Stat 134 lec 33

warmup 1:00-1:10

Let $X, Y \overset{iid}{\sim} Exp(1)$. $Z = \frac{Y}{X}$

Find $f_Z(z)$.

Hint: use convolution formula

$$f_Z(z) = \int_{x=0}^{\infty} \underbrace{f_X(x)}_{e^{-x}} \underbrace{f_Y(zx)}_{e^{-zx}} x \, dx$$

$X \sim Gamma(r, \lambda)$

$f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$

$$= \int_0^{\infty} e^{-x} e^{-zx} x \, dx$$

$$= \int_0^{\infty} \underbrace{x e^{-(1+z)x}}_{\text{variable part of Gamma } (r=2,\ \lambda=1+z)} dx$$

$$= \frac{\Gamma(2)}{(1+z)^2} = \boxed{\frac{1}{(1+z)^2} \text{ for } 0 < z < \infty}$$

last time

Next week:
M: Sec 6.2
W: review
F: midterm 2 — Chap 4 (not 4.3), 5.1 – 5.4, and MGF

Uniform spacing (see #13 p 355)

You randomly throw n dart at $[0,1]$.

For $0 < a < a+k \le n$, $U_{(a+k)} - U_{(a)} \stackrel{d}{=} U_{(k)\text{ out of } n} \sim \boxed{Beta(k, n-k+1)}$

sec 5.4 convolution

$X>0, Y>0$

let $Z = \frac{Y}{X}$, then $f_Z(z) = \int_{x=0}^{x=\infty} f(x, zx)\, x \, dx \stackrel{\text{if } X,Y \text{ indep.}}{=} \int_{x=0}^{x=\infty} f_X(x) f_Y(zx)\, x\, dx$ convolution formula.

ex $X, Y \sim U(0,1)$, $Z = \frac{Y}{X}$
Find $f_Z(z)$

Cases

$0 \le z < 1$: $f_Z(z) = \int_0^1 f_X(x) f_Y(zx)\, x\, dx = \int_0^1 x\, dx = \boxed{\tfrac{1}{2} \text{ for } 0 < z < 1}$

since $f_X(x) = 0$ for $x > 1$

$z \ge 1$ (ex suppose $z = 100$)

$f_Z(100) = \int_0^1 f_X(x) f_Y(100x)\, x\, dx = \int_0^{1/100} f_X(x) f_Y(100x)\, x\, dx$

x must be less than $\frac{1}{100}$

$= \int_0^{1/100} 1 \cdot 1 \cdot x\, dx = \frac{1}{2}\left(\frac{1}{100}\right)^2$

(more generally $f_Z(z) = \int_0^{1/z} f_X(x) f_Y(xz)\, x\, dx = \boxed{\frac{1}{2z^2} \text{ for } z \ge 1}$

Today

1. Uniform spacing continued
2. Sec 6.1 Conditional Distribution: Discrete case.
3. Sec 6.2 Conditional expectation: $E(T \mid X = x)$

① Uniform spacing continued

let $U_1, \ldots, U_{10} \overset{iid}{\sim} \text{Unif}(0,1)$

and $U_{(1)}, \ldots, U_{(10)}$ be ordered standard uniform.

let $Y \sim U_{(7)}$, $X \sim U_{(9)}$

Let $Z = \frac{Y}{X}$

What distribution is Z?

$$0 = \frac{0}{U_{(9)}}, \frac{U_{(1)}}{U_{(9)}}, \frac{U_{(2)}}{U_{(9)}}, \ldots, \frac{U_{(9)}}{U_{(9)}} = 1$$

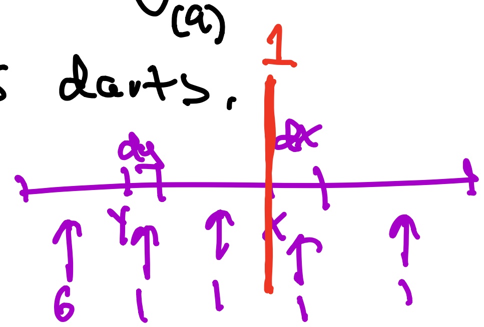
Since $U_{(9)}$ is constant $\frac{U_{(1)}}{U_{(9)}}, \ldots, \frac{U_{(8)}}{U_{(9)}}$ is

an ordering of 8 points between 0 and 1.

So $Z = \frac{Y}{X} = \frac{U_{(7)}}{U_{(9)}}$ is the 7^{th} order statistic

out of 8 darts.

Picture



7^{th} ordered uniform out of 8 darts before new 1 ex

Throw down 5 darts on $(0,1)$.

$X = U_{(2)}$ $Y = U_{(4)}$

Find $P(Y > 4X)$

$$\frac{X}{Y} = \frac{U_{(2)} \text{ out of } 5}{U_{(4)} \text{ out of } 5} = U_{(2)} \text{ out of } 3 \sim \text{Beta}(2,2)$$

$$\begin{array}{cc} k & n-k+1 \\ \| & \| \quad \| \\ 2 & 3 \quad 2 \end{array}$$

$$P(Y > 4X) = P\left(\underbrace{\frac{X}{Y}}_{Z} < \frac{1}{4}\right)$$

$Z \sim \text{Beta}(2,2)$

$$f_Z(z) = \frac{\Gamma(4)}{\Gamma(2)\Gamma(2)} z(1-z) = 6z(1-z)$$

$$P(Z < 1/4) = 6\int_0^{1/4} z\,dz - 6\int_0^{1/4} z^2\,dz = \boxed{10/64}$$

ex

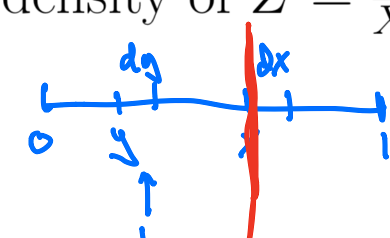
Let $Y \sim U_{(1)}$ and $X \sim U_{(2)}$ for 10 iid U(0,1). The variable part of the joint density is $(1-x)^8$. The density of $Z = \frac{Y}{X}$ is:

a $1/(2z)$

b 1 for $0<z<1$!

c $1/(2z^2)$

d none of the above



$Z = U_{(1)}$ out of 1 $= U(0,1)$

w/ density 1

② Sec 6.1 Conditional Distribution: Discrete case.

let X, N discrete RVs w/ joint distribution $P(X=x, N=n)$.

Bayes rule

$$P(X=x \mid N=n) = \frac{P(X=x, N=n)}{P(N=n)}$$

$$\Rightarrow \quad P(X=x, N=n) = P(X=x \mid N=n)P(N=n)$$

Rule of average conditional probabilities

marginal prob of X

$$P(X=x) = \sum_n P(X=x, N=n)$$

$$= \sum_n P(X=x \mid N=n)P(N=n)$$

ex

buses in 1 min

Let N have Poisson (λ) distribution. Let X be a random variable with the following property: for every n, the conditional distribution of X given $(N = n)$ is binomial (n, p). Find the unconditional distribution of X and state its parameter(s). Show all your work for full credit.

red buses in min

$(X \sim \text{Pois}(p\lambda))$

$$P(N=n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

$$P(X=x \mid N=n) = \frac{n!}{x!(n-x)!}p^x q^{n-x}$$

Find $P(X=x)$

$$P(X=x) = \sum_{n=x}^{\infty} P(X=x \mid N=n)P(N=n)$$

$$= \sum_{n=x}^{\infty} \frac{\cancel{n!}}{x!(n-x)!}p^x q^{n-x} \frac{e^{-\lambda}\lambda^n}{\cancel{n!}}$$

$\lambda^x \cdot \lambda^{n-x}$

$$= \frac{e^{-\lambda}\lambda^x p^x}{x!} \boxed{\sum_{n=x}^{\infty} \frac{\lambda^{n-x} q^{n-x}}{(n-x)!}}$$

Finish

$$\frac{e^{-\lambda}\lambda^x p^x}{x!}\underbrace{\left(1+\lambda q+\frac{(\lambda q)^2}{2!}+\cdots\right)}_{\substack{\shortparallel \\ e^{\lambda q}}}$$

$$= \frac{e^{-\lambda\overbrace{(1-q)}^{p}}(\lambda p)^x}{x!} \Rightarrow X \sim \mathrm{Pois}(\lambda p) \checkmark$$

③ **Sec 6.2** Conditional Expectation (discrete case)

Bayes rule:

recall $P(T=t \mid S=s) = \dfrac{P(T=t, S=s)}{P(S=s)}$

ex (T,S) is joint distribution below,

Find $P(T=3 \mid S=7)$

$= \dfrac{P(T=3, S=7)}{P(S=7)} = \dfrac{.3}{.4} = \boxed{.75}$

	T=3	T=4	Sum ← marginal of S
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1.0

marginal of T

Find $P(T=4 \mid S=7)$

$\dfrac{P(T=4, S=7)}{P(S=7)} = \dfrac{.1}{.4} = .25$

Find $E(T \mid S=7)$

$\sum_{t \in T} t\,P(T=t \mid S=7) = 3P(T=3 \mid S=7) + 4P(T=4 \mid S=7)$

$= 3(.75) + 4(.25) = 3.25$

Find $E(T \mid S=6)$

$3P(T=3 \mid S=6) + 4 \cdot P(T=4 \mid S=6)$

$3\left(\frac{.2}{.4}\right) + 4\left(\frac{.2}{.4}\right) = 3.5$

	T=3	T=4	Sum
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1.0

marginal of S

marginal of T

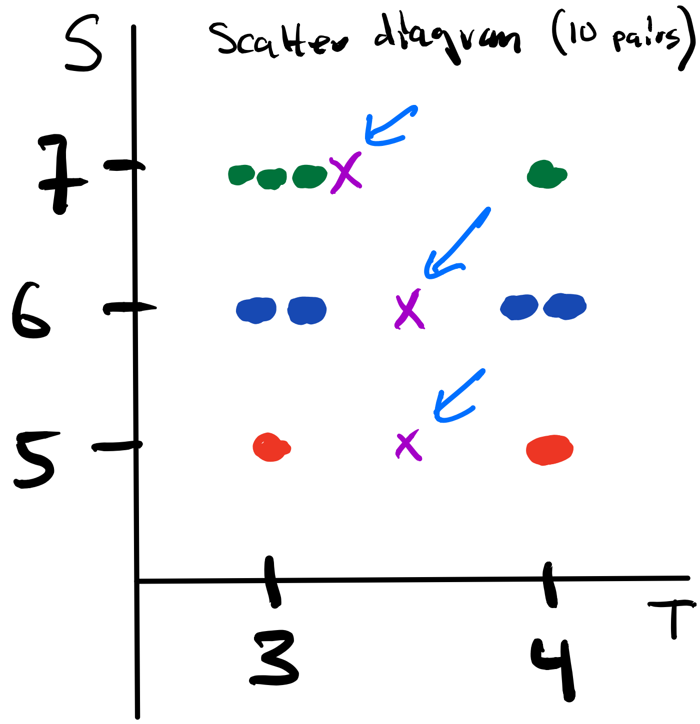
$$\left.\begin{aligned} E(T \mid S=7) &= 3.25 \\ E(T \mid S=6) &= 3.5 \\ E(T \mid S=5) &= 3.5 \end{aligned}\right\} \text{function of } S$$

Picture

joint distribution

	T=3	T=4	sum
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
sum	0.6	0.4	1

$\longleftrightarrow$



Two main points:

1. $E(T|S)$ is a function of S.
2. $E(T|S)$ is a RV so it has an expectation.

Next we explore the expectation of $E(T|S)$.

Let $g(S) = E(T|S)$

$g(7) = E(T|S=7) = 3.25$

$g(6) = 3.5$

$g(5) = 3.5$

$$E(g(S)) = \sum_{s \in S} g(s) P(S=s)$$

$$= 3.25(.4) + 3.5(.4) + 3.5(.2)$$

$$= 3.4$$

	T=3	T=4	Sum
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1

Find $E(T) = \sum_{t \in T} t P(T=t) = 3(.6) + 4(.4)$

$$= 3.4$$

In other words,

$$E(E(T|S)) = E(T)$$

This is called the property of iterated expectations.

Intuitively,

If you have a class that is $2/3$ girls and $1/3$ boys and the girls weigh on average 100 lbs and boys weigh 200 lbs then the average weight of the class should be

$\frac{2}{3}(100) + \frac{1}{3}(200)$, i.e. we take the weighted average of the averages.

Rule of average conditional expectations

For any random variable T with finite expectation and any discrete RV S,

$$E(T) = \sum_{\text{all } s} E(T \mid S=s) \cdot P(S=s)$$

(see end of this lecture for a formal proof)

Appendix

Iterated Expectation

We show $E(Y) = E(E(Y|X))$:

$$E(Y) = \sum_{\text{all } y} y P(Y=y)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} P(X=x, Y=y)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} \frac{P(X=x, Y=y)}{P(X=x)} P(X=x)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} P(Y=y | X=x) P(X=x)$$

$$= \sum \quad \sum y P(Y=y | X=x) \cdot P(X=x)$$

$$= \sum_{\text{all } x} E(Y | X=x) \cdot P(X=x)$$

$$= E(E(Y|X))$$

$\square$