Stat 134 lec 17

For Wednesday review, write down questions in discussion board on b-courses.

Last time sec 3.5 Poisson distribution

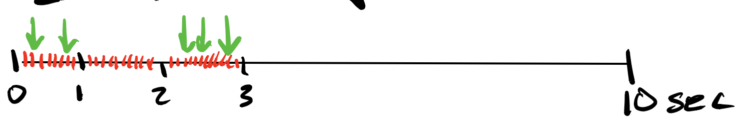
$X \sim Pois(\mu)$

$P(X=k) = \frac{e^{-\mu}\mu^k}{k!}$

$E(X) = Var(X) = \mu.$

Poisson Process or Poisson Random Scatter (PRS):

ex radioactive decay of Americium 241 in 10 seconds




Assumptions

① no two particles arrive at the same time. (this allows us to divide 10 sec into n small time intervals each with at most one arrival.

② X is a sum of n ← large iid Bernoulli (p) trials.
↖ small

$\mu = np$ is avg # of arrivals in 10 sec.

$\lambda = \mu/10$ is the arrival rate per second.

X = # arrivals in 10 seconds.

Suppose $\lambda = 4$ arrivals/sec

then $\mu = \lambda \cdot 10 = 40 \Rightarrow X \sim Pois(40)$

Americium has a long half life.

Y = # arrivals in 12070 sec.

$Y \sim Pois(\underset{\nwarrow 4}{\lambda} \cdot 12070)$

Today ① go over student comments of concept test

② Finish sec 3.5 Poisson Thinning

③ Start midterm review.

① Responses

1. Which of the following can be modeled as a Poisson Random Scatter with intensity $\lambda > 0$?

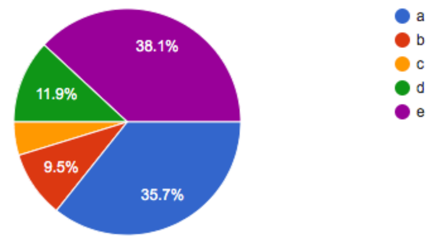
   a The number of blueberries in a 3 cubic inch blueberry muffin

   **b** The number of patients entering a doctor's office in a 24 hour period.

   **c** The number of times a day a person feels hungry

   **d** The number of air pulses counted every second from cars driving over an empty rubber hose lying across a highway between noon and 1pm.

   **e** more than one of the above




| | |
|---|---|
| a | B isnt possible since the distribution would not be iid over a 24 hour period (who goes to the doctor at 4 am). C isnt possible since the trials are not independent. D isnt possible since you could feasibly have two or more cars run over the hose in one second A fits all of the criteria of PRS |

No. Can make shorter time interval so no two pulses in same time interval

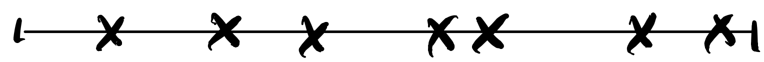
| | |
|---|---|
| a | The number of blueberries in a cubic muffin would apply because we are looking at the volume as a distribution of trials (analogous to time). |

| | |
|---|---|
| a | The distribution is not fully random for patients entering a doctor's office over a 24 hour period and hunger pangs. Air pulses come in pairs which doesn't match poisson scatter unless you count pairs of hits. |

## ② Sec 3.5 Poisson thinning

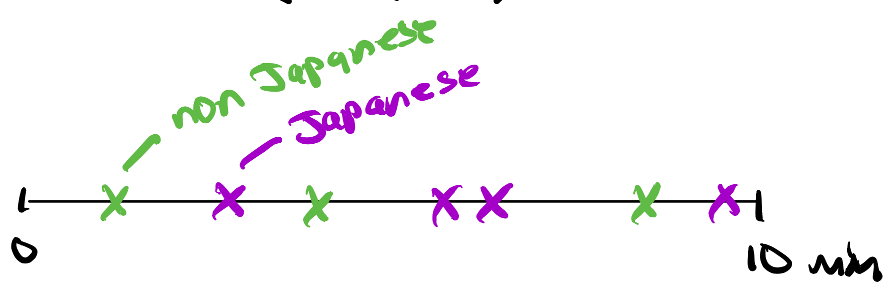
Cars arrive at a toll booth according to a Poisson process at a rate $\lambda = 3$ arrivals/min

$X$ = # cars arriving at a toll booth in 10 min. $X \sim \text{Pois}(\underbrace{\lambda \cdot 10}_{30})$



0 10 min

Of cars arriving, it is known, over the long term, that 60% are Japanese imports.




Call Japanese cars a success and non Japanese a failure.

$$\#\text{ cars} \sim \text{Pois}(\lambda \cdot 10) = \text{Pois}(30)$$

$$\#\text{ Japanese imports} \sim \text{Pois}(p\lambda \cdot 10) = \text{Pois}(18)$$

$$\#\text{ non Japanese} \sim \text{Pois}(q\lambda \cdot 10) = \text{Pois}(12)$$

Ex What is the probability that in a given 10 min interval, 15 cars arrive at the booth and 10 are Japanese imports?

$X$ = # cars in 10 min $\sim Pois(\underset{\lambda t}{30})$

$J$ = # Japanese cars in 10 min $\sim Pois(18)$

$nJ$ = # nonJapanese $\sim Pois(12)$

indep

$$P(X=15, J=10) = P(nJ=5, J=10)$$

$$= P(nJ=5)P(J=10)$$

$$= \frac{e^{-12}\,12^{5}}{5!} \cdot \frac{e^{-18}\,18^{10}}{10!}$$

③ midterm review

Which distributions are related to Bernoulli $(p)$ ?

## Discrete

| | name and range | $P(k) = P(X = k)$ for $k \in$ range | mean | variance |
|---|---|---|---|---|
| | uniform on $\{a, a+1, \ldots, b\}$ $\{1,2,\ldots,n\}$ | $\frac{1}{b-a+1}$ | $\frac{a+b}{2}$ $\frac{n+1}{2}$ | $\frac{(b-a+1)^2-1}{12}$ $\frac{n^2-1}{12}$ |
| ✓ | Bernoulli $(p)$ on $\{0, 1\}$ | $P(1) = p$; $P(0) = 1-p$ | $p$ | $p(1-p)$ |
| ✓ | binomial $(n, p)$ on $\{0, 1, \ldots, n\}$ | $\binom{n}{k} p^k (1-p)^{n-k}$ | $np$ | $np(1-p)$ |
| ✓ | Poisson $(\mu)$ on $\{0, 1, 2, \ldots\}$ | $\frac{e^{-\mu}\mu^k}{k!}$ | $\mu$ | $\mu$ |
| ✓ | hypergeometric $(n, N, G)$ on $\{0, \ldots, n\}$ | $\frac{\binom{G}{k}\binom{N-G}{n-k}}{\binom{N}{n}}$ | $\frac{nG}{N}$ | $n\left(\frac{G}{N}\right)\left(\frac{N-G}{N}\right)\left(\frac{N-n}{N-1}\right)$ |
| ✓ | geometric $(p)$ on $\{1, 2, 3 \ldots\}$ | $(1-p)^{k-1}p$ | $\frac{1}{p}$ | $\frac{1-p}{p^2}$ |
| ✓ | geometric $(p)$ on $\{0, 1, 2 \ldots\}$ | $(1-p)^k p$ | $\frac{1-p}{p}$ | $\frac{1-p}{p^2}$ |
| ✓ | negative binomial $(r, p)$ on $\{0, 1, 2, \ldots\}$ | $\binom{k+r-1}{r-1} p^r (1-p)^k$ | $\frac{r(1-p)}{p}$ | $\frac{r(1-p)}{p^2}$ |
| ✓ | normal | $\Phi(x)$ | $\mu$ | $\sigma$ |

For Poisson and normal we are thinking of them as a limit of binomial.

**8.** Let $X_1$ and $X_2$ be independent random variables such that for $i = 1, 2$, the distribution of $X_i$ is Poisson $(\mu_i)$. Let $m$ be a fixed positive integer. Find the distribution of $X_1$ given that $X_1 + X_2 = m$. Recognize this distribution as one of the famous ones, and provide its name and parameters.

$$\left.\begin{aligned} X_1 &\sim \text{Pois}(\mu_1) \\ X_2 &\sim \text{Pois}(\mu_2) \end{aligned}\right\} \text{ indep}$$

$X_1 \mid X_1 + X_2 = m$ takes values $0, 1, 2, \ldots, m$

$$P(X_1 = k \mid X_1 + X_2 = m) = \frac{P(X_1 = k, X_2 = m-k)}{P(X_1 + X_2 = m)}$$

$$= \frac{P(X_1 = k) P(X_2 = m-k)}{P(X_1 + X_2 = m)}$$

$$= \frac{\frac{\cancel{e^{-\mu_1}} \mu_1^k}{k!} \cdot \frac{\cancel{e^{-\mu_2}} \mu_2^{m-k}}{(m-k)!}}{\frac{\cancel{e^{-(\mu_1+\mu_2)}} (\mu_1+\mu_2)^m}{m!}}$$

$$= \binom{m}{k} \left(\frac{\mu_1}{\mu_1+\mu_2}\right)^k \left(\frac{\mu_2}{\mu_1+\mu_2}\right)^{m-k}$$

$$\Rightarrow X_1 \mid X_1 + X_2 = m \sim \text{Bin}\left(m, \frac{\mu_1}{\mu_1+\mu_2}\right)$$

ex

**8.** Let $m$ be a positive integer and let $v$ and $w$ be two probabilities strictly between 0 and 1. Let $X$ have the binomial $(m, v)$ distribution. Let $Y$ be independent of $X$ and suppose the distribution of $Y$ is given by $P(Y=j) = (1-w)^j w$ for $j \geq 0$. Fill in the blank with a formula in terms of $v$ and $w$; show your work below.

$$P(X=Y) = w(\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)^m$$

$$P(X=Y) = P(X=0, Y=0) + P(X=1, Y=1) + \cdots + P(X=m, Y=m)$$

$$= P(X=0)P(Y=0) + P(X=1)P(Y=1) + \cdots + P(X=m)P(Y=m)$$

$$P(X=j) = \binom{m}{j} v^j (1-v)^{m-j}$$

$$P(Y=j) = (1-w)^j w$$

$$P(X=Y) = \binom{m}{0}(1-v)^m w + \binom{m}{1} v (1-v)^{m-1} w + \binom{m}{2} v^2 (1-v)^{m-2} (1-w)^2 w + \cdots + \binom{m}{m} v^m (1-w)^m w$$

$$= w\left[\binom{m}{0}(1-v)^m + \binom{m}{1} v (1-v)^{m-1}(1-w)^1 + \binom{m}{2} v^2 (1-v)^{m-2}(1-w)^2 + \cdots + \binom{m}{m} v^m (1-w)^m\right]$$

$$= w\left[\binom{m}{0}\underbrace{(1-v)}_{=a}{}^m \underbrace{[(1-w)v]}_{=b}{}^0 + \binom{m}{1}(1-v)^{m-1}[(1-w)v]^2 + \binom{m}{2}(1-v)^{m-2}[(1-w)v]^3 + \cdots + \binom{m}{m}(1-v)^0[(1-w)v]^m\right]$$

$$= w\left[(1-v) + (1-w)v\right]^m \text{ by binomial theorem}$$

recall

$$(a+b)^m = \binom{m}{0} a^0 b^m + \binom{m}{1} a^1 b^{m-1} + \cdots + \binom{m}{m} a^m b^0$$

$p = \frac{1}{10}$

**1. 10 people throw their hats into a box and randomly redistribute the hats among themselves. Assume every permutation of the hats is equally likely. Let $N$ be the number of people who get their own hats back. Find the following:**

$I_2 = \begin{cases} 1 & 2^{nd} \text{ person gets his own hat back} \\ 0 & \text{else} \end{cases}$

**(a)** $\mathbb{E}[N^2]$

$N = I_1 + I_2 + \dots + I_{10}$

chance $1^{st}$ two people get hat back $= \frac{1}{10} \cdot \frac{1}{9}$

$E(N^2) = E((I_1 + \dots + I_{10})^2) = 10 \cdot p + 10 \cdot 9 \cdot p_{12}$

$= 1 + 1 = 2$

**(b)** $P(N = 8)$

c) $P(N=0)$