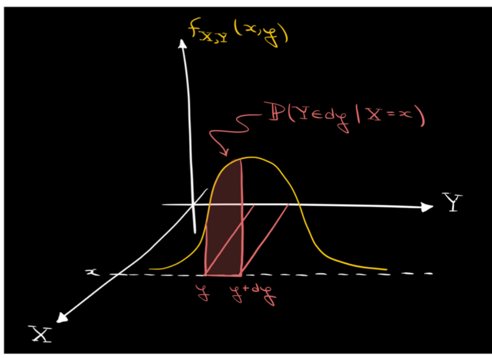
Stat 134 lec 35

Last time

sec 6.3 Conditional probability (continuous case).

$$f_{Y|X=x}(y) = \frac{f_{(X,Y)}(x,y)}{f_X(x)}$$ ← constant

conditional density



ex

$$P(Y \in dy \mid X = x) = f_{Y|X=x}(y)\, dy$$

$$P(Y \in dy) = \int_{x \in X} P(Y \in dy \mid X = x) f_X(x)\, dx$$

integral conditioning formula.

Bayesian Statistics.

ex $X \sim \text{Unif}(0,1)$

$I_1 | X = x,\ I_2 | X = x \overset{iid}{\sim} \text{Ber}(x)$

CAUTION X is continuous and I_1 is discrete.

$$P(I_1 = 1) = \int_{x=0}^{x=1} \underbrace{P(I_1 = 1 \mid X = x)}_{x} \cdot \underbrace{f_X(x)}_{1}\, dx = \int_0^1 x\, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\underbrace{f_{X|I_1=1}(x)}_{\text{Posterior}} = \frac{\overbrace{P(I_1 = 1 \mid X = x)}^{\text{likelihood}}\ \overbrace{f(x = x)}^{\text{Prior}}}{\underbrace{P(I_1 = 1)}_{\text{constant.}}}$$

$$= \frac{x \cdot 1}{1/2} = \boxed{2x}$$

Today
1. Sec 6.3 Integral conditioning formula.
2. Bayesian statistics
3. Conjugate Pairs

① Sec 6.3 Integral conditioning formula

ex $X \sim \text{Unif}(0,1)$

$I_1 | X=x, I_2 | X=x \overset{iid}{\sim} \text{Ber}(x)$

Find $P(I_1=1, I_2=1)$

$$P(I_1=1, I_2=1) = \int_{x \in X} P(I_1=1, I_2=1 | X=x) f_X(x)\,dx$$

$$P(I_1=1, I_2=1 | X=x) = P(I_1=1 | X=x) P(I_2=1 | X=x), \quad P(I_1=1|X=x) = x, \quad P(I_2=1|X=x) = x$$

$$= \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = 1/3$$

ex

Find $P(I_2=1 | I_1=1)$?

Are I_1 and I_2 independent?

$$P(I_2=1 | I_1=1) = \frac{P(I_2=1, I_1=1)}{P(I_1=1)} = \boxed{2/3}, \quad P(I_2=1, I_1=1) = 1/3, \quad P(I_1=1) = 1/2$$

$$P(I_2=1 | I_1=1) \overset{?}{=} P(I_2=1) \Rightarrow \boxed{\text{not indep}}$$

$$P(I_2=1 | I_1=1) = 2/3, \quad P(I_2=1) = 1/2$$

② Sec 6.3 Bayesian Stats

Review ~~Beta~~ Distribution

variable part,

$X \sim Beta(r, s)$

$$f_X(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}, \quad 0 < x < 1$$

where for $r \in \mathbb{Z}^+$, $\Gamma(r) = (r-1)!$

ex For $0<x<1$ if

$$f_X(x) \propto 1 \Rightarrow X \sim Beta(1,1)$$

$$f_X(x) \propto x \Rightarrow X \sim Beta(2,1)$$

$$f_X(x) \propto x(1-x) \Rightarrow X \sim Beta(2,2)$$

ex $X \sim Unif(0,1)$

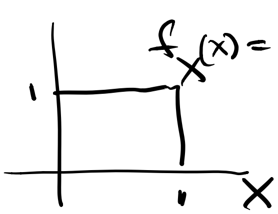
$I_1 | X=x, I_2 | X=x \overset{iid}{\sim} Ber(x)$

$$f_X(x) = 1 \Rightarrow X \sim Beta(1,1)$$

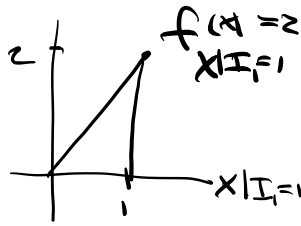
see 1st page

$$f_{X|I_1=1}(x) = 2x \Rightarrow X | I_1=1 \sim Beta(2,1)$$

Prior $X \sim Unif(0,1)$



Posterior



ex Let A be an event and let $X \sim U(0,1)$.
Suppose that conditional of $X=x$, A has probability x.
Find the conditional density of X given A^c.

soln

$$f_{X|A^c}(x) \propto \underbrace{\text{likelihood}}_{P(A^c|X=x) = 1-x} \cdot \underbrace{\text{prior}}_{f_X(x) = 1} = 1-x$$

$$\Rightarrow X|A^c \sim \text{Beta}(1,2) \Rightarrow f_{X|A^c}(x) = \frac{\overbrace{\Gamma(3)}^{2}}{\underbrace{\Gamma(2)}_{1}\underbrace{\Gamma(1)}_{1}}(1-x)$$

tinyurl: http://tinyurl.com/april23-pt1

http://tinyurl.com/april23-pt2

Stat 134

Monday April 23 2019

1. Let A and B be events and let X be a random variable uniformly distributed on (0,1). Suppose that conditional on X=x, A and B are independent each with probability x. The conditional density of X given that A occurs and B doesn't is:

   a $beta(2,1)$

   b $beta(2,2)$

   c $beta(3,2)$

   d none of the above

$f_{X|AB^c}(x) \propto$ likelihood · prior

$\propto P(AB^c|X=x) f_X(x)$

$= P(A|X=x)\,P(B^c|X=x)\cdot 1$

$P(A|X=x) = x$, $P(B^c|X=x) = 1-x$

2. Let A and B be events and let X be a random variable uniformly distributed on (0,1). Suppose that conditional on X=x, A and B are independent each with probability x. The conditional density of X given that A and B occurs is:

a $beta(2, 1)$

b $beta(2, 2)$

c $beta(3, 2)$

d none of the above

$f_{X|AB}(x) \propto x^2$

$Beta(3,1)$

③ Sec 6.3 conjugate pairs

When the prior and the posterior belong the same distribution family we say that the prior and the likelihood are conjugate. The prior and likelihood being conjugate means their variable parts are similar.

ex prior $X \sim \text{beta}(r, s)$

likelihood $Y \sim \text{Bin}(n, x)$

$$\text{Posterior} \propto \text{likelihood} \cdot \text{Prior}$$

$$f_{X|Y=j}(x) \propto P(Y=j \mid X=x)\, f_X(x)$$

similar

$$x^{j+r-1}(1-x)^{n-j+s-1} \propto x^{j}(1-x)^{n-j} \cdot x^{r-1}(1-x)^{s-1}$$

$$\Rightarrow X \mid Y=j \sim \text{Beta}(j+r,\ n-j+s)$$

Conclusion: beta for prior and Binomial for likelihood is a conjugate pair since the posterior is beta.

Suppose $\Theta \sim \text{Gamma}(r, \lambda)$ with r, λ known.

Let $(N_1 | \Theta = \theta, N_2 | \Theta = \theta, N_3 | \Theta = \theta) \overset{iid}{\sim} \text{Pois}(\theta)$.

Find the posterior distribution of Θ.

$$f_{\Theta | N_1 = n_1, N_2 = n_2, N_3 = n_3}(\theta) \propto \text{likelihood} \cdot \text{prior}$$

$$\propto \underbrace{P(N_1 = n_1, N_2 = n_2, N_3 = n_3 | \Theta = \theta)} \cdot f_\Theta(\theta)$$

$$\propto \frac{e^{-\theta} \theta^{n_1}}{n_1!} \frac{e^{-\theta} \theta^{n_2}}{n_2!} \frac{e^{-\theta} \theta^{n_3}}{n_3!} \cdot \theta^{r-1} e^{-\lambda\theta}$$

$$\propto \theta^{(n_1+n_2+n_3)+r-1} e^{-(3+\lambda)\theta}$$

$$\sim \text{Gamma}(n_1+n_2+n_3+r, 3+\lambda)$$

Conclusion: Gamma and Poisson are conjugate pairs with posterior gamma