Stat 134 lec 32

Warmup 8:00 – 8:10 AM

Uniform spacing:

You randomly throw n dart at $[0,1]$.
What distribution is $U_{(a+k)} - U_{(a)}$ where $0 \le a < a+k \le n$.

last time we saw

$U_{(9)} - U_{(7)}$ out of 10 is $U_{(2)}$ out of 10

which is $\text{Beta}(\underset{k}{2}, \underset{n-k+1}{9})$

So $U_{(a+k)} - U_{(a)} = U_{(k)}$ out of n

which is $\boxed{\text{Beta}(k, n+k-1)}$

Last time

sec 5.4 Convolution formula for density of sum

Let X, Y be RVs and $S = X + Y$.

if $X>0, Y>0$

if X, Y independent

$$f_S(s) = \int_{x=0}^{x=s} f(x, s-x)\,dx = \int_{x=0}^{x=s} f_X(x) f_Y(s-x)\,dx$$

ex (Uniform spacing) (see #13 p 355)

Today

1. Go over student comments from concept test last time.

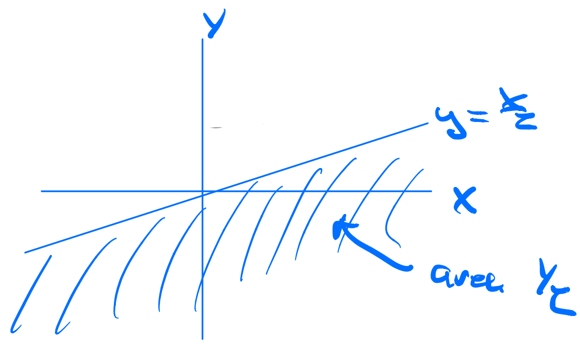
sec 5.4

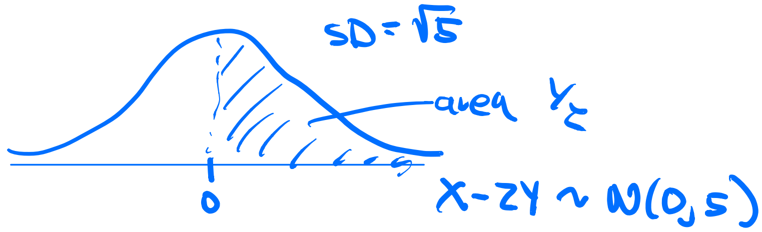
2. triangular density

3. Convolution formula of ratio Y/X

① similar to concept test from last time.

Let $X \sim N(0,1)$ and $Y \sim N(0,1)$ be independent. $P(X > 2Y)$ equals ?





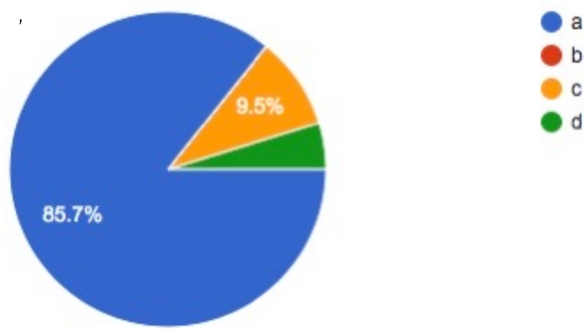
Let $X \sim N(68, 3^2)$ and $Y \sim N(66, 2^2)$ be independent. $P(X > Y)$ equals

a $1 - \Phi(\frac{0-2}{\sqrt{3^2+2^2}})$

b $1 - \Phi(\frac{0-2}{3^2+2^2})$

c $1 - \Phi(\frac{68-66}{\sqrt{3^2+2^2}})$

d none of the above

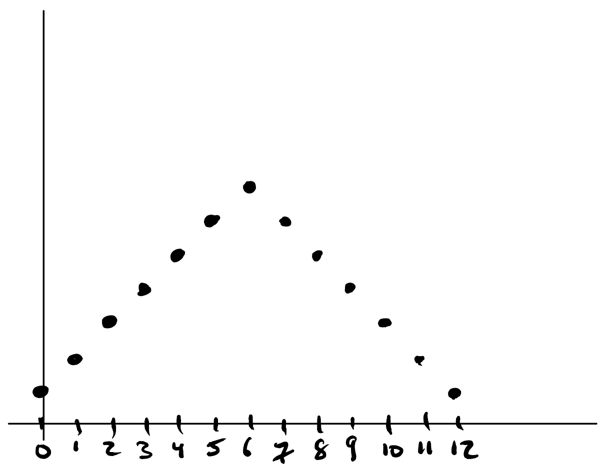


a	This is P(X-Y>0). X-Y~N(68-66, 3^2 +2^2). When you standardize this, you get 0-2 over std which is the sqrt of the variance. To get the area greater than this z-score you subtract the cdf to that point from 1.

② Triangular density

Let $X \sim \text{Unif}\{0,1,2,\ldots,6\}$
$Y \sim \text{Unif}\{0,1,2,\ldots,6\}$ } indep.

The distribution of $S = X + Y$ looks like



True ?

False ?

$$P(S=6) = \underset{\underset{P(0)P(6)}{\shortparallel}}{P(0,6)} + P(1,5) + \cdots + P(6,0) = \frac{7}{49} = 1/7$$

$$P(S=s) = \sum_{x=0}^{x=s} P(x)P(s-x) \quad , \quad 0 \le s \le 6$$

$$P(S=7) = P(1,6) + P(2,5) + \cdots + P(6,1) = 6/49$$

$$P(S=s) = \sum_{x=s-6}^{x=6} P(x)P(s-x) \qquad 6 \le s \le 12$$

Continuous case:

$$\left.\begin{array}{l} X \sim U(0,1) \\ Y \sim U(0,1) \end{array}\right\} \text{ indep}$$

Find density of $S = X + Y$

for $0 < s < 1$

$$f_S(s) = \int_{x=0}^{x=s} f_X(x) f_Y(s-x)\,dx = \int_0^s 1 \cdot 1\,dx = s$$

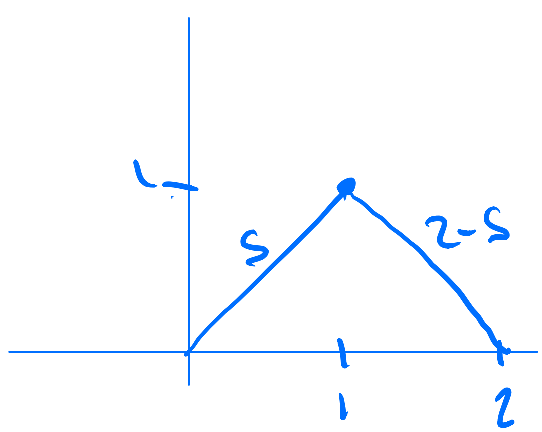
for $1 < s < 2$

$$f_S(s) = ?$$

$$f_S(s) = \int_{x=s-1}^{x=1} f_X(x)\, f_Y(s-x) = \int_{s-1}^{1} 1 \cdot 1\,dx$$

$$= 1 - (s-1)$$

$$= 2 - s$$



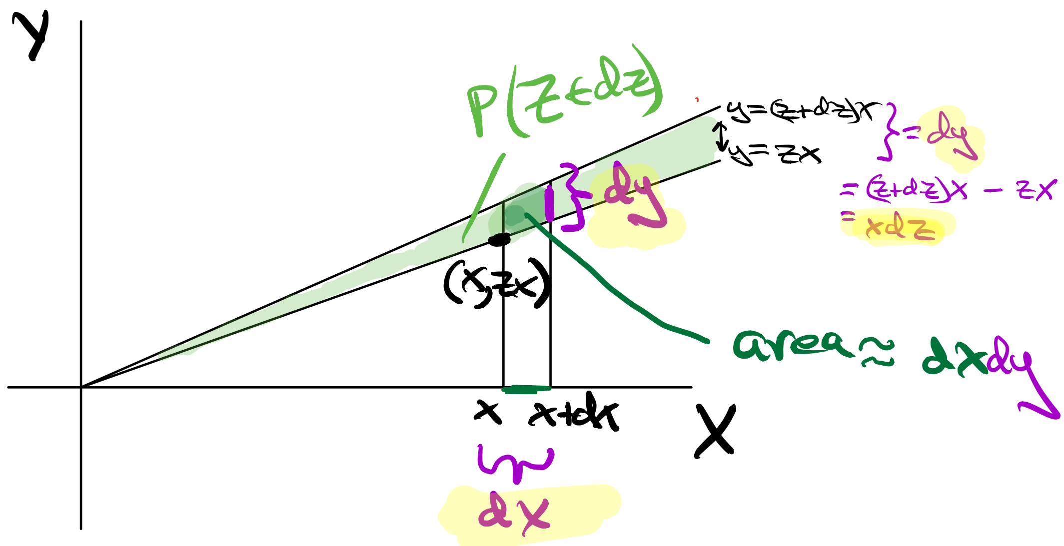
③ Convolution formula for density of ratio Y/X

$X > 0,\ Y > 0$

let $Z = \dfrac{Y}{X}$.

Find $f_Z(z)$.

Picture $\quad Y = ZX$ — slope



$$P(Z\in dz) = \int_{x=0}^{x=\infty} P(Z\in dz, X\in dx)$$

$\parallel$

$f_Z(z)\,dz$

$$= \int_{x=0}^{x=\infty} f(x,zx)\,dy\,dx \qquad dy = x\,dz$$

Convolution formula.

$$\Rightarrow f_Z(z) = \int_{x=0}^{x=\infty} f(x,zx)\,x\,dx \overset{\text{if } X,Y \text{ indep.}}{=} \int_{x=0}^{x=\infty} f_X(x)\,f_Y(zx)\,x\,dx$$

ex

Recall gamma density

$X \sim \text{Gamma}(r, \lambda)$

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$$

variable part

so

$$\boxed{\int_0^\infty x^{r-1} e^{-\lambda x} = \frac{\Gamma(r)}{\lambda^r}}$$

Let $X, Y \overset{iid}{\sim} \text{Exp}(1)$. $Z = \frac{Y}{X}$

Find $f_Z(z)$.

soln

$$f_Z(z) = \int_{x=0}^\infty \overset{e^{-x}}{f_X(x)} \overset{e^{-zx}}{f_Y(zx)}\, x\, dx$$

$$= \int_{x=0}^\infty e^{-x} e^{-zx} x\, dx$$

$$= \int_{x=0}^\infty x\, e^{-(1+z)x} dx$$

gamma $(r=2, \lambda = 1+z)$

$$= \frac{\Gamma(2)}{(1+z)^2} = \boxed{\frac{1}{(1+z)^2} \text{ for } 0 < z < \infty.}$$

tinyurl http://tinyurl.com/nov18-pt1

Stat 134

Monday April 15 2019

1. Let $Y \sim U_{(1)}$ and $X \sim U_{(2)}$ for 10 iid U(0,1). The variable part of the joint density is $(1-x)^8$. The density of $Z = \frac{Y}{X}$ is:

   a $1/(2z)$

   b 1

   c $1/(2z^2)$

   d none of the above

$f(x,y) = c(1-x)^8$

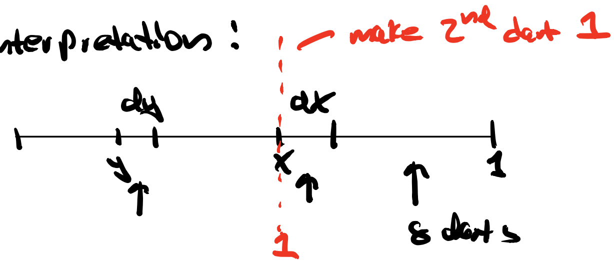
$f_Z(z) = \int_0^1 f(x, zx)\, x\, dx$

$f_Z(z) = c\int_0^1 (1-x)^8 x\, dx$ ← not a function of z.

$\Rightarrow f_Z(z)$ is constant on [0,1] with area 1 so it must take value 1.

i.e $Z \sim U(0,1)$

Interpretation:



$$\frac{U_{(1)} \text{ out of } 10}{U_{(2)} \text{ out of } 10} = U_{(1)} \text{ out of } 1$$

density 1 on $[0,1]$.

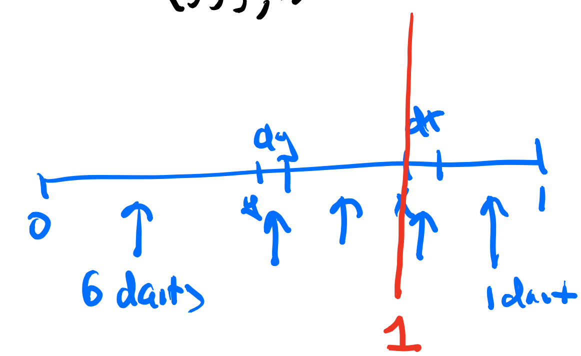
Notice that $U_{(1)} = \text{Beta}(1,1)$.

ex: Let $Y \sim U_{(7)}$, $X \sim U_{(9)}$ for 10 iid $U(0,1)$.

The joint density $f_{X,Y}(x,y) = \binom{10}{6,1,1,1,1} y^6 (x-y)(1-x)$

Let $Z = \frac{Y}{X}$

What distribution is Z?



$$\frac{U_{(7)} \text{ out of } 10}{U_{(9)} \text{ out of } 10} = U_{(7)} \text{ out of } 8$$

$$= \text{Beta}(\overset{7}{k}, \overset{8}{n}-\overset{7}{k}+1) = \boxed{\text{Beta}(7,2)}$$