*Stat 134: Conditional Probabilities, Distributions, & Expectations Review*

*Adam Lucas*

*December 11th, 2019*

*Problem 1*

Let $X_1 \sim$ Geom $(p_1)$, $X_2 \sim$ Geom $(p_2)$, $X_1 \perp X_2$, both on $\{1,2,\ldots\}$. Find:

a. $P(X_1 \leq X_2)$;

b. $P(X_1 = x \mid X_1 \leq X_2)$. Recognize $X_1 | X_1 \leq X_2$ as a named distribution, and state the parameter(s).

a) $P(X_1 = x, X_2 \geq x) = q_1^{x-1} p_1 \, q_2^{x-1} = p_1 (q_1 q_2)^{x-1}$

$P(X_1 \leq X_2) = \sum_{x=1}^{\infty} P(X_1 = x, X_2 \geq x) = \sum_{x=1}^{\infty} p_1 (q_1 q_2)^{x-1}$

$= \frac{p_1}{1 - q_1 q_2}$

b) $P(X_1 = x \mid X_1 \leq X_2) = \frac{P(X_1 = x, X_1 \leq X_2)}{P(X_1 \leq X_2)} = \frac{p_1 (q_1 q_2)^{x-1}}{\frac{p_1}{1-q_1 q_2}} = (q_1 q_2)^{x-1} (1 - q_1 q_2)$

$\hookrightarrow$ Geom$(1 - q_1 q_2)$

*Problem 2*

Let $Y \sim$ Beta $(r,s)$. Conditioned on $Y = y$, let $X \sim$ Geometric $(y)$ on $\{0,1,2,\ldots\}$ For simplicity, assume $r,s > 1$.

a. Find $\mathbb{E}(X)$.

b. $P(X = x, Y \in dy)$

c. Find $P(X = x)$, for $x \in \{0,1,2,\ldots\}$.

a) $E(X) = E(E(Y|X)) = E\left(\frac{1-Y}{Y}\right) = \int_0^1 \left(\frac{1-y}{y}\right) \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} y^{r-1} (1-y)^{s-1} \, dy$

$= \frac{s}{r-1}$

b) $P(X = x, Y \in dy) = P(Y \in dy) \, P(X = x \mid Y = y)$

$= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} y^{r-1} (1-y)^{s-1} \, dy \quad (1-y)^x y$

c) $P(X = x) = \int_0^1 P(X = x, Y \in dy) = \frac{\Gamma(r+s)\,\Gamma(r+1)\,\Gamma(s+x)}{\Gamma(r)\,\Gamma(s)\,\Gamma(r+s+x+1)}$

*Problem 3*

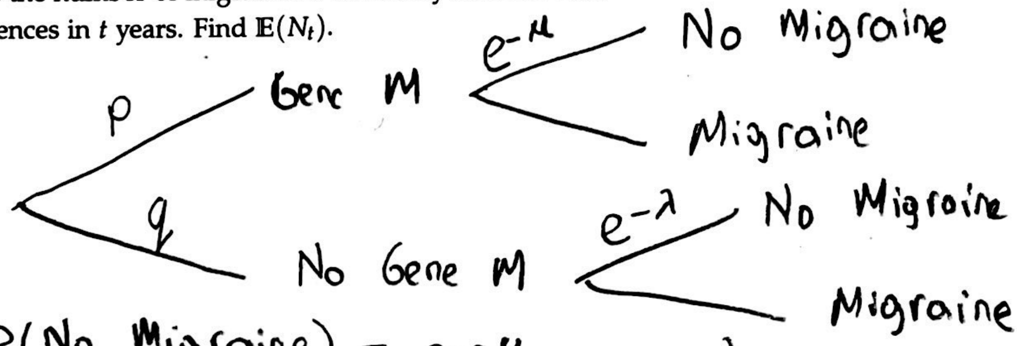
Suppose a proportion $p$ of a population has a gene $m$ that makes them predisposed to migraines. Of these people, the number of migraines they experience in a year follows a Poisson process with rate $\mu$ per year, whereas the rest of the population experiences migraines according to a Poisson process with rate $\lambda$.

a. What is the probability that a randomly selected individual experiences no migraines in a given year?

Hint: Condition on whether the individual has gene $m$.

b. Let $N_t$ denote the number of migraines a randomly selected individual experiences in $t$ years. Find $\mathbb{E}(N_t)$.

a)




$P(\text{No Migraine}) = pe^{-\mu} + qe^{-\lambda}$

b) Let $I$ be indicator that the person has gene $m$

$E[N_t] = P(I=1)\,E(N_t \mid I=1) + P(I=0)\,E(N_t \mid I=0)$

$= p\mu t + q\lambda t$

*Problem 4*

Let $X, Y$ have joint density $f_{X,Y}(x,y) = 2\lambda^2 e^{-\lambda(x+y)}, 0 < x < y$. It can be shown that $f_X(x) = 2\lambda e^{-2\lambda x}, x > 0$. Find:

a. The conditional density of $Y$, given $X = x$;

b. $\mathbb{E}(Y|X = x)$.

a) $f_{Y|X}(y) = \dfrac{f(x,y)}{f(x)} = \dfrac{2\lambda^2 e^{-\lambda(x+y)}}{2\lambda e^{-2\lambda x}} = \lambda e^{-\lambda(y-x)}$

on $x < y < \infty$

b) $E(Y|X=x) = \int_x^\infty y f_{Y|X}(y)\,dy$

$= \int_x^\infty y\lambda e^{-\lambda(y-x)}\,dy = x + \frac{1}{\lambda}$