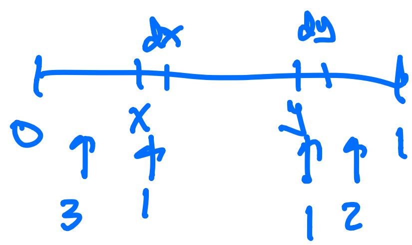
Stat 134 <u>lec 35</u> (MT2 review)

Warmup 1:00-1:10

Let (X, Y) have joint density $f_{X,Y}(x, y) = 420x^3(1-y)^2$ for $0 < x < y < 1$.

Fill in the blanks: X and Y represent the ___4___ smallest and ___5___ smallest of ___7___ i.i.d. Unif (0,1) random variables, respectively.



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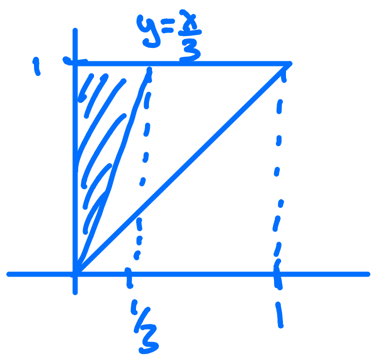
Yesterday

For joint densities, can we go over what should make you choose to integrate over x first and then over y vs. what should make you choose to integrate over y first and then over x?

Let (X, Y) have joint density $f_{X,Y}(x, y) = 420x^3(1-y)^2$ for $0 < x < y < 1$.

Find $P(3X < Y)$;



$$\int_{y=0}^{1} \int_{x=0}^{y/3} 420x^3(1-y)^2 \, dx \, dy$$

$$= 420 \int_{y=0}^{1} (1-y)^2 \underbrace{\int_{x=0}^{y/3} x^3 \, dx}_{\left.\frac{x^4}{4}\right|_0^{y/3} = \frac{y^4}{4 \cdot 3^4}} \, dy$$

$$= \frac{420}{4 \cdot 3^4} \int_0^1 \underbrace{(1-y)^2}_{1-2y+y^2} y^4 \, dy$$

$$= \frac{420}{4 \cdot 3^4} \left[\int_0^1 y^4 \, dy - 2\int_0^1 y^5 \, dy + \int_0^1 y^6 \, dy \right]$$

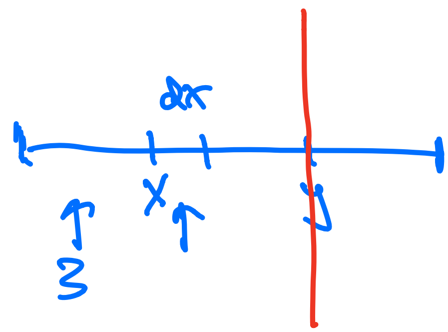
$$= \left(\frac{1}{3}\right)^4$$

Method 2:

$P\left(\frac{X}{Y} < \frac{1}{3}\right)$

$X \sim U_{(4)}$ out of 7

$Y \sim U_{(5)}$ out of 7



$\frac{X}{Y} \sim U_{(4)}$ out of 4

$P\left(U_{(4)} < 1/3\right) = \left(\frac{1}{3}\right)^4$.

ex

Let X, Y have joint density given by

$$f_{X,Y}(x,y) = \frac{\lambda}{y} e^{-\lambda y}, \quad 0 < x < y.$$

Find the marginal distribution of Y.

$$f_Y(y) = \int_0^y f_{X,Y}(x,y)\,dx$$

$$= \int_0^y \frac{\lambda}{y} e^{-\lambda y}\,dx$$

$$= \frac{\lambda e^{-\lambda y}}{y}\, x \Big|_{x=0}^{x=y}$$

$$= \boxed{\lambda e^{-\lambda y}, \quad y > 0}$$

$$\Rightarrow Y \sim \text{Exp}(\lambda),$$

The joint density of X and Y is

$$f(x,y) = \begin{cases} \frac{4y}{x} & \text{for } 0 < x < 1 \text{ and } 0 < y < x \\ 0 & \text{otherwise} \end{cases}$$

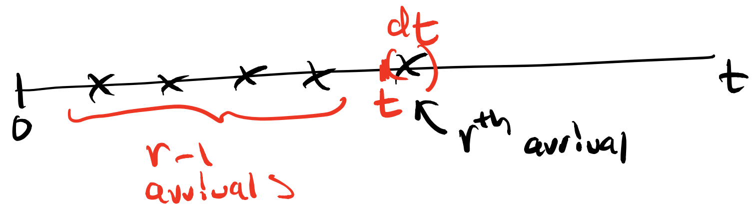
Find the following:

(a) $E(XY)$

(b) The marginal density of X

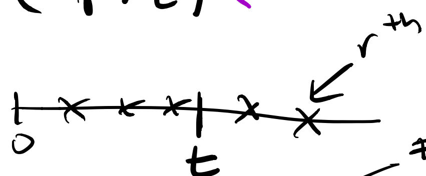
Sec 4.2 Gamma(r,λ) = distribution for r^{th} arrival time of a Poisson (λ) process.

T_r = arrival time of r^{th} call.



Let $T_r \sim \text{gamma}(r, \lambda)$

Find $P(T_r > t)$ (Right tail probability)



arrivals in time t
$X_t \sim \text{Pois}(\lambda t)$

$$P(T_r > t) = P(X_t \le r-1)$$

$$= P(X_t = 0) + P(X_t = 1) + \dots + P(X_t = r-1)$$

$$= e^{-\lambda t} + e^{-\lambda t}\lambda t + \frac{e^{-\lambda t}(\lambda t)^2}{2!} + \dots + \frac{e^{-\lambda t}(\lambda t)^{r-1}}{(r-1)!}$$

ex

let $T \sim \text{Gamma}(r=4, \lambda=2)$

Find $P(T>7)$

$P(T>7) = P(N_7 \leq 3)$ r-1

where $N_7 \sim \text{Pois}(14)$ 2·7

$$P(T>7) = e^{-14}\left(1 + \frac{14}{1} + \frac{14^2}{2!} + \frac{14^3}{3!}\right)$$

5. Travelers arrive at an airport Information desk according to a Poisson process at the rate of 15 per hour. Assume that each traveler arriving at the desk has a 60% chance of being male and a 40% chance of being female, independent of all other travelers.

a) Fill in the blank with a number: The fifth male traveler is expected to arrive at the desk ________ minutes after the first male traveler.

$T \sim \text{Pois}\left(15 \cdot \frac{1}{60}\right)$

$M \sim \text{Pois}\left(15(.6)\left(\frac{1}{60}\right)\right) = \text{Pois}\left(9/60\right)$

$F \sim \text{Pois}\left(15(.4)\left(\frac{1}{60}\right)\right) = \text{Pois}\left(6/60\right)$

W_5 = wait time of 5^{th} male $\sim$ Gamma$\left(5, \frac{9}{60}\right)$

W_1 = wait time of 1^{st} male $\sim$ Gamma$\left(1, 9/60\right)$

$$E(W_5 - W_1) = \underset{\frac{5}{9/60}}{\underset{\parallel}{E(W_5)}} - \underset{\frac{1}{9/60}}{\underset{\parallel}{E(W_1)}} = \frac{4 \cdot 60}{9} = \boxed{\frac{80}{3} \text{ min}}$$

5. Travelers arrive at an airport Information desk according to a Poisson process at the rate of 15 per hour. Assume that each traveler arriving at the desk has a 60% chance of being male and a 40% chance of being female, independent of all other travelers.

b) Find the chance that the fifth male traveler arrives at the desk more than 30 minutes after the first male traveler.

$M \sim \mathrm{Pois}(9/60)$
$W \sim \mathrm{Pois}(6/60)$

$$P(W_5 - W_1 > 30) = P(W_4 > 30)$$
$$= P(N_{30} \leq 3)$$

$$= P(N_{30}=0) + P(N_{30}=1) + P(N_{30}=2) + P(N_{30}=3)$$
$$= \boxed{e^{-4.5}\left(1 + 4.5 + \frac{4.5^2}{2!} + \frac{4.5^3}{3!}\right)}$$

since,

$$N_{30} \sim \mathrm{Pois}\left(\frac{9}{60} \cdot 30\right) = \mathrm{Pois}(4.5)$$

5. Travelers arrive at an airport Information desk according to a Poisson process at the rate of 15 per hour. Assume that each traveler arriving at the desk has a 60% chance of being male and a 40% chance of being female, independent of all other travelers.

c) Find the expected number of female travelers who arrive at the desk before the fifth male traveler.

$X \sim \text{NegBin}(r, p)$ on $0, 1, 2, \ldots$

is the number of failures until your r^{th} success

$$E(X) = r \cdot \frac{1-p}{p}$$

Here let X = # female before the 5^{th} male

$X \sim \text{NegBin}(5, .6)$ on $0, 1, 2, \ldots$

$$E(X) = 5\left(\frac{.4}{.6}\right) = \boxed{10/3}$$

Review MGF $\quad M_X(t) = E(e^{tX})$

Main properties

1. $M_X(0) = 1$

2. $M_{aX}(t) = M_X(at)$

3. $M_X'(0) = E(X)$

   $M_X''(0) = E^2(X)$

   $\vdots$

   $M_X^{(k)}(0) = E^{(k)}(X)$

4. If $X_1, \ldots, X_n$ are independent then

   $M_{X_1 + \cdots + X_n}(t) = M_{X_1}(t) \cdots M_{X_n}(t)$

5. $M_X(t)$ is unique for t in a neighborhood of 0. So if $M_X(t) = e^{t^2/2}$ for all t then $X \sim N(0,1)$.

ex

CLT

Let $X_1, \ldots, X_n \overset{iid}{\sim} F$, mean μ, SD σ ← any distribution

$S_n = \sum_{i=1}^{n} X_i$

$S_n \longrightarrow N(n\mu, n\sigma^2)$ as $n \to \infty$

Pf/ We show that

$$\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \longrightarrow Z \sim N(0,1) \quad \text{as } n \to \infty$$

Let $Y_i = \dfrac{X_i - \mu}{\sigma}$

$$\sum_{i=1}^{n} Y_i = \frac{\overbrace{\sum_{i=1}^{n} X_i}^{S_n} - n\mu}{\sigma}$$

So $$\sum_{i=1}^{n} \frac{Y_i}{\sqrt{n}} = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$$

We will show that for n large,

$\sum_{i=1}^{n} \frac{Y_i}{\sqrt{n}}$ and Z have the same MGF.

Note that

$$E(Y_i) = E\left(\frac{X_i - \mu}{\sigma}\right) = \frac{1}{\sigma} E(X_i - \mu) = 0$$

$$Var(Y_i) = \frac{1}{\sigma^2} Var(X_i - \mu) = \frac{1}{\sigma^2} \cdot \sigma^2 = 1$$

So $E^2(Y_i) = Var(Y_i) + E(Y_i)^2 = 1$

Make a Taylor series of $M_{\frac{Y_i}{\sqrt{n}}}(t)$ around 0:

$$M_{\frac{Y_i}{\sqrt{n}}}(t) = M_{Y_i}\left(\frac{t}{\sqrt{n}}\right) = M_{Y_i}(0) + M'_{Y_i}(0)\frac{t}{\sqrt{n}} + \frac{M''_{Y_i}(0)\frac{t^2}{n}}{2!} + \cdots$$

$$= 1 + \underbrace{E(Y_i)}_{=0}\frac{t}{\sqrt{n}} + \frac{\overbrace{E(Y_i^2)}^{=1}}{2!}\frac{t^2}{n} + \cdots$$

$$= 1 + \frac{1}{n}\left[\frac{t^2}{2} + \frac{t^3 M'''(0)}{3!\, n^{1/2}} + \cdots\right]$$

$\to 0$

all terms $\to 0$ since have n in denom,

$$\to 1 + \frac{1}{n}\frac{t^2}{2} \quad \text{as } n \to \infty$$

So

$\frac{Y_i}{\sqrt{n}}$ are independent,

$$M_{\frac{S_n - n\mu}{\sqrt{n}\sigma}}(t) = M_{\frac{Y_1}{\sqrt{n}}}(t) \cdots M_{\frac{Y_n}{\sqrt{n}}}(t)$$

$$\frac{S_n - n\mu}{\sqrt{n}\sigma} = \sum_{i=1}^{n} \frac{Y_i}{\sqrt{n}}$$

$$\to \left[1 + \frac{1}{n}\frac{t^2}{2}\right]^n \approx e^{\frac{t^2}{2}}$$

which is MGF of $N(0,1)$

Hence $\frac{S_n - n\mu}{\sqrt{n}\sigma} \to N(0,1)$

$\square$