Stat 134 lec 26

Last time

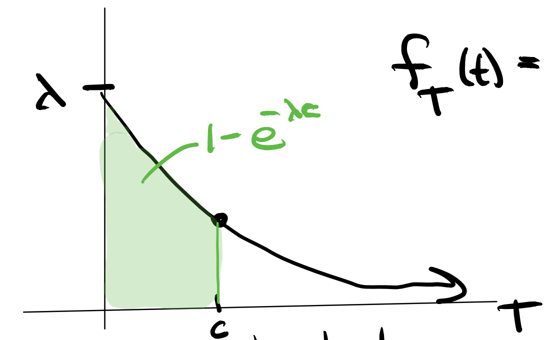
Sec 4.5 CDF $F_X(x) = P(X \le x)$

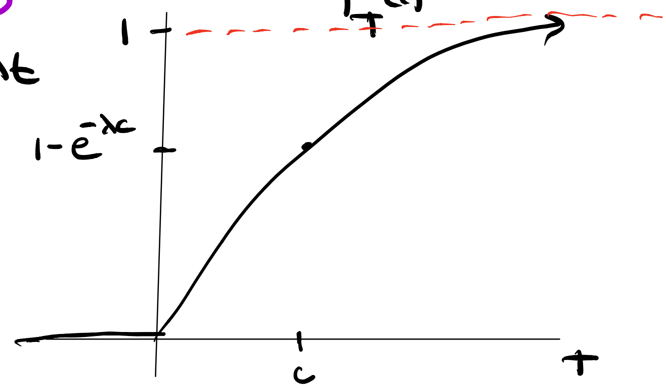
ex $T \sim Exp(\lambda)$ — values $[0, \infty)$

$c > 0$

$f_T(t) = \lambda e^{-\lambda t}$

$F_T(t) = 1 - e^{-\lambda t}$



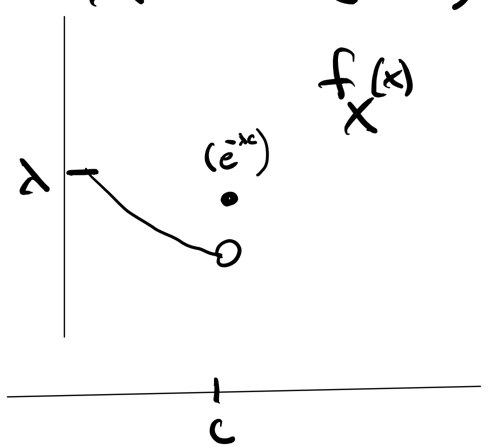


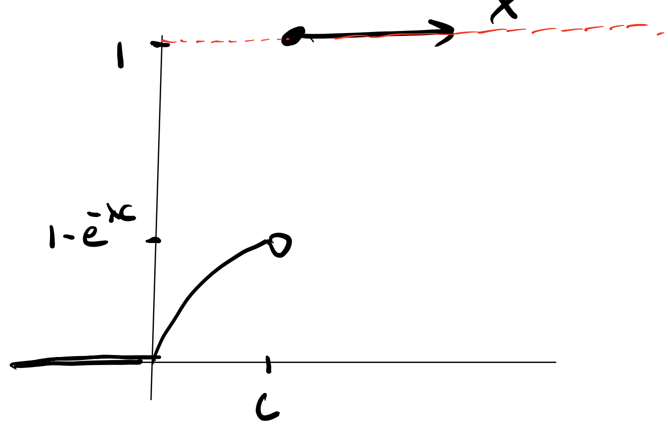
ex mixed distribution

$X = \min(T, c)$ — values $[0, c]$

$f_X(x)$

$F_X(x)$

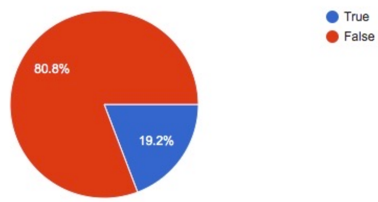


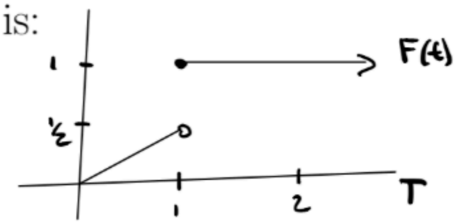


Today

1. review student responses, mixed distribution
2. review MGF
3. sec 4.5 Using CDF to find $E(X)$

Suppose you are trying to discretly leave a party. Your time to leave is uniform from 0 to 2 minutes. However, if your walk to the exit takes more than 1 minute, you run into a friend at the door and and must spend the full 2 minutes to leave. Let T represent the time it takes you to leave. True or false, the graph of the cdf of T is:

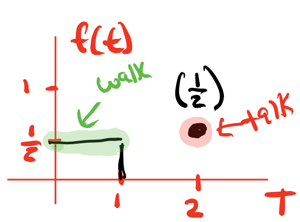


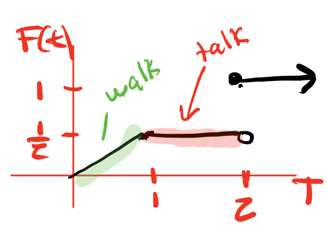


TRUE	From 0-1 minutes, the increase of CDF is linear, which is indicative of a uniform increase, and then it flattens out from 1-2, which represents the condition that after 1 minute, there is a set amount of time you have to spend

FALSE	Slope should be 1 for 0 to 1 min

FALSE	The beginning of the graph is right since it is half of the uniform cdf, but at 1, the graph should continue flat until 2 where it jumps to 1.You cannot leave between 1 and 2 and at two you will leave.





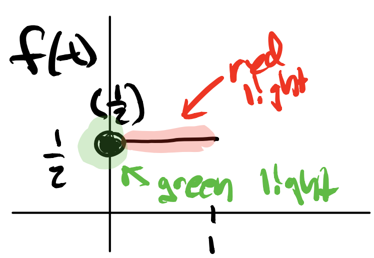
Another mixed distribution:

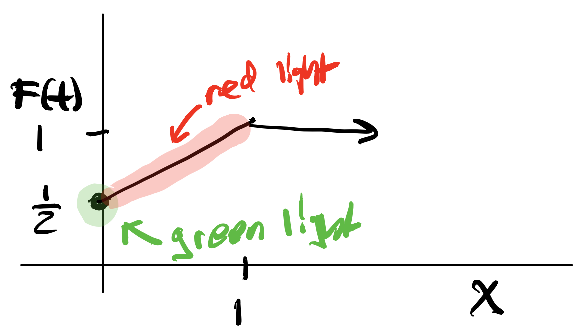
ex

Suppose stop lights at an intersection alternately show green for one minute, and red for one minute (no yellow). Suppose a car arrives at the lights at a time distributed uniformly from 0 to 2 minutes. Let X be the delay of the car at the lights (assuming there is only one car on the road). Graph the density and the cdf.

soln

mixed density





② Review MGF

X RV, $t \in \mathbb{R}$

$$M_X(t) = E(e^{tX})$$

ex $X \sim \text{Gamma}(r, \lambda)$

variable part

$$f_X(x) = \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}, \quad x > 0$$

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r, \quad t < \lambda$$

ex $X \sim \text{Unif}(0,1)$

$$f_X(x) = 1_{(x \in (0,1))}$$

Find $M_X(t)$

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} 1_{x \in (0,1)} \, dx$$

$$= \int_0^1 e^{tx} \cdot 1 \, dx = \frac{1}{t} e^{tx} \Big|_{x=0}^{x=1}$$

$$= \boxed{\frac{1}{t}(e^t - 1)}$$

Property of MGF:

$M_X(t) = M_Y(t)$ for all t in a neighborhood of zero, iff $X \overset{d}{=} Y$ (i.e. X and Y have the same distribution)

ex $X \sim \text{Unif}(0,1)$

$Y = 1 - X$

by change of variable rule $(g(x) = 1 - X)$

$$f_Y(y) = \frac{1}{\underbrace{|g'(x)|}_{=-1}} \underbrace{f_X(x)}_{1_{x \in (0,1)}} \Big|_{x = 1-y} = 1_{(y \in (0,1))}$$

$1 - y \in (0,1) \Rightarrow y \in (0,1)$

so $Y \sim \text{Unif}(0,1)$.

Hence $X \overset{d}{=} Y$ equal distributions but not equal RVs

Find $M_Y(t)$:

$$M_Y(t) = \int_0^1 e^{ty} 1 \, dy = \frac{1}{t} e^{ty} \Big|_{y=0}^{y=1} = \boxed{\frac{1}{t}(e^t - 1)}$$

Knowing the distribution uniquely specifies the MGF

And

Knowing the MGF uniquely specifies the distribution tinyurl.com/nov1-pt1

Stat 134

Monday April 1 2019

1. Let X have density $f(x) = xe^{-x}$ for $x > 0$.
   The MGF is?

   a $M_X(t) = \frac{1}{1-t}$ for $t < 1$

   b $M_X(t) = \frac{1}{(1-t)^2}$ for $t < 1$

   c $M_X(t) = \frac{1}{(1+t)^2}$ for $t > -1$

   d none of the above

Variable part Gamma $x^{r-1}e^{-\lambda x}$

$\Rightarrow X \sim \text{Gamma}(2,1)$

we know MGF of Gamma(r,λ) is $\left(\frac{\lambda}{\lambda-t}\right)^r$, $t<\lambda$

so $\boxed{M_X(t) = \frac{1}{(1-t)^2} \text{ for } t<1}$

Recall, $M_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^r$, $t<\lambda$ is the MGF for Gamma(r,λ)

ex If $M_X(t) = \frac{1}{\sqrt{1-t}}$ for $t<1$

what distribution is X?

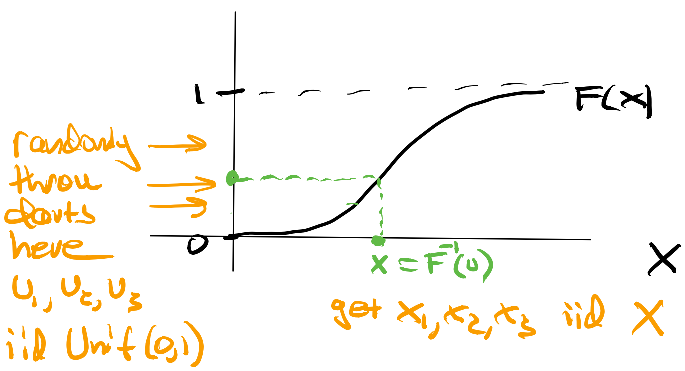
$X \sim \text{Gamma}(r=\frac{1}{2}, \lambda=1)$

see #9 p324

③ Sec 4.5 Using CDF to find $E(X)$ for $X \geq 0$

Inverse distribution function, $F^{-1}(u)$

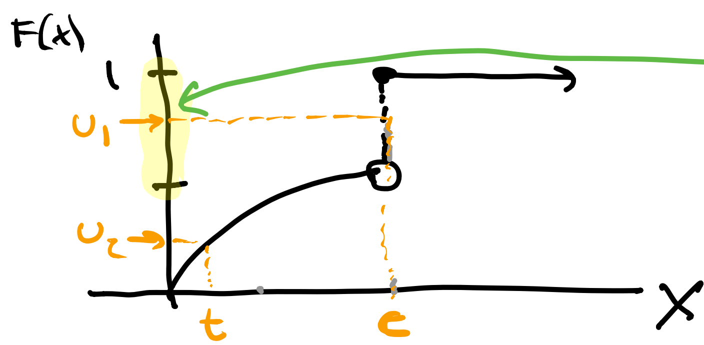
Let X have CDF $F(x)$.



$$\boxed{F^{-1}(u) = x}$$

Note: doesn't have to be continuous RV.

ex $X = \min(T, c)$, $T \sim Exp(\lambda)$



This yellow region gets assigned the single value c.

Thm (P322) — Proof at end of lecture.

Let X have CDF F.

Then the RV $F^{-1}(U) = X$

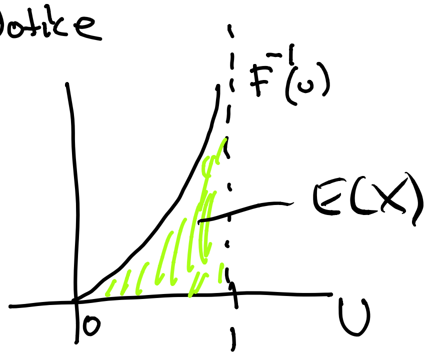
How is this useful to us finding $E(X)$?

It is sometimes easier to calculate $E(X)$ using the cdf (avoid doing integration by parts):

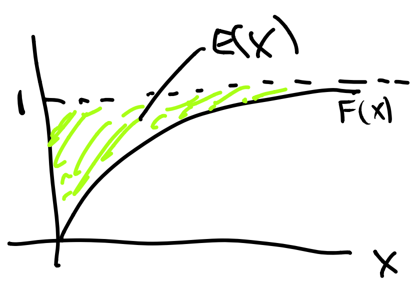
1 since $U \sim \text{Unif}(0,1)$

$$E(X) = E\left(F^{-1}(U)\right) = \int_0^1 F^{-1}(u) f_U(u)\,du$$

Notice



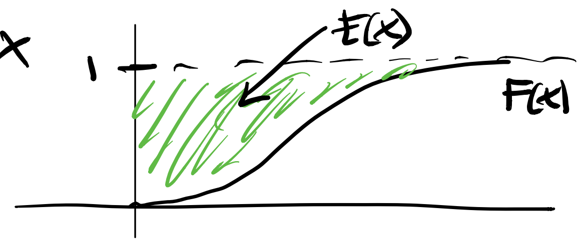
Now reflect the above graph about the diagonal $y = x$



We can find the shaded region by integrating $1 - F(x)$ with respect to x:

Thm Let X be a pos. random variable, with CDF F. (continuous, discrete, mixed),

$$E(X) = \int_0^\infty (1 - F(x))\,dx$$



ex $T \sim \text{expon}(\lambda)$

$$F_T(t) = 1 - e^{-\lambda t}$$

Calculate $E(T)$.

$$E(T) = \int_0^\infty (1 - F(t))\,dt = \int_0^\infty e^{-\lambda t}\,dt$$

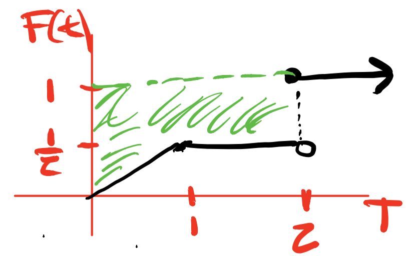
$$= -\frac{1}{\lambda} e^{-\lambda t} \Big|_0^\infty = \boxed{\frac{1}{\lambda}}$$

Wow that was easy!

ex

Suppose you are trying to discretly leave a party. Your time to leave is uniform from 0 to 2 minutes. However, if your walk to the exit takes more than 1 minute, you run into a friend at the door and and must spend the full 2 minutes to leave. Let T represent the time it takes you to leave. Find $E(T)$

$\boxed{E(t) = 5/4}$



ex

For $k = 1, 2, 3, \ldots$, let X_k be uniformly distributed on $\{0, \frac{1}{10^k}, \frac{2}{10^k}, \ldots, \frac{9}{10^k}\}$. Verify that the MGF of X_k is

$$M_{X_k}(t) = \begin{cases} \frac{1}{10} \cdot \frac{1-e^{t/10^{k-1}}}{1-e^{t/10^k}} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

(Hint: Recall that $\sum_{i=0}^{j} r^i = \frac{1-r^{j+1}}{1-r}$.)

$$M_{X_k}(t) = E\left(e^{tX_k}\right)$$

$$= \sum_{i=0}^{9} e^{t\frac{i}{10^k}} \cdot \frac{1}{10} = \frac{1}{10} \sum_{i=0}^{9} \left(e^{t/10^k}\right)^i$$

$$= \frac{1}{10} \cdot \frac{1 - \left(e^{t/10^k}\right)^{10}}{1 - e^{t/10^k}} \qquad \left(e^{t/10^k}\right)^{10} = e^{10t/10^k} \qquad 10t/10^k = t/10^{k-1}$$

$$= \boxed{\frac{1}{10} \, \frac{1 - e^{t/10^{k-1}}}{1 - e^{t/10^k}}}$$

Appendix

See p 322 in book

Claim for any CDF F

$X = F^{-1}(U)$ is a RV with cdf F.

U ~ Unif$(0,1)$

Proof/ let $X = F^{-1}(U)$

$F_X(x) = P(X \le x)$ we will show $F_X = F$

$= P(F^{-1}(U) \le x)$

$= P(F \cdot F^{-1}(U) \le F(x))$ since F is increasing

$= P(U \le F(x))$

$= F(x)$ since $P(U \le u) = u$

$\square$