Stat 134 lec 33

Warmup 8:00 – 8:10 AM

uniform spacing!

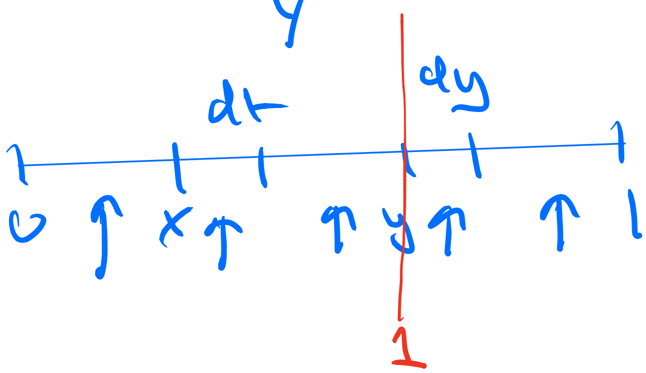
Throw down 5 darts on $(0,1)$.

$X = U_{(2)}$ $Y = U_{(4)}$

Find $P(Y > 4X)$

Note $\frac{X}{Y}$ is a beta

(Compare with solution in lecture 28)



$$\frac{U_{(2)} \text{ out of } 5}{U_{(4)} \text{ out of } 5} = U_{(2)} \text{ out of } 3 \sim \text{Beta}(2, 2)$$

k, $n-k+1$ (3, 2)

$P(Y > 4X) = P\left(\frac{X}{Y} < \frac{1}{4}\right)$

$Z \sim \text{Beta}(2,2)$ $\quad f_Z(z) = \frac{\Gamma(4)}{\Gamma(2)\Gamma(2)} z(1-z) = 6z(1-z)$

$P(Z < 1/4) = 6\int_0^{1/4} z\,dz - 6\int_0^{1/4} z^2\,dz = 10/64$

Last time:

Sec 5.4 Convolution formula for density of Y/X

$Z = Y/X$, $X>0$, $Y>0$

$$f(z) = \int_{x=0}^{x=\infty} f(x, zx)\, x\, dx$$

ex More uniform spacing (see #13 p 355)

$U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$

$$\frac{U_{(k)} \text{ in } n}{U_{(k+\ell)} \text{ in } n} \overset{D}{=} U_{(k)} \text{ in } k+\ell-1 \sim \text{Beta}(k, \underbrace{\ell}_{k+\ell-1-k+1})$$

Today

① Sec 5.4 Review student explanation from current text.

② Sec 6.1 Conditional Distribution: Discrete case.

③ Sec 6.2

ⓐ Conditional expectation: $E(T \mid X = x)$

ⓑ Rule of average conditional expectations,

$$E(T) = E(E(T \mid X))$$

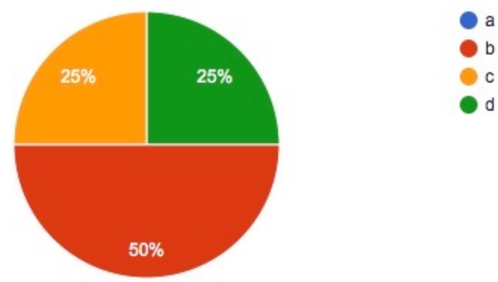
Let $Y \sim U_{(1)}$ and $X \sim U_{(2)}$ for 10 iid U(0,1). The variable part of the joint density is $(1-x)^8$. The density of $Z = \frac{Y}{X}$ is:

a $1/(2z)$

b 1

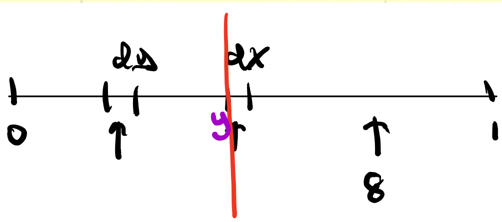
c $1/(2z^2)$

d none of the above



Discuss with your neighbor for 1 minute

b	Since all the hits occur before x, this is a Beta(1,1) so it is a uniform distribution



$$Z = \frac{U_{(1)}}{y = U_{(2)}} \sim U_{(1)} \text{ out of } 1 \sim \text{Beta}(1,1) \text{ which has density } 1.$$

② Sec 6.1 Conditional Distribution: Discrete case.

let X, N discrete RVs w/ joint distribution $P(X=x, N=n)$.

Bayes rule

$$P(X=x \mid N=n) = \frac{P(X=x, N=n)}{P(N=n)}$$

$$\Rightarrow \quad P(X=x, N=n) = P(X=x \mid N=n) P(N=n)$$

Rule of average conditional probabilities

marginal prob of X

$$P(X=x) = \sum_n P(X=x, N=n)$$

$$= \sum_n P(X=x \mid N=n) P(N=n)$$

ex

cars in 1 min

$X \mid N=n$ is # red cars out of n

Let N have Poisson (λ) distribution. Let X be a random variable with the following property: for every n, the conditional distribution of X given $(N=n)$ is binomial (n,p). Find the unconditional distribution of X and state its parameter(s). Show all your work for full credit.

X = # red cars in 1 min.

$$P(N=n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

$$P(X=x \mid N=n) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

Find $P(X=x)$

$$P(X=x) = \sum_{n=x}^{\infty} P(X=x \mid N=n) P(N=n)$$

$$= \sum_{n=x}^{\infty} \frac{\cancel{n!}}{x!(n-x)!} p^x q^{n-x} \frac{e^{-\lambda}\lambda^n}{\cancel{n!}} \qquad \lambda^x \cdot \lambda^{n-x}$$

$$= \frac{e^{-\lambda}\lambda^x p^x}{x!} \sum_{n=x}^{\infty} \frac{\lambda^{n-x} q^{n-x}}{(n-x)!}$$

Finish

$$= \frac{e^{-\lambda} \lambda^x p^x}{x!} \underbrace{\left(1 + \lambda q + \frac{(\lambda q)^2}{2!} + \cdots\right)}_{= e^{\lambda q}}$$

$$= \frac{e^{-\lambda \overbrace{(1-q)}^{p}} (\lambda p)^x}{x!} \Rightarrow X \sim \text{Poi}(\lambda p)$$

Sec 6.2 Conditional Expectation (discrete case)

Bayes rule:

recall $P(T=t \mid S=s) = \dfrac{P(T=t, S=s)}{P(S=s)}$

ex (T,S) is joint distribution below,

Find $P(T=3 \mid S=7)$

$$= \frac{P(T=3, S=7)}{P(S=7)} = \frac{.3}{.4} = .75$$

	T=3	T=4	Sum
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1.0

← marginal of S

marginal of T

Find $P(T=4 \mid S=7)$

$$\frac{P(T=4, S=7)}{P(S=7)} = \frac{.1}{.4} = .25$$

Find $E(T) = \sum_{t \in T} t P(T=t) = 3(.6) + 4(.4) = 3.4$

Find $E(T \mid S=7)$

$$\sum_{t \in T} t P(T=t \mid S=7) = 3P(T=3 \mid S=7) + 4P(T=4 \mid S=7)$$

$= 3(.75) + 4(.25) = 3.25$

Find $E(T \mid S=6)$

$3P(T=3 \mid S=6) + 4 \cdot P(T=4 \mid S=6)$

$3\left(\frac{.2}{.4}\right) + 4\left(\frac{.2}{.4}\right) = 3.5$

	T=3	T=4	Sum ← marginal of S
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1.0

marginal of T

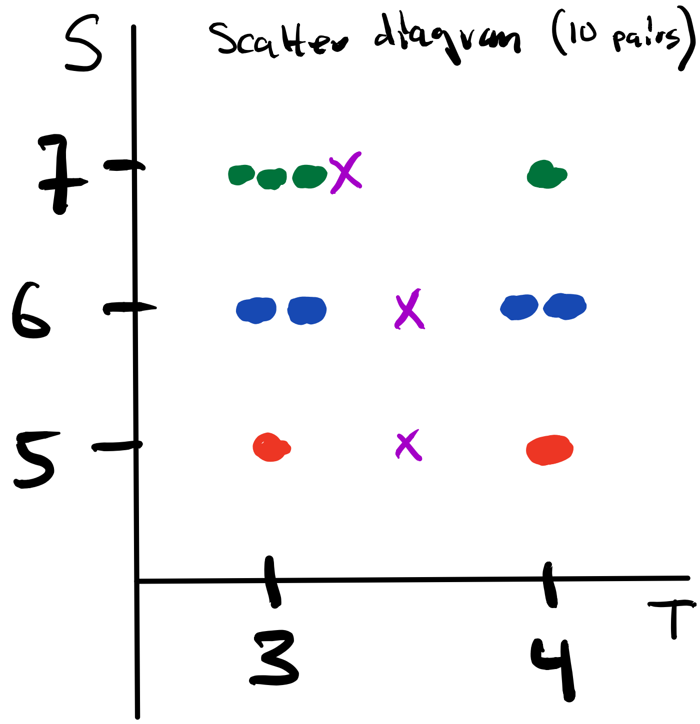
$$\left.\begin{aligned} E(T \mid S=7) &= 3.25 \\ E(T \mid S=6) &= 3.5 \\ E(T \mid S=5) &= 3.5 \end{aligned}\right\}\ \text{function of } S$$

Picture

joint distribution

	T=3	T=4	Sum
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1

$\longleftrightarrow$



Two main points:

1. E(T|S) is a function of S.
2. E(T|S) is a RV so it has an expectation.

Next we explore the expectation of E(T|S).

Let $g(S) = E(T|S)$

$g(7) = E(T|S=7) = 3.25$

$g(6) = 3.5$

$g(5) = 3.5$

$$E(g(S)) = \sum_{s \in S} g(s) P(S=s)$$

$$= 3.25(.4) + 3.5(.4) + 3.5(.2)$$

$$= 3.4$$

← this is $E(T)$.

it is the weighted average of all of the group averages.

In other words,

$$E(E(T|S)) = E(T)$$

This is called the property of iterated expectations.

Intuitively,

If you have a class that is $2/3$ girls and $1/3$ boys and the girls weigh on average 100 lbs and boys weigh 200 lbs then the average weight of the class should be $\frac{2}{3}(100) + \frac{1}{3}(200)$, i.e. we take the weighted average of the averages.

Rule of average conditional expectations

For any random variable T with finite expectation and any discrete RV S,

$$E(T) = \sum_{\text{all } s} E(T \mid S=s) \cdot P(S=s)$$

(see end of this lecture for a formal proof)

ex

8 transistors (type 1) are distributed $\text{Exp}\left(\frac{1}{100}\right)$ and 4 transistors (type 2) are $\text{Exp}\left(\frac{1}{200}\right)$.

Let T be the lifetime of a randomly picked transistor.

Find $E(T)$.

soln

Let X = type of transistor

$$E(T) = E(E(T|X))$$

$$= E(T|X=1)\cdot P(X=1) + E(T|X=2)P(X=2)$$

$$= 100 \cdot \frac{8}{12} + 200 \cdot \frac{4}{12} = 133.3$$

ex

(2 pts) Let $X \sim \text{Geom}(p)$ on $\{1, 2, \ldots\}$. Suppose Y is a variable such that given $X = x$, Y is uniformly distributed on $\{1, 2, \ldots, x\}$. Find $\mathbb{E}(Y)$. (Hint: for $U \sim \text{Unif}\ \{a, a+1, \ldots, b\}$, $\mathbb{E}(U) = \frac{a+b}{2}$.)

$$E(Y) = E(E(Y|X))$$

$$Y|X = \text{Unif}(1, 2, \ldots, X)$$

$$E(Y|X) = \frac{1+X}{2}$$

$$E(Y) = E(E(Y|X)) = E\left(\frac{1+X}{2}\right) = E\left(\frac{1}{2}\right) + E\left(\frac{X}{2}\right)$$

$$= \boxed{\frac{1}{2} + \frac{1}{2p}}$$

Appendix

Iterated Expectation

We show $E(Y) = E(E(Y|X))$:

$$E(Y) = \sum_{\text{all } y} y P(Y=y)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} P(X=x, Y=y)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} \frac{P(X=x, Y=y)}{P(X=x)} P(X=x)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} P(Y=y \mid X=x) P(X=x)$$

$$= \sum_{\text{all } x} \left(\sum_{\text{all } y} y P(Y=y \mid X=x) \right) \cdot P(X=x)$$

$$= \sum_{\text{all } x} E(Y \mid X=x) \cdot P(X=x)$$

$$= E(E(Y|X))$$

$\square$