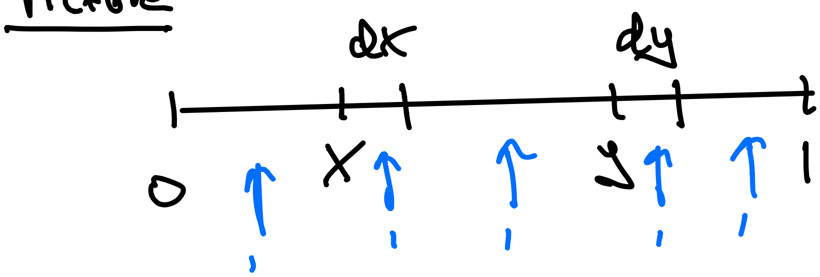
Stat 155 Lec 28

Warmup 1:00–1:10

ex Throw down 5 darts on $(0,1)$.
Find the joint density, $f(x,y)$,
$X = U_{(2)}$ and $Y = U_{(4)}$.

Picture



$$f(x,y)\,dx\,dy = \binom{5}{1,1,1,1,1} x\,dx\,(y-x)\,dy\,(1-y)$$

$$\Rightarrow f(x,y) = \binom{5}{1,1,1,1,1} x(y-x)(1-y)$$

$\binom{5}{1,1,1,1,1} = 5!$ $\qquad 0<x<y<1$

$$\Rightarrow \boxed{f(x,y) = 5!\,x(y-x)(1-y) \text{ for } 0<x<y<1}$$

Quiz 5 Wednesday sec 4.4–4.6, MGF and 5.1, 5.2

Last time

Sec 4.6 Uniform order statistic

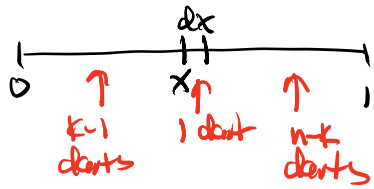
$U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$

$U_{(1)}, \dots, U_{(n)}$ order statistics

$P(U_{(k)} \in dx) = f(x)dx$

ex $U_1 = .2$ $U_2 = .7$ $U_3 = .1$

$\Rightarrow U_{(1)} = .1,\ U_{(2)} = .2,\ U_{(3)} = .7$



Note

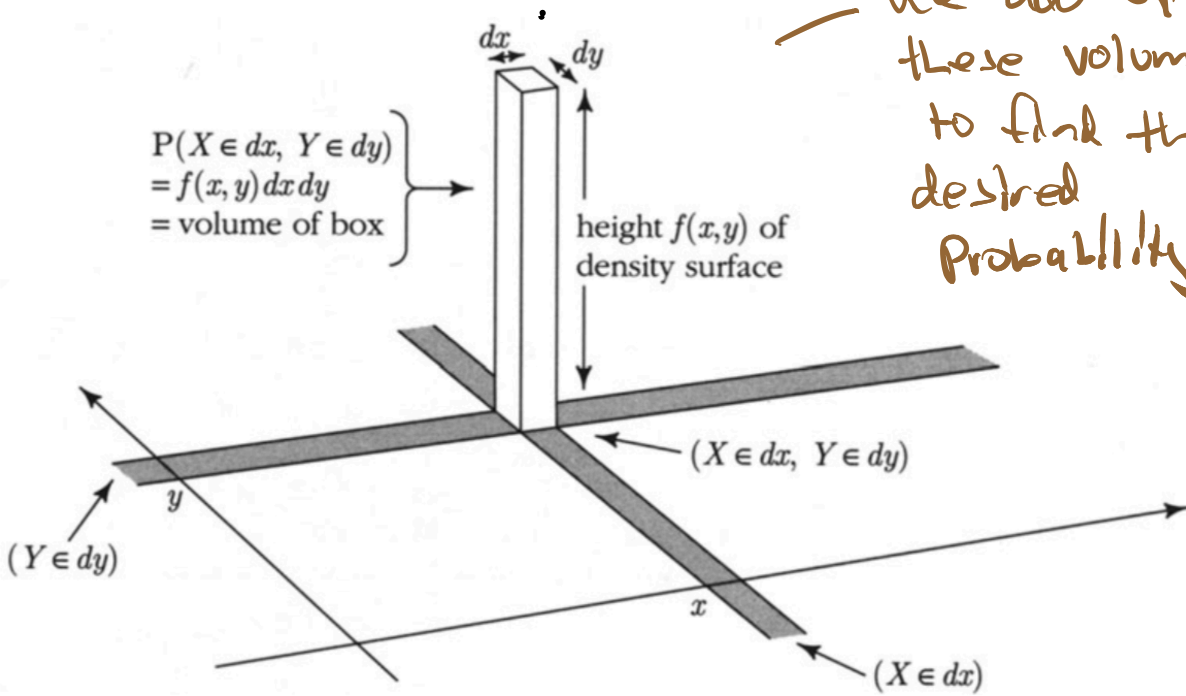
$$\binom{n}{a,b,c} = \frac{n!}{a!b!c!} = \binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}$$

$$f_{U_{(k)}}(x) = \binom{n}{k-1,1,n-k} x^{k-1} (1-x)^{(n-k+1)-1} \quad \text{on } 0<x<1$$

today
1. sec 5.1, 5.2 Continuous Joint Distribution
2. sec 4.6 Beta distribution
3. sec 5.1, 5.2 Calculate probabilities with $f(x,y)$.
4. sec 5.1, 5.2 Independence

① Sec 5.1, 5.2 Joint Density

$P(X \in dx, Y \in dy) \approx f(x,y)\,dx\,dy$



we add up these volumes to find the desired probability.

$$\int_y \int_x f(x,y)\,dx\,dy = \int_x \int_y f(x,y)\,dy\,dx = 1$$

http://tinyurl.com/oct28-pt1

Stat 134

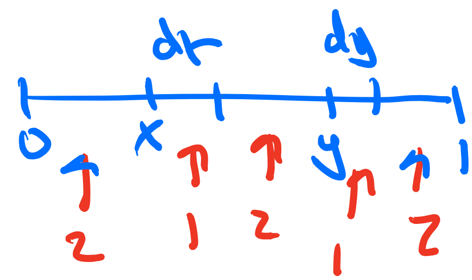
1. I throw down 8 darts on $(0,1)$. The variable part of the joint density of $X = U_{(3)}$ and $Y = U_{(6)}$ is:

a $x(y-x)^5(1-y)^2$

b $x^2(y-x)^2(1-y)^2$

c $x^4(y-x)^2(1-y)^2$

d none of the above



Let (X, Y) have joint density $f_{X,Y}(x, y) = 420x^3(1 - y)^2$ for $0 < x < y < 1$.

Fill in the blanks: X and Y represent the 4th smallest and 5th smallest of 7th i.i.d. Unif (0,1) random variables, respectively.

② Sec 4.6 Beta distribution.

$X \sim \text{Beta}(r,s)$ for $r>0, s>0$ is a distribution often used to model physical processes that take values between 0 and 1.

ex the proportion of defective items in a shipment.

Let $r,s>0$

Defn $X \sim \text{Beta}(r,s)$ if

$$f(x) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} x^{r-1}(1-x)^{s-1} \quad \text{for } 0<x<1.$$

where $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$ Gamma function for $r>0$

or $\Gamma(r) = (r-1)!$ if $r \in \mathbb{Z}^+$

Notice if $r=1, s=1$, $f(x) = 1_{x\in(0,1)}$

$\Rightarrow \text{Beta}(1,1) = \text{Unif}(0,1)$.

ex Let $U_1, \ldots, U_n \overset{iid}{\sim} U(0,1)$

$$f_{U_{(k)}}(x) = \binom{n}{k-1,1,n-k} x^{k-1}(1-x)^{(n-k+1)-1} \quad \text{on } 0<x<1$$

compare with,

$$f_{\text{Beta}(r,s)}(x) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} x^{r-1}(1-x)^{s-1} \quad \text{for } 0<x<1.$$

Notice that $f_{U_{(k)}}(x)$ and $f_{Beta(r,s)}(x)$ have the same variable part of their density when $r = k$

$$s = n-k+1$$

then $\Gamma(s+r) = \Gamma(n-k+1+k) = \Gamma(n+1) = n!$

$$\Gamma(r) = (k-1)!$$

$$\Gamma(s) = (n-k)!$$

$$\Rightarrow \frac{\Gamma(s+r)}{\Gamma(r)\Gamma(s)} = \binom{n}{k-1,\ 1,\ n-k}$$

$\Rightarrow$ Standard uniform ordered statistics are beta!

see appendix of notes

<u>Thm</u> $X \sim Beta(r, s)$

$$E(X) = \frac{r}{r+s}$$

Hence if $X \sim U_{(k)}$

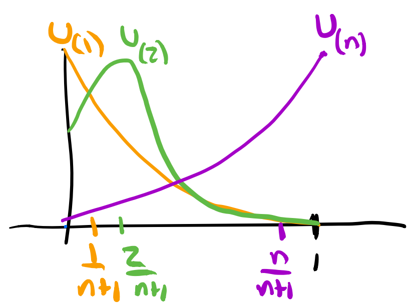
$$E(X) = \frac{k}{n-k+1+k} = \boxed{\frac{k}{n+1}}$$

$$E(U_{(1)}) = \frac{1}{n+1}$$

$$E(U_{(2)}) = \frac{2}{n+1}$$

$$\vdots$$

$$E(U_{(n)}) = \frac{n}{n+1}$$



ex (Bayesian Statistics)

Let P be the chance a coin lands head. Suppose the prior distribution of P is

$$f_P(p) = \begin{cases} c(1-p)^4 & \text{for } 0 \leq p \leq 1 \\ 0 & \text{else} \end{cases}$$

a) Is this a beta distribution? If so, what are the parameters? $f_{Beta(r,s)}(p) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} p^{r-1}(1-p)^{s-1}$ for $0<p<1$.

yes, compare $(1-p)^4$ and $p^{r-1}(1-p)^{s-1}$ $\Rightarrow P \sim Beta(1,5)$

b) Calculate the constant c

$$c = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} = \frac{\Gamma(6)}{\Gamma(1)\Gamma(5)} = \frac{5!}{0!4!} = 5$$

c) What is the mean of P?

$$E(P) = \frac{r}{r+s} = \frac{1}{1+5} = 1/6$$

If $X \sim \text{Beta}(r,s)$

$$f(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}$$

Since $\int_0^1 f(x)\,dx = 1 \Rightarrow \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$

ex Let $X \sim \text{Beta}(3,4)$

Compute $E(7X - 5X^6)$

$$\left(\text{so } f(x) = \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} x^2 (1-x)^3\right)$$

$$E(7x - 5x^6) = \underbrace{7E(x)}_{= 7\cdot 3/7} - 5E(x^6)$$

$$E(x^6) = \int_0^1 x^6 \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} x^2 (1-x)^3\,dx$$

$$= \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} \int_0^1 x^8 (1-x)^3\,dx$$

$$\int_0^1 x^8 (1-x)^3\,dx = \frac{\Gamma(9)\Gamma(4)}{\Gamma(13)}$$

$$= \frac{\Gamma(9)\Gamma(7)}{\Gamma(3)\Gamma(13)}$$

$$\Rightarrow E(7X - 5X^6) = \boxed{3 - \frac{5\Gamma(9)\Gamma(7)}{\Gamma(3)\Gamma(13)}}$$

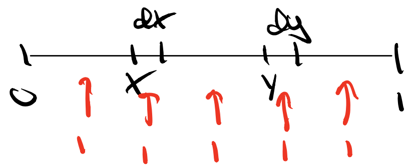
③ sec 5.1, 5.2 Calculate probabilities with $f(x,y)$.

Throw down 5 darts on $(0,1)$.

ex $X = U_{(2)}$ $Y = U_{(4)}$

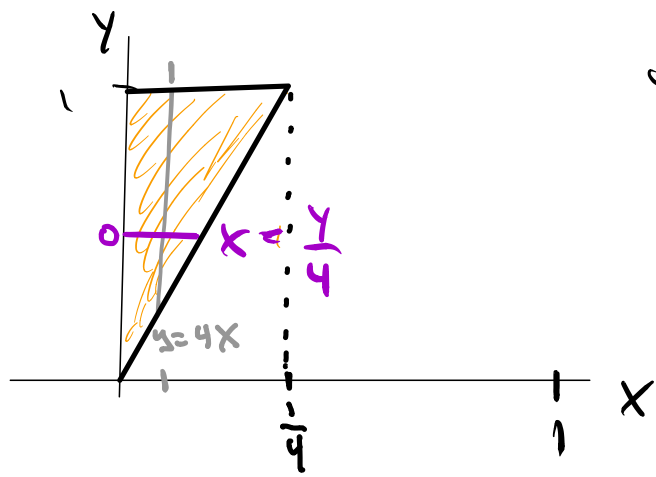
Find $P(Y > 4X)$

recall,



$$f(x,y) = \binom{5}{1,1,1,1,1} x\,(y-x)(1-y)$$

$$= \boxed{5!\, x\,(y-x)\,(1-y)}$$



$$P(Y > 4X) = \int_{y=0}^{y=1} \int_{x=0}^{x=\frac{y}{4}} 5!\, x\,(y-x)(1-y)\,dx\,dy$$

(or)

$$P(Y > 4X) = \int_{x=0}^{x=1/4} \int_{y=4x}^{y=1} 5!\, x\,(y-x)(1-y)\,dy\,dx$$

details:

$$P(Y > 4x) = \int_{y=0}^{y=1} \int_{x=0}^{y/4} 120\,x\,(y-x)(1-y)\,dx\,dy$$

$$= 120 \int_{y=0}^{y=1} (1-y) \int_{x=0}^{x=y/4} (xy - x^2)\,dx\,dy$$

$$= \int_{y=0}^{y=1} 120(1-y)\left(\frac{x^2 y}{2} - \frac{x^3}{3}\right)\Bigg|_{x=0}^{x=y/4} dy$$

$$= \int_0^1 120\,(1-y)\left(\frac{y^3}{32} - \frac{y^3}{3\cdot 64}\right) dy$$

$$= \frac{5}{192} \cdot 120 \int_0^1 y^3 - y^4\,dy = \frac{5 \cdot 120}{192}\left(\frac{y^4}{4} - \frac{y^5}{5}\right)\Bigg|_0^1$$

$$= \frac{5 \cdot 120}{192}\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{30}{192} = .156$$

④ Sec 5.1, 5.2

Independent RVs

def^n X and Y are independent if

$$P(X \in dx, Y \in dy) = P(X \in dx)\, P(Y \in dy)$$

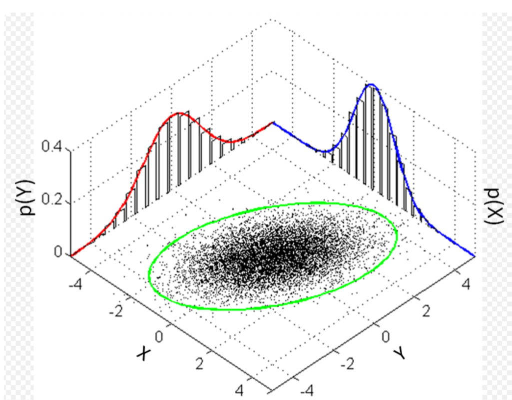
$$\overset{\shortparallel}{f(x,y)\,dx\,dy} \qquad \overset{\shortparallel}{f(x)\,dx} \qquad \overset{\shortparallel}{f(y)\,dy}$$

$$\Longleftrightarrow \boxed{f(x,y) = f(x)\,f(y)}$$

ex $X, Y \overset{iid}{\sim} N(0,1)$

$$f(x,y) = \phi(x)\,\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$



Not a great picture because the oval in green should be a circle. This is the picture of a correlated bivariate normal from Chapter 6 instead of an uncorrelated bivariate normal.

ex If $X, Y \overset{iid}{\sim} U(0,1)$

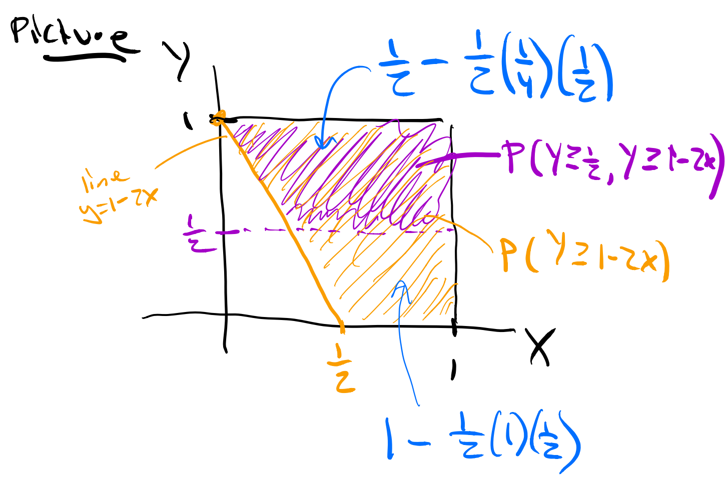
Find $P(Y \geq \frac{1}{2} \mid Y \geq 1-2X)$

soln

$$f(x,y) = \underset{1}{f(x)}\,\underset{1}{f(y)} = 1 \quad \text{for } 0<x,y<1$$

$$P(Y \geq \tfrac{1}{2} \mid Y \geq 1-2X) = \frac{P(Y\geq\frac{1}{2},\, Y\geq 1-2X)}{P(Y\geq 1-2X)} \qquad \text{Baye's rule}$$

Picture



so,

$$\frac{P(Y\geq\frac{1}{2},\, Y\geq 1-2X)}{P(Y\geq 1-2X)} = \frac{\frac{1}{2} - \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)}{1 - \frac{1}{2}(1)\left(\frac{1}{2}\right)} = \frac{7/16}{3/4} = \boxed{7/12}$$

Appendix

Let $X \sim \text{Beta}(r,s)$

then $E(X) = \dfrac{r}{r+s}$.

Pf/ Note that $\displaystyle\int_0^1 f(x)\,dx = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = 1$

$$\Rightarrow \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$$

$$E(X) = \int_0^1 x f(x)\,dx = \frac{\Gamma(r+s)}{\Gamma(r)\cancel{\Gamma(s)}} \underbrace{\int_0^1 \overbrace{x\, x^{r-1}}^{x^{r+1-1}} (1-x)^{1-s}\,dx}_{= \frac{\cancel{\Gamma(s)}\Gamma(r+1)}{\Gamma(s+r+1)}}$$

$$= \frac{(r+s-1)!\; r!}{(s+r)!} = \boxed{\frac{r}{r+s}}$$

$\square$