Stat 134 lec 20 (no lec 19)

last time — tough midterm!
Great job!

today

sec 4.1 Continuous distributions
① probability density
② expectation and variance.

(1) <u>sec 4.1</u> Probability density.

let X be a continuous RV

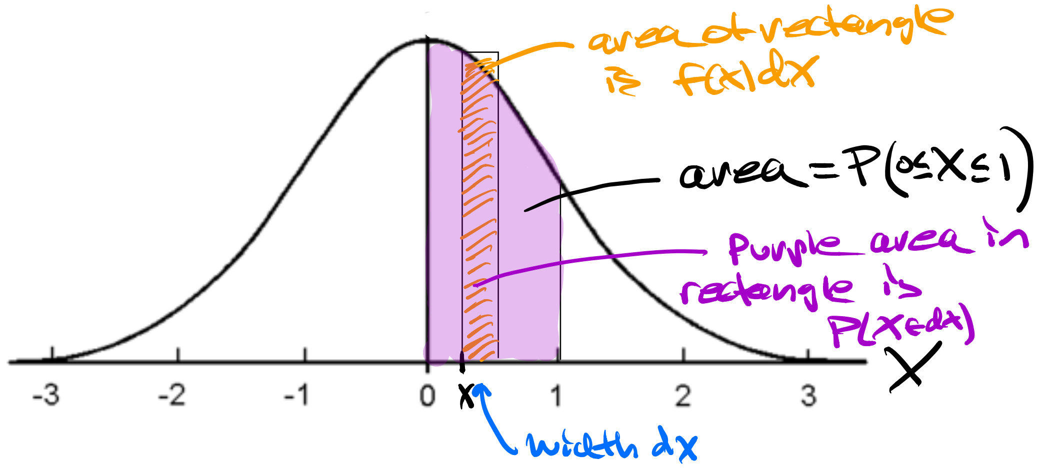
The probability density of X is described by a <u>prob density function</u>

$$f(x) \geq 0 \text{ for } x \in X$$

and $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

ex The <u>standard normal distribution</u>

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$ is a prob density function



Throwing a dart randomly at the histogram the x coordinate of your dart is most likely to be near zero.

The probability of getting an x coordinate of x is written $P(X \in dx)$.

we see from the rectangle in the picture,

$P(X \in dx) \approx f(x)dx$ (notice purple and orange area not same)

here dx = tiny interval around x and also the length of the interval

For all a, b $P(a \le X \le b) = \int_a^b P(X \in dx) = \int_a^b f(x)dx$

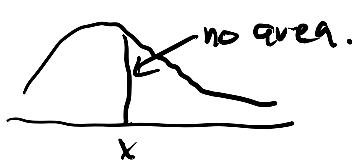
Note $f(x)$ is not a probability.

$f(x)dx$ is a probability

$f(x) \approx \frac{P(X \in dx)}{dx}$

units of f? — Prob per unit length hence "prob density"

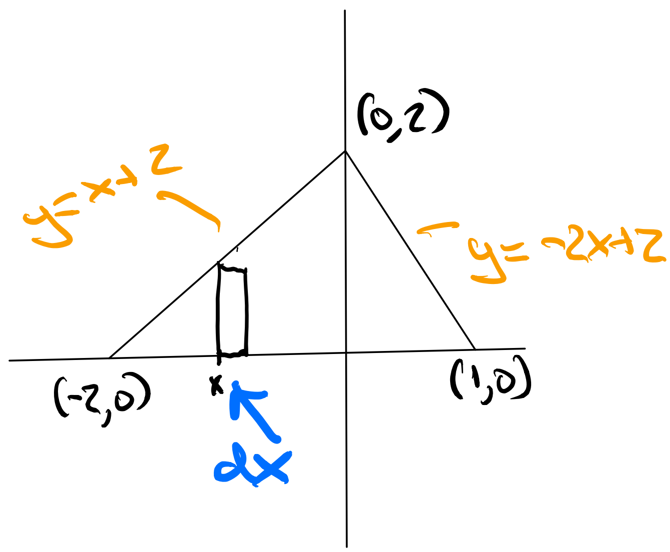
$P(X=x) = 0$



Hence $P(a \le X \le b) = P(a < X < b)$
(we don't have to worry about endpts).

ex 4.1.12 b

You randomly throw darts at the triangle:



a) Why is this not a density?

Area = $\frac{1}{2}$ height · width

= $\frac{1}{2}$ 2 · 3 = 3 ← not 1.

b) How would you change the shape of the triangle so that it is a density?

change vertex (0,2) to $(0, \frac{2}{3})$

c) Find $P(x \in dx)$ approximately (for x shown in picture)

$$\boxed{P(x \in dx) \approx \frac{x+2}{3} dx}$$

d) $$f(x) = \begin{cases} \frac{x+2}{3} & \text{for } -2 < x < 0 \\ \frac{-2x+2}{3} & \text{for } 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

(2)

Expectation and Variance

For discrete,

$$E(g(X)) = \sum_{x \in X} g(x) P(X=x)$$

$\left(\text{assuming } E(|g(X)|) < \infty \right)$

For continuous,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) P(X \in dx) = \int_{-\infty}^{\infty} g(x) f(x) \, dx$$

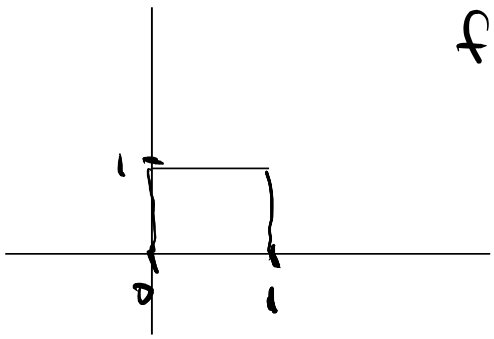
$\left(\text{assuming } E(|g(X)|) < \infty \right)$

$$E(X) = \int_{-\infty}^{\infty} x f(x) \, dx$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx$$

$$Var(X) = E(X^2) - E(X)^2$$

ex U : unif$(0,1)$ — standard uniform



$$f(u) = \begin{cases} 1 & \text{if } 0<u<1 \\ 0 & \text{else} \end{cases}$$

$$E(U) = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$E(U^2) = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$Var(U) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

Change of scale

To calculate $E(X)$, $Var(X)$, $P(X \in dx)$ we sometimes make a change of scale
$Y = c + dX$ where c, d are constants

Y hopefully has a simpler density.

we can recover $E(X)$, $Var(X)$, $P(X \in dx)$

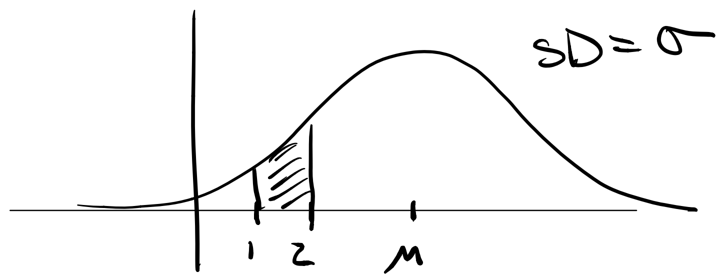
from $E(Y)$, $Var(Y)$, $P(Y \in dY)$.

ex

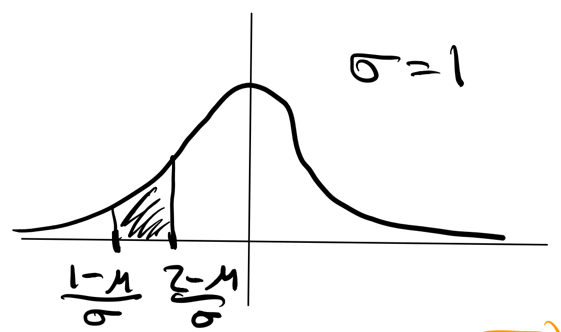
$$Var(Y) = Var(c + dX) = d^2\, Var(X)$$

$$\Rightarrow Var(X) = \frac{Var(Y)}{d^2}$$

ex Let $X \sim N(\mu, \sigma^2)$
Find $P(1 < X < 2)$



Lets make a change of scale so X is transformed to the std normal



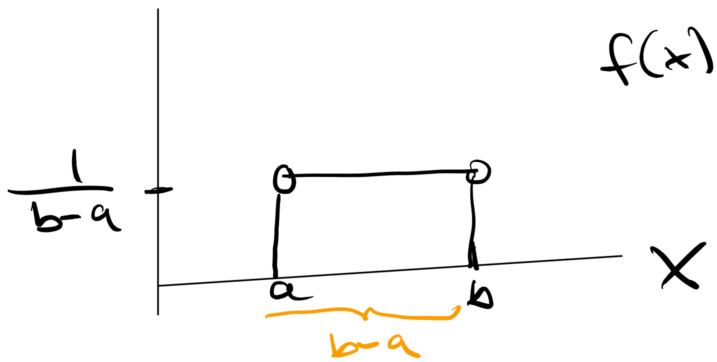
note
$$\operatorname{Var}(Z) = \operatorname{Var}\left(-\frac{\mu}{\sigma} + \frac{1}{\sigma}X\right) = \frac{1}{\sigma^2}\operatorname{Var}(X) = \frac{\sigma^2}{\sigma^2} = 1$$

change of scale
$$Z = \frac{X - \mu}{\sigma} = \underbrace{-\frac{\mu}{\sigma}}_{= c} + \underbrace{\frac{1}{\sigma}}_{= d} X$$

$\rightsquigarrow$ $X = \mu + \sigma Z$

$$P(1 < X < 2) = P(1 < \mu + \sigma Z < 2)$$
$$= P\left(\frac{1-\mu}{\sigma} < Z < \frac{2-\mu}{\sigma}\right)$$
$$= \boxed{\Phi\left(\frac{2-\mu}{\sigma}\right) - \Phi\left(\frac{1-\mu}{\sigma}\right)}$$

ex: let $X \sim \text{Unif}(a,b)$



$$f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{else} \end{cases}$$

a) You should change the scale of X to?

$$U = \frac{X-a}{b-a} = \underbrace{\frac{-a}{b-a}}_{=c} + \underbrace{\frac{1}{b-a}}_{=d} X$$

b) Find $E(X)$

$$X = a + (b-a)U \Rightarrow E(X) = a + (b-a)\frac{1}{2}$$
$$= \frac{a+b}{2}$$

c) Find $Var(X)$.

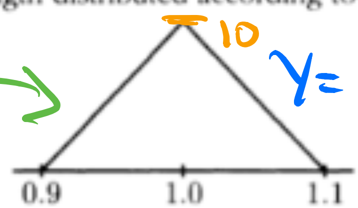
$$Var(X) = var\left((b-a)U + a\right)$$
$$= (b-a)^2 \underbrace{Var(U)}_{=1/12} = \boxed{\frac{(b-a)^2}{12}}$$

Concept test

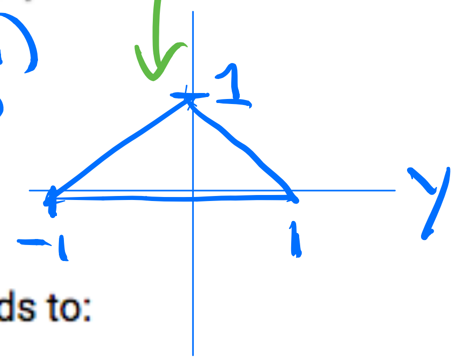
to be density height is 1

Suppose a manufacturing process designed to produce rods of length 1 inch exactly, in fact produces rods with length distributed according to the density graphed below.

to be density height is 10



$y = 10(x-1)$



You should change the scale of X= the length of rods to:

- a: X-1
- b: .1(X-1)
- c: 10X-1
- d: none of the above

More convenient to have rod lengths from -1 to 1. To change $(.9, 1.1)$ to $(-1,1)$ use change of scale $y = 10(x-1)$ since 1 maps to 0 and $.9$ maps to -1.

Our density is now

$$f_Y(y) = \begin{cases} -y+1 & \text{if } 0 \le y \le 1 \\ y+1 & \text{if } -1 \le y \le 0 \\ 0 & \text{else} \end{cases}$$