Stat 134 lec 22

next Wednesday Q4 covers 4.1, 4.2

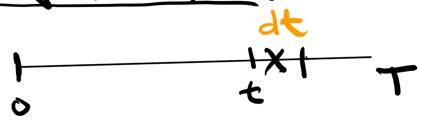
last time sec 4.2 exponential distribution.

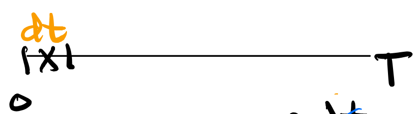
survival function

$T \sim Exp(\lambda)$ $\quad f(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$ or $P(T>t) = e^{-\lambda t}$

T = time until 1^{st} success (arrival)

Memoryless Property : $P(T \in dt \mid T > t) = P(0 < T < 0 + dt)$





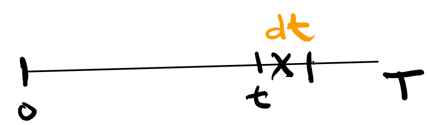
$$P(T \in dt \mid T > t) = \frac{P(T \in dt)}{P(T > t)} \approx \frac{\lambda \cancel{e^{-\lambda t}}\, dt}{\cancel{e^{-\lambda t}}} = \boxed{\lambda dt}$$

$\boxed{\lambda dt} = P(0 < T < 0 + dt)$

conclusions:

① So long as a lightbulb is still functioning it is as good as new

② $\lambda = \dfrac{P(T \in dt \mid T > t)}{dt}$



instantaneous rate of success. This is always the same (not dependent on t).

Relationship w/ $X \sim Geom(p)$

$pX \to Exp(1)$ as $p \to 0$

Today sec 4.2

⓪ $Exp(1)$ is the limit of $p \cdot Geom(p)$.

① Expectation and Variance of $Exp(\lambda)$.

② Gamma distribution.

(0) Sec 4.2 relationship between $\text{Exp}(\lambda)$ and $\text{Geom}(p)$.

Review Taylor series for e^{-p}:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \approx 1 + x \text{ for small } x$$

so $$e^{-p} = 1 - p + \frac{p^2}{2!} - \frac{p^3}{3!} + \dots \approx 1 - p \text{ for small } p$$

$\Rightarrow$ $1 - p \approx e^{-p}$ for small p.

Let $X \sim \text{Geom}(p)$, then

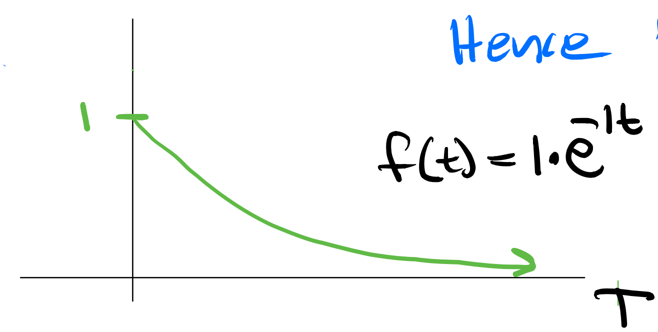
$$P(pX > pk) = P(X > k) = (1-p)^k \approx (e^{-p})^k = e^{-pk}$$

for small p

let $t = pk$, then

$$P(pX > t) \approx e^{-t}$$

Hence $pX \to \text{Exp}(1)$ as $p \to 0$



①

Sec 4.2 Expectation and Variance of Exp(λ)

$T \sim Exp(\lambda)$

see end of lec notes.

$$E(T) = \int_0^\infty t f(t) dt = \lambda \int_0^\infty t e^{-\lambda t} dt = \boxed{\frac{1}{\lambda}}$$

see end of lec notes.

$$E(T^2) = \int_0^\infty t^2 f(t) dt = \lambda \int_0^\infty t^2 e^{-\lambda t} dt = \frac{2}{\lambda^2}$$

$$var(T) = E(T^2) - E(T)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

check for $\lambda = 1$

Recall that for $X \sim Geom(p)$

$E(X) = \frac{1}{p}$

$var(X) = \frac{q}{p^2}$

$E(pX) = pE(X) = \frac{p}{p} = 1 \to 1$ as $p \to 0$

$pX \to T \sim Exp(1)$ as $p \to 0$

$E(pX) \to E(T)$ as $p \to 0$

$\Rightarrow E(T) = 1$ ✓

$var(pX) = p^2 var(X) = \frac{p^2 q}{p^2} = q \to 1$ as $p \to 0$.

$var(pX) \to var(T)$ as $p \to 0$

$\Rightarrow var(T) = 1$ ✓

Interpretation of λ:

$\lambda = \dfrac{1}{E(T)}$ (one over average arrival time)

This makes sense since the instantaneous rate of success is constant by the memoryless property of $Exp(\lambda)$.

So if it takes on avg 5 min to have a success then $\lambda = 1/5$.

ex Let T = waiting time (min) until a blue Honda enters a toll gate starting at noon. The waiting time is on average 2 minutes. Find $P(T > 3)$

soln

$E(T) = 2$ min

$\lambda = \frac{1}{2}$

$T \sim Exp(\frac{1}{2})$

$P(T > 3) = e^{-\frac{1}{2}3} = \boxed{e^{-3/2}}$

② Gamma Distribution

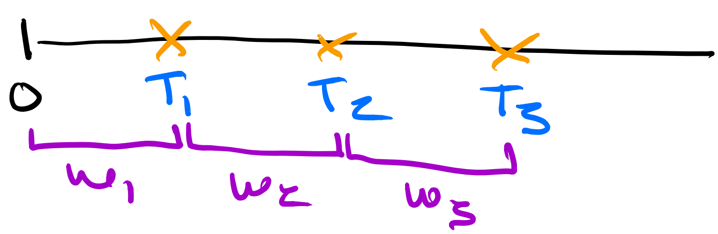
A gamma distribution, $T_r \sim \text{Gamma}(r, \lambda)$, $\lambda > 0$, is a sum of r iid $\text{Exp}(\lambda)$.

$$T_r = W_1 + W_2 + \dots + W_r, \quad W_i \sim \text{Exp}(\lambda)$$

ex T_r = time of the r^{th} arrival of a Poisson Process $(\text{Pois}(\lambda t))$

ex T_r = time when the r^{th} lightbulb dies.

Picture



$$\left.\begin{array}{l} T_1 \sim \text{Gamma}(1, \lambda) \\ T_2 \sim \text{Gamma}(2, \lambda) \\ T_3 \sim \text{Gamma}(3, \lambda) \end{array}\right\}$$

dependent but W_1, W_2, W_3 are independent, $\text{Exp}(\lambda)$

$$T_1 = W_1$$
$$T_2 = W_1 + W_2$$
$$T_3 = W_1 + W_2 + W_3$$

Review Poisson Process (a.k.a PRS)

$N_t \sim Pois(\lambda t)$ $\leftarrow \mu$

N_t = # arrivals in time t where the rate of arrivals is λ.

$$P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

ex Let T_1 = waiting time (min) until a blue Honda enters a toll gate starting at noon. The waiting time is on average 2 minutes. Find the probability 3 blue Hondas arrive in 8 minutes.

soln

$\lambda = 1/2$

$N_8 \sim Pois(4)$ $\leftarrow \frac{1}{2} \cdot 8$

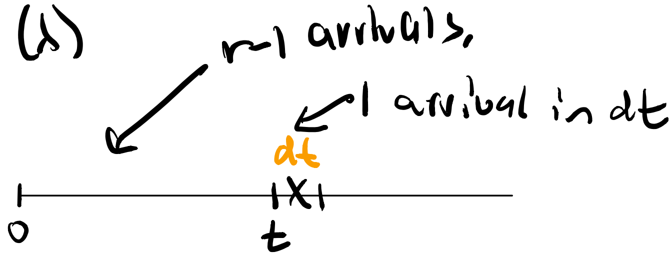
$$P(N_8 = 3) = \frac{e^{-4} 4^3}{3!}$$

Back to gamma distribution: $T_r \sim \text{Gamma}(r, \lambda)$

Let $N_t \sim \text{Pois}(\lambda t)$

$W \sim \text{Exp}(\lambda)$

$P(T_r \in dt)$ means



so

$$P(T_r \in dt) = P(N_t = r-1)\,P(W \in dt \mid W > t)$$

$$\approx \frac{e^{-\lambda t}(\lambda t)^{r-1}}{(r-1)!} \cdot \lambda\, dt$$

$$= \underbrace{\frac{1}{(r-1)!} \lambda^r t^{r-1} e^{-\lambda t}}_{f(t)}\, dt$$

<u>Gamma function $\Gamma(r)$, $r > 0$</u>

If $r \in \mathbb{Z}^+$ define $\Gamma(r) = (r-1)!$ where $\Gamma(r)$ is called <u>the gamma function</u>

For $\lambda = 1$, $f(t) = \underbrace{\frac{1}{\Gamma(r)}}_{\text{const}} \underbrace{t^{r-1} e^{-t}}_{\text{variable}}$

and $\int_0^\infty f(t)\,dt = 1 \Rightarrow$

$$\frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-t}\,dt = 1$$

$$\Rightarrow \boxed{\Gamma(r) = \int_0^\infty t^{r-1} e^{-t}\,dt}$$

This allows us to define

$T \sim \text{Gamma}(r, \lambda)$ for $r > 0$ as

having density $f(t) = \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}$ $t > 0$.

ex Let $T_r \sim \text{Gamma}(r=4, \lambda=2)$

Find $f(t)$

$$\frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-t} = \frac{1}{\Gamma(4)} 2^4 t^3 e^{-2t}$$

$\Gamma(4) = 3 \cdot 2$

$$= \boxed{\frac{8}{3} t^3 e^{-2t}}$$

The formula $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} \, dt$ is the definition of the gamma function.

You can show that $\Gamma(r) = (r-1)!$

for $r \in \mathbb{Z}^+$

Now, $T_r \sim \text{Gamma}(r, \lambda)$ has density

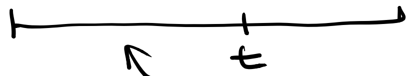
$$f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}, & t > 0 \\ 0, & \text{else} \end{cases}$$

Let $T_r \sim \text{Gamma}(r, \lambda)$

$$P(T_r > t) = \int_t^{\infty} \frac{1}{\Gamma(r)} \lambda^r s^{r-1} e^{-\lambda s}\, ds$$

we won't use the density.

$N_t \sim \text{Pois}(\lambda t)$



$\leq r-1$ arrivals here.

$$P(T_r > t) = P(N_t < r)$$

$$= \sum_{i=0}^{r-1} P(N_t = i)$$

$$= \boxed{\sum_{i=0}^{r-1} \frac{e^{-\lambda t} (\lambda t)^i}{i!}}$$

ex let $T_4 \sim \text{Gamma}(r=4, \lambda=2)$

Find $P(T_4 > 7)$

$= P(N_7 \leq 3)$ where $N_7 \sim \text{Pois}(\overset{2\cdot 7}{14})$

$$= e^{-14}\left(1 + \frac{14}{1!} + \frac{14^2}{2!} + \frac{14^3}{3!}\right)$$

Suppose customers are arriving at a ticket booth at rate of five per minute, according to a Poisson arrival process. Find the probability that:

(a) Starting from time 0, the $9th$ customer doesn't arrive within 5 minutes;

(b) At least one customer arrives within 40 seconds after the arrival of the $13th$ customer.

Soln

a) $P(T_9 > 5) = P(N_5 < 9) = \boxed{\sum_{k=0}^{8} e^{-25} \frac{25^k}{k!}}$

b) $P\left(W_{14} < \frac{2}{3}\right) = 1 - P\left(W_{14} > \frac{2}{3}\right)$

$P\left(W_{14} > \frac{2}{3}\right) = e^{-5 \cdot 2/3}$

$= \boxed{1 - e^{-5 \cdot 2/3}}$

Appendix

Expectation and Variance of exponential

let $T \sim \text{Exp}(\lambda)$

$f(t) = \lambda e^{-\lambda t}$ density

$$E(T) = \lambda \int_0^\infty t e^{-\lambda t} \, dt$$

I recommend using the tabular method for integration by parts. This works well when the function you are integrating is the product of two expressions, where the n^{th} derivative of one expression is zero.

ex $t^4 e^{3t}$ ✓ good

ex $\sin t \, e^{3t}$ ✗ bad.

To find $\lambda \int_0^\infty t e^{-3t}$:

$\frac{d}{dt}$		$\int$
t	$+$	$e^{-\lambda t}$
1	$-$	$\frac{-e^{-\lambda t}}{\lambda}$
0		$\frac{e^{-\lambda t}}{\lambda^2}$

$$E(T) = \lambda\int_0^\infty t e^{-\lambda t}\,dt = \lambda\left(t\left(-\frac{e^{-\lambda t}}{\lambda}\right) - 1\cdot\frac{e^{-\lambda t}}{\lambda^2}\right)\Bigg|_0^\infty$$

$$= 0 - \left(0 - \frac{1}{\lambda}\right)$$

$$= \boxed{\frac{1}{\lambda}}$$

Next find $Var(T) = E(T^2) - E(T)^2$:

$$E(T^2) = \lambda\int_0^\infty t^2 e^{-\lambda t}\,dt$$

$\frac{d}{dt}$		$\int$
t^2	$+$	$e^{-\lambda t}$
$2t$	$-$	$-\frac{1}{\lambda}e^{-\lambda t}$
2	$+$	$\frac{1}{\lambda^2}e^{-\lambda t}$
0		$-\frac{1}{\lambda^3}e^{-\lambda t}$

$$E(T^2) = \lambda\left(t^2\left(-\frac{1}{\lambda}e^{-\lambda t}\right) - 2t\left(\frac{1}{\lambda^2}e^{-\lambda t}\right) + 2\left(-\frac{1}{\lambda^3}e^{-\lambda t}\right)\right)\Bigg|_0^\infty$$

$$= \frac{2}{\lambda^2}$$

$$\Rightarrow Var(T) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \boxed{\frac{1}{\lambda^2}}$$