

next Monday Q4 covers 4.1, 4.2

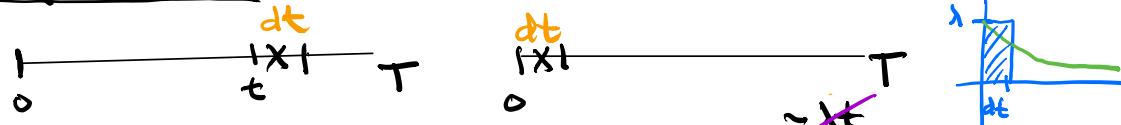
last time sec 4.2 exponential distributions.

survival function

$$T \sim \text{Exp}(\lambda) \quad f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases} \quad \text{or } P(T > t) = e^{-\lambda t}$$

$T = \text{time until 1st success (arrival)}$

Memoryless Property : $P(T \in dt | T > t) = P(0 < T < 0+dt)$



$$P(T \in dt | T > t) = \frac{P(T \in dt)}{P(T > t)} \approx \frac{\lambda e^{-\lambda t} dt}{e^{-\lambda t}} = \boxed{\lambda dt}$$

$\stackrel{''}{=} P(0 < T < 0+dt)$

conclusions:

① So long as a lightbulb is still functioning, it is as good as new

② $\lambda = \frac{P(T \in dt | T > t)}{dt}$

instantaneous rate of success. This is always the same

(not dependent on t)

Today sec 4.2

① $\text{Exp}(1)$ is the limit of $P \cdot \text{geom}(p)$.

② Expectation and Variance of $\text{Exp}(\lambda)$.

③ Gamma distributions,

① sec 4.2 relationship between $\text{Exp}(\lambda)$ and $\text{Geom}(p)$

We will show, for $X \sim \text{Geom}(p)$,

$$pX \rightarrow \text{Exp}(1) \text{ as } p \rightarrow 0$$

Review Taylor series for e^{-p} :

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \approx 1 + x \text{ for small } x$$

$$\text{so } e^{-p} = 1 - p + \frac{p^2}{2!} - \frac{p^3}{3!} + \dots \approx 1 - p \text{ for small } p$$

$$\Rightarrow 1 - p \approx e^{-p} \text{ for small } p.$$

Let $X \sim \text{Geom}(p)$, then

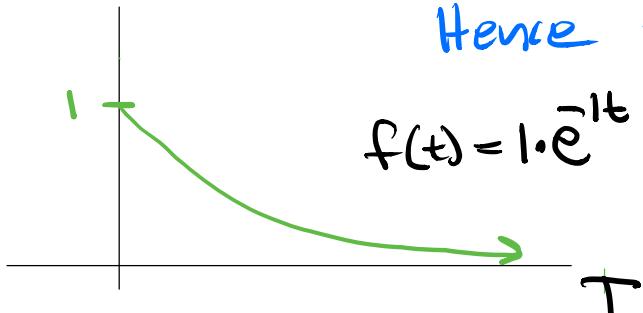
$$P(pX > pk) = P(X > k) = q^k = (1-p)^k \approx (e^{-p})^k = e^{-pk}$$

let $t = pk$, then

$$P(pX > t) \approx e^{-t}$$

Hence $pX \rightarrow \text{Exp}(1)$

$$f(t) = 1 \cdot e^{-1t} \quad \text{as } p \rightarrow 0$$



(2)

Sec 4.2 Expectation and Variance of $\text{Exp}(\lambda)$

$$T \sim \text{Exp}(\lambda)$$

$$E(T) = \int_0^\infty t f(t) dt = \lambda \int_0^\infty t e^{-\lambda t} dt =$$


see end of
lec notes.

$$E(T^2) = \int_0^\infty t^2 f(t) dt = \lambda \int_0^\infty t^2 e^{-\lambda t} dt =$$


see end of
lec notes.

$$\text{Var}(T) = E(T^2) - E(T)^2 = \frac{\lambda^2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

check for $\lambda=1$

Recall that for $X \sim \text{Geom}(p)$

$$E(X) = \frac{1}{p}$$

$$\text{Var}(X) = \frac{q}{p^2}$$

$$E(pX) = pE(X) = \frac{p}{p} = 1 \rightarrow 1 \text{ as } p \rightarrow 0$$

$$pX \rightarrow T \sim \text{Exp}(1) \text{ as } p \rightarrow 0$$

$$E(pX) \rightarrow E(T) \text{ as } p \rightarrow 0$$

$$\Rightarrow \boxed{E(T) = 1}$$

$$\text{Var}(pX) = p^2 \text{Var}(X) = \frac{p^2 q}{p^2} = q \rightarrow 1 \text{ as } p \rightarrow 0.$$

$$\text{Var}(pX) \rightarrow \text{Var}(T) \text{ as } p \rightarrow 0$$

$$\Rightarrow \boxed{\text{Var}(T) = 1}$$

Interpretation of λ :

$$\lambda = \frac{1}{E(T)} \text{ (one over average arrival time)}$$

This makes sense since the instantaneous rate of success is constant by the memoryless property of $\text{Exp}(\lambda)$.

So if it takes on avg 5 min to have a success then $\lambda = \frac{1}{5}$,

e.g. Let T = waiting time (min) until a blue Honda enters a toll gate starting at noon.

The waiting time is on average 2 minutes,

Find $P(T > 3)$

$$E(T) = 2 \text{ min}$$

$$\lambda = \frac{1}{2}$$

$$T \sim \text{Exp}\left(\frac{1}{2}\right)$$
$$P(T > 3) = e^{-\frac{1}{2} \cdot 3} = e^{-\frac{3}{2}}$$

③ Gamma Distribution

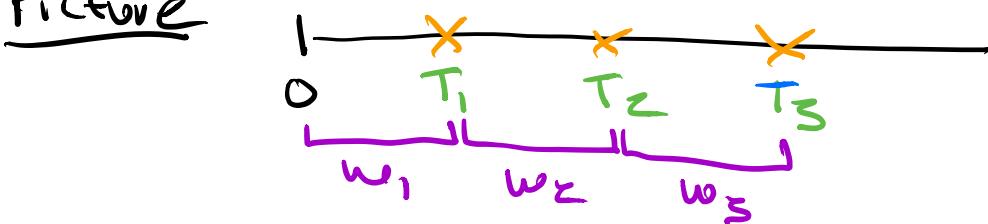
A gamma distribution, $T_r \sim \text{Gamma}(r, \lambda)$, $\lambda > 0$, is a sum of r iid $\text{Exp}(\lambda)$.

$$T_r = w_1 + w_2 + \dots + w_r, \quad w_i \sim \text{Exp}(\lambda)$$

$\underline{\text{Ex}}$ T_r = time of the r^{th} arrival of a Poisson process ($\text{Pois}(\lambda t)$)

$\underline{\text{Ex}}$ T_r = time when the r^{th} lightbulb dies,

Picture



$T_1 \sim \text{Gamma}(1, \lambda)$
 $T_2 \sim \text{Gamma}(2, \lambda)$
 $T_3 \sim \text{Gamma}(3, \lambda)$

} dependent but
 w_1, w_2, w_3
 are independent,
 $\text{Exp}(\lambda)$

$$T_1 = w_1$$

$$T_2 = w_1 + w_2$$

$$T_3 = w_1 + w_2 + w_3$$

Review Poisson Process (a.k.a PRS)

$$N_t \sim \text{Pois}(\lambda t)$$

N_t = # arrivals in time t where the rate of arrivals is λ .

$$P(N_t = n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

\cong Let T_i = waiting time (min) until a blue Honda enters a toll gate starting at noon. The waiting time is on average 2 minutes. Find the probability 3 blue Hondas arrive in 8 minutes.

$$\lambda = \frac{1}{2} \leftarrow E(T)$$

$$N_8 \sim \text{Pois}\left(4 \overbrace{\leftarrow \frac{1}{2} \cdot 8}^{\lambda t}\right)$$

$$P(N_8 = 3) = \underbrace{e^{-4}}_{3!} \underbrace{4^3}_{3!}$$

Back to gamma distribution: $T_r \sim \text{Gamma}(r, \lambda)$

Let $N_t \sim \text{Pois}(\lambda t)$
 $W \sim \text{Exp}(\lambda)$

$P(T_r \in dt)$ means $\frac{1}{t} \int_0^t \dots$

$$\begin{aligned} \text{so } P(T_r \in dt) &= P(N_t = r-1) P(W \in dt | W > t) \\ &\approx \frac{e^{-\lambda t} (\lambda t)^{r-1}}{(r-1)!} \cdot \lambda dt \\ &= \underbrace{\frac{1}{(r-1)!} \lambda^r t^{r-1} e^{-\lambda t}}_{f(t)} dt \end{aligned}$$

Gamma function $\Gamma(r)$, $r > 0$

If $r \in \mathbb{Z}^+$ define $\Gamma(r) = (r-1)!$. where
 $\Gamma(r)$ is called the gamma function

$$\text{For } \lambda = 1, \quad f(t) = \frac{1}{\Gamma(r)} t^{r-1} e^{-t}$$

const variable

$$\text{and } \int_0^\infty f(t) dt = 1 \Rightarrow$$

$$\frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-t} dt = 1$$

$$\Rightarrow \boxed{\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt}.$$

This allows us to define

$T \sim \text{Gamma}(r, \lambda)$ for $r > 0$ as

having density $f(t) = \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}$

$t > 0$

ex Let $T_r \sim \text{Gamma}(r=4, \lambda=2)$

$$\begin{aligned} \text{Find } f(t) &= \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} \\ &= \frac{1}{\Gamma(4)} 2^4 t^3 e^{-2t} \\ &= \boxed{\frac{8}{3} t^3 e^{-2t}} \end{aligned}$$

The formula $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$,
is the definition of the gamma function.

You can show that $\Gamma(r) = (r-1)!$

for $r \in \mathbb{Z}^+$

Now, $T_r \sim \text{Gamma}(r, \lambda)$ has density

$$f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}, & t > 0 \\ 0, & \text{else} \end{cases}$$

Let $T_r \sim \text{Gamma}(r, \lambda)$

$$P(T_r > t) = \int_t^\infty \frac{1}{\Gamma(r)} \lambda^r s^{r-1} e^{-\lambda s} ds$$

$N_t \sim \text{Pois}(\lambda t)$

$$\begin{aligned} P(T_r > t) &= P(N_t < r) \\ &= \sum_{i=0}^{r-1} P(N_t = i) \\ &= \boxed{\sum_{i=0}^{r-1} \frac{e^{-\lambda t} (\lambda t)^i}{i!}} \end{aligned}$$

Ex Let $T_4 \sim \text{Gamma}(r=4, \lambda=2)$

Find $P(T_4 > 7) \quad \leftarrow < 4$

$$= P(N_7 \leq 3) \text{ where } N_7 \sim \text{Pois}(14)$$

$$= e^{-14} \left(1 + \frac{14}{1!} + \frac{14^2}{2!} + \frac{14^3}{3!} \right)$$

2.7
↓

Suppose customers are arriving at a ticket booth at rate of five per minute, according to a Poisson arrival process. Find the probability that:

- (a) Starting from time 0, the 9th customer doesn't arrive within 5 minutes;
- (b) At least one customer arrives within 40 seconds after the arrival of the 13th customer.

$$\text{Q) } P(T_9 > 5) = P(N_5 < 9) = \boxed{\sum_{k=0}^8 \frac{e^{-\lambda t} \cdot \lambda^k}{k!}}$$

$\lambda = 5$, $t = 5$, $\lambda t = 25$

Appendix

Expectation and Variance of exponential

let $T \sim \text{Exp}(\lambda)$

$f(t) = \lambda e^{-\lambda t}$ density

$$E(T) = \lambda \int_0^\infty t e^{-\lambda t} dt$$

I recommend using the tabular method for integration by parts. This works well when the function you are integrating is the product of two expressions, where the n^{th} derivative of one expression is zero.

$$\text{ex } t^4 e^{3t} \quad \checkmark \quad \text{good}$$

$$\text{ex } \sin t e^{3t} \quad \times \quad \text{bad.}$$

To find $\lambda \int_0^\infty t e^{-\lambda t} dt$:

$\frac{d}{dt}$	\int
t	$e^{-\lambda t}$
1	$-\frac{e^{-\lambda t}}{\lambda}$
0	$\frac{e^{-\lambda t}}{\lambda^2}$

$$E(\tau) = \lambda \int_0^\infty t e^{-\lambda t} dt = \lambda \left(t \left(-\frac{e^{-\lambda t}}{\lambda} \right) - 1 \cdot \frac{e^{-\lambda t}}{\lambda^2} \right) \Big|_0^\infty$$

$$= 0 - (0 - \frac{1}{\lambda})$$

$$= \boxed{\frac{1}{\lambda}}$$

Next find $\text{Var}(\tau) = E(\tau^2) - E(\tau)^2$:

$$E(\tau^2) = \lambda \int_0^\infty t^2 e^{-\lambda t} dt$$

$$\begin{array}{c} \frac{d}{dt} \\ \hline t^2 & \int \\ & e^{-\lambda t} \\ 2t & - \frac{1}{\lambda} e^{-\lambda t} \\ 2 & \frac{1}{\lambda^2} e^{-\lambda t} \\ 0 & - \frac{1}{\lambda^3} e^{-\lambda t} \end{array}$$

$$E(\tau^2) = \lambda \left(t^2 \left(-\frac{1}{\lambda} e^{-\lambda t} \right) - 2t \left(\frac{1}{\lambda^2} e^{-\lambda t} \right) + 2 \left(-\frac{1}{\lambda^3} e^{-\lambda t} \right) \right) \Big|_0^\infty$$

$$= \frac{2}{\lambda^2}$$

$$\Rightarrow \text{Var}(\tau) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda} \right)^2 = \boxed{\frac{1}{\lambda^2}}$$