

Special lecture

Moment Generating Function of X

Not in book

Next time Finish Sec 4.5, Start Sec 4.6.

Moment Generating Function (MGF) of X

The k^{th} moment of a RV X is the number

$$\mu_k = E(X^k) \quad \text{defined for } k=0, 1, 2, 3, \dots$$

$$\mu_0 = E(X^0) = E(1) = 1$$

$$\mu_1 = E(X)$$

$$\mu_2 = E(X^2)$$

:

$$\text{Note } \text{Var}(x) = E(X^2) - E(X)^2 = \mu_2 - \mu_1^2$$

} moments describe your distribution
 ex 1st moment is mean
 2nd moment relates to variance
 3rd moment relates to how skewed the distribution is,

Computing the moments of a RV is sometimes hard, because it involves computing complicated

integrals or sums

$$E(X) = \begin{cases} \int_{-\infty}^{\infty} x f(x) dx & \text{if } X \text{ continuous} \\ \sum_{-\infty}^{\infty} x P(x) & \text{if } X \text{ discrete} \end{cases}$$

The MGF allows one to compute the moments by computing derivatives, which is easier.

Ex $X \sim \text{Ber}(p)$

$$\text{we know } E(X) = 1 \cdot p + 0 \cdot q = p$$

$$E(X^2) = 1^2 \cdot p + 0^2 \cdot q = p$$

The moment generating function for X is
 $M_X(t) = 1 + p(e^t - 1)$ for all t . — see next page.

$$\frac{d}{dt} M_X(t) \Big|_{t=0} = p e^t \Big|_{t=0} = p \quad \leftarrow E(X)$$

$$\frac{d^2}{dt^2} M_X(t) \Big|_{t=0} = \frac{d}{dt} p e^t \Big|_{t=0} = p \quad \leftarrow E(X^2)$$

You can easily see that $E(X^k) = p$.

To find moments we take derivatives of the MGF and evaluate at $t=0$.

What is the MGF?

$$\text{Recall } E(g(x)) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

We define the MGF of X to be

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx \text{ if } X \text{ continuous}$$

$M_X(t)$ is sometimes written $\psi(t)$ in HW9

$$\begin{cases} \sum_{x=-\infty}^{\infty} e^{tx} P(x) & \text{if } X \text{ discrete} \end{cases}$$

$$\begin{aligned}
 & \text{Let } X \sim \text{Ber}(p) \quad X = \begin{cases} 1 & \text{if } X=1 \\ 0 & \text{if } X=0 \end{cases} \quad \text{so } tX = \begin{cases} t & \text{if } X=1 \\ 0 & \text{if } X=0 \end{cases} \\
 & \text{Find } M_X(t) \quad M_X(t) = E(e^{tX}) \\
 & = e^{t \cdot 0} p(0) + e^{t \cdot 1} p(1) \\
 & = 1 \cdot (1-p) + e^t \cdot p \\
 & = \boxed{1 + p(e^t - 1)} \quad \text{for all } t
 \end{aligned}$$

$$\text{Why is } \left. M_X^{(k)}(t) \right|_{t=0} = E(X^k) ?$$

Recall the Taylor series for e^y :

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots$$

let $t \in \mathbb{R}$, X RV.

You can do this for the RV tX

$$e^{tX} = 1 + tX + \frac{(tX)^2}{2!} +$$

$$\begin{aligned}
 E(e^{tX}) &= E(1) + E(tX) + E\left(\frac{(tX)^2}{2!}\right) + \dots \\
 M_X(t) &= E(1) + tE(X) + \frac{t^2}{2!} E(X^2) + \dots
 \end{aligned}$$

$$\left. \frac{d}{dt} M_X(t) \right|_{t=0} = E(X)$$

$$\left. \frac{d^2}{dt^2} M_X(t) \right|_{t=0} = E(X^2)$$

more

generally,

$$\left. \frac{d^k}{dt^k} M_X(t) \right|_{t=0} = \left. \frac{d^k}{dt^k} E(e^{tx}) \right|_{t=0}$$

Let's not worry about why this step is true. Ask me for details if interested.

$$\begin{aligned} & \stackrel{\text{---!!!}}{=} E\left(\left. \frac{d^k}{dt^k} e^{tx} \right|_{t=0}\right) \\ & = E(X^k) \end{aligned}$$

Summary, the MGF $M_X(t) = E(e^{tx})$ contains info about all of the moments of X . By taking the k^{th} derivative and evaluating at 0 you get the k^{th} moment.

Note An MGF doesn't always exist in an interval around zero. (see end of lecture notes for an example).

What makes MGF so helpful is :

Then If a MGF exists in an interval around zero, $\left. M^{(k)}(t) \right|_{t=0} = E(X^k)$

Ex let $X \sim \text{Gamma}(r, \lambda)$

$$f(x) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}, & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Recall $\Gamma(r) = \int_0^\infty u^{r-1} e^{-u} du$

Find $M_X(t)$.

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{xt} \left(\frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \right) dx$$

$$= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^{r-1} e^{(t-\lambda)x} dx$$

which converges for $(t-\lambda) < 0$
i.e. $t < \lambda$

Substitution
 $U = \underbrace{-(t-\lambda)}_{>0} x$

$$\text{so } X = -\frac{U}{t-\lambda}, \quad dx = -\frac{1}{t-\lambda} dU$$

$$= \frac{\lambda^r}{\Gamma(r)} \int_{U=0}^{U=\infty} \frac{U^{r-1}}{(\lambda-t)^{r-1}} e^{-U} \frac{1}{\lambda-t} dU$$

$$= \frac{\lambda^r}{\Gamma(r)} \frac{1}{(\lambda-t)^r} \int_0^\infty u^{r-1} e^{-u} du$$

$$= \frac{\lambda^r}{\Gamma(r)} \frac{r!}{(\lambda-t)^r} = \left(\frac{\lambda}{\lambda-t} \right)^r \quad \text{for } t < \lambda$$

Recall If a MGF exists in an interval

around zero, $M^{(x)}(t) \Big|_{t=0} = E(X^x)$

ex

Let $X \sim \text{Gamma}(r, \lambda)$

Find $\frac{d}{dt} M_X(t) \Big|_{t=0}$

and

$$\frac{d^2}{dt^2} M_X(t) \Big|_{t=0}$$

$$M(t) = \left(\frac{\lambda}{\lambda-t}\right)^r = \lambda^r (\lambda-t)^{-r}$$

$$M'(t) = \lambda^r (-r)(\lambda-t)^{-r-1}(-1) = \frac{r\lambda}{(\lambda-t)^{r+1}}$$

$$M'(0) = \frac{r}{\lambda} \Rightarrow \boxed{E(X) = \frac{r}{\lambda}}$$

$$\begin{aligned} M''(t) &= r\lambda(-r-1)(\lambda-t)^{-r-2}(-1) \\ &= r(r+1)\lambda \frac{1}{(\lambda-t)^{r+2}} \end{aligned}$$

$$\Rightarrow M''(0) = r(r+1)\lambda \frac{1}{\lambda^{r+2}} = \frac{r(r+1)}{\lambda^2}$$

$$\Rightarrow \boxed{E(X^2) = \frac{r(r+1)}{\lambda^2}}$$

$$\begin{aligned} \Rightarrow \text{Var}(X) &= E(X^2) - E(X)^2 = \frac{r(r+1)}{\lambda^2} - \frac{r^2}{\lambda^2} \\ &= \frac{r^2}{\lambda^2} + \frac{r}{\lambda^2} - \frac{r^2}{\lambda^2} = \boxed{\frac{r}{\lambda^2}} \end{aligned}$$

$\stackrel{def}{=} A$ RV X takes values 1, 2, 3
with prob y_1, y_2, y_3 .

Find $M_{X(t)}$ = $\sum_{x=1}^3 e^{tx} p(x)$

$$= \left\{ e^{t \cdot 1} + e^{t \cdot 2} + e^{t \cdot 3} \right\}_{\text{for all } t}$$

Ex $X \sim \text{Geom}\left(\frac{1}{3}\right)$

$$P(X=k) = \left(\frac{2}{3}\right)^{k-1} \left(\frac{1}{3}\right) \quad k=1, 2, 3, \dots$$

Find $M_X(t) \equiv E(e^{tX})$

$$\begin{aligned} M_X(t) &= E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} \left(\frac{2}{3}\right)^{x-1} \left(\frac{1}{3}\right) \\ &= \sum_{x=1}^{\infty} e^t (e^t)^{x-1} \left(\frac{2}{3}\right)^{x-1} \left(\frac{1}{3}\right) \\ &= \frac{1}{3} e^t \sum_{x=1}^{\infty} \left(e^t \frac{2}{3}\right)^{x-1} \end{aligned}$$

$$\sum_{x=1}^{\infty} \left(e^t \frac{2}{3}\right)^{x-1} = 1 + \left(e^t \frac{2}{3}\right) + \left(e^t \frac{2}{3}\right)^2 + \dots$$

$$= \frac{1}{1 - \left(e^t \frac{2}{3}\right)} \quad \text{if } e^t \frac{2}{3} < 1$$

$$\text{or} \\ t < \log\left(\frac{3}{2}\right)$$

so
$$M_X(t) = \frac{1}{3} e^t \cdot \frac{1}{1 - \left(e^t \frac{2}{3}\right)} \quad \text{if } t < \log\left(\frac{3}{2}\right)$$

Appendix:

Let X be a discrete RV with probability mass function $P(X) = \begin{cases} \frac{6}{\pi^2 x^2} & x=1, 2, \dots \\ 0 & \text{else} \end{cases}$

The MGF, $M_X(t)$, only exists at $t=0$, and hence doesn't exist on an interval around zero.

pf/

It is known that the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \text{ converges to } \frac{\pi^2}{6}.$$

Then $P(X) = \begin{cases} \frac{6}{\pi^2 x^2} & x=1, 2, 3, \dots \\ 0 & \text{else.} \end{cases}$

is the prob function of a RV X ,

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} P(x) \\ = \sum_{x=1}^{\infty} \frac{6e^{tx}}{\pi^2 x^2}$$

The ratio test can be used to show that it diverges if $t > 0$. Hence this RV only has an MGF at $t=0$.

□