Stat 134 <u>lec 24</u>

Warmup: 11:00–11:10

Let $X \sim Ber(p)$, $X = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{with prob } 1-p \end{cases}$

a) Find

$$\left.\begin{aligned} E(X) &= 1\cdot p + 0\cdot(1-p) = p \\ E(X^2) &= 1^2\cdot p + 0^2(1-p) = p \\ &\vdots \\ E(X^k) &= 1^k\cdot p + 0^k(1-p) = p \end{aligned}\right\} \text{called moments of } X$$

think of e^{tX} as a function $g(X)$ of $X \sim Ber(p)$

b) Find $E(e^{tX})$, $t \in \mathbb{R}$

$$= e^{t\cdot 1}\cdot p + \underbrace{e^{t\cdot 0}}_{=1}(1-p)$$

$$= e^{t\cdot 1}p + 1-p$$

$$= pe^t + 1-p = \boxed{1 + p(e^t-1)} \leftarrow \text{for all } t\in\mathbb{R}$$

$$\frac{d}{dt}\left(E(e^{tX})\right)\Big|_{t=0} = \frac{d}{dt}\left(1+p(e^t-1)\right)\Big|_{t=0} = pe^t\Big|_{t=0} = p$$

$$\frac{d^2}{dt^2}\left(E(e^{tX})\right)\Big|_{t=0} = \frac{d}{dt}\left(pe^t\right)\Big|_{t=0} = p$$

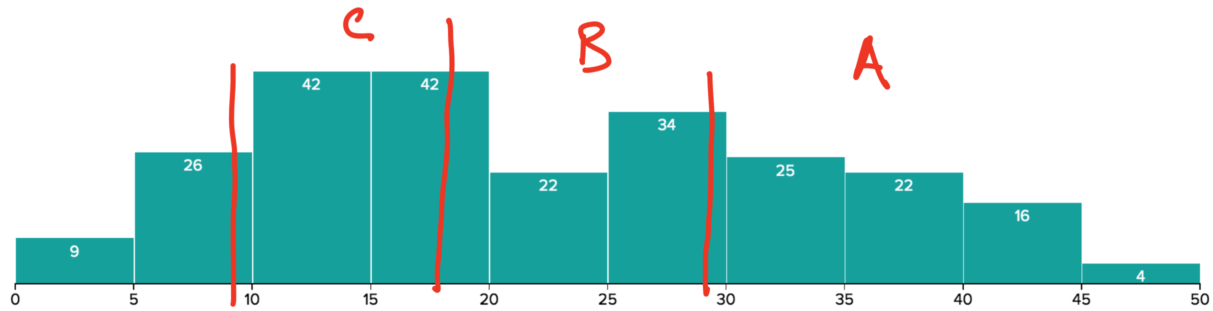
$$\vdots$$

$$\frac{d^k}{dt^k}\left(E(e^{tX})\right)\Big|_{t=0} = p$$

To find the moments of X we take the derivatives of $E(e^{tX})$ and evaluate at $t=0$.

For X a RV, $M_X(t) = E(e^{tX})$ is called the moment generating function (MGF) of X.

Midterm results



MINIMUM	MEDIAN	MAXIMUM	MEAN	STD DEV
0.0	20.0	49.0	21.94	11.41

Special lecture

Moment Generating Function of X

Not in book.

Next time Finish MGF, Sec 4.4, start Sec 4.5

Moment Generating Function (MGF) of X

The k^{th} moment of a RV X is the number

$E(X^k)$ defined for $k = 0, 1, 2, 3, \ldots$

$E(X^0) = E(1) = 1$

$E(X)$

$E(X^2)$

moments describe your distribution
ex 1^{st} moment is mean
2^{nd} moment relates to variance
3^{rd} moment relates to how skewed the distribution is.

Note $Var(X) = E(X^2) - E(X)^2$

Computing the moments of a RV is sometimes hard, because it involves computing complicated integrals or sums

$$E(X) = \begin{cases} \int_{-\infty}^{\infty} x f(x)\,dx & \text{if } X \text{ continuous} \\ \sum_{-\infty}^{\infty} x P(x) & \text{if } X \text{ discrete} \end{cases}$$

The MGF allows one to compute the moments by computing derivatives, which is easier.

Recall $E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$

We define the MGF of X to be

$$M_X(t) = E(e^{tX}) = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{if } X \text{ continuous} \\ \sum_{x=-\infty}^{\infty} e^{tx} p(x) & \text{if } X \text{ discrete} \end{cases}$$

$M_X(t)$ is sometimes written $\psi_X(t)$ in HW 9

In the warmup saw for $X \sim \text{Ber}(p)$

k^{th} derivative evaluated at $t=0$

$$M_X^{(k)}(t)\Big|_{t=0} = E(X^k)$$

Lets show that is true for most RV X.

Recall the Taylor series for e^y:

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

let $t \in \mathbb{R}$, X RV.

You can do this ~~for~~ the RV tX

$$e^{tX} = 1 + tX + \frac{(tX)^2}{2!} +$$

$$E(e^{tX}) = E(1) + E(tX) + E\left(\frac{(tX)^2}{2!}\right) + \cdots$$

$M_X(t)$

$$= E(1) + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots$$

$$\frac{d}{dt} M_X(t) \Big|_{t=0} = E(X)$$

$$\frac{d^2}{dt^2} M_X(t) \Big|_{t=0} = E(X^2)$$

more generally,

$$\frac{d^k}{dt^k} M_X(t) \Big|_{t=0} = \frac{d^k}{dt^k} E(e^{tX}) \Big|_{t=0}$$

Lets not worry about why this step is true. See Leibniz rule. — !!!

$$= E\left(\frac{d^k}{dt^k} e^{tX} \Big|_{t=0}\right)$$

$$= E(X^k)$$

Summary, the MGF $M_X(t) = E(e^{tX})$ contains info about all of the moments of X. By taking the k^{th} derivative and evaluating at 0 you get the k^{th} moment.

Thm If a MGF exists in an interval around zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

Note An MGF doesn't always exist in an interval around zero. (see appendix for an example)

ex let $X \sim \text{Gamma}(r, \lambda)$

$$f(x) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}, & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Recall $\Gamma(r) = \int_0^\infty u^{r-1} e^{-u} du$

Find $M_X(t)$.

Step 1 write $M_X(t)$ as an integral

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \left(\frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \right) dx$$

$$= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^{r-1} e^{(t-\lambda)x} dx \quad \text{for } t < \lambda$$

which converges for $(t-\lambda)<0$ i.e $\boxed{t<\lambda}$

Step 2 Solve the integral

Hint:

make a u substitution

let

$$u = \underbrace{-(t-\lambda)}_{>0} x$$

so $x = -\frac{u}{t-\lambda} = \frac{u}{\lambda - t}$, $dx = \frac{1}{\lambda - t} du$

$$\frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^{r-1} e^{(t-\lambda)x} dx =$$

$$= \frac{\lambda^r}{\Gamma(r)} \int_{u=0}^{u=\infty} \frac{u^{r-1}}{(\lambda-t)^{r-1}} e^{-u} \frac{1}{\lambda - t} du$$

$$= \frac{\lambda^r}{\Gamma(\cancel{r})} \frac{1}{(\lambda-t)^r} \int_0^\infty u^{r-1} e^{-u}\, du \quad \Gamma(r)$$

recall $\Gamma(r) = \int_0^\infty u^{r-1} e^{-u}\, du$

$$= \frac{\lambda^r}{(\lambda-t)^r} = \left(\frac{\lambda}{\lambda-t}\right)^r \text{ for } t<\lambda$$

Recall If a MGF exists in an interval around zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

ex

Let $X \sim \text{Gamma}(r, \lambda)$

Find $\frac{d}{dt} M_X(t)\Big|_{t=0}$

Recall $M_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^r$ for $t<\lambda$

$$M_X(t) = \lambda^r (\lambda-t)^{-r}$$

$$M_X'(t) = \lambda^r (-r)(\lambda-t)^{-r-1}(-1)$$

$$= \frac{r\lambda^r}{(\lambda - t)^{r+1}}$$

$$M'(0) = \frac{r}{\lambda} \Rightarrow \boxed{E(X) = \frac{r}{\lambda}} \checkmark$$

Similarly,

Find: $\frac{d^2}{dt^2} M_X(t) \Big|_{t=0}$

$$M''(t) = r\lambda^r(-r-1)(\lambda - t)^{-r-2}(-1)$$

$$= r(r+1)\lambda^r \frac{1}{(\lambda - t)^{r+2}}$$

$$\Rightarrow M''(0) = r(r+1)\lambda^r \frac{1}{\lambda^{r+2}} = \frac{r(r+1)}{\lambda^2}$$

$$\Rightarrow \boxed{E(X^2) = \frac{r(r+1)}{\lambda^2}}$$

$$\Rightarrow \operatorname{Var}(X) = E(X^2) - E(X)^2 = \frac{r(r+1)}{\lambda^2} - \frac{r^2}{\lambda^2}$$

$$= \frac{r^2}{\lambda^2} + \frac{r}{\lambda^2} - \frac{r^2}{\lambda^2} = \boxed{\frac{r}{\lambda^2}}$$

ex: A RV X ~~takes~~ takes values 1, 2, 3 with prob 1/2, 1/3, 1/6.

Find $M_X(t)$.

$$= E(e^{tx}) = \sum_{x=1}^{3} e^{tx} p(x)$$

$$= \boxed{e^{t\cdot 1} \cdot \frac{1}{2} + e^{2t} \cdot \frac{1}{3} + e^{3t} \cdot \frac{1}{6} \quad \text{for all } t}$$

ex $X \sim Geom(\frac{1}{3})$

$P(X=x) = (\frac{2}{3})^{x-1}(\frac{1}{3}) \qquad x=1,2,3,\ldots$

Find $M_X(t) = E(e^{tX})$

$$= \sum_{x=1}^{\infty} e^{tx}\left(\frac{2}{3}\right)^{x-1}\left(\frac{1}{3}\right)$$

$$= \sum_{x=1}^{\infty} e^{t}\left(e^{t}\right)^{x-1}\left(\frac{2}{3}\right)^{x-1}\left(\frac{1}{3}\right)$$

$$= \frac{1}{3}e^{t}\sum_{x=1}^{\infty}\left(e^{t}\cdot\frac{2}{3}\right)^{x-1}$$

$$\sum_{x=1}^{\infty}\left(e^{t}\cdot\frac{2}{3}\right)^{x-1} = 1 + e^{t}\frac{2}{3} + \left(e^{t}\frac{2}{3}\right)^{2} + \cdots$$

$$= \frac{1}{1-\left(e^{t}\frac{2}{3}\right)} \quad \text{if } e^{t}\frac{2}{3} < 1$$

or $t < \log\left(\frac{3}{2}\right)$

$$\Rightarrow M_X(t) = \frac{1}{3}e^{t}\cdot\frac{1}{1-e^{t}\cdot\frac{2}{3}} \quad \text{if } t < \log\left(\frac{3}{2}\right)$$

Appendix:

Let X be a discrete RV with probability mass function

$$p(x) = \begin{cases} \frac{6}{\pi^2 x^2} & x = 1, 2, \ldots \\ 0 & \text{else} \end{cases}$$

The MGF, $M_X(t)$, only exists at $t=0$, and hence doesn't exist on an interval around zero.

Pf/

It is known that the series

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ converges to $\pi^2/6$.

Then

$$p(x) = \begin{cases} \frac{6}{\pi^2 x^2} & x = 1, 2, 3, \ldots \\ 0 & \text{else.} \end{cases}$$

is the prob function of a RV X,

$$M_X(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} \frac{6e^{tx}}{\pi^2 x^2}$$

The ratio test can be used to show this diverges if $t > 0$. Hence this RV only has an MGF at $t=0$.

$\square$