stat 134 lec 30

Earlier material

$T \sim Exp(\lambda)$, $cT \sim Exp\left(\frac{\lambda}{c}\right)$

Competing exponentials

$T_1 \sim Exp(\lambda_1)$, $T_2 \sim Exp(\lambda_2)$ $\Rightarrow P(T_1 < T_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$

Properties of std normal Z	Proved it
$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$	no
$E(Z) = 0$	no
$SD(Z) = 1$	no

Even though Z is symmetric around zero it is possible $E(X)$ is undefined
ex Cauchy distribution

Let $X, Y \overset{iid}{\sim} N(0,1)$ with density $\phi(x) = ce^{-\frac{1}{2}x^2}$, $c>0$

$f(x,y) = \phi(x)\phi(y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$ for $c>0$

we still need to show that $c = \frac{1}{\sqrt{2\pi}}$, $E(X)=0$, $SD(X)=1$.

Last time

sec 5.2 Marginal density $f_Y(y) = \int_{x=-\infty}^{x=\infty} f(x,y)\,dx$

Today

sec 5.3

1. Solutions to concept test
2. Rayleigh distribution
3. Sums of independent normal RVs is normal.

1

Stat 134

Friday November 8 2019

1. S and T are i.i.d. Exp(λ). $X = \text{Min}(S, T)$ and $Y = \text{Max}(S, T)$. The joint density is $f(x, y) = 2\lambda^2 e^{-\lambda(x+y)}$. The marginal density of Y is:

   a $\lambda(1 - e^{-\lambda y})e^{-\lambda y}$ for $y > 0$

   b $2\lambda(1 - e^{-\lambda y})e^{-\lambda y}$ for $y > 0$

   c $2\lambda(1 - e^{-\lambda y})$ for $y > 0$

   d none of the above

since $0 < x < y < \infty$

method 1

$$f_Y(y) = \int_0^y 2\lambda^2 e^{-\lambda(x+y)}\,dx$$

$$= 2\lambda^2 e^{-\lambda y}\int_0^y e^{-\lambda x}\,dx$$

$$= 2\lambda^2 e^{-\lambda y}\left(\frac{e^{-\lambda x}}{-\lambda}\right)\Bigg|_0^y$$

$$= \boxed{2\lambda(1-e^{-\lambda y})(e^{-\lambda y})}$$

method 2

$$F(y) = P(Y<y) = P(S<y, T<y)$$

$$= P(S<y)^2 = (1-e^{-\lambda y})^2$$

$$f(y) = \frac{d}{dy}F(y) = 2(1-e^{-\lambda y})\cdot(e^{-\lambda y})\cdot\lambda$$

$$= \boxed{2\lambda(1-e^{-\lambda y})(e^{-\lambda y})}$$

②

Sec 5.3 Rayleigh Distribution

let $T \sim \text{Exp}(\frac{1}{2})$, $f(t) = \frac{1}{2}e^{-\frac{1}{2}t}$, $t > 0$

$R = \sqrt{T}$ ← R is called the Rayleigh Distribution

write $R \sim \text{Ray}$

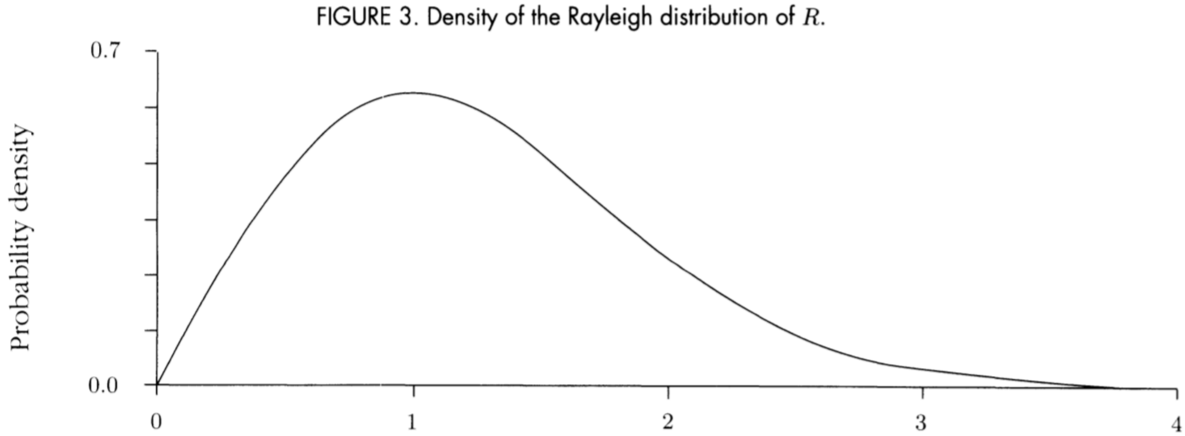
Find $f_R(r)$.

$$f_R(r) = \frac{1}{|(\sqrt{t})'|} \cdot f_T(t) \Big|_{t=r^2}$$

$$= \frac{1}{\frac{1}{2\sqrt{t}}} \cdot \frac{1}{2} e^{-\frac{1}{2}t} \Big|_{t=r^2}$$

$$= \boxed{r e^{-\frac{1}{2}r^2}, \quad r > 0}$$

FIGURE 3. Density of the Rayleigh distribution of R.



Note:

$$P(R > r) = P(R^2 > r^2) = P(\underset{\sim Exp(\frac{1}{2})}{T} > r^2)$$

$$= e^{-\frac{1}{2}r^2}$$

So $\boxed{F_R(r) = 1 - e^{-\frac{1}{2}r^2}, \quad r > 0}$

$$f_R(r) = \frac{d}{dr} F_R(r)$$

and $\boxed{f_R(r) = r e^{-\frac{1}{2}r^2}, \quad r > 0}$

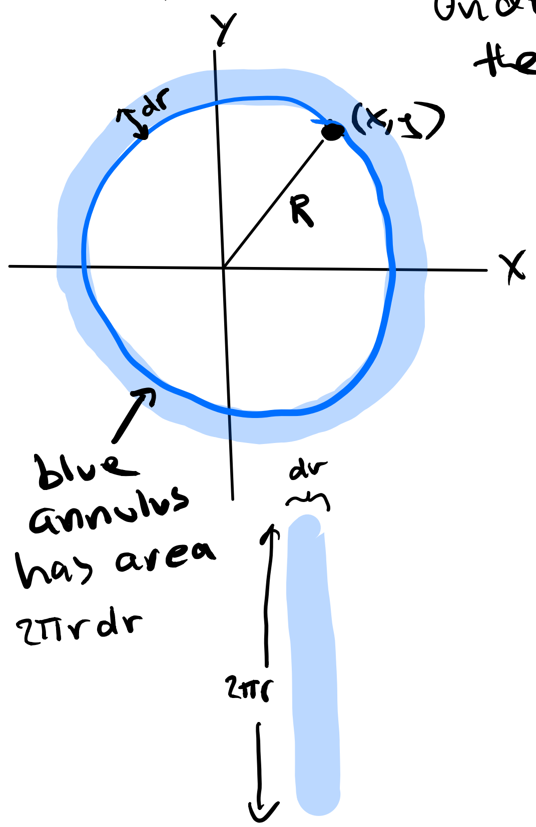
The Rayleigh distribution will help us find the density of the standard normal:

For $X, Y \overset{iid}{\sim} N(0,1)$

Let $R = \sqrt{X^2 + Y^2}$

We will show that $R \sim \text{Ray}$

$P(R \in dr)$ is the volume of the cylinder under $f(x,y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$ over the shaded blue annulus.



$$P(R \in dr) \approx \text{height of } f(x,y) \text{ above blue annulus} \cdot \text{area of blue annulus}$$

$$P(R \in dr) = f_R(r)\,dr$$

$$= c^2 e^{-\frac{1}{2}r^2} \cdot (2\pi r\,dr)$$

$$= c^2 2\pi \underbrace{r e^{-\frac{1}{2}r^2}}_{\text{Rayleigh Density}} dr$$

$$1 = \int_{r=0}^{r=\infty} P(R \in dr) = c^2 2\pi \underbrace{\int_0^\infty r e^{-\frac{1}{2}r^2} dr}_{=1}$$

$$\Rightarrow c^2 \cdot 2\pi = 1$$

$$\Rightarrow \boxed{c = \frac{1}{\sqrt{2\pi}}}$$

Conclusions

① For $X, Y \stackrel{iid}{\sim} N(0,1)$ and $T \sim Exp(\frac{1}{2})$

$R=\sqrt{X^2+Y^2}$ and $R=\sqrt{T}$

are both the Rayleigh distribution

② $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ is density of the standard normal.

(i.e $C=\frac{1}{\sqrt{2\pi}}$)

— see end of lecture notes

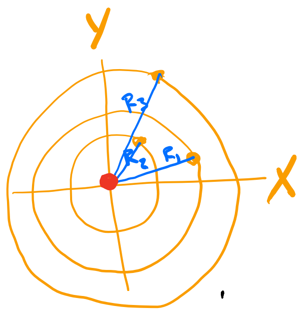
③ $E(X)=0$ and $SD(X)=1$

ex Suppose 3 shots are fired at a target. Assume for each shot, both X and Y are standard normals. Let W be the closest distance among 3 shots to the bullseye. Find $f_W(w)$

distance to Bullseye

$$R = \sqrt{X^2+Y^2}$$

Picture



$$W = \min(R_1, R_2, R_3)$$

where $R_1, R_2, R_3 \overset{iid}{\sim} \text{Ray}$

$$F_{R_1}(r) = 1 - e^{-\frac{1}{2}r^2}$$

method 1 (cdf)

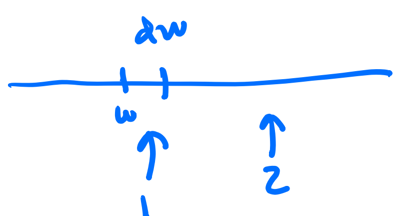
$$P(W>w) = P(R_1>w, R_2>w, R_3>w) = \left[P(R_1>w)\right]^3$$

$$= \left(e^{-\frac{1}{2}w^2}\right)^3 = e^{-\frac{3}{2}w^2}$$

$$\Rightarrow F(w) = 1 - e^{-\frac{3}{2}w^2} \Rightarrow f_W(w) = \frac{d}{dw}F_W(w)$$

$$\Rightarrow \boxed{f_W(w) = 3we^{-\frac{3}{2}w^2}}$$

method 2 (Rayleigh ordered statistics)



$$P(W \in dw) = \binom{3}{1,2} we^{-\frac{1}{2}w^2}dw \cdot \left(e^{-\frac{1}{2}w^2}\right)^2$$

$$= 3we^{-\frac{3}{2}w^2}dw$$

$$\Rightarrow f_W(w) = 3we^{-\frac{3}{2}w^2}$$

If $R_1 \sim$ Ray, $f_{R_1}(r) = re^{-\frac{1}{2}r^2}$

Find density of $W = \frac{1}{\sqrt{3}} R_1$

$$f_W(w) = \frac{1}{1/\sqrt{3}} re^{-\frac{1}{2}r^2} \quad r = \sqrt{3}w$$

$$= \sqrt{3}\sqrt{3}\, w e^{-\frac{3}{2}w^2} = \boxed{3we^{-\frac{3}{2}w^2}}$$

So $\min(R_1, R_2, R_3) \sim \frac{1}{\sqrt{3}}$ Ray

Let $R_4 \sim$ Ray

Find $P(W < R_4)$ — min of first 3 darts are closer to bullseye than 4th dart.

$\frac{1}{\sqrt{3}}$ Ray

hint: competing exponentials.

$$P(W < R_4) = P(W^2 < R_4^2)$$

$$W^2 \sim \left(\frac{1}{\sqrt{3}}\text{Ray}\right)^2 = \left(\frac{1}{\sqrt{3}}\right)^2 \text{Ray}^2 = \frac{1}{3}\text{Ray}^2 = \text{Exp}\left(\frac{1/2}{1/3}\right)$$

$\text{Ray}^2 \sim \text{Exp}\left(\frac{1}{2}\right)$, $\frac{1/2}{1/3} = \frac{3}{2}$

$$P(\underset{\text{Exp}(3/2)}{W^2} < \underset{\text{Exp}(1/2)}{R_4^2}) = \frac{3/2}{3/2 + 1/2} = 3/4$$

Sec 5.3 Sum of independent normals

Definition of the Normal (μ, σ^2) distribution:

We know the density of the std normal

$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$. Make a change of scale $X = \mu + \sigma Z$. By the change of variable rule you can show

$$f_X(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

MGF Review (lecture 24)

Recall the MGF of a RV X is $M(t) = E(e^{Xt})$.

MGF of std normal

$X \sim N(0,1)$, $M(t) = e^{\frac{t^2}{2}}$ for all t

Properties of MGF

if X, Y are independent $M_{X+Y}(t) = M_X(t) M_Y(t)$

for t in an interval contain

$M_X(t) = M_Y(t)$ iff $X \overset{d}{=} Y$ ← same distribution

$M_{b+aX}(t) = e^{bt} M_X(at)$

ex Use the properties of MGF to find the MGF of $X \sim N(\mu, \sigma^2)$

$Z \sim N(0,1)$, $\mu \in \mathbb{R}$, $\sigma > 0$

$X = \mu + \sigma Z$

$a = \mu$
$b = \sigma$

$$M_X(t) = M_{\mu + \sigma Z} = e^{\mu t} e^{\frac{1}{2}(\sigma t)^2}$$

$$= \boxed{e^{\mu t} e^{\frac{\sigma^2 t^2}{2}}}$$

So for $X \sim N(0, 2)$

$$M_X(t) = e^{t^2}$$

ex Use MGF to prove that the sum of two independent std normals is $N(0,2)$.

hint

$$X \sim N(\mu,\sigma^2) \Rightarrow M_X(t) = e^{\mu t} e^{\frac{\sigma^2 t^2}{2}}$$

$$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$$

$$M_{Z_1+Z_2}(t) = M_{Z_1}(t) M_{Z_2}(t)$$

$$= e^{\frac{t^2}{2}} \cdot e^{t^2/2} = e^{t^2}$$

MGF of $N(0,2)$

Thm Let $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ indep.

then $aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

Pf/ $M_{X_1}(t) = e^{\mu_1 t} e^{\sigma_1^2 \frac{t^2}{2}}$

$M_{aX_1}(t) = M_{X_1}(at) = e^{\mu_1 a t} e^{\sigma_1^2 a^2 \frac{t^2}{2}}$

then $M_{aX_1 + bX_2}(t) = M_{aX_1}(t) M_{bX_2}(t)$

$= e^{\mu_1 a t} e^{\sigma_1^2 a^2 \frac{t^2}{2}} \cdot e^{\mu_2 b t} e^{\sigma_2^2 b^2 \frac{t^2}{2}}$

$= e^{(\mu_1 a + \mu_2 b)t} e^{(\sigma_1^2 a^2 + \sigma^2 b^2)\frac{t^2}{2}}$

by uniqueness of MGF

$aX_1 + bX_2 \sim N(\mu_1 a + \mu_2 b, \sigma_1^2 a^2 + \sigma^2 b^2)$

$\square$

Appendix

Claim Let $X \sim N(0,1)$. We show that $E(X)=0$ and $SD(X)=1$.

Let $X \sim N(0,1)$

Show $E(X)=0$

$\phi(x)$ is symmetric about zero so all we have to show is that $E(X)$ converges absolutely.

(i.e $E(|X|) < \infty$).

$$E(|X|) = \int_{-\infty}^{\infty} |x| \phi(x) dx$$

$$= 2 \int_{0}^{\infty} x \phi(x) dx$$

$$= 2 \int_{0}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}} \underbrace{\int_{0}^{\infty} x e^{-\frac{x^2}{2}} dx}_{=1}$$

Rayleigh density

$$= \sqrt{\frac{2}{\pi}} < \infty.$$

$\Rightarrow E(X)=0$ since $\phi(x)$ is semetric around zero

Next we show $SD(X) = 1$:

We know $X^2 + Y^2 \sim Exp\left(\frac{1}{2}\right)$ from above
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{\lambda}$

so $E(X^2+Y^2) = \frac{1}{\frac{1}{2}} = 2$

$\frac{1}{2}$ λ rate of X^2+Y^2

$\Rightarrow E(X^2) + \underset{E(X^2)}{\underset{\shortparallel}{E(Y^2)}} = 2$

$\Rightarrow 2E(X^2) = 2 \Rightarrow E(X^2) = 1$

but $SD(X) = \sqrt{E(X^2) - E(X)^2}$

$= \sqrt{1 - 0} = 1$ ✓