Stat 134: Section X

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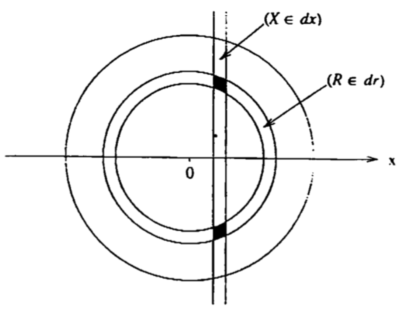
Problem 1

Let (X, Y) be picked uniformly from the unit disc $R^2 \leq 1$, where $R^2 = X^2 + Y^2$. Find

1. the joint desity of R and X;
2. Optional: repeat a) for a point (X, Y, Z) picked at random from the inside the unit sphere $R^2 \leq 1$, where now $R^2 = X^2 + Y^2 + Z^2$.

Ex 5.2.17 in Pitman's Probability

17. a) Note that $-R \leq X \leq R$. Let $0 \leq r \leq 1$, and $-r \leq x \leq r$. If $X \in dx$ and $R \in dr$, then the point (X, Y) lies in one of two (almost) parallelograms:



So

$$P(X \in dx, R \in dr) = \frac{2 \times \text{area of parallelogram}}{\text{area of circle}}$$
$$= 2 \times dx \times \frac{dr}{\sqrt{1-(x/r)^2}} \Big/ \pi = \frac{2}{\pi} \frac{r}{\sqrt{r^2 - x^2}} dx dr$$

b) Note again that $-R \leq X \leq R$. Let $0 \leq r \leq 1$, and $-r \leq x \leq r$. If $X \in dx$ and $R \in dr$, then the point (X, Y, Z) lies in an "inner tube" formed by rotating the parallelogram in (a) about the x-axis. Hence

$$P(X \in dx, R \in \quad = 2 \times \pi \times (\text{distance to } x\text{-axis}) \times (\text{area of parallelogram}) \Big/ (\text{volum} \quad \text{here})$$
$$= 2 \times \pi \times \sqrt{r^2 - x^2} \times \frac{dx dr}{\sqrt{1-(x/r)^2}} \Big/ \frac{4}{3}\pi = \frac{3}{2} r dx dr$$

So $f(x, r) = \frac{3}{2} r, \quad 0 \leq r \leq 1, \quad -r \leq x \leq r.$

Problem 2

Let X be exponentially distributed with rate λ, independent of Y, which is exponentially distributed with rate μ. Find $P(X \geq 3Y)$.
Ex 5.2.5 in Pitman's Probability

Since X, Y are independent, we have

$$\begin{aligned}
P(X \geq 3Y) &= \int_0^\infty f_Y(y) \int_{3y}^\infty f_X(x) dx dy \\
&= \int_0^\infty \mu e^{-\mu y} \int_{3y}^\infty \lambda e^{-\lambda x} dx dy \\
&= \int_0^\infty \mu e^{-\mu y} e^{-3\lambda y} dx dy \\
&= \mu / (\mu + 3\lambda)
\end{aligned}$$

Problem 3

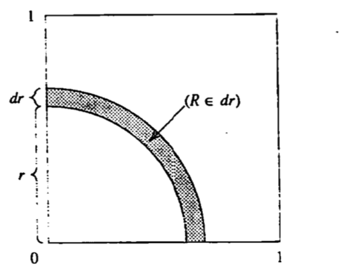
Let X and Y be independnet and uniform $(0, 1)$ and let $R = \sqrt{X^2 + Y^2}$. Answer the following questions:

1. Find out the density $f_R(r)$.

2. Find out the CDF $F_R(r)$.

Ex 5.2.20 in Pitman's Probability

20. Since (X, Y) has uniform distribution on the unit square, it follows that the probability that (X, Y) lies in a given subset of the unit square is the area of that subset.



a) If $0 < r < 1$ then

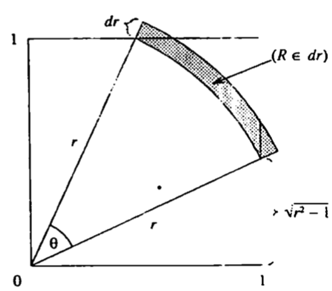
$$f_R(r)dr = P(R \in dr) = r \times \frac{\pi}{2} \times dr$$

(Recall that arc length = radius × angle subtended) while if $1 < r < \sqrt{2}$ then

$$f_R(r) = P(R \in dr) = r \times \theta \times dr,$$

where $\frac{\pi}{2} = \theta + 2\arccos\frac{1}{r}$ (arccos has range $[0, \pi]$). So

$$f_R(r) = \begin{cases} (\pi/2)r & 0 < r < 1 \\ r\,(\pi/2 - 2\arccos(1/r)) & 1 < r < \sqrt{2} \end{cases}$$



b) Integrate f_R, or observe that if $0 < r < 1$ then

$$F_R(r) = P(R \le r) = P((X, Y) \text{ is within } r \text{ of } (0, 0)) = \frac{1}{4}\pi r^2$$

and if $1 < r < \sqrt{2}$ then

$$\begin{aligned} F_R(r) &= \text{area of sector of circle} + \text{area of 2 triangles} \\ &= \frac{\theta}{2\pi} \times \pi r^2 + 2 \times \frac{1}{2}\sqrt{r^2 - 1} \\ &= \left(\frac{\pi}{4} - \arccos\frac{1}{r}\right) r^2 + \sqrt{r^2 - 1}. \end{aligned}$$