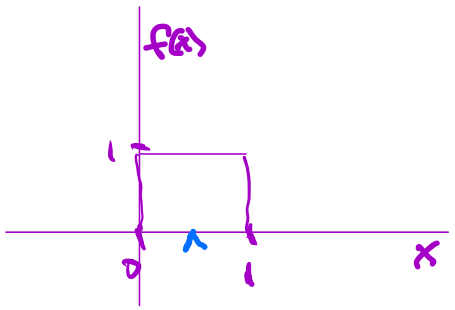
Stat 134 lec 19 (no lec 18)

Warmup 9:00–9:10

Let $X \sim \text{Unif}(0,1)$ be the standard uniform distribution with histogram (density)

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \le 1 \\ 0 & \text{else} \end{cases}$$

Picture



Define

$$E(X) = \int_{x=-\infty}^{\infty} x f(x)\,dx$$

Find $E(X)$, $E(X^2)$, and $\text{Var}(X)$.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} x \cdot 0\,dx + \int_{0}^{1} x \cdot 1\,dx + \int_{1}^{\infty} x \cdot 0\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \boxed{1/2}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \boxed{1/3}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = 1/3 - (1/2)^2 = \boxed{1/12}$$

Last time

Congratulations on finishing midterm 1 !

today

Sec 4.1 Continuous Distributions

1. probability density
2. change of scale
3. expectation and variance.

(1) Sec 4.1 Probability density.

Let X be a continuous RV

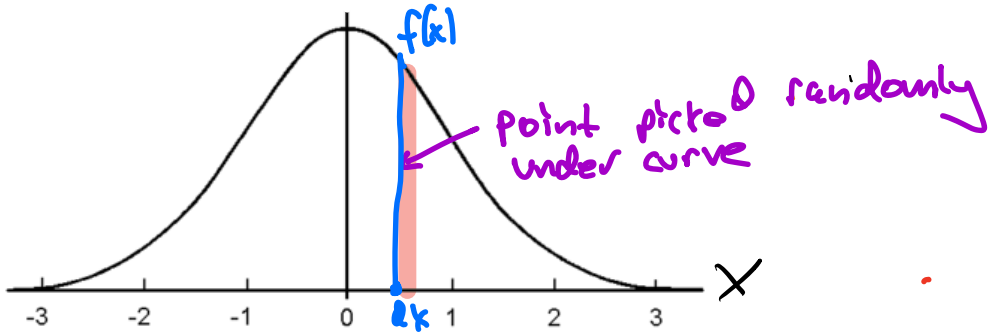
The probability density (histogram) of X is described by a prob density function

$$f(x) \geq 0 \quad \text{for } x \in X$$

$$\text{and } \int_{-\infty}^{\infty} f(x)\,dx = 1$$

ex The standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$

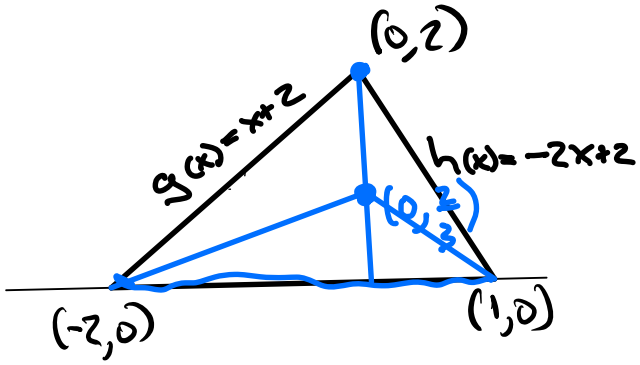


By geometry, the chance of picking a pt under the curve in the red strip above is $f(x)dx$, where x is the x coordinate of the point, and dx is the width of the strip.

If $f(x)$ isn't a density, to make it a density divide it by the area under $f(x)$.

ex 4.1.12 b

Consider a point picked uniformly at random from the area inside the following triangle

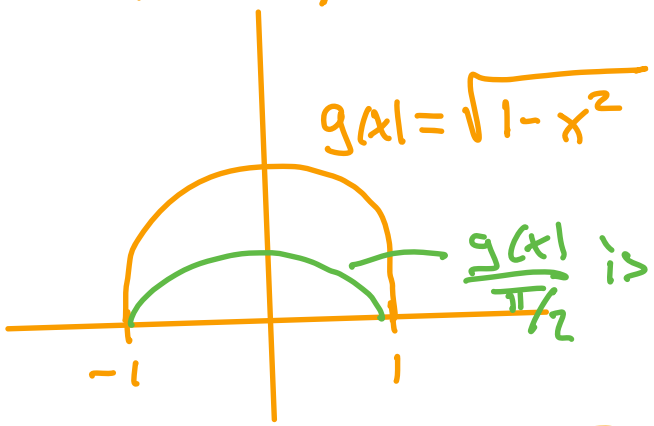


Find the density function of the x-coordinate $f(x)$

$A = \frac{1}{2} \cdot 2 \cdot 3 = 3$

$$f(x) = \begin{cases} \frac{x+2}{3} & -2 \le x \le 0 \\ \frac{-2x+2}{3} & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

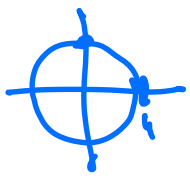
Note there is nothing special about the shape being a triangle. It could be a half circle with radius 1 for example.



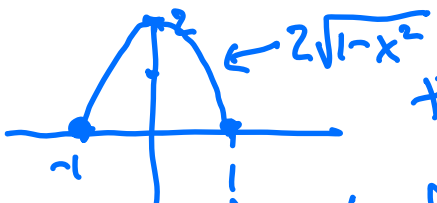
Here the area is $\frac{\pi}{2}$. To make g into a density divide it by $\pi/2$

$$f(x) = \frac{\sqrt{1-x^2}}{\pi/2}$$

Suppose the shape was a full circle radius 1



Now part of shape is under the x axis.
If you flip the bottom semicircle and add it to top you get a shape

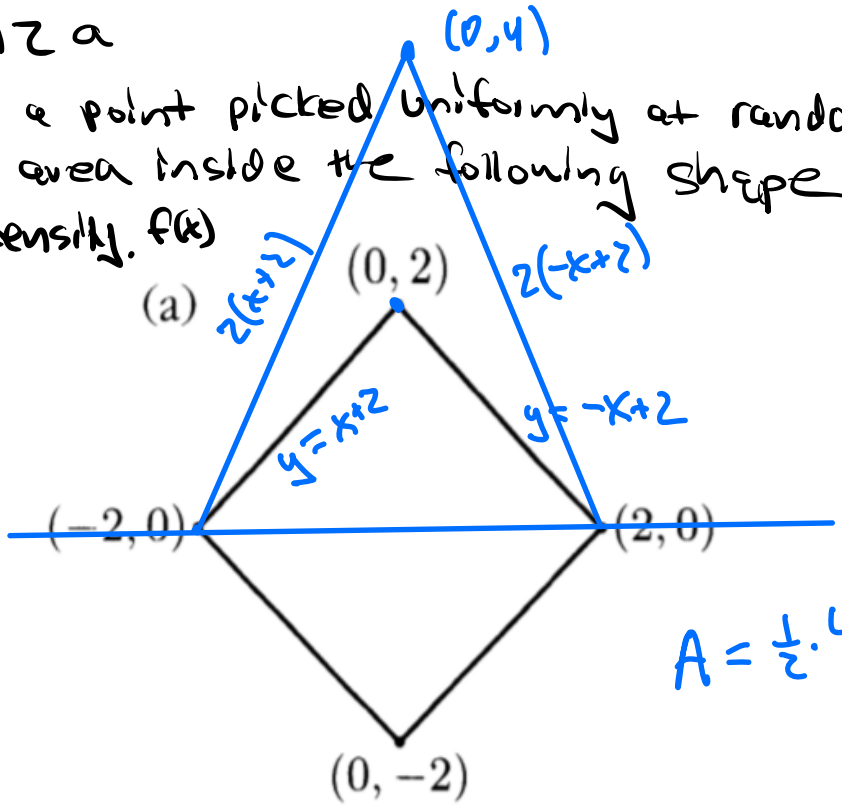


that is easier to think about
This is not a density. To make a density divide by π.

$$f(x) = \begin{cases} \frac{2\sqrt{1-x^2}}{\pi} & -1 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

Ex 4.1.12 a

Consider a point picked uniformly at random from the area inside the following shape. Find the density, $f(x)$.



(0,4)

$2(x+2)$

$2(-x+2)$

$y = x+2$

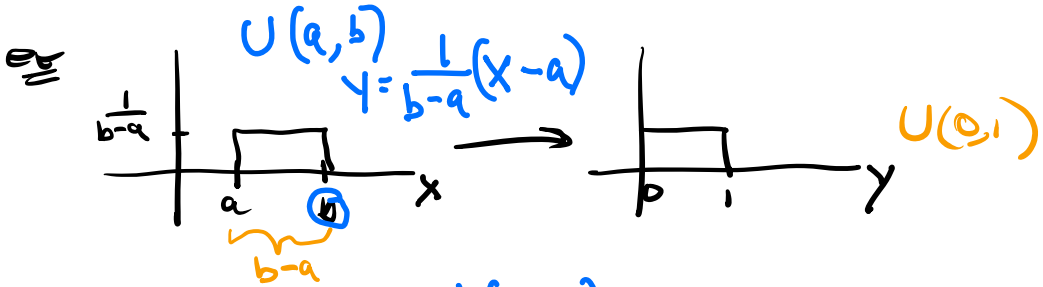
$y = -x+2$

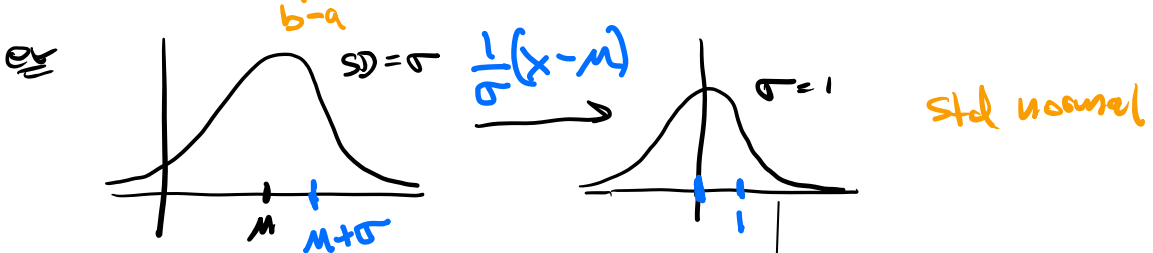
$A = \frac{1}{2} \cdot 4 \cdot 4 = 8$

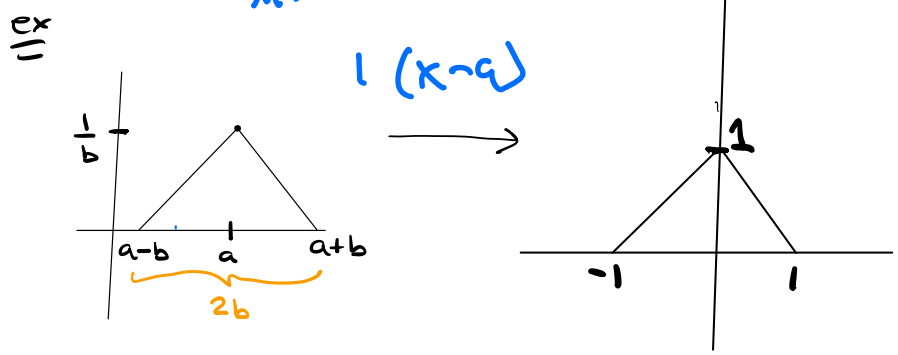
$$f(x) = \begin{cases} \frac{2(x+2)}{8} & -2 \le x \le 0 \\ \frac{2(-x+2)}{8} & 0 \le x \le 2 \\ 0 & \text{else} \end{cases}$$

② Change of scale

A change of scale is a transformation $Y = m + nX$, (constants.) of X. The purpose is that it makes it easier to calculate $E(X)$ and $\text{Var}(X)$. It maps one density to another.

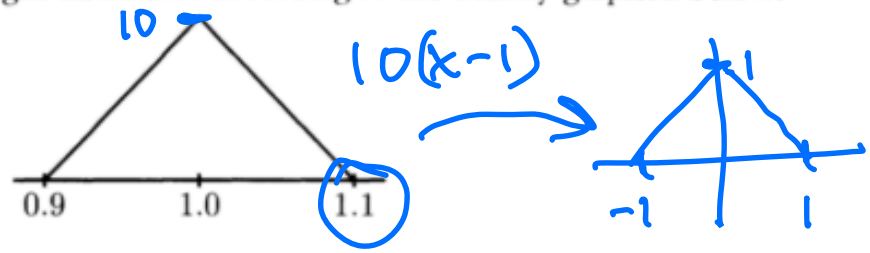






tinyurl.com/oct7-2022

Suppose a manufacturing process designed to produce rods of length 1 inch exactly, in fact produces rods with length distributed according to the density graphed below.



You should change the scale of X= the length of rods to:

- a: X-1
- b: .1(X-1)
- c: 10X-1
- d: none of the above

3

Expectation and Variance

For discrete,

$$E(g(X)) = \sum_{x \in X} g(x) P(X = x)$$

For continuous,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) P(X \in dx) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

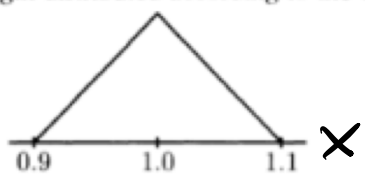
$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$Var(X) = E(X^2) - E(X)^2$$

See Wawmby for example,

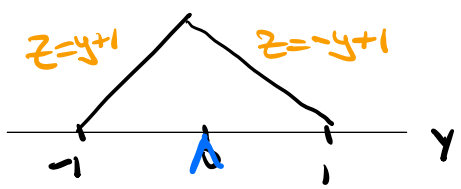
Suppose a manufacturing process designed to produce rods of length 1 inch exactly, in fact produces rods with length distributed according to the density graphed below.



Find the variance of the length of the rods.

$Y = 10(X-1)$ change of scale.

$Var(Y) = 100\,Var(X) \Rightarrow Var(X) = \frac{Var(Y)}{100}$ ← easier to find.



Find the density of Y:

Find Var(X)

$$Var(Y) = E(Y^2) - (E(Y))^2$$

$$E(Y^2) = \int_{-1}^{0} y^2(y+1)\,dy + \int_{0}^{1} y^2(-y+1)\,dy = 1/6$$

$$E(Y) = 0$$

$$Var(Y) = 1/6$$

$$Var[X] = \frac{1/6}{100} = \boxed{1/600}$$