Stat 134: Section 17

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Conceptual Review

Please discuss these short questions with those around you in section. These problems are intended to highlight concepts that will be relevant for today's problems.

a. What is a joint distribution? Can you come up with examples where the joint distribution is jointly independent and dependent?

b. What is the joint density function and marginal density?

c. Describe the joint uniform distribution.

Problem 1

Let X, Y be independently distributed and uniform on $(0, 1)$. Find:

$$P(Y \geq \frac{1}{2} | Y \geq 1 - 2X).$$

Ex 5.1.3 in Pitman's Probability

Solution:

We use the Bayes rule:

$$P(Y \geq \frac{1}{2} | Y \geq 1 - 2X) = \frac{P(Y \geq 1 - 2X | Y \geq \frac{1}{2}) \cdot P(Y \geq \frac{1}{2})}{P(Y \geq 1 - 2X)}.$$

Now we use the independent property to derive the probability above. Conditional on $Y \geq \frac{1}{2}$, $p(y) = 2$ on $[1/2, 1]$. Then:

$$\begin{aligned} P(Y \geq 1 - 2X | Y \geq \frac{1}{2}) &= \int_{y=1/2}^{1} p(y) P(y \geq 1 - 2X) dy \\ &= \int_{y=1/2}^{1} 2 \cdot P(X \leq \frac{1-y}{2}) dy \\ &= \frac{7}{8}. \end{aligned} \tag{1}$$

Similarly, we can compute:

$$P(Y \geq 1 - 2X) = \int_{y=0}^{1} p(y) P(y \geq 1 - 2X) dy = 3/4.$$

Hence the result is $7/12$.

Problem 2

For a striaght stick, we pick two points uniformly and independently. What is the probability that the three parts can form a triangle?
Ex 5.1.9 in Pitman's Probability

Solution. Let X, Y be the first and second points, and let A be the set of coordinates of X, Y such that the three pieces form a triangle. The first point can be in either the first half or the second half:

$$P((X,Y) \in A) = \int_{x=0}^{1/2} p(x)P((x,Y) \in A)dx + \int_{x=1/2}^{1} p(x)P((x,Y) \in A)dx.$$

Note that x is a number while X is a random variable. Then $p(x) = 1$ for all x, and we consider the first half. For $x \in [0, 1/2]$, the range of Y for which a triangle can be formed is $(1/2, x + 1/2)$. Thus $P((x,Y) \in A) = x$. Hence:

$$\int_{x=0}^{1/2} p(x)P((x,Y) \in A)dx = \int_{x=0}^{1/2} xdx = \frac{1}{8}.$$

By symmetry the second half is also $1/8$. Hence the final result is $1/4$.

Problem 3

Suppose X_1, X_2, X_3 are independent exponential distributions with $\lambda_1, \lambda_2, \lambda_3$. Find $P(X_1 < X_2 < X_3)$.
Ex 5.2.16 in Pitman's Probability

Solution. Recall that the exponential distribution has a pdf $\lambda e^{-\lambda x}$ whose integral is $-e^{-\lambda x}$. Then:

$$\begin{aligned} P(X_1 < X_2 < X_3) &= \int_{x_1=0}^{\infty} \int_{x_2=x_1}^{\infty} \int_{x_3=x_2}^{\infty} p(x_1, x_2, x_3) dx_3 dx_2 dx_1 \\ &= \int_{x_1=0}^{\infty} \int_{x_2=x_1}^{\infty} \int_{x_3=x_2}^{\infty} \lambda_1 \lambda_2 \lambda_3 e^{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3} dx_3 dx_2 dx_1 \\ &= \int_{x_1=0}^{\infty} \int_{x_2=x_1}^{\infty} \lambda_1 \lambda_2 e^{-\lambda_1 x_1 - (\lambda_2 + \lambda_3) x_2} dx_2 dx_1 \\ &= \int_{x_1=0}^{\infty} \lambda_1 \frac{\lambda_2}{\lambda_2 + \lambda_3} e^{-(\lambda_1 + \lambda_2 + \lambda_3) x_1} dx_1 \\ &= \frac{\lambda_1 \lambda_2}{(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)}. \end{aligned} \tag{2}$$