# Stat 134 lec 41

**Warmup 1:00-1:10**

The single and multivariate MGF is defined as:

$$M_Y(t) = E\left(e^{tY}\right)$$ single variable MGF

$$M_{(X,Y)}(s,t) = E\left(e^{sX+tY}\right)$$ multivariate MGF

Show that $M_{(X,Y)}(s,t) = M_X(s)\, M_Y(t)$ iff X, Y are independent.

$$M_{(X,Y)}(s,t) = E\left(e^{sX+tY}\right)$$ multivariate MGF

$$= E\left(e^{sX} \cdot e^{tY}\right)$$

iff X, Y indep.

$$\overset{\text{iff } X,Y \text{ indep.}}{=} E\left(e^{sX}\right)E\left(e^{tY}\right) = M_X(s)\, M_Y(t)$$

## last time

Please post questions in advance on b-courses / discussion for wednesday review

Sec 6.5. Bivariate Normal

Def$^n$ (Standard Bivariate Normal Distribution)

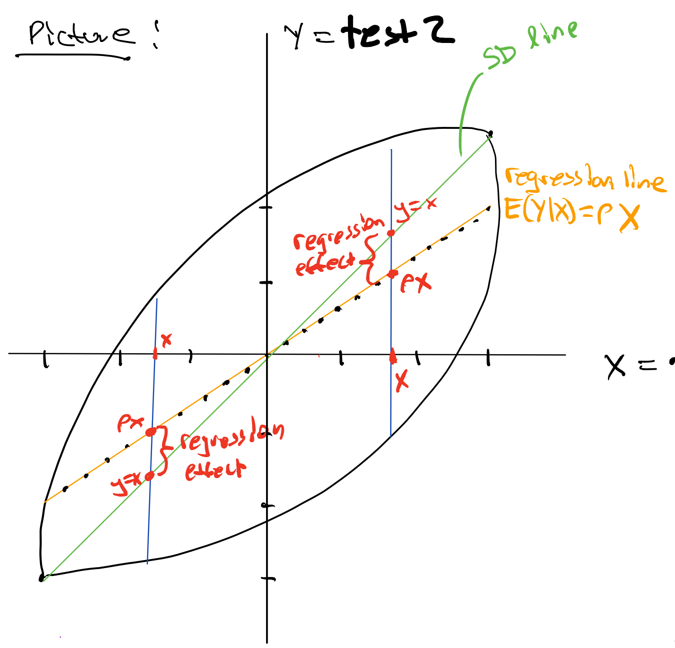
let $X, Z$ iid $N(0,1)$, $-1 \le \rho \le 1$

$$Y = \rho X + \sqrt{1-\rho^2}\, Z \sim N(0,1)$$

$$\text{Corr}(X,Y) = \rho$$

regression effect

Picture:




Regression effect,

Corr (test 1, test 2) = .6

If 1 SD above mean on test 1 then on average you will be less than 1 SD above average on test 2.

(regression line is less steep than SD line).

Today
1. MGF of bivariate normal
2. sec 6.5 properties of bivariate normal
3. sec 6.4 Total Variance Decomposition

① MGF of bivariate normal

**Thm** Let $(X,Y)$ be standard bivariate normal.

The MGF of $(X,Y)$ is

$$M_{(X,Y)}(s,t) = e^{\frac{s^2}{2} + \frac{t^2}{2} + st\rho}$$

Pf/
$$\begin{aligned}
M_{X,Y}(s,t) &= E\left[e^{sX+tY}\right] \\
&= E\left[e^{sX+t(\rho X+\sqrt{1-\rho^2}Z)}\right] \\
&= E\left[e^{(s+t\rho)X} \cdot e^{t\sqrt{1-\rho^2}Z}\right] \\
&\overset{\text{independence}}{=} E\left[e^{(s+t\rho)X}\right] E\left[e^{t\sqrt{1-\rho^2}Z}\right] \\
&= M_X(s+t\rho) \cdot M_Z(t\sqrt{1-\rho^2})
\end{aligned}$$

Recall that $X \sim N(0,1)$ so $M_X(a) = e^{\frac{a^2}{2}}$

Finish proof.

$$\begin{aligned}
&= e^{\frac{(s+t\rho)^2}{2}} \cdot e^{\frac{(t\sqrt{1-\rho^2})^2}{2}} \\
&= e^{\frac{s^2}{2} + st\rho + \frac{t^2\rho^2}{2}} \cdot e^{\frac{t^2}{2} - \frac{t^2\rho^2}{2}} \\
&= \boxed{e^{s^2/2 + t^2/2 + st\rho}}
\end{aligned}$$

$\square$

## ② Properties of Bivariate Normal

Recall that if 2 RVs $X, Y$ are independant, $Cov(X,Y)=0$

$\Rightarrow Corr(X,Y)=0$

However the converse isn't true in generall. ($Corr(X,Y)=0 \not\Rightarrow X,Y$ indep.)

Let $X \sim N(0,1)$

$Y = X^2$

X and Y are dependent

X and Y are uncorrelated.

we show $X, Y$ are uncorrelated.

$$Cov(X,Y) = \underset{\underset{E(X^3)}{\parallel}}{E(XY)} - \underset{\underset{0}{\parallel}}{E(X)}E(Y)$$

Find $E(X^3)$

$X \sim N(0,1)$

$M_X(t) = e^{t^2/2}$

$$M_X'''(t)\Big|_{t=0} = te^{t^2/2} + 2te^{t^2/2} + t^3 e^{t^2/2}\Big|_{t=0} = 0$$

$\Rightarrow E(X^3) = 0$

$\Rightarrow Corr(X,Y) = 0$

Thm If $(X,Y)$ is std bivariate normal then
$\rho = \text{Corr}(X,Y) = 0$ iff $X,Y$ are independent.

Pf/ From the warm up we know
for any joint distribution $(X,Y)$,
$M_{(X,Y)}(s,t) = m_X(s)\, m_Y(t)$ iff $X,Y$ are independent.

Since $(X,Y)$ is std bivariate normal,
$$M_{(X,Y)}(s,t) = e^{s^2/2 + t^2/2 + st\rho}$$

$X,Y$ indep $\Rightarrow$ $M_{(X,Y)}(s,t) = m_X(s)\, m_Y(t)$ $\Rightarrow \rho = 0$

$$e^{s^2/2 + t^2/2 + st\rho} = e^{s^2/2}\, e^{t^2/2}$$

Conversely if $\rho = 0$ $\quad e^{s^2/2 + t^2/2 + st\rho} = e^{s^2/2}\, e^{t^2/2} \Rightarrow X,Y$ indep.

Recall from lec 30 that the sum of independent normal random variables is normal.
What about dependent normal random variables?

recall,

$M_Y(t) = E(e^{tY}) = M_{tY}(1)$ single variable MGF

$M_{(X,Y)}(s,t) = E(e^{sX+tY}) = M_{sX+tY}(1)$ multivariable MGF

recall,

$Z \sim N(\mu, \sigma^2)$ iff $M_Z(a) = e^{\mu a} e^{\sigma^2 \frac{a^2}{2}}$

Thm Let $X, Y \sim N(0,1)$ and $\mathrm{corr}(X,Y) = \rho$.
$(X,Y)$ is std bivariate normal iff
$sX + tY$ is normal for all constants $s, t$.

Pf/

$\Rightarrow$: Suppose $X, Y$ std bivariate normal.
(i.e. $Y = \rho X + \sqrt{1-\rho^2}\, Z$ for $X, Z$ iid $N(0,1)$)

$$sX + tY = sX + t(\rho X + \sqrt{1-\rho^2}\, Z)$$
$$= (s + t\rho)X + t\sqrt{1-\rho^2}\, Z$$

is normal since $X, Z$ are independent normals,

$\Leftarrow$: Suppose $sX+tY$ is normal,

note $E(sX+tY) = sE(X) + tE(Y) = 0$

$Var(sX+tY) = Var(sX) + Var(tY) + \underbrace{2Cov(sX,tY)}_{= 2st\,Cov(X,Y)}$

$= s^2 + t^2 + 2st\rho$

$Cov(X,Y) = corr(X,Y) = \rho$

hence, $sX+tY \sim N(0, s^2+t^2+2st\rho)$

Recall,

For $Z \sim N(\mu, \sigma^2) \Rightarrow M_Z(a) = e^{\mu a} e^{\sigma^2 \frac{a^2}{2}}$

Then $M_{(X,Y)}(s,t) = M_{sX+tY}(1) = e^{(s^2+t^2+2st\rho)\frac{1^2}{2}}$

normal

$\sigma^2$

$= e^{\frac{s^2}{2} + \frac{t^2}{2} + st\rho}$ ← MGF of $(X,Y)$

$\Rightarrow (X,Y)$ is standard bivariate normal $\square$

We dont need to restrict ourselves to $X, Y \sim N(0,1)$

**Corollary** — proof at end of lecture

Let $U \sim N(\mu_U, \sigma_U^2)$
$V \sim N(\mu_V, \sigma_V^2)$
and $Corr(U,V) = \rho$

$(U,V) \sim BV(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$ iff $sU + tV$ is normal for all constants $s,t$.

ex

Let M be a students score on the midterm of a class and F the student's score on the final of the same class.

Suppose $(M, F) \sim BV(70, 65, 8^2, 10^2, 0.6)$.

$\mu_M$, $\mu_F$, $\sigma_M^2$, $\sigma_F^2$, $\rho$.

Find the chance that the student scores higher on the final than the midterm.

Hint $P(F > M) = P(F - M > 0)$

what distribution is $F-M$? — Normal!

(Since $(M,F)$ is Bivariate Normal)

$E(F-M) = 65 - 70 = -5$

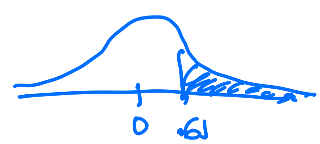
$Var(F-M) = Var(F) + Var(M) - 2\underbrace{Cov(F,M)}_{= Corr(F,M)\,SD(M)\,SD(F)}$

$= 10^2 + 8^2 - 2(.6)(8)(10) = 68$

$SD(F-M) = \sqrt{68}$

$\Rightarrow F - M \sim N(-5, 68)$

$\Rightarrow \dfrac{0 - E(F-M)}{SD(F-M)} = \dfrac{0-(-5)}{\sqrt{68}} = \boxed{.61}$




$P(F > M) = P(F - M > 0) = 1 - \Phi(.61) = .27$

# (3) Total Variance decomposition — Sec 6.4

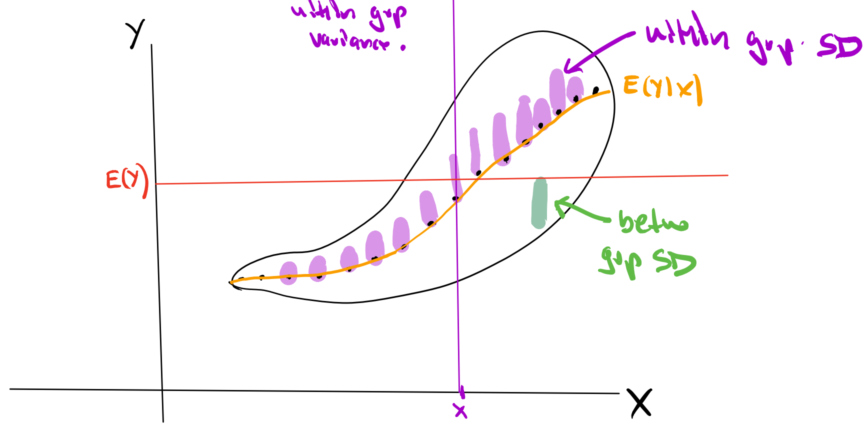
Properties (from lec 34)

1. $E(Y) = E(E(Y|X))$ iterated expectation
2. $E(aY+b|X) = aE(Y|X)+b$
3. $E(Y+Z|X) = E(Y|X)+E(Z|X)$
4. $E(g(X)|X) = g(X)$
5. $E(g(X)Y|X) = g(X)E(Y|X)$
6. $Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$ total variance decomposition

$Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$ (sec. 6.2.18)

tot var = avg$(\ )^2$ + $(\ )^2$ var of $E(Y|X)$ (between group variance)

within grp variance.



ex. Let $U \sim \text{Unif}(0,1)$

Given $U=u$, $X$ is exponential with mean $u$

i.e. $X|U=u \sim \text{Exp}\left(\frac{1}{u}\right)$

recall if $Y \sim \text{Exp}(\lambda)$

$E(Y)=\frac{1}{\lambda}$

$\text{Var}(Y)=\frac{1}{\lambda^2}$

$E(X|U)=U$

$\text{Var}(X|U)=U^2$

Find $E(X)$, $E(UX)$ and $\text{Var}(X)$.

$E(X)=E(E(X|U))=E(U)=1/2$

$E(UX)=E(E(UX|U))=E(U\underbrace{E(X|U)}_{=U})$

$=E(U^2)=\underbrace{\text{Var}(U)}_{=1/12}+\underbrace{E(U)^2}_{=(1/2)^2}=1/3$

$\text{Var}(X)=\underbrace{E(\underbrace{\text{Var}(X|U)}_{=U^2})}_{=1/3}+\underbrace{\text{Var}(\underbrace{E(X|U)}_{=U})}_{=1/12}=5/12$

## Appendix

### Corollary

Let $U \sim N(\mu_U, \sigma_U^2)$

$V \sim N(\mu_V, \sigma_V^2)$

and $\mathrm{Corr}(U,V) = \rho$

$(U,V) \sim BV(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$ iff

$sU + tV$ is normal for all constants $s, t$.

Pf/ let $X = \dfrac{U-\mu_U}{\sigma_U}$ $\quad (U = \mu_U + X\sigma_U)$

$Y = \dfrac{V-\mu_V}{\sigma_V}$ $\quad (V = \mu_V + Y\sigma_V)$

$(U,V) \sim BV(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$

iff $(X,Y) \sim BV(0, 0, 1, 1, \rho)$

$$sU + tV = s(\mu_U + X\sigma_U) + t(\mu_V + Y\sigma_V)$$
$$= \underbrace{s\sigma_U}_{=a} X + \underbrace{t\sigma_V}_{=b} Y + \underbrace{s\mu_U + t\mu_V}_{\text{constant}}$$

this is normal for all constants $s, t$

iff $aX + bY$ is normal for all constants $a, b$.

$\square$