Stat 134 lec 34

Last time

Sec 6.2 X discrete.

conditional expectation

$$E(Y|X=x) = \sum_{y \in Y} y \cdot P(Y|X=x)$$

$E(Y|X)$ is a RV (function of x)

law of iterated expectation

$$E(Y) = E(E(Y|X))$$

Today

1. Sec 6.2 properties of conditional expectation.
2. Sec 6.3

   a) Conditional densities.
   b) multiplication rule
   c) rule of average conditional probabilities

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sec 6.2 Properties of conditional expectation

For a fixed conditioning event A, conditional expectation has all the familiar properties of expectation

$$E(X+Y) = E(X) + E(Y)$$

$$E(X+Y\,|\,A) = E(X|A) + E(Y|A)$$

since $X+Y|A$ is a restriction of $X+Y$ to a smaller outcome space.

So suppose A is the event $X=x$

What is $E(X+Y\,|\,X=x)$?

$$= E(X|X=x) + E(Y|X=x)$$

$$= x + E(Y|X=x)$$

So we write,

equality of RV.

$$E(X+Y|X) = X + E[Y|X]$$

Properties

1. $E(X) = E(E(X|Y))$ equality of numbers
2. $E(aY+b|X) = aE(Y|X) + b$
3. $E(Y+Z\,|\,X) = E(Y|X) + E(Z|X)$
4. $E(g(X)|X) = g(X)$
5. $E(g(X)Y|X) = g(X)\,E(Y|X)$

equality of RV

(2)

(a) <u>sec 6.3</u> <u>Conditional density</u>:

Let A be an event.
If X is a discrete RV.

$$P(A \mid X=x) = \frac{P(A, X=x)}{P(X=x)}$$ by Bayes' rule.

If X is continuous,

$$P(A \mid X \in dx) = \frac{P(A, X \in dx)}{P(X \in dx)}$$

we define

$$P(A \mid X=x) = \lim_{dx \to 0} \frac{P(A, X \in dx)}{P(X \in dx)}$$

If Y is continuous

$$P(Y \in dy \mid X=x) = \lim_{dx \to 0} \frac{P(Y \in dy, X \in dx)}{P(X \in dx)}$$

$$\parallel$$

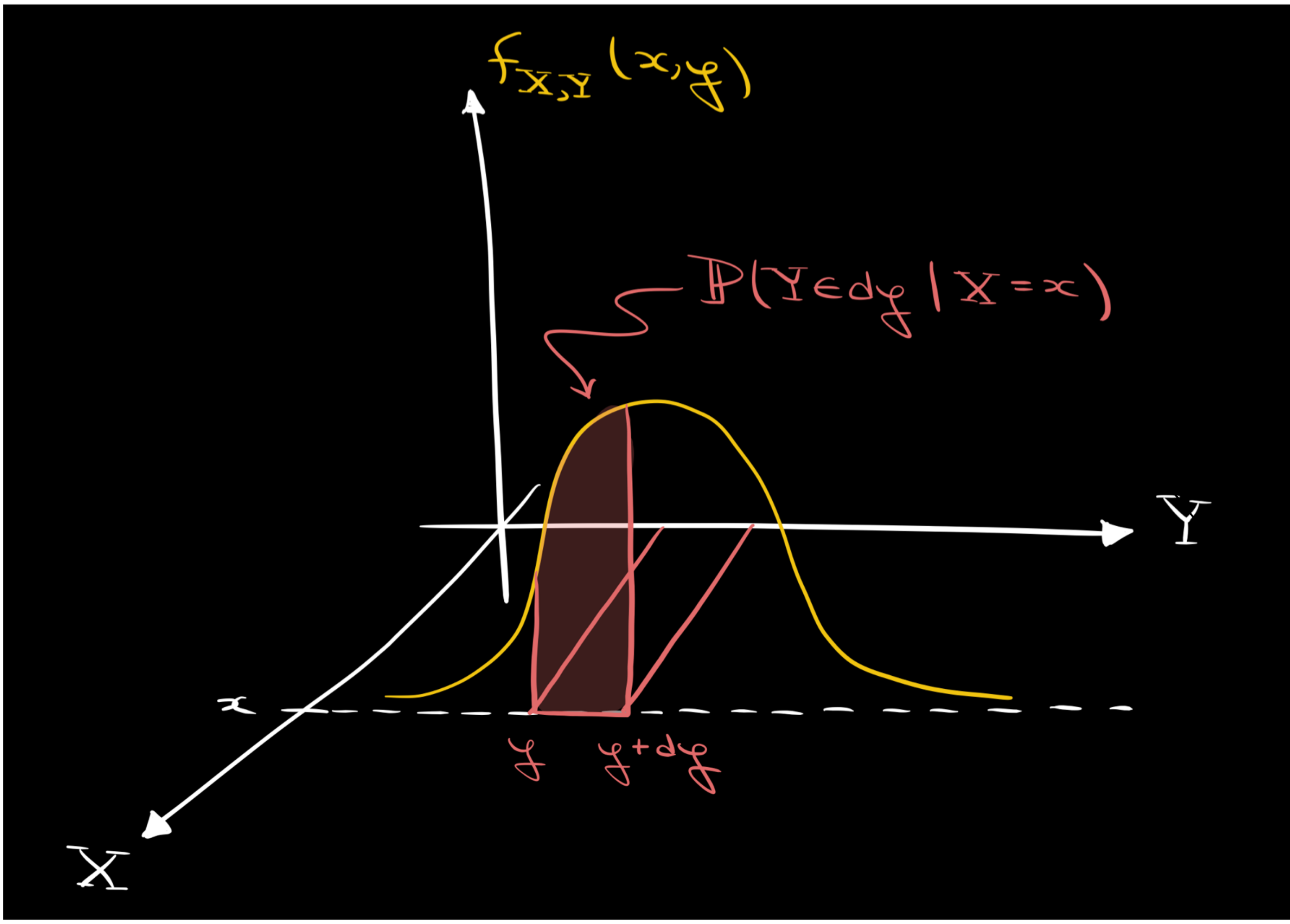
$$f_{Y \mid X=x}(y)\,\cancel{dy} = \lim_{dx \to 0} \frac{f_{X,Y}(x,y)\,\cancel{dx}\,\cancel{dy}}{f_X(x)\,\cancel{dx}}$$

$$\Rightarrow \boxed{f_{Y \mid X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)}}$$

<u>conditional density</u>
<u>of Y given $X=x$</u>

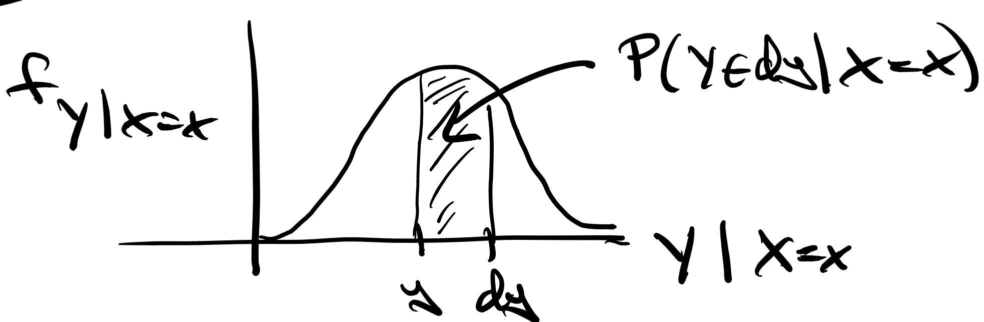
constant (not a function of y).

Picture (Courtesy of John Michael Laurel from our class!)



$f_{Y|X=x}(y)$ consists of a slice of $f(x,y)$ through $X=x$.

I will draw the distribution of $Y|X=x$ as



ex $f(x,y) = K(y-x)^8, \quad 0<x<y<1$

a) find K

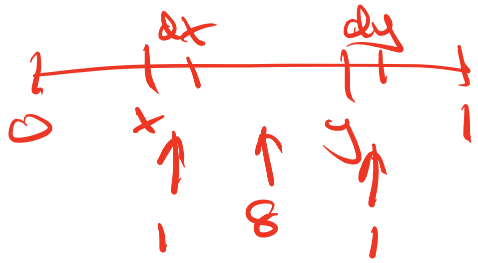
b) find the marginal distribution of x.

Soln a :

Recognize $f(x,y)$ as joint density of two order statistics,

$X = U_{(1)}$

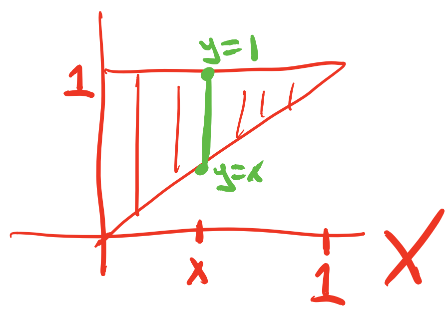
$Y = U_{(10)}$



$$\Rightarrow f(x,y) = \binom{10}{1,8,1}(y-x)^8$$

$$\Rightarrow K = \frac{10!}{1!\,8!\,1!} = 90.$$

Soln b : method 1 (integrate out y)



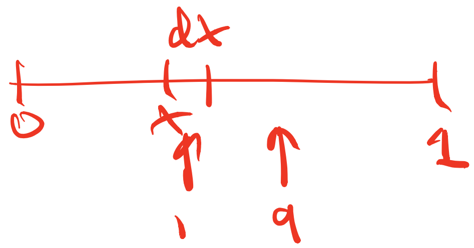
$$f_X(x) = \int_{y=x}^{y=1} f(x,y)\,dy$$

$$= \int_{y=x}^{y=1} 90\,(y-x)^8\,dy \qquad \text{let } U = y-x, \; dU = dy$$

$$= \int_{U=0}^{U=1-x} 90\,U^8\,dU = \frac{90\,U^9}{9}\Big|_0^{1-x} = \boxed{10(1-x)^9, \quad 0<x<1}$$

(or) Method 2 (Find $P(X \in dx)$)

$$P(X \in dx) = f_X(x)\,dx$$



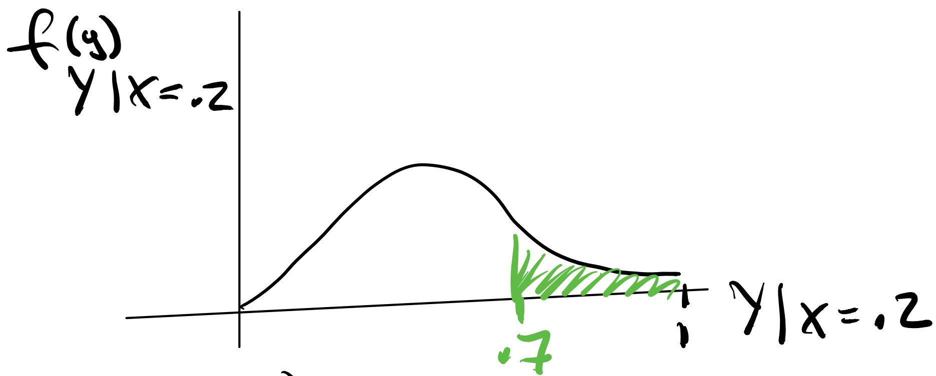
$$\Rightarrow f_X(x) = \binom{10}{1,9}(1-x)^9$$

$$= \boxed{10(1-x)^9, \quad 0<x<1}$$

c) find $P(y > .7 \mid x = .2)$

soln

Method 1 (integrate the conditional density)



$$P(Y > .7 \mid X = .2) = \int_{y=.7}^{y=1} f_{Y|X=.2}(y)\,dy$$

$$f_{Y|X=.2}(y) = \frac{f(.2,y)}{f_X(.2)} = \left.\frac{90(y-x)^8}{10(1-x)^9}\right|_{x=.2}$$

$$= \frac{90(y-.2)^8}{10(.8)^9}$$

$$P(y>.7 \mid x=.2) = \frac{9}{(.8)^9}\int_{y=.7}^{y=1} (y-.2)^8\,dy$$

$u = y-.2$
$du = dy$

$$= \frac{9}{(.8)^9}\int_{u=.5}^{u=.8} u^8\,du = \frac{9}{(.8)^9}\frac{u^9}{9}\Big|_{.5}^{.8}$$

$$= \frac{9}{(.8)^9}\left(\frac{.8^9}{9} - \frac{(.5)^9}{9}\right)$$

$$= \boxed{1 - \left(\frac{.5}{.8}\right)^9}$$

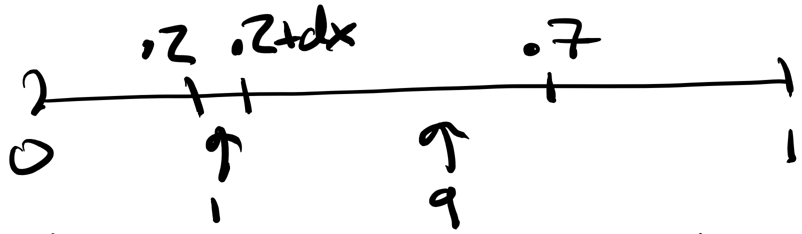
<u>Method 2</u> (use fact that $X = U_{(1)}$ and $Y = U_{(10)}$)

$$P(Y>.7 \mid X=.2) = 1 - P(Y<.7 \mid X=0.2)$$

By Bayes' rule,

$$P(Y<.7 \mid X=0.2) = \frac{P(Y<.7,\ X=.2)}{P(X=.2)}$$

<u>Picture</u>



$P(Y<.7, X=.2)$ is the chance that the remaining 9 darts land between .2 and .7. This is $(.7-.2)^9 = (.5)^9$.
$P(X=.2)$ is the chance that the remaining 9 darts land between .2 and 1. This is $(1-.2)^9 = (.8)^9$

Hence,

$$P(Y>.7 \mid X=.2) = \boxed{1 - \left(\frac{.5}{.8}\right)^9}.$$

(b) Multiplication rule

discrete case:

$$P(X=x, Y=y) = P(Y=y \mid X=x) \cdot P(X=x)$$

continuous case:

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} \Rightarrow \boxed{f(x,y) = f_{Y|X=x}(y)\, f_X(x)}$$

ex $X \sim \text{Gamma}(2, \lambda)$
given $X=x$, $Y \sim U(0,x)$ (i.e. $Y|X=x \sim U(0,x)$)
Find $f(x,y)$

Soln

$$f(x,y) = f_{Y|X=x}(y)\, f_X(x)$$

where

$$f_{Y|X=x}(y) = \frac{1}{x}, \quad 0<y<x<\infty$$

$$f_X(x) = \lambda^2 x e^{-\lambda x}$$

$$\Rightarrow f(x,y) = \frac{1}{x} \lambda^2 x e^{-\lambda x} = \boxed{\lambda^2 e^{-\lambda x}}$$

(c) The rule of average conditional probabilities

discrete case (X discrete):

$$P(A) = \sum_{x \in X} P(A \mid X = x) P(X = x)$$

ex Let A = event that 2^{nd} card in deck is K of hearts.

X = 1^{st} card in deck

There are 52 possible values of X each with probability $\frac{1}{52}$.

average of all possible conditionals.

$$P(A) = \underbrace{P(A \mid X = \text{K of hearts})}_{= 0} \underbrace{P(X = \text{K of hearts})}_{= 1/52} +$$

$$\underbrace{P(A \mid X = \text{5 of spades})}_{= \frac{1}{51}} \underbrace{P(X = \text{5 of spades})}_{= 1/52} + \cdots$$

$$= 51 \left(\frac{1}{51} \cdot \frac{1}{52} \right) = 1/52$$

Continuous case (X continuous):

$$\boxed{P(A) = \int_{x \in X} P(A \mid X = x) f_X(x)\, dx}$$

integral conditioning formula

ex $X \sim \text{Unif}(0,1)$

given $X=p$, let $I_1, I_2 \overset{iid}{\sim} \text{Ber}(p)$

Find the probability the first toss is heads (i.e $I_1=1$).

soln

$A =$ event $I_1 = 1$

$P(A \mid X=p) = p$

$f_X(p) = 1$

$$P(A) = \int_{p=0}^{p=1} P(A \mid X=p) f_X(p)\, dp$$

$$= \int_0^1 p\, dp = \frac{p^2}{2}\Big|_0^1 = \frac{1}{2}$$

So not knowing anything about the coin, the chance it will land heads is $1/2$. If $I_1=1$ then we will see that the chance $I_2=1$ is $2/3$.

(i.e $P(I_2=1 \mid I_1=1) = 2/3$)

To see this we need to know about Bayesian statistics.

Bayesian Statistics

$$\text{Posterior} \propto \text{likelihood} \cdot \text{prior}.$$

proportional.

ex: $X \sim \text{Unif}(0,1)$

given $X = p$, let $I_1, I_2 \overset{iid}{\sim} \text{Ber}(p)$

given that we have a p-coin, the chance it lands head is p.

likelihood : $P(I_1 = 1 \mid X = p) = p$

posterior : $f(X = p \mid I_1 = 1)$ ← computed below

prior : $f_X(p) = 1$ ← $X \sim U(0,1)$ says p is equally likely to be any value $0<p<1$.

$$\underbrace{f(X=p \mid I_1 = 1)}_{\text{posterior}} = \frac{\overbrace{P(I_1 = 1 \mid X = p)}^{\text{likelihood}} \cdot \overbrace{f_X(x)}^{\text{prior}}}{\underbrace{P(I_1 = 1)}_{\text{constant}}}$$

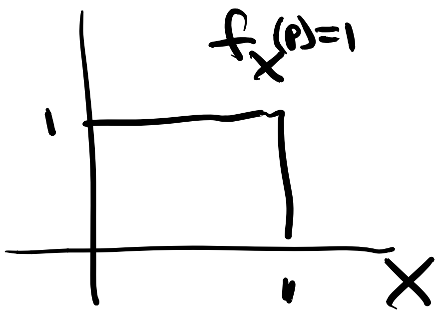
since $f(X = p, I_1 = 1) = f(X = p \mid I_1 = 1)\, P(I_1 = 1)$

$$\|$$

$$f(I_1 = 1, X = p) = P(I_1 = 1 \mid X = p)\, f_X(p)$$

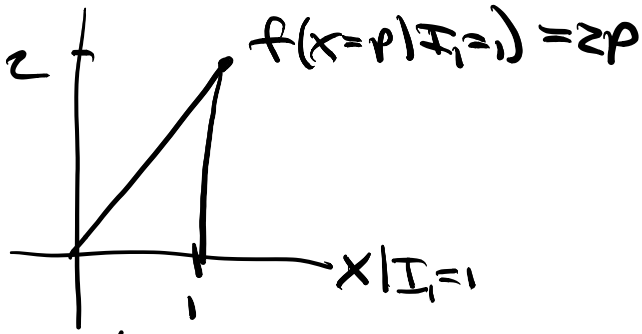
Hence,
$$\underbrace{f(X = p \mid I_1 = 1)}_{\text{posterior}} = \frac{\overbrace{p}^{\text{likelihood}} \cdot \overbrace{1}^{\text{prior}}}{\underbrace{\tfrac{1}{2}}_{\text{constant}}} = \boxed{2p.}$$

Prior



no idea what is probability coin lands heads,

Posterior



based on our 1st ~~toss~~ this is the updated distribution of the probability the coin lands heads,

We will make this posterior our new prior when we flip the coin a second time.

To be continued next time.