Stat 134: Section 21 Solution

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April 12, 2020

Problem 1

Let $N \sim \text{Pois}(\lambda)$, and X be a random variable such that $X|N = n$ is binomial (n, p) distributed.

a Show that the unconditional distribution of X is Poisson and find the parameter.

b It is known that X-rays produce chromosome breakages in cells. The number of such breakages usually follows a Poisson distribution quite closely, where the parameter depends on the time of exposure, etc. For a particular dosage and time of exposure, the number of breakages follows the Poisson (0.4) distribution. Assume that each breakage heals with probability 0.2, independently of the others. Find the chance that after such an X-ray, there are 4 healed breakages.

Ex 6.1.7 in Pitman's Probability

Soln: Let $k \in \mathbb{N}$, and consider the probability that $X = k$:

$$\begin{aligned}
P(X = k) &= \sum_{n=0}^{\infty} P(X = k|N = n) \cdot P(N = n) \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot e^{-\lambda} \lambda^n / n! \\
&= \sum_{n=k}^{\infty} p^k (1-p)^{n-k} \cdot e^{-\lambda} \lambda^n \cdot \frac{1}{k!(n-k)!} \\
&= \sum_{n=0}^{\infty} p^k (1-p)^{n} \cdot e^{-\lambda} \lambda^{n+k} \cdot \frac{1}{k!\; n!} \\
&= p^k \lambda^k e^{-\lambda p} / k!
\end{aligned}$$

Hence X follows a Poisson distribution with parameter λp. For part b, simply plug in $\lambda = 0.4, p = 0.2$ to get the final result as

$$e^{-0.08} \cdot (0.08)^4 / 4!.$$

Problem 2

Let $A_1, \ldots ,A_{20}$ be independent events each with probability 1/2. Let X be the number of events among the first 10 which occur and let Y be the number of events among the last 10 which occur. Find the conditional probability that X = 5, given that X +Y = 12.
Ex 6.1.4 in Pitman's Probability
Soln: The computation is quite straightforward by applying the binomial distributions:

$$\begin{aligned} P(X = 5|X + Y = 12) &= P(X = 5, X + Y = 12)/P(X + Y = 12) \\ &= P(X = 5, Y = 7)/P(X + Y = 12) \\ &= \binom{10}{5}\binom{10}{7}\bigg/\binom{20}{12}. \end{aligned}$$

Problem 3

Random variables X and Y are called conditionally independent given Z if given the value of Z, X, and Y are independent. That is,

$$P(X = x, Y = y \mid Z = z) = P(X = x|Z = z)P(Y = y|Z = z) \tag{1}$$

for all possible values x, y, and z. Prove that X and Yare conditionally independent given Z if and only if the conditional distribution of Y given $X = x$ and $Z = z$ is a distribution which depends only on z:

$$P(Y = y|X = x, Z = z) = P(Y = y|Z = z) \tag{2}$$

for all possible values x, y, and z. Give a further equivalent condition in terms of the conditional distribution of X given $Y = y$ and $Z = z$.
Ex 6.1.9 in Pitman's Probability
Soln: We first prove the forward direction. By conditional probability,

$$P(Y = y|X = x, Z = z) = P(Y = y, X = x \mid Z = z)/P(X = x \mid Z = z),$$

by equation (1), the result follows. The backward direction is similar. By symmetry, we can give another condition:

$$P(X = x|Y = y, Z = z) = P(X = x|Z = z) \tag{3}$$