Stat 134 lec 23

Warm up 1:00–1:10

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$\parallel$

Suppose customers are arriving at a ticket booth at rate of five per minute, according to a Poisson arrival process. Find the probability that:

t

$\Downarrow$

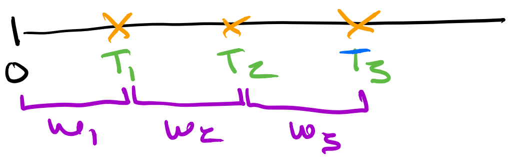
(a) Starting from time 0, the $9th$ customer doesn't arrive within 5 minutes;

$$P(T_9 > 5) = P(\underbrace{N_5}_{\sim Pois(25)} < 9)$$

$$= \sum_{k=0}^{8} e^{-25} \cdot \frac{25^k}{k!}$$

Wednesday lecture on Moment Generating Functions (not in book)

Last time sec 4.2 Gamma Distribution



$T_1 \sim Exp(\lambda)$, $\lambda > 0$ $\qquad f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$ ← variable part

$T_r \sim Gamma(r, \lambda)$, $\lambda > 0$, $r > 0$ $\qquad f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$

where $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t}\, dt$ Gamma function

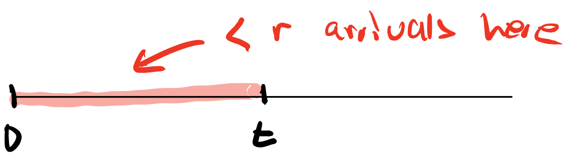
Notes about $T_r \sim Gamma(r, \lambda)$:

$\Gamma(r) = (r-1)!$ for $r \in \mathbb{Z}^+$

If $r \in \mathbb{Z}^+$, T_r = time to r^{th} arrival

$Exp(\lambda) = Gamma(r=1, \lambda)$

$P(T_r > t) = P(N_t < r)$ where $N_t \sim Pois(\lambda t)$



Today Sec 4.4 (skip 4.3)

0. Gamma example
1. Change of Variable formula for densities.
2. Recognizing a distribution from the variable part of its density

(6) Gamma example

Suppose customers are arriving at a ticket booth at rate of five per minute, according to a Poisson arrival process. Find the probability that:

(a) Starting from time 0, the $9th$ customer doesn't arrive within 5 minutes;

(b) At least one customer arrives within 40 seconds after the arrival of the $13th$ customer.

$$T_{14} < T_{13} + 2/3 \iff \underbrace{T_{14} - T_{13}}_{\substack{\shortparallel \\ W_{14} \sim Exp(5)}} < 2/3$$

$$P(\underbrace{T_{14} - T_{13}}_{W_{14}} < 2/3) = 1 - P(W_{14} > \tfrac{2}{3})$$

$$= \boxed{1 - e^{-5 \cdot 2/3}}$$ using survival formula for exponential.

Alternatively, since W_{14} and W_1 both have distribution $Exp(5)$, and $T_1 = W_1$ since $Gamma(r=1, \lambda=5) = Exp(5)$.

$$P(\underbrace{T_{14} - T_{13}}_{\substack{\shortparallel \\ W_1 \\ \shortparallel \\ T_1}} < 2/3) = 1 - P(T_1 > 2/3)$$

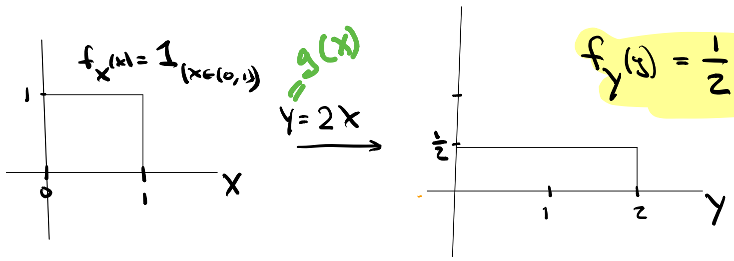
$$= 1 - P(\overset{\shortparallel}{N_{2/3}} < 1) = 1 - P(N_{2/3} = 0)$$

$$= \boxed{1 - e^{-5 \cdot 2/3}}$$ using Poisson formula for $N_{2/3} \sim Pois(5 \cdot \tfrac{2}{3})$ ← # arrivals in 2/3 minutes.

① Sec 4.4 Change of Variable formula for densities

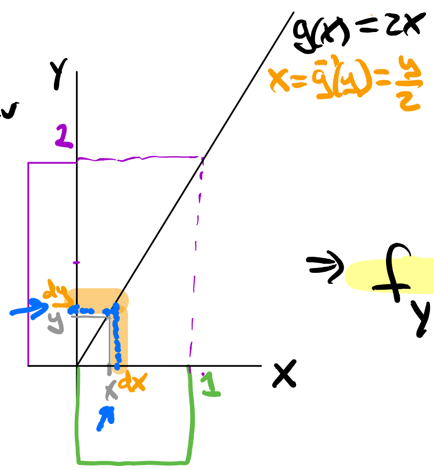
ex The density of $X \sim U(0,1)$ is $f_X(x) = 1_{(x \in (0,1))}$

What is density of $Y = 2X$



$$f_Y(y) = \frac{1}{2} 1_{(y \in (0,2))}$$

Another view:



$$P(Y \in dy) = P(X \in dx)$$

$$f_Y(y)dy = f_X(x)dx$$

$$\Rightarrow f_Y(y) = f_X(x) \frac{dx}{dy}$$

$$= \frac{1}{\frac{dy}{dx}} f_X(x) = \frac{1}{2} \cdot 1_{(x \in (0,1))}$$

$\frac{dy}{dx} = g'(x) = 2$

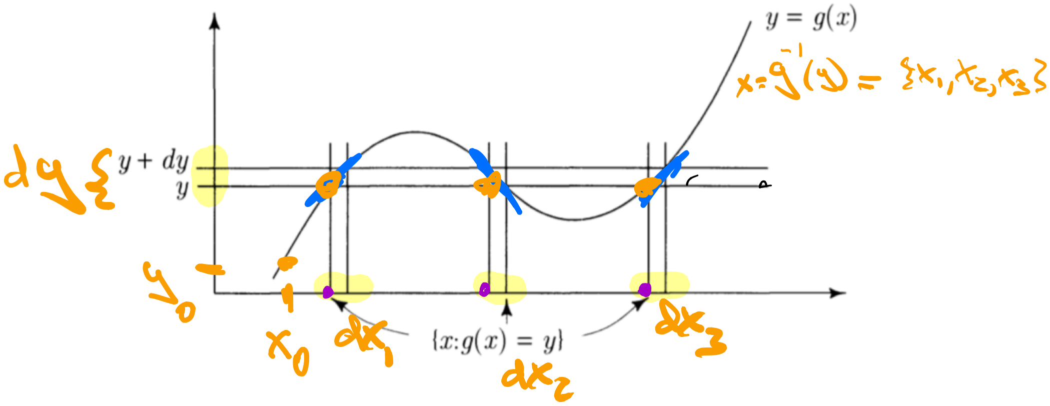
evaluated at $x = g^{-1}(y)$

$$= \frac{1}{2} 1_{\left(\frac{y}{2} \in (0,1)\right)}$$

$$= \frac{1}{2} 1_{(y \in (0,2))}$$

Here the transformation of X to $Y = g(X)$ is linear and one to one.

What if $Y = g(X)$ isn't linear and one-one?



mutually exclusive events.

$Y \in dy$ iff $X \in dx_1$ or $X \in dx_2$ or $X \in dx_3$

$$P(Y \in dy) = P(X \in dx_1) + P(X \in dx_2) + P(X \in dx_3)$$

$$f_Y(y)\,dy = f_X(x_1)\,dx_1 + f_X(x_2)\,dx_2 + f_X(x_3)\,dx_3$$

$$f_Y(y) = f_X(x_1)\frac{dx_1}{dy} + f_X(x_2)\frac{dx_2}{dy} + f_X(x_3)\frac{dx_3}{dy}$$

$$= \frac{f_X(x_1)}{g'(x_1)} + \frac{f_X(x_2)}{|g'(x_2)|} + \frac{f_X(x_3)}{g'(x_3)}$$

evaluated at $x = g^{-1}(y)$

$P(X \in dx_2) \geq 0$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{g'(x)}$$

Thm (P 307)

Let X be a continuous RV with density $f_X(x)$.

Let $Y = g(X)$ have a derivative that is zero at only finitely many pts.

then $$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \frac{f_X(x)}{|g'(x)|}\Bigg|_{x=g^{-1}(y)}$$

evaluated at $x = g^{-1}(y)$.

ex

let $X = N(0,1)$, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

Find the density of $\underset{g(x)}{\underset{\parallel}{Y}} = \sigma X + \mu$ where $\sigma > 0$.

$y = g(x) = \sigma x + \mu$

$g'(x) = \sigma$

$x = g^{-1}(y) = \frac{y-\mu}{\sigma}$ since $y = \sigma x + \mu$

$\Rightarrow y - \mu = \sigma x$

$\Rightarrow \frac{y-\mu}{\sigma} = x$

$$f_Y(y) = \frac{f_X(x)}{|g'(x)|}\Bigg|_{x=\frac{y-\mu}{\sigma}} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}{\sigma}\Bigg|_{x=\frac{y-\mu}{\sigma}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$$

Note:

$E(Y) = E(\sigma X + \mu) = \sigma \underset{0}{\underset{\parallel}{E(X)}} + \mu = \mu$

$Var(Y) = Var(\sigma X + \mu) = \sigma^2 \underset{1}{\underset{\parallel}{Var(X)}} = \sigma^2$

We will see later in the semester that Y is normal.

$\Rightarrow \boxed{Y \sim N(\mu, \sigma^2)}$

Extra Example:

I didn't have time to do this in class. Here is an example where the change of variable formula has two terms.

Change of variable formula:

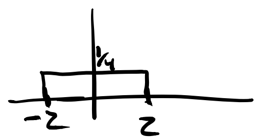
$$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \left.\frac{f_X(x)}{|g'(x)|}\right|_{x=g^{-1}(y)}$$

ex $X \sim \text{Unif}(-2, 2)$, $f_X(x) = \frac{1}{4}\mathbb{1}_{(x\in(-2,2))}$

Find density of $Y = X^2$

note: $g(x) = x^2$

$x = g^{-1}(y) = \pm\sqrt{y}$



$$f_Y(y) = \left.\frac{\frac{1}{4}\mathbb{1}_{(x\in(-2,2))}}{2x}\right|_{x=\sqrt{y}} + \left.\frac{\frac{1}{4}\mathbb{1}_{(x\in(-2,2))}}{|2x|}\right|_{x=-\sqrt{y}}$$

$$= \frac{\frac{1}{4}\mathbb{1}_{(y\in(0,4))}}{2\sqrt{y}} + \frac{\frac{1}{4}\mathbb{1}_{(y\in(0,4))}}{|-2\sqrt{y}|}$$

$$= \frac{\frac{1}{4}\mathbb{1}_{(y\in(0,4))}}{\sqrt{y}} = \begin{cases} \frac{1}{4\sqrt{y}} & 0<y<4 \\ 0 & \text{else} \end{cases}$$

graph of $f_Y(y)$ =