Stat 134 lec 2

Last time

Addition rule (OR) if A, B mutually exclusive sets

$$P(A \text{ or } B) = P(A) + P(B).$$

Inclusion exclusion (OR)

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

Today

① sec 1.3 Distributions

② sec 1.4 Conditional Probability

③ sec 1.5 Baye's rule

Some easy facts about sets:

$$A_1 \cup A_2 \cup A_3 = (A_1 \cup A_2) \cup A_3 \quad \Rightarrow \quad \bigcup_{i=1}^{m+1} A_i = \left(\bigcup_{i=1}^{m} A_i\right) \cup A_{m+1}$$

$$A_1A_3 \cup A_2A_3 = (A_1 \cup A_2)A_3 \quad \Rightarrow \quad \bigcup_{i=1}^{m} (A_i A_{m+1}) = \left(\bigcup_{i=1}^{m} A_i\right) A_{m+1}$$

These are proven by induction. I sketch such a proof for the second one below:

Show

$$\bigcup_{i=1}^{m}(A_i A_{m+1}) = \left(\bigcup_{i=1}^{m} A_i\right) A_{m+1}$$

base case $m=1$

$$(A_1 A_2) = (A_1) A_2$$

assume true for $m \leq 4$

$$\bigcup_{i=1}^{4}(A_i B) = \left(\bigcup_{i=1}^{4} A_i\right) B$$

Show true for $m = 5$

$$\bigcup_{i=1}^{5}(A_i B) = \underbrace{\bigcup_{i=1}^{4}(A_i B)}_{=} \cup A_5 B$$

by induction $(m=4)$

$$\left(\bigcup_{i=1}^{4} A_i\right) B$$

$$= \left(\bigcup_{i=1}^{4} A_i \cup A_5\right) B$$

by induction $(m=2)$

$$= \left(\bigcup_{i=1}^{5} A_i\right) B \quad \checkmark$$

Sec 1.3 Distributions

generalized inclusion exclusion:

see #12 p31

ex Let $A_1, \dots, A_{n+1}$ be events.

Show $P\left(\bigcup_{i=1}^{n+1} A_i\right) = P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1}) - P\left(\bigcup_{i=1}^{n} (A_i A_{n+1})\right)$

Pf/ We prove by induction.

① base case is true, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 A_2)$ by inclusion exclusion rule.

② Assume true for union of $\leq n$ events and show true for union of $n+1$ events:

$$P\left(\bigcup_{i=1}^{n+1} A_i\right) = P\left(\bigcup_{i=1}^{n} A_i \cup A_{n+1}\right)$$

$$= P\left(\bigcup A_i\right) + P(A_{n+1}) - P\left(\left(\bigcup_{i=1}^{n} A_i\right) A_{n+1}\right)$$

by inclusion exclusion rule ($n=1$ case)

$$= P\left(\bigcup A_i\right) + P(A_{n+1}) - P\left(\bigcup_{i=1}^{n} (A_i A_{n+1})\right)$$

By the principle of mathematical induction we have shown the claim for all n.

$\square$

Named distribution

Uniform distribution on a finite set $\{x_1, \dots, x_n\}$:

Imagine you have numbers $x_1, \dots, x_n$ in a hat.

ex $\{1, 1, 2\}$ with x_1, x_2, x_3

Let X be a random draw of one of these numbers (i.e. $P(X = x_i) = \frac{1}{n}$ for all i)

ex $P(X=1) = P(X = x_1 \text{ or } x_2) = \frac{2}{3}$, $P(X=2) = \frac{1}{3}$

We write that $X \sim \text{Unif}(\{x_1, \dots, x_n\})$

ex Suppose (7) a (1) word (4) is (2) randomly (8) picked (6) from (4) this (4) sentance (8).

What is the distribution of the length of the word picked?

answ $\text{unif}(\{7, 1, 4, 2, 8, 6, 4, 4, 8\})$

tinyurl.com/august30-pt1

tinyurl.com/august30-pt2

Stat 134

1. A deck of cards is shuffled. What is the chance that the top card is the **king** of spades **or** the bottom card is the **king** of spades

a $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \times \frac{1}{52}$

b $\frac{1}{52} + \frac{1}{51}$

c $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \times \frac{1}{51}$

d none of the above

addition rule

$P(KS\ top) + P(KS\ bot)$

$= \frac{1}{52} + \frac{1}{52}$

It is not possible to have KS as top card and bottom card simultaneously, Hence use the addition formula.

not $\frac{1}{52} + \frac{1}{51}$. $P(KS\ bot) = \frac{1}{52}$ since this is an unconditional probability.

2. Two separate decks of cards are shuffled. What is the chance that the top card of the first deck is the **king** of spades **or** the bottom card of the second deck is the **king** of spades

P(KS top 1^{st} deck)

P(KS bot 2^{nd} deck

a $\frac{1}{52} + \frac{1}{52} - \underbrace{\frac{1}{52} \times \frac{1}{52}}_{P(\text{KS top } 1^{st} \text{ deck and KS bot } 2^{nd} \text{ deck})}$

b $\frac{1}{52} + \frac{1}{51}$

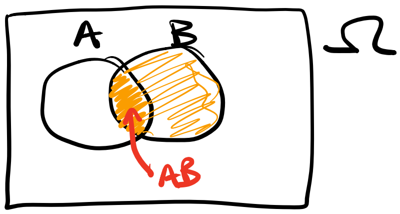
c $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \times \frac{1}{51}$

d none of the above

There are 2 seperate 52 card decks so it is possible to have KS as top of 1^{st} deck and bottom of 2^{nd} deck. Use inclusion exclusion formula.

Sec 1.4 Conditional Probability and Independence

Let A, B be subsets of Ω (i.e events).



given

Baye's rule says $P(A|B) = \dfrac{P(AB)}{P(B)}$

$\Leftrightarrow$ $\boxed{P(AB) = P(A|B)P(B)}$ multiplication rule. (AND)

A and B

We say A and B are <u>independent</u> if

$P(A|B) = P(A)$

or equivalently if $P(AB) = P(A)P(B)$

ex A = last card is queen of spades
B = 1^{st} card is king of spades

A and B are dependent

or $P(A|B)P(B)$

$$P(AB) = \underset{\frac{1}{52}}{\underset{\|}{P(B)}}\underset{\frac{1}{51}}{\underset{\|}{P(A|B)}} = \boxed{\frac{1}{52}\cdot\frac{1}{51}}$$

Sec 1.5 Baye's rule

ex A factory produces 2 models of cell phones.

Given $P(1) = .3$

$P(\text{good} \mid 1) = .8$

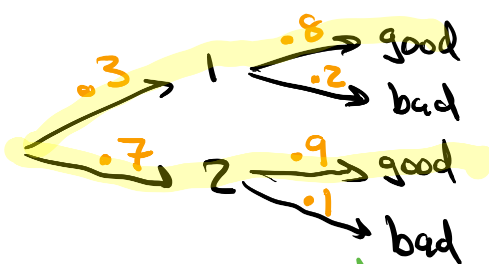
$P(\text{good} \mid 2) = .9$

$P(A \text{ and } B)$

$P(AB)$

$P(A,B)$

$P(A \cap B)$



Find $P(1, \text{good}) = (.3)(.8) = .24$ (and)

$P(\text{good}) = (.3)(.8) + (.7)(.9) = .87$

$P(1 \mid \text{good}) = \dfrac{P(1, \text{good})}{P(\text{good})} = \dfrac{.24}{.87} = .28$

Another way to write Bayes rule:

$$P(1 \mid \text{good}) = \frac{P(1, \text{good})}{P(\text{good})} = \frac{P(\text{good} \mid 1)P(1)}{P(\text{good} \mid 1)P(1) + P(\text{good} \mid 2)P(2)}$$

$P(1 \mid \text{good}) = .28$ ↑ posterior conditional probability

$P(1) = .3$ prior

$P(\text{good} \mid 1) = .8$ likelihood conditional probability