Stat 134: Section 19

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Conceptual Review

a. What is the general convolution formula?

b. $X \sim Gamma(r, \lambda)$, what is $f_X(x)$?

c. $Z \sim Beta(r, s)$, what is $f_Z(z)$?

d. What is the C.D.F. of a $Beta(r, s)$ distribution?

Problem 1

Let $X = UV$ for independent uniform $(0, 1)$ variables U and V. Find the density of X.
Ex. 5.4.9 in Pitman's Probability

Problem 2

Let X and Y be independent variables with $Gamma(r, \lambda)$ and $Gamma(s, \lambda)$ distributions, respectively. Using convolution, show that $Z = \frac{X}{X+Y}$ follows a $Beta(r, s)$ distribution.

Hint: rewrite integrand to be one of the known distributions after taking out constants.

Problem 3

Suppose $U_1, U_2 \overset{\text{i.i.d.}}{\sim}$ Unif $(0, 1)$. Let $Z = Y - X$, where $X = U_{(1)}$, $Y = U_{(2)}$. Note that Z represents the range of our random variables.

a. Find the joint density $f(x, y)$ of X, Y.

b. Find the C.D.F. of Z, $F_Z(z)$.

Hint: Draw the region of interest. It may be easier to work with $P(Z \geq z)$.

c. Use part (b) to find the density of Z.

d. It can be shown that for the range $Z_n = U_{(n)} - U_{(1)}$ of n i.i.d. Unif $(0, 1)$ random variables, the CDF of Z_n is given by $F_{Z_n}(z) = z^n + nz^{n-1}(1 - z)$. Using what we know about order statistics, explain why this is the case.

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