Bivariate Normal.

a. $(X,Y) \sim N(\mu, \Sigma)$ $\quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left\{\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right\}\right)$$

b. $X, Y \overset{iid}{\sim} N(0,1) \Rightarrow (X,Y)$ is bivariate normal?

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}y^2\right)$$

$$= \frac{1}{2\pi}\exp\left(-\frac{1}{2}(x^2+y^2)\right)$$

$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

c. $\mathbb{X} = (X_1, \cdots, X_n)'$ $\quad \mathbb{X} \sim N(\mu, \Sigma)$

μ: $(\mu_1, \cdots, \mu_n)'$

Σ: $n \times n$ Variance Matrix.

$\exists\ A\mathbb{X} \sim N(A\mu, A\Sigma A^t)$

for $i = 1, \cdots, n$ $\quad Z_i \sim N(\mu_i, \sigma_i^2)$

$$\mathbb{Z} = (Z_1, \cdots, Z_n) \sim N\left(\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{pmatrix}\right)$$

$$a_1Z_1 + \cdots + a_nZ_n = (a_1, \cdots, a_n)\mathbb{Z} \sim N(a_1\mu_1 + \cdots + a_n\mu_n,\ a_1^2\sigma_1^2 + \cdots + a_n^2\sigma_n^2)$$

d. $(X,Y) \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right)$

then X and Y are indep $\Leftrightarrow \rho = 0$

$\operatorname{cov}(a_1Z_1 + \cdots + a_nZ_n,\ b_1Z_1 + \cdots + b_nZ_n) = 0$

$$\operatorname{cov}(a_1Z_1 + \cdots + a_nZ_n,\ b_1Z_1 + \cdots + b_nZ_n) = \sum_{i,j} a_ib_j \operatorname{cov}(Z_i, Z_j)$$

$$= \underbrace{\sum_i a_ib_i\sigma_i^2}_{} = 0.$$

1. $X, Y \overset{iid}{\sim} N(0,1)$ $\quad (X,Y) \sim N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix}\right)$

a. k $\quad \mathbb{P}(X > kY) = \mathbb{P}(X - kY > 0) = \frac{1}{2}$

$\mathbb{E}(X-kY) = \mathbb{E}X - k\mathbb{E}Y = 0$

$X - kY \sim N(0, ?)$

b. $\mathbb{P}(U > kV)$ $\qquad U = \sqrt{3}X + Y, \quad V = X - \sqrt{3}Y$

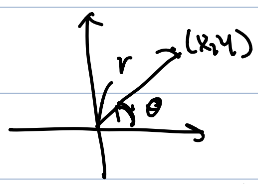
$= \mathbb{P}(U - kV > 0) = \mathbb{P}((\sqrt{3}-k)X + (1+\sqrt{3}k)Y > 0) = \frac{1}{2}$

$\mathbb{E}((\sqrt{3}-k)X + (1+\sqrt{3}k)Y) = (\sqrt{3}-k)\mathbb{E}X + (1+\sqrt{3}k)\mathbb{E}Y = 0$

$(\sqrt{3}-k)X + (1+\sqrt{3}k)Y \sim N(0, ?)$

c. $\mathbb{P}(U^2+V^2 < 1) = \mathbb{P}((\sqrt{3}X+Y)^2 + (X-\sqrt{3}Y)^2 < 1)$

$= \mathbb{P}(X^2+Y^2 < \frac{1}{4}) = \mathbb{P}(\sqrt{X^2+Y^2} < \frac{1}{2})$



$(X,Y) \longleftrightarrow (r, \theta)$ $\qquad x = r\cos\theta, \; y = r\sin\theta \qquad r = \sqrt{x^2+y^2}$

$f_{X,Y}(x,y) = \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right)$

$u(r,\theta) = (r\cos\theta, r\sin\theta) \qquad |J_u| = \begin{vmatrix}\cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta\end{vmatrix} = r$

$f_{r,\theta}(r,\theta) = \frac{1}{2\pi} r\exp\left(-\frac{r^2}{2}\right) I_{(0<r)} I_{(0<\theta<2\pi)}$

$f_r(r) = r\exp\left(-\frac{r^2}{2}\right)$

$\mathbb{P}(r < \frac{1}{2}) = \int_0^{\frac{1}{2}} r e^{-\frac{r^2}{2}} dr = 1 - e^{-1/8}.$

d. $(X,Y) \sim N\left((\mu_1,\mu_2)', \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\right)$ $\qquad X|Y=y \sim N\left(\mu_1 + \frac{\sigma_1}{\sigma_2}\rho\cdot(y-\mu_2), (1-\rho^2)\sigma_1^2\right)$

$f_{X,Y}(x,y)/f_Y(y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp(\sim) \Big/ \frac{1}{\sqrt{2\pi}\sigma_2}\exp\left(-\frac{1}{2}\cdot\frac{(y-\mu_2)^2}{\sigma_2^2}\right)$

(X,V) $\qquad \underset{0}{\mathbb{E}X}, \underset{0}{\mathbb{E}V}, \underset{1}{\mathrm{Var}X}, \underset{\mathrm{Var}(X-\sqrt{3}Y)=4}{\mathrm{Var}V}, \mathrm{cov}(X,V) = \mathrm{cov}(X, X-\sqrt{3}Y) = \mathrm{cov}(X,X) - \sqrt{3}\,\mathrm{cov}(X,Y)$

$= 1$

$(X,V)' \sim N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1&1\\1&4\end{pmatrix}\right)$

$X|V=v \sim N\left(\frac{1}{4}v, \frac{3}{4}\right)$

2. (a) $W \sim N(\mu, \sigma^2)$. $Z|W=w \sim N(aw+b, \tau^2)$

$$f(z,w) = f_W(w) \cdot f_{Z|W=w}(z) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(w-\mu)^2}{2\sigma^2}\right)\frac{1}{\sqrt{2\pi}\tau}\exp\left(-\frac{(z-aw-b)^2}{2\tau^2}\right)$$

$$= \frac{1}{2\pi\sigma\tau}\exp\left(-\frac{1}{2}\underbrace{\left\{\frac{(w-\mu)^2}{\sigma^2} + \frac{(z-aw-b)^2}{\tau^2}\right\}}\right)$$

$$\left(\frac{1}{\sigma^2}+\frac{a^2}{\tau^2}\right)w^2 + \frac{1}{\tau^2}z^2 - \frac{2a}{\tau^2}zw - \left(\frac{2\mu}{\sigma^2} - \frac{2ab}{\tau^2}\right)w - \left(\frac{2b}{\tau^2}\right)z + C$$

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2}\left\{\frac{(z-\mu_1)^2}{\sigma_1^2} + \frac{(w-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(z-\mu_1)(w-\mu_2)}{\sigma_1\sigma_2}\right\}\frac{1}{(1-\rho^2)}\right)$$

$EZ = \mu_1$ (= $a\mu+b$), $EW = \mu_2$ (= μ), $\operatorname{Var} Z = \sigma_1^2$, $\operatorname{Var} W = \sigma_2^2$ (= σ^2), $\operatorname{cov}(Z,W) = \rho\sigma_1\sigma_2$

$EZ = E(E(Z|W)) = E(aw+b) = a\mu+b$

Z^2: $\frac{1}{(1-\rho^2)}\frac{1}{\sigma_1^2} = \frac{1}{\tau^2}$ — ①

w^2: $\frac{1}{(1-\rho^2)}\cdot\frac{1}{\sigma_2^2} = \frac{1}{\sigma^2} + \frac{a^2}{\tau^2}$ — ②

zw: $\frac{\rho}{1-\rho^2}\frac{1}{\sigma_1\sigma_2} = \frac{a}{\tau^2}$ — ③

$\frac{③}{\sqrt{①②}} \leftarrow \rho = \frac{a\sigma}{\sqrt{\tau^2+a^2\sigma^2}}$

$\sigma_1^2 = a^2\sigma^2+\tau^2$

$$(W,Z)' \sim N\left(\begin{pmatrix}\mu \\ a\mu+b\end{pmatrix}, \begin{pmatrix}\sigma^2 & a\sigma^2 \\ a\sigma^2 & a^2\sigma^2+\tau^2\end{pmatrix}\right)$$

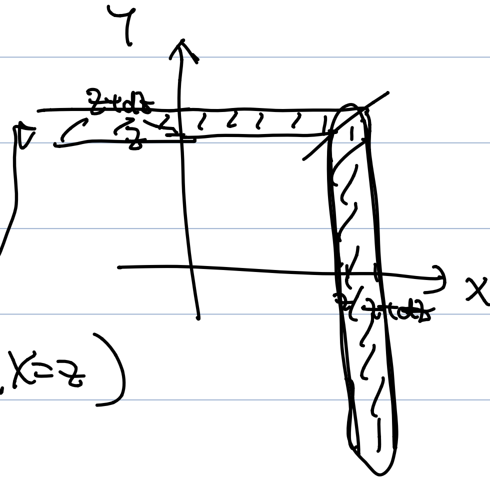
(b) $Z \sim N(a\mu+b, a^2\sigma^2+\tau^2)$

(c) $W|Z=z \sim N\left(\mu + \frac{a\sigma^2}{a^2\sigma^2+\tau^2}(z-a\mu-b), \frac{\tau^2\sigma^2}{\tau^2+a^2\sigma^2}\right)$

3. $(X,Y) \sim N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1&\rho\\\rho&1\end{pmatrix}\right)$

$\mathbb{E}\max(X,Y)$

$Z = \max(X,Y)$



$$\mathbb{P}(z \le Z \le z+dz)$$
$$\approx 2\,\mathbb{P}(z \le X \le z+dz)\,\mathbb{P}(Y \le z \mid X=z)$$
$$= 2\,\phi(z)\,dz \cdot \Phi\left(\frac{z-\rho z}{\sqrt{1-\rho^2}}\right)$$

$$f_Z(z) = 2\phi(z)\,\Phi\left(\frac{1-\rho}{\sqrt{1-\rho^2}}z\right)$$

$$\int_{-\infty}^{\infty} z \cdot 2\phi(z)\,\Phi\left(\frac{1-\rho}{\sqrt{1-\rho^2}}z\right)dz$$
$$= \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\,\underbrace{ze^{-\frac{1}{2}z^2}}_{(-e^{-\frac{1}{2}z^2})'}\,\Phi\left(\frac{1-\rho}{\sqrt{1-\rho^2}}z\right)dz$$
$$= \underbrace{\left[-\sqrt{\frac{2}{\pi}}\,e^{-\frac{1}{2}z^2}\,\Phi\left(\frac{1-\rho}{\sqrt{1-\rho^2}}z\right)\right]_{-\infty}^{\infty}}_{=0} + \underbrace{\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\,e^{-\frac{1}{2}z^2}\left(\Phi\left(\frac{1-\rho}{\sqrt{1-\rho^2}}z\right)\right)'dz}$$
$$= \int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\,e^{-\frac{1}{2}z^2}\,\frac{1-\rho}{\sqrt{1-\rho^2}}\,\phi\left(\frac{1-\rho}{\sqrt{1-\rho^2}}z\right)dz$$
$$= \sqrt{\frac{1-\rho}{\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{1+\rho}}\,\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2 - \frac{1}{2}\left(\frac{(1-\rho)^2}{1-\rho^2}z^2\right)\right)dz$$
$$= \sqrt{\frac{1-\rho}{\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\cdot\frac{2}{1+\rho}z^2\right)\sqrt{\frac{2}{1+\rho}}\,dz \qquad w = \sqrt{\frac{2}{1+\rho}}z$$
$$= \sqrt{\frac{1-\rho}{\pi}}\underbrace{\int_{-\infty}^{\infty}\underbrace{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{w^2}{2}\right)}_{N(0,1)}dw}_{=1}$$

$$= \sqrt{\frac{1-\rho}{\pi}}$$