Suppose we have n envelopes & n letters, each associated with a corresponding letter/envelope respectively. We randomly sort the letters into the envelopes.

a) $\mathbb{P}(\text{letter } i \text{ in correct envelope}) = \frac{1}{n}$ } Letter i equally likely to be in any envelope.

$\mathbb{P}(\text{letters } i,j \text{ correct}) = \mathbb{P}(i \text{ correct}) \cdot \mathbb{P}(j \text{ correct} \mid i \text{ correct})$

$= \frac{1}{n} \cdot \frac{1}{n-1}$ } Given i correct, j is equally likely to be in any of the remaining $n-1$ envelopes.

And so,

$\mathbb{P}(k \text{ specified letters all correct})$

$= \frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} \cdot \ldots \cdot \frac{1}{n-(k-1)} = \frac{(n-k)!}{n!}$ } Will be useful later.

b) $\mathbb{P}(\text{at least one letter correct}) =$

$\mathbb{P}\left(\bigcup_{i=1}^{n} \{\text{letter } i \text{ correct}\}\right)$

Inclusion-exclusion formula for n events.

$$= \sum_{i=1}^{n} \mathbb{P}(\text{letter } i \text{ correct}) - \sum_{1 \le i < j \le n} \mathbb{P}(\text{letters } i,j \text{ correct})$$

$$+ \sum_{1 \le i < j < k \le n} \mathbb{P}(\text{letters } i,j,k \text{ correct}) + \ldots + (-1)^{n+1} \mathbb{P}(\text{all letters correct})$$

$$= \sum_{i=1}^{n} \frac{1}{n} - \sum_{i<j} \frac{1}{n(n-1)} + \sum_{i<j<k} \frac{(n-3)!}{n!} + \dots + \frac{1}{n!}$$

$$= \binom{n}{1}\frac{1}{n} - \binom{n}{2}\frac{(n-2)!}{n!} + \binom{n}{3}\frac{(n-3)!}{n!} + \dots + \binom{n}{n}\frac{1}{n!}$$

All $\binom{n}{2}$ unique pairs of letters

Unique triplets of letters

$$= \frac{n}{n} - \frac{\cancel{n!}}{2!\,\cancel{(n-2)!}} \cdot \frac{\cancel{(n-2)!}}{\cancel{n!}} + \frac{\cancel{n!}}{3!\,\cancel{(n-3)!}} \cdot \frac{\cancel{(n-3)!}}{\cancel{n!}} + \dots + (-1)^{n+1}\frac{1}{n!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{(-1)^{n+1}}{n!}$$

$$\boxed{= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}}$$

c) To approximate this, we must do a trick:

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} - 1 + 1$$

$$= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^{n+1}}{n!}$$

$$= 1 - \sum_{k=0}^{n} \frac{(-1)^{n+1}}{n!} \sim \boxed{1 - e^{-1}} \text{ for large } n.$$