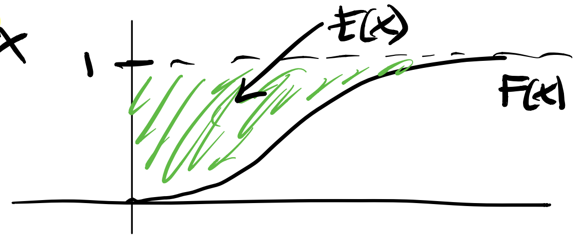
Now reflect the above graph about the diagonal $y = x$



we can find the shaded region by integrating $1 - F(x)$ with respect to x:

Thm Let X be a pos. random variable, with CDF F. (continuous, discrete, mixed).

$$E(X) = \int_0^\infty (1 - F(x))\,dx$$



ex $T \sim \text{expon}(\lambda)$

$$F_T(t) = 1 - e^{-\lambda t}$$

Calculate $E(T)$.

$$E(T) = \int_0^\infty (1 - F(t))\,dt = \int_0^\infty e^{-\lambda t}\,dt = -\frac{1}{\lambda} e^{-\lambda t}\Big|_0^\infty$$

$$= \boxed{\frac{1}{\lambda}}$$

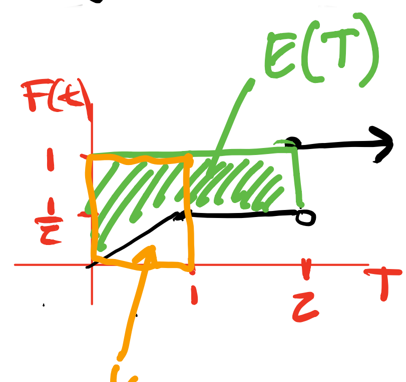
So easy!

It is sometimes easier to calculate $E(X)$ using the cdf (avoid doing integration by parts): $E(T) = \int_0^\infty t\lambda e^{-\lambda t}\,dt$

ex

Suppose you are trying to discretly leave a party. Your time to leave is uniform from 0 to 2 minutes. However, if your walk to the exit takes more than 1 minute, you run into a friend at the door and and must spend the full 2 minutes to leave. Let T represent the time it takes you to leave. Find $E(T)$

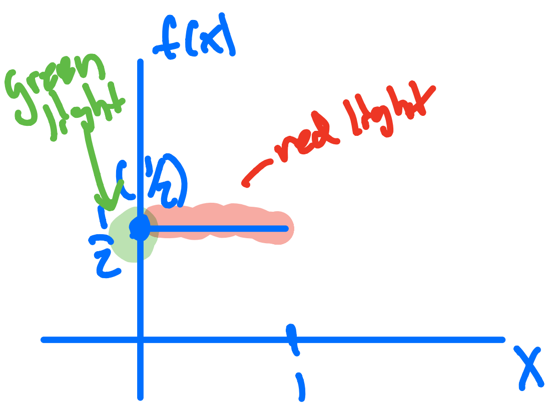
$$E(T) = \frac{3}{4} + \frac{1}{2} = \boxed{5/4}$$



ex

1/4

2. Suppose stop lights at an intersection alternately show green for one minute, and red for one minute (no yellow). Suppose a car arrives at the lights at a time distributed uniformly from 0 to 2 minutes. Let X be the delay of the car at the lights (assuming there is only one car on the road). Graph the density and the cdf of X.



to be continued.

Appendix

See p 322 in book

Claim for any CDF F

$X = F^{-1}(U)$ is a RV with cdf F.

Unif(0,1)

Proof/ let $X = F^{-1}(U)$

$$F_X(x) = P(X \le x)$$

we will show $F_X = F$

$$= P(F^{-1}(U) \le x)$$

$$= P(F \cdot F^{-1}(U) \le F(x))$$

since F is increasing

$$= P(U \le F(x))$$

$$= F(x)$$

since $P(U \le u) = u$

$\square$