Stat 134: Change of Variable and Operations Review Section

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Change of Variable and Operations Review

a. List the steps for finding the density of $Y = g(X)$, when X has a known density.

b. What is the convolution formula for a sum of two random variables: $Z = X + Y$?.

c. What is the general convolution formula for a function of two random variables: $Z = g(X, Y)$?

a. See Pitman's Probaility pages 304 to 307.

b.

$$f_{X+Y}(z) = \int_X f_{X,Y}(x, z-x)dx$$

c.

$$f_Z(z) = \int_X f_{X,Y}(x, y)|dy/dz|dx = \int_Y f_{X,Y}(x, y)|dx/dz|dy$$

Problem 1

Suppose a particle is fired from the origin of the (x,y)-plane in a straight line in a direction at a random angle Θ to the x-axis, where $\Theta \sim Unif(0, \pi)$. Let Y be the y-coordinate of the place where the particle crosses either the line $x = 1$ or the line $x = -1$. Find the distribution of Y.

Inspired by ex 4.4.6 in Pitman's Probability

We will use the steps from review question (a), but let's first write the transformation mathematically:

$$\tan(\Theta) = \begin{cases} Y/1, \text{ if } \Theta > \pi/2 \\ Y/(-1), \text{ if } \Theta < \pi/2 \end{cases}$$

Therefore,

$$\Theta = \begin{cases} \arctan(Y), \text{ if } \Theta > \pi/2 \\ \arctan(-Y), \text{ if } \Theta < \pi/2 \end{cases}$$

1. The range of Y is $(0, \infty)$.

2. The transformation is not 1-1 because for every value of $y \in (0, \infty)$, there is a value of θ in $(0, \pi/2)$ and another value of θ in $(\pi/2, \pi)$ resulting in $Y = y$. We will have to account for this when computing the density of Y.

3. We could calculate $dy/d\theta$, but in this case it's easier to calculate $d\theta/dy$. $d\theta/dy = \frac{1}{1+y^2}$ if $\theta < \pi/2$, and is $-\frac{1}{1+y^2}$ if $\theta > \pi/2$. Conveniently, $|d\theta/dy| = \frac{1}{1+y^2}$ in both cases.

4. $$f_Y(y) = \sum_{\theta:\theta \text{ results in } y} \frac{f_\theta(\theta)}{|dy/d\theta|} = \sum_{\theta:\theta \text{ results in } y} f_\theta(\theta)|d\theta/dy| = 2 \cdot \frac{1}{\pi}\frac{1}{1+y^2} = \frac{2}{\pi(1+y^2)}$$

5. In this case there is no need to eliminate θ from the density because we used $|d\theta/dy|$ in step 4 and the density of Θ is constant. Hence, the density of Y is $\frac{2}{\pi(1+y^2)}$ for $y \in (0, \infty)$.

Problem 2

The time before a light bulb begins to flicker is exponentially distributed with rate 10. However, once it starts to flicker, it continues to flicker for an additional amount of time before it fully burns out. This additional amount of time is randomly distributed according to a standard uniform distribution and is independent of the amount of time the light bulb took to start flickering. Find the distribution of the amount of time it takes for the light bulb to fully die out.

Let T denote the time it takes for the light bulb to fully flicker out. $T = X + Y$, where $X \sim Unif(0,1)$ and $Y \sim Exp(10)$. The range of Y is $(0,\infty)$. Note that the joint density of X and Y is $f_{X,Y}(x,y) = \lambda e^{-\lambda y}$, where $\lambda = 10$.

for $t \in (0,1)$

$$\begin{aligned} f_T(t) &= \int_0^t f_{X,Y}(x, t-x)dx \\ &= \int_0^t \lambda e^{-\lambda(t-x)}dx \\ &= 1 - e^{-\lambda t} \end{aligned}$$

for $t \in (1,\infty)$

$$\begin{aligned} f_T(t) &= \int_0^1 f_{X,Y}(x, t-x)dx \\ &= \int_0^1 \lambda e^{-\lambda(t-x)}dx \\ &= e^{-\lambda t}(e^{\lambda} - 1) \end{aligned}$$

Plugging in $\lambda = 10$, the density of T is

$$f_T(t) = \begin{cases} 1 - e^{-10t}, t \in (0,1) \\ e^{-10t}(e^{10} - 1), t \in (1,\infty) \\ 0, else \end{cases}$$

Problem 3

Let $V \sim Unif(0,1)$ and $W \sim Gamma(2,\lambda)$.

a. Find the distribution of Z, where $Z = VW$.

b. Find the distribution of $(XY)^V$, where X, Y and V are all independent standard normal random variables.

a. The joint density of V and W is

$$f_{V,W}(v,w) = \Gamma(2)^{-1}e^{w\lambda}w\lambda^2 = e^{w\lambda}w\lambda^2$$

Then,

$$\begin{aligned}
f_Z(z) = \int_X f_{V,W}(v,w)|dw/dz|dv &= \int_X f_{V,W}(v,z/v)\cdot\frac{1}{v}dv \\
&= \int_0^1 e^{\frac{z}{v}\lambda}\frac{z}{v}\lambda^2\cdot\frac{1}{v}dv \\
&= \int_0^1 e^{\frac{z}{v}\lambda}\frac{z}{v}\lambda^2\cdot\frac{1}{v}dv \\
&= \int_{u=\infty}^1 e^{uz\lambda}u^2z\lambda^2\cdot\frac{-1}{u^2}du,\ \text{using the transformation } u = 1/v \\
&= \int_{u=1}^\infty e^{uz\lambda}z\lambda^2du \\
&= \lambda\int_{u=1}^\infty e^{uz\lambda}z\lambda du,\ \text{notice the integrand is an } Exp(z\lambda) \text{ density} \\
&= \lambda e^{-z\lambda\cdot 1},\ \text{using the survival of an exponential}
\end{aligned}$$

So $Z \sim Exp(\lambda)$.

b. Let $T = (XY)^V$.

$$\begin{aligned}
P(T < t) &= P((XY)^V < t) \\
&= P(-\log(XY)^V > -\log(t)) \\
&= P(V(-\log(X) - \log(Y)) > -\log(t))
\end{aligned}$$

Recall that $-\log(X) \sim Exp(1)$, so $-\log(X) - \log(Y) \sim Gamma(2,1)$.

Then, using part (a), $V(-\log(X) - \log(Y)) \sim Exp(1)$.

So, $P(V(-\log(X) - \log(Y)) > -\log(t)) = e^{-1(-\log(t))} = t$.
Therefore, $(XY)^V \sim Unif(0,1)$.