WED, JAN 29

MORE ON BAYES' RULE:

$$\boxed{\mathbb{P}(B_j \mid A) = \frac{\mathbb{P}(A \mid B_j)\,\mathbb{P}(B_j)}{\sum_{i=1}^{n} \mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)}}$$ (P.49)

IMPORTANT DISCUSSION, BOTTOM OF P.48:

1^{st} STAGE: ONE OF $B_1, \ldots, B_n$ OCCURS (UNKNOWN)

2^{nd} STAGE: RESULT A IS OBSERVED.

$\mathbb{P}(B_i \mid A) \longleftarrow$ POSTERIOR PROBS.

$\mathbb{P}(B_i) \longleftarrow$ PRIOR PROBS.

$\mathbb{P}(A \mid B_i) \longleftarrow$ LIKELIHOODS.

OFTEN EASIER TO CALCULATE.

§1.6 SEQUENCES OF EVENTS.

* OFTEN RANDOM EXPERIMENTS/ PROCESSES HAVE MULTIPLE STEPS/STAGES.

<u>GROUPS</u> LET A, B, C BE EVENTS.

SHOW THAT:

$$\boxed{\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B \mid A)\,\mathbb{P}(C \mid A \cap B)}$$

<u>ANS</u>

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A \cap B)\,\mathbb{P}(A \cap B \cap C \mid A \cap B)$$

$$= \mathbb{P}(A \cap B)\,\mathbb{P}(C \mid A \cap B)$$

$$= \mathbb{P}(A)\,\mathbb{P}(B \mid A)\,\mathbb{P}(C \mid A \cap B).$$

IN GENERAL:

$$\boxed{\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = \mathbb{P}(A_1)\,\mathbb{P}(A_2 \mid A_1)\,\mathbb{P}(A_3 \mid A_1 \cap A_2) \cdots \mathbb{P}\left(A_n \,\middle|\, \bigcap_{i=1}^{n-1} A_i\right).}$$

EG GEOMETRIC(p) DISTRIBUTION.

COIN: $\mathbb{P}(T) = p$, $\mathbb{P}(H) = 1-p$.

X = # OF FLIPS UNTIL A T.

WRITE T_i AND H_i FOR EVENT THAT

iTH FLIP IS A T/H.

*FLIPS INDEP.

$$\mathbb{P}(X=k) = \mathbb{P}(H_1 H_2 \cdots H_{k-1} T_k)$$
$$= \mathbb{P}(H_1)\mathbb{P}(H_2)\cdots\mathbb{P}(H_{k-1})\mathbb{P}(T_k)$$
$$= (1-p)^{k-1} p$$

$(k \geq 1)$.

INDEPENDENCE, IN GENERAL:

A, B, C ARE INDEPENDENT IF

1. $\mathbb{P}(B \mid A) = \mathbb{P}(B \mid A^c) = \mathbb{P}(B)$
2. $\mathbb{P}(C \mid AB) = \mathbb{P}(C \mid A^c B) = \mathbb{P}(C \mid AB^c)$
$= \mathbb{P}(C \mid A^c B^c) = \mathbb{P}(C)$.

$$\Rightarrow \mathbb{P}(ABC) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C).$$

* THIS GENERALIZES TO $n \geq 2$ EVENTS
$A_1, \ldots, A_n$

→ SEE DISCUSSION ON p. 67.

PAIRWISE INDEPENDENCE:

$A_1, \ldots, A_n$ SUCH THAT A_i, A_j ARE INDENDENT (i.e. $P(A_i A_j) = P(A_i)P(A_j)$) FOR ALL $1 \leq i \neq j \leq n$.

GROUPS FIND AN EXAMPLE THAT SHOWS THAT

PW INDEP. $\neq$ INDEP.

ANS FLIP A FAIR COIN TWICE

S = EVENT THAT BOTH FLIPS ARE THE SAME.

① CLEARLY S, 1^{st} FLIP, 2^{nd} FLIP ARE NOT INDEPENDENT.

② BUT

$$\mathbb{P}(H_1 \cap S) = \mathbb{P}(H_1 \cap H_2) = \frac{1}{4} = \mathbb{P}(H_1)\mathbb{P}(S)$$

SO S, 1^{st} FLIP ARE INDEP.

(SEE P. 70 FOR ALL DETAILS)

EG (GAMBLER'S RUIN)

SUPPOSE YOU PLAY n INDEPENDENT GAMES. IN EACH YOU HAVE A $1/N$ CHANCE OF WINNING.

HOW MANY TIMES n MUST YOU PLAY TO HAVE $\geq 50\%$ CHANCE OF WINNING AT LEAST ONCE?

(AN OLD GAMBLERS' RULE OF THUMB SAYS $n \approx \frac{2}{3}N$.)

ANS (p.60)

$$\mathbb{P}(\text{WIN} \geq \text{ONCE}) = 1 - \mathbb{P}(\text{NO WINS})$$

$$= 1 - \left(1 - \frac{1}{N}\right)^n \overset{\text{(SET)}}{=} \frac{1}{2}.$$

$$\Rightarrow \quad \frac{1}{2} = \left(1 - \frac{1}{N}\right)^n$$

$$\Rightarrow \quad \log\left(\frac{1}{2}\right) = n \log\left(1 - \frac{1}{N}\right)$$

$$\Rightarrow \quad n = \log(1/2) / \log(1 - 1/N)$$

$$\Rightarrow \quad n \approx N \log(2) \quad \left[\log(1+z) \approx z, \ z \to 0\right]$$

$$\approx 0.69 N \approx \frac{2}{3} N.$$

<u>GROUPS</u> DECK OF CARDS IS WELL-SHUFFLED, AND 5 CARDS ARE DEALT.

$\mathbb{P}$(FLUSH)

= $\mathbb{P}$(ALL 5 OF SAME SUIT) = ?

ANS LET S_i, H_i, D_i, C_i BE EVENT THAT iTH CARD IS A S, H, D, C.

TREE DIAGRAM

$$\bullet \xrightarrow[\frac{13}{52}]{} S_1 \xrightarrow[\frac{12}{51}]{} S_2 \xrightarrow[\frac{11}{50}]{} S_3 \xrightarrow[\frac{10}{49}]{} S_4 \xrightarrow[\frac{9}{48}]{} S_5$$

$$\mathbb{P}(\text{SPADE FLUSH}) = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{C(13,5)}{C(52,5)}$$

$$\mathbb{P}(\text{FLUSH}) = 4 \cdot \mathbb{P}(\text{SPADE FLUSH})$$

$$= 4 \cdot \frac{C(13,5)}{C(52,5)} \approx 0.198\% .$$