Stat 134 lec 43

Warmup 1:00–1:10

Let $N \sim Pois(\mu)$ $\quad P(N=n) = \frac{\mu^n e^{-\mu}}{n!}$

Let $U_1, U_2, \dots U_N \overset{iid}{\sim} U(0,1)$, independent of N

Let $M = \min(U_1, U_2, \dots, U_N)$.

If $N=0$, define $M=1$.

What distribution is $M|N$?

$M|N=3 = U_{(1)}$ of 3 $\sim Beta(1,3)$

$U_{(1)}$ of N $\sim Beta(1,N)$

Find $E(M|N)$ and $E(M)$

$$\boxed{E(M|N) = \frac{1}{1+N}}$$

$$E(M) = E(E(M|N)) = E\left(\frac{1}{1+N}\right) = \sum_{n=0}^{\infty} \frac{1}{1+n} \frac{e^{-\mu}\mu^n}{n!}$$

$$= \frac{1}{\mu} e^{-\mu} \underbrace{\sum_{n=0}^{\infty} \frac{\mu^{n+1}}{(n+1)!}}_{= \mu + \frac{\mu^2}{2!} + \cdots = e^{\mu}-1}$$

$$= \boxed{\frac{1}{\mu}\left(1-e^{-\mu}\right)}$$

Jiangyue Chen

Yesterday

Can we go over question 1 in Review_Indicators?

Reply

Problem 1

In a bin, there are r red balls and b blue balls. Suppose I take the balls out, one by one (i.e. without replacement), until there are no more red balls in the bin. Let X denote the number of balls taken out. Find:

a. $\mathbb{E}(X)$;

b. $\text{Var}(X)$.

a) $X =$ # balls taken out until no more red

$= r + I_1 + \cdots + I_b$ $\qquad I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ blue ball taken out} \\ 0 & \text{else} \end{cases}$ $\quad r/r+1$

$$\boxed{E(X) = r + b\left(\frac{r}{r+1}\right)}$$

— red_1 — red_2 — .... — red_r —

b) $Var(X) = Var(r + I_1 + \cdots + I_b) = Var(I_1 + \cdots + I_b)$

$= b\,Var(I_1) + b(b-1)\,Cov(I_1, I_2)$

$\frac{r}{\cancel{r+1}} \cdot \frac{\cancel{r+1}}{r+2} = \frac{r}{r+2}$

$I_1 I_2 = I_{12} = \begin{cases} 1 & \text{if both } b_1 \text{ and } b_2 \text{ taken out} \\ 0 & \text{else} \end{cases}$

— red_1 — b_1 — red_2 — .... — red_r —

$\uparrow$

$Cov(I_1, I_2) = E(I_1 I_2) - E(I_1)E(I_2)$

$$= \boxed{\frac{r}{r+2} - \left(\frac{r}{r+1}\right)^2}$$

$Var(I_1) = \left(\frac{r}{r+1}\right)\left(\frac{1}{r+1}\right)$

$$\boxed{Var(X) = b\left(\frac{r}{r+1}\right)\left(\frac{1}{r+1}\right) + b(b-1)\left[\frac{r}{r+2} - \left(\frac{r}{r+1}\right)^2\right]}$$

ex Expectation / Variance with indicators.

6. A drawer contains s black socks and s white socks, where s is a positive integer. I pull two socks out at random without replacement and call that my first pair. Then I pull two socks out at random without replacement from the remaining socks in the drawer, and call that my second pair. I proceed in this way till I have s pairs and the drawer is empty.

Let D be the number of pairs in which the two socks are of different colors.

a) Find $E(D)$

any of 2s poss. socks

$\frac{2s}{2s} \cdot \frac{s}{2s-1}$ — any of s poss socks remaining

$= \frac{s}{2s-1}$

$D = I_1 + \cdots + I_s$

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ pair diff color} \end{cases}$

$E(D) = s \cdot \left(\frac{s}{2s-1}\right) = \boxed{\frac{s^2}{2s-1}}$

6. A drawer contains s black socks and s white socks, where s is a positive integer. I pull two socks out at random without replacement and call that my first pair. Then I pull two socks out at random without replacement from the remaining socks in the drawer, and call that my second pair. I proceed in this way till I have s pairs and the drawer is empty.

Let D be the number of pairs in which the two socks are of different colors.

b) Find $\mathrm{Var}(D)$.

$I_1, \ldots, I_s$ exchangeable, since draw socks w/o replacement from drawer.

$$D = I_1 + \cdots + I_s$$

$$\mathrm{Var}(D) = s\,\mathrm{Var}(I_1) + s(s-1)\,\mathrm{Cov}(I_1, I_2)$$

$$\mathrm{Cov}(I_1, I_2) = E(\overset{I_1 I_2}{I_{12}}) - E(I_1)E(I_2)$$

$$\frac{2s}{2s}\,\frac{s}{2s-1}\,\frac{2s-2}{2s-2}\,\frac{s-1}{2s-3} = \frac{s}{2s-1}\cdot\frac{s-1}{2s-3}$$

$$I_{12} = \begin{cases} 1 & \text{if } 1^{st} \text{ and } 2^{nd} \text{ pair diff color} \\ 0 & \text{else} \end{cases}$$

$$s(s-1)\,\mathrm{Cov}(I_1, I_2) = s(s-1)\left[\frac{s}{2s-1}\cdot\frac{s-1}{2s-3} - \left(\frac{s}{2s-1}\right)^2\right]$$

$$s\cdot\mathrm{Var}(I_1) = s\cdot\left[\left(\frac{s}{2s-1}\right)\left(1 - \frac{s}{2s-1}\right)\right]$$

$$\mathrm{Var}(D) = s\cdot\mathrm{Var}(I_1) + s(s-1)\,\mathrm{Cov}(I_1, I_2)$$

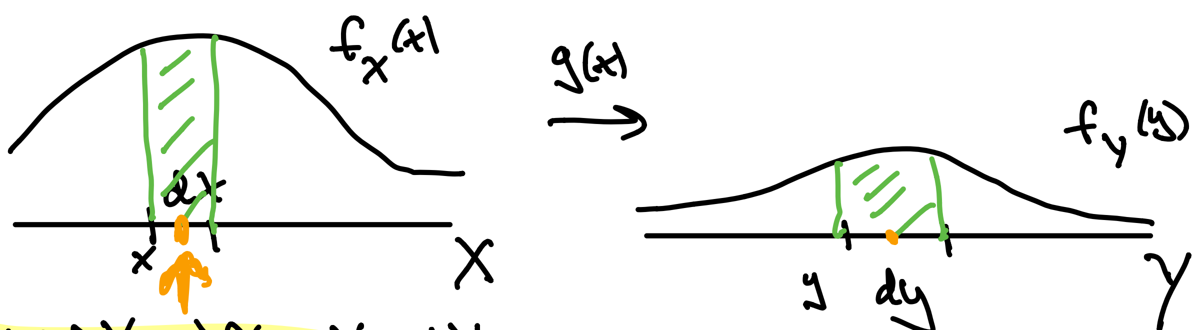
$\square$

Change of Variable rule

Let X be RV and $Y=g(X)$ for a 1-1 function g

$$f_Y(y) = \frac{1}{|g'(x)|} f_X(x) \Big|_{x=g^{-1}(y)}$$

Picture



$X \in dX$ iff $Y \in dY$

$\Rightarrow P(X \in dX) = P(Y \in dY)$

$\Rightarrow f_X(x)dx = f_Y(y)dy$

$$\Rightarrow f_Y(y) = \frac{1}{\left|\frac{dy}{dx}\right|} f_X(x) \Big|_{x=g^{-1}(y)}$$

Lily Yang

Yesterday

Could we go over 5.2.9b from the textbook?

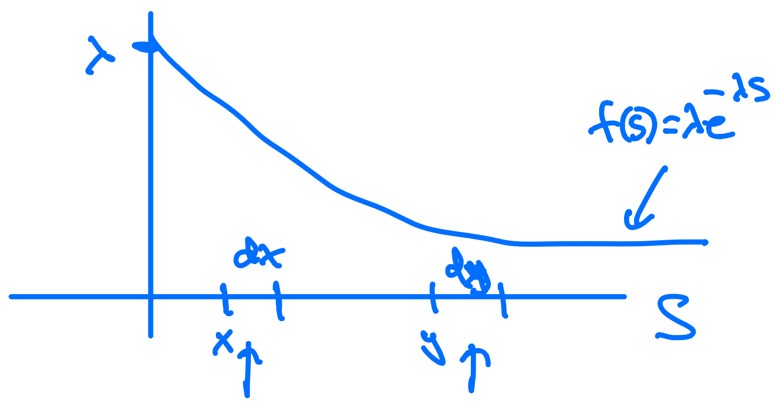
Reply

9. Minimum and maximum of two independent exponentials. Let $X = \min(S,T)$ and $Y = \max(S,T)$ for independent exponential(λ) variables S and T. Let $Z = Y - X$.

a) Find the joint density of X and Y. Are X and Y independent?

b) Find the joint density of X and Z. Are X and Z independent?

c) Identify the marginal distributions of X and Z.



$$P(X \in dx, Y \in dy) = f(x,y)\,dx\,dy$$

$$= \binom{2}{1} f(x)dx \cdot \binom{1}{1} f(y)dy$$

$$= 2\lambda e^{-\lambda x} \cdot \lambda e^{-\lambda y}\,dx\,dy$$

$$= \boxed{\underbrace{2\lambda^2 e^{-\lambda(x+y)}}_{f(x,y)}}\,dx\,dy$$

$$F_Y(y) = P(Y<y) = P(S<y, T<y) = P(S<y)P(T<y)$$
$$= (1-\lambda e^{-\lambda y})^2$$

$$f(y) = F'(y) = \boxed{2(1-\lambda e^{-\lambda y})(\lambda^2 e^{-\lambda y})}$$

$$F_X(x) = P(X<x) = 1 - P(X>x)$$
$$= 1 - P(S>x, T>x)$$
$$= 1 - P(S>x)P(T>x)$$
$$= 1 - e^{-\lambda x} \cdot e^{-\lambda x}$$
$$= 1 - e^{-2\lambda x}$$

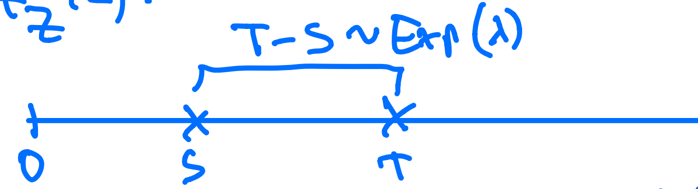
$$\boxed{f_X(x) = 2\lambda e^{-2\lambda x}} \Rightarrow X \sim \text{Exp}(2\lambda)$$

$\Rightarrow X, Y$ not indep.

b) $Z = Y - X$

First find $f_Z(z)$:

Note if $T>S$



Poisson random scatter w/ rate λ.

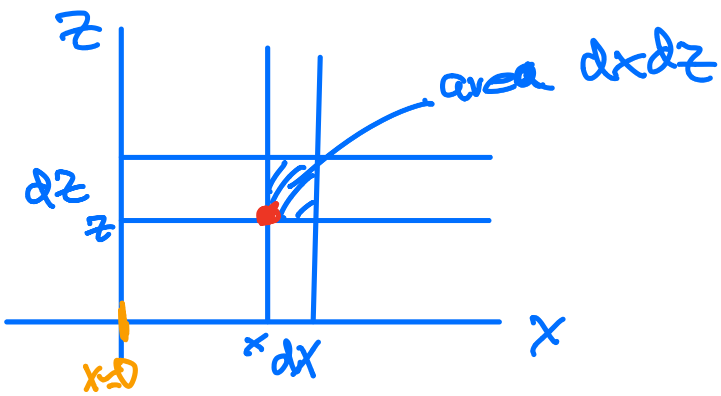
Similarly if $S<T$ then $S-T \sim \text{Exp}(\lambda)$.

$$P(Z > z) = \underbrace{P(Z > z \mid T > S)}_{= e^{-\lambda z}} \underbrace{P(T > S)}_{= 1/2} + \underbrace{P(Z > z \mid T < S)}_{= e^{-\lambda z}} \underbrace{P(T < S)}_{= 1/2} = e^{-\lambda z}$$

$$\Rightarrow \boxed{Z \sim \text{Exp}(\lambda)}$$

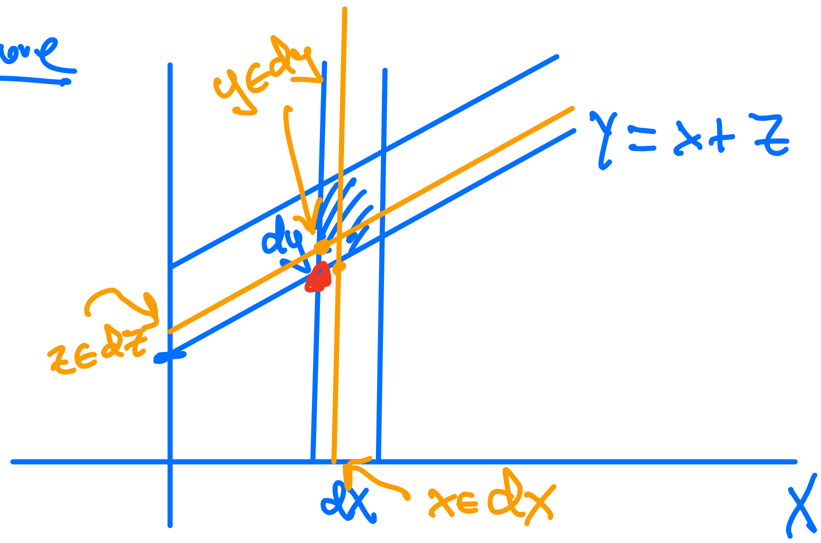
Next find $f_{X,Z}(x,z)$

$P(X \in dx, Z \in dz)$ is the volume under a surface $f_{X,Z}(x,z)$ over $dx dz$



If transform second coordinate $z \Rightarrow x+z$ we have the following:

Picture



The key is to notice that if $x\in dx$ and $z\in dz$ in previous picture then $x\in dx$ and $y\in dy$ in new picture.

Furthermore $dx\,dz = dx\,dy$ so

$$P(X\in dx, Y\in dy) = P(X\in dx, Z\in dz)$$

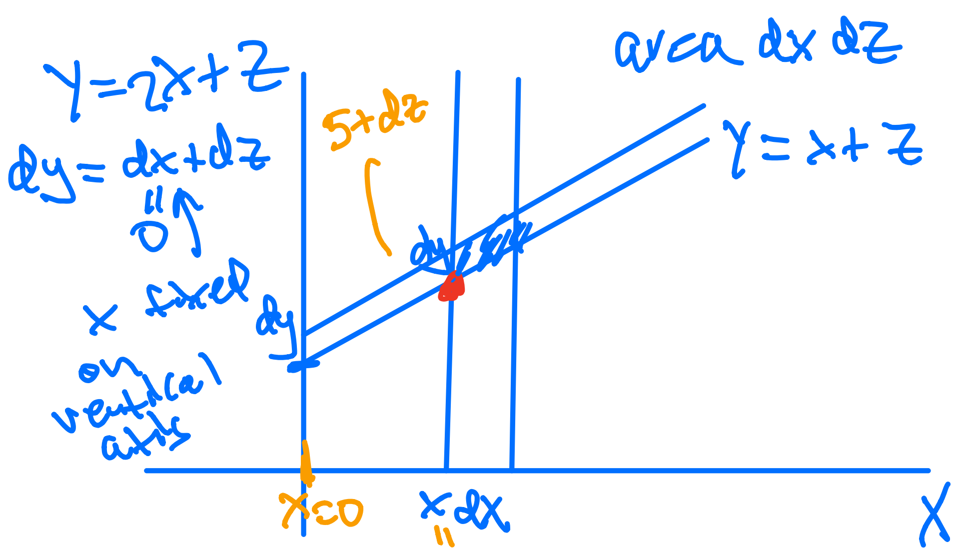
$$\Rightarrow f_{X,Y}(x, x+z)\,dx\,dy = f_{X,Z}(x,z)\,dx\,dz$$

$$\Rightarrow \boxed{f_{X,Z}(x,z) = f_{X,Y}(x, x+z)}$$

$$= 2\lambda e^{-\lambda(2x+z)} \text{ from part a.}$$

$$= 2\lambda e^{-2\lambda x} \cdot \lambda e^{-\lambda z}$$

$$= f_X(x) \quad f_Z(z) \Rightarrow \boxed{x, z \text{ indep.}}$$



Y=2X+Z
dy=dx+dz
0
x fixed
on vertical axis
dy
5+dz
dy
area dx dz
Y=x+Z
x=0
x dx
X

$$z = 2y - x$$

$$y = \frac{1}{2}z + \frac{1}{2}x$$

$$dy = \frac{1}{2}dz$$