stat 134 lec 30

Q7 next week is cancelled. Too much stress! We want you to learn and be happy ☺

Last time

sec 5.2 Marginal density $f_Y(y) = \int_{x=-\infty}^{x=\infty} f(x,y)\,dx$

sec 5.3 independent normal variables:

Let $X, Y \overset{iid}{\sim} N(0,1)$ with density $\phi(x) = ce^{-\frac{1}{2}x^2}$, $c>0$

$f(x,y) = \phi(x)\phi(y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$ for $c>0$.

we still need to show that $c = \frac{1}{\sqrt{2\pi}}$, $E(X)=0$, $SD(X)=1$.

Today

sec 5.3

① Student explanation to concept test

② Rayleigh distribution $f_R(r) = re^{-\frac{1}{2}r^2}$, $r>0$

③ Sums of independent standard normal RVs is normal.

① Student explanation

1. S and T are i.i.d. Exp(λ). $X = \text{Min}(S, T)$ and $Y = \text{Max}(S, T)$. The joint density is $f(x, y) = 2\lambda^2 e^{\lambda(x+y)}$. The marginal density of X is:

   a $2\lambda e^{-2\lambda x}$ for $x > 0$

   b $2\lambda e^{-\lambda x}$ for $x > 0$

   c $\lambda e^{-\lambda x}$ for $x > 0$

   d none of the above

10/29/201	a	a	Important part is that y goes from x to inf because y represents the maximum. At that point take the integral of the joint density with respect to x from x to inf to get this answer.	
10/29/2018 9:51:5	a	a		
10/29/2018 9:52:0	a	a		
10/29/201	a	a	Integrate f(x,y) from x to infinity since y>x	
10/29/2018 9:52:2	d	a	Oh no	

Method 1

Find marginal

$$f_X(x) = \int_x^\infty 2\lambda^2 e^{-\lambda(x+y)}\, dy$$

$$= 2\lambda^2 e^{-\lambda x} \int_x^\infty e^{-\lambda y}\, dy$$

$$\frac{e^{-\lambda y}}{-\lambda}\Bigg|_x^\infty = \frac{e^{-\lambda x}}{\lambda}$$

$$= \boxed{2\lambda e^{-2\lambda x}}$$

Sec 5.3 Rayleigh Distribution

let $T \sim \text{Exp}(\frac{1}{2})$, $f(t) = \frac{1}{2}e^{-\frac{1}{2}t}$, $t > 0$

$R = \sqrt{T}$

Find $f_R(r)$.

Soln

By the change of variable formula

$$f_R(r) = \frac{1}{\left|(\sqrt{t})'\right|} \cdot f_T(t)\Big|_{t=r^2}$$

$$f_R(r) = \frac{1}{\frac{1}{2\sqrt{t}}} \cdot \frac{1}{2}e^{-\frac{1}{2}t}\Big|_{t=r^2}$$

$$= re^{-\frac{1}{2}r^2}, \quad r > 0$$

R is called the Rayleigh distribution.

Write $R \sim \text{Rayleigh}$.

Note:

$$P(R > r) = P(R^2 > r^2) = P(\underset{\sim \text{Exp}(\frac{1}{2})}{T} > r^2)$$

$$= e^{-\frac{1}{2}r^2}$$

So $\boxed{F_R(r) = 1 - e^{-\frac{1}{2}r^2}}$

For $X, Y \overset{iid}{\sim} N(0,1)$ $g(X,Y)$

we could show that $R = \sqrt{X^2+Y^2}$

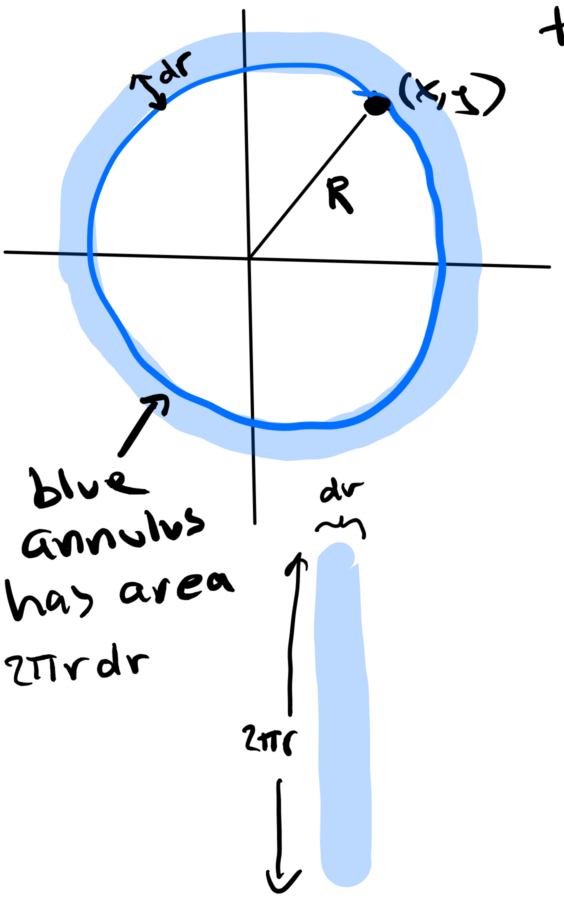
Rayleigh

using a multivariable change of variable formula.

Instead we will do it another way.

Let $R = \sqrt{X^2+Y^2}$

$P(R \in dr)$ is the volume of the cylinder under $f(x,y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$ over the shaded blue annulus.



$P(R \in dr)$ = height of $f(x,y)$ above blue annulus $\cdot$ area of blue annulus

$\|$

$f_R(r)\,dr$

$= c^2 e^{-\frac{1}{2}r^2} \cdot (2\pi r\,dr)$

$= c^2 2\pi\, r e^{-\frac{1}{2}r^2} dr$

Rayleigh Density

$$1 = \int_{r=0}^{r=\infty} P(R \in dr) = c^2 2\pi \underbrace{\int_0^\infty r e^{-\frac{1}{2}r^2} dr}_{=1}$$

$$\Rightarrow c^2 \cdot 2\pi = 1$$

$$\Rightarrow c = \frac{1}{\sqrt{2\pi}}$$

Conclusions

① For $X, Y \overset{iid}{\sim} N(0,1)$ and $T \sim Exp(\frac{1}{2})$

$R = \sqrt{X^2+Y^2}$ and $R = \sqrt{T}$

are both the Rayleigh distribution

$$\Rightarrow X^2+Y^2 \sim Exp(\tfrac{1}{2}) = gamma(1, \tfrac{1}{2})$$

② $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ is density of the standard normal.

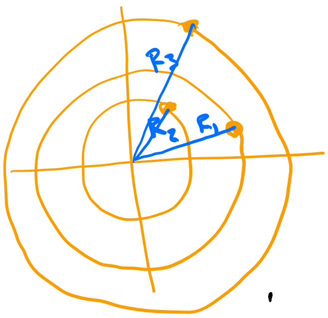
(i.e. $c = \frac{1}{\sqrt{2\pi}}$)

– see end of lecture notes

③ $E(X) = 0$ and $SD(X) = 1$

$\underline{\underline{ex}}$ Suppose 3 shots are fired at a target. Assume for each shot, both X and Y are standard normals. Let W be the closest distance among 3 shots to the bulls eye. Find $f_W(w)$

Soln
Picture!



$$W = \min(R_1, R_2, R_3)$$

where $R_1, R_2, R_3 \overset{iid}{\sim}$ Rayleigh.

Method 1: (use CDF)

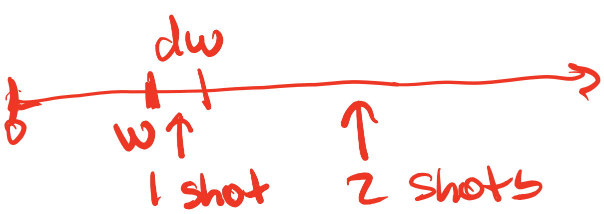
$$P(W>w) = P(R_1>w, R_2>w, R_3>w)$$
$$= P(R_1>w)^3$$
$$= \left(e^{-\frac{1}{2}w^2}\right)^3 = e^{-\frac{3}{2}w^2}$$

$$\Rightarrow F(w) = 1 - e^{-\frac{3}{2}w^2}$$

$$\boxed{f_W(w) = 3we^{-\frac{3}{2}w^2}}$$

Method 2: (Rayleigh ordered statistic)

$$P(W\in dw) = P(R\in dw, R>w, R>w) = P(R\in du)\cdot P(R>u)^2$$



$P(R>w)^2$

$$P(W\in du) = \binom{3}{1,2} we^{-\frac{1}{2}w^2}dw \cdot \left(e^{-\frac{1}{2}w^2}\right)^2 = 3we^{-\frac{3}{2}w^2}du$$

$$\Rightarrow \boxed{f_W(w) = 3we^{-3/2\,u^2}}$$

Sum of independent standard normals

warm up 1:

$X \sim N(0,1)$

$Y = g(X) = -X$

find f_Y

$$f_Y(y) = \frac{1}{|g'(x)|} f_X(x) \Big|_{x=-y}$$

$$= f_X(x)\Big|_{x=-y} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\Big|_{x=-y}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}.$$

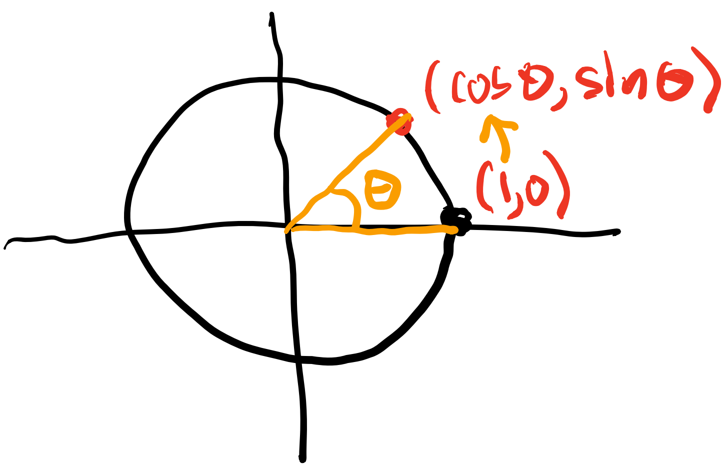
Note if g doesn't stretch the line (i.e. g is a rigid motion)

then $\boxed{f_Y(y) = f_X(x)}$

warm up 2:

rotation matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$



Find the rotation matrix for rotation by $-\theta$

$\cos -\theta = \cos\theta$
$\sin -\theta = -\sin\theta$

$$\rightsquigarrow \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Linear combinations and rotations

We will show that if you add 2 independent standard normals you will get a normal.

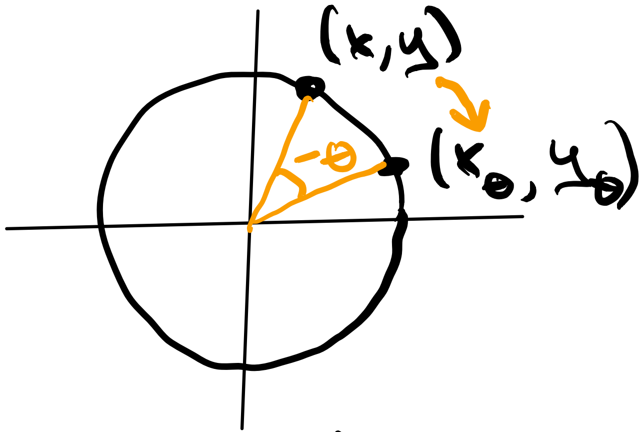
First we will show that if $X, Y \overset{iid}{\sim} N(0,1)$ then $\cos\theta X + \sin\theta Y \sim N(0,1)$.

let $X, Y \overset{iid}{\sim} N(0,1)$

and (x,y) a pt in the plane.

Know $f_{(X,Y)}^{(x,y)} = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$

let (x_θ, y_θ) be rotation of (x,y) clockwise by $-\theta$



$$x^2+y^2 = x_\theta^2 + y_\theta^2$$

What is joint density of (X_θ, Y_θ)?

No stretching of the plane so

$$f_{(X_\theta,Y_\theta)}(x_\theta, y_\theta) = f_{(X,Y)}(x,y)$$

The change of variables rule for joint densities is beyond the scope of this class but for rigid motions it's super simple.

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \qquad x^2+y^2 = x_\theta^2+y_\theta^2$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(x_\theta^2+y_\theta^2)}$$

$\Rightarrow$ compute marginal density

$$\boxed{f_{X_\theta}(x_\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_\theta^2}}$$

$$\Rightarrow \boxed{X_\theta \sim N(0,1)}$$

Next, write X_θ as a function of X and Y using the rotation matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

i.e. Find

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta X + \sin\theta Y \\ -\sin\theta X + \cos\theta Y \end{bmatrix}$$

$$\Rightarrow \boxed{X_\theta = \cos\theta X + \sin\theta Y}$$

It follows that for any α, B with $\alpha^2 + B^2 = 1$ and X, Y iid $N(0,1)$

$$\alpha X + BY \sim N(0,1)$$

For example $\underbrace{\cos\frac{\pi}{4}}_{\text{"}1/\sqrt{2}} X + \underbrace{\sin\frac{\pi}{4}}_{\text{"}1/\sqrt{2}} Y \sim N(0,1)$

Normal distribution

For $Z \sim N(0,1)$ using change of variable rule you can show

$$X = \mu + \sigma Z \sim N(\mu, \sigma^2)$$

Normal Distribution

For $Z \sim N(0,1)$ make a change of scale $X = \mu + \sigma Z$.

Normal (μ, σ^2) Distribution

If Z has standard normal distribution and μ and σ are constants with $\sigma \geq 0$, then

$$X = \mu + \sigma Z$$

has mean μ, standard deviation σ, and variance σ^2. The distribution of X is called the *normal distribution with mean μ and variance σ^2*, abbreviated normal (μ, σ^2). So X has normal (μ, σ^2) distribution if and only if the standardized variable

$$Z = (X - \mu)/\sigma$$

has normal $(0, 1)$ or *standard normal* distribution. To find $P(c < X < d)$, change to standard units and use the standard normal table

$$P(c < X < d) = P(a < Z < b) = \Phi(b) - \Phi(a)$$

where $\quad a = (c - \mu)/\sigma \qquad Z = (X - \mu)/\sigma \qquad b = (d - \mu)/\sigma$

Formula for the normal (μ, σ^2) density. For $\sigma > 0$, the formula is

$$f_X(x) = \frac{1}{\sigma}\phi((x - \mu)/\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} \qquad (-\infty < x < \infty).$$

It follows that for $X, Y \overset{iid}{\sim} N(0,1)$,

$$\frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Y = Z \sim N(0,1)$$

$$\Rightarrow \boxed{X+Y = \sqrt{2}\,Z \sim N(0,2)}$$

Hence, if you add two indep std normals you get a normal.

Appendix

Claim Let $X \sim N(0,1)$. We show that $E(X)=0$ and $SD(X)=1$.

Let $X \sim N(0,1)$

Show $E(X)=0$

$\phi(x)$ is symmetric about zero so all we have to show is that $E(X)$ converges absolutely.

(i.e $E(|X|) < \infty$).

$$E(|X|) = \int_{-\infty}^{\infty} |x|\phi(x)\,dx$$

$$= 2\int_0^{\infty} x\,\phi(x)\,dx$$

$$= 2\int_0^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$= 2\cdot\frac{1}{\sqrt{2\pi}} \int_0^{\infty} x e^{-\frac{x^2}{2}}\,dx$$

$= 1$

Rayleigh density

$$= \sqrt{\frac{2}{\pi}} < \infty.$$

$\Rightarrow E(x) = 0$ since $\Phi(x)$ is semetric around zero

<u>Next</u> we show $SD(X) = 1$:

We know $X^2 + Y^2 \sim \text{expon}\left(\frac{1}{2}\right)$ from above

$$\frac{1}{2} = \lambda$$

so $E(x^2+y^2) = \dfrac{1}{\frac{1}{2}} = 2$

$\frac{1}{2}$ λ rate of $x^2 + y^2$

$$\Rightarrow E(x^2) + \underset{\overset{\parallel}{E(x^2)}}{E(Y^2)} = 2$$

$\Rightarrow 2E(x^2) = 2 \Rightarrow E(x^2) = 1$

but $SD(x) = \sqrt{E(x^2) - E(x)^2}$

$$= \sqrt{1 - 0} = 1$$