— FRI, JAN 24 —

In a random experiment/process:

* Ω = <u>outcome space</u>
* $\omega \in \Omega$ <u>outcomes</u>
* $A \subset \Omega$ <u>events</u>

<u>EG</u> Flip a coin twice:

4 possible outcomes

$\Omega = \{HH, HT, TH, TT\}$

$A = \{$get at least one T$\} = \{HT, TH, TT\}$.

* A <u>probability function</u> $\mathbb{P} : \Omega \to [0,1]$ assigns to each $\omega \in \Omega$ a number between 0 and 1 such that

AXIOM 1 : $\mathbb{P}(A) \geq 0$ for all $A \subset \Omega$

AXIOM 2 : $\mathbb{P}(\Omega) = 1$

AXIOM 3 : If $A, B \subset \Omega$ are disjoint (i.e. $A \cap B = \emptyset$) then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$$

↑ The Addition Rule

The simplest example is when all outcomes $\omega \in \Omega$ are equally likely.] §1.1

Then

$$\mathbb{P}(\omega) = \frac{1}{\#\Omega} \quad \text{for all } \omega \in \Omega$$

[Assume here $\#\Omega < \infty$.]

$$\Rightarrow \mathbb{P}(A) = \frac{\# A}{\# \Omega} \quad \text{for all } A \subset \Omega.$$

EG In a deck of playing cards:

4 suits: $\underbrace{H, D}_{\text{RED}}, \underbrace{C, S}_{\text{BLACK}}$ (2 colors)

13 ranks: A, 1 – 10, J, Q, K

52 cards

GROUPS Deal 2 cards from the deck.

$\mathbb{P}$(At least one Ace) = ?

ANS

$$\Omega = \{ (\overset{52}{*}, \overset{51}{*}) \}, \qquad \# \Omega = 52 \cdot 51$$

$$E = \{ (\underset{4}{A}, \underset{3}{A}), (\underset{48}{\text{not } A}, \underset{4}{A}), (\underset{4}{A}, \underset{48}{\text{not } A}) \}$$

So $\mathbb{P}(E)$

$$= \frac{4(3 + 2 \cdot 48)}{52 \cdot 51} = 0.149$$

i.e. a 14.9% chance.

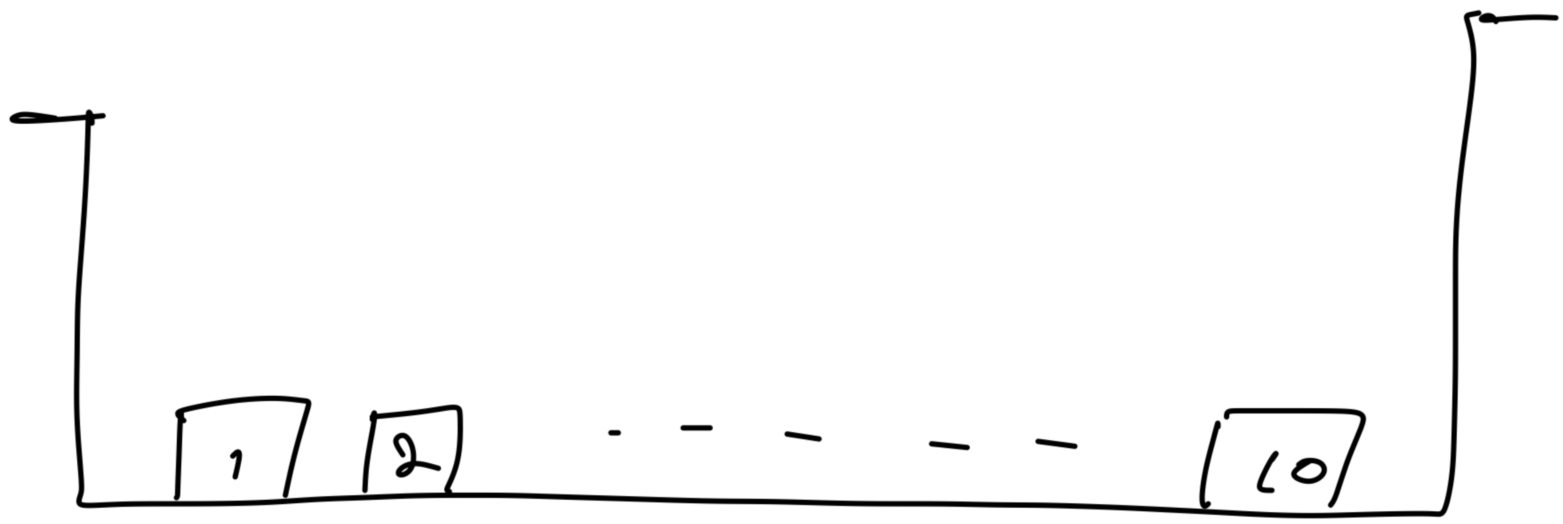
OR Let $E^c = \Omega \setminus E = \{\text{no Ace}\}$

$$\mathbb{P}(E^c) = \frac{48 \cdot 47}{52 \cdot 51}.$$

So $\mathbb{P}(E) = 1 - \mathbb{P}(E^c) = 1 - \dfrac{48 \cdot 47}{52 \cdot 51} = 0.149$

GROUPS

A box has 10 cards, labelled 1–10 :



Select 2 cards with replacement.

P(2nd card is larger than twice the 1st) = ?

__ANS__ $\#\Omega = 10 \cdot 10 = 100$

$E = 2^{nd}$ card $> 2 \cdot 1^{st}$ card

$$= \left\{ \begin{array}{l} (1, >2) \\ (2, >4) \\ (3, >6) \\ (4, >8) \\ (5, >10) \\ \vdots \end{array} \right\} \begin{array}{l} \longleftarrow 8 \text{ ways} \\ \longleftarrow 6 \\ \longleftarrow 4 \\ \longleftarrow 2 \\ \longleftarrow 0 \\ \vdots \end{array}$$

$$\therefore \mathbb{P}(E) = \frac{8+6+4+2}{100} = \frac{20}{100} = 0.2$$

i.e. a 20% chance.

§1-2 Interpretations of probability:
(Frequentist vs Bayesian)

(1) Flip a coin. It comes up H w.p. $p \in [0,1]$ and T w.p. $1-p$. Observe the relative frequency of H's after n flips:

$$\frac{\# \text{ H in trials 1 to } n}{n}$$

This converges (by Law of Large Numbers) to p as $n \to \infty$.

(2) <u>But</u> sometimes this repetition of trials does not apply / make sense:

<u>Ex</u> $\mathbb{P}$(A given patient survives a given operation) = ?

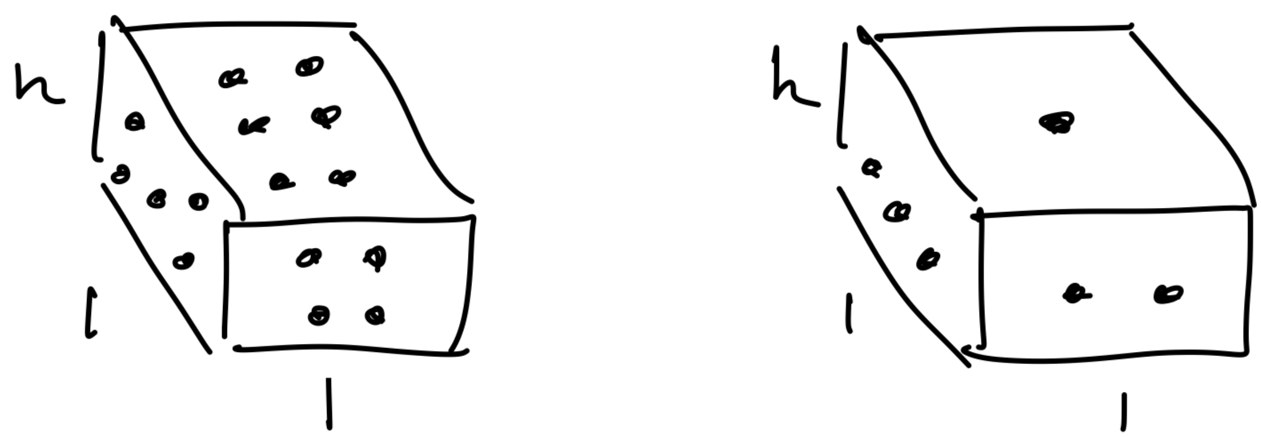
This value may change over time as you gain more information about the patient, etc.

[And whether it actually exists is a philosophical problem: Is something random vs modeling it with randomness?]

§1.3 Probability distributions.

Often we do not have $\mathbb{P}(\omega) \equiv p$ for all ω.

GROUPS Imagine the following die, with an unknown height h:



Suppose $\mathbb{P}(1 \text{ or } 6) = 2/3$.

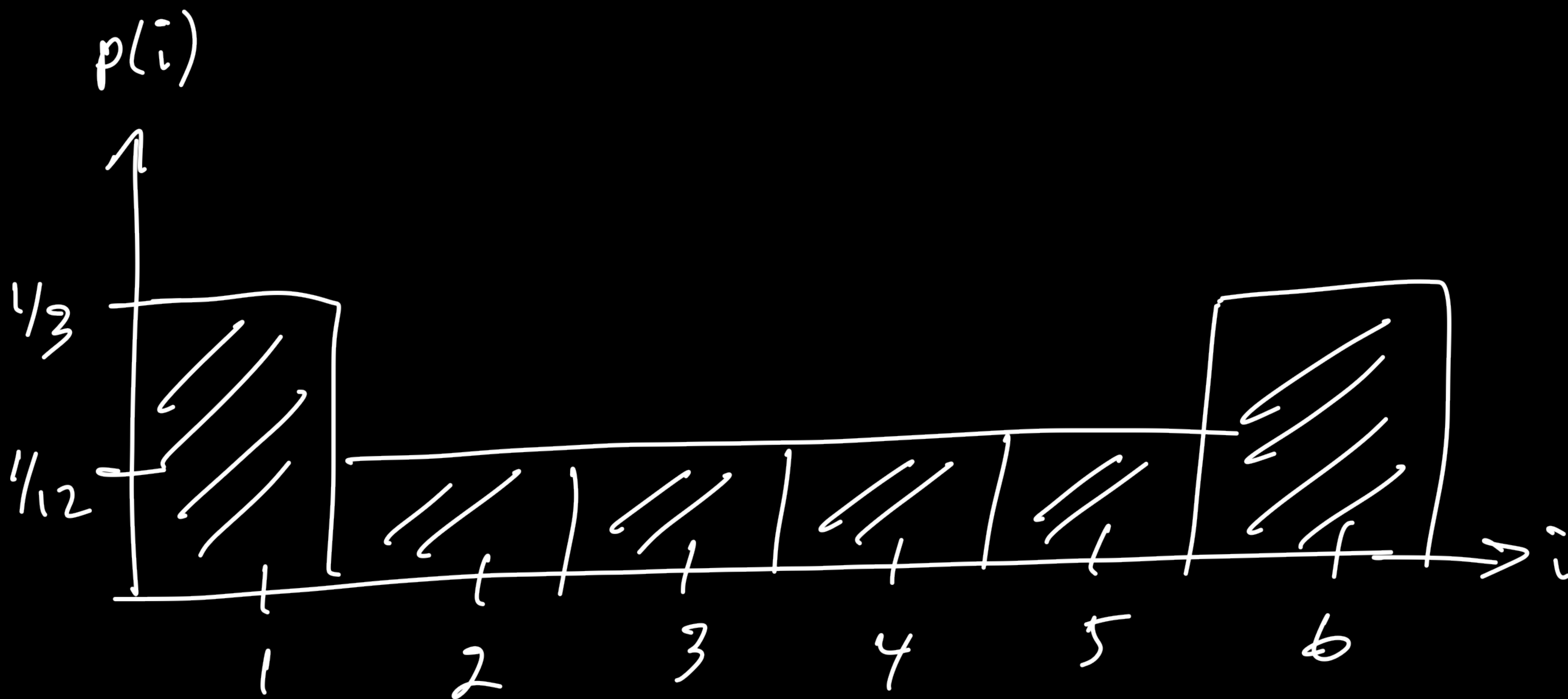
Find the probability distribution (i.e. $\mathbb{P}(\omega)$ for all $\omega \in \Omega = \{1, 2, \dots, 6\}$) and draw a diagram to represent it.

ANS By symmetry, $\mathbb{P}(2) = \cdots = \mathbb{P}(5) = x$.

$$2/3 + 4x = 1 \Rightarrow x = 1/12$$

So $\mathbb{P}(1) = \mathbb{P}(6) = 1/3$

$\mathbb{P}(2) = \cdots = \mathbb{P}(5) = 1/12$



Set Theory review:

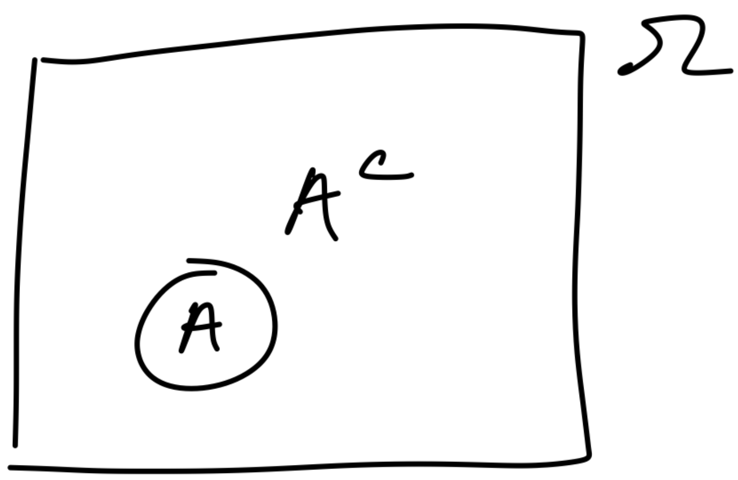
$$A^c = \Omega \setminus A = \{\omega \in \Omega : \omega \notin A\}$$

$$A \cap B = \{\omega \in \Omega : \omega \in A \text{ and } \omega \in B\} \; [= AB]$$

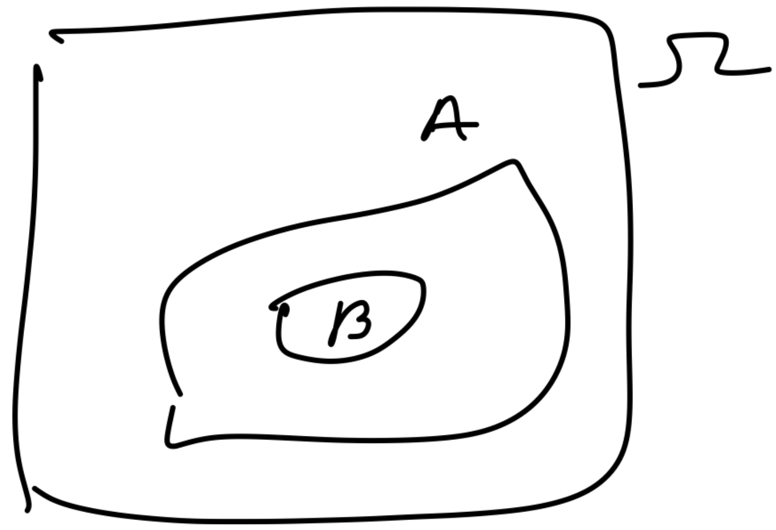
$$A \cup B = \{\omega \in A : \omega \in A \text{ or } \omega \in B\}.$$

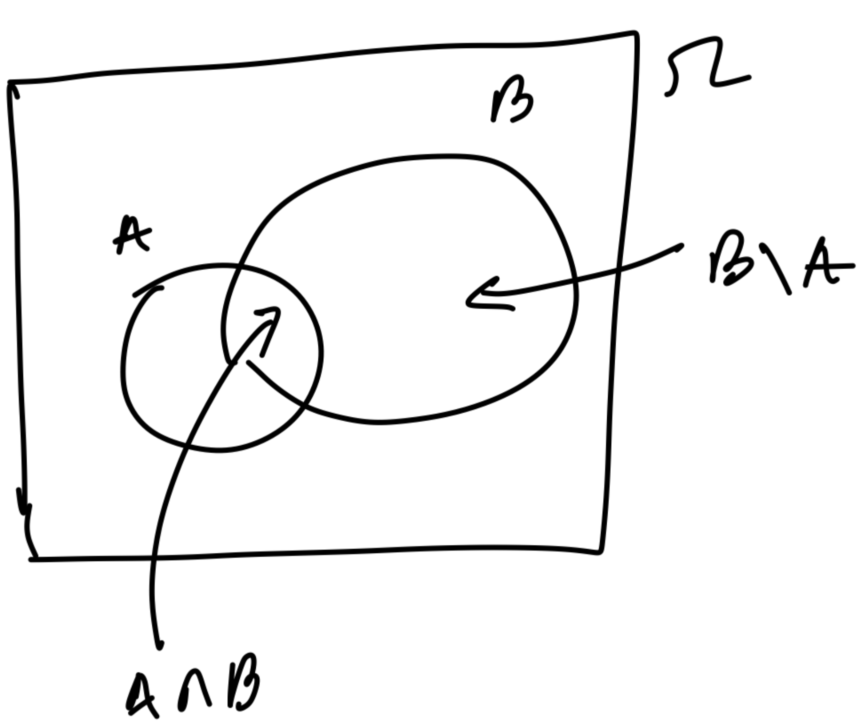
$$A \setminus B = A \cap B^c.$$

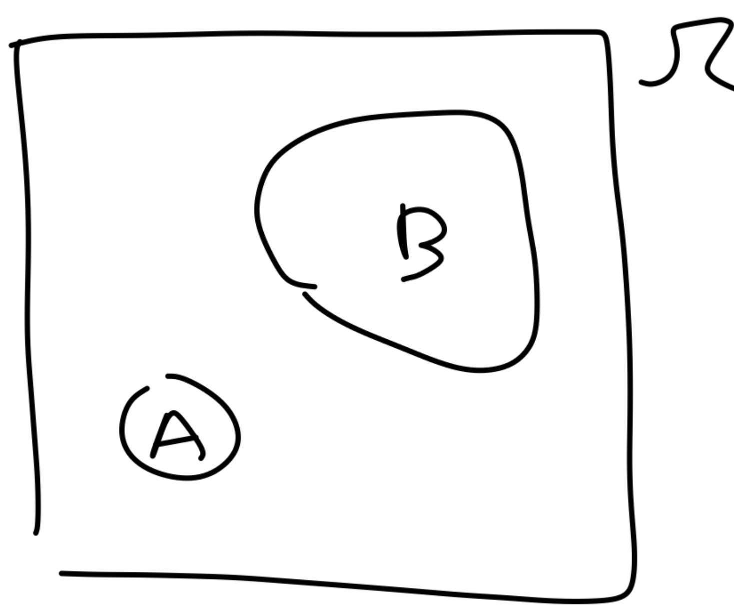
Venn diagrams:



$B \subset A$







$A \cap B = \emptyset$

A, B disjoint

Difference Rule:

Suppose $B \subset A$. Then

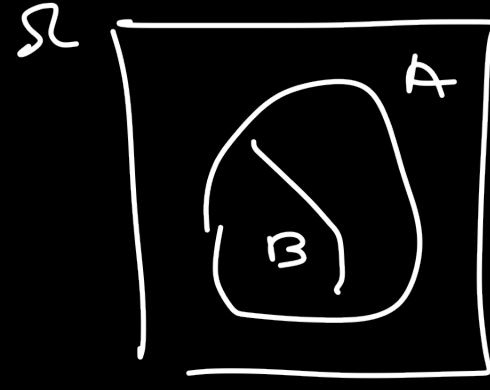
$$\mathbb{P}(A \setminus B) = \mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(B).$$

Special case (complement rule):

$$\mathbb{P}(A^c) = \mathbb{P}(\Omega \setminus A) = 1 - \mathbb{P}(A).$$

GROUPS Prove this by splitting up A into 2 disjoint events, and then using the addition rule.

<u>ANS</u> $A = B \cup (A \setminus B)$



$\therefore \mathbb{P}(A) = \mathbb{P}(B) + \mathbb{P}(A \setminus B)$

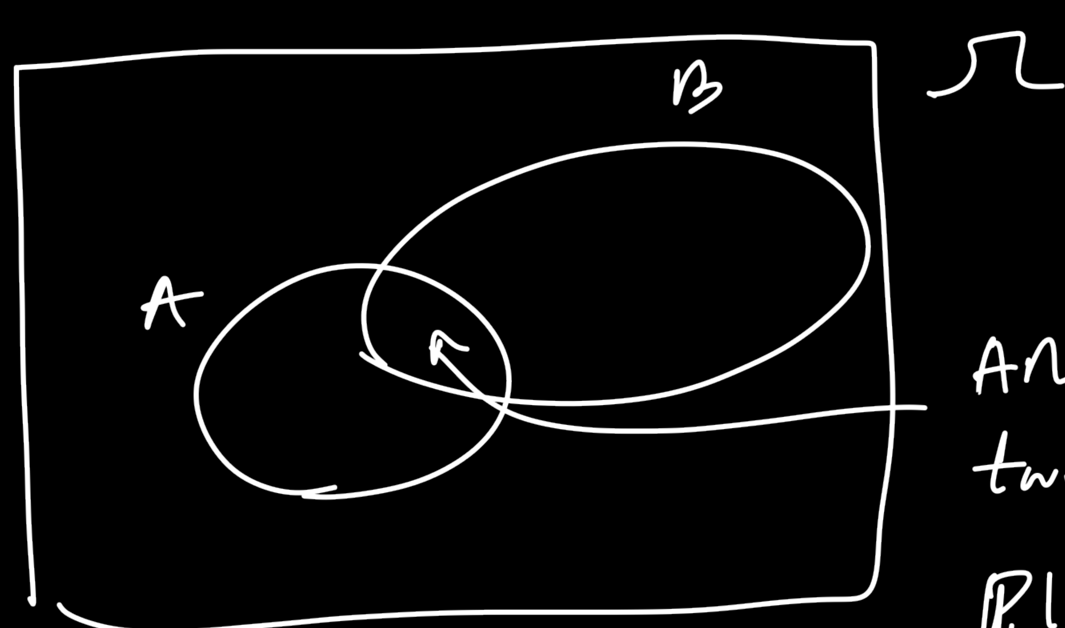
$\Rightarrow \mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(B).$

By a similar argument, can prove the:

<u>Inclusion-Exclusion Rule</u>

For <u>any</u> $A, B \subset \Omega$,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$



$A \cap B$ added twice when $\mathbb{P}(A) + \mathbb{P}(B)$.

A <u>partition</u> of the outcome space Ω is a disjoint collection of events that cover the space. i.e, $A_1, \ldots, A_n$ s.t.

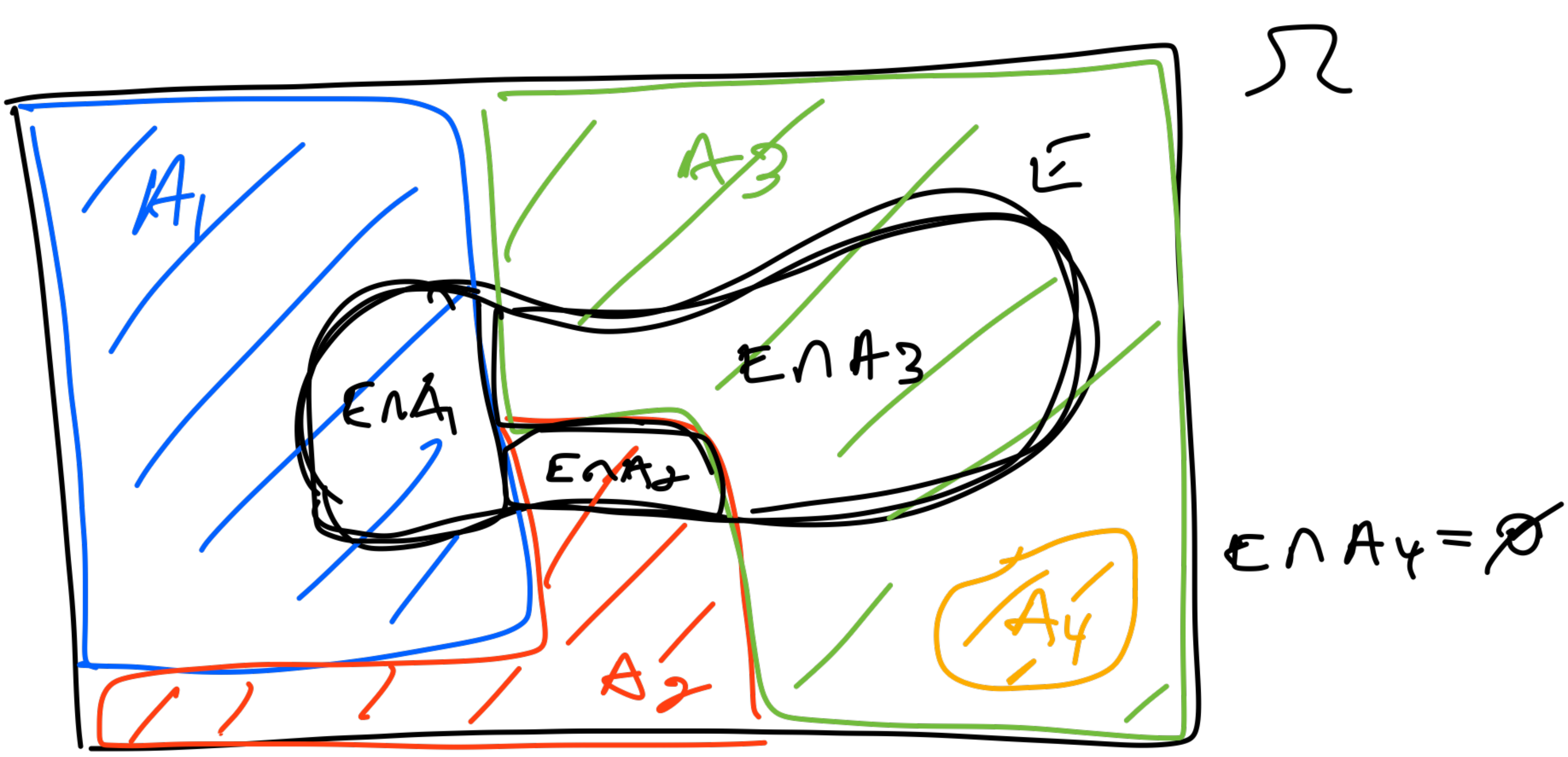
* $A_i \cap A_j = \emptyset$ for all $i \neq j$
* $\bigcup_{i=1}^{n} A_i = \Omega$.

Law of Total Probability

Let $E \subset \Omega$ be an event and $A_1, \ldots, A_n$ be a partition of Ω.

Then

$$\mathbb{P}(E) = \sum_{i=1}^{n} \mathbb{P}(E \cap A_i)$$



Two <u>named distributions</u>:

① Bernoulli (p) distribution on $\Omega = \{0,1\}$:

$$\mathbb{P}(1) = p$$
$$\mathbb{P}(0) = 1-p.$$

<u>EG</u> Let X = result of flipping a fair coin. If we identify $1 \equiv T$ and $0 \equiv H$, then $X \sim \text{Bernoulli}(\frac{1}{2})$.
[<u>Think</u>: X an experiment, $p = \mathbb{P}(\text{success})$.]

② Uniform distribution on points $\Omega = \{x_1, \dots, x_n\}$:

$$\mathbb{P}(x_i) = \frac{1}{n} \text{ for all } i = 1, \dots, n.$$

<u>EG</u> $X \sim \text{Uniform}(\{a, a+1, \dots, b\})$, then

$$\mathbb{P}(X = i) = \frac{1}{b-a+1} \text{ for } i \in \{a, \dots, b\}$$

We will learn many more distributions as the semester progresses. See p 476-477 for a useful list.

GROUPS A deck of cards is shuffled. What is the probability that the top OR bottom card is the King of Spades?

(A) $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \cdot \frac{1}{52}$

(B) $\frac{1}{52} + \frac{1}{51}$

(C) $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \cdot \frac{1}{51}$

(D) None of the above.

ANS The events that king of Spades is on top / bottom are disjoint, and each have prob. $\frac{1}{52}$.

$\therefore$ $\mathbb{P}$(King of Spades on top or bottom)

$$= \frac{1}{52} + \frac{1}{52}.$$

[using $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ when $A \cap B = \emptyset$.]

So (D).

GROUPS Two separate decks are shuffled. What is the probability that King of Spades is the top card on 1^{st} deck **OR** the bottom card of 2^{nd} deck?

(A) $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \cdot \frac{1}{52}$

(B) $\frac{1}{52} + \frac{1}{51}$

(C) $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \cdot \frac{1}{51}$

(D) None of the above.

ANS

$\mathbb{P}$(KS on top of 1st OR KS on bottom of 2nd)

$= \mathbb{P}$(KS on top of 1st) + $\mathbb{P}$(KS on bottom of 2nd)

$-\mathbb{P}$(KS on top of 1st AND KS on bottom of 2nd)

$$= \frac{1}{52} + \frac{1}{52} - \frac{1}{52} \cdot \frac{1}{52}.$$

[Using $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$]

So (A).