FRI, JAN 29

§2 REPEATED TRIALS AND SAMPLING

* WE CONSIDER AN EXPERIMENT THAT CONSISTS OF A SERIES OF INDEPENDENT BERNOULLI TRIALS.

* LET $X = \begin{cases} 1 & \text{IF TRIAL SUCCEEDS} \\ 0 & \text{TRIAL FAILS} \end{cases}$.

$X \sim \text{Bernoulli}(p)$, i.e.

$$\mathbb{P}(X = k) = \begin{cases} p & k = 1 \\ q = 1-p & k = 0 \\ 0 & \text{o/w} \end{cases}.$$

* RECALL, IF N = # OF TRIALS UNTIL A SUCCESS, THEN $N \sim \text{Geometric}(p)$:

$$\mathbb{P}(N=k) = \begin{cases} q^{k-1}p & k=1,2,\dots \\ 0 & \text{o/w} \end{cases}.$$

* ANOTHER NATURAL QUESTION: SUPPOSE WE RUN n TRIALS. LET Y = # OF n TRIALS THAT ARE SUCCESSES. THEN,

(§2.1) $\quad Y \sim \text{Binomial}(n,p)$

$$\mathbb{P}(Y=k) = \begin{cases} \binom{n}{k} p^k q^{n-k} & k=0,1,\dots,n \\ 0 & o/w \end{cases}.$$

[SEE APPENDIX 1:

$\binom{n}{k}$ = "n CHOOSE k" $= \dfrac{n!}{k!\,(n-k)!}$

$m!$ = "m FACTORIAL" $= m(m-1)\cdots 2\cdot 1$]

ONE WAY TO SEE THAT $\binom{n}{k}$ IS # WAYS TO CHOOSE k FROM n OBJECTS IS BY INDUCTION, NOTING:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Choose 1st and k-1 from rest

Don't choose 1st and k from rest.

GROUPS

A FAIR COIN IS TOSSED 20 TIMES. GIVEN THAT THERE WERE 12 H'S, WHAT IS THE CHANCE THAT THE 1ST TOSS WAS A H?

ANS: $Y = \#\, H$'s. $Y \sim \text{Binomial}(20, \frac{1}{2})$.

$$\mathbb{P}(H_1 \mid Y = 12)$$

$$= \frac{\mathbb{P}(H_1 \cap Y = 12)}{\mathbb{P}(Y = 12)}$$

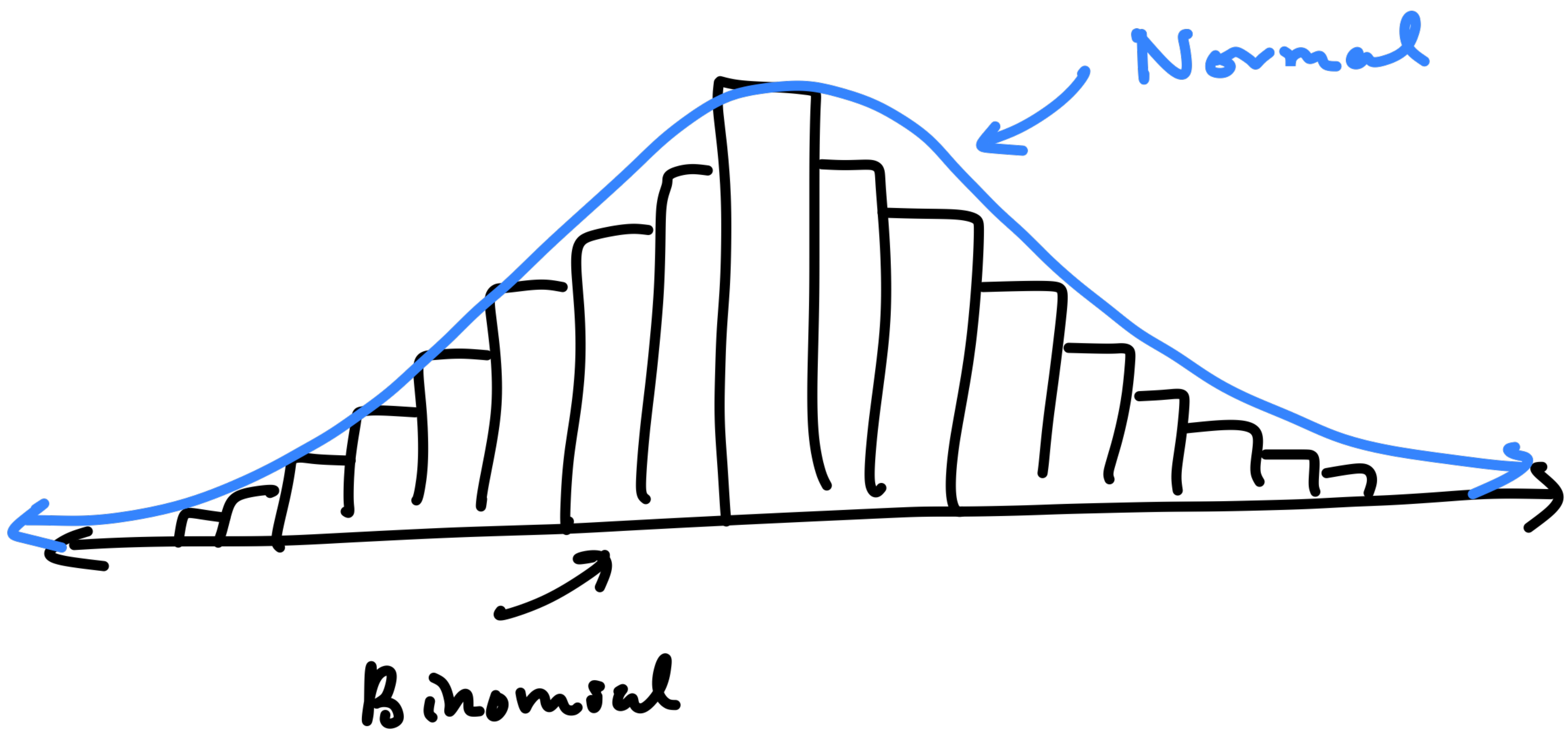
$$= \frac{(\frac{1}{2})\, \mathbb{P}(Y' = 11)}{\mathbb{P}(Y = 12)}, \quad Y' \sim \text{Binomial}(19, \tfrac{1}{2})$$

$$= \frac{(\frac{1}{2}) \binom{19}{11} (\frac{1}{2})^{11} (\frac{1}{2})^{19-11}}{\binom{20}{12} (\frac{1}{2})^{12} (\frac{1}{2})^{20-12}}$$

$$= \binom{19}{11} \Big/ \binom{20}{12}.$$

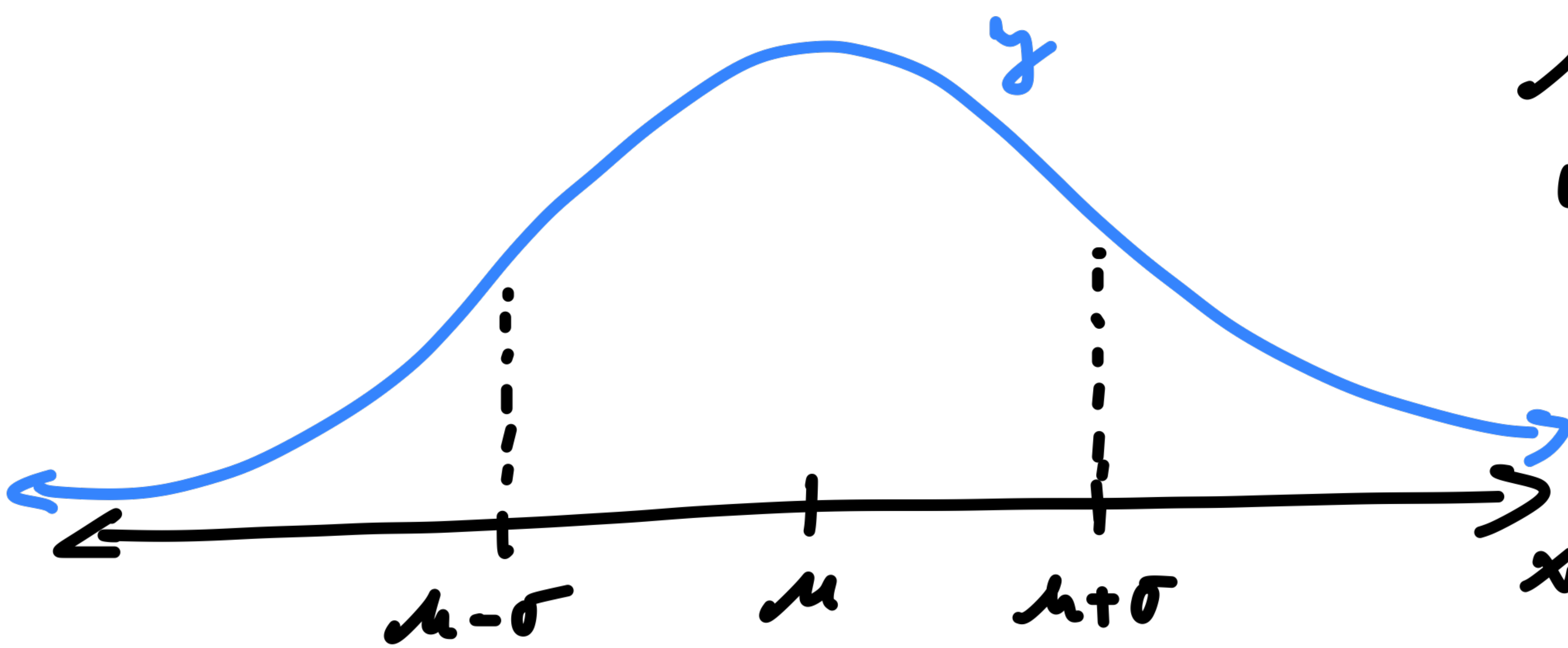
§2.1 NORMAL APPROXIMATION.

SEE p 88-89: THE Binomial (n, p) DISTRIBUTION HAS A (DISCRETE) "BELL-CURVED" SHAPE.



* WHEN Binomial (n,p) HISTOGRAM IS PROPERLY RESCALED, AND AS $n \to \infty$, IT APROACHES THE NORMAL BELL CURVE:

$$y = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{\sigma^2}}, \quad -\infty < x < \infty.$$



μ = mean

σ = SD

* THIS ALLOWS US TO APROXIMATE

$\mathbb{P}(\text{Bin}(n,p) = k)$ WHEN n IS

LARGE.

STANDARD NORMAL: $\mu = 0$

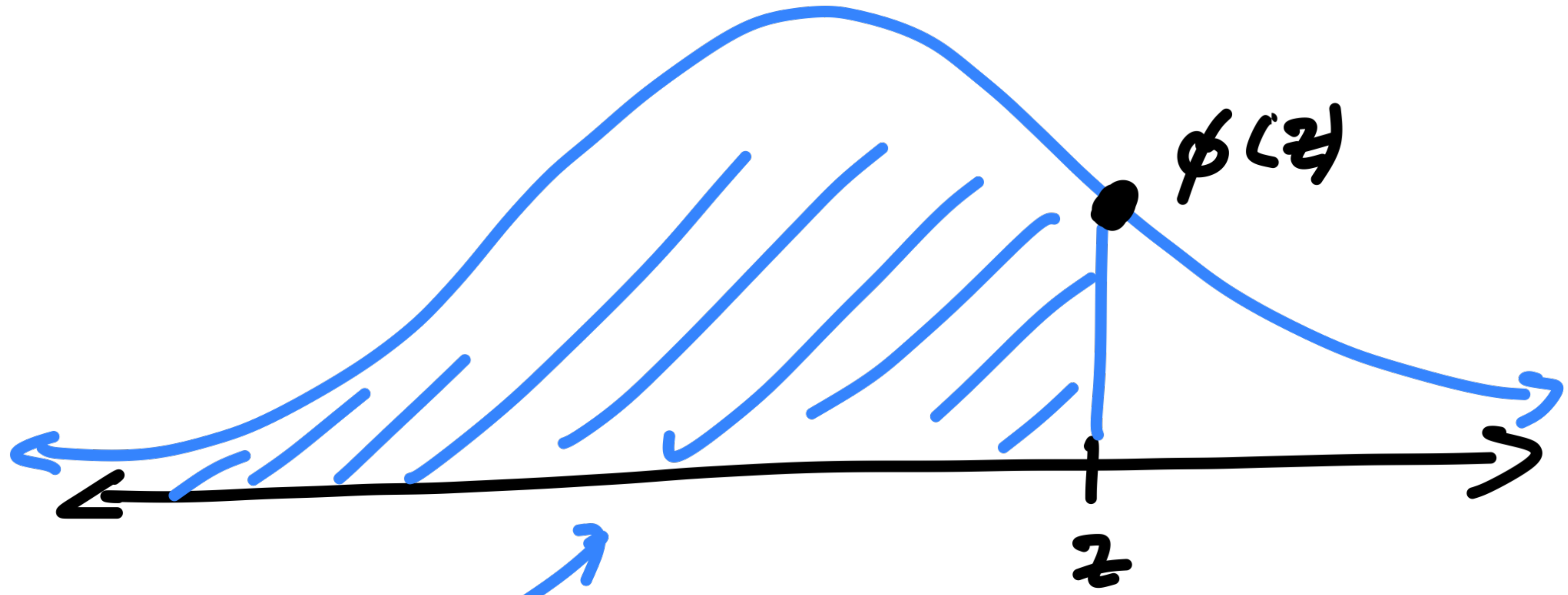
$\sigma = 1$

Density Function

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Cumulative Distribution Function (CDF)

$$\Phi(z) = \int_{-\infty}^{z} \phi(y)\,dy$$



$$\Phi(z) = \int_{-\infty}^{z} \phi(y)\, dy$$

BY CHANGE OF VARIABLES

(LET $z = \frac{x-\mu}{\sigma}$),

$$P(a \leq \text{NORMAL}(\mu,\sigma) \leq b)$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

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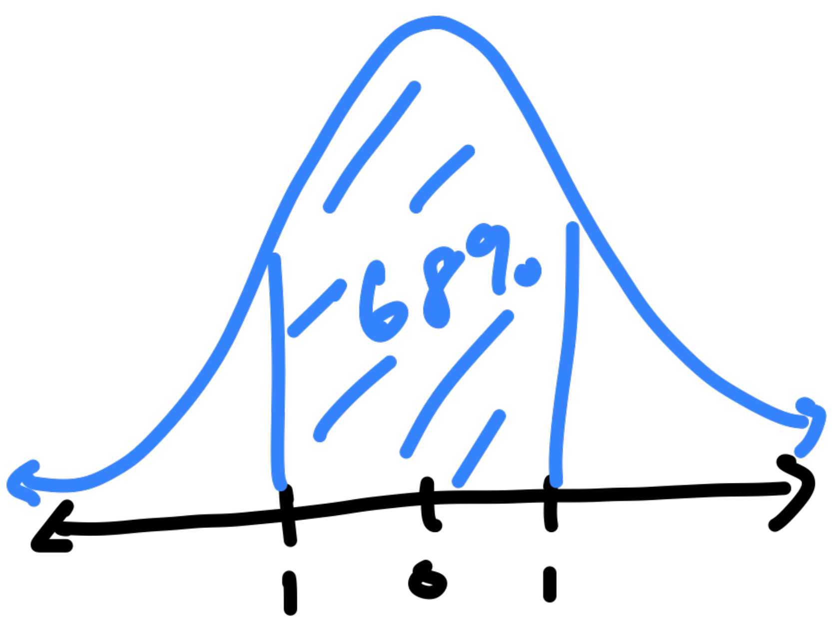
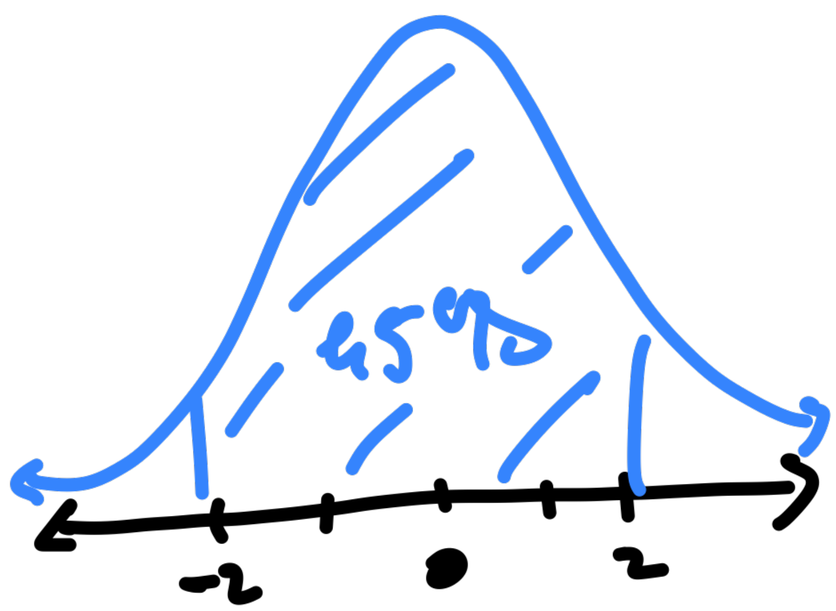
* NO CLOSED FORM FOR $\Phi(z)$, BUT VALUES ARE TABULATED IN APPENDIX 5. [FOR $z \geq 0$, BUT BY SYMMETRY, $\Phi(-z) = 1 - \Phi(z)$.]

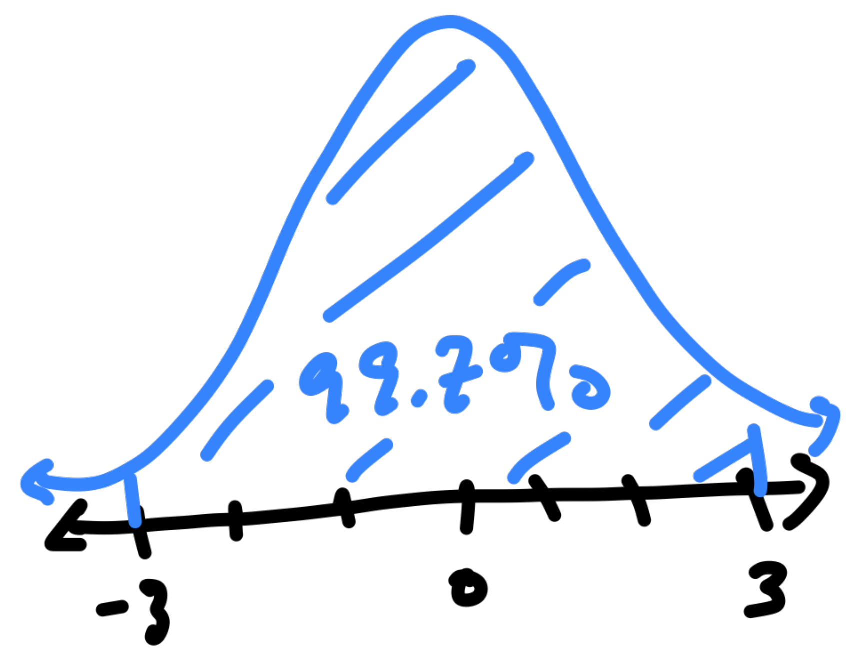
AN IMPORTANT RULE OF THUMB:

$$\Phi(a,b) = \Phi(b) - \Phi(a)$$

$\Phi(-1,1) \approx 68\%$ $\Phi(-2,2) = 95\%$ $\Phi(-3,3) = 99.7\%$







* WE'LL SEE IN § 3 THAT THE EXPECTED VALUE = MEAN OF $X \sim \text{Binomial}(n, p)$ IS $\mathbb{E}X = np$.

* AND ITS $SD(X) = \sqrt{npq}$.

IT TURNS OUT (SEE § 2.3 – NOT COVERED) THAT IF $\sqrt{npq}$ IS LARGE, THEN

$$\text{Binomial}(n,p) \approx \text{Normal}(\mu, \sigma)$$

WHERE $\mu = np$ AND $\sigma = \sqrt{npq}$.

NORMAL APPROXIMATION:

LET X = # SUCCESSES IN n Bernoulli(p) TRIALS.

$$\boxed{\mathbb{P}(a \le X \le b) \approx \Phi\left(\frac{b+\frac{1}{2}-\mu}{\sigma}\right) - \Phi\left(\frac{a-\frac{1}{2}-\mu}{\sigma}\right)}$$

WHERE $\mu = \mathbb{E}(X) = np$, $\sigma = SD(X) = \sqrt{npq}$.

* THE "$\frac{1}{2}$" IS A "CONTINUITY CORRECTION" THAT MAKES THE APPROX BETTER WHEN $\sqrt{npq}$ IS SMALL.

* RECALL $\Phi(-3,3) = 99.7\%$

* $\Phi(-4,4) = 99.99\%$, ...

LAWS THAT HOLD FOR LARGE n:

<u>SQUARE ROOT LAW:</u> # SUCCESSES IN n Bernoulli (p) TRIALS WILL, WITH HIGH PROBABILITY, BE IN A SMALL INTERVAL OF WIDTH $O\left(\frac{1}{\sqrt{n}}\right)$ CENTERED AROUND np.

<u>LAW OF LARGE NUMBERS:</u> THE PROPORTION OF SUCCESSES, WITH HIGH PROB., WILL BE VERY CLOSE TO p.