Stat 134: Section

Brett Kolesnik

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Conceptual Review

Please discuss these short questions with those around you in section. These problems are intended to highlight concepts that will be relevant for today's problems.

a. How do you compute the distribution of the sum of two random variables?

b. How do you compute the distribution of the ratio of two random variables?

Problem 1

Let S_3 be the sum of 3 independent uniform (0,1) random variables. Find $P(S_3 \leq 1.5)$.
Ex 5.4.2 in Pitman's Probability

Recall that the sum of two uniform random variables, denoted S_2, have density z for $z \in [0,1]$ and $2-z$ for $z \in [1,2]$. Then, we can compute:

$$P(S_3 \leq 1.5) = \int_{z=0}^{1.5} \int_{x=0}^{1.5-z} p(z,x) dx dz$$

Divide the integral into the part where $z \leq 0.5$ and its complement. Then, by independence of S_2 and X we find that:

$$\int_{z=0}^{0.5} \int_{x=0}^{1.5-z} p(z,x) dx dz = \int_{z=0}^{0.5} p(z) dz = \frac{1}{8}.$$

For $z > 0.5$, $\int_{x=0}^{1.5-z} p(z,x) dx = 1.5 - z$, so:

$$\int_{z=0.5}^{1.5} \int_{x=0}^{1.5-z} p(z,x) dx dz = \int_{z=0.5}^{1.5} (1.5-z) p(z) dz = \frac{3}{8}.$$

Hence, the result is one half. Alternatively, one can use the symmetry property to directly argue that the result is one half but this approach does not work for other numbers.

Problem 2

Find the density of $Z = X - Y$, where X, Y are independent exponential (λ) variables.
Ex 5.4.13 in Pitman's Probability

First we assume that z is nonnegative. Direct computation gives:

$$f_Z(z) = \int_{x=z}^{\infty} p_X(x) \cdot p_Y(x-z)dx,$$

which leads to the result $\frac{\lambda e^{-\lambda z}}{2}$. The case where z is negative follows by a similar computation and we conclude the result to be $\frac{\lambda e^{-\lambda|z|}}{2}$, and one may notice the symmetric property of this variable Z.

Problem 3

Suppose $X_1, \cdots, X_n$ are independent gamma distributions with parameters (r_i, λ). What is the distribution of $X_1 + X_2 + \cdots + X_n$?
Ex 5.4.6 in Pitman's Probability

Just consider the case of two independent Gamma with (r_1, λ) and (r_2, λ). Then,

$$\begin{aligned} f_{X+Y}(z) &= \int_{x=0}^{z} f_X(x) f_Y(z-x)dx \\ &\propto \int_{x=0}^{z} x^{r_1-1} e^{-x/\lambda} (z-x)^{r_2-1} e^{-(z-x)/\lambda} dx \\ &= e^{-z/\lambda} \int_{x=0}^{z} x^{r_1-1} (z-x)^{r_2-1} dx \\ &= e^{-z/\lambda} z^{r_1+r_2-1} \int_{u=0}^{1} u^{r_1-1} (1-u)^{r_2-1} dx \\ &\propto e^{-z/\lambda} z^{r_1+r_2-1} \end{aligned}$$

where $\propto$ means 'is proportional to'. Note that the resulting density is exactly proportional to the density of a Gamma distribution with $(r_1 + r_2, \lambda)$, and since the have the same domain, it follows that the sum of two Gammas should be Gamma with $(r_1 + r_2, \lambda)$. By induction, it follows that $X_1 + \cdots + X_n$ should be a Gamma distribution with parameters $(r_1 + \cdots + r_n, \lambda)$.