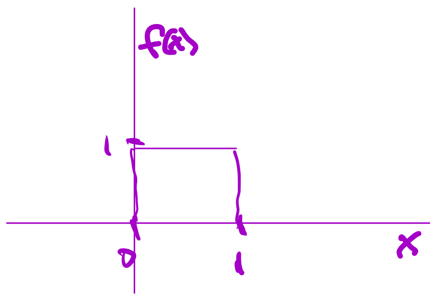
Stat 134 lec 20 (no lec 19)

warmup 1:00-1:10

Let $X \sim Unif(0,1)$ be the standard uniform distribution with histogram (density)

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \le 1 \\ 0 & \text{else} \end{cases}$$

Picture



Define

$$E(X) = \int_{x=-\infty}^{\infty} x f(x)\,dx$$

Find $E(X)$, $E(X^2)$, and $Var(X)$.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x \cdot 1\,dx = \frac{x^2}{2}\Big|_0^1 = \boxed{\frac{1}{2}}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \boxed{1/3}$$

$$Var(X) = E(X^2) - (E(X))^2 = 1/3 - (1/2)^2 = \boxed{1/12}$$

Last time

Congratulations on finishing midterm 1 !

today

Sec 4.1 Continuous Distributions

1. probability density
2. expectation and variance.
3. change of scale

① sec 4.1 Probability density.

let X be a continuous RV

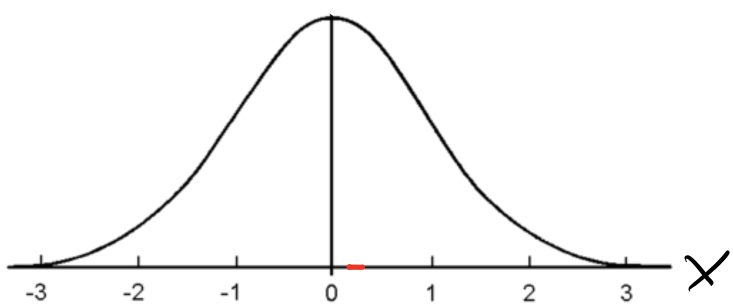
The probability density (histogram) of X is described by a prob density function

$$f(x) \geq 0 \text{ for } x \in X$$

and $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

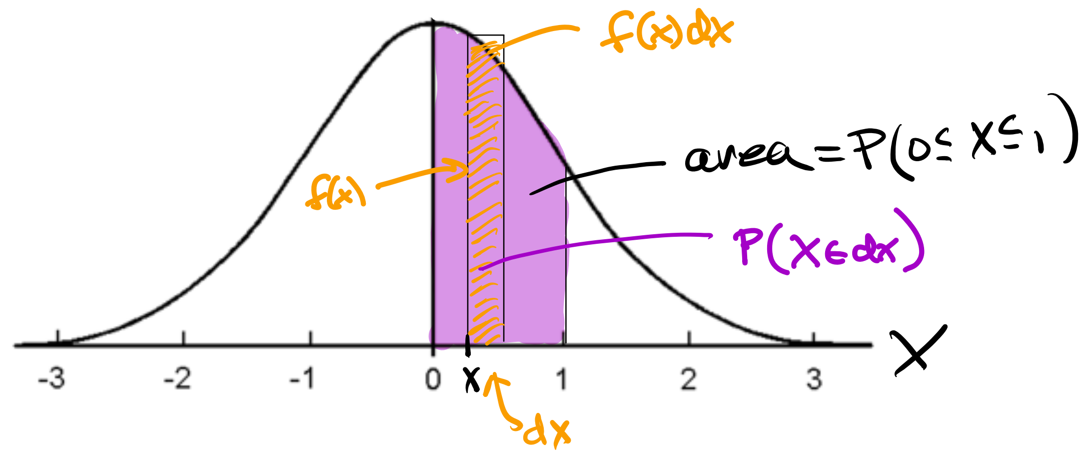
ex The standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$



Throwing a dart randomly at the histogram the x coordinate of your dart is most likely to be near zero.

The probability of getting an x coordinate in a small neighborhood of x is written $P(X \in dx)$.



we see from the rectangle in the picture,

$P(X \in dx) \approx f(x)dx$ (notice purple and orange area not same)

here dx = tiny interval around x and also the length of the interval

For all a, b $P(a \le X \le b) = \int_a^b P(X \in dx) \approx \int_a^b f(x)dx$

Note $f(x)$ is not a probability.
$f(x)dx$ is a probability.

$f(x) \approx \dfrac{P(X \in dx)}{dx}$ ← probability ← length

units of f? — probability / unit length ← probability density

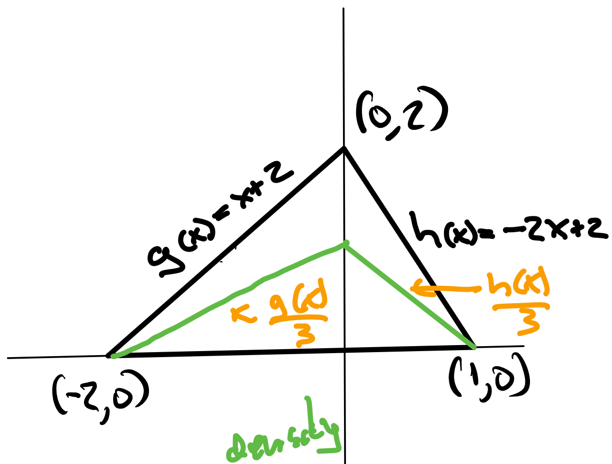
$P(X = x) = 0$

Hence $P(a \le X \le b) = P(a < X < b)$
(we don't have to worry about endpts).

ex 4.1.12 b

Consider a point picked uniformly at random from the area inside the following triangle

Find the density function of the x-coordinate $f(x)$

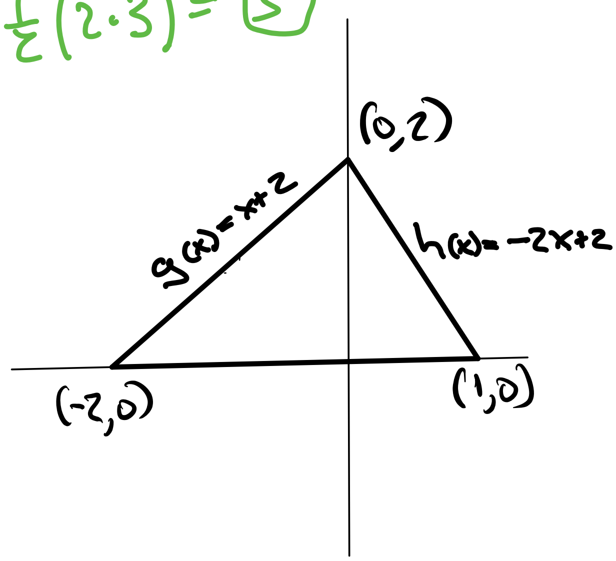


idea $\int_{-2}^{1} f(t)\,dt = \int_{-2}^{0} \frac{g(t)}{\text{area}} + \int_{0}^{1} \frac{h(t)\,dt}{\text{area}}$

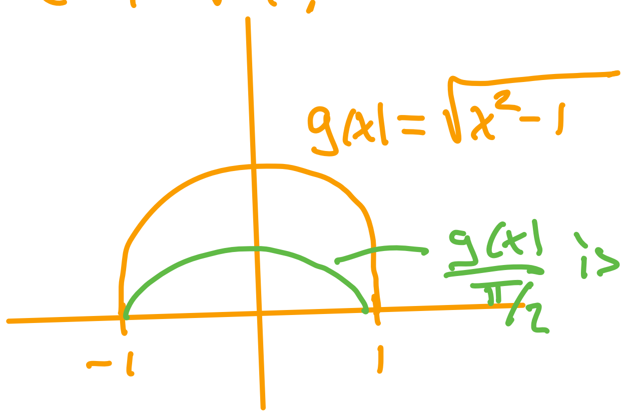
$$f(x) = \begin{cases} \frac{x+2}{3}, & -2 \le x \le 0 \\ \frac{-2x+2}{3}, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

$\frac{1}{2}(2 \cdot 3) = \boxed{3}$

Area of $\triangle$ = ?



Note there is nothing special about the shape being a triangle, It could be a half circle with radius 1 for example,



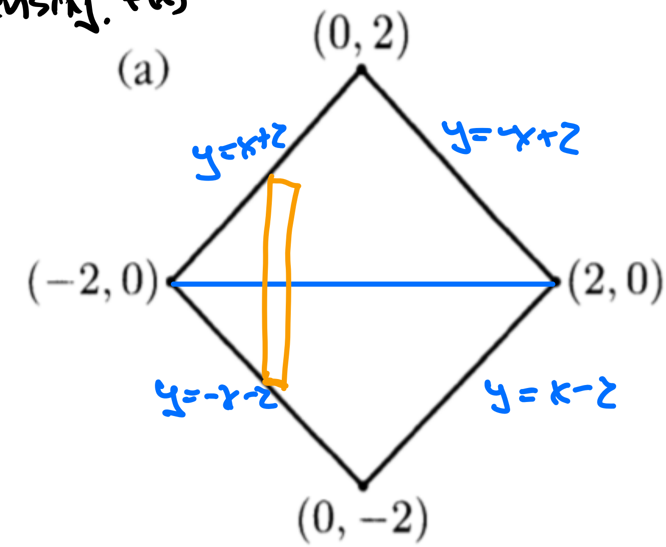
$\frac{g(x)}{\pi/2}$ is a density,

Here the area is $\frac{\pi}{2}$, To make g into a density divide it by π

$$f(x) = \frac{\sqrt{x^2 - 1}}{\pi}$$

ex 4.1.12 a

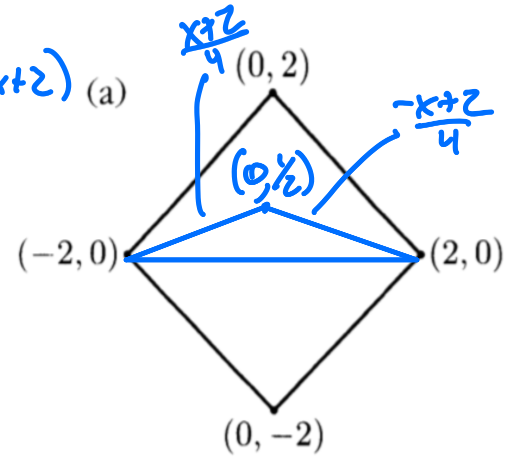
Consider a point picked uniformly at random from the area inside the following shape. Find the density. $f(x)$



$$f(x) = \begin{cases} \frac{x+2}{4}, & -2 \le x \le 0 \\ \frac{-x+2}{4}, & 0 \le x \le 0 \\ 0 & \text{else} \end{cases}$$

$$\text{Area} = 2\left(\tfrac{1}{2} \cdot 4 \cdot 2\right) = 8$$

$$\int_{-2}^{2} f(x)\,dx = \underset{\substack{\| \\ 8}}{\frac{1}{\text{Area}}} \cdot \int_{-2}^{0} 2(x+2)\,dx + \int_{0}^{2} 2(-x+2)$$



(2)

Expectation and Variance

For discrete,

$$E(g(X)) = \sum_{x \in X} g(x) P(X=x)$$

For continuous,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) P(X \in dx) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$Var(X) = E(X^2) - E(X)^2$$

See Wamby for example,

③

Change of scale

To calculate $E(X)$, $Var(X)$, $P(X \in dx)$ we sometimes make a linear change of scale

$Y = c + bX$ where c, b are constants

Y hopefully has a simpler density function.

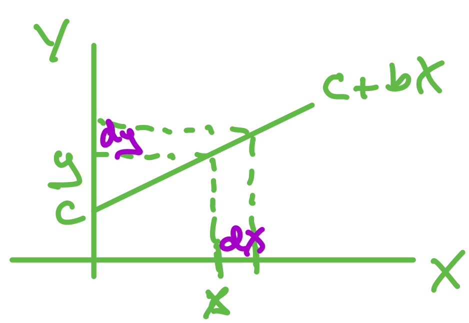
We can recover $E(X)$, $Var(X)$, $P(X \in dx)$ from $E(Y)$, $Var(Y)$, $P(Y \in dY)$.

$$E(Y) = E(bX + c) = bE(X) + c$$

$$\Rightarrow E(X) = \frac{E(Y) - c}{b}$$

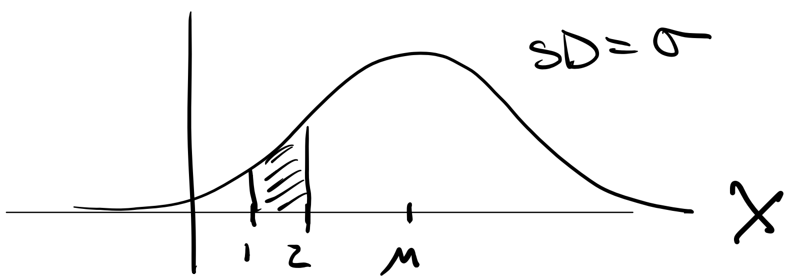
$$Var(Y) = Var(bX + c) = b^2 Var(X)$$

$$\Rightarrow Var(X) = \frac{Var(Y)}{b^2}$$



$$P(X \in dx) = P(Y \in dY)$$

ex Let $X \sim N(\mu, \sigma^2)$
Find $P(1 < X < 2)$

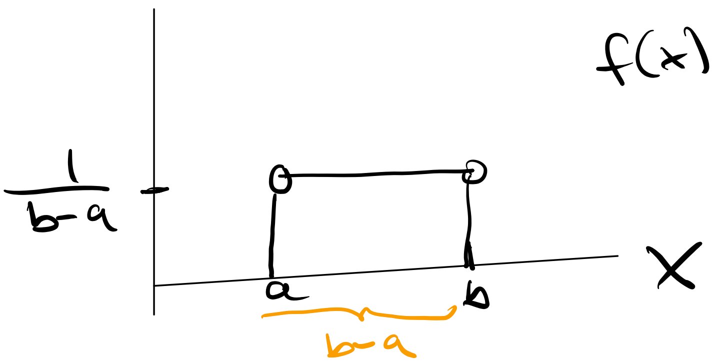


$$Z = \frac{X - \mu}{\sigma} = \underbrace{\frac{1}{\sigma}}_{b} X - \underbrace{\frac{\mu}{\sigma}}_{c}$$

Change of scale

$$P(1 < X < 2) = \Phi\left(\frac{2-\mu}{\sigma}\right) - \Phi\left(\frac{1-\mu}{\sigma}\right)$$

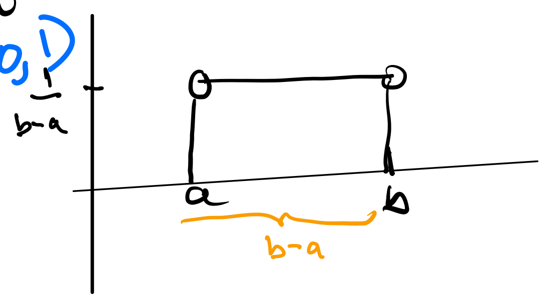
ex: let $X \sim \text{Unif}(a,b)$



$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{else} \end{cases}$$

a) You should change the scale of X to?

$U = \dfrac{X-a}{b-a} \sim \text{Unif}(0,1)$



b) Find $E(X)$

$\frac{1}{2} = E(U) = E\left(\frac{1}{b-a}(x-a)\right) = \frac{1}{b-a} E(x-a)$

$= \frac{1}{b-a}\left(E(x) - a\right)$

$\boxed{E(x) = \frac{a+b}{2}}$

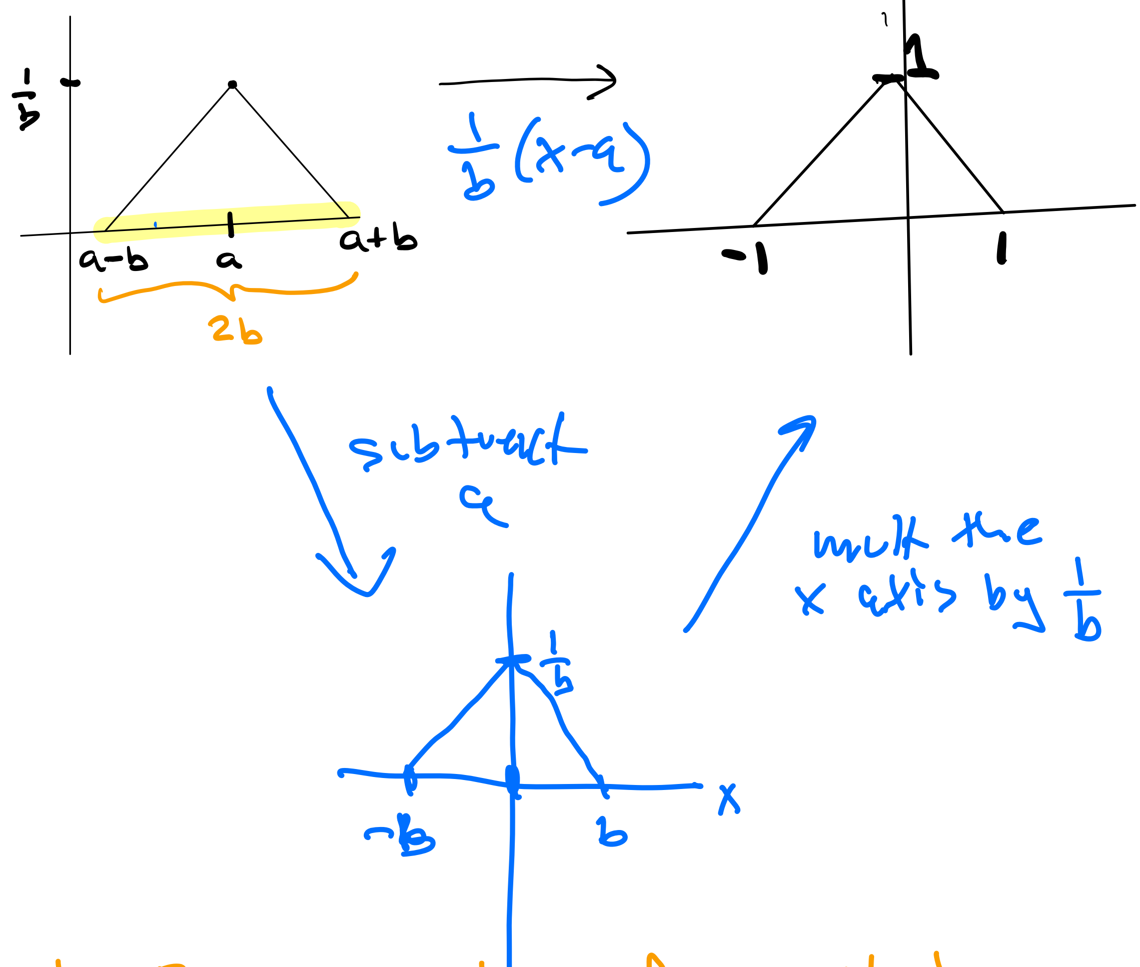
c) Find $\text{Var}(X)$.

$\frac{1}{12} = \text{Var}(U) = \frac{1}{(b-a)^2} \text{Var}(x-a)$

$\Rightarrow \boxed{\text{Var}(x) = \frac{(b-a)^2}{12}}$

What change of scale do have here?

we are changing the values X takes. Because it is a linear change of scale a triangle shaped density maps to the same shape.



Note Because the end result is a density with area 1 the height of the triangle is 1 not $\left(\frac{1}{b}\right)^2$.