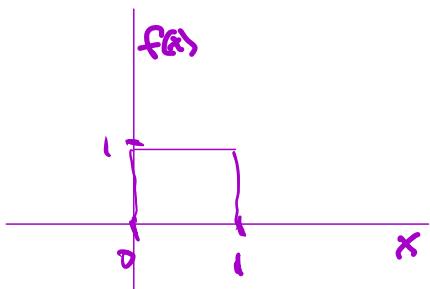


Stat 134 Lec 20 (no lec 19)

warming 1:00-1:10

Let $X \sim \text{Unif}(0, 1)$ be the standard uniform distribution with

Picture



$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

Define

$$E(X) = \int_{x=-\infty}^{\infty} x f(x) dx$$

Find $E(x)$, $E(x^2)$, and $\text{Var}(x)$.

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 x \cdot 1 dx = \frac{x^2}{2} \Big|_0^1 = \boxed{\frac{1}{2}}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^1 x^2 \cdot 1 dx = \frac{x^3}{3} \Big|_0^1 = \boxed{\frac{1}{3}}$$

$$\text{Var}(x) = E(x^2) - (E(x))^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \boxed{\frac{1}{12}}$$

Last time

Congratulations on finishing midteam 1 !

today

Sec 4.1 Continuous Distributions

- ① Probability density
- ② expectation and variance,
- ③ change of scale

① sec 4.1 Probability density.

let X be a continuous RV

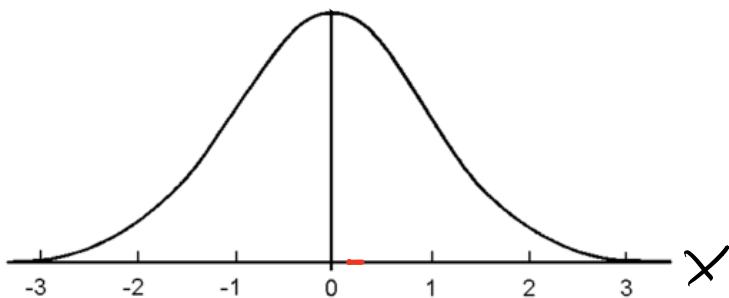
The probability density (histogram) of X is described by a prob density function

$$f(x) \geq 0 \text{ for } x \in X$$

$$\text{and } \int_{-\infty}^{\infty} f(x) dx = 1$$

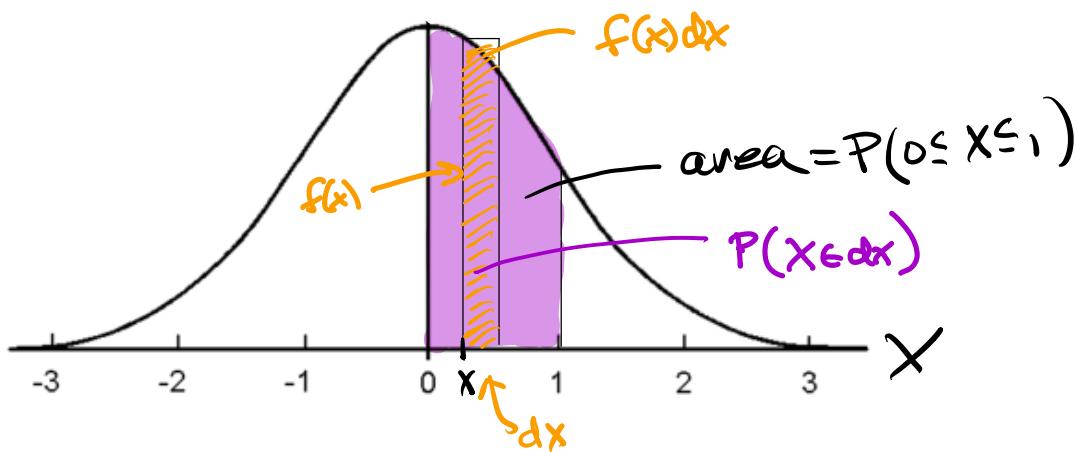
e.g. the standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$



Throwing a dart randomly at the histogram the x coordinate of your dart is most likely to be near zero.

The probability of getting an x coordinate in a small neighborhood of x is written $P(X \in dx)$.



we see from the rectangle in the picture,

$$P(X \in dx) \approx f(x)dx \quad (\text{notice purple and orange area not same})$$

here $dx = \text{tiny interval around } x$ and also the length of the interval

$$\text{For all } a, b \quad P(a \leq X \leq b) = \int_a^b P(X \in dx) \approx \int_a^b f(x)dx$$

Note $f(x)$ is not a probability.

$f(x)dx$ is a probability.

$$f(x) \approx \frac{P(X \in dx)}{dx} \leftarrow \text{Probability}$$

units of f ? — probability / unit length \leftarrow probability density

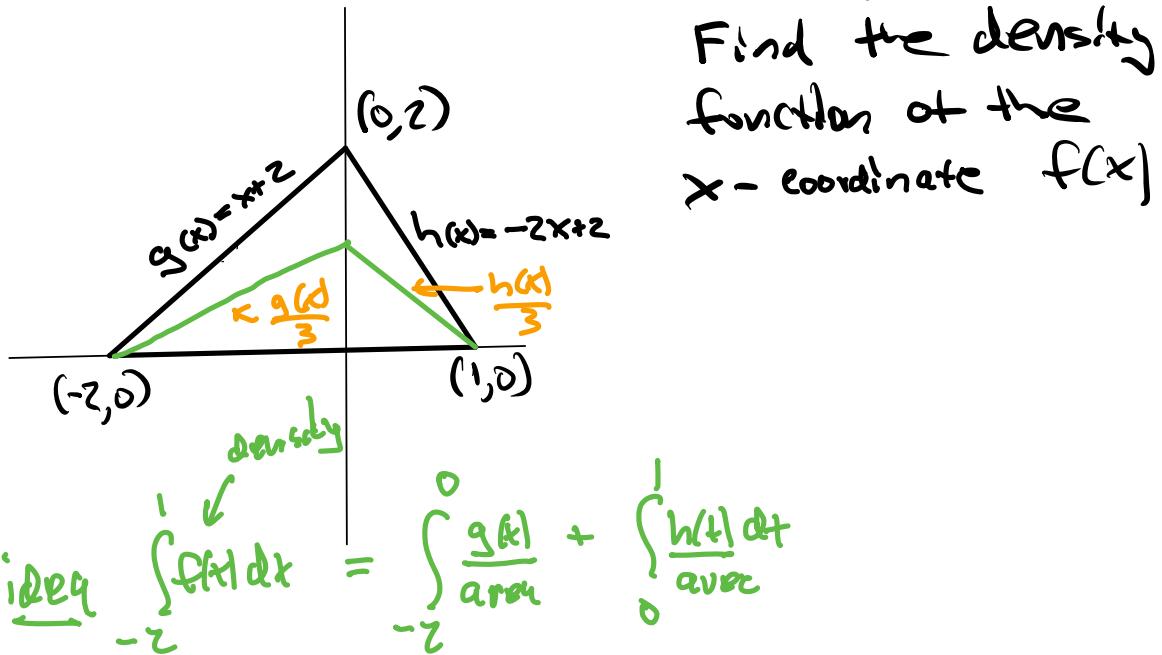
$$P(X=x) = 0$$

$$\text{Hence } P(a \leq X \leq b) = P(a < X < b)$$

(we don't have to worry about endpoints),

ex 4.1.12 b

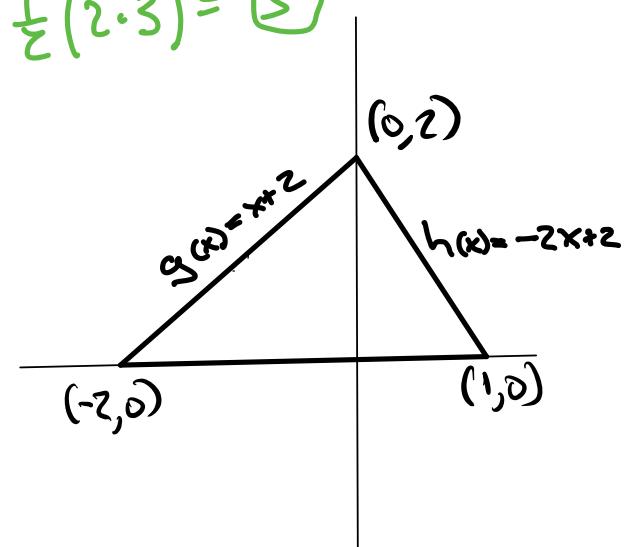
Consider a point picked uniformly at random from the area inside the following triangle.



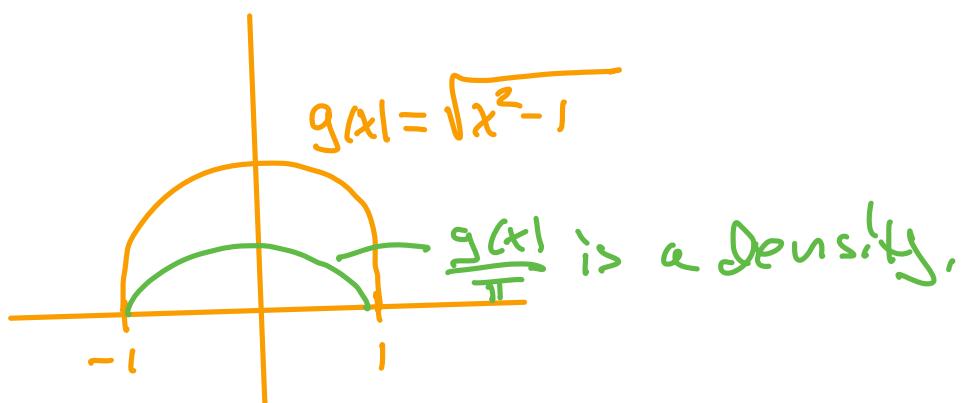
Find the density function of the x-coordinate $f(x)$

$$f(x) = \begin{cases} \frac{x+2}{3}, & -2 \leq x \leq 0 \\ \frac{-2x+2}{3}, & 0 \leq x \leq 1 \\ 0, & \text{else} \end{cases}$$

Area of $\triangle = ?$



Note there is nothing special about the shape being a triangle, It could be a half circle for example



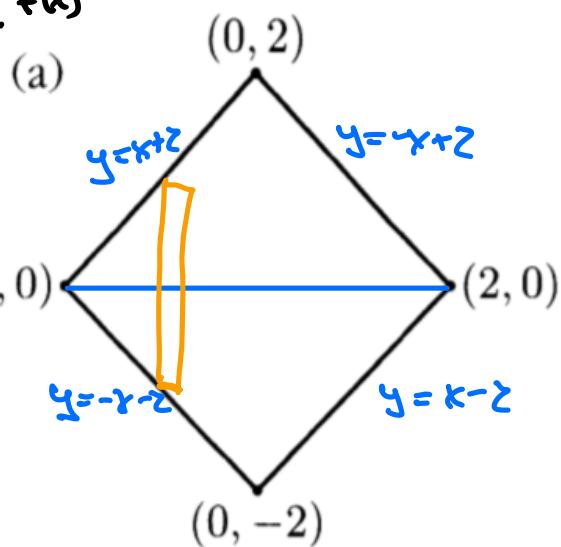
Here the area is π , To make g into a density divide it by π

$$f(x) = \frac{\sqrt{x^2 - 1}}{\pi}$$

Ex 4.1.12 a

Consider a point picked uniformly at random from the area inside the following shape.

Find the density, $f(x)$

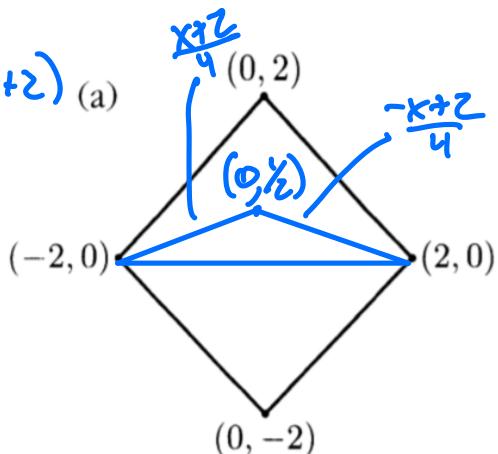


$$f(x) = \begin{cases} \frac{x+z}{4}, & -2 \leq x \leq 0 \\ \frac{-x+z}{4}, & 0 \leq x \leq 0 \\ 0, & \text{else} \end{cases}$$

$$\text{Area} = 2 \left(\frac{1}{2} \cdot 4 \cdot 2 \right) = 8$$

$$\int_{-2}^2 f(x) dx = \frac{1}{\text{Area}} \cdot \int_{-2}^0 2(x+z) dx + \int_{-2}^0 2(-x+z) dx$$

$$= \frac{1}{8} \cdot \left[2x^2 + 2xz \Big|_{-2}^0 + 2x^2 - 2xz \Big|_{-2}^0 \right]$$



(2)

Expectation and Variance

For discrete,

$$E(g(x)) = \sum_{x \in X} g(x) P(X=x)$$

For continuous,

$$E(g(x)) = \int_{-\infty}^{\infty} g(x) P(X=x) dx = \int_{-\infty}^{\infty} g(x) f(x) dx$$

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$\text{Var}(x) = E(x^2) - E(x)^2$$

See Wamby for example,

(3)

Change of scale

To calculate $E(X)$, $\text{Var}(X)$, $P(X \leq x)$ we sometimes make a linear change of scale

$Y = c + bX$ where c, b are constants

Y hopefully has a simpler density function.

We can recover $E(X)$, $\text{Var}(X)$, $P(X \leq x)$

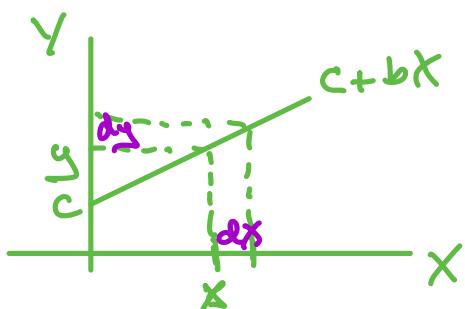
from $E(Y)$, $\text{Var}(Y)$, $P(Y \leq y)$.

$$E(Y) = E(bX + c) = bE(X) + c$$

$$\Rightarrow E(X) = \frac{E(Y) - c}{b}$$

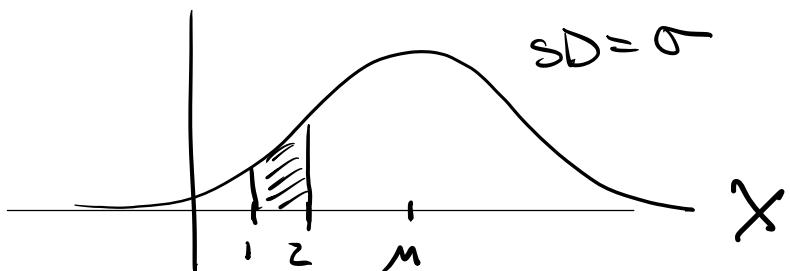
$$\text{Var}(Y) = \text{Var}(bX + c) = b^2 \text{Var}(X)$$

$$\Rightarrow \text{Var}(X) = \frac{\text{Var}(Y)}{b^2}$$



$$P(X \leq x) = P(Y \leq y)$$

Ex Let $X \sim N(\mu, \sigma^2)$
 Find $P(1 < X < 2)$

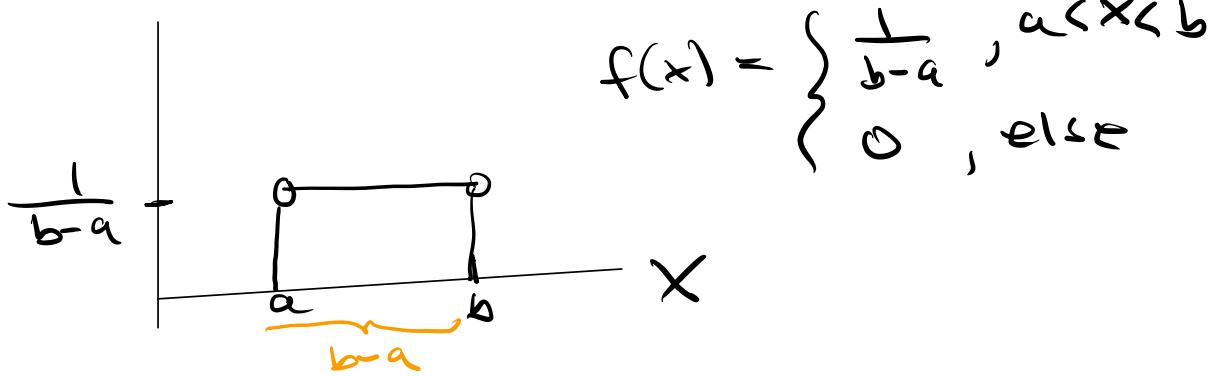


$$Z = \frac{X - \mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma} \uparrow c$$

Change of scale

$$P(1 < X < 2) = \Phi\left(\frac{2-\mu}{\sigma}\right) - \Phi\left(\frac{1-\mu}{\sigma}\right)$$

$\stackrel{ex}{=}$ Let $X \sim \text{Unif}(a, b)$



a) You should change the scale of X to?

$$U = \frac{X-a}{b-a} \sim \text{Unif}(0, 1)$$

b) Find $E(X)$

$$\begin{aligned} E[X] &= E[U] = E\left(\frac{1}{b-a}(X-a)\right) = \frac{1}{b-a} E(X-a) \\ &= \frac{1}{b-a} (E(X) - a) \end{aligned}$$

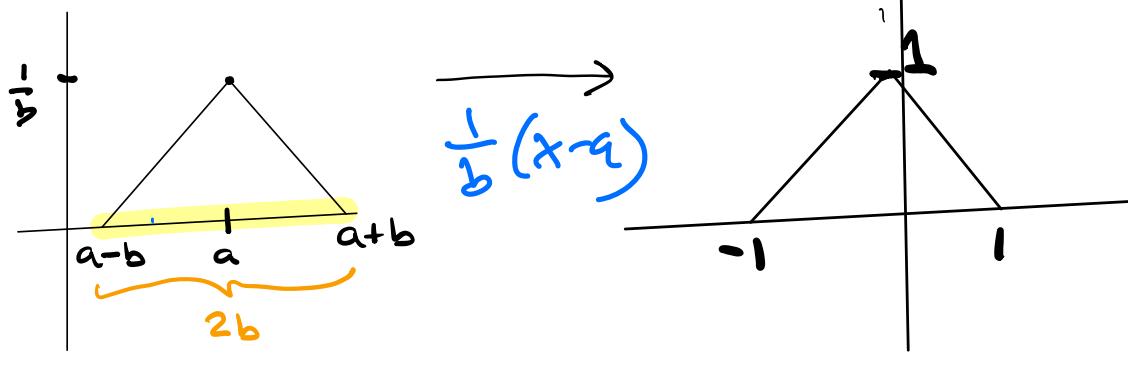
c) Find $\text{Var}(X)$.

$$E(X) = \frac{a+b}{2}$$

$$\begin{aligned} \frac{1}{2} &= \text{Var}(U) = \frac{1}{(b-a)^2} \text{Var}(X-a) \\ \Rightarrow \text{Var}(X) &= \frac{(b-a)^2}{12} \end{aligned}$$

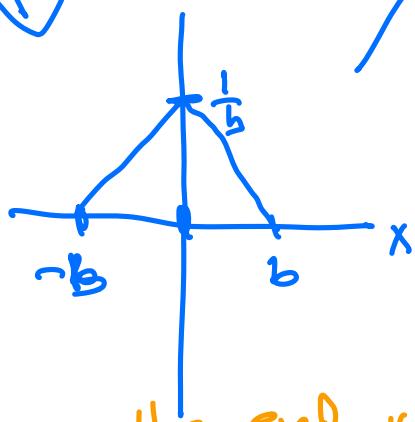
What change of scale do we have here?

We are changing the values x takes.
Because it's a linear change of scale
a trapezoid shaped density maps to
the same shape.



subtract a

mult the x axis by $\frac{1}{b}$



Note Because the end result is a density with area 1 the height of the triangle is 1 not $(\frac{1}{b})^2$.

