Stat 134 lec 41 (dead week review)

Plan

Monday 4/30

Class Review Gamma and Poisson Process
Conditional expectation

Section sec 6.5 bivariate normal

Wednesday 5/1

No class but Adam OH 10-11 SLC

Section Final review

Friday 5/3

Class Go over 2016 final

Adam OH 1-2 SLC

Poisson Scatter and Poisson arrival times (Gamma)

Sec 3.5 Poisson Dist.

$X \sim \text{Pois}(\mu)$

$P(X=k) = \dfrac{e^{-\mu}\mu^k}{k!} \quad k=0,1,2,\ldots$

$E(X) = \mu$

$\text{Var}(X) = \mu.$

Think of $\text{Pois}(\mu)$ as limit of $\text{Bin}(n, p_n)$ where $n \cdot p_n \to \mu$.

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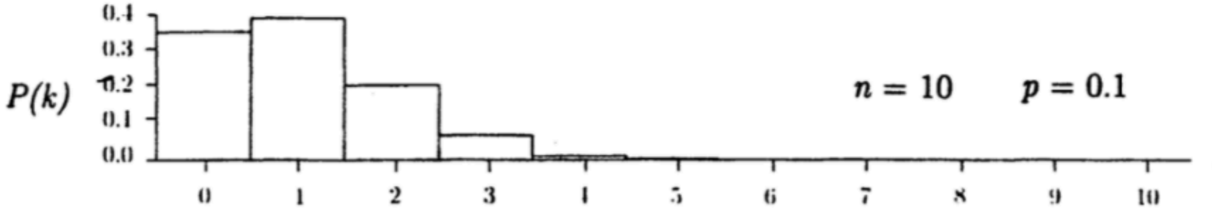
Example 1. The binomial (10, 1/10) distribution.

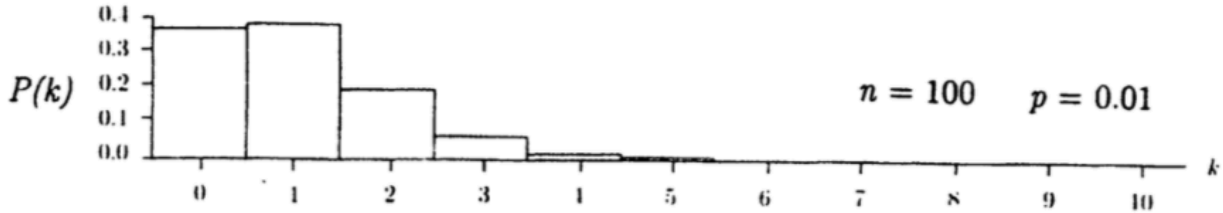
This is the distribution of the number of black balls obtained in 10 random draws with replacement from a box containing 1 black ball and 9 white ones.

Poisson($\mu=1$) is a limit of binomials Bin$(n, \frac{1}{n})$



Example 2. The binomial (100, 1/100) distribution.

This is the distribution of the number of black balls obtained in 100 random draws with replacement from a box containing 1 black ball and 99 white ones.

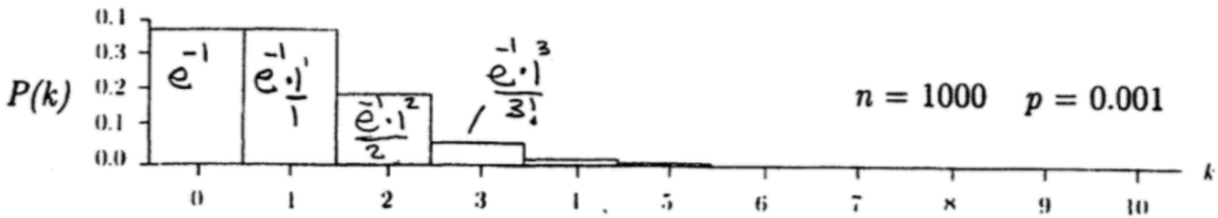


Example 3. The binomial (1000, 1/1000) distribution.

Now take 1000 random draws with replacement from a box with 1 black ball and 999 white ones. This is the distribution of the number of black balls drawn:

$X \sim \text{Pois}(1)$

$P(X=k) = \frac{e^{-1}}{k!}$



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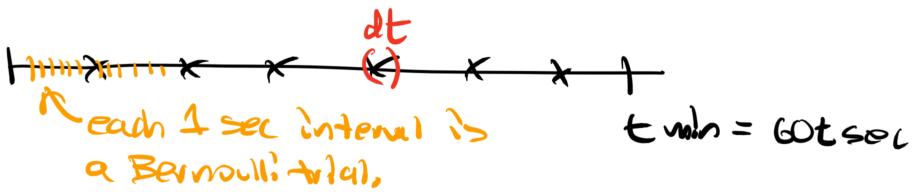
Poisson process w/ intensity λ

ex X_t = # calls coming into a hotel reservation center in t min with intensity rate $\lambda = \frac{1}{2}$ call/min

$X_t \sim \text{Pois}(\underbrace{\lambda t}_{\mu})$ where λ = # calls/minute.

μ = avg # calls in t min.

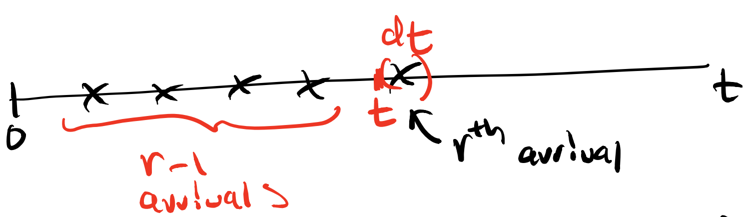


$X_{dt} \sim \text{Pois}(\lambda dt)$

$$P(\text{one call in } dt) = P(X_{dt}=1) = \frac{e^{-\lambda dt}(\lambda dt)^1}{1!} = e^{-\lambda dt} \cdot \lambda dt \approx \lambda dt \text{ for small } dt$$

Sec 4.2 Gamma(r,λ) = distribution for r^{th} arrival time of a Poisson (λ) process.

T_r = arrival time of r^{th} call.



$P(T_r \in dt) = P(X_t = r-1,\ X_{dt} = 1)$ where

$X_t \sim \text{Pois}(\lambda t)$

$X_{dt} \sim \text{Pois}(\lambda dt)$

$$= P(X_t = r-1)\,P(X_{dt} = 1)$$ since the interval $[0,t]$ and $[t, t+d]$ are disjoint

$$= \frac{e^{-\lambda t}(\lambda t)^{r-1}}{(r-1)!} \cdot \lambda dt$$

so $f_{T_r}(t) = \frac{\lambda^r}{(r-1)!} t^{r-1} e^{-\lambda t}$ is density of gamma(r, λ)

$T_1 \sim \exp(\lambda)$

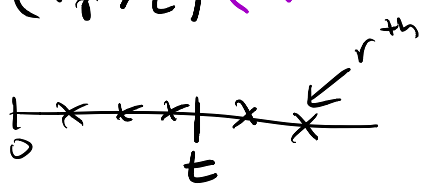
<u>Fact</u> $E(T_1) = \frac{1}{\lambda}$, $Var(T_1) = \frac{1}{\lambda^2}$

$T_r = \underbrace{T_1 + \cdots + T_1}_{r \text{ indep}} \Rightarrow E(T_r) = \frac{r}{\lambda}$, $Var(T_r) = \frac{r}{\lambda^2}$

<u>ex</u> Let $T_r \sim$ gamma(r, λ)

Find $P(T_r > t)$ (Right tail probability)

<u>Soln</u>



$$P(T_r > t) = P(X_t \le r-1)$$

$$= P(X_t = 0) + P(X_t = 1) + \cdots + P(X_t = r-1)$$

$$= e^{-\lambda t} + e^{-\lambda t}\lambda t + \frac{e^{-\lambda t}(\lambda t)^2}{2!} + \cdots + \frac{e^{-\lambda t}(\lambda t)^{r-1}}{(r-1)!}$$

ex

Let $r>0$ be an integer, and $\lambda>0$.

a) Use what you know about a Poisson process with rate λ to find for every $t>0$

$$\int_0^t \frac{\lambda^r}{(r-1)!} s^{r-1} e^{-\lambda s}\, ds .$$

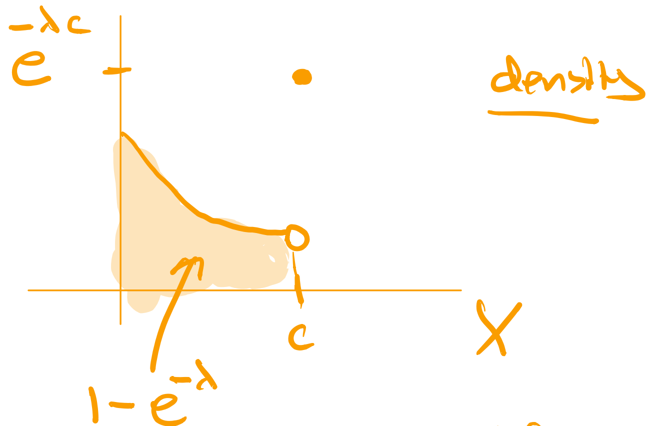
$F(s) = P(T_r < t)$

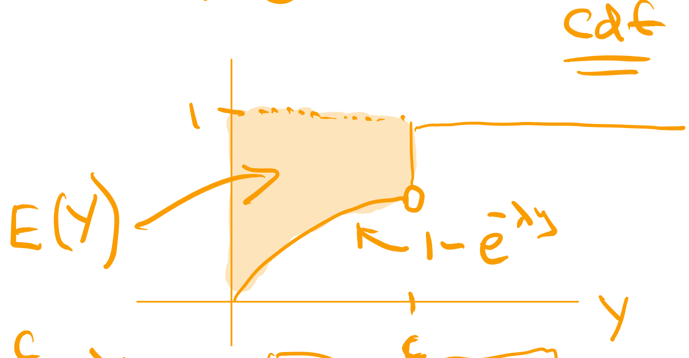
T_r

$= 1 - P(T_r > t)$

$$= \boxed{1 - e^{-\lambda t}\left(1 + \lambda + \frac{(\lambda t)^2}{2!} + \dots + \frac{(\lambda t)^{r-1}}{(r-1)!}\right)}$$

b) Let $T \sim \exp(\lambda)$ and let $c>0$ be a constant. let $Y = \min(T, c)$ be "T killed at c". Find $E(Y)$





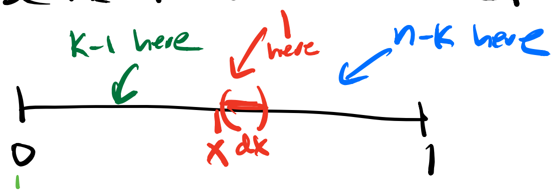
See lec 26 for details

$$E(Y) = \int_0^c e^{-\lambda y}\, dy = \boxed{\frac{1}{\lambda} - \frac{1}{\lambda} e^{-\lambda c}}$$

sec 4.6 Review of beta(r, s).

let $U_1, \dots, U_n \overset{iid}{\sim}$ Unif(0,1) and

$U_{(k)}$ be the k^{th} order statistic.



$$P(U_{(k)} \in dx) = \binom{n}{1} 1 dx \cdot \binom{n-1}{k-1} x^{k-1} \cdot (1-x)^{n-k}$$

$$\Rightarrow \quad f_{U_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} x^{\overset{r-1}{k-1}} (1-x)^{(n-k+1)-1 \; s-1}$$

$$X \sim beta(r,s) \Rightarrow f_X(x) = \frac{n!}{(r-1)!(s-1)!} x^{r-1} (1-x)^{s-1}$$

$$= \frac{\Gamma(n)}{\Gamma(r)\Gamma(s)}$$

ex Fix $\lambda > 0$ and

let $P(X = n) = c \frac{\lambda^n}{n!}$ for $n = 1, 2, 3, \dots$

Let $T \mid X = n$ have density

proportional to $(1-t)^{n-1}$ for $0 < t < 1$.

a) Find the value of c and compute the density of T.

Hint: write $P(T \in dt) = \sum_{n=1}^{\infty} P(T \in dt \mid X = n) P(X = n)$

b) Find the expectation and variance of both T and X.

soln

a) c is constant part of $X \sim \text{Pois}(\lambda)$

$\Rightarrow c = e^{-\lambda}$

$T \mid X = n \sim \text{beta}(1, n)$ since

$t^{1-1}(1-t)^{n-1}$ is variable part of beta(1,n),

$$P(T \in dt) = \sum_{n=1}^{\infty} P(T \in dt \mid X = n)\, P(X = n)$$

$$= \sum_{n=1}^{\infty} \underbrace{\frac{\Gamma(n+1)}{\Gamma(n)\Gamma(1)}}_{= n} (1-t)^{n} e^{-\lambda} \frac{\lambda^n}{n!}\, dt$$

$$\Rightarrow f_T(t) = \sum_{n=1}^{\infty} n\, (1-t)^{n} e^{-\lambda} \frac{\lambda^n}{n!}$$

$$= \sum_{n=1}^{\infty} \lambda^n \frac{(1-t)^{n-1}}{(n-1)!} e^{-\lambda}$$

$$= \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{(\lambda(1-t))^{n-1}}{(n-1)!}$$

$$= \lambda e^{-\lambda} \underbrace{\sum_{n=0}^{\infty} \left(\frac{\lambda(1-t)^n}{n!} \right)}_{= e^{-\lambda(1-t)}}$$

$$= \lambda e^{-\lambda} e^{\lambda(1-t)} = \lambda e^{-\lambda t}$$

$\Rightarrow T \sim \exp(\lambda)$

b) $E(T) = \frac{1}{\lambda}$ since $T \sim \exp(\lambda)$

$\text{Var}(T) = \frac{1}{\lambda^2}$

$E(X) = \lambda$ since $X \sim \text{Pois}(\lambda)$

$\text{Var}(X) = \lambda$

Review of conditional expectation

Properties (from lec 34)

1. $E(Y) = E(E(Y|X))$ iterated expectation
2. $E(aY+b|X) = aE(Y|X)+b$
3. $E(Y+Z|X) = E(Y|X)+E(Z|X)$
4. $E(g(X)|X) = g(X)$
5. $E(g(X)Y|X) = g(X)E(Y|X)$
6. $Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$ total variance decomp.

ex Let $U \sim Unif(0,1)$

Given $U=u$, X is exponential with mean u — You

Find $E(X)$, $E(UX)$ and $Var(X)$.

Soln

Know $E(X|U=u) = u$

$$E(X|U) = U$$

$$E(X) = E[E(X|U)] = E[U] = 1/2$$

$$E(UX) = E(E(UX|U)) = E[U\underbrace{E(X|U)}_{=U}]$$

$$= \underbrace{E(U^2)}_{=Var(U)+E(U)^2} = 1/3 = \tfrac{1}{12} + \tfrac{1}{4}$$

$$Var(X) = \underbrace{E(Var(X|U))}_{=1/3} + \underbrace{Var(E(X|U))}_{=Var(U)\,=\,1/12}$$

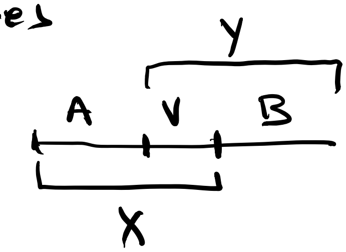
$$= 1/3 + 1/12$$

$$= 5/12$$

ex: Toss a fair coin 30 times

$X = \#$ heads first 20

$Y = \#$ heads 2nd 20



$$\left.\begin{array}{l} A \sim Bin(10, 1/2) \\ V \sim Bin(10, 1/2) \\ B \sim Bin(10, 1/2) \end{array}\right\} \text{ indep.}$$

$X = A + V$

$Y = V + B$

Find $Corr(X, Y)$?

$$Corr(X,Y) = \frac{Cov(X,Y)}{SD(X)SD(Y)}$$

$$= Cov(A+V, V+B) = Cov(V, V)$$

$$= Var(V) = 10\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)$$

$$= \frac{10}{4}$$

$$Corr(X,Y) = \frac{10/4}{20/4} = 1/2$$

Find $E(Y|X) = E(V+B \mid X)$

$$= E(V|X) + E(B|X)$$

$$E(B|X) = E(B) = 10 \cdot 1/2 = 5$$

$V|X$ is hypergeom $(N=20, n=10, G=X)$

$$\boxed{\begin{array}{l} X \text{ heads} \\ N-X \text{ tails} \end{array}}$$

$\downarrow$ draw n w/o replacement

$V|X$ is number of good in sample size n

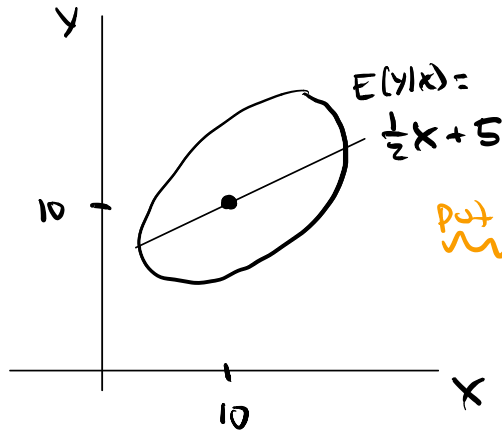
$$E(V|X) = n \cdot \frac{G}{N} = 10 \cdot \frac{X}{20} = \tfrac{1}{2}X$$

$$E(Y|X) = \frac{1}{2}X + 5$$

$$\left.\begin{array}{l} X \sim N(10,5) \\ Y \sim N(10,5) \end{array}\right\} \text{binomial is approximately normal by CLT}$$

$$\mathrm{Corr}(X,Y) = 1/2$$

<u>Picture</u>



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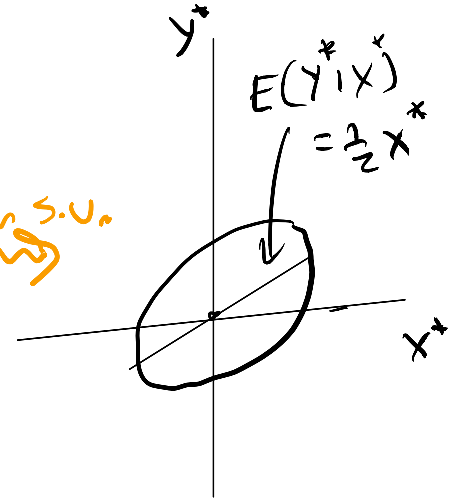

(x, y) bivariate normal

(x^*, y^*) std bivariate normal