stat 134 lec 18

(there is no video for this review lecture)

Midterm Review

Justin Gerwien

6:06pm

Could you provide us with a list of all of the geometric series sums that we must know for the midterm/the class in general?

① $S = 1 + z + z^2 + z^3 + \cdots$ for $|z| < 1$ geometric sum.

$zS = z + z^2 + z^3 + \cdots$

$\underbrace{S - zS}_{S(1-z)} = 1 \Rightarrow \boxed{S = \frac{1}{1-z} \text{ for } |z|<1}$

② $\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$ Taylor series for e^x

also $e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$

$\boxed{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots}$ Taylor series for $\log(1+x)$

Tiffany Yu

8:57pm

Types of questions to expect on the midterm (concepts that will definitely be tested!!)

(1 like)

1. 10 people throw their hats into a box and randomly redistribute the hats among themselves. Assume every permutation of the hats is equally likely. Let N be the number of people who get their own hats back. Find the following:

(a) $\mathbb{E}[N^2]$

Let N_i be the indicator for the event that the i-th person gets their own hat back.

$$\mathbb{E}[N^2] = \mathbb{E}[(\sum_{i=1}^{10} N_i)^2] = \sum_{i=1}^{10}\sum_{j=1}^{10} \mathbb{E}[N_i N_j] = 90(1/90) + 10(1/10) = 2$$

(b) $P(N = 8)$

If 8 people got their own hat back, that means only 2 people have hats that are not their own. There are $\binom{10}{2}$ ways to pick such a pair.

$$P(N = 8) = \frac{\binom{10}{2}}{10!}$$

The numerator needs some explanation.
If the first 8 hats go to their owner then there is only one ordering of 2 hats that dont go to their owner. There are $\binom{10}{2}$ ways to pick 2 of the hats that dont go to their owner.

Note $P(N=7) \neq \frac{\binom{10}{3}}{10!}$ since if the first 7 hats go to their owner and the last 3 dont, there are multiple ways the 3 hats dont go to their owner.

we have $P(N=7) = \frac{\binom{10}{3} \cdot 2}{10!}$ ← # ways last 3 hats can not belong to their owner.

In this way you can see that

$$P(N=0) = \frac{\text{\# ways no hats belong to owner}}{10!}$$

(c) $P(N = 0)$

Let A_i represent the event that person i got their own hat back.

$$P(N = 0) = 1 - P(A_1 \cup A_2 \cup ... \cup A_{10})$$

By the Principle of Inclusion Exclusion:

$$P(A_1 \cup A_2 \cup ... \cup A_{10}) = \sum_{i=1}^{10} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - ...$$
$$= 10\frac{1}{10} - \binom{10}{2}\frac{1}{10 * 9} + \binom{10}{3}\frac{1}{10 * 9 * 8} - ... = 1 - \frac{1}{2!} + \frac{1}{3!} - ... - \frac{1}{10!}$$

Thus

$$P(N = 0) = 1 - P(A_1 \cup A_2 \cup ... \cup A_{10}) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!}$$

Ravenna Collver

8:46pm

More practice/explanation for harder hypergeometric/counting problems would be helpful (eg still not clear on 2.5.8 or section 6 problem 2b)

2.5.8 P 128

8. In a raffle with 100 tickets, 10 people buy 10 tickets each. If there are 3 winning tickets drawn at random find the probability that:

a) one person gets all 3 winning tickets;

b) there are 3 different winners;

c) some person gets two winners and someone else gets just one.

Soln

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8. Let the outcome space be the set of all 3-element subsets from the set $\{1, \ldots, 100\}$, so that, e.g., the 3-set $\{23, 21, 1\}$ means that the winning tickets were tickets #1, #21, and #23. Each of the $\binom{100}{3}$ such 3-sets is equally likely.

a) The event (one person gets all three winning tickets) is the disjoint union of the 10 equally likely events (person i gets all three tickets). The event (person i gets all three tickets) corresponds to all 3-sets consisting entirely of tickets bought by person i. There are $\binom{10}{3}$ such 3-sets, so the desired probability is

$$10 \times \frac{\binom{10}{3}}{\binom{100}{3}} = .007421.$$

b) The event (there are three different winners) is the disjoint union of the $\binom{10}{3}$ equally likely events (the winning tickets were bought by persons i, j, and k) (i, j, k all different). The event (the winning tickets were bought by persons i, j, and k) consists of $\binom{10}{1} \times \binom{10}{1} \times \binom{10}{1}$ 3-sets, so the desired probability is

$$\binom{10}{3} \times \frac{\binom{10}{1}\binom{10}{1}\binom{10}{1}}{\binom{100}{3}} = .742115.$$

c) By subtraction, .250464. Just to be sure, use the above technique to obtain the desired probability:

$$\binom{10}{1}\binom{9}{1} \times \frac{\binom{10}{2}\binom{10}{1}}{\binom{100}{3}} = .250464.$$

2. (5 pts) Five (5) golden tickets are randomly distributed among 100 total candy bars, which are then sold in 20 packs of 5. (Multiple golden tickets might appear in a single pack of 5.) Let X represent the number of packs you must buy to find a golden ticket. For $k \in \{1, 2, \ldots, 20\}$, find:

(a) $P(X > k)$;

(b) $P(X = k)$.

(a) Observe $X > k$ is the same as drawing $5k$ bars of which none contain a golden ticket. Then,

$$P(X > k) = \frac{\binom{5}{0}\binom{95}{5k}}{\binom{100}{5k}}$$

(b) There are two ways to proceed here. The fastest way is to use $P(X = k) = P(X > k - 1) - P(X > k)$ and part (a). For this, we must also note $P(X > 0) = 1$ to obtain $P(X = 1)$. (The formula from (a) does yield $P(X > 0) = 1$, so students are fine without this technical remark.)

Alternately, we may compute the probability directly: Let Y_i denote the number of gold tickets appearing in pack i. Then,

$$\begin{aligned}
P(X = k) &= P(Y_1 = 0, Y_2 = 0, \ldots, Y_{k-1} = 0) \cdot P(Y_k \geq 1 \mid Y_1 = 0, \ldots, Y_{k-1} = 0) \\
&= P(\text{no gold in } 5(k-1) \text{ draws})(1 - P(0 \text{ gold in } 5 \mid \text{no gold in } 5(k-1) \text{ draws})) \\
&= \frac{\binom{95}{5(k-1)}}{\binom{100}{5(k-1)}} \left(1 - \frac{\binom{95-5(k-1)}{5}}{\binom{100-5(k-1)}{5}}\right)
\end{aligned}$$

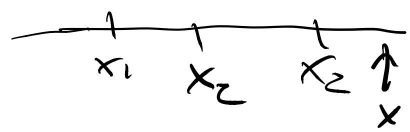
Armin Gholami

Yesterday

Can we go over the different ways to write out certain (or all) distributions when we want to find P(X>x), P(X=x), P(X<x) for RV x? I know P(X=x) is just plug and chugging the value into the formula, but any help to conceptualize the boundary cases would be appreciated!

Let X_1, X_2, X_3 be i.i.d.

let $X = \max(X_1, X_2, X_3)$



$$P(x > X) = P(x > X_1, x > X_2, x > X_3)$$
$$= P(x > X_1)^3$$

$$P(X = x) = P(X > x-1) - P(X > x)$$

A without replacement version of the negative binomial

6. Draw cards from a standard deck until three Aces have appeared.
Let X = number of cards drawn. Find:

i. $P(X > x)$
ii. $P(X = x)$
iii. $\mathbb{E}X$ as a simple fraction
iv. $\mathrm{Var}(X)$ using the method of indicators

(i)

$$P(X>x) = P(3^{rd} \text{ ace after } x^{th} \text{ draw})$$
$$= P\left(\begin{matrix}\text{2 aces in}\\ x \text{ draws}\end{matrix}\right) + P\left(\begin{matrix}\text{1 ace in}\\ x \text{ draws}\end{matrix}\right) + P\left(\begin{matrix}\text{0 ace in}\\ x \text{ draws}\end{matrix}\right)$$
$$= \boxed{\frac{\binom{4}{2}\binom{48}{x-2}}{\binom{52}{x}} + \frac{\binom{4}{1}\binom{48}{x-1}}{\binom{52}{x}} + \frac{\binom{4}{0}\binom{48}{x}}{\binom{52}{x}}}$$

(ii) $P(X=x) = P(X>x-1) - P(X>x)$

and use above answer

(or)

$$P(X=x) = P\left(\begin{matrix}\text{2 aces in}\\ x-1 \text{ draws}\end{matrix}, \begin{matrix}\text{ace on}\\ x^{th} \text{ draw}\end{matrix}\right)$$
$$= P\left(\begin{matrix}\text{2 aces in}\\ x-1 \text{ draws}\end{matrix}\right) P\left(\begin{matrix}\text{ace on}\\ x^{th} \text{ draw}\end{matrix} \,\middle|\, \begin{matrix}\text{2 aces}\\ \text{in } x-1 \text{ draws}\end{matrix}\right)$$
$$= \boxed{\frac{\binom{4}{2}\binom{48}{x-3}}{\binom{52}{x-1}} \cdot \frac{2}{52-x+1}}$$

(iii) label non aces $1, 2, 3, \ldots, 48$

$$X = 3 + I_1 + I_2 + \cdots + I_{48}$$

non ace can go in any of 5 slots,

$$\_\_ A_1 \_\_ A_2 \_\_ A_3 \_\_ A_4 \_\_$$

where,

$p_1 = 3/5$

$$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ non ace before } 3^{rd} \text{ ace} \\ 0 & \text{else} \end{cases}$$

$$\boxed{E(X) = 3 + 48\left(3/5\right) = 28.8}$$

(iv)

$$\begin{aligned} \mathrm{var}(X) &= \mathrm{var}(3 + I_1 + \cdots + I_{48}) \\ &= \mathrm{var}(I_1 + \cdots + I_{48}) \\ &= E\left((I_1 + \cdots + I_{48})^2\right) - E(I_1 + \cdots + I_{48})^2 \end{aligned}$$

$$E\left((I_1 + \cdots + I_{48})^2\right) = 48 E(I_1) + 48 \cdot 47 E(I_{12})$$

$E(I_1)$: " p_1

where

2nd non ace can go in any of 4 out of 6 slots

$$\_\_ A_1 \_\_ A_2 \_\_ nA_1 \_\_ A_3 \_\_ A_4 \_\_$$

$p_{12} = 3/5 \; 4/6$

$$I_{12} = \begin{cases} 1 & \text{if } 1^{st} \text{ and } 2^{nd} \text{ non ace before } 3^{rd} \text{ ace} \\ 0 & \text{else} \end{cases}$$

$$\mathrm{var}(X) = 48 p_1 + 48 \cdot 47 q_{12} - (48 p_1)^2 = \boxed{101.76}$$

Tiffany Yu

3. Phone calls arrive into a telephone exchange according to a Poisson arrival process at rate λ per unit time. This exchange serves three regions, and an incoming call gets routed to region i with probability p_i, for $i = 1, 2, 3$. Note that $p_1 + p_2 + p_3 = 1$.

   (a) Let N_t^i denote the numbers of calls routed to region i in time t starting from time 0. Find $P(N_t^1 = j, N_t^2 = k)$.

Soln

$$\left.\begin{array}{l} N_t^1 \sim \text{Pois}(\lambda p_1 t) \\ N_t^2 \sim \text{Pois}(\lambda p_2 t) \end{array}\right\} \text{ indep.}$$

$$P(N_t^1 = j, N_t^2 = k) = P(N_t^1 = j)P(N_t^2 = k)$$

$$= \boxed{\frac{e^{-\lambda p_1 t}(\lambda p_1 t)^j}{j!} \frac{e^{-\lambda p_2 t}(\lambda p_2 t)^k}{!k}}$$