Stat 134 lec 9

warmup 11:00–11:10

Hypergeometric/Counting

You and your friend are playing Scrabble. Suppose that there are 10 copies of each letter of the alphabet in the bag of letters. Each of you choose 7 letters at random from the bag (i.e. you collectively draw 14 letters).

1. What is the chance that k Zs are drawn collectively? (Your answer will depend on k.)

2. What is the chance that you get 3 Zs and your friend gets 2 Zs?

i.) $$\frac{\binom{10}{k}\binom{250}{14-k}}{\binom{260}{14}}$$

ii.) P(you get 3 Zs) · P(friend gets 2 Zs given you get 3 Zs)

$$= \frac{\binom{10}{3}\binom{250}{4}}{\binom{260}{7}} \cdot \frac{\binom{7}{2}\binom{246}{5}}{\binom{253}{7}}$$

Last time Harder hypergeometric problems.

today ① sec 2.5 Binomial approx to hypergeometric.

② sec 3.1 – random variables (RV)

joint distribution of 2 RVs and independence

① **Sec 2.5** Binomial approx to hypergeometric.

Binomial — independent trials
Hypergeometric — dependent trials.

ex 100 person Class with a grade distribution:

A grade : 70 students
B grade : 30 students.

Sample 5 students at random w/o replacement (SRS).

Find $P(3A\text{'s}, 2B\text{s})$

exact
hypergeometric
$$= \frac{\binom{70}{3}\binom{30}{2}}{\binom{100}{5}} = \binom{5}{3}\frac{70}{100}\frac{69}{99}\frac{68}{98}\frac{30}{97}\frac{29}{96} = .316$$

approx
binomial
$$= \binom{5}{3}(.7)^3(.3)^2 = .309$$

when N is large relative to n, $HG(5, 100, 70) \approx Bin(5, .7)$

why?

$$HG(n, N, G) \approx Bin\left(n, \frac{G}{N}\right)$$

Summary of approximations

$HG(n, N, G)$

↓ approx by binomial
N large, n small
$p = \frac{G}{N}$

binomial (n, p)

↙ approx by Poisson
$p \to 0$, $n \to \infty$, $np \to \mu$

Poisson(μ)

↘ approx by normal
n large
$\mu = np$, $\sigma = \sqrt{npq}$
$0 < \mu \pm 3\sigma < n$
use continuity correction

Normal(μ, σ^2)

② Sec 3.1 Intro to Random Variables (RV)

A RV, X, is the outcome of an experiment.

What distribution is the following RV?

X = The number of aces in 5 cards drawn from a standard deck?

$X \sim HG(5, 52, 4)$

belongs to

ex flip a prob p coin 2 times

$X = \#$ heads

$X = 1$ is an event

$P(X=1) = \binom{2}{1} p^1 (1-p)^2$ binomial formula

we write $X \sim Bin(2, p)$

More precisely,

outcome space

$X: \Omega \longrightarrow \mathbb{R}$ is a function

$HH \longmapsto 2$

$HT \longmapsto 1$

$TH \longmapsto 1$

$TT \longmapsto 0$

so $X=1$ means $\{HT, TH\} \subseteq \Omega$

X has a probability distribution

X	0	1	2
$P(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

P(HT or TH)

You can find the distribution of $g(X) = |X-1|$?

Y ‖ function of a RV

HT, TH   TT, HH

$Y = g(X)$	0	1
$P(g(X))$	$\frac{1}{2}$	$\frac{1}{2}$

$Y \sim Unif\{0, 1\}$

$Y \sim Ber(\frac{1}{2})$ or $Bin(1, \frac{1}{2})$

Joint Distribution

Let (X,Y) be the joint outcome of 2 RVs X, Y.

The event $(X=x, Y=y)$ is the intersection of events $X=x$ and $Y=y$.

ex X: one draw from $\boxed{1}\ \boxed{2}\ \boxed{2}\ \boxed{3}$

Given $X=x$, Y = number of heads in x coin tosses.

$$P(X=x, Y=y) = P(Y=y \mid X=x) \cdot P(X=x)$$

$$P(X=1, Y=1) = \underbrace{P(Y=1 \mid X=1)}_{= 1/2} \cdot \underbrace{P(X=1)}_{= 1/4} = 1/8$$

What the range of values of X? – 1, 2, 3

Y? – 0, 1, 2, 3

Find,

$P(1,0) = \underbrace{P(Y=0 \mid X=1)}_{= 1/2} \underbrace{P(X=1)}_{= 1/4} = 1/8$

$P(1,1) = \frac{1}{2} \cdot \frac{1}{4} = 1/8$

$P(2,0) = P(Y=0 \mid X=2) P(X=2) = 1/4 \cdot 1/2 = 1/8$

$P(2,1) = 1/2 \cdot 1/2 = 1/4$

$P(2,2) = \frac{1}{4} \cdot \frac{1}{2} = 1/8$

$P(3,0)$ $\quad 1/8 \cdot 1/4 = 1/32$

$P(3,1) = \frac{3}{8} \cdot \frac{1}{4} = \frac{3}{32}$

$P(3,2) = \frac{3}{8} \cdot \frac{1}{4} \quad \frac{3}{32}$

$P(3,3) = \frac{1}{8} \cdot \frac{1}{4} \quad \frac{1}{32}$

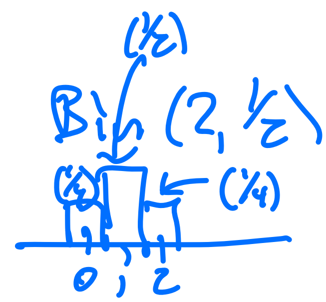
marginal prob of X

$P(x) = \sum_{y \in Y} P(x,y)$

Y \ X	1	2	3	
3	0	0	$\frac{1}{32}$	$\frac{1}{32}$
2	0	$\frac{1}{8}$	$\frac{3}{32}$	$\frac{7}{32}$
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{32}$	$\frac{15}{32}$
0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{9}{32}$
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

marginal prob of Y

$P(y) = \sum_{x \in X} P(x,y)$



$X-1 \sim \text{Bin}(2, \frac{1}{2})$

Y not a named distribution.

Is X, Y dependent? — yes.

ex $P(Y=0 \mid X=1) = \frac{1}{2}$

$P(Y=0) = 9/32$ $\Big\}$ $\Rightarrow$ X, Y dep

<u>Def^n</u> Two RVs are <u>independent</u> if

$$P(Y=y \mid X=x) = P(Y=y) \quad \text{for all } x \in \mathcal{X},\ y \in \mathcal{Y}$$

By the multiplication rule,

if X, Y are indep,

$$P(X=x, Y=y) = \underbrace{P(Y=y \mid X=x)}_{= P(Y=y)} P(X=x)$$

so $$\boxed{P(X=x, Y=y) = P(X=x)P(Y=y)}.$$

A fair coin is tossed twice.
Let $X = \#$ heads on the first toss.
Let $Y = \#$ heads on the first 2 tosses.

Y \ X	0	1	P(y)
2	0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
0	$\frac{1}{4}$	0	$\frac{1}{4}$
P(x)	$\frac{1}{2}$	$\frac{1}{2}$	

$P(x=0, Y=0) = 1/4$ ✓

To confirm, for example, the top left cell is zero

$$P(x=0, Y=2) = P(x=0)\,P(Y=2 \mid x=0) = 0$$
$$P(x=0) = 1/2, \quad P(Y=2 \mid x=0) = 0$$ ✓

Do this for every cell.

a The table above is correct

b $Y \sim Bin(2, \frac{1}{2})$ ✓

c More than one of the above

d None of the above