Stat 134 lec 22

Warmup: 1:00-1:00

Let $X \sim Geom(p)$ with values $1, 2, 3, \ldots$

a) Find $P(X > 3)$ — q^3

b) Find $P(X > 13 \mid X > 10) = \frac{P(X>13, X>10)}{P(X>10)} = \frac{P(X>13)}{P(X>10)}$

$$= \frac{q^{13}}{q^{10}} = q^3$$

Says, the geom distribution is has no memory what happened before the present time. If you got 10 tails in a row, the chance you get more than 13 tails in a row is the same as getting more than 3 tails in a row in a new game.

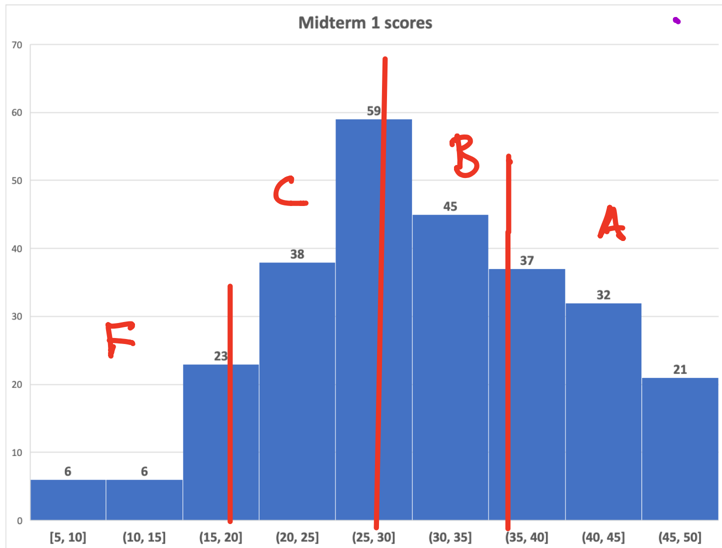
More generally,

$P(X > k+j \mid X > j) = P(X > k)$

This is called the memoryless property of Geom(p).

next Wednesday Q4 covers 4.1, 4.2

Great job on midterm!



Mean 31.4

SD 9.6

Final grade will be based on previous year cutoffs

last time sec 4.2 exponential distribution.

$T \sim \text{Exp}(\lambda)$ $\quad f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$ or $P(T > t) = e^{-\lambda t}$ survival function

T = time until 1st success (arrival)

Today sec 4.2

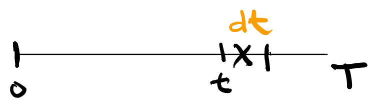
1. Memoryless property of Exponential
2. Expectation and Variance of $\text{Exp}(\lambda)$.
3. Gamma distribution.

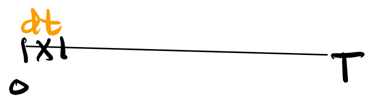
① sec 4.2 **Memoryless property of Exponential**

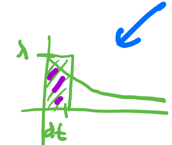
$T \sim \text{Exp}(\lambda)$ $\quad f(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$ $\quad$ or $\quad P(T > t) = e^{-\lambda t}$ ← survival function

T = time until 1^{st} success (arrival)

Memoryless Property : $P(T \in dt \mid T > t) = P(0 < T < 0 + dt)$







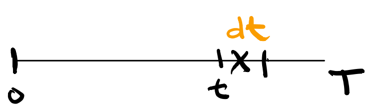
$$P(T \in dt \mid T > t) = \frac{P(T \in dt)}{P(T > t)} \approx \frac{\lambda \cancel{e^{-\lambda t}}\, dt}{\cancel{e^{-\lambda t}}} = \boxed{\lambda dt}$$

$\boxed{\lambda dt}$ = $P(0 < T < 0 + dt)$

conclusions :

① So long as a lightbulb is still functioning it is as good as new

② $\lambda = \dfrac{P(T \in dt \mid T > t)}{dt}$



instantaneous rate of success. This is always the same (not dependent on t).

Fact:

Only 2 distributions are memoryless:

For discrete $(X = 1, 2, 3, \ldots)$ – Geometric

For continuous $(T \geq 0)$ – Exponential

②

Sec 4.2 Expectation and Variance of Exp(λ)

$T \sim \text{Exp}(\lambda)$

$$E(T) = \int_0^\infty t f(t)\,dt = \lambda \int_0^\infty t e^{-\lambda t}\,dt = \boxed{\frac{1}{\lambda}}$$

see end of lec notes.

$$E(T^2) = \int_0^\infty t^2 f(t)\,dt = \lambda \int_0^\infty t^2 e^{-\lambda t}\,dt = \frac{2}{\lambda^2}$$

see end of lec notes.

$$\text{var}(T) = E(T^2) - E(T)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

Interpretation of λ:

$\lambda = \dfrac{1}{E(T)}$ (one over average arrival time)

This makes sense since the instantaneous rate of success is constant by the memoryless property of Exp(λ).

So if it takes on avg 5 min to have a success then $\lambda = 1/5$.

ex Let T = waiting time (min) until a blue Honda enters a toll gate starting at noon. The waiting time is on average 2 minutes. Find $P(T > 3)$

$E(T) = 2$ min $\Rightarrow \lambda = 1/2$

$T \sim \text{Exp}(1/2)$

$$P(T > 3) = e^{-\frac{1}{2}\cdot 3} = \boxed{e^{-3/2}}$$

(3) Gamma Distribution

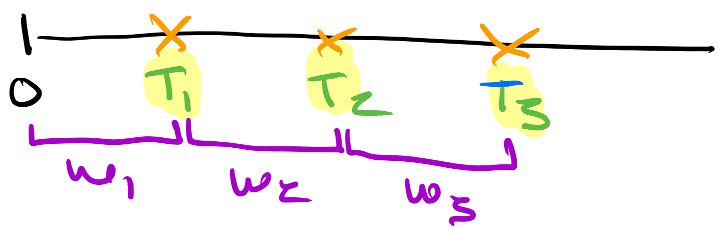
A gamma distribution, $T_r \sim \text{Gamma}(r, \lambda)$, $\lambda > 0$, is a sum of r iid $\text{Exp}(\lambda)$.

$$T_r = W_1 + W_2 + \dots + W_r \quad , \quad W_i \sim \text{Exp}(\lambda)$$

ex T_r = time of the r^{th} arrival of a Poisson Process ($\text{Pois}(\lambda t)$)

ex T_r = time when the r^{th} lightbulb dies.

Picture



$$\left.\begin{aligned} T_1 &\sim \text{Gamma}(1, \lambda) \\ T_2 &\sim \text{Gamma}(2, \lambda) \\ T_3 &\sim \text{Gamma}(3, \lambda) \end{aligned}\right\}$$

dependent but W_1, W_2, W_3 are independent, $\text{Exp}(\lambda)$

$$T_1 = W_1$$
$$T_2 = W_1 + W_2$$
$$T_3 = W_1 + W_2 + W_3$$

Review Poisson Process (a.k.a PRS)

$N_t \sim Pois(\lambda t)$

N_t = # arrivals in time t where the rate of arrivals is λ.

$$P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

ex Let T_1 = waiting time (min) until a blue Honda enters a toll gate starting at noon. The waiting time is on average 2 minutes. Find the probability 3 blue Hondas arrive in 8 minutes.

$N_8 \sim Pois\left(\frac{1}{2} \cdot 8\right) = Pois(4)$

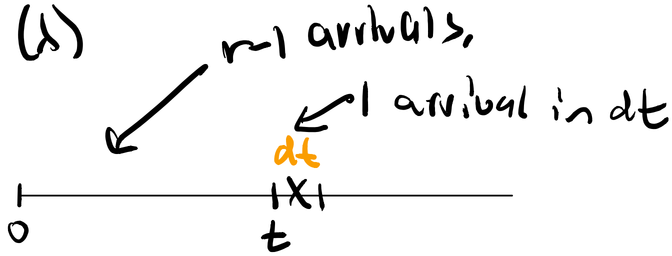
$$P(N_8 = 3) = \boxed{\frac{e^{-4} 4^3}{3!}}$$

Back to gamma distribution: $T_r \sim \text{Gamma}(r, \lambda)$

Let $N_t \sim \text{Pois}(\lambda t)$

$W \sim \text{Exp}(\lambda)$

$P(T_r \in dt)$ means



so $P(T_r \in dt) = P(N_t = r-1) P(W \in dt \mid W > t)$

$$\approx \frac{e^{-\lambda t} (\lambda t)^{r-1}}{(r-1)!} \cdot \lambda \, dt$$

$$= \underbrace{\underbrace{\frac{1}{(r-1)!}}_{= \frac{1}{\Gamma(r)}} \lambda^r t^{r-1} e^{-\lambda t}}_{f(t)} \, dt$$

<u>Gamma function $\Gamma(r)$, $r > 0$</u>

If $r \in \mathbb{Z}^+$ define $\Gamma(r) = (r-1)!$ where $\Gamma(r)$ is called <u>the gamma function</u>

For $\lambda = 1$, $f(t) = \underbrace{\frac{1}{\Gamma(r)}}_{\text{const}} \underbrace{t^{r-1} e^{-t}}_{\text{variable}}$

and $\int_0^\infty f(t)\,dt = 1 \Rightarrow$

$$\frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-t}\, dt = 1$$

$$\Rightarrow \Gamma(r) = \int_0^\infty t^{r-1} e^{-t}\, dt$$

This allows us to define

$T \sim \text{Gamma}(r, \lambda)$ for $r > 0$ as

having density $f(t) = \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}$

$t \geq 0$

ex Let $T_r \sim \text{Gamma}(r=4, \lambda=2)$

Find $f(t) = \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}$

$= \frac{1}{\Gamma(4)} 2^4 t^3 e^{-2t} = \boxed{\frac{8}{3} t^3 e^{-2t}}$

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The formula $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} \, dt$.

is the definition of the gamma function.

You can show that $\Gamma(r) = (r-1)!$

for $r \in \mathbb{Z}^+$

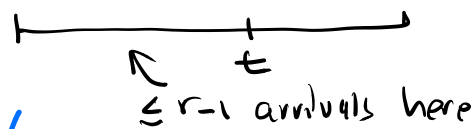
Now, $T_r \sim \text{Gamma}(r, \lambda)$ has density

$$f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}, & t \geq 0 \\ 0, & \text{else} \end{cases}$$

Let $T_r \sim \text{Gamma}(r, \lambda)$

$$P(T_r > t) = \int_t^\infty \frac{1}{\Gamma(r)} \lambda^r s^{r-1} e^{-\lambda s}\, ds$$

$N_t \sim \text{Pois}(\lambda t)$



$$P(T_r > t) = P(N_t < r) = P(N_t \leq r-1)$$

$$= \sum_{i=0}^{r-1} P(N_t = i)$$

$$= \boxed{\sum_{i=0}^{r-1} \frac{e^{-\lambda t}(\lambda t)^i}{i\,!}}$$

ex let $T_4 \sim \text{Gamma}(r=4, \lambda=2)$

Find $P(T_4 > 7)$

$$P(T_4 > 7) = P(N_7 < 4) = P(N_7 \leq 3)$$

$N_7 \sim \text{Pois}(2 \cdot 7)$

$$= e^{-14}\left(1 + \frac{14}{1!} + \frac{14^2}{2!} + \frac{14^3}{3!}\right)$$

Appendix

Expectation and Variance of exponential

let $T \sim Exp(\lambda)$

$f(t) = \lambda e^{-\lambda t}$ density

$$E(T) = \lambda \int_0^\infty t e^{-\lambda t} dt$$

I recommend using the tabular method for integration by parts. This works well when the function you are integrating is the product of two expressions, where the n^{th} derivative of one expression is zero.

ex $t^4 e^{3t}$ ✓ good

ex $\sin t \, e^{3t}$ ✗ bad.

To find $\lambda \int_0^\infty t e^{-3t}$:

$\frac{d}{dt}$		$\int$
t	$+$	$e^{-\lambda t}$
1	$-$	$-\frac{e^{-\lambda t}}{\lambda}$
0		$\frac{e^{-\lambda t}}{\lambda^2}$

$$E(T) = \lambda \int_0^\infty t e^{-\lambda t} dt = \lambda \left(t \left(-\frac{e^{-\lambda t}}{\lambda} \right) - 1 \cdot \frac{e^{-\lambda t}}{\lambda^2} \right) \Big|_0^\infty$$

$$= 0 - \left(0 - \frac{1}{\lambda}\right)$$

$$= \boxed{\frac{1}{\lambda}}$$

Next find $Var(T) = E(T^2) - E(T)^2$:

$$E(T^2) = \lambda \int_0^\infty t^2 e^{-\lambda t} dt$$

$\frac{d}{dt}$		$\int$
t^2	$+$	$e^{-\lambda t}$
$2t$	$-$	$-\frac{1}{\lambda} e^{-\lambda t}$
2	$+$	$\frac{1}{\lambda^2} e^{-\lambda t}$
0		$-\frac{1}{\lambda^3} e^{-\lambda t}$

$$E(T^2) = \lambda \left(t^2 \left(-\frac{1}{\lambda} e^{-\lambda t} \right) - 2t \left(\frac{1}{\lambda^2} e^{-\lambda t} \right) + 2 \left(-\frac{1}{\lambda^3} e^{-\lambda t} \right) \right) \Big|_0^\infty$$

$$= \frac{2}{\lambda^2}$$

$$\Rightarrow Var(T) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \boxed{\frac{1}{\lambda^2}}$$