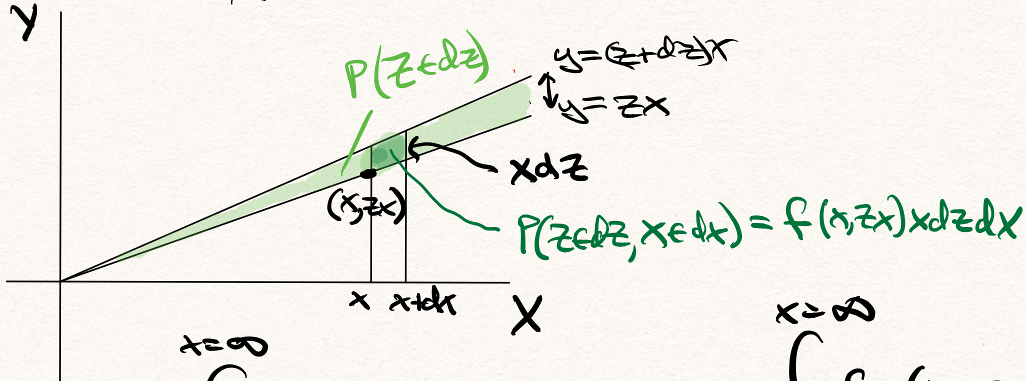
stat 134 lec 35

Last time

Sec 5.4 Convolution formula ratio

For ratio $Z = \frac{Y}{X}$, $X>0$, $Y>0$, indep.

$$Z = \frac{Y}{X} \Rightarrow Y = ZX$$ ← slope.



$$P(Z\in dz) = \int_{x=0}^{x=\infty} P(Z\in dz, X\in dx) = \int_{x=0}^{x=\infty} f_{(X,Y)}(x, zx)\, x\, dz\, dx$$

$$\parallel$$

$$f(z)\,dz$$

$$f(z) = \int_{x=0}^{x=\infty} f(x) f(zx)\, x\, dx \quad \text{convolution formula}$$

Today

① Sec 5.4 Review student explanation from current text.

② Sec 6.2 (read 6.1 at home).

ⓐ Conditional expectation: $E(T \mid X=x)$

ⓑ Rule of average conditional expectations,

$$E(T) = E(E(T \mid X))$$

① Sec 5.4

Concept test

Stat 134

Monday November 5 2018

1. Let $U \sim U(0,1)$ and $V \sim U(0,1)$ be independent. The density of $Y = U/V$ for $0 < y < 1$ is:

   a $1/(2y)$

   b $1/2$

   c $1/(2y^2)$

   d none of the above

Discuss with neighbor your solution for 1 min.

Soln

$Y = U/V$

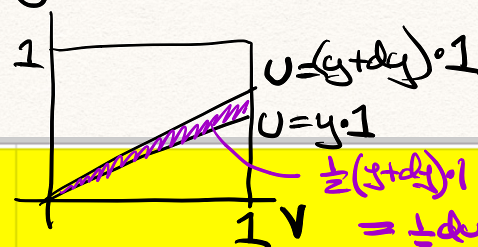
since $f_V(v)$ must be >0.

need $v<1$

need $yv<1$ ✓

$$f_Y(y) = \int_{v=0}^{v=1} f_V(v) f_U(yv)\, v\, dv$$

$$= \int_0^1 1 \cdot 1\, v\, dv = \frac{v^2}{2}\Big|_0^1 = \boxed{\tfrac{1}{2} \text{ for } 0<y<1}$$

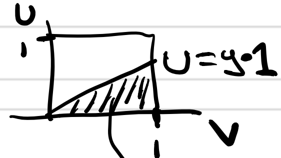


$\frac{1}{2}(y+dy)\cdot 1 - \frac{1}{2}y\cdot 1$

$= \frac{1}{2}dy$ ← f(y).

11/5/2018	b	b	Draw a pair of lines that are near each other, and I find that the area they cover is 0.5dz.
11/5/2018	b	b	P(Y<y) =P(u/v < y) =P(u<vy)= 0.5y
11/5/2018 9:44:01	b	b	
11/5/2018 9:44:05	b	b	Density functions for u and v are just 1; integral from 0 to 1 of y would just be 1/2
11/5/2018 9:44:09	b	b	Properties of uniform. Density of y would be y^2/2. From 0-1, it'd be 1/2
11/5/2018 9:44:33	b	b	
11/5/2018 9:44:44	b	b	Integral of 1*1 from 0 to 1 after transformation out of x and into U and V
11/5/2018 9:44:49	b	b	Joint density is 1 since U and V are independent and the density of U and V is 1. Integral from 0 to 1 of xdx is (1/2)x^2, evaluate at 1 and 0 and you get (1/2)
11/5/2018 9:45:02	b	b	Integral of v from 0 to 1 = .5
11/5/2018	b	b	The pdf's of U and V are 1. So fv(v) = 1 and fu(yv) = 1. Use convolution formula to find fy(y) = integral from 0 to 1 of (1*1*v*dv) = 1/2
11/5/2018 9:45:23	b	b	
11/5/2018 9:45:27	b	b	
11/5/2018	b	b	Mike said I was right
11/5/2018	d	d	the integral is from 0 to y of vdv using the convolution formula and fu(u)=1=fv(v), which evaluates to (y^2)/2

cdf method



$F(y) = \frac{1}{2}y$

$f(y) = \frac{1}{2}$

only for convolution formula for sum.

Sec 6.2 Conditional Expectation (discrete case).

Recall Bayes rule:

$$P(T=t \mid S=s) = \frac{P(T=t, S=s)}{P(S=s)}$$

ex (T,S) is joint distribution below.

Find $P(T=3 \mid S=7)$

$$= \frac{P(T=3, S=7)}{P(S=7)} = \frac{.3}{.4} = .75$$

	T=3	T=4	Sum
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1.0

marginal of S

marginal of T

Find $P(T=4 \mid S=7)$

$$= \frac{P(T=4, S=7)}{P(S=7)} = \frac{.1}{.4} = .25$$

Find $E(T) = \sum_{t \in T} t\,P(T=t) = 3(.6) + 4(.4)$

$$= 3.4$$

Find $E(T \mid S=7)$

$$= \sum_{t \in T} t\,P(T=t \mid S=7) = 3 \cdot P(T=3 \mid S=7) + 4 \cdot P(T=4 \mid S=7)$$

$$= 3 \cdot (.75) + 4 \cdot (.25) = 3.25$$

Find $E(T \mid S=6)$

$$= 3 \cdot P(T=3 \mid S=6) + 4 \cdot P(T=4 \mid S=6)$$

$$= 3 \cdot \left(\frac{.2}{.4}\right) + 4 \cdot \left(\frac{.2}{.4}\right) = 3.5$$

	T=3	T=4	Sum ← marginal of S
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1.0

marginal of T

$$\left.\begin{aligned} E(T \mid S=7) &= 3.25 \\ E(T \mid S=6) &= 3.5 \\ E(T \mid S=5) &= 3.5 \end{aligned}\right\} \text{function of } S$$

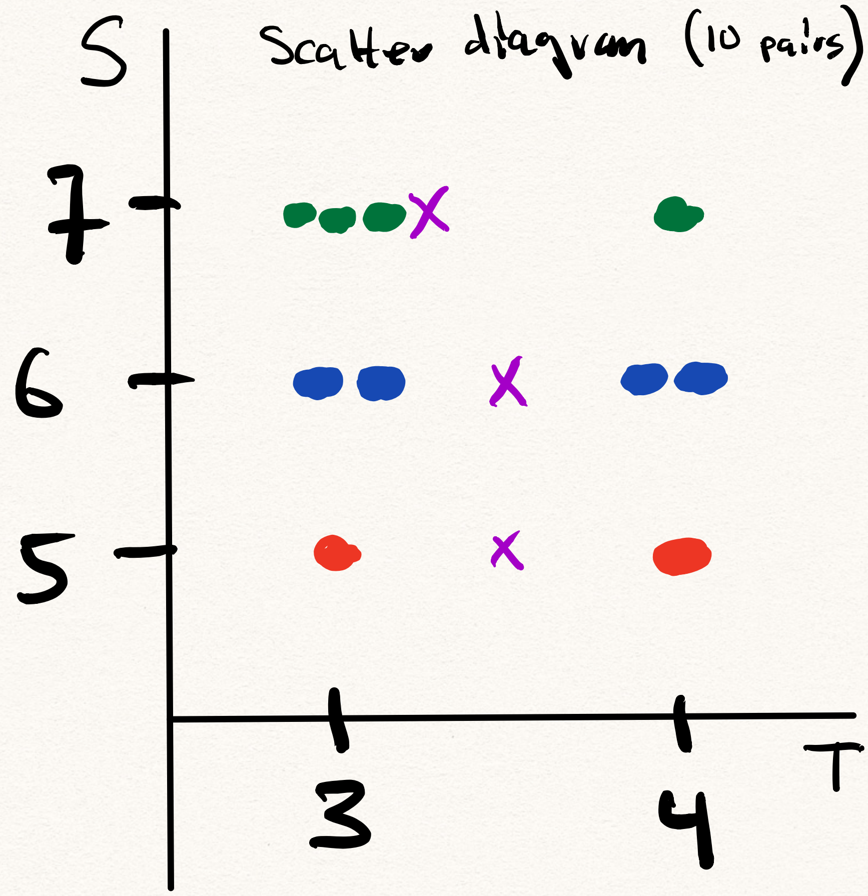
Picture

joint distribution

	T=3	T=4	Sum
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1

$\longleftrightarrow$

Scatter diagram (10 pairs)



Two main points:

① E(T|S) is a function of S.

② E(T|S) is a RV so it has an expectation.

Next we explore the expectation of E(T|S).

Let $g(S) = E(T|S)$

$g(7) = E(T|S=7) = 3.25$

$g(6) = 3.5$

$g(5) = 3.5$

$$E(g(S)) = \sum_{s \in S} g(s) P(S=s)$$

$$= 3.25(.4) + 3.5(.4) + 3.5(.2)$$

$$= 3.4$$

← this is $E(T)$.

it is the weighted average of all of the group averages.

In other words,

$$E(E(T|S)) = E(T)$$

This is called the property of iterated expectations.

Intuitively,

If you have a class that is $2/3$ girls and $1/3$ boys and the girls weigh on average 100 lbs and boys weigh 200 lbs then the average weight of the class should be $\frac{2}{3}(100) + \frac{1}{3}(200)$. i.e. we take the weighted average of the averages.

Rule of average conditional expectations

For any random variable Y with finite expectation and any discrete RV X,

$$E(Y) = \sum_{\text{all } x} E(Y \mid X=x) \cdot P(X=x)$$

(see end of this lecture for a formal proof) ex

8 transistors (type 1) are distributed $Exp\left(\frac{1}{100}\right)$ and 4 transistors (type 2) are $Exp\left(\frac{1}{200}\right)$.

Let T be the lifetime of a randomly picked transistor.

Find $E(T)$.

soln

Let X = type of transistor

$$E(T) = E(E(T|X))$$

$$= E(T|X=1)\cdot P(X=1) + E(T|X=2)P(X=2)$$

$$= 100 \cdot \frac{8}{12} + 200 \cdot \frac{4}{12} = 133,3$$

Stat 134

Wednesday November 7 2018

1. Let N have a Poisson(μ) distribution. Suppose that given $N = n$, random variable X follows a Binomial(n,p) distribution. E(X) is:

   a np

   b μ

   c $n\mu$

   d none of the above

$X | N=n \sim Bin(n,p)$

$E(X|N=n) = np$

$E(X|N) = Np$

$E(E(X|N)) = E(N)p = \mu p$

2. Let N have a Geometric(p) distribution on $1, 2, 3 \ldots$. Suppose that given $N = n$, random variable X follows a Binomial(n,p) distribution. E(X) is:

a 1

b $1/p$

c p

d none of the above

$X \mid N=n \sim Bin(n,p)$

$E(X \mid N=n) = np$

$E(X \mid N) = Np$

$E(E(X \mid N)) = E(N)p = \frac{1}{p} \cdot p = 1$

Appendix

Iterated Expectation

We show $E(Y) = E(E(Y|X))$:

$$E(Y) = \sum_{\text{all } y} y P(Y=y)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} P(X=x, Y=y)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} \frac{P(X=x, Y=y)}{P(X=x)} P(X=x)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} P(Y=y | X=x) P(X=x)$$

$$= \sum_{\text{all } x} \left(\sum_{\text{all } y} y P(Y=y | X=x) \right) \cdot P(X=x)$$

$$= \sum_{\text{all } x} E(Y | X=x) \cdot P(X=x)$$

$$= E(E(Y|X))$$

$\square$