Warmup 1:00 – 1:10

A fair coin is tossed twice.
Let $X = \#$ heads on the first toss.
Let $Y = \#$ heads on the first 2 tosses.

$Y \backslash X$	0	1	$P(y)$
2	0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
0	$\frac{1}{4}$	0	$\frac{1}{4}$
$P(x)$	$\frac{1}{2}$	$\frac{1}{2}$	

a The table above is correct

b $Y \sim Bin(2, \frac{1}{2})$

c More than one of the above

d None of the above

Is X and Y indep? ← NO b/c have zero in table w/ nonzero marginal.

Last time

Sec 3.1 Random Variables

The event $(X=x, Y=y)$ is the intersection of events $X=x$ and $Y=y$. sometimes written (x,y)

The probability X and Y satisfies some condition (i.e $P(X+Y=s)$) is the sum of $P(x,y)$ that satisfy that condition.

ex $$P(X+Y=s) = \sum_{(x,y):\, x+y=s} P(x,y) = \sum_{\text{all } x} P(x, s-x)$$

Independence of (X,Y,Z) means

$P(X=x, Y=y, Z=z) = P(X=x)P(Y=y)P(Z=z)$ for all $x \in X$, $y \in Y$, $z \in Z$.

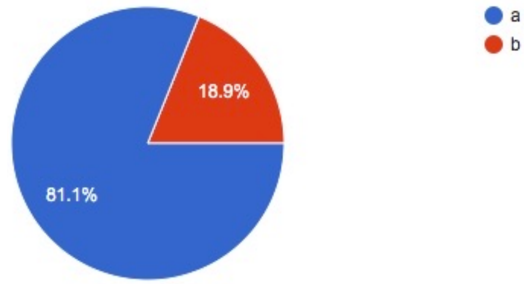
Today

1. Student responses from Concept test last time.
2. Sec 3.1 Sums of independent Poissons is Poisson
3. Sec 3.2 Expectation of a RV.

① student responses from Concept test last time.

The joint distribution of X and Y is drawn below:

	$3/8$	$1/2$	$1/8$	P(x) \ P(y)
1	$1/4$	$1/3$	$1/12$	$2/3$
0	$1/8$	$1/6$	$1/24$	$1/3$
Y \ X	0	1	2	



a X and Y are independent

b X and Y are dependent since the two rows $Y = 0$ and $Y = 1$ give different probabilities for each value of X.

a	Requirement for independence is that p(xy) = p(x)p(y) for all x and y, which is satisfied here

a	Calculate the probability of y is 0 given x is 1 and if it is equal to the probability that y is 0 then the variable are independent. They are both 1/3 so they are independent from each other

need to check for all cells.

② Sum of independent Poisson is Poisson

informal arguement:

$$\left.\begin{array}{l} X_1 \sim Bin\left(1000, \frac{1}{1000}\right) \approx Pois(1) \\ X_2 \sim Bin\left(2000, \frac{1}{1000}\right) \approx Pois(2) \end{array}\right\} \text{indep}$$

$X_1 + X_2 \sim ?$ $Bin\left(3000, \frac{1}{1000}\right) \approx Pois(3)$

$X_1 + X_2$ = # heads in $1000 + 2000 = 3000$ $p = \frac{1}{1000}$ coin tosses.

So sum of two indep binomials with the same p is Binomial and this example suggest that sum of 2 indep Poisson is Poisson.

Lets prove this rigorously:

Recall binomial theorem

$$(a+b)^3 = \binom{3}{3} a^3 b^0 + \binom{3}{2} a^2 b^1 + \binom{3}{1} a^1 b^2 + \binom{3}{0} a^0 b^3$$
$$= a^3 + 3a^2 b + 3ab^2 + b^3$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Recall $X \sim Pois(\mu)$

$$P(X=k) = \frac{e^{-\mu} \mu^k}{k!}$$

<u>Claim</u> If $X \sim \text{Pois}(\mu)$ and $Y \sim \text{Pois}(\lambda)$ are independent then

$S = X + Y \sim \text{Pois}(\mu + \lambda)$.

Pf/

$$P(S=s) = P(X=0, Y=s) + P(X=1, Y=s-1) + \dots P(X=s, Y=0)$$

addition rule

$$= \sum_{k=0}^{s} P(X=k, Y=s-k)$$

independence of X, Y

$$= \sum_{k=0}^{s} P(X=k) P(Y=s-k)$$

$$= \sum_{k=0}^{s} \frac{e^{-\mu} \mu^k}{k!} \cdot \frac{e^{-\lambda} \lambda^{s-k}}{(s-k)!}$$

$\frac{s!}{s!} = 1$

$$= e^{-(\lambda+\mu)} \frac{1}{s!} \sum_{k=0}^{s} \frac{s!}{k!(s-k)!} \mu^k \lambda^{s-k}$$

$= \binom{s}{k}$

binomial theorem

$$= e^{-(\lambda+\mu)} \frac{1}{s!} (\mu+\lambda)^s$$

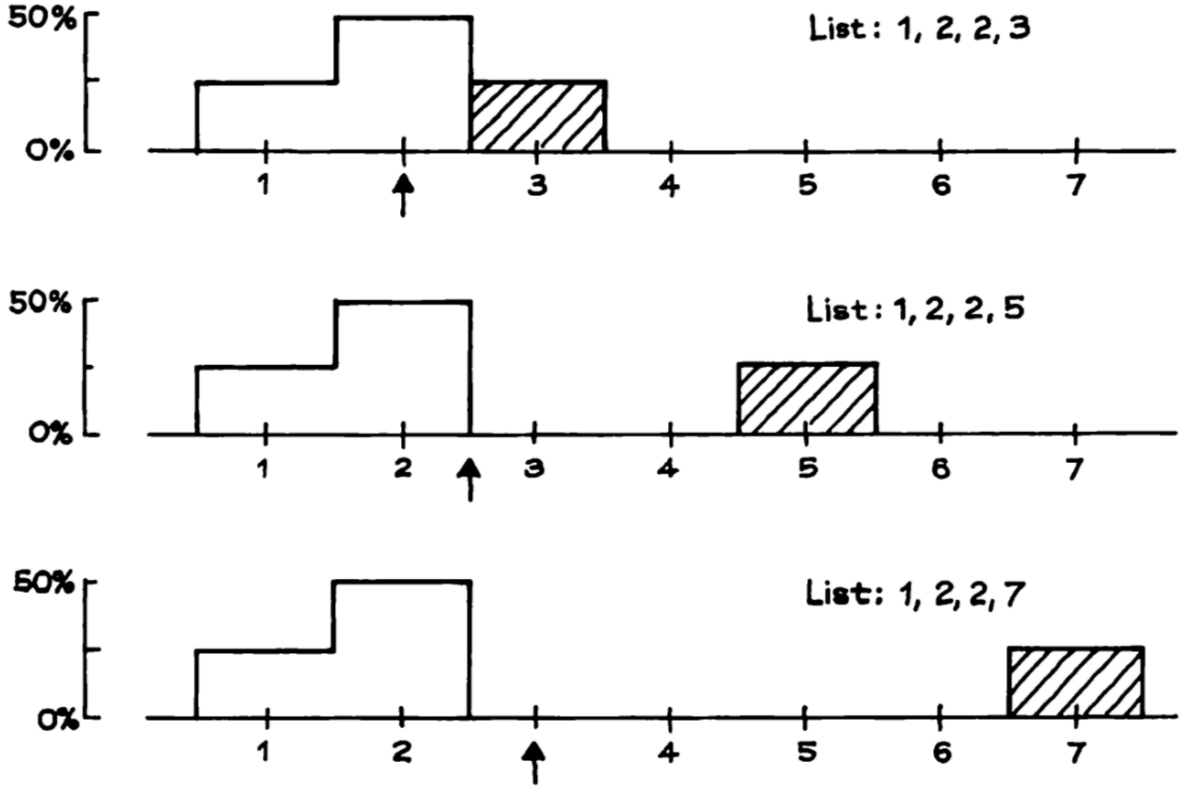
$$\Rightarrow S \sim \text{Pois}(\mu+\lambda).$$

$\square$

③ sec 3.2 Expectation

$$E(X) = \sum_{x \in X} x \cdot P(X=x)$$

$$E(X) = 1 \cdot \tfrac{1}{4} + 2 \cdot \tfrac{1}{2} + 3 \cdot \tfrac{1}{4} = 2$$



Properties of Expectation — P167 Pitman

① $E(c) = c$

② $E(X+Y) = E(X) + E(Y)$ (X, Y <u>don't</u> need to be independent)

③ $E(aX+b) = aE(X) + b$

Indicators

An indicator is a RV that has only 2 values 1 (w/ prob p) and 0 (with prob 1-p).

$$I = \begin{cases} 1 & \text{prob } p \\ 0 & \text{prob } 1-p \end{cases}$$ — same as a Bernoulli p trial.

$$E(I) = 1 \cdot p + 0 \cdot (1-p) = p$$

RV that are counts can often be written as a sum of indicators.

ex $X \sim Bin(n,p)$

successes in n Bernoulli p trials,

ex $X = $ # heads in n flips of p coin

$$X = I_1 + I_2 + \dots + I_n$$

where $I_j = \begin{cases} 1 & \text{if } j^{th} \text{ trial success} \\ 0 & \text{else} \end{cases}$ — p

$$E(X) = \underbrace{E(I_1)}_{p} + \dots + \underbrace{E(I_n)}_{p} = \boxed{np}$$

indicators are independent since trials are indep.

Ex X = # aces in a poker hand from a deck of cards

$X \sim \text{Hyper}(n, N, G)$

$N = 52$

$G = 4$

$n = 5$

a) What are the range of values of X?

$0, 1, 2, 3, 4$

b) Write X as a sum of indicators

$X = I_1 + I_2 + I_3 + I_4 + I_5$

c) How is I_2 defined?

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ card is an ace} \\ 0 & \text{else} \end{cases}$

<u>note</u> X is a sum of dep. Bernoulli trials w/ prob G/N of success.

d) Find $E(I_2)$

$E(I_2) = P(2^{nd} \text{ card is an ace}) = \boxed{\frac{4}{52}}$

e) Find $E(X)$

$E(X) = 5 \cdot E(I_1) = \boxed{5 \cdot \left(\frac{4}{52}\right)}$

Note

You may define $I_2 = \begin{cases} 1 & \text{if get 2 aces} \\ 0 & \text{else} \end{cases}$

so $X = I_1 + 2 \cdot I_2 + 3 \cdot I_3 + 4 \cdot I_4$

This is also correct but more complicated that my solution.

We have

$$E(I_1) = \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} \qquad E(I_3) = \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}$$

$$E(I_2) = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} \qquad E(I_4) = \frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$$

so $$E(X) = \frac{1}{\binom{52}{5}}\left[\binom{4}{1}\binom{48}{4} + 2 \cdot \binom{4}{2}\binom{48}{3} + 3\binom{4}{3}\binom{48}{2} + 4\binom{4}{4}\binom{48}{4}\right]$$

wow!

$$= 5 \cdot \left(\frac{4}{52}\right)$$ ← I checked this using R

ex Suppose a fair die is rolled 10 times. Let X = number of different faces that appear in 10 rolls.

ex if roll 2, 3, 4, 2, 3, 5, 2, 3, 3, 2 then $X=4$

a) what are the range of values of X?

1, 2, 3, 4, 5, 6

b) write X as a sum of indicators

$X = I_1 + I_2 + I_3 + I_4 + I_5 + I_6$

c) How is I_2 defined?

$I_2 = \begin{cases} 1 & \text{if 2 appears at least once,} \\ 0 & \text{else} \end{cases}$

d) Find $E(I_2)$

$= 1 - P(2 \text{ never appears}) = 1 - \left(\frac{5}{6}\right)^{10}$

e) Find $E(X)$

$\boxed{6 \cdot \left(1 - \left(\frac{5}{6}\right)^{10}\right)}$

A die is rolled 12 times. Find the expectation of the number of faces that don't appear

a) what are the range of values of X?

$X = 1, 2, 3, 4, 5$

b) write X as a sum of indicators

$X = I_1 + I_2 + I_3 + I_4 + I_5 + I_6$

c) How is I_2 defined?

$P = \left(\frac{5}{6}\right)^{12}$

$I_2 = \begin{cases} 1 & \text{if 2 doesn't appear} \\ 0 & \text{else} \end{cases}$

d) Find $E(I_2)$

$\left(\frac{5}{6}\right)^{12}$

$=$

P(2 never appears)

e) Find $E(X)$ $= \boxed{6 \left(\frac{5}{6}\right)^{12}}$