

Warmup 8:00 - 8:10 AM

rule of average conditional probability:

Suppose $Y|X=x \sim \text{Pois}(x)$ where $X \sim \text{Exp}(\lambda)$

Show $Y \sim \text{Geom}\left(\frac{\lambda}{\lambda+1}\right)$ on $0, 1, 2, \dots$

$$P(Y=y) = \int_0^\infty P(Y=y|X=x) f_X(x) dx$$

$$= \int_0^\infty e^{-x} x^y \lambda e^{-\lambda x} \lambda e^{-\lambda x} dx$$

$$= \frac{\lambda}{(\lambda+1)^{y+1}} \int_0^\infty \frac{(\lambda+1)^{y+1}}{y!} x^{y+1-1} e^{-\lambda x} dx$$

$$= \left(\frac{1}{\lambda+1}\right)^y \frac{\lambda}{\lambda+1}$$

$$\Rightarrow Y \sim \text{Geom}\left(\frac{\lambda}{\lambda+1}\right),$$

recall if $Y \sim \text{Geom}\left(\frac{\lambda}{\lambda+1}\right)$ on $0, 1, 2, \dots$

$$P(Y=y) = \left(\frac{1}{\lambda+1}\right)^y \left(\frac{\lambda}{\lambda+1}\right)$$

$$P(Y=y|X=x) = \frac{e^{-x} x^y}{y!}$$

$$f_X(x) = \lambda e^{-\lambda x}$$

recall

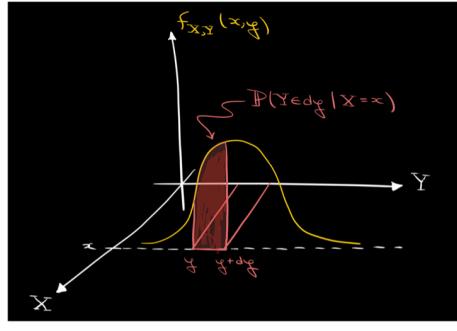
$$X|_{Y=y} \sim \text{Geom}\left(y+1, \frac{\lambda}{\lambda+1}\right)$$

$$f_{X|Y=y}(x) = \frac{(\lambda+1)^{y+1}}{\Gamma(y+1)} x^{y+1-1} e^{-\lambda x}$$

Last time

sec 6.3 Conditional density

X, Y continuous RVs.



$$P(Y \in dy | X=x) = \lim_{dx \rightarrow 0} \frac{P(Y \in dy, X \in dx)}{P(X \in dx)}$$

$$f(y) dy = \lim_{dx \rightarrow 0} \frac{f(x, y) dx dy}{f_X(x) dx}$$

$$f(y) = \frac{f(x, y)}{f_X(x)} \quad \text{Bayes' rule}$$

$$f(x, y) = f_Y(y) f_X(x) \quad \text{mult rule.}$$

Average conditional Probabilities

X discrete, Y discrete

$$P(Y=y) = \sum_{x \in X} P(Y=y | X=x) P(X=x)$$

X continuous Y continuous

$$P(Y \in dy) = \int_{x \in X} P(Y \in dy | X=x) f_X(x) dx$$

X discrete Y continuous

$$P(Y \in dy) = \sum_{x \in X} P(Y \in dy | X=x) P(X=x)$$

X continuous Y discrete

$$P(Y=y) = \int_{x \in X} P(Y=y | X=x) f_X(x) dx$$

Today ① sec 6.3 more practice

② Covariance and the Variance of sum

① More practice

Let $Y|X=x \sim \text{Pois}(x)$ where $X \sim \text{Exp}(\lambda)$
 We see that $Y \sim \text{Geom}\left(\frac{\lambda}{\lambda+1}\right)$ on $0, 1, 2, \dots$

Show that $X|Y=y$ is Gamma and find parameters,

recall if $Y \sim \text{Geom}\left(\frac{\lambda}{\lambda+1}\right)$ on $0, 1, 2, \dots$

$$P(Y=y) = \left(\frac{1}{\lambda+1}\right)^y \left(\frac{\lambda}{\lambda+1}\right)$$

$$P(Y=y|X=x) = \overbrace{e^{-\lambda} \lambda^y}^{f_X(x)}$$

$$f_X(x) = \lambda e^{-\lambda} \overbrace{\lambda^x}{y!}$$

$$\begin{aligned} P(X \in dx | Y=y) &= \frac{P(X \in dx, Y=y)}{P(Y=y)} \quad \text{Because we} \\ &= \frac{P(Y=y | X=x) f_X(x)}{P(Y=y)} \quad \leftarrow \text{mult rule} \end{aligned}$$

$$= \frac{\overbrace{e^{-\lambda} \lambda^y}{y!} \cdot \lambda e^{-\lambda} dx}{\left(\frac{1}{\lambda+1}\right)^y \frac{\lambda}{\lambda+1}}$$

$$\left\{ \begin{aligned} &\propto \lambda^y e^{-(\lambda+1)x} \\ &X|Y=y \sim \text{Gamma}(y+1, \lambda+1) \end{aligned} \right.$$

② Sec 6.1 Covariance and variance of a sum

$$X, Y, S = X+Y$$

$$\text{mean } M_X, M_Y, M_S = M_X + M_Y$$

$$\text{Var}(S) = E((S - M_S)^2) = E(D_S^2)$$

D_S deviation from mean

$$\begin{aligned} D_S &= S - M_S \\ &= X + Y - (M_X + M_Y) \\ &= D_X + D_Y \end{aligned}$$

$$\begin{aligned} \text{Var}(S) &= E((D_X + D_Y)^2) \\ &= E(D_X^2 + D_Y^2 + 2D_X D_Y) \\ &= E(D_X^2) + E(D_Y^2) + 2E(D_X D_Y) \\ &\quad \text{Var}(X) \quad \text{Var}(Y) \quad \text{Corr}(X, Y) \end{aligned}$$

Defn The covariance of X and Y is

$$\text{Cov}(X, Y) = E((X - M_X)(Y - M_Y))$$

Bilinearity Properties

Proved end of lecture.

(a) $\text{Cov}(X+Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$

(b) $\text{Cov}(aX, bY) = ab \text{Cov}(X, Y)$

More generally

$$\begin{aligned} \text{Cov}\left(\sum_{i=1}^n a_i X_i, \sum_{j=1}^m b_j Y_j\right) \\ = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \text{Cov}(X_i, Y_j) \end{aligned}$$

Proved end of lecture,

Thm $\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$

Easy facts

$$\text{Cov}(X, X) = E(X^2) - E(X)^2 = \text{Var}(X)$$

$$\text{Cov}(X, X) = \text{Var}(X)$$

$$\text{Cov}(X, Y) = \text{Cov}(Y, X)$$

$$\text{Cov}(X, c) = 0$$

Constant

if

Simplify

$$\text{Cov}(x - 5y, 3x + y - z + 10)$$

$$= 3\text{Var}(x) + \text{Cov}(x, y) - \text{Cov}(x, z) + 0$$

$$- 15\text{Cov}(x, y) - 5\text{Var}(y) + 5\text{Cov}(y, z) + 0$$

Recall x, y independent

$$\Rightarrow E(x+y) = E(x)E(y)$$

if x, y indep
 what \downarrow
 $\text{Cov}(x+y, y)$
 " "
 $\text{Var}(y)$

$\text{Cov}(x, y) = 0$ if x, y independent.

Hence if x, y indep,

$$\begin{aligned}
 \text{Var}(x+y) &= \text{Var}(x) + \text{Var}(y) + 2\text{Cov}(x, y) \\
 &= \text{Var}(x) + \text{Var}(y)
 \end{aligned}$$

Stat 134

Wednesday April 24 2019

1. Consider a Poisson(λ) process. Let $T_r \sim \text{gamma}(r, \lambda)$ be the rth arrival time. $\text{Cov}(T_1, T_3)$ equals:

- a** λ
 - b** λ^2
 - c** $1/\lambda^2$
 - d** none of the above

$$\text{Cov}(\tau_1, \tau_{3-\tau_1}) = 0$$

$$\text{Cov}(T_1, T_2) - \text{Var}(T_1) = 0$$

$$\text{Cov}(\tau_1, \tau_2) = \text{Var}(\tau_1) = \boxed{\frac{1}{\sigma^2 n}}$$

ex Let x_1, \dots, x_n be identically distributed

$$\text{Var}\left(\sum_{i=1}^n x_i\right) = \text{Cov}\left(\sum_{i=1}^n x_i, \sum_{j=1}^n x_j\right)$$

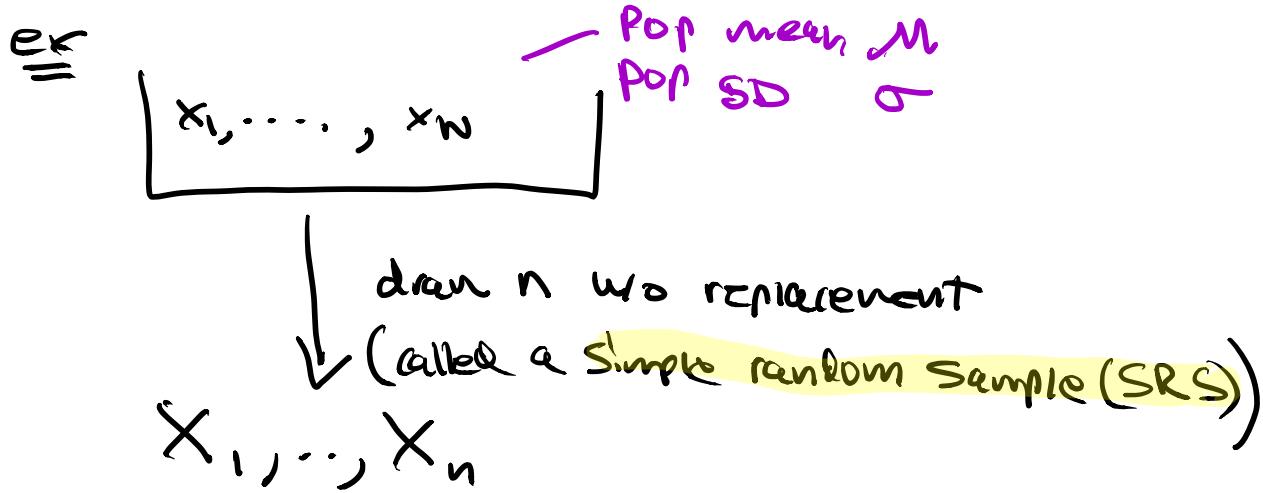
Def

The Variance-covariance matrix has all n^2 terms

$$\begin{matrix} & x_1 & x_2 & \cdots & x_n \\ x_1 & \text{Cov}(x_1, x_1) & \text{Cov}(x_1, x_2) & & \\ x_2 & & \text{Cov}(x_2, x_2) & & \\ \vdots & & & \ddots & \\ \vdots & & & & \text{Cov}(x_n, x_n) \\ x_n & & & & n \times n \end{matrix}$$

$$\text{Var}\left(\sum_{i=1}^n x_i\right) = n \text{Var}(x_1) + n(n-1) \text{Cov}(x_1, x_2)$$

↑ diagonal ↖ off diagonal



let $S_n = X_1 + \dots + X_n$ be the sample sum

Find $E(S_n)$

$\text{Var}(S_n)$,

$$E(S_n) = E(X_1 + \dots + X_n) = nE(X_1) = n\mu$$

$$\text{Var}(S_n) = n \text{Var}(X_1) + n(n-1) \text{Cov}(X_1, X_2)$$

σ^2

By the symmetry of sampling without replacement
 (see P238-239) we have $\text{Cov}(X_i, X_j) = \text{Cov}(X_1, X_2)$.

trick to find $\text{Cov}(X_1, X_2)$

take a census $n=N$

$$\text{Var}(S_N) = \text{Var}(S_N) = 0$$

$$\Rightarrow 0 = N\sigma^2 + N(N-1) \text{Cov}(X_1, X_2)$$

$$\Rightarrow \text{Cov}(X_i, X_j) = -\frac{N\sigma^2}{N(N-1)} = \boxed{-\frac{\sigma^2}{N-1}}$$

$$\begin{aligned} \text{So } \text{Var}(S_n) &= n\sigma^2 + n(n-1)\left(-\frac{\sigma^2}{N-1}\right) \\ &= n\sigma^2 \left[\frac{N-n}{N-1} \right] \end{aligned}$$

↖ correction factor < 1

(note if draw w replacement)

$$\text{Var}(S_n) = n\sigma^2.$$

Recall, we have seen this correction factor before (p241)

$$X \sim \text{Bin}(n, p) \Rightarrow \text{Var}(X) = (n)pq$$

$$X \sim \text{hypergeom}(n, N, 6) \Rightarrow \text{Var}(X) = n \left(\frac{6}{N} \right) \left(\frac{N-6}{N} \right) \left(\frac{N-n}{N-1} \right).$$

\uparrow \uparrow \uparrow
 p q Correction
 factor

Appendix Bilinearity Properties

Thm

$$(a) \text{Cov}(X+Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$$

$$(b) \text{Cov}(aX, bY) = ab \text{Cov}(X, Y)$$

Pf
a)

$$\text{Cov}(X+Y, Z) = E((X+Y - \mu_{X+Y})(Z - \mu_Z))$$

$$= E((X - \mu_X) + (Y - \mu_Y))(Z - \mu_Z)$$

$$= E((X - \mu_X)(Z - \mu_Z) + (Y - \mu_Y)(Z - \mu_Z))$$

$$= E((X - \mu_X)(Z - \mu_Z)) + E((Y - \mu_Y)(Z - \mu_Z))$$

$$= \text{Cov}(X, Z) + \text{Cov}(Y, Z).$$

□

$$b) \text{Cov}(aX, bY) = E((aX - \mu_{aX})(bY - \mu_{bY}))$$

$$= E((aX - a\mu_X)(bY - b\mu_Y))$$

$$= E(ab(X - \mu_X)(Y - \mu_Y))$$

$$= ab E((X - \mu_X)(Y - \mu_Y))$$

$$= ab \text{Cov}(X, Y)$$

□

Appendix

Thm $\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$

Pf $\text{Cov}(X, Y) = E(D_X D_Y) = E((X - \mu_X)(Y - \mu_Y))$

$$\begin{aligned} &= E(XY - \mu_X Y - X\mu_Y + \mu_X \mu_Y) \\ &= E(XY) - \mu_X \mu_Y - \mu_X \mu_Y + \mu_X \mu_Y \\ &= E(XY) - E(X)E(Y) \end{aligned}$$

□