Jihee Yoon
Tuesday

Stat 134 lec 18

3. (3 pts) A coin is weighted so that its probability of landing on heads is 20%. Suppose the coin is flipped 20 times. Find an upper bound for the probability it lands on heads at least 16 times. Note: No credit will be given for exact probabilities. You have to use error bounds.

SOLUTION:
E[X] = 4 $(20(\frac{1}{5}) = 4)$
Var[X] = (0.2)(0.8)(20) = 3.2

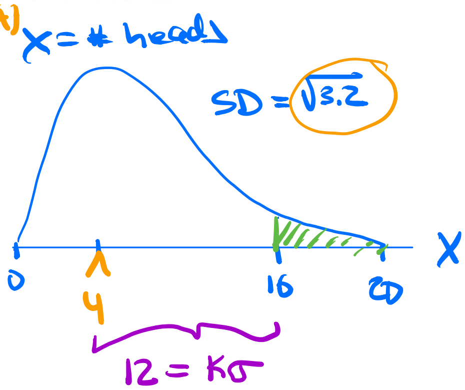
$\le \frac{E((x-\mu)^2)}{12^2}$ ← Var(x)

$P((x-\mu)^2 \ge 12^2)$ $12 = k\sigma$ $= \frac{1}{(12/\sqrt{3.2})^2}$

Chebyshev's (+1) :
P(X >= 16) = P(|X-E[X]| >= 12) <= $\frac{Var(X)}{12^2}$ = $\frac{3.2}{144}$

$\mu + k\sigma$

$\Rightarrow k = \frac{12}{\sqrt{3.2}} = 6.70$



Markov's (+1)
P(X >= 16) <= $\frac{E[X]}{16}$ = $\frac{4}{16}$

The tightest upper bound is $\frac{3.2}{144}$ (+1)

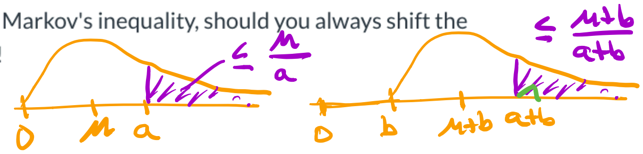
Connie Mi

Yesterday

1b. from the 2019 Spring midterm - can you explain why you must create a "substitute" random variable in order to shift the range of values to start at 0? When using Markov's inequality, should you always shift the distribution so that the range of values starts at 0? Thanks!

Reply (4 likes)

Yes,



$\frac{\mu}{a} \le \frac{\mu+b}{a+b}$

iff $\mu(a+b) \le a(\mu+b)$

iff $\cancel{\mu a} + \mu b \le \cancel{a\mu} + ab$

iff $\mu \le a$

(only use Markov if $\mu \le a$)

1. (5 pts)

a Suppose that there is a machine that gives out a random number Y between 0 and 80. You are also told that $\mathbb{E}[Y] = 20$. Now someone proposes you a game where you win if the number that shows up is strictly smaller than 40. Assume that you always play games when you have a chance of at least $\frac{1}{2}$ of winning. Given the information you have, can you determine whether you should agree to play the game?

b Suppose that there is a machine that gives out a random number X between 20 and 100. You are also told that $\mathbb{E}[X] = 40$. Now someone proposes you a game where you win if the number that shows up is strictly smaller than 60. Assume that you always play games when you have a chance of at least $\frac{1}{2}$ of winning. Given the information you have, can you determine whether you should agree to play the game?

a Using Markov's inequality we get

$$\mathbb{P}(Y \ge 40) \le \frac{\mathbb{E}[Y]}{40} = \frac{20}{40} = \frac{1}{2}.$$

This implies that

$$\mathbb{P}(Y < 40) = 1 - \mathbb{P}(Y \ge 40) \ge 1 - \frac{1}{2} = \frac{1}{2}$$

hence you should agree to play the game.

b Substitute $Y = X - 20$ into your conclusion from part a and you get

$$\mathbb{P}(X < 60) = \mathbb{P}(X - 20 < 40) = \mathbb{P}(Y < 40) \ge \frac{1}{2}$$

hence you should agree to play the game

Tessa Williams
Tuesday

Could you go over the cookie problem (prob 3) from the discussion section 10 worksheet? Thanks.

Reply (2 likes)

Problem 3

There are c chocolate chip cookies, o oatmeal raisin cookies, and p peanut butter cookies in a jar. You draw cookies from the jar without replacement until you get 3 chocolate chip cookies. Let X be the number of cookies until you get your 3rd chocolate chip cookie. Find Var(X).

Soln

X = # cookies until you get your 3^{rd} chocolate chip cookie

$$X = \sum_{i=1}^{o+p} I_i + 3, \quad \text{where} \qquad P_1 = \frac{3}{c+1}$$

$$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ non chocolate chip cookie is before } 3^{rd} \text{ choc c chip cookie.} \\ 0 & \text{else} \end{cases}$$

$$E(X) = 3 + (o+p)\cdot P_1 = \boxed{3 + (o+p)\left(\frac{3}{c+1}\right)}$$

$$\mathrm{Var}(X) = \mathrm{Var}\left(\sum_{i=1}^{o+p} I_i + 3\right) = \mathrm{Var}\left(\sum_{i=1}^{o+p} I_i\right)$$

$$= E\left(\left(\sum_{i=1}^{o+p} I_i\right)^2\right) - \left[E\left(\sum_{i=1}^{o+p} I_i\right)\right]^2$$

will find next

$$\left[E\left(\sum_{i=1}^{o+p} I_i\right)\right]^2 = \left[(o+p)\,P_1\right]^2$$

$$\begin{array}{c|ccc} & I_1 & I_2 & \cdots\ I_{o+p} \\ \hline I_1 & I_1 & I_1 I_2 & \\ \vdots & & \ddots & \\ I_{o+p} & & & I_{o+p} \end{array}$$

$$E\left(\left(\sum_{i=1}^{o+p} I_i\right)^2\right) = \underbrace{(o+p)\,E(I_1)}_{\text{diagonal}} + \underbrace{(o+p)(o+p-1)\,E(I_{12})}_{\text{off diagonal}}$$

$E(I_1) = P_1$, $E(I_{12}) = P_{12}$

where,

$$I_{12} = \begin{cases} 1 & \text{if } 1^{st} \text{ and } 2^{nd} \text{ non chocolate chip cookie is before } 3^{rd} \text{ chocolate chip cookie.} \\ 0 & \text{else} \end{cases}$$

3rd cc

$$P_{12} = \frac{3}{c+1} \cdot \frac{4}{c+2}$$

so

$$Var(X) = (o+p)P_1 + (o+p)(o+p-1)P_{12} - \left((o+p)P_1\right)^2$$

where $P_1 = \frac{3}{c+1}$

$$P_{12} = \frac{3}{c+1} \cdot \frac{4}{c+2}$$

2. (3 pts) Each person is STAT 134 has a lucky number from {1, 2, ..., k}. Students walk into a classroom one at a time. Let X be the number of students who have to walk into the classroom until two people share the same lucky number. Find $\mathbb{E}(X)$.

SOLUTION:
P(X = x) = $\frac{k-1}{k} * \frac{k-2}{k} * ... \frac{k-x+2}{k} * \frac{x-1}{k}$ (+2)
E[X] = $\sum_{x=2}^{k+1} xP(X=x)$ (+1)

Dont know how many students.
Cant break students into one of two types

for $2 \le x \le k+1$,
$X = x$ means the first $x-1$ students have different numbers and the last student has one of the $x-1$ numbers.

$$P(X=x) = \underbrace{\frac{k}{k} \cdot \frac{k-1}{k} \cdots \frac{k-x+2}{k}}_{x-1 \text{ students}} \cdot \underbrace{\frac{x-1}{k}}_{\text{last student}}$$

Then $E[X] = \sum_{x=2}^{k+1} x P(X=x)$

Problem from lec 17:

1. 10 people throw their hats into a box and randomly redistribute the hats among themselves. Assume every permutation of the hats is equally likely. Let N be the number of people who get their own hats back. Find the following:

(a) $\mathbb{E}[N^2]$

Let N_i be the indicator for the event that the i-th person gets their own hat back.

$$\mathbb{E}[N^2] = \mathbb{E}[(\sum_{i=1}^{10} N_i)^2] = \sum_{i=1}^{10}\sum_{j=1}^{10} \mathbb{E}[N_i N_j] = 90(1/90) + 10(1/10) = 2$$

(b) $P(N = 8)$

If 8 people got their own hat back, that means only 2 people have hats that are not their own. There are $\binom{10}{2}$ ways to pick such a pair.

$$P(N=8) = \frac{\binom{10}{2}}{10!} \leftarrow \text{\# ways 2 hats don't go to owner.}$$

a b c
b c a
c a b
a c b

The numerator needs some explanation. If the first 8 hats go to their owner then there is only one ordering of 2 hats that don't go to their owner. There are $\binom{10}{2}$ ways to pick 2 of the hats that don't go to their owner.

Note $P(N=7) \neq \frac{\binom{10}{3}}{10!}$ since if the first 7 hats go to their owner and the last 3 don't, there are multiple ways the 3 hats don't go to their owner.

we have $P(N=7) = \frac{\binom{10}{3}\cdot 2}{10!} \leftarrow$ # ways last 3 hats can not belong to their owner.

In this way you can see that

$$P(N=0) = \frac{\text{\# ways no hats belong to owner}}{10!}$$

$$P(N=8) = \frac{\quad}{10!}$$

(c) $P(N=0)$

Let A_i represent the event that person i got their own hat back.

$$P(N=0) = 1 - P(A_1 \cup A_2 \cup ... \cup A_{10})$$

By the Principle of Inclusion Exclusion:

$$\begin{aligned} P(A_1 \cup A_2 \cup ... \cup A_{10}) &= \sum_{i=1}^{10} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - ... \\ &= 10\frac{1}{10} - \binom{10}{2}\frac{1}{10*9} + \binom{10}{3}\frac{1}{10*9*8} - ... = 1 - \frac{1}{2!} + \frac{1}{3!} - ... - \frac{1}{10!} \end{aligned}$$

Thus

$$P(N=0) = 1 - P(A_1 \cup A_2 \cup ... \cup A_{10}) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!}$$

Thu Nguyen

Yesterday

On discussion 3, problem 3: The matching problem, part b), is it incorrect to write the probability that at least one letter is put in a correctly addressed envelope as:

$$1 - P\,(no\ letter\ is\ put\ in\ correct\ envelope) = 1 - \underbrace{\frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \ldots \cdot \frac{1}{2} \cdot \frac{1}{1}} = 1 - \frac{(n-1)!}{n!} = 1 - \frac{1}{n}$$

If it is incorrect, then why?

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Reply (1 like)

Out of n hats including mine randomly pick $n-1$ hats.
Chance I dont get my hat is

$$\frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \ldots \cdot \frac{1}{2} = \frac{(n-1)!}{n!} = \frac{1}{n}$$

equivalently out of n hats randomly pick one hat not to be selected. There is a $1/n$ chance I pick my hat.

ex $n=4$

Correct order a b c d

$4! = 24$ possible orderings

orderings w/ no matches

b a d c

3x b d a c

b c d a

$\Rightarrow \frac{9}{24}$ is chance no one gets hat back

$\uparrow \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}$

$\left(\text{not } 1 - \frac{1}{4}\right)$

Qingsong Ji

Wednesday

Roll a 20 sided die 10 times, what is P(get 1 triple, 3 double, and 1 single)? Thank you!

Reply (2 likes)

Suppose first that get 1 triple, 3 double and 1 single in that order. (i.e. $\underbrace{ttt}\,\underbrace{dd}\,\underbrace{dd}\,\underbrace{dd}\,s$)

The chance of this is

$$\left(\frac{20}{20}\cdot\frac{1}{20}\cdot\frac{1}{20}\right)\cdot\left(\frac{19}{19}\cdot\frac{1}{19}\right)\left(\frac{18}{18}\cdot\frac{1}{18}\right)\left(\frac{17}{17}\cdot\frac{1}{17}\right)\cdot\left(\frac{16}{16}\right)$$

Because the triple, doubles and single can occur in different positions we must multiply by

$$\underbrace{\binom{10}{3}}_{\text{triple}}\underbrace{\binom{7}{2}\binom{5}{2}\binom{3}{2}}_{\text{doubles}}\underbrace{\binom{1}{1}}_{\text{single}}$$

So final answer is

$$\boxed{\binom{10}{3}\binom{7}{2}\binom{5}{2}\binom{3}{2}\binom{1}{1}\cdot\left(\frac{20}{20}\cdot\frac{1}{20}\cdot\frac{1}{20}\right)\cdot\left(\frac{19}{19}\cdot\frac{1}{19}\right)\left(\frac{18}{18}\cdot\frac{1}{18}\right)\left(\frac{17}{17}\cdot\frac{1}{17}\right)\cdot\left(\frac{16}{16}\right)}$$

Could you provide us with a list of all of the geometric series sums that we must know for the midterm/the class in general?

① $S = 1+z+z^2+z^3+\cdots$ for $|z|<1$ geometric sum.

$zS = z+z^2+z^3+\cdots$

$\underbrace{S-zS}_{S(1-z)} = 1 \Rightarrow \boxed{S = \frac{1}{1-z} \text{ for } |z|<1}$

② $\boxed{e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}$ Taylor series for e^x

also $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

$\boxed{\log(1+x) = x - \frac{x^2}{2}+\frac{x^3}{3}-\cdots}$ Taylor series for $\log(1+x)$

ex

$X \sim \text{Pois}(\mu)$

$P(X=k) = e^{-\mu}\frac{\mu^k}{k!} \quad k=0,1,2,\ldots$

Show that $E(X)=\mu$

Recall $e^{\mu} = 1+\mu+\frac{\mu^2}{2!}+\cdots = \sum_{k=0}^{\infty}\frac{\mu^k}{k!}$ Taylor series.

$E(X) = \sum_{k=0}^{\infty} k\cdot P(X=k) = \sum_{k=0}^{\infty} k e^{-\mu}\frac{\mu^k}{k!}$

$= \sum_{k=1}^{\infty} \cancel{k} e^{-\mu}\frac{\mu^{k-1}\mu}{(k-1)!\cancel{k}}$ (note $0\cdot e^{-\mu}\frac{\mu^0}{0!} = 0$)

$= \mu e^{-\mu}\sum_{k=1}^{\infty}\frac{\mu^{k-1}}{(k-1)!}$

$= \mu e^{-\mu}\underbrace{\left(1+\mu+\frac{\mu^2}{2!}+\cdots\right)}_{e^{\mu}} = \boxed{\mu}$