Stat 134 lec 31

Last time

sec 5.3 independent normal variables

a) $T \sim Exp(\frac{1}{2}) \Rightarrow R = \sqrt{T}$ has density $f(r) = re^{-\frac{1}{2}r^2}, \ r > 0$

$\leftarrow$ called Rayleigh RV

b) Using Rayleigh distribution we showed if $Z_1, Z_2 \overset{iid}{\sim} N(0,1)$

① $Z_1^2 + Z_2^2 = T \sim Exp(\frac{1}{2})$.

② $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$, $E(Z) = 0$, $SD(Z) = 1$

c) Definition

If $Z \sim N(0,1)$ then $X = \mu + \sigma Z \sim N(\mu, \sigma^2)$

with density $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

d)

Thm Let $\left.\begin{array}{l} X_1 \sim N(\mu_1, \sigma_1^2) \\ X_2 \sim N(\mu_2, \sigma_2^2) \end{array}\right\}$ inden.

then $aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

Today

sec 5.3

① A linear combination of independent normals is normal

② Chi square distribution

sec 5.4

③ Convolution formula for the density of $X+Y$

tinyurl http://tinyurl.com/april12-pt1

http://tinyurl.com/april12-pt2

Stat 134

Friday November 2 2018

1. Let $X \sim N(0,1)$ and $Y \sim N(0,1)$ be independent. $P(X > 2Y)$ equals

   a $1-\Phi(0)$

   b $1-\Phi(\frac{1}{\sqrt{3}})$

   c $1-\Phi(\sqrt{3})$

   d none of the above

soln

$Y \sim N(0,1) \Rightarrow 2Y \sim N(0,2)$

and $X-2Y \sim N(0,5)$

$P(X>2Y) = P(X-2Y>0)$

normalize $= P\left(\frac{(X-2Y)-0}{\sqrt{5}} > \frac{0-0}{\sqrt{5}}\right)$

$= P(Z>0) = 1-\Phi(0) = \frac{1}{2}$

2. Let $X \sim N(68, 3^2)$ and $Y \sim N(66, 2^2)$ be independent. $P(X > Y)$ equals

a $1 - \Phi(\frac{0-2}{\sqrt{3^2+2^2}})$

b $1 - \Phi(\frac{0-2}{3^2+2^2})$

c $1 - \Phi(\frac{68-66}{\sqrt{3^2+2^2}})$

d none of the above

soln

$X-Y \sim N(68-66, 3^2+2^2) = N(2, 13).$

$$P(X>Y) = P(X-Y>0)$$

$$= P\Big(\underbrace{\frac{(X-Y)-2}{\sqrt{13}}}_{=Z} > \frac{0-2}{\sqrt{13}}\Big)$$

$$= 1 - P\Big(\underbrace{\frac{(X-Y)-2}{\sqrt{13}}}_{=Z} < \frac{-2}{\sqrt{13}}\Big)$$

$$= 1 - \Phi\Big(\frac{-2}{\sqrt{13}}\Big)$$

Sec 5.3 Chi-square distribution

Fact — see end of lecture notes for proof.

if $Z \sim N(0,1)$ then

$$Z^2 \sim Gamma\left(\tfrac{1}{2},\tfrac{1}{2}\right)$$

Recall that if $T \sim Gamma(r,\lambda)$, $M_T(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$, $t < \lambda$

Hence $M_{Z^2}(t) = \left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right)^{1/2}$, $t < 1/2$

Let $Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$

$$M_{Z_1^2+\dots+Z_n^2}(t) = \left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right)^{1/2 \cdot n}, \quad t < \frac{1}{2}$$

By uniqueness of MGF

$$Z_1^2 + \dots + Z_n^2 \sim Gamma\left(\tfrac{n}{2},\tfrac{1}{2}\right)$$

called chi square distribution, χ^2_n, with n degrees of freedom

ex Let $X, Y \overset{iid}{\sim} N(0,1)$

Let $R = \sqrt{X^2+Y^2}$ be Raleigh distribution.

Then $R^2 = X^2 + Y^2 \sim \chi^2_2$ (we showed last time $R^2 \sim Exp(\frac{1}{2})$ $= Gamma(1,\frac{1}{2})$)

Sec 5.4 The Density Convolution Formula

eg Let $X \sim \text{Unif}\{0,1,2,3,4,5,6\}$
$Y \sim \text{Unif}\{0,1,2,3,4,5,6\}$

Let $S = X + Y$.

Find the probability mass function of S.

$$P(S=3) = P(X=0, Y=3-0) + P(X=1, Y=3-1) + P(X=2, Y=3-2) + P(X=3, Y=3-3)$$

$$P(S=s) = \sum_{x=0}^{s} P(X=x, Y=s-x)$$

convolution formula.

Let $X > 0$, $Y > 0$ be continuous RVs.

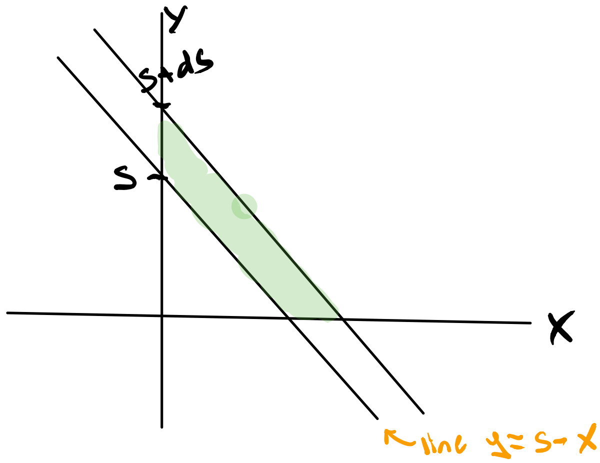
Let $S = X + Y$

Find the density of S

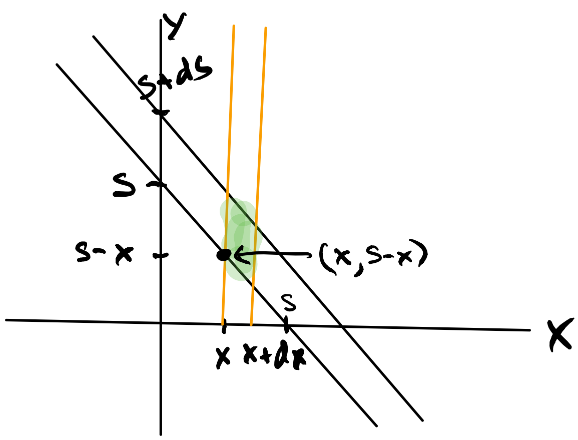
$s = x + y$
$y = s - x$

y intercept.

$P(S \in ds)$ is the volume under $f(x,y)$ over the green region.



$P(S \in ds,\ X \in dx)$ is the volume under $f(x,y)$ over the green region.



$$\underbrace{P(S \in ds)}_{f(s)\,ds} = \int_{x=0}^{x=s} \overbrace{P(S \in ds,\ X \in dx)}^{f(x,s-x)\,dx\,ds}$$

$$= \underbrace{\int_{x=0}^{x=s} f(x, s-x)\,dx}_{f(s)}\, ds$$

$x=s$

$$\Rightarrow f(s) = \int_{x=0}^{\infty} f(x, s-x)\,dx$$

convolution formula for densities.

Ex $X, Y \overset{iid}{\sim} \text{expon}(\lambda)$ $S = X + Y$

$$f_S(s) = \int_0^s f(x, s-x)\,dx$$

independence

$$= \int_0^s f_X(x)\, f_Y(s-x)\,dx$$

$$= \int_0^s \lambda e^{-\lambda x} \lambda e^{-\lambda(s-x)}\,dx$$

$$= \int_0^s \lambda^2 e^{-\lambda s}\,dx = \lambda^2 e^{-\lambda s} x \Big|_0^s$$

$$= \boxed{\lambda^2 e^{-\lambda s} \cdot s}$$

variable part of gamma$(2, \lambda)$

$\Rightarrow S \sim \text{gamma}(2, \lambda)$.

ex Let $X \sim U_{(7)}$, $Y \sim U_{(9)}$ for 10 iid $U(0,1)$.
The joint density $f_{X,Y}(x,y) = \binom{10}{6,1,1,1,1} x^6 (y-x)(1-y)$

for $0<x<y<1$.

Let $Z = Y - X$

a) For a fixed z, what is the smallest and largest value of x?

$z = y - x$
$x = y - z$ where $y \in [0,1]$

x must be >0 and <1

smallest: 0 (let $y = z$)
largest: $1-z$ (let $y = 1$)

so convolution formula for $Z = Y - X$ is

$$f_Z(z) = \int_{x=0}^{x=1-z} f_{X,Y}(x, x+z)\,dx$$

ex Let $X \sim U_{(7)}$, $Y \sim U_{(9)}$ for 10 iid $U(0,1)$.

The joint density $f_{X,Y}(x,y) = \binom{10}{6,1,1,1,1} x^6 (y-x)(1-y)$

for $0<x<y<1$.

Prove via the convolution formula that the density of $Z = Y - X$ is a Beta distribution and find the parameters.

Let $c = \binom{10}{6,1,1,1,1}$

By convolution formula,

$$f_Z(z) = \int_0^{1-z} f_{X,Y}(x, x+z)\,dx$$

$$= \int_0^{1-z} c x^6 (x+z-x)(1-(x+z))\,dx$$

$$= cz \int_0^{1-z} \left((1-z)x^6 - x^7\right) dx$$

$$= cz\, (1-z)\frac{x^7}{7} - \frac{x^8}{8} \Bigg|_{x=0}^{x=1-z}$$

$$= cz \left(\frac{(1-z)^8}{7} - \frac{(1-z)^8}{8} \right)$$

Ignoring constants, $f_Z(z) \propto z(1-z)^8$, $0<z<1$ $\Rightarrow \boxed{Z \sim Beta(2,9)}$

Appendix

If $Z \sim N(0,1)$, then $Z^2 \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

Proof/

$\lambda > 0$, pos integer r,

gamma (r, λ) density

$$f(t) = \frac{\lambda^r}{\Gamma(r)} t^{r-1} e^{-\lambda t}, \quad t > 0$$

let Z = std normal

$X = Z^2$ change of variable rule.

$$\text{Find } f_X(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-\frac{1}{2}x}, \quad x > 0$$

$$= \frac{\left(\frac{1}{2}\right)^{1/2}}{\cancel{\sqrt{\pi}}\ \Gamma(\frac{1}{2})} x^{\frac{1}{2}-1} e^{-\frac{1}{2}x}$$

$\Rightarrow X \sim \text{gamma}(\frac{1}{2}, \frac{1}{2})$

$\square$