Last time — Stat 134 — lec 27

sec 4.5 Expectation of a nonnegative RV using CDF

$$E(X) = \int_0^\infty (1 - F(x))\,dx$$

ex let $X \sim \text{Geom}\left(\frac{1}{2}\right)$

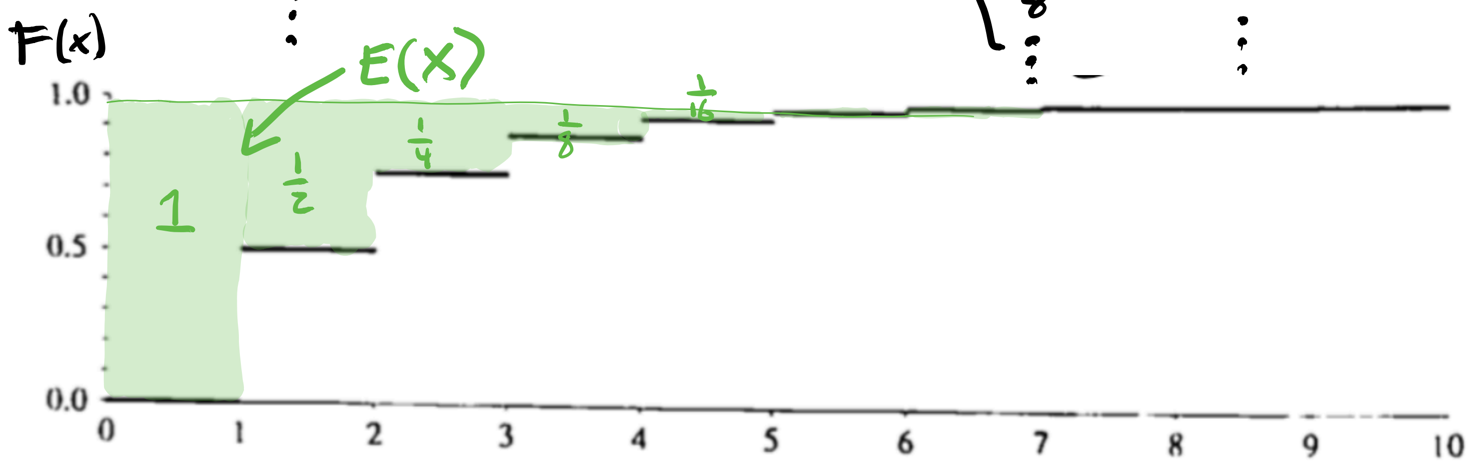
$P(X=1) = 1/2$

$P(X=2) = \frac{1}{2}\cdot\frac{1}{2} = 1/4$

Picture $P(X=3) = \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} = 1/8$

$\vdots$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 1 \\ 1/2 & 1 \le x < 2 \\ \frac{3}{4} & 2 \le x < 3 \\ \frac{7}{8} & 3 \le x < 4 \\ \vdots & \vdots \end{cases}$$



$$E(X) = \int_0^\infty (1 - F(x))\,dx$$

$$E(X) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \boxed{2}$$

p

$$= \sum_{j=0}^{\infty} \underbrace{P(X > j)}_{= q^j} = \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j \leftarrow \text{tail sum formula.}$$

Today

1. Review student explanations of MGF concept test.
2. sec 4.6 order statistics
3. sec 5.1 continuous joint distributions.

① student responses

Let X have density $f(x) = xe^{-x}$ for $x > 0$.
The MGF is?

a $\mathrm{M_X(t)} = \frac{1}{1-t}$ for $t < 1$

b $\mathrm{M_X(t)} = \frac{1}{(1-t)^2}$ for $t < 1$

c $\mathrm{M_X(t)} = \frac{1}{(1+t)^2}$ for $t > -1$

d none of the above

b	This is Gamma(2,1), so the MGF is (1/1-t)^2

match xe^{-x} with $x^{r-1}e^{-\lambda x}$

$\Rightarrow X \sim \text{Gamma}(2,1)$

Know $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r,\ t < \lambda$

b	Using wolfram alpha I know that the integral of x*e^(tx -x) dx from zero to infinity is 1/(1-t)^2 for t <1 thus b is the answer

b	Computing the moment generating function: M_{X}(t) = E(e^{tX}) = \int_{\mathbb{R}^+} e^{tX} \cdot xe^{-x}dx = \int_{\mathbb{R}^+} xe^{x(t-1)}dx. Rewriting the integral, we produce the following: \frac{1}{1-t} \int_{\mathbb{R}^+} x(1-t)e^{-(1-t)x}dx. Looking closely at the above expression, we can clearly see that it is the expectation of an Exponential Distribution with \lambda = 1-t (or equivalently Gamma Distribution with r = 1 and \lambda = 1-t), multiplied by a factor \frac{1}{1-t}. Using this knowledge, we can conclude that M_{X}(t) = (\frac{1}{1-t})^2 with t < 1. The constraint of t < 1 is enforced to preserve convergence.

$M_X(t) = E(e^{tX})$

$= \int_0^\infty e^{tx} x e^{-x} dx$

$= \int_0^\infty x e^{x(t-1)} dx$

$= \frac{1}{1-t}\int_0^\infty x(1-t)e^{-(1-t)x} dx$ $\quad (1-t) = \lambda$

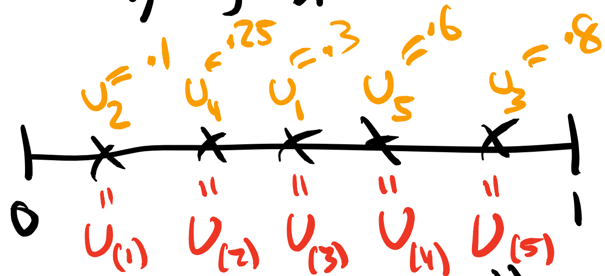
$= \frac{1}{1-t}\int_0^\infty x \lambda e^{-\lambda x}$

expectation of Exp(1-t)

$= \frac{1}{1-t} \cdot \frac{1}{1-t}$

② **Sec 4.6** order statistic of $U(0,1)$

let $U_1, \dots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$



throw 5 darts at $[0,1]$.

let $U_{(k)}$ = called the k^{th} order statistic

$= k^{th}$ largest value of $U_1, \dots, U_n$

(assuming no ties)

ex

$U_{(1)} = \min(U_1, \dots, U_n)$

$U_{(n)} = \max(U_1, \dots, U_n)$

Review counting

You have 3 red, 2 green and 5 blue marbles, How many orderings of these 10 marbles are there?

ex rrrggbbbbb
grrrgbbbbb
ggrrrbbbbb
⋮

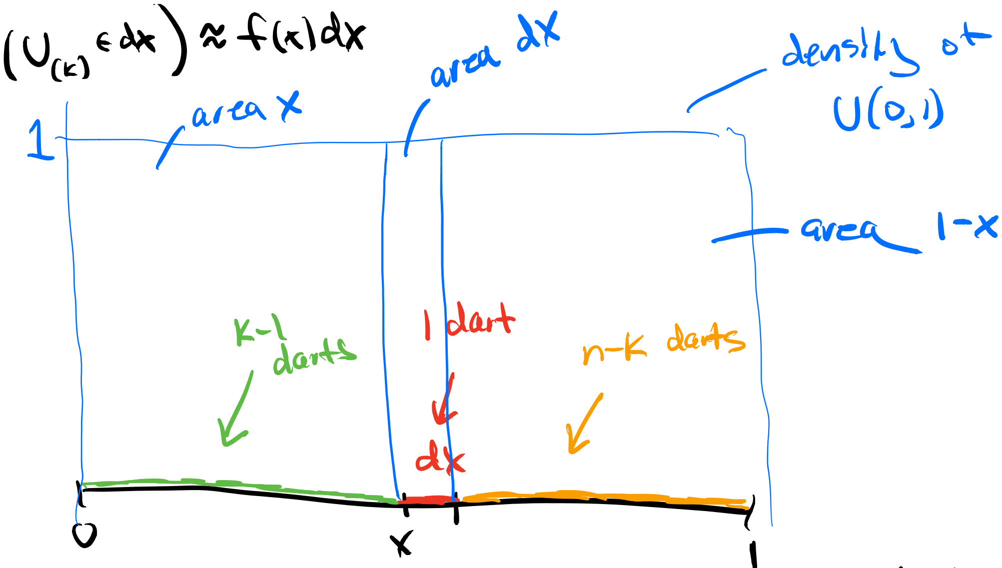
Answ

$$\binom{10}{3,2,5} = \binom{10}{3}\binom{7}{2}\binom{5}{5} = \frac{10!}{3!\,2!\,5!}$$

Next, find density of $U_{(k)}$

write $P(U_{(k)} \in dx) \approx f(x)\,dx$

Picture



$U_{(k)} \in dx$ means that $k-1$ darts are between 0 and x,

and one is in dx, and $n-k$ darts are between x and 1

$$P(U_{(k)} \in dx) = P\big(k\text{-}1 \text{ darts} \in (0,x),\ 1 \text{ dart} \in dx,\ n\text{-}k \text{ darts} \in (x,1)\big)$$

$$= P\big(k\text{-}1 \text{ darts} \in (0,x)\big) \cdot P\big(1 \text{ dart} \in dx \mid k\text{-}1 \text{ darts} \in (0,x)\big)$$

$$\cdot P\big(n\text{-}k \text{ darts} \in (x,1) \mid 1 \text{ dart} \in dx,\ k\text{-}1 \text{ darts} \in (0,x)\big)$$

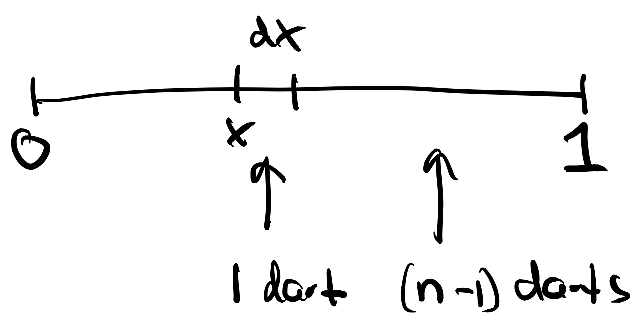
$$= \binom{n}{k-1} x^{k-1} \binom{n-k+1}{1} 1\,dx \binom{n-k}{n-k} (1-x)^{n-k}$$

$$= \underbrace{\binom{n}{k-1,\ 1,\ n-k} x^{k-1} (1-x)^{(n-k+1)-1}}_{f_{U_{(k)}}(x)}\ dx$$

$$\Rightarrow f_{U_{(k)}}(x) = \binom{n}{k-1,\ 1,\ n-k} x^{k-1} (1-x)^{(n-k+1)-1} \quad \text{for } 0<x<1$$

Ex Let $U_1, \dots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$

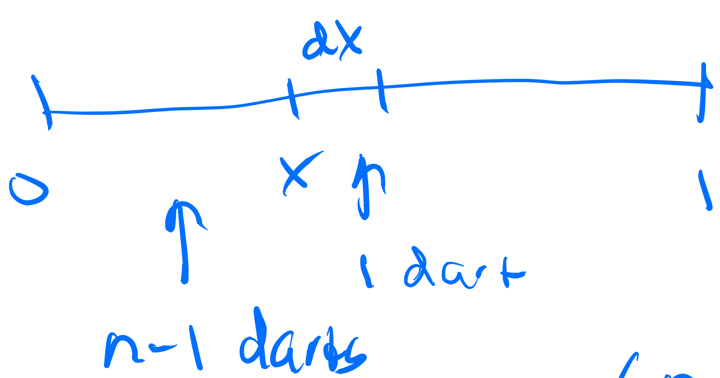
Find the density of $U_{(1)}$



$$f_{U_{(1)}}(x)dx = \binom{n}{1,\ n-1} dx\,(1-x)^{n-1}$$
$$= n(1-x)^{n-1}\,dx$$

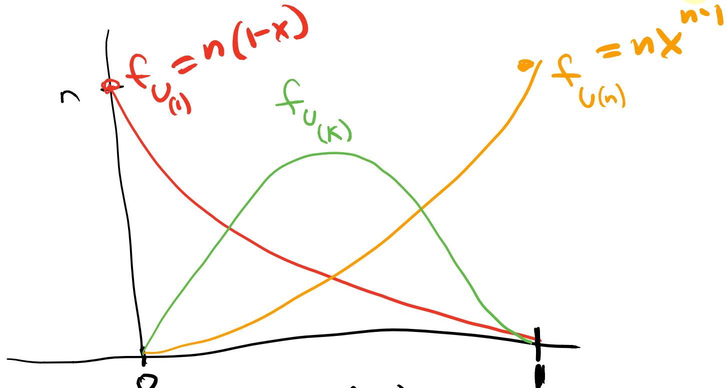
Ex Let $U_1, \dots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$

Find the density of $U_{(n)}$



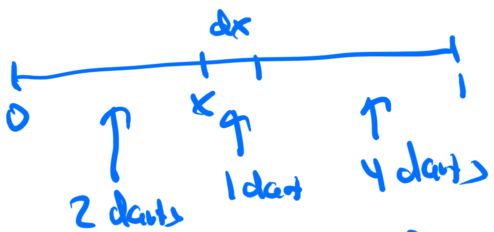
$$f_{U_{(n)}}(x)\,dx \approx \binom{n}{n-1,\,1} x^{n-1}\,dx = n x^{n-1}\,dx$$

<u>Picture</u>



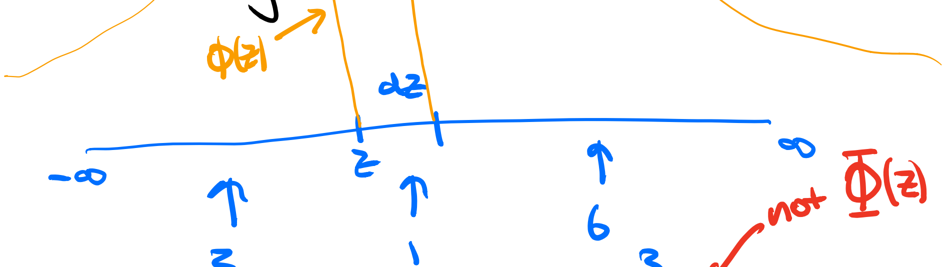
Order statistic of $U(0,1)$ provides a family of densities on the unit interval.

ex $x^2(1-x)^4$ for $0<x<1$ is the variable part of what RV? How many darts do you throw?



$P(U_{(3)} \in dx) = f(x)dx$ $\quad f_{U_{(3)}}(x) = \binom{7}{2,1,4} x^2(1-x)^4$

Ex Let $Z_{(1)}, \dots, Z_{(10)}$ be the values of 10 independent standard normal variables arranged in increasing order. Find the density of $Z_{(4)}$



$$f(z)\,dz = \binom{10}{3,1,6}\left(\Phi(z)\right)^3 \phi(z)\,dz \left(1-\Phi(z)\right)^6$$

$$\Rightarrow f(z) = \binom{10}{3,1,6}\left(\Phi(z)\right)^3 \phi(z)\left(1-\Phi(z)\right)^6$$

Chap 5 Continuous Joint Dist

sec 5.1, 5.2

X, Y have joint density $f(x,y)$
means f must satisfy

$f(x,y) \geq 0$ ← think of this as a surface over the plane

$$\int_y \int_x f(x,y)\,dx\,dy = \int_x \int_y f(x,y)\,dy\,dx = 1$$

the total volume under the surface is 1

Let A be a subset of the plane

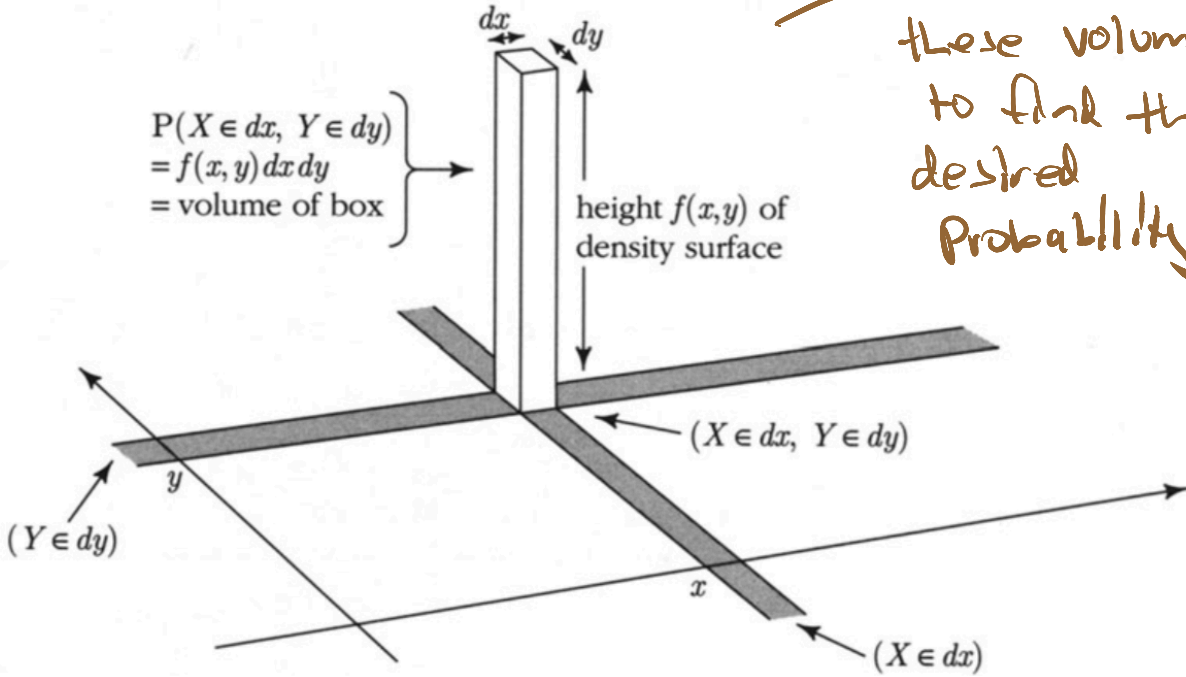
$$P((X,Y) \in A) = \iint_A f(x,y)\,dx\,dy$$

the volume of the surface over region A. this is a number between 0 and 1.

$$P(X \in dx, Y \in dy) = f(x,y)\,dx\,dy$$

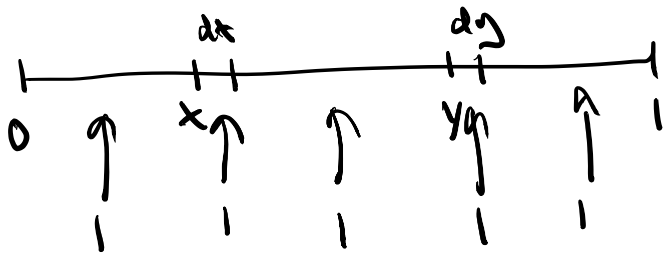
the volume of the surface over a little rectangle in the plane.

Picture



we add up the these volumes to find the desired probability.

ex Throw down 5 darts on $(0,1)$. Find the joint density of $X = U_{(2)}$ and $Y = U_{(4)}$.



$$f(x,y)\,dx\,dy = \binom{5}{1,1,1,1,1} x\,dx\,(y-x)\,dy\,(1-y)$$

$$= \underbrace{\binom{5}{1,1,1,1,1}}_{=5!} x(y-x)(1-y) \qquad 0<x<y<1$$

$$\Rightarrow \boxed{f(x,y) = 5!\,x(y-x)(1-y) \quad \text{for } 0<x<y<1}$$

tinyurl

http://tinyurl.com/april3-pt1

http://tinyurl.com/april3-pt2

Stat 134

Wednesday April 3 2019

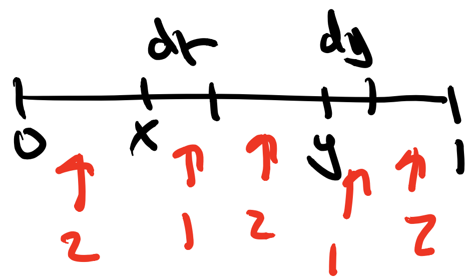
1. I throw down 8 darts on $(0, 1)$. The variable part of the joint density of $X = U_{(3)}$ and $Y = U_{(6)}$ is:

   a $x(y-x)^5(1-y)^2$

   b $x^2(y-x)^2(1-y)^2$

   c $x^4(y-x)^2(1-y)^2$

   d none of the above



2. Is $f(x,y) = \binom{6}{1,4,1}(y-x)^4$ on $0 < x < y < 1$ a joint density function?

a yes

b no

c not enough info to decide

$X = U_{(1)}$

$Y = U_{(6)}$