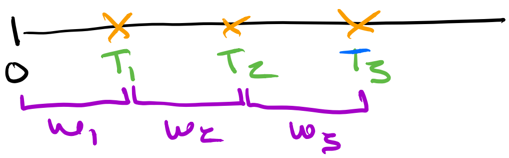
Stat 134 lec 23

Monday lecture on Moment Generating Functions (not in book)

Last time sec 4.2 Gamma Distribution



$T \sim Exp(\lambda),\ \lambda > 0$ $\qquad f(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$ ← variable part

$T_r \sim Gamma(r, \lambda),\ r, \lambda > 0$ $\qquad f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$

where $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t}\, dt$ Gamma function

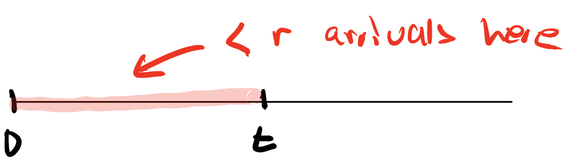
Notes about $T_r \sim Gamma(r, \lambda)$:

$\Gamma(r) = (r-1)!$ for $r \in \mathbb{Z}^+$

If $r \in \mathbb{Z}^+$, T_r = time to r^{th} arrival

$Exp(\lambda) = Gamma(r=1, \lambda)$

$P(T_r > t) = P(N_t < r)$ where $N_t \sim Pois(\lambda t)$



Today Sec 4.4 (skip 4.3)

⓪ Finish Gamma example from last time.

① Change of variable formula for densities.

② Recognizing a distribution from the variable part of its density

⓪ Finish Gamma example from last time.

λ

Suppose customers are arriving at a ticket booth at rate of five per minute, according to a Poisson arrival process. Find the probability that:

t

(a) Starting from time 0, the $9th$ customer doesn't arrive within 5 minutes;

(b) At least one customer arrives within 40 seconds after the arrival of the $13th$ customer.

soln

$\sim \text{Pois}(5\cdot 5)$

a) $P(T_9 > 5) = P(N_5 < 9) = \boxed{\sum_{k=0}^{8} e^{-25} \frac{25^k}{k!}}$

b) $P(T_{14} < T_{13} + 2/3) = P(\underbrace{T_{14} - T_{13}}_{W_{14}} < 2/3)$

$= 1 - P(W_{14} > 2/3)$

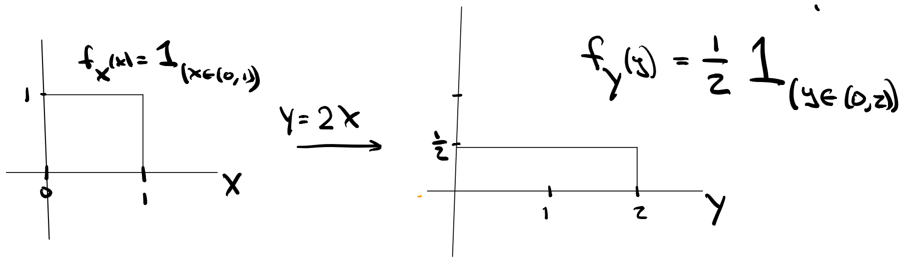
$= \boxed{1 - e^{-5(2/3)}}$

$= P(N_{2/3} = 0)$

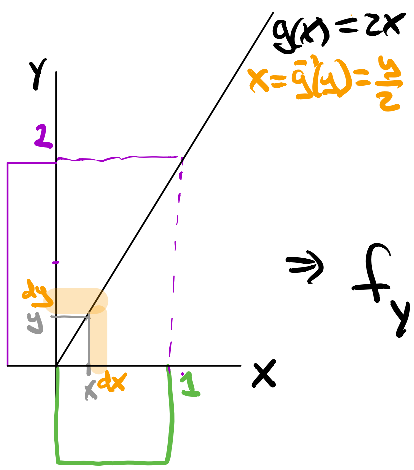
① Sec 4.4 Change of Variable formula for densities

ex The density of $X \sim U(0,1)$ is $f_X(x) = \mathbb{1}_{(x \in (0,1))}$

What is density of $Y = 2X$



Another view:



$$P(Y \in dy) = P(X \in dx)$$

$$f_Y(y)\,dy = f_X(x)\,dx$$

$$\Rightarrow f_Y(y) = f_X(x)\frac{dx}{dy}$$

$$= \frac{1}{\frac{dy}{dx}} f_X(x) = \frac{1}{2}\cdot \mathbb{1}_{(x\in(0,1))}$$

$$\frac{dy}{dx} = g'(x) = 2$$

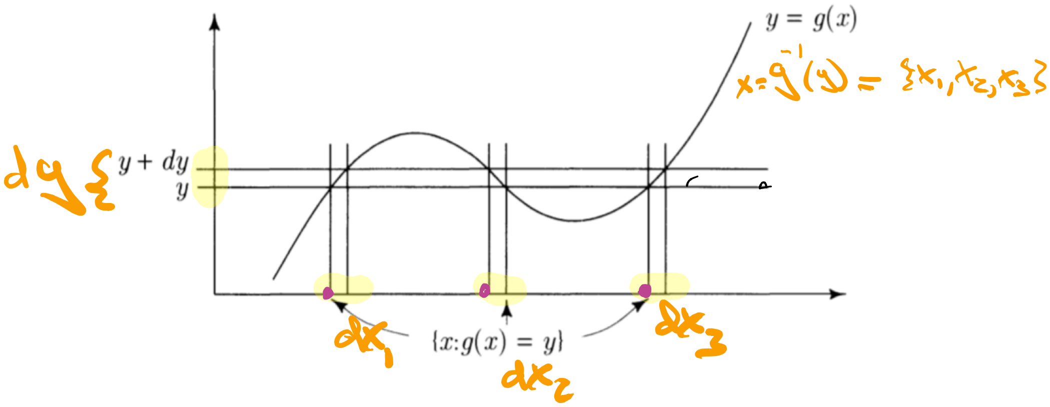
evaluated at $x = g^{-1}(y)$

$$= \frac{1}{2}\mathbb{1}_{(\frac{y}{2}\in(0,1))}$$

$$= \frac{1}{2}\mathbb{1}_{(y\in(0,2))}$$

Here the transformation of X to $Y = g(X)$ is linear and one to one.

What if $Y = g(X)$ isn't linear and one-one?



mutually exclusive events.

$Y \in dy$ iff $X \in dx_1$ or $X \in dx_2$ or $X \in dx_3$

$$P(Y \in dy) = P(X \in dx_1) + P(X \in dx_2) + P(X \in dx_3)$$

$$f_Y(y)\,dy = f_X(x_1)\,dx_1 + f_X(x_2)\,dx_2 + f_X(x_3)\,dx_3$$

$$f_Y(y) = f_X(x_1)\frac{dx_1}{dy} + f_X(x_2)\frac{dx_2}{dy} + f_X(x_3)\frac{dx_3}{dy}$$

$$= \frac{f_X(x_1)}{g'(x_1)} + \frac{f_X(x_2)}{|g'(x_2)|} + \frac{f_X(x_3)}{g'(x_3)}$$

evaluated at $x = g^{-1}(y)$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{g'(x)}$$

$P(X \in dx_2) \geq 0$

Thm (P 307)

Let X be a continuous RV with density $f_X(x)$.

Let $Y = g(X)$ have a derivative that is zero at only finitely many pts,

then $$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \frac{f_X(x)}{|g'(x)|}\Bigg|_{x = g^{-1}(y)}$$

evaluated at $x = g^{-1}(y)$

ex

let $X = N(0,1)$, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

Find the density of $Y = \sigma X + \mu$ where $\sigma > 0$.

$y = g(x) = \sigma x + \mu$

$g'(x) = \sigma$

$x = g^{-1}(y) = \frac{y-\mu}{\sigma}$

$y = \sigma x + \mu$

$y - \mu = \sigma x$

$\frac{y-\mu}{\sigma} = x$

$$f_Y(y) = \frac{f_X(x)}{|g'(x)|}\Bigg|_{x = \frac{y-\mu}{\sigma}} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}{\sigma}\Bigg|_{x = \frac{y-\mu}{\sigma}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$$

Note:

$E(Y) = E(\sigma X + \mu) = \sigma \underset{0}{E(X)} + \mu = \mu$

$Var(Y) = Var(\sigma X + \mu) = \sigma^2 \underset{1}{Var(X)} = \sigma^2$

We will see later in the semester that Y is normal.

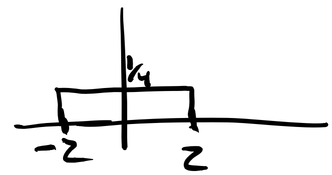
$\Rightarrow \boxed{Y \sim N(\mu, \sigma^2)}$

Change of variable formula:

$$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \frac{f_X(x)}{|g'(x)|}\Bigg|_{x=g^{-1}(y)}$$

ex $X \sim \text{Unif}(-2,2)$, $f_X(x) = \frac{1}{4}\mathbb{1}_{(x\in(-2,2))}$

Find density of $Y = X^2$



note: $g(y) = x^2$

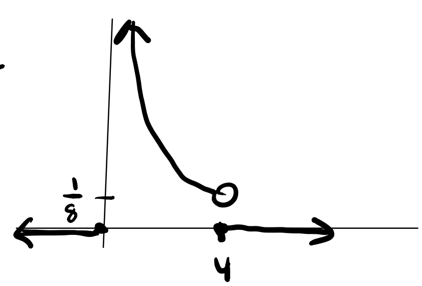
$x = g^{-1}(y) = \pm\sqrt{y}$

$$f_Y(y) = \frac{\frac{1}{4}\mathbb{1}_{(x\in(-2,2))}}{2x}\Bigg|_{x=\sqrt{y}} + \frac{\frac{1}{4}\mathbb{1}_{(x\in(-2,2))}}{|2x|}\Bigg|_{x=-\sqrt{y}}$$

$$= \frac{\frac{1}{4}\mathbb{1}_{(y\in(0,4))}}{2\sqrt{y}} + \frac{\frac{1}{4}\mathbb{1}_{(y\in(0,4))}}{|-2\sqrt{y}|}$$

$$= \frac{\frac{1}{4}\mathbb{1}_{(y\in(0,4))}}{\sqrt{y}} = \begin{cases} \frac{1}{4\sqrt{y}} & 0<y<4 \\ 0 & \text{else} \end{cases}$$

graph of $f_Y(y)$ =



ex (extra change of variable example – not covered in class)

(3 pts) Suppose the random variable X, which measures the magnitude of an earthquake (on the Richter scale) in the Bay Area, follows the Exponential (λ) distribution. Since the Richter scale is logarithmic, we want to study the distribution of the total energy of earthquakes. Find the distribution of $Y = e^X$.

Change of variable formula:

$$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \frac{f_X(x)}{|g'(x)|} \quad \text{evaluated at } x = g^{-1}(y).$$

$X \sim \text{Exp}(\lambda) \quad f_X(x) = \lambda e^{-\lambda x} \quad x>0$

$g(x) = e^x$ is one to one

$g'(x) = e^x \quad x = \log y$

$$f_Y(y) = \left.\frac{\lambda e^{-\lambda x}}{e^x}\right|_{x=\log y} = \left.\lambda e^{(-\lambda-1)x}\right|_{x=\log y}$$

$$= \lambda e^{(-\lambda-1)\log y} = \boxed{\lambda y^{(-\lambda-1)}, \ y>1}$$

(2) Recognizing a distribution from the variable part of its density.

A density can be written as

$$f(t) = c\,h(t)$$

constant (c); variable part ($h(t)$).

$$1 = \int_{-\infty}^{\infty} f(t)\,dt = c\int_{-\infty}^{\infty} h(t)\,dt \Rightarrow c = \frac{1}{\int_{-\infty}^{\infty} h(t)\,dt}$$

So you can figure out the density from its variable part.

List of densities. Please circle their variable parts:

ex $T \sim \text{Gamma}(r,\lambda)$ $\quad f(t) = \frac{1}{\Gamma(r)}\lambda^r t^{r-1} e^{-\lambda t}$, $t > 0$

$T \sim \text{Normal}(\mu,\sigma^2)$ $\quad f(t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$

$T \sim \text{Unif}(a,b)$ $\quad f(t) = \frac{1}{b-a}\mathbb{1}_{(t\in(a,b))}$

$T_r \sim \text{Gamma}(r, \lambda),\quad r, \lambda > 0 \qquad f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$

$T \sim \text{Exp}(\lambda),\ \lambda > 0 \qquad f(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$ ← variable part

ex Name the distribution with the following variable part **ex** $\text{Gamma}(r = \tfrac{1}{2}, \lambda = 3)$

a) $h(t) = t^3 e^{-\frac{1}{2}t}$ Gamma $(r = 4, \lambda = \tfrac{1}{2})$

b) $h(t) = e^{-\frac{1}{2}t^2}$ Normal $(\mu = 0, \sigma = 1)$

c) $h(t) = e^{-3t}$ Gamma $(r = 1, \lambda = 3) = \text{Exp}(\lambda = 3)$

d) $h(t) = t^{-\frac{1}{2}} e^{-t}$ Gama $(r = \tfrac{1}{2}, \lambda = 1)$

e) $h(t) = \mathbb{1}_{(t \in (0,1))}$ Unit $(0,1)$

tinyurl.com/oct25-pt1

Stat 134

Friday October 25 2019

1. Let Z be a standard normal RV (with variable part $e^{-\frac{z^2}{2}}$). The variable part of the distribution $X = Z^2$ is?

   $r=1/2, \lambda=1/2$

   a Gamma $x^{-\frac{1}{2}}e^{-\frac{x}{2}}$

   b Gamma $x^{\frac{1}{2}}e^{-\frac{x}{2}}$

   c Exponential $e^{-\frac{x}{2}}$

   d Normal $e^{-\frac{x^2}{2}}$

   e none of the above

$g(z) = z^2$

$g'(z) = 2z$

$f_X(x) \propto \frac{1}{|2z|} e^{-\frac{z^2}{2}}$ | $z = \pm\sqrt{x}$

proportional

$\propto \frac{1}{\sqrt{x}} e^{-\frac{x}{2}}$ — from the variable part of $f_X(x)$ I recognize this is Gamma $(r=1/2, \lambda=1/2)$

$x^{1-1/2}$