

Warmup 8:00 - 8:10 AM

Let  $(X, Y)$  be bivariate normal. Then  $(2X+3Y+4, 6X-Y-4)$  is bivariate normal.

- a** true
  - b** false
  - c** not enough info to decide

$$a(2x+3y+4) + b(6x-y-4) \\ = \underbrace{(2a+6b)x + (3a-4b)y + 4a-4b}_{\begin{array}{l} \text{normal since} \\ (x,y) \text{ is a unit normal} \end{array}} \Rightarrow (2x+3y+4, \\ 6x-y-4) \in \text{horizontal normal}$$

## Final review RRR week

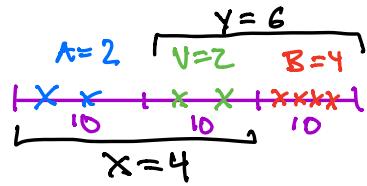
### Today

- ① Sec 6.5 Regression and bivariate normal (lec 39)
- ② Sec 4.2 Gamma / Poisson Thinning (lec 22)
- ③ Expectation / Variance w/ Indicators (lec 14)

Ex Toss a fair coin 30 times

$X = \# \text{ heads first } 20$

$Y = \# \text{ heads last } 20$



$$\left. \begin{array}{l} A \sim \text{Bin}(10, \frac{1}{2}) \\ V \sim \text{Bin}(10, \frac{1}{2}) \\ B \sim \text{Bin}(10, \frac{1}{2}) \end{array} \right\} \text{indep.} \quad \begin{array}{l} X = A + V \\ Y = V + B \end{array}$$

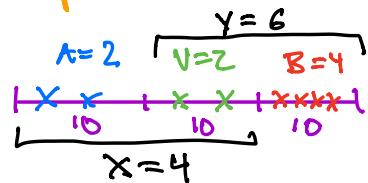
a) Find  $\text{Corr}(X, Y)$ ?

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\text{SD}(X)\text{SD}(Y)}$$

$$\begin{aligned} \text{Cov}(X, Y) &= \text{Cov}(A+V, V+B) = \text{Cov}(V, V) \\ &= \text{Var}(V) = 10 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{10}{4} \end{aligned}$$

$$\text{Cov}(X, Y) = \frac{\frac{10}{4}}{\frac{20}{4}} = \frac{1}{2}$$

$\text{SD}(X)\text{SD}(Y) = \sqrt{\frac{20}{4}} \cdot \sqrt{\frac{20}{4}}$



b) Find  $E(Y|X) = E(V+B|X)$

$$= E(V|X) + E(B|X)$$

$$E(B) = 10 \cdot \frac{1}{2} = 5$$

What distribution is  $V|X$ ?

hint use Bayes rule to find

$$\begin{aligned} P(V=v | X=x) &= \frac{P(V=v, X=x)}{P(X=x)} \\ &= \frac{P(V=v, A=x-v)}{P(X=x)} \\ &= \frac{\binom{20}{v} \left(\frac{1}{2}\right)^{10} \binom{10}{x-v} \left(\frac{1}{2}\right)^{10}}{\binom{20}{x} \left(\frac{1}{2}\right)^{20}} \end{aligned}$$

$$\Rightarrow V|X \sim \text{Hyper}(N=20, G=x, n=10)$$

$$E(V|X) = n \cdot \frac{G}{N} = 10 \cdot \frac{x}{20} = \frac{1}{2}x$$

$$E(Y|X) = E(V|X) + E(B|X) = \frac{1}{2}x + 5$$

Note that  $(X, Y)$  is approximately bivariate normal:

$$X \sim \text{Bin}(20, \frac{1}{2}) \approx N(10, 5)$$

$$Y \sim \text{Bin}(20, \frac{1}{2}) \approx N(10, 5)$$

$$\text{Corr}(X, Y) = \frac{1}{2}$$

$(X, Y) \sim BV(10, 10, 5, 5, \frac{1}{2})$  since

$$\begin{aligned} aX + bY &= a(A + V) + b(V + B) \\ &= aA + (a+b)V + bB \end{aligned}$$

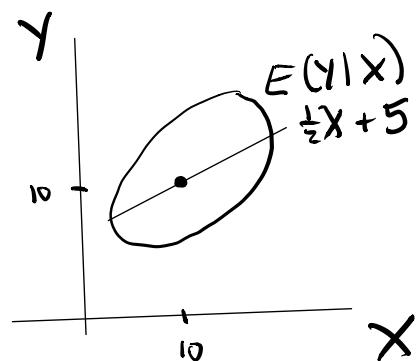
is normal since  $A, V, B$  are independent normals.

eqn of regression line  $E(Y|X)$ :

$$\frac{\hat{Y} - \mu_Y}{\sigma_Y} = \frac{1}{\sqrt{2}} \frac{X - \mu_X}{\sigma_X}$$

$$\frac{\hat{Y} - 10}{\sqrt{5}} = \frac{1}{\sqrt{2}} \left( \frac{X - 10}{\sqrt{5}} \right)$$

$$\boxed{\hat{Y} = \frac{1}{2}X + 5}$$

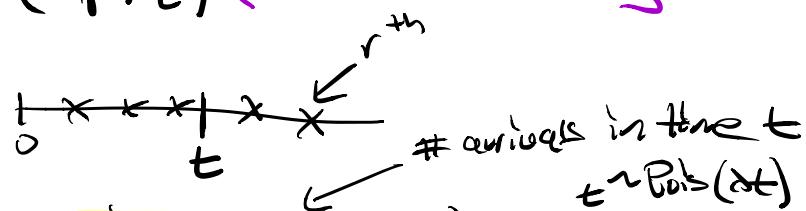


(2)

Sec 4.2 Gamma( $r, \lambda$ ) = distribution for  $r^{\text{th}}$  arrival time of a Poisson( $\lambda$ ) process.

$T_r = \text{arrival time of } r^{\text{th}} \text{ call.} \sim \text{Gamma}(r, \lambda)$

Find  $P(T_r > t)$  (right tail probability)



$$\begin{aligned} P(T_r > t) &= P(N_t \leq r-1) \\ &= P(N_t = 0) + P(N_t = 1) + \dots + P(N_t = r-1) \\ &= e^{-\lambda t} + e^{-\lambda t} \lambda t + \frac{e^{-\lambda t} (\lambda t)^2}{2!} + \dots + \frac{e^{-\lambda t} (\lambda t)^{r-1}}{(r-1)!} \end{aligned}$$

Ex

Let  $T \sim \text{Gamma}(r=4, \lambda=2)$

Find  $P(T > 7)$

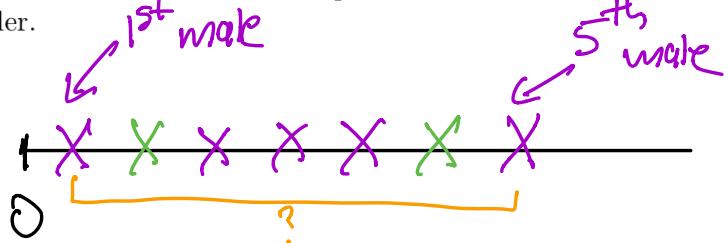
$$P(T > 7) = P(N_7 \leq 3)$$

where  $N_7 \sim \text{Pois}(14)$

$$P(T > 7) = \boxed{e^{-14} \left( 1 + \frac{14}{1!} + \frac{14^2}{2!} + \frac{14^3}{3!} \right)}$$

5. Travelers arrive at an airport Information desk according to a Poisson process at the rate of 15 per hour. Assume that each traveler arriving at the desk has a 60% chance of being male and a 40% chance of being female, independent of all other travelers.

- a) Fill in the blank with a number: The fifth male traveler is expected to arrive at the desk \_\_\_\_\_ minutes after the first male traveler.



$$N \sim \text{Pois}(15 \cdot \frac{1}{60}) \quad \text{male and female}$$

$$M \sim \text{Pois}\left(15 \cdot .6 \cdot \left(\frac{1}{60}\right)\right) = \text{Pois}\left(\frac{9}{60}\right) \quad \text{male}$$

$$F \sim \text{Pois}\left(15 \cdot .4 \cdot \left(\frac{1}{60}\right)\right) = \text{Pois}\left(\frac{6}{60}\right) \quad \text{female}$$

$$T_5 = \text{wait time for } 5^{\text{th}} \text{ male} \sim \text{Gamma}\left(5, \frac{9}{60}\right)$$

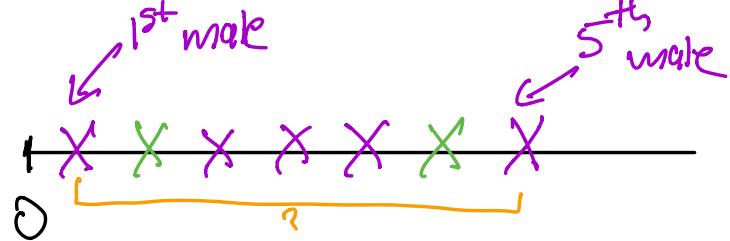
$$T_1 = \text{“ “ “ } 1^{\text{st}} \text{ male} \sim \text{Gamma}\left(1, \frac{9}{60}\right)$$

$$E(T_5 - T_1) = E(W_4) = \frac{4}{9/60} = \frac{4 \cdot 60}{9} = \boxed{\frac{80}{3} \text{ min}}$$

$\uparrow$   
Gamma  $(4, \frac{9}{60})$

5. Travelers arrive at an airport Information desk according to a Poisson process at the rate of 15 per hour. Assume that each traveler arriving at the desk has a 60% chance of being male and a 40% chance of being female, independent of all other travelers.

- b) Find the chance that the fifth male traveler arrives at the desk more than 30 minutes after the first male traveler.



$$N \sim \text{Pois} \left( 15 \cdot \frac{1}{60} \right) \text{ male and female}$$

$$M \sim \text{Pois} \left( 15 \cdot 0.6 \left( \frac{1}{60} \right) \right) = \text{Pois} \left( \frac{9}{60} \right) \text{ male}$$

$$F \sim \text{Pois} \left( 15 \cdot 0.4 \left( \frac{1}{60} \right) \right) = \text{Pois} \left( \frac{6}{60} \right) \text{ female}$$

let  $T_5$  = arrival time for 5<sup>th</sup> male

$T_1$  = arrival time 1<sup>st</sup> male

$$\begin{aligned} P(T_5 - T_1 > 30) &= P(T_4 > 30) \\ &= P(N_{30} \leq 3) \end{aligned}$$

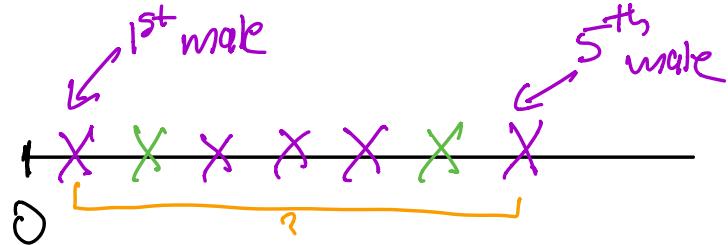
$$= P(N_{30} = 0) + P(N_{30} = 1) + P(N_{30} = 2) + P(N_{30} = 3)$$

$$N_{30} \sim \text{Pois} \left( \frac{9}{60} \cdot 30 \right) = \text{Pois} (4.5)$$

$$\Rightarrow P(N_{30} \leq 3) = e^{-4.5} \left( 1 + 4.5^1 + \frac{4.5^2}{2!} + \frac{4.5^3}{3!} \right)$$

5. Travelers arrive at an airport Information desk according to a Poisson process at the rate of 15 per hour. Assume that each traveler arriving at the desk has a 60% chance of being male and a 40% chance of being female, independent of all other travelers.

- c) Find the expected number of female travelers who arrive at the desk before the fifth male traveler.



$$N \sim \text{Pois}(15 \cdot \frac{1}{60}) \quad \text{male and female}$$

$$M \sim \text{Pois}\left(15 \cdot 0.6 \cdot \left(\frac{1}{60}\right)\right) = \text{Pois}\left(\frac{9}{60}\right) \quad \text{male}$$

$$F \sim \text{Pois}\left(15 \cdot 0.4 \cdot \left(\frac{1}{60}\right)\right) = \text{Pois}\left(\frac{6}{60}\right) \quad \text{female}$$

Recall  $X \sim \text{NegBin}(r, p)$  on  $0, 1, 2, \dots$   
 is the number of failures before the  
 $r^{\text{th}}$  success

$$E(X) = r \cdot \frac{1-p}{p}$$

Here  $X = \# \text{ female before the } 5^{\text{th}} \text{ male}$

$$X \sim \text{NegBin}(5, 0.6) \text{ on } 0, 1, 2, \dots$$

$$E(X) = 5 \left( \frac{1-0.6}{0.6} \right) = \boxed{10/3}$$

### ③ Expectation / Variance with Indicators.

6. A drawer contains  $s$  black socks and  $s$  white socks, where  $s$  is a positive integer. I pull two socks out at random without replacement and call that my first pair. Then I pull two socks out at random without replacement from the remaining socks in the drawer, and call that my second pair. I proceed in this way till I have  $s$  pairs and the drawer is empty.

Let  $D$  be the number of pairs in which the two socks are of different colors.

a) Find  $E(D)$

$$D = I_1 + \dots + I_s$$

$$I_2 = \begin{cases} 1 & \text{if 2nd pair diff color} \\ 0 & \text{otherwise} \end{cases}$$

$$E(D) = s \cdot \left( \frac{s}{2s-1} \right) = \boxed{\frac{s^2}{2s-1}}$$

any of  $2s$  poss. socks  
 $\frac{2s}{2s}, \frac{s}{2s-1}$   
 any of  $s$  poss. socks remaining

6. A drawer contains  $s$  black socks and  $s$  white socks, where  $s$  is a positive integer. I pull two socks out at random without replacement and call that my first pair. Then I pull two socks out at random without replacement from the remaining socks in the drawer, and call that my second pair. I proceed in this way till I have  $s$  pairs and the drawer is empty.

Let  $D$  be the number of pairs in which the two socks are of different colors.

b) Find  $\text{Var}(D)$ ,

$$D = I_1 + \dots + I_s$$

$$\text{Var}(D) = s \text{Var}(I_1) + s(s-1) \text{Cov}(I_1, I_2)$$

$$\text{Cov}(I_1, I_2) = E(I_{12}) - E(I_1)E(I_2)$$

$$\frac{s}{2s} \cdot \frac{s}{2s-1} \cdot \frac{2s-2}{2s-2} \cdot \frac{s-1}{2s-3}$$

$$= \frac{s}{2s-1} \cdot \frac{s-1}{2s-3}$$

$$I_{12} = \begin{cases} 1 & \text{if } 1^{\text{st}} \text{ and } 2^{\text{nd}} \text{ pair diff color} \\ 0 & \text{else} \end{cases}$$

$$s(s-1) \text{Cov}(I_1, I_2) = s(s-1) \left[ \frac{s}{2s-1} \cdot \frac{s-1}{2s-3} - \left( \frac{s}{2s-1} \right)^2 \right]$$

$$s \cdot \text{Var}(I_1) = s \cdot \left[ \left( \frac{s}{2s-1} \right) \left( 1 - \frac{s}{2s-1} \right) \right]$$

$$\text{Var}(D) = s \cdot \text{Var}(I_1) + s(s-1) \text{Cov}(I_1, I_2)$$

$I_1, \dots, I_s$

exchangeably,

since drawer

Socks w/o

replacement from  
drawer.

