Stat 134 lec 30

Warmup: 1:00–1:10

Stat 134

Friday November 8 2019

1. S and T are i.i.d. Exp(λ). $X = \text{Min}(S, T)$ and $Y = \text{Max}(S, T)$. The joint density is $f(x, y) = 2\lambda^2 e^{-\lambda(x+y)}$. The marginal density of Y is:

   a $\lambda(1 - e^{-\lambda y})e^{-\lambda y}$ for $y > 0$

   b $2\lambda(1 - e^{-\lambda y})e^{-\lambda y}$ for $y > 0$

   c $2\lambda(1 - e^{-\lambda y})$ for $y > 0$

   d none of the above

Solve this another way that doesn't use $f(x,y)$.

Find $F(y) = P(Y < y)$ and from this find $f(y)$.

$$F(y) = P(Y<y) = P(S<y, T<y) = \left(P(S<y)\right)^2$$
$$= \left(1-e^{-\lambda y}\right)^2$$

$$f(y) = \frac{d}{dy}F(y) = 2\left(1-e^{-\lambda y}\right)\cdot\left(e^{-\lambda y}\right)\cdot\lambda$$ by power rule.

$$= \boxed{2\lambda\left(1-e^{-\lambda y}\right)\left(e^{-\lambda y}\right)}$$

Earlier material

$T \sim \text{Exp}(\lambda)$, $cT \sim \text{Exp}\left(\frac{\lambda}{c}\right)$

Competing exponentials

$T_1 \sim \text{Exp}(\lambda_1), T_2 \sim \text{Exp}(\lambda_2) \Rightarrow P(T_1 < T_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$

Properties of std normal Z	Proved it
$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$	no
$E(Z) = 0$ — Even though Z is symmetric around zero it is possible $E(Z)$ is undefined ex Cauchy distribution	no
$SD(Z) = 1$	no

Let $X, Y \overset{iid}{\sim} N(0,1)$ with density $\phi(x) = ce^{-\frac{1}{2}x^2}$, $c>0$

$f(x,y) = \phi(x)\phi(y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$ for $c>0$

we still need to show that $c = \frac{1}{\sqrt{2\pi}}$, $E(X)=0$, $SD(X)=1$.

Last time

sec 5.2 Marginal density $f_Y(y) = \int_{x=-\infty}^{x=\infty} f(x,y)\,dx$

Today

1. sec 5.2 Expectation $E(g(X,Y))$
2. sec 5.3 Rayleigh distribution
3. sec 5.3 Sums of independent normal RVs is normal.

(1) Sec 5.2 Expectation $E(g(X,Y))$

Let (X,Y) have joint density $f(x,y)$. and $g(X,Y)$ be a function of X, Y.

Define

$$E(g(x,y)) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} g(x,y) f(x,y)\,dx\,dy.$$

ex

joint density

$$f(x,y) = \begin{cases} 30\,(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$

$Y = U_{(6)}$

Find

$$E(Y) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} y\, f(x,y)\,dx\,dy$$

$g(X,Y) = Y$

we know $Y = U_{(6)} \sim \text{Beta}(6,1) \Rightarrow E(Y) = \frac{6}{6+1} = \boxed{\frac{6}{7}}$

k $\quad n-k+1 = 6-6+1 = 1$

See appendix to notes.

② Sec 5.3 Rayleigh Distribution

let $T \sim \text{Exp}(\frac{1}{2})$, $f(t) = \frac{1}{2}e^{-\frac{1}{2}t}$, $t > 0$

$R = \sqrt{T}$ ⟵ R is called the Rayleigh Distribution

write $R \sim \text{Ray}$

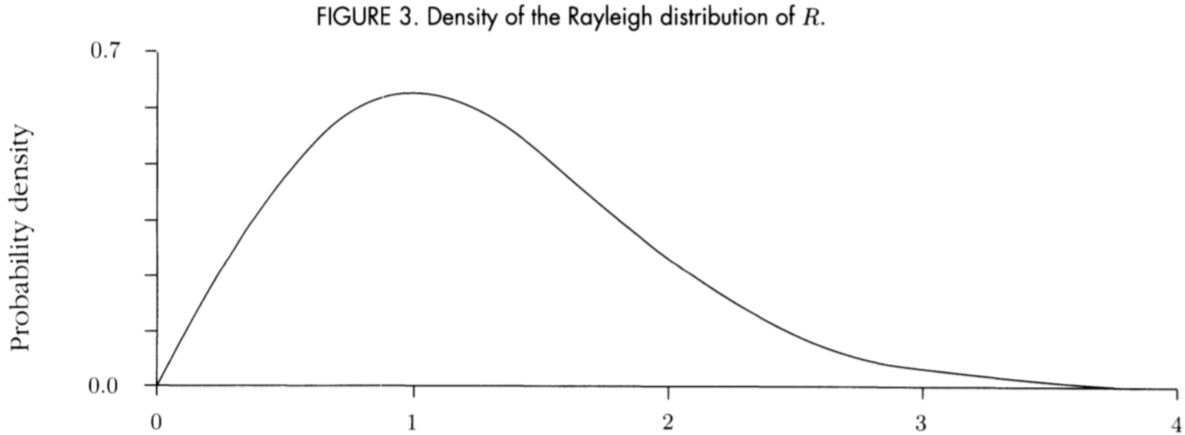
Find $f_R(r)$.

$$f_R(r) = \frac{1}{|(\sqrt{t})'|} \cdot f_T(t) \Big|_{t=r^2}$$

$$= \frac{1}{\frac{1}{2\sqrt{t}}} \cdot \frac{1}{2}e^{-\frac{1}{2}t} \Big|_{t=r^2}$$

$$= \boxed{re^{-\frac{1}{2}r^2}, \quad r > 0}$$

FIGURE 3. Density of the Rayleigh distribution of R.



Note:

$$P(R > r) = P(R^2 > r^2) = P(\underset{\sim \text{Exp}(\frac{1}{2})}{T} > r^2) = e^{-\frac{1}{2}r^2}$$

$$= e^{-\frac{1}{2}r^2}$$

So $\boxed{F_R(r) = 1 - e^{-\frac{1}{2}r^2}, \quad r > 0}$

$$f_R(r) = \frac{d}{dr} F_R(r)$$

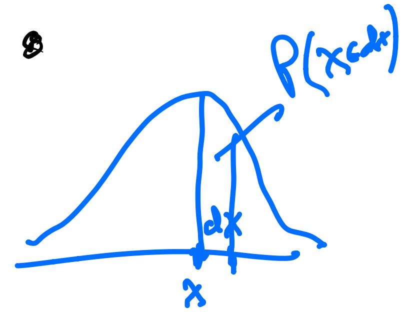
$$f_R(r) = r e^{-\frac{1}{2}r^2}, \quad r > 0$$

The Rayleigh distribution will help us find the density of the standard normal:

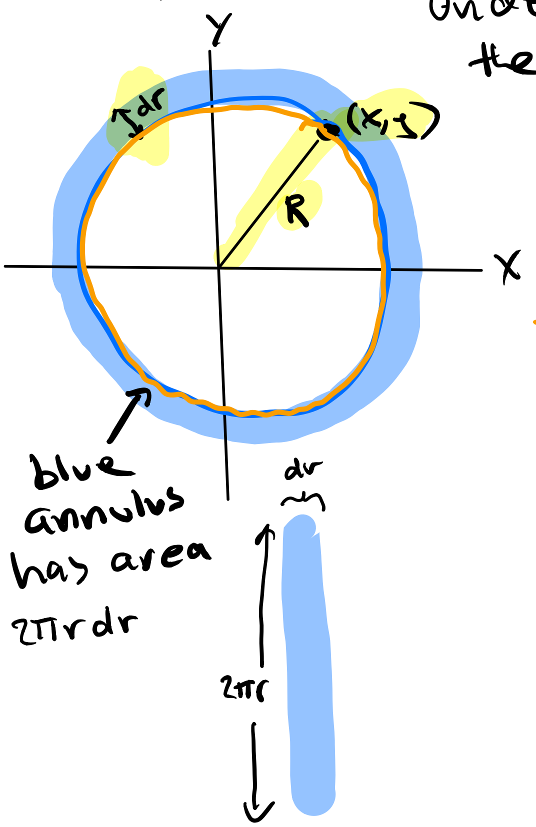
For $X, Y \overset{iid}{\sim} N(0,1)$

Let $R = \sqrt{X^2 + Y^2}$

we will show that $R \sim \text{Ray}$



$P(R \in dr)$ is the volume of the cylinder under $f(x,y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$ over the shaded blue annulus.



$P(R \in dr) = f_R(r)\,dr \approx$ height of $f(x,y)$ above blue annulus

$\cdot$ area of blue annulus

$$= \underbrace{c^2 e^{-\frac{1}{2}r^2}} \cdot (2\pi r\,dr)$$

$$= c^2 2\pi\, r e^{-\frac{1}{2}r^2}\,dr$$

Rayleigh Density

$$1 = \int_{r=0}^{r=\infty} P(R \in dr) = c^2 2\pi \underbrace{\int_0^\infty r e^{-\frac{1}{2}r^2}\,dr}_{=1}$$

$$\Rightarrow c^2 \cdot 2\pi = 1$$

$$\Rightarrow c = \frac{1}{\sqrt{2\pi}}$$

$$P(R \in dr) = \left(\frac{1}{2\pi}\right) \cdot 2\pi r e^{-\frac{1}{2}r^2} dr$$
$$= r e^{-\frac{1}{2}r^2} dr$$

Conclusions

(1) For $X, Y \overset{iid}{\sim} N(0,1)$ and $T \sim \text{Exp}(\frac{1}{2})$

$R = \sqrt{X^2+Y^2}$ and $R = \sqrt{T}$

are both the Rayleigh distribution

(2) $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ is density of the standard normal.

(i.e $c = \frac{1}{\sqrt{2\pi}}$)

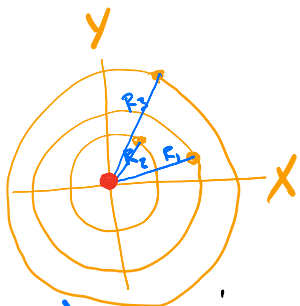
- see end of lecture notes

(3) $E(X) = 0$ and $SD(X) = 1$

ex Suppose 3 shots are fired at a target. Assume for each shot, both X and Y are standard normals. Let W be the closest distance among 3 shots to the bullseye. Find $f_W(w)$

$$R = \sqrt{X^2+Y^2}$$ distance to Bullseye

Picture



$W = \min(R_1, R_2, R_3)$

where $R_1, R_2, R_3 \overset{iid}{\sim} \text{Ray}$

$F_{R_1}(r) = 1 - e^{-\frac{1}{2}r^2}$

(cdf approach)

$$P(W > w) = P(R_1 > w,\ R_2 > w,\ R_3 > w) = \left(P(R_1 > w)\right)^3$$

$$= \left(e^{-\frac{1}{2}w^2}\right)^3 = e^{-\frac{3}{2}w^2}$$

$$\Rightarrow F(w) = 1 - e^{-\frac{3}{2}w^2}$$

$$\boxed{f_W(w) = 3w\, e^{-\frac{3}{2}w^2}}$$

Note: you could also do this this using Rayleigh ordered statistics since W is the min of 3 Rayleighs.

If $R_1 \sim \text{Ray}$, $f_{R_1}(r) = re^{-\frac{1}{2}r^2}$

Find density of $W = \frac{1}{\sqrt{3}} R_1$

$$f_W(w) = \frac{1}{1/\sqrt{3}} \, r e^{-\frac{1}{2}r^2} \Big|_{r = \sqrt{3}\, w}$$

$$= \sqrt{3} \cdot \sqrt{3}\, w e^{-\frac{3}{2}w^2} = \boxed{3we^{-\frac{3}{2}w^2}}$$

So $\min(R_1, R_2, R_3) \sim \frac{1}{\sqrt{3}} \text{Ray}$

Appendix

Expectation $E(g(X,Y))$

ex

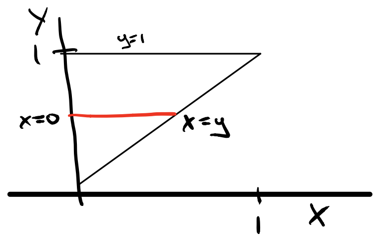
$$E(g(x,y)) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} g(x,y) f(x,y)\,dx\,dy$$

joint density

$$f(x,y) = \begin{cases} 30(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$

$Y = U_{(6)}$



Find

$$E(Y) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} y\, f(x,y)\,dx\,dy$$

$g(x,y) = Y$

$$= \int_{y=0}^{y=1} y \int_{x=-\infty}^{x=\infty} f(x,y)\,dx\,dy$$

$\parallel$

$f_Y(y) = 6y^5$

showed this in lecture 29

$$= 6\int_{y=0}^{y=1} y^6\,dy = \frac{6y^7}{7}\Big|_0^1 = \boxed{6/7}$$

Appendix

Claim Let $X \sim N(0,1)$. We show that $E(X)=0$ and $SD(X)=1$.

Let $X \sim N(0,1)$

Show $E(X)=0$

$\phi(x)$ is symmetric about zero so all we have to show is that $E(X)$ converges absolotely.

(i.e $E(|X|) < \infty$).

$$E(|X|) = \int_{-\infty}^{\infty} |x| \phi(x) dx$$

$$= 2 \int_{0}^{\infty} x \phi(x) dx$$

$$= 2 \int_{0}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x e^{-\frac{x^2}{2}} dx$$

$\uparrow$ Rayleigh density $= 1$

$$= \sqrt{\frac{2}{\pi}} < \infty.$$

$\Rightarrow E(X)=0$ since $\phi(x)$ is semetric around zero

Next we show $SD(X) = 1$:

We know $X^2 + Y^2 \sim Exp\left(\underbrace{\tfrac{1}{2}}_{=\lambda}\right)$ from above

so $E(X^2+Y^2) = \dfrac{1}{\frac{1}{2}} = 2$

λ rate of X^2+Y^2

$\Rightarrow E(X^2) + \underbrace{E(Y^2)}_{=E(X^2)} = 2$

$\Rightarrow 2E(X^2) = 2 \Rightarrow E(X^2) = 1$

but $SD(X) = \sqrt{E(X^2) - E(X)^2}$

$= \sqrt{1 - 0} = 1$