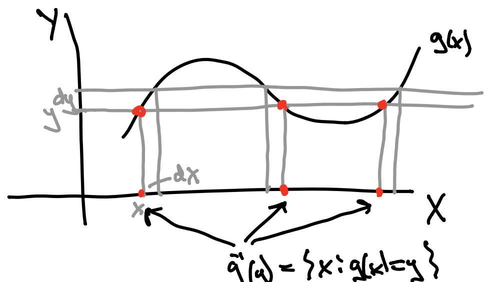
Stat 134 lec 25

Last 2 times

① sec 4.4 Change of Variable rule

many to one g:



$f_Y(y)dy = f_X(x_1)dx_1 + f_X(x_2)dx_2 + f_X(x_3)dx_3$

$f_Y(y) = f_X(x_1)\frac{dx_1}{dy} + f_X(x_2)\frac{dx_2}{dy} + f_X(x_3)\frac{dx_3}{dy}$

$= \frac{f_X(x_1)}{g'(x_1)} + \frac{f_X(x_2)}{|g'(x_2)|} + \frac{f_X(x_3)}{g'(x_3)}$

$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{g'(x)}$

$P(x \in dx_2) \geq 0$

② $X \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

$f_X(x) = \frac{1}{\Gamma(\frac{1}{2})}\left(\frac{1}{2}\right)^{\frac{1}{2}} x^{-\frac{1}{2}} e^{-\frac{1}{2}x}$

constant part, variable part

③ MGF (not in book)

$M_X(t) = E(e^{tX})$

Thm If a MGF exists in an interval around zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

ex $X \sim \text{Pois}(\mu)$

$M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk}\frac{\mu^k e^{-\mu}}{k!} = e^{-\mu}\sum_{k=0}^{\infty}\frac{(\mu e^t)^k}{k!}$

$= e^{-\mu}e^{\mu e^t} = \boxed{e^{\mu(e^t-1)} \text{ for all } t}$

then $E(X) = \frac{d}{dt}M_X(t)\Big|_{t=0} = e^{\mu(e^t-1)} \cdot \mu e^t \Big|_{t=0} = \mu$ ✓

Today

① Key properties of MGF
② Review concept test responses from lec 23.
③ Sec 4.5 CDF of a mixed distribution.

① Key Properties of MGF

ⓐ If an MGF exists in an interval containing zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

last time

ⓑ If X and Y are independent RVs,

$M_{X+Y}(t) = M_X(t)M_Y(t)$.

Proved in MGF HW.

ⓒ $M_X(t) = M_Y(t)$ for all t in an interval around 0 then $F_X(z) = F_Y(z)$ (i.e X and Y have the same distribution).

Skip proof — we can invert a MGF to get the CDF.

ex If $M_X(t) = \frac{1}{2}e^{1t} + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}$, $= E(e^{tX})$

e^{xt} tells us the value of X and the associated coefficients tell us the probabilities

(i.e $X = 1, 2, 3$ w/ prob $1/2, 1/3, 1/6$.)

so MGF ⇒ distribution of X when X has finite # values,

Property ⓐ is useful to find $E(X)$, $Var(X)$,

Property ⓑ and ⓒ allow us to prove for example that sum of independent Poisson is Poisson.

ex

$$\left.\begin{array}{l} X_1 \sim \text{Pois}(\mu_1) \\ X_2 \sim \text{Pois}(\mu_2) \end{array}\right\} \text{ independent.}$$

Show that $X_1 + X_2 \sim \text{Pois}(\mu_1 + \mu_2)$.

$$M_{X_1}(t) = e^{\mu_1(e^t - 1)} \quad \text{for all } t$$

$$M_{X_2}(t) = e^{\mu_2(e^t - 1)} \quad \text{for all } t$$

$$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t) = \boxed{e^{(\mu_1+\mu_2)(e^t - 1)}}$$

MGF of Pois $(\mu_1 + \mu_2)$ for all t.

$$\Rightarrow X_1 + X_2 \sim \text{Pois}(\mu_1 + \mu_2)$$

ex Let X be a RV and a a constant.

Show that $M_{aX}(t) = M_X(at) \leftarrow E(e^{Xat})$

hint

$$\begin{aligned} M_{aX}(t) &= E(e^{aXt}) \\ &= E(e^{Xat}) \\ &= M_X(at). \end{aligned}$$

For $X \sim \text{Gamma}(r, \lambda)$

recall $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$ for $t < \lambda$

ex Let $X \sim \text{Exp}(\lambda)$ and $a > 0$.

Show that $Y = aX$ is also exponential, and specify the new parameter.

$$M_Y(t) = M_{aX}(t) = M_X(at) = \left(\frac{\lambda}{\lambda - at}\right)^1 \quad \text{for } at < \lambda$$

$$= \frac{\frac{\lambda}{a}}{\left(\frac{\lambda}{a} - t\right)} \quad \text{for } t < \frac{\lambda}{a}$$

> 0

this is MGF of $\text{Exp}\left(\frac{\lambda}{a}\right)$.

2. Let Z be a standard normal RV (with variable part $e^{-\frac{z^2}{2}}$). The variable part of the distribution $X = Z^2$ is?

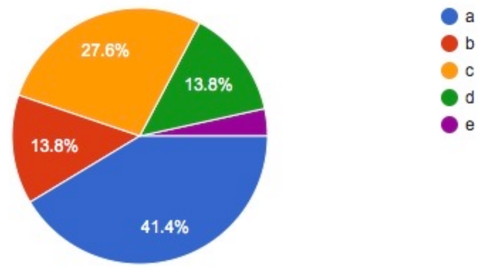
a Gamma $x^{-\frac{1}{2}}e^{-\frac{x}{2}}$

b Gamma $x^{\frac{1}{2}}e^{-\frac{x}{2}}$

c Exponential $e^{-\frac{x}{2}}$

d Normal $e^{-\frac{x^2}{2}}$

e none of the above



Change of variable formula:

$$f_X(x) = \sum_{\{z \mid g(z)=x\}} \frac{f_Z(z)}{|g'(z)|} \quad \text{evaluated at } z = g^{-1}(x)$$

c	X = Z^2 so therefore Z = sqrtx. Plugging that into the variable part for Z, you get e^(-x/2)

$$e^{-z^2/2} \longrightarrow e^{-x/2}$$

$g(z)$ $g'(z)$ $g^{-1}(x)$

a	by applying the rule for change of variable, get gz, g'z, g-1z then get a

$$g(z) = z^2$$
$$g'(z) = 2z$$
$$x = z^2 \Rightarrow z = \pm\sqrt{x} \leftarrow g^{-1}(x)$$

$$e^{-z^2/2} \longrightarrow \frac{1}{2\sqrt{x}}e^{-x/2} + \frac{1}{2\sqrt{x}}e^{-x/2} = x^{-1/2}e^{-x/2}$$

(2) Sec 4.5 Cummulative Distribution Function (CDF)

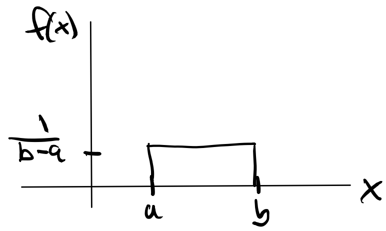
defn X RV

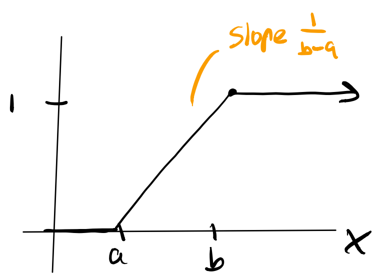
$$F_X(x) = P(X \le x)$$

Use Describes a distribution

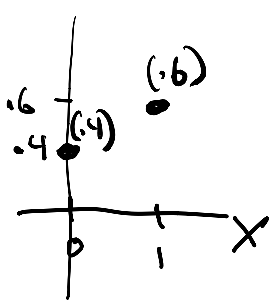
(equivalent to a density or probability mass function)

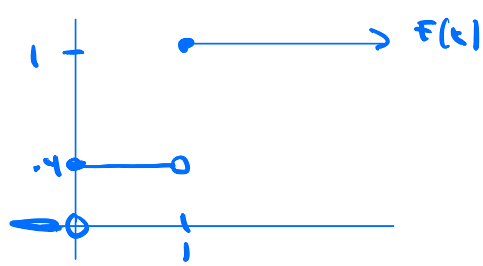
ex $X \sim \text{Unif}(a,b)$



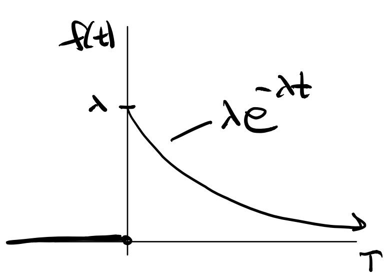


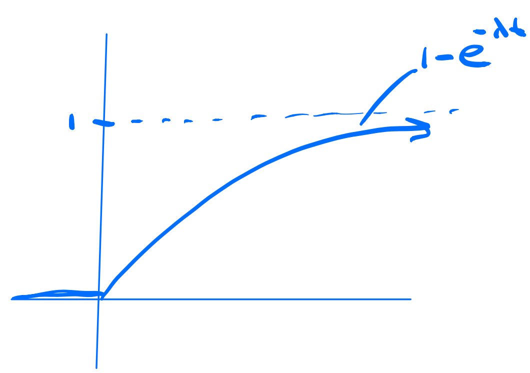
ex $X \sim \text{Bernoulli}(p=.6)$





ex $T \sim \text{Exp}(\lambda)$





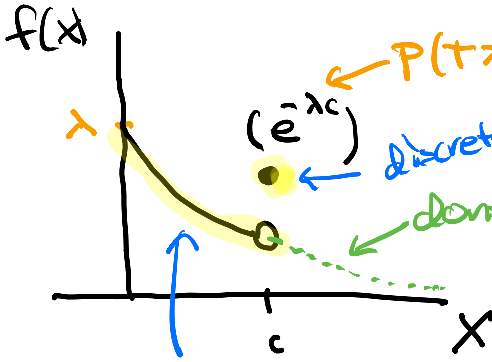
Ex (Mixed distribution)

$T \sim Exp(\lambda)$

$c > 0$

$$X = \begin{cases} T & \text{if } x < c \\ c & \text{if } x = c \end{cases}$$

$\leftarrow X = \min(T, c)$
"T killed by c"



$\leftarrow P(T > c)$

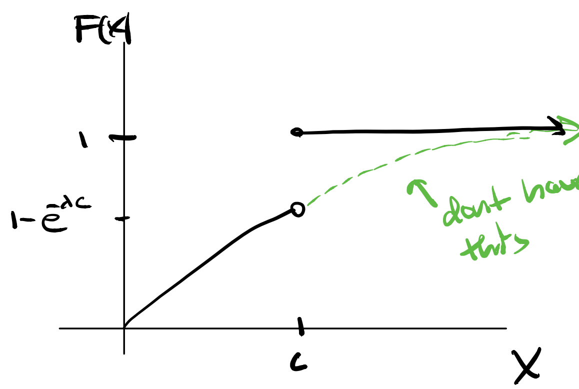
$(e^{-\lambda c})$

$\leftarrow$ discrete part with probability $e^{-\lambda c}$

$\leftarrow$ dont have this

continuous part with probability

$P(T < c) = 1 - e^{-\lambda c}$



dont have this

Note: continuous distributions dont have cdfs with jumps.

tinyurl.com/oct30-pt1

Stat 134

Friday March 22 2019

1. Suppose you are trying to discretly leave a party. Your time to leave is uniform from 0 to 2 minutes. However, if your walk to the exit takes more than 1 minute, you run into a friend at the door and and must spend the full 2 minutes to leave. Let T represent the time it takes you to leave. True or false, the graph of the cdf of T is: