

Quiz 5 Wednesday Sec 4.4-4.6, MGF and
5.1, 5.2 up to today's lecture.

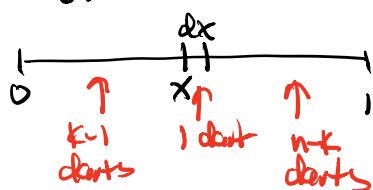
Test time

Sec 4.6 Uniform order statistic

$$U_1, \dots, U_n \stackrel{iid}{\sim} U(0,1)$$

$U_{(1)}, \dots, U_{(n)}$ order statistics

$$P(U_{(k)} \in dx) = f(x)dx$$



Note
 $\binom{n}{a,b,c} = \frac{n!}{a!b!c!}$

$$= \binom{n}{a} \binom{n-a}{b} \binom{n-a-b}{c}$$

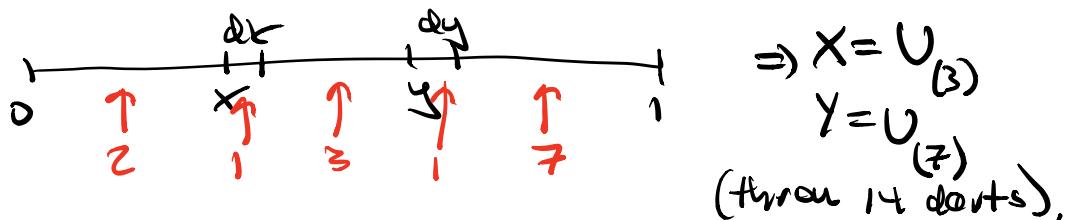
$$f_{U_{(k)}}(x) = \binom{n}{k-1, 1, n-k} x^{k-1} (1-x)^{n-k-1} \quad \text{on } 0 < x < 1$$

Sec 5.1, 5.2 Continuous Joint Distribution

$$P(X \in dx, Y \in dy) = f(x, y) dx dy.$$

Ex What joint density, $f(x, y)$ has
variable part $x^2(y-x)^3(1-y)^7$ on $0 \leq x \leq y \leq 1$.

expect X and Y are unit ordered statistics.



today ① Review student responses to concept test.

② Sec 4.6 Beta Distribution

③ Sec 5.1, 5.2 Calculate probabilities with $f(x, y)$.

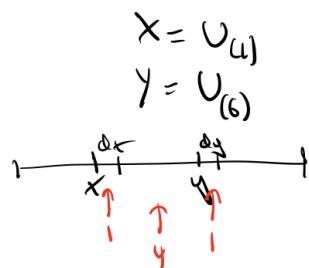
① Student responses

Is $f(x, y) = \binom{6}{1,4,1}(y-x)^4$ on $0 < x < y < 1$ a joint density function?

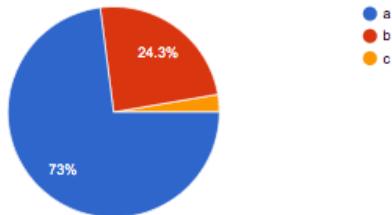
a) yes

b) no

c) not enough info to decide



37 responses

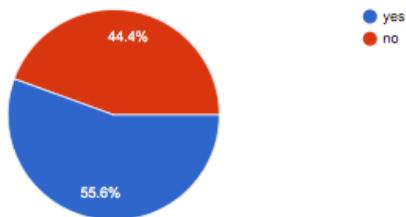


a
b
c

feedback

Did a student explanation from part 1 help you?

36 responses



yes
no

② Sec 4.6 Beta distribution.

$X \sim \text{Beta}(r, s)$ for $r > 0, s > 0$ is a distribution often used to model physical processes that take values between 0 and 1,
 ex: the proportion of defective items in a shipment.

Defⁿ Let $r, s > 0$
 $X \sim \text{Beta}(r, s)$ if

$$f(x) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} x^{r-1} (1-x)^{s-1} \quad \text{for } 0 < x < 1.$$

where $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$ Gamma function for $r > 0$
 or $\Gamma(r) = (r-1)! \quad r \in \mathbb{Z}^+$

Notice if $r=1, s=1, f(x) = 1_{x \in (0,1)}$
 $\Rightarrow \text{Beta}(1, 1) = \text{Unif}(0,1).$

ex Let $U_1, \dots, U_n \stackrel{iid}{\sim} U(0,1)$

$$f_{U_k}(x) = \binom{n}{k, 1, n-k} x^{k-1} (1-x)^{(n-k)-1} \quad \text{on } 0 < x < 1$$

Notice that $f_{U_{(k)}}(x)$ and $f_{\text{Beta}(s,s)}(x)$ have the same variable part of their density when $r = k$

$$s = n - k + 1$$

then $\Gamma(s+r) = \Gamma(n-k+1+k) = \Gamma(n+k) = n!$

$$\Gamma(r) = (k-1)!$$

$$\Gamma(s) = (n-k)!$$

$$\Rightarrow \frac{\Gamma(s+r)}{\Gamma(r)\Gamma(s)} = \binom{n}{k-1, 1, n-k}$$

\Rightarrow Uniform ordered statistics are beta !

Thm *see appendicit notes* $X \sim \text{Beta}(r, s)$

$$E(X) = \frac{r}{r+s}$$

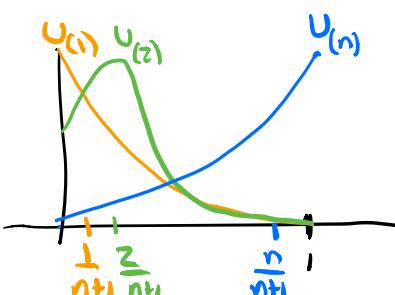
Hence if $X \sim U_{(k)}$

$$E(X) = \frac{k}{n-k+1+k} = \boxed{\frac{k}{n+1}}$$

$$E(U_{(1)}) = \frac{1}{n+1}$$

$$E(U_{(2)}) = \frac{2}{n+1}$$

$$\vdots \\ E(U_{(n)}) = \frac{n}{n+1}$$



Ex (Bayesian Statistics)

The chance a coin lands heads is modeled by the RV w/ density

$$f_p(p) = \begin{cases} C(1-p)^r & \text{for } 0 \leq p \leq 1 \\ 0 & \text{else} \end{cases}$$

a) Is this a beta distribution? If so, what are the parameters?

Yes, compare $(1-p)^r$ and $p^{r-1}(1-p)^{s-1} \Rightarrow P \sim \text{Beta}(r, s)$

b) Calculate the constant C

$$C = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} = \frac{\Gamma(s)}{\Gamma(1)\Gamma(s)} = \frac{s!}{0!s!} = \textcircled{s}$$

c) What is the mean of P?

$$E(P) = \frac{r}{r+s} = \frac{1}{1+5} = \textcircled{1/6}$$

If $X \sim \text{Beta}(r, s)$

$$f(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1} (1-x)^{s-1}$$

Since $\int_0^1 f(x) dx = 1 \Rightarrow \int_0^1 x^{r-1} (1-x)^{s-1} dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$

ex Let $X \sim \text{Beta}(3, 4)$

Compute $E(7x - 5x^6)$

$$E(7x - 5x^6) = 7E(x) - 5E(x^6)$$

$$\begin{aligned} E(x^6) &= \int_0^1 x^6 \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} x^2 (1-x)^3 dx \\ &= \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} \left[\int_0^1 x^8 (1-x)^3 dx \right] \end{aligned}$$

$$= \frac{\Gamma(9)\Gamma(7)}{\Gamma(3)\Gamma(13)}$$

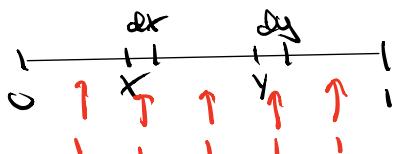
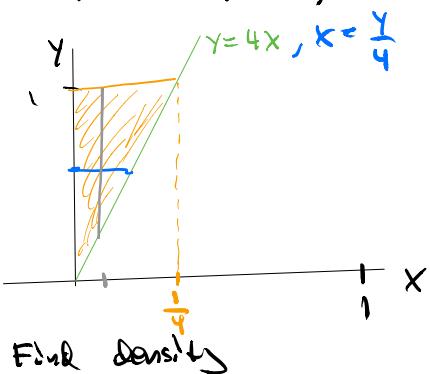
$$\Rightarrow E(7x - 5x^6) = \boxed{3 - 5 \frac{\Gamma(9)\Gamma(7)}{\Gamma(3)\Gamma(13)}}$$

③ See 5.1, 5.2 Calculate probabilities with $f(x, y)$.

Throw down 5 darts on $(0, 1)$.

$$\Leftrightarrow X = U(1), Y = U(4)$$

$$\text{Find } P(Y > 4X)$$



$$f(x, y) = \binom{5}{1, 1, 1, 1} \times (y-x)(1-y)$$

$$= 5! \times (y-x)(1-y)$$

$$P(Y > 4X) = \int_{y=1}^{y=1} \int_{x=\frac{y}{4}}^{x=1} 5! \times (y-x)(1-y) dx dy$$

(or)

$$y=0 \quad x=0$$

$$x=\frac{y}{4} \quad y=1$$

$$P(Y > 4X) = \int_{x=0}^{x=1} \int_{y=4x}^{y=1} 5! \times (y-x)(1-y) dy dx$$

details:

$$\begin{aligned} P(Y > 4x) &= \int_{y=0}^{y=1} \int_{x=0}^{x=y/4} 120 \times (y-x)(1-y) dx dy \\ &= 120 \int_{y=0}^{y=1} \int_{x=0}^{x=y/4} (xy - x^2) dx dy \\ &= \int_{y=0}^{y=1} \left[120(1-y) \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \right]_{x=0}^{x=y/4} dy \\ &= \int_0^1 120(1-y) \left(\frac{y^2}{32} - \frac{y^3}{3 \cdot 64} \right) dy \\ &= \frac{5}{192} \cdot 120 \int_0^1 y^3 - y^4 dy = \frac{5 \cdot 120}{192} \left(\frac{y^4}{4} - \frac{y^5}{5} \right) \Big|_0^1 \\ &= \frac{5 \cdot 120}{192} \left(\frac{1}{4} - \frac{1}{5} \right) = \frac{30}{192} = \textcircled{156} \end{aligned}$$

independent RVs

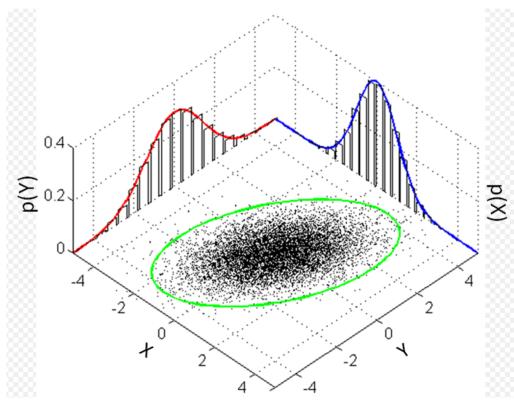
Defn X and Y are independent if

$$P(X \in dx, Y \in dy) = P(X \in dx) P(Y \in dy)$$
$$f(x,y) dx dy \quad " \quad f(x) dx \quad " \quad f(y) dy$$

$$\Leftrightarrow f(x,y) = f(x)f(y)$$

ex $X, Y \stackrel{iid}{\sim} N(0,1)$

$$f(x,y) = \phi(x)\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$
$$= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$



Not a great picture because the oval in green should be a circle. This is the picture of a correlated bivariate normal from chapter 6 instead of an uncorrelated bivariate normal.

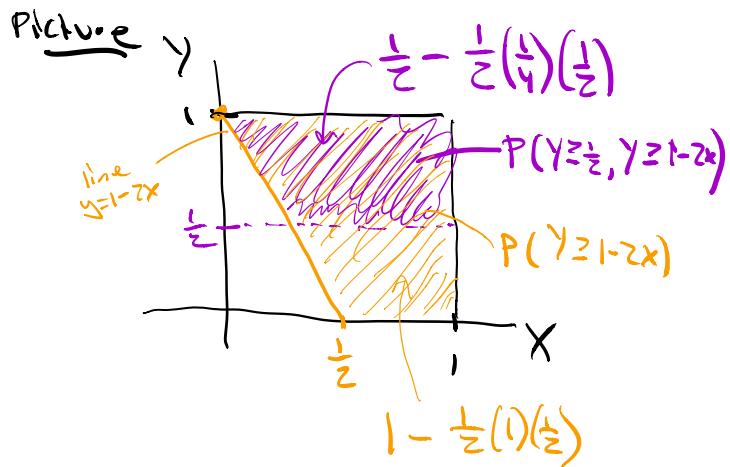
ex If $X, Y \sim i.i.d U(0, 1)$

$$\text{Find } P(Y \geq \frac{1}{2} \mid Y \geq 1 - 2x)$$

soln

$$f(x, y) = f(x)f(y) = 1 \quad \text{for } 0 < x, y < 1$$

$$P(Y \geq \frac{1}{2} \mid Y \geq 1 - 2x) = \frac{P(Y \geq \frac{1}{2}, Y \geq 1 - 2x)}{P(Y \geq 1 - 2x)}$$

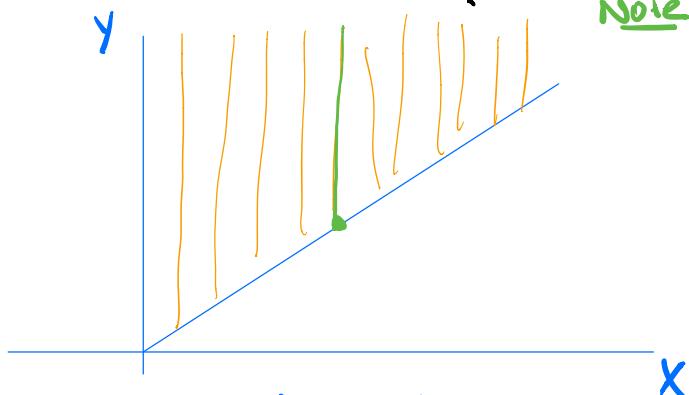


so,

$$\frac{P(Y \geq \frac{1}{2}, Y \geq 1 - 2x)}{P(Y \geq 1 - 2x)} = \frac{\frac{1}{2} - \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)}{1 - \frac{1}{2}(1)\left(\frac{1}{2}\right)} = \frac{\frac{7}{16}}{\frac{3}{4}} = \boxed{\frac{7}{12}}$$

or
 Let $X \sim \text{Exp}(\lambda)$, $Y \sim \text{Exp}(m)$
 be independent lifetimes of two bulbs.

Find $P(Y > X)$.



Note $\int_{y=x}^{y=\infty} e^{-\lambda x} dy = \frac{e^{-\lambda x}}{\lambda}$

$$\begin{aligned}
 f(x,y) &= \lambda e^{-\lambda x} m e^{-\lambda y} \\
 P(X > Y) &= \lambda m \int_{x=0}^{\infty} e^{-\lambda x} \int_{y=x}^{y=\infty} e^{-\lambda y} dy dx \\
 &\quad \left[-\frac{e^{-\lambda y}}{\lambda} \right]_{y=x}^{y=\infty} = \frac{e^{-\lambda x}}{\lambda} \\
 &= \lambda \int_{x=0}^{\infty} e^{-(\lambda+m)x} dx = \boxed{\frac{\lambda}{\lambda+m}}
 \end{aligned}$$

Appendix

Let $X \sim \text{Beta}(r, s)$

then $E(X) = \frac{r}{r+s}$,

Pf/ Note that $\int_0^1 f(x) dx = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 x^{r-1} (1-x)^{s-1} dx = 1$

$$\Rightarrow \int_0^1 x^{r-1} (1-x)^{s-1} dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$$

$$E(X) = \int_0^1 x f(x) dx = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 x \cancel{x^{r-1}} (1-x)^{s-1} dx$$

$$\frac{\cancel{\Gamma(s)\Gamma(r+1)}}{\Gamma(s+r+1)}$$

$$= \frac{(r+s-1)! \cdot r!}{(s+r)!} = \boxed{\frac{r}{r+s}}$$

□