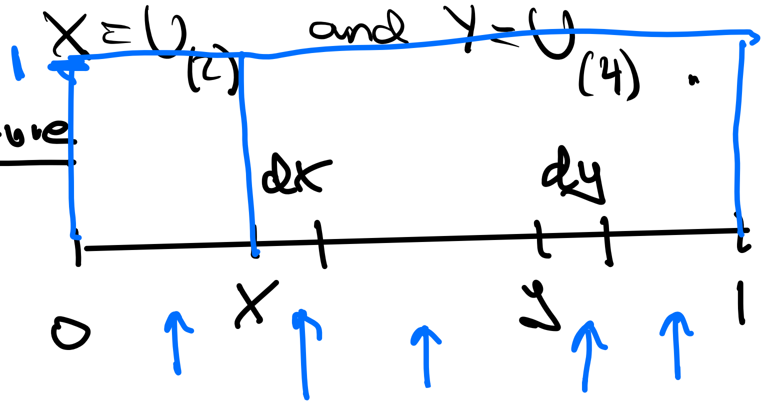
Stat 134 lec 27

warmup 9:00–9:10

ex Throw down 5 darts on $(0,1)$.
Find the joint density, $f(x,y)$, of
$X = U_{(2)}$ and $Y = U_{(4)}$.

Hint: Find $P(X\in dx, Y \in dy) \approx f(x,y)\,dx\,dy$

Picture



$$P(X\in dx, Y\in dy) = \binom{5}{1} x \binom{4}{1} dx \binom{3}{1}(y-x)\binom{2}{1} dy \binom{1}{1}(1-y)$$

$$= \underbrace{\binom{5}{1,1,1,1,1} x (y-x)(1-y)}_{f(x,y) \text{ for } 0\le x<y\le 1}\,dx\,dy$$

Last time

Sec 4.6 uniform order statistic

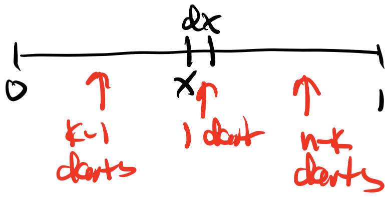
$U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$

$U_{(1)}, \dots, U_{(n)}$ order statistics

ex
$U_1 = .2 \quad U_2 = .7 \quad U_3 = .1$

$\Rightarrow U_{(1)} = .1, \ U_{(2)} = .2, \ U_{(3)} = .7$

$P(U_{(k)} \in dx) = f(x)dx$



Note

$$\binom{n}{a,b,c} = \frac{n!}{a!\,b!\,c!} = \binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}$$

$$f_{U_{(k)}}(x) = \binom{n}{k-1,1,n-k} x^{k-1}(1-x)^{(n-k+1)-1} \quad \text{on } 0<x<1$$

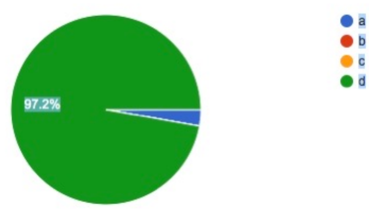
. $x^2(1-x)^4$ for $0 < x < 1$ is the variable part of the density of what random variable?

a $U_{(3)}$ of n=6 darts

b $U_{(2)}$ of n=7 darts

c $U_{(1)}$ of n=7 darts

d none of the above

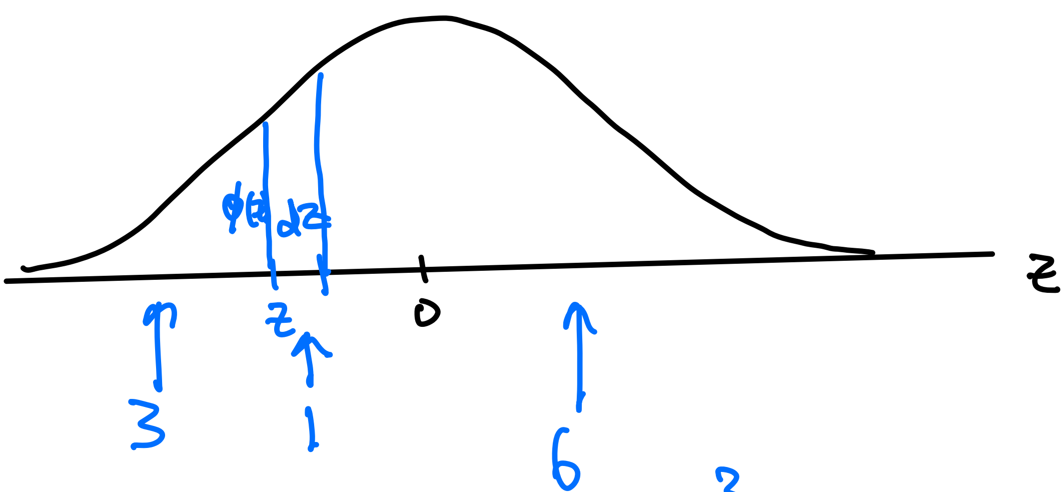


d	U(3) of n=7 darts. The x^2 tell us there are 2 before dx, so dx must be the third. The (1-x)^4 tells us there are 4 after. So, there is a total of 7.

ex

normal order statistic

Let $Z_{(1)}, \dots, Z_{(10)}$ be the values of 10 independent standard normal variables arranged in increasing order. Find the density of $Z_{(4)}$



$$P(Z_{(4)} \in dz) = \binom{10}{3,1,6} (\Phi(z))^3 \, \phi(z)dz \, (1-\Phi(z))^6$$

??

$f(z)dz$

$$\boxed{f(z) = \binom{10}{3,1,6} (\Phi(z))^3 \phi(z) (1-\Phi(z))^6}$$

$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ $\quad \Phi(z) = P(Z_{(4)} < z)$

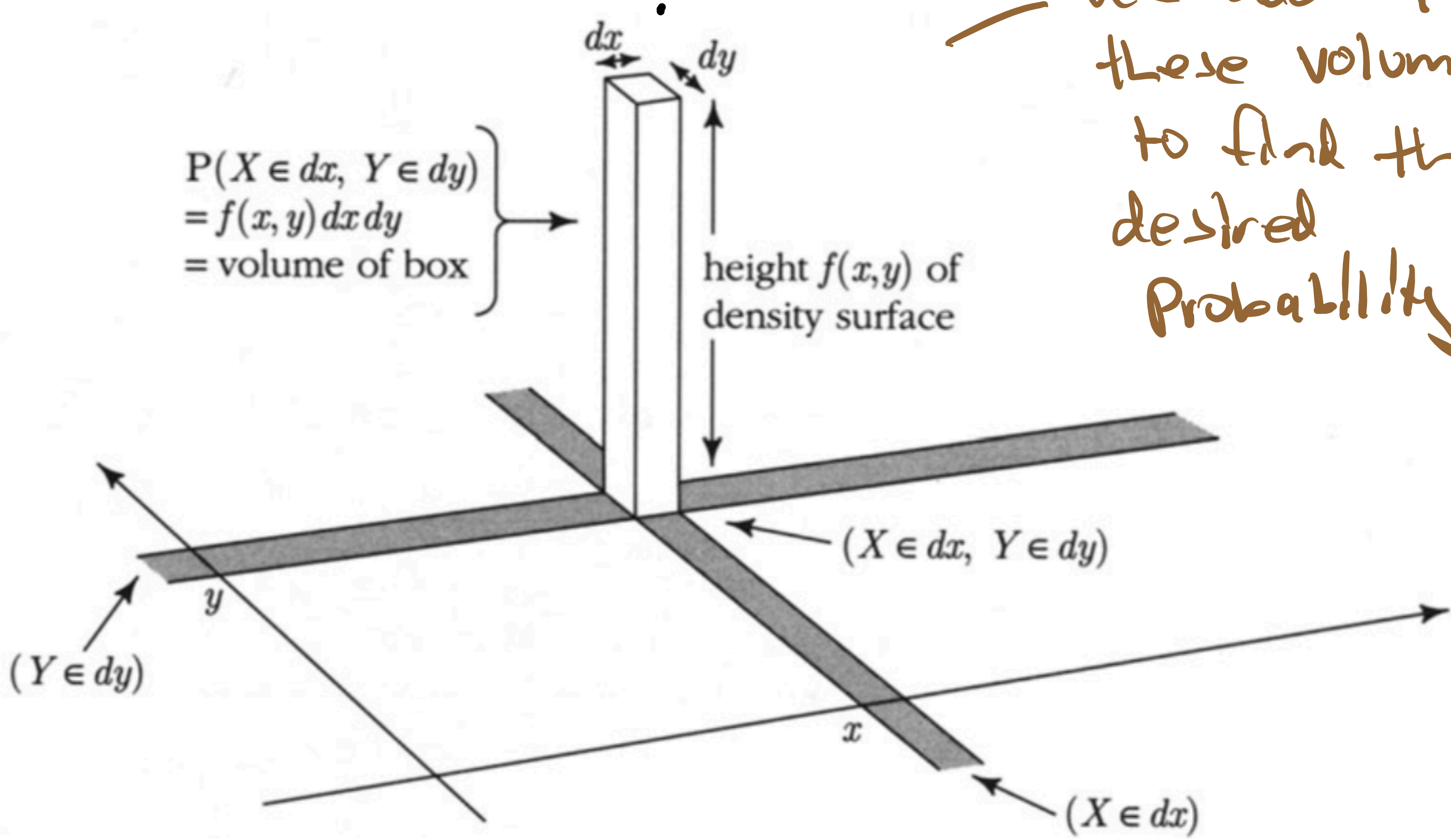
today ① sec 5.1, 5.2 Continuous Joint Distribution

② sec 4.6 Beta distribution

③ sec 5.1, 5.2 Calculate probabilities with $f(x, y)$.

① Sec 5.1, 5.2 Joint Density

$P(X \in dx, Y \in dy) \approx f(x,y)\,dx\,dy$



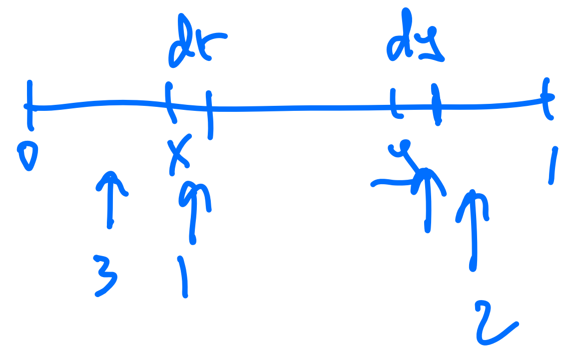
we add up these volumes to find the desired probability.

$$\int_y \int_x f(x,y)\,dx\,dy = \int_x \int_y f(x,y)\,dy\,dx = 1$$

Let (X, Y) have joint density $f_{X,Y}(x, y) = 420x^3(1-y)^2$ for $0 < x < y < 1$.

Fill in the blanks: X and Y represent the 4th smallest and 5 smallest of 7 i.i.d. Unif (0,1) random variables, respectively.

$X = U_{[4]}$



② Sec 4.6 Beta distribution.

$X \sim Beta(r,s)$ for $r>0, s>0$ is a distribution often used to model physical processes that take values between 0 and 1.

ex the proportion of defective items in a shipment.

Let $r,s>0$

Defn $X \sim Beta(r,s)$ if

$$f(x) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} x^{r-1}(1-x)^{s-1} \quad \text{for } 0 \leq x < 1.$$

where $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$ Gamma function for $r>0$

or $\Gamma(r) = (r-1)!$ if $r \in \mathbb{Z}^+$

Notice if $r=1, s=1$, $f(x) = 1_{x\in(0,1)}$

$\Rightarrow Beta(1,1) = Unif(0,1)$.

ex Let $U_1, \ldots, U_n \overset{iid}{\sim} U(0,1)$

$$f_{U_{(k)}}(x) = \binom{n}{k-1, 1, n-k} x^{k-1}(1-x)^{(n-k+1)-1} \quad \text{on } 0<x<1$$

compare with,

$$f_{Beta(r,s)}(x) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} x^{r-1}(1-x)^{s-1} \quad \text{for } 0 \leq x < 1.$$

Notice that $f_{U_{(k)}}(x)$ and $f_{Beta(r,s)}(x)$ have the same variable part of their density when $r = k$

$s = n-k+1$

then $\Gamma(s+r) = \Gamma(n-k+1+k) = \Gamma(n+1) = n!$

$\Gamma(r) = (k-1)!$

$\Gamma(s) = (n-k)!$

$$\Rightarrow \frac{\Gamma(s+r)}{\Gamma(r)\Gamma(s)} = \binom{n}{k-1,\ 1,\ n-k}$$

$\Rightarrow$ Standard uniform ordered statistics are beta!

see appendix of notes

Thm $X \sim Beta(r,s)$

$$E(X) = \frac{r}{r+s}$$

Hence if $X \sim U_{(k)}$

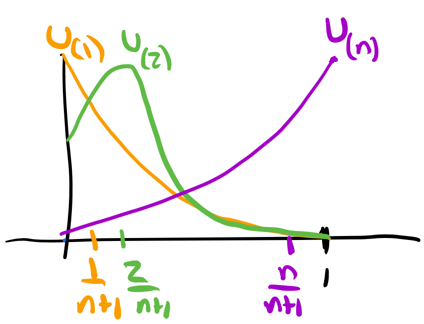
$$E(X) = \frac{k}{n-k+1+k} = \boxed{\frac{k}{n+1}}$$

$E(U_{(1)}) = \frac{1}{n+1}$

$E(U_{(2)}) = \frac{2}{n+1}$

$\vdots$

$E(U_{(n)}) = \frac{n}{n+1}$



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Stat 134

Friday October 21 2022

1. Let P be the chance a coin lands head. Suppose the prior distribution of P is $f_P(p) = c(1-p)^4$ for $0 \le p \le 1$ for some constant c. Which of the following is true:

   a $P \sim Beta(1, 4)$

   b $c = 5$

   c $E(P) = \frac{1}{5}$

   d more than one of the above

Compare $(1-p)^4$ and $p^{r-1}(1-p)^{s-1}$ $\Rightarrow$ $r=1$, $s=5$

$P \sim Beta(1,5)$

$$c = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} = \frac{\Gamma(6)}{\Gamma(1)\Gamma(5)} = \frac{5!}{0!\,4!} = \boxed{5}$$

$$E(P) = \frac{r}{r+s} = \frac{1}{1+5} = 1/6$$

If $X \sim Beta(r, s)$

$$f(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}$$

Since $\int_0^1 f(x)\,dx = 1 \Rightarrow \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$

ex Let $X \sim Beta(3,4)$

$\left(\text{so } f(x) = \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} x^2(1-x)^3 \right)$

Compute $E(7X - 5X^6)$

$$= 7\underset{\overset{\|}{3/7}}{E(X)} - 5E(X^6)$$

$$E(x^6) = \int_0^1 x^6 \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} x^2(1-x)^3\,dx$$

$$= \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} \underbrace{\int_0^1 x^8(1-x)^3\,dx}_{\frac{\Gamma(9)\Gamma(4)}{\Gamma(13)}} = \frac{\Gamma(9)\Gamma(7)}{\Gamma(3)\Gamma(13)}$$

$$\Rightarrow E(7x - 5x^6) = \boxed{3 - 5\frac{\Gamma(9)\Gamma(7)}{\Gamma(3)\Gamma(13)}}$$

Appendix

Let $X \sim Beta(r,s)$

then $E(X) = \dfrac{r}{r+s}$.

Pf/ Note that $\displaystyle\int_0^1 f(x)\,dx = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = 1$

$$\Rightarrow \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$$

$$E(X) = \int_0^1 x f(x)\,dx = \frac{\Gamma(r+s)}{\Gamma(r)\cancel{\Gamma(s)}} \underbrace{\int_0^1 \overbrace{x\, x^{r-1}}^{x^{r+1-1}} (1-x)^{1-s}\,dx}_{=\ \frac{\cancel{\Gamma(s)}\Gamma(r+1)}{\Gamma(s+r+1)}}$$

$$= \frac{(r+s-1)!\; r!}{(s+r)!} = \boxed{\frac{r}{r+s}}$$

$\square$