Stat 134 lec 7

Warmup: 11:00-11:10

Suppose you and I each have a box of 600 marbles. In my box, 4 of the marbles are black, while 3 of your marbles are black. We each draw 300 marbles **with replacement** from our own boxes. **Approximately**, what is the chance you and I draw the same number of black marbles?

Def^n Poisson (μ)

$$P(k) = \frac{e^{-\mu}\mu^k}{k!} \text{ for } k=0,1,2,\dots$$

Hint: Using a Poisson Approx to the Binomial what is the chance you get k blacks?

$$\frac{e^{-1.5}(1.5)^k}{k!}$$

$$\left(\text{exactly } \binom{300}{k}\left(\frac{3}{600}\right)^k\left(\frac{597}{600}\right)^{300-k}\right)$$

X = # black marbles I draw — $p_x = \frac{4}{600}$

Y = # black marbles you draw — $p_y = 3/600$

$\mu_x = 300 \cdot \frac{4}{600} = 2$

$\mu_y = 300 \frac{3}{600} = 1.5$

$$P(X=Y) = \sum_{k=0}^{300} P(X=k, Y=k) \quad \text{by addition rule}$$

$$= \sum_{k=0}^{300} P(X=k)P(Y=k) \quad \text{by independence}$$

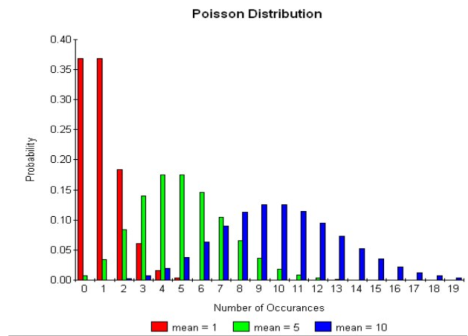
$$\approx \boxed{\sum_{k=0}^{300} \frac{e^{-2}\,2^k}{k!} \cdot \frac{e^{-1.5}\,1.5^k}{k!}}$$

Announcement: Q1 next Wednesday in section
covers 1.1 – 1.6 and 2.1

Last time

sec 2.4 **Poisson Distribution**

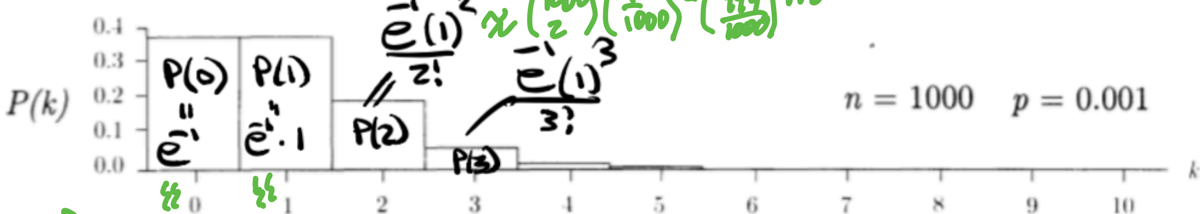
$$P(k) = \frac{e^{-\mu}\mu^k}{k!}, \quad k = 0, 1, 2, \ldots$$



We saw that Pois(μ) is a limit of binomials for $n \to \infty$, $p \to 0$ and $np \to \mu$.

The binomial (1000, 1/1000) distribution.

Now take 1000 random draws with replacement from a box with 1 black ball and 999 white ones. This is the distribution of the number of black balls drawn:



$P(0) = e^{-1} \approx \left(\frac{1}{1000}\right)^0 \left(\frac{999}{1000}\right)^{1000}$

$P(1) = e^{-1}\cdot 1 \approx 1000 \cdot \left(\frac{1}{1000}\right)\left(\frac{999}{1000}\right)^{999}$

$P(2) = \frac{e^{-1}(1)^2}{2!} \approx \binom{1000}{2}\left(\frac{1}{1000}\right)^2\left(\frac{999}{1000}\right)^{998}$

$P(3) = \frac{e^{-1}(1)^3}{3!}$

mode of Bin(n,p):

$m = \lfloor np + p \rfloor$

$$\text{mode} = \begin{cases} m & \text{if } np+p \notin \mathbb{Z} \\ m-1, m & \text{if } np+p \in \mathbb{Z} \end{cases}$$

mode of Pois(μ):

$m = \lfloor \mu \rfloor$ since $np + p \to \mu + p \approx \mu$

$$\text{mode} = \begin{cases} m & \text{if } \mu \notin \mathbb{Z} \\ m-1, m & \text{if } \mu \in \mathbb{Z} \end{cases}$$

Today

① **sec 2.5** Random sampling

independent trials (draw w/ replacement)
- binomial distribution — 2 outcome trial
- multinomial distribution — k outcome trial

dependent trials (draw w/o replacement)
- hypergeometric distribution — 2 outcome trial
- multivariate hypergeometric distribution — k outcome trial

(1) Sec 2.5

Random sampling with replacement

ex Class 100 students

grade distribution:

A 50 students

B 30 students

C 15 students

D 5 students

You sample 10 students with replacement.

a) What is the chance you get

AAAABBBCCD ?

$(.5)^4(.3)^3(.15)^2(.05)^1$

AAABBBCCDA

$(.5)^3(.3)^3(.15)^2(.05)^1(.5)^1$

$\shortparallel$

$(.5)^4(.3)^3(.15)^2(.05)^1$

b) Find $P(4A's, 3B's, 2C's, 1D)$

$$\frac{10!}{4!\,3!\,2!\,1!}(.5)^4(.3)^3(.15)^2(.05)^1$$

$\leftrightarrow$ AAAABBBCCD

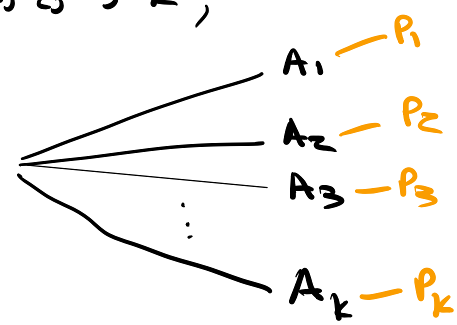
$\shortparallel$

$$\binom{10}{4,3,2,1} = \binom{10}{4}\cdot\binom{6}{3}\binom{3}{2}\cdot\binom{1}{1}$$

Written $\text{Multi}(n, p_1, \dots, p_k)$

<u>Def</u> <u>Multinomial Distribution</u>

If you have n independent trials, where each trial has k possible outcomes, $A_1, A_2, \dots, A_k$ with probabilities $p_1, p_2, \dots, p_k$,



then the probability you get n_1 outcome A_1, n_2 outcome A_2, …, n_k outcome A_k is

$$P(n_1, n_2, \dots, n_k) = \binom{n}{n_1, n_2, \dots, n_k} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$$

$$\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

<u>Note</u> Binomial distribution is a special case with $k = 2$.

independent trials (draw w/ replacement)
- binomial distribution — 2 outcome trial
- multinomial distribution — k outcome trial

random sample without replacement

ex In a very student friendly class with 100 students the grade distribution is:

A 70 students
B 30 students

You sample 5 students at random without replacement (called a simple random sample (SRS))

a) Find the chance you get

A A A B B

$$\frac{70}{100}\cdot\frac{69}{99}\cdot\frac{68}{98}\cdot\frac{30}{97}\cdot\frac{29}{96}$$

A A B B A

$$\frac{70}{100}\cdot\frac{69}{99}\cdot\frac{30}{98}\cdot\frac{29}{97}\cdot\frac{68}{96}$$

$$= \frac{70}{100}\cdot\frac{69}{99}\cdot\frac{68}{98}\cdot\frac{30}{97}\cdot\frac{29}{96}$$

b) Find P(3A's, 2B's).

$$\frac{5!}{3!\,2!}\,\frac{70}{100}\cdot\frac{69}{99}\cdot\frac{68}{98}\cdot\frac{30}{97}\cdot\frac{29}{96}$$

$$\frac{5!}{3!\,2!} = \binom{5}{3,2}$$

(if drawn w/ replacement $\binom{5}{3,2}(.7)^3(.3)^2$)

$$\frac{5!}{3!\,2!}\,\frac{70}{100}\cdot\frac{69}{99}\cdot\frac{68}{98}\cdot\frac{30}{97}\cdot\frac{29}{96} = \frac{\frac{70\cdot 69\cdot 68}{3!}\cdot\frac{30\cdot 29}{2!}}{\frac{100\cdot 99\cdot 98\cdot 97\cdot 96}{5!}}$$

$$= \frac{\binom{70}{3}\binom{30}{2}}{\binom{100}{5}}$$

A B

hypergeometric formula

written

Def^n hypergeometric distribution $HG(n, N, G)$

Suppose a population of size N contains G good and B bad elements ($N = G + B$).

A sample, size n, with g good and b bad elements ($n = g + b$) is chosen at random without replacement.

$$P(g \text{ good and } b \text{ bad}) = \frac{\binom{G}{g}\binom{B}{b}}{\binom{N}{n}}$$

this generalizes to the multivariate hypergeometric distribution

Now instead of 2 types of elements we have k with sizes $G_1, G_2, \ldots, G_k$ ($N = G_1 + \cdots + G_k$) and in our sample we have $n = g_1 + \cdots + g_k$.

$$P(g_1, g_2, \ldots, g_k) = \frac{\binom{G_1}{g_1}\binom{G_2}{g_2}\cdots\binom{G_k}{g_k}}{\binom{N}{n}}$$

ex Class 100 students

grade distribution:

A	50 students
B	30 students
C	15 students
D	5 students

You sample 10 students without replacement (SRS)

Find $P(4 \text{ A's}, 3 \text{ B's}, 2 \text{ C's}, 1 \text{ D}) = \dfrac{\binom{50}{4}\binom{30}{3}\binom{15}{2}\binom{5}{1}}{\binom{100}{10}}$

ex: A 5 card poker hand consists of a SRS of 5 cards from a 52 card deck. There are $\binom{52}{5}$ poker hands.

a) Find P(poker hand has 4 aces and a king)

$$\frac{\binom{4}{4}\binom{4}{1}\binom{44}{0}}{\binom{52}{5}} \quad \binom{44}{0}=1$$

b) Find P(poker hand has 4 aces).

$$\frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} \quad \text{or} \quad \frac{\binom{12}{1}\binom{4}{4}\binom{4}{1}}{\binom{52}{5}}$$

choosing your non ace.

c) Find P(poker hand has 4 of a kind)

ranks (aaaab)
a ≠ b

$$\frac{\binom{13}{1}\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} \quad \text{or} \quad \frac{\binom{13}{1}\binom{12}{1}\binom{4}{4}\binom{4}{1}}{\binom{52}{5}}$$

Stat 134

Chapter 2

1. The probability of being dealt a three of a kind poker hand (ranks $aaabc$ where $a \neq b \neq c$) is:

a $\binom{4}{3}\binom{4}{1}\binom{44}{1}/\binom{52}{5}$

b $\binom{13}{1}\binom{12}{2}\binom{4}{3}\binom{4}{1}\binom{4}{1}/\binom{52}{5}$

c $\binom{13}{1}\binom{12}{1}\binom{4}{3}\binom{4}{1}\binom{44}{1}/\binom{52}{5}$

d none of the above

$\binom{44}{1} = \binom{11}{1}\binom{4}{1}$

aaabc = aaacb

$\binom{12}{2} = \frac{12\cdot 11}{2} = \frac{\binom{12}{1}\binom{11}{1}}{2}$

Choose a rank for 3 of a kind, then choose 2 ranks for the singles.
We have $\binom{12}{2} = \frac{12\cdot 11}{2}$ instead of $12\cdot 11$ since the order we get the singles doesn't matter. (i.e. aaabc = aaacb in our poker hand).

Test your understanding:
Find the prob of getting aaabbc in a 6 card poker hand.

Answer $\binom{13}{1}\binom{12}{1}\binom{11}{1}\binom{4}{3}\binom{4}{2}\binom{4}{1}/\binom{52}{6}$