Stat 134 lec 28

Last time

sec 4.6 uniform order statistic

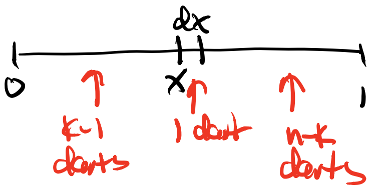
$U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$

$U_{(1)}, \dots, U_{(n)}$ order statistics

ex $U_1 = .2$ $U_2 = .7$ $U_3 = .1$

$\Rightarrow U_{(1)} = .1,\ U_{(2)} = .2,\ U_{(3)} = .7$

$P(U_{(k)} \in dx) = f(x)dx$



Note

$\binom{n}{a,b,c} = \frac{n!}{a!b!c!}$

$= \binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}$

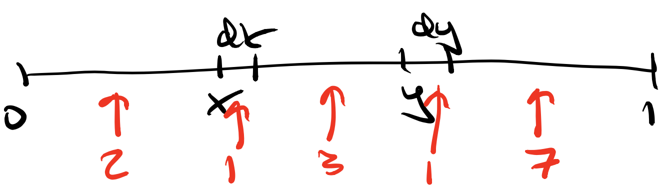
$$f_{U_{(k)}}(x) = \binom{n}{k-1,1,n-k} x^{k-1}(1-x)^{(n-k+1)-1} \quad \text{on } 0<x<1$$

sec 5.1, 5.2 Continuous Joint Distribution

$$P(X \in dx, Y \in dy) = f(x,y)\,dx\,dy.$$

ex What joint density, $f(x,y)$ has variable part $x^2(y-x)^3(1-y)^7$ on $0<x<y<1$.

expect X and Y are unif ordered statistics.



$\Rightarrow X = U_{(3)}$

$Y = U_{(7)}$

(throw 14 darts).

today ① ~~Review student responses to concept test~~

② sec 4.6 Beta distribution

③ sec 5.1, 5.2 Calculate probabilities with $f(x,y)$.

④ sec 5.1, 5.2 Independence

① Review student responses to concept test

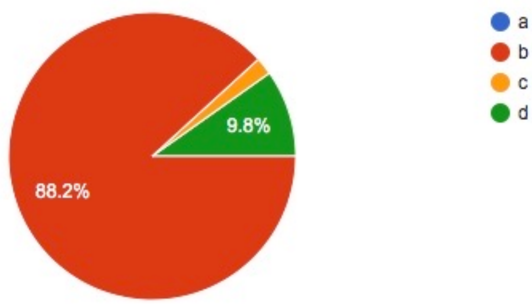
I throw down 8 darts on $(0,1)$. The variable part of the joint density of $X = U_{(3)}$ and $Y = U_{(6)}$ is:

a $x(y-x)^5(1-y)^2$

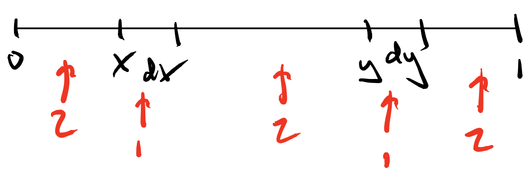
b $x^2(y-x)^2(1-y)^2$

c $x^4(y-x)^2(1-y)^2$

d none of the above



b	2 darts land in x region 1 dart lands in dx region 2 darts land between the X-th and the Yth dart 1 dart lands in dy 2 darts land in region between y and 1 (the probability space)



d	Multiply b. by (8 choose 2,1,2,1,2)

Let (X, Y) have joint density $f_{X,Y}(x, y) = 420x^3(1-y)^2$ for $0 < x < y < 1$.

Fill in the blanks: X and Y represent the 4th smallest and 5th smallest of 7 i.i.d. Unif (0,1) random variables, respectively.

(2) sec 4.6 Beta distribution.

$X \sim Beta(r,s)$ for $r>0$, $s>0$ is a distribution often used to model physical processes that take values between 0 and 1.

ex the proportion of defective items in a shipment.

Let $r,s>0$

Def^n $X \sim Beta(r,s)$ if

$$f(x) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} x^{r-1}(1-x)^{s-1} \quad \text{for } 0<x<1.$$

where $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$ Gamma function for $r>0$

or $\Gamma(r) = (r-1)!$ if $r \in \mathbb{Z}^+$

Notice if $r=1, s=1$, $f(x) = 1_{x\in(0,1)}$

$\Rightarrow Beta(1,1) = Unif(0,1)$.

ex Let $U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$

$$f_{U_{(k)}}(x) = \binom{n}{k-1,1,n-k} x^{k-1}(1-x)^{(n-k+1)-1} \quad \text{on } 0<x<1$$

compare with,

$$f_{Beta(r,s)}(x) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} x^{r-1}(1-x)^{s-1} \quad \text{for } 0<x<1.$$

Notice that $f_{U_{(k)}}(x)$ and $f_{Beta(r,s)}(x)$ have the same variable part of their density when $r = k$

$s = n-k+1$

then $\Gamma(s+r) = \Gamma(n-k+1+k) = \Gamma(n+1) = n!$

$\Gamma(r) = (k-1)!$

$\Gamma(s) = (n-k)!$

$$\Rightarrow \frac{\Gamma(s+r)}{\Gamma(r)\Gamma(s)} = \binom{n}{k-1,\,1,\,n-k}$$

$\Rightarrow$ uniform ordered statistics are beta!

see appendix of notes

<u>Thm</u> $X \sim Beta(r,s)$

$$E(X) = \frac{r}{r+s}$$

Hence if $X \sim U_{(k)}$

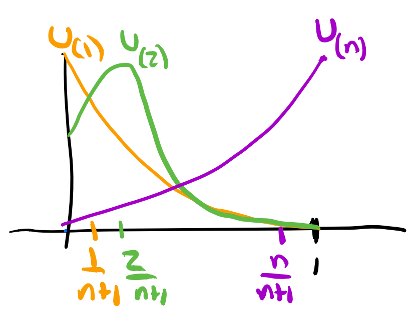
$$E(X) = \frac{k}{n-k+1+k} = \boxed{\frac{k}{n+1}}$$

$$E(U_{(1)}) = \frac{1}{n+1}$$

$$E(U_{(2)}) = \frac{2}{n+1}$$

$\vdots$

$$E(U_{(n)}) = \frac{n}{n+1}$$



ex (Bayesian Statistics)

Let P be the chance a coin lands head. Suppose the prior distribution of P is

$$f_P(p) = \begin{cases} c(1-p)^4 & \text{for } 0 \le p \le 1 \\ 0 & \text{else} \end{cases}$$

a) Is this a beta distribution? If so, what are the parameters? $f_{Beta(r,s)}(p) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} p^{r-1}(1-p)^{s-1}$ for $0<p<1$.

yes, compare $(1-p)^4$ and $p^{r-1}(1-p)^{s-1} \Rightarrow P \sim Beta(1,5)$

b) Calculate the constant c

$$c = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} = \frac{\Gamma(6)}{\Gamma(1)\Gamma(5)} = \frac{5!}{0!4!} = 5$$

c) What is the mean of P?

$$E[P] = \frac{r}{r+s} = \frac{1}{1+5} = 1/6$$

If $X \sim \text{Beta}(r,s)$

$$f(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}$$

Since $\int_0^1 f(x)\,dx = 1 \Rightarrow \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$

ex Let $X \sim \text{Beta}(3,4)$ $\left(\text{so } f(x) = \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} x^2(1-x)^3\right)$

Compute $E(7X - 5X^6)$

$$E(7x-5x^6) = \underbrace{7E(x)}_{= 7\cdot 3/7} - 5E(x^6)$$

$$E(x^6) = \int_0^1 x^6 \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} x^2(1-x)^3\,dx$$

$$= \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} \int_0^1 x^8(1-x)^3\,dx \qquad \int_0^1 x^8(1-x)^3\,dx = \frac{\Gamma(9)\Gamma(4)}{\Gamma(13)}$$

$$= \frac{\Gamma(9)\Gamma(7)}{\Gamma(3)\Gamma(13)}$$

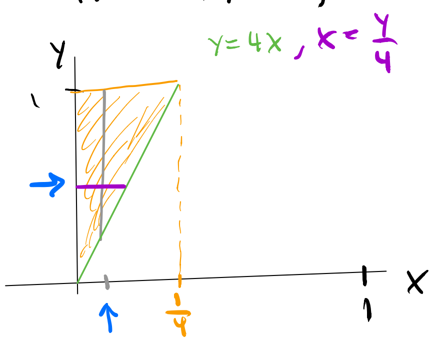
$$\Rightarrow E(7X-5X^6) = \boxed{3 - \frac{5\Gamma(9)\Gamma(7)}{\Gamma(3)\Gamma(13)}}$$

③ Sec 5.1, 5.2 Calculate probabilities with $f(x,y)$.

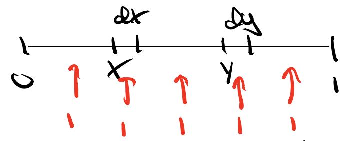
Throw down 5 darts on $(0,1)$.

ex $X = U_{(2)}$ $Y = U_{(4)}$

Find $P(Y > 4X)$



Find density



$$f(x,y) = \binom{5}{1,1,1,1,1} x(y-x)(1-y) = \boxed{5!\, x(y-x)(1-y)}$$

$$P(Y>4X) = \int_{y=0}^{y=1} \int_{x=0}^{x=\frac{y}{4}} 5!\, x(y-x)(1-y)\,dx\,dy$$

(or)

$$P(Y>4X) = \int_{x=0}^{x=1/4} \int_{y=4x}^{y=1} 5!\, x(y-x)(1-y)\,dy\,dx$$

details:

$$P(Y > 4x) = \int_{y=0}^{y=1} \int_{x=0}^{y/4} 120\, x\, (y-x)(1-y)\, dx\, dy$$

$$= 120 \int_{y=0}^{y=1} (1-y) \int_{x=0}^{x=y/4} (xy - x^2)\, dx\, dy$$

$$= \int_{y=0}^{y=1} 120(1-y)\left(\frac{x^2 y}{2} - \frac{x^3}{3}\right)\Bigg|_{x=0}^{x=y/4} dy$$

$$= \int_0^1 120\,(1-y)\left(\frac{y^3}{32} - \frac{y^3}{3\cdot 64}\right) dy$$

$$= \frac{5}{192} \cdot 120 \int_0^1 y^3 - y^4\, dy = \frac{5 \cdot 120}{192}\left(\frac{y^4}{4} - \frac{y^5}{5}\right)\Bigg|_0^1$$

$$= \frac{5 \cdot 120}{192}\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{30}{192} = .156$$

④ Sec 5.1, 5.2

Independent RVs

def^n X and Y are independent if

$$P(X \in dx, Y \in dy) = P(X \in dx)\, P(Y \in dy)$$

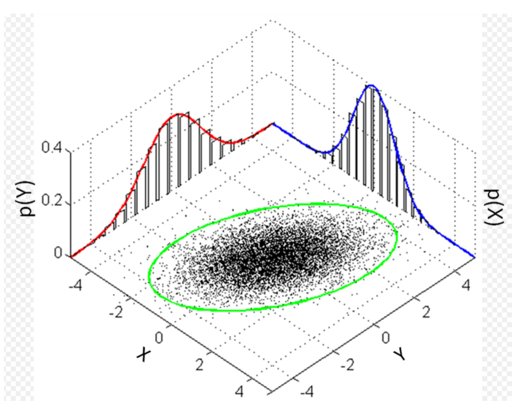
$$\overset{\shortparallel}{f(x,y)\,dx\,dy} \qquad \overset{\shortparallel}{f(x)\,dx} \qquad \overset{\shortparallel}{f(y)\,dy}$$

$$\iff \boxed{f(x,y) = f(x)f(y)}$$

ex $X, Y \overset{iid}{\sim} N(0,1)$

$$f(x,y) = \phi(x)\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$



Not a great picture because the oval in green should be a circle. This is the picture of a correlated bivariate normal from chapter 6 instead of an uncorrelated bivariate normal.

ex

If $X, Y \overset{iid}{\sim} U(0,1)$

Find $P(Y \geq \frac{1}{2} \mid Y \geq 1-2X)$

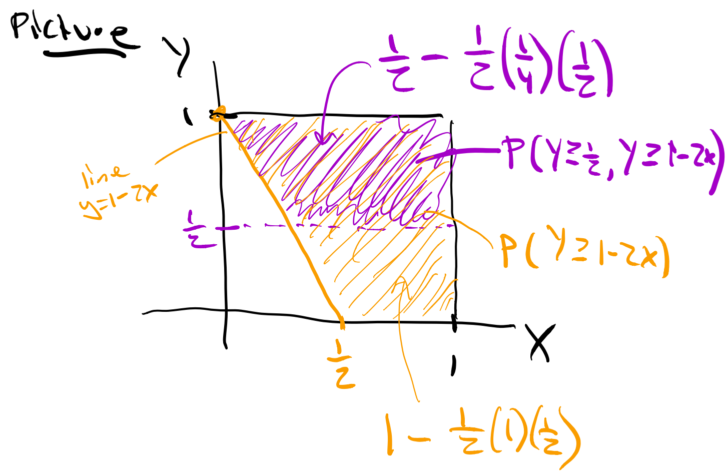
soln

$f(x,y) = f(x)f(y) = 1$ for $0 < x, y < 1$

$$f(x,y) = \underset{1}{\underset{\shortparallel}{f(x)}}\,\underset{1}{\underset{\shortparallel}{f(y)}} = 1$$

$$P(Y \geq \tfrac{1}{2} \mid Y \geq 1-2X) = \frac{P(Y \geq \frac{1}{2}, Y \geq 1-2X)}{P(Y \geq 1-2X)}$$ Baye's rule

Picture



so,

$$\frac{P(Y \geq \frac{1}{2}, Y \geq 1-2X)}{P(Y \geq 1-2X)} = \frac{\frac{1}{2} - \frac{1}{2}(\frac{1}{4})(\frac{1}{2})}{1 - \frac{1}{2}(1)(\frac{1}{2})} = \frac{7/16}{3/4}$$

$$= 7/12$$

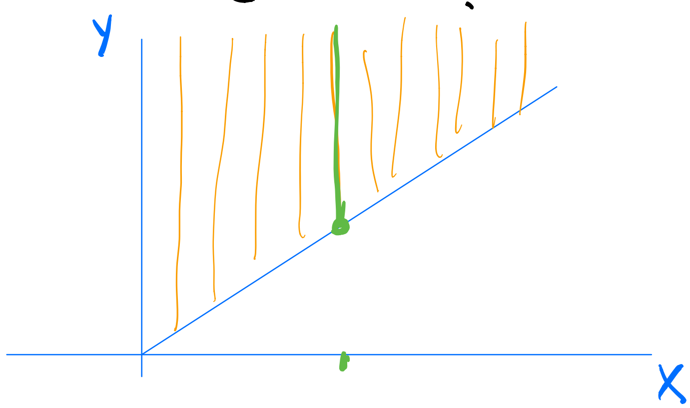
ex

Let $X \sim \text{Exp}(\lambda)$, $Y \sim \text{Exp}(\mu)$

(recall, $f_X(x) = \lambda e^{-\lambda x}$)

be independent lifetimes of two bulbs.

Find $P(Y > X)$.



$$f(x,y) = \lambda e^{-\lambda x} \mu e^{-\mu y}$$

$$P(Y > X) = \lambda \mu \int_{x=0}^{x=\infty} e^{-\lambda x} \underbrace{\int_{y=x}^{y=\infty} e^{-\mu y} \, dy}_{} \, dx$$

$$= \frac{e^{-\mu y}}{\mu} \Bigg|_{y=x}^{y=\infty} = \frac{e^{-\mu x}}{\mu}$$

$$= \lambda \int_{x=0}^{\infty} e^{-(\lambda+\mu)x} \, dx = \boxed{\frac{\lambda}{\lambda+\mu}}$$

Appendix

Let $X \sim \text{Beta}(r,s)$

then $E(X) = \dfrac{r}{r+s}$.

Pf/ Note that $\int_0^1 f(x)\,dx = \dfrac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = 1$

$$\Rightarrow \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$$

$$E(X) = \int_0^1 x f(x)\,dx = \frac{\Gamma(r+s)}{\Gamma(r)\cancel{\Gamma(s)}} \underbrace{\int_0^1 \overbrace{x\,x^{r-1}}^{x^{r+1-1}}(1-x)^{1-s}\,dx}_{= \frac{\cancel{\Gamma(s)}\Gamma(r+1)}{\Gamma(s+r+1)}}$$

$$= \frac{(r+s-1)!\; r!}{(s+r)!} = \boxed{\frac{r}{r+s}}$$

$\square$