Stat 134 lec 23

Wednesday Lecture on Moment Generating Functions (not in book)

Last time sec 4.2 Gamma Distribution

$T \sim Exp(\lambda),\ \lambda > 0$

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$$

variable part

$T_r \sim Gamma(r, \lambda),\ r, \lambda > 0$

$$f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$$

where $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t}\, dt$ Gamma function

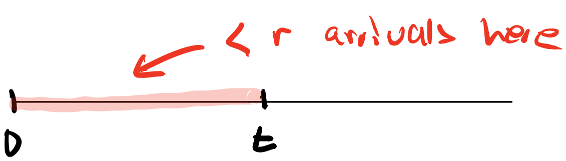
Notes about $T_r \sim Gamma(r, \lambda)$:

$\Gamma(r) = (r-1)!$ for $r \in \mathbb{Z}^+$

If $r \in \mathbb{Z}^+$, T_r = time to r^{th} arrival

$Exp(\lambda) = Gamma(r=1, \lambda)$

$P(T_r > t) = P(N_t < r)$ where $N_t \sim Pois(\lambda t)$



Today Sec 4.4 (skip 4.3)

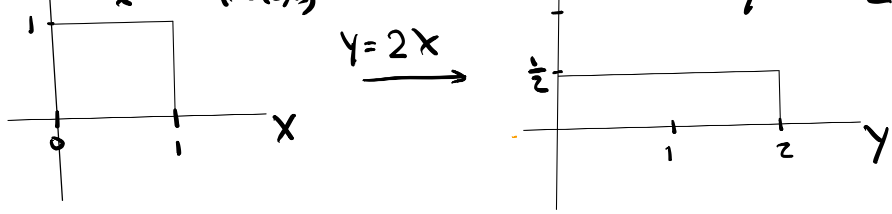
1. Change of variable formula for densities.
2. Recognizing a distribution from the variable part of its density

① Sec 4.4 Change of Variable formula for densities

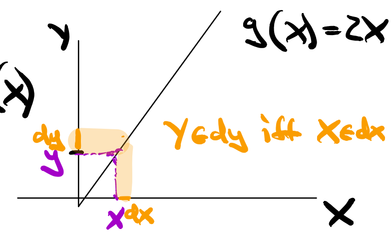
ex The density of $X \sim U(0,1)$ is $f_X(x) = \mathbb{1}_{(x \in (0,1))}$

What is density of $Y = 2X$

$f_X(x) = \mathbb{1}_{(x\in(0,1))}$ $\qquad f_Y(y) = \frac{1}{2}\mathbb{1}_{(y\in(0,2))}$

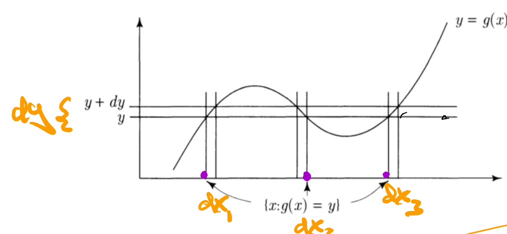


Here the transformation of X to $Y = g(X)$ is linear.



What if $Y = g(X)$ isn't linear?

Picture $\quad g^{-1}(y) = \{x_1, x_2, x_3\}$



mutually exclusive events.

$Y \in dy$ iff $X \in dx_1$ or $X \in dx_2$ or $X \in dx_3$

$P(Y \in dy) = P(X \in dx_1) + P(X \in dx_2) + P(X \in dx_3)$

$f_Y(y)\,dy = f_X(x_1)\,dx_1 + f_X(x_2)\,dx_2 + f_X(x_3)\,dx_3$

$f_Y(y) = f_X(x_1)\frac{dx_1}{dy} + f_X(x_2)\frac{dx_2}{dy} + f_X(x_3)\frac{dx_3}{dy}$

$= \frac{f_X(x_1)}{g'(x_1)} + \frac{f_X(x_2)}{|g'(x_2)|} + \frac{f_X(x_3)}{g'(x_3)}$

$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{g'(x)}$ $\qquad P(X \in dx_2) \geq 0$

Thm (P 307)

Let X be a continuous RV with density $f_X(x)$.

Let $Y = g(X)$ have a derivative that is zero at only finitely many pts,

then $f_Y(y) = \sum_{\{x \mid g(x) = y\}} \frac{f_X(x)}{|g'(x)|}$ evaluated at $x = g^{-1}(y)$.

ex

let $X = N(0,1)$, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

Find the density of $Y = \sigma X + \mu$ where $\sigma > 0$.

Note $x = g^{-1}(y) = \frac{y - \mu}{\sigma}$

$$f_Y(y) = \left.\frac{f_X(x)}{\sigma}\right|_{x = \frac{y-\mu}{\sigma}} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$$

$E(Y) = E(\sigma X + \mu) = \sigma \underset{\overset{\shortparallel}{0}}{E(X)} + \mu = \mu$

$Var(Y) = Var(\sigma X + \mu) = \sigma^2 \underset{\overset{\shortparallel}{1}}{Var(X)} = \sigma^2$

We will see later in the semester that Y is normal.

$\Rightarrow \boxed{Y \sim N(\mu, \sigma^2)}$

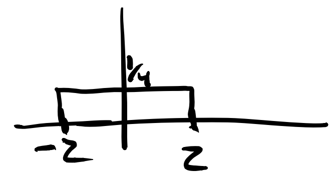
Change of variable formula:

$$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \frac{f_X(x)}{|g'(x)|} \quad \text{evaluated at } x = g^{-1}(y).$$

ex $X \sim \text{Unif}(-2,2)$, $f_X(x) = \frac{1}{4}\mathbb{1}_{(x\in(-2,2))}$

Find density of $Y = x^2$

note: $x = g^{-1}(y) = \pm\sqrt{y}$

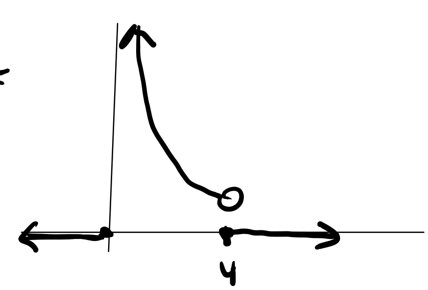


$$f_Y(y) = \left.\frac{\frac{1}{4}\mathbb{1}_{(x\in(-2,2))}}{2x}\right|_{x=\sqrt{y}} + \left.\frac{\frac{1}{4}\mathbb{1}_{(x\in(-2,2))}}{|2x|}\right|_{x=-\sqrt{y}}$$

$$= \frac{\frac{1}{4}\mathbb{1}_{(y\in(0,4))}}{2\sqrt{y}} + \frac{\frac{1}{4}\mathbb{1}_{(y\in(0,4))}}{|-2\sqrt{y}|}$$

$$= \frac{\frac{1}{4}\mathbb{1}_{(y\in(0,4))}}{\sqrt{y}} = \begin{cases} \frac{1}{4\sqrt{y}} & 0<y<4 \\ 0 & \text{else} \end{cases}$$

graph of $f_Y(y)$ =



$\underline{\underline{ex}}$

(3 pts) Suppose the random variable X, which measures the magnitude of an earthquake (on the Richter scale) in the Bay Area, follows the Exponential (λ) distribution. Since the Richter scale is logarithmic, we want to study the distribution of the total energy of earthquakes. Find the distribution of $Y = e^X$.

Change of variable formula:

$$f_Y(y) = \sum_{\{x \mid g(x) = y\}} \frac{f_X(x)}{|g'(x)|} \text{ evaluated at } x = g^{-1}(y).$$

$X \sim \text{Exp}(\lambda)$ $\quad f_X(x) = \lambda e^{-\lambda x}$ $\quad x > 0$

$g(x) = e^x$ is one to one

$g'(x) = e^x$ $\quad x = \log y$

$$f_Y(y) = \left.\frac{\lambda e^{-\lambda x}}{e^x}\right|_{x = \log y} = \left.\lambda e^{(-\lambda - 1)x}\right|_{x = \log y}$$

$$= \lambda e^{(-\lambda-1)\log y} = \boxed{\lambda y^{(-\lambda-1)}, \quad y > 1}$$

(2) Recognizing a distribution from the variable part of its density.

A density can be written as

$$f(t) = c\, h(t)$$

c ← constant; $h(t)$ ← variable part.

$$1 = \int_{-\infty}^{\infty} f(t)\,dt = c\int_{-\infty}^{\infty} h(t)\,dt \Rightarrow c = \frac{1}{\int_{-\infty}^{\infty} h(t)\,dt}$$

So you can figure out the density from its variable part.

Often it is advantageous to ignore the constant part of a density and just work with the variable part.

List of densities. Please circle their variable parts:

ex

$T \sim \text{Gamma}(r, \lambda)$ $\quad f(t) = \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}, \; t > 0$

$T \sim \text{Normal}(\mu, \sigma^2)$ $\quad f(t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$

$T \sim \text{Unif}(a, b)$ $\quad f(t) = \frac{1}{b-a} \mathbb{1}_{(t \in (a,b))}$

$\underline{\underline{ex}}$ Name the distribution with the following variable part $\underline{\underline{ex}}$ Gamma$(r=1/2, \lambda=3)$

a) $h(t) = t^3 e^{-\frac{1}{2}t}$ Gamma$(r=4, \lambda=1/2)$

b) $h(t) = e^{-\frac{1}{2}t^2}$ Normal $(\mu=0, \sigma=1)$

c) $h(t) = e^{-3t}$ Gamma $(r=1, \lambda=3)$

d) $h(t) = t^{-1/2} e^{-t}$ Gamma $(r=1/2, \lambda=1)$

e) $h(t) = \mathbb{1}_{(t\in(0,1))}$ Unif$(0,1)$

Stat 134

Monday March 18 2019

1. Let V be a standard normal RV. The distribution of $X = V^2$ is?

   a Gamma

   b Uniform

   c Normal

   d none of the above

$V \sim N(0,1)$

$f_V(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}$

$f_X(x) = \left.\frac{\frac{1}{\sqrt{2\pi}} e^{-v^2/2}}{2v}\right|_{v=\sqrt{x}} + \left.\frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}}{|2v|}\right|_{v=-\sqrt{x}}$

$\propto \frac{e^{-\frac{x}{2}}}{\sqrt{x}} = x^{-1/2} e^{-1/2 x}$

Gamma $(r = 1/2, \lambda = 1/2)$

$f_X(x) = \frac{1}{\Gamma(\frac{1}{2})} \left(\frac{1}{2}\right)^{1/2} x^{-1/2} e^{-1/2 x}$

variable part

constant part.