Stat 134: Bivariate Normal

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Dec 4th, 2019

1. $\binom{X}{Y} \sim N\left(\binom{\mu_X}{\mu_Y}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}\right)$

$f(x,y) = \frac{1}{2\pi \cdot \sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot \exp\left[\frac{1}{-2(1-\rho^2)} \cdot \left(\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right)\right]$

Conceptual Review

2. WLOG, assume $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$

$Y = \rho X + \sqrt{1-\rho^2} Z$ where $X, Z \overset{iid}{\sim} N(0,1)$

1. The definition of the bivariate normal distribution.
2. The construction of the bivariate normal random variable (the decomposition representation).
3. Conditional distributions for the bivariate normal distribution.
4. Linear transformation of the multivariate normal distribution.

3. $X \mid Y=y \sim N\left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y), \ \sigma_X^2(1-\rho^2)\right)$

$Y \mid X=x \sim N\left(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X), \ \sigma_Y^2(1-\rho^2)\right)$

4. $Z_i \overset{indpt}{\sim} N(\mu_i, \sigma_i^2)$

$X = \sum a_i Z_i, \quad Y = \sum b_i Z_i$

$\binom{X}{Y} \sim N\left(\begin{pmatrix} \sum a_i\mu_i \\ \sum b_i\mu_i \end{pmatrix}, \begin{pmatrix} \sum a_i^2\sigma_i^2 & \sum a_i b_i \sigma_i^2 \\ \sum a_i b_i \sigma_i^2 & \sum b_i^2\sigma_i^2 \end{pmatrix}\right)$

Problem 1

Let X and Y have bivariate normal distribution with parameters μ_X, μ_Y, σ_X^2, σ_Y^2, and ρ.

1. Predict Y given $X = x$.
2. Find $\mathbb{P}(Y > y \mid X = x)$.
3. Find $\mathbb{P}(Y > \mu_Y, X > \mu_X)$.
4. Find $\mathbb{E}(Y \mid a < X < b)$, where $a < b$.

1. $Y \mid X=x \sim N\left(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X), \ \sigma_Y^2(1-\rho^2)\right)$

So Predict Y by $\mu_Y + \rho \cdot \frac{\sigma_Y}{\sigma_X}(x-\mu_X)$

2. $P(Y > y \mid X=x) = 1 - \Phi\left(\frac{y - (\mu_Y + \rho\cdot\frac{\sigma_Y}{\sigma_X}(x-\mu_X))}{\sigma_Y\sqrt{1-\rho^2}}\right)$

3. Let $\tilde{Y} = \frac{Y-\mu_Y}{\sigma_Y}$, $\tilde{X} = \frac{X-\mu_X}{\sigma_X}$, then $\tilde{X} \sim N(0,1)$, $\tilde{Y} \sim N(0,1)$

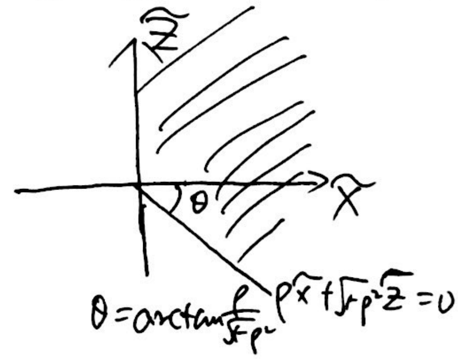
$Cov(\tilde{X}, \tilde{Y}) = \rho$. The decomposition representation $\Longrightarrow$

$\tilde{Y} = \rho\tilde{X} + \sqrt{1-\rho^2}\tilde{Z}$ where $\tilde{Z} \perp\!\!\!\perp \tilde{X}$, $\tilde{Z} \sim N(0,1)$

$P(Y > \mu_Y, X > \mu_X) = P(\tilde{Y} > 0, \tilde{X} > 0)$

$= P(\rho\tilde{X} + \sqrt{1-\rho^2}\tilde{Z} > 0, \tilde{X} > 0)$

$= \frac{1}{4} + \arctan\frac{\rho}{\sqrt{1-\rho^2}}$



4. WLOG, $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$.

$\mathbb{E}(Y \mid X=x) = \rho x$, $\quad f(x \mid a<X<b) = \frac{\frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})}{\Phi(b) - \Phi(a)}$

$\mathbb{E}(Y \mid a<X<b) = \int_a^b \mathbb{E}(Y \mid X=x) f(x \mid a<X<b)\,dx$

$= \int_a^b \rho x \cdot \frac{\frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})}{\Phi(b)-\Phi(a)}\,dx = \frac{\frac{\rho}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}a^2} - e^{-\frac{1}{2}b^2}\right)}{\Phi(b) - \Phi(a)}$

Problem 2

Let $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$.

1. If X and Y have bivariate normal distribution with correlation ρ, show that $\rho = 0$ if and only if X, Y are independent.
2. Find a counter-example such that X and Y are uncorrelated but they are not independent.

1. ① X, Y indpt $\Rightarrow \rho = 0$

② WLOG, assum $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$

$$\rho = 0 \Rightarrow f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \cdot \exp\left(\frac{x^2+y^2-2\rho xy}{-2(1-\rho^2)}\right)$$

$$= \frac{1}{2\pi} \cdot \exp\left(\frac{x^2+y^2}{-2}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)$$

$$= f(x) \cdot f(y) \Rightarrow \text{indpt}$$

2. Let $\begin{cases} X \sim N(0,1) \\ W = \begin{cases} 1 & \text{w.p } \frac{1}{2} \\ -1 & \text{w.p } \frac{1}{2} \end{cases} \end{cases}$ X, W indpt $Y = XW$

① $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}X\,\mathbb{E}Y = \mathbb{E}W\,\mathbb{E}X^2 - (\mathbb{E}X)^2\mathbb{E}W = 0$

② $P(Y \le x) = P(Y \le x \mid W=1)P(W=1) + P(Y \le x \mid W=-1)P(W=-1)$

$= P(X \le x)P(W=1) + P(-X \le x)P(W=-1)$

$= \frac{1}{2}\Phi(x) + \frac{1}{2}\Phi(x)$

$= \Phi(x) \Rightarrow Y \sim N(0,1)$

Here $X \sim N(0,1)$, so $-X \sim N(0,1)$

③ $|X| = |Y| \Rightarrow X, Y$ not indpt.

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