lec 40 stat 134

Last time sec 6.5

we write (U,V) bivariate normal w/ $Corr(U,V)=\rho$ as

$$(U,V) \sim BV(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$$

std bivariate normal.

Thm Let $(X,Y) \sim BV(0,0,1,1,\rho)$
normal.
The MGF of (X,Y) is

$$M_{(X,Y)}(s,t) = e^{\frac{s^2}{2}+\frac{t^2}{2}+st\rho}$$

Today
1. review concept test from last time
2. Review MGF
3. sec 6.5 Properties of Bivariate normal

1. A test score in Math and Physics is bivariate normal, $\rho > 0$. The average is 60 on both tests and the SDs are the same. Of students scoring 75 on the Math test:

a about half scored over 75 on Physics

b more than half scored over 75 on Physics

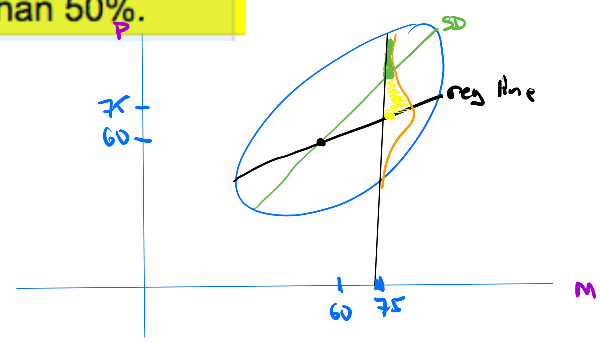
c less than half scored over 75 on Physics

discuss how you did this for 1 minute.

c	If rho=1, it would be exactly half

c	Due to the regression effect, those who scored 75 on math should expect to regress to the mean for physics since both tests are independent

c	The scores of students who scored 7 75 on math is normal centered on the regression line. 75 is above the regression line, so the proportion of students above 75 is less than 50%.



② Review MGF $M_X(t) = E(e^{tX})$

Main properties

ⓐ $M_{aX}(t) = M_X(at)$

ⓑ X, Y indep iff $M_{X+Y}(t) = M_X(t) M_Y(t)$

ⓒ $M_X'(0) = E(X)$

$M_X''(0) = E(X^2)$

$\vdots$

$M_X^{(k)}(0) = E(X^k)$

ⓓ $M_X(t)$ is unique for t in a neighborhood of 0. So if $M_X(t) = e^{t^2/2}$ for all t then $X \sim N(0,1)$

Multivariate MGF

$M_{(X,Y)}(s,t) = E(e^{sX+tY})$ multivariate MGF

Note that

$M_{(X,Y)}(s,t) = E(e^{sX+tY}) = M_{sX+tY}(1)$

X, Y indep iff $M_{(X,Y)}(s,t) = M_X(s) M_Y(t)$

③ Properties of Bivariate normal

Recall that if 2 RVs X, Y are independent, $Cov(X,Y)=0$

$\Rightarrow Corr(X,Y)=0$

However the converse isn't true in general. ($Corr(X,Y)=0 \not\Rightarrow X,Y$ indep.)

Let $X \sim N(0,1)$

$Y = X^2$

X and Y are dependent

we show X, Y are uncorrelated.

$$Cov(X,Y) = \underbrace{E(XY)}_{= E(X^3) = 0} - \underbrace{E(X)}_{=0}E(Y)$$

Find $E(X^3)$

$X \sim N(0,1)$

$M_X(t) = e^{t^2/2}$

$$M'''_X(t)\Big|_{t=0} = te^{t^2/2} + 2te^{t^2/2} + t^3e^{t^2/2}\Big|_{t=0} = 0$$

$\Rightarrow E(X^3) = 0$

$\Rightarrow Corr(X,Y) = 0$

Thm If (X,Y) is std bivariate normal then

$\rho = \text{Corr}(X,Y) = 0$ iff X, Y are independent.

Pf/

$$M_{(X,Y)}(s,t) = M_{sX+tY}(1) = M_{sX}(1) \cdot M_{tY}(1) = M_X(s) M_Y(t)$$

iff X and Y are independent.

Since (X,Y) is std bivariate normal,

$$M_{(X,Y)}(s,t) = e^{s^2/2 + t^2/2 + st\rho}$$

Finish proof.

$= M_X(s) M_Y(t)$ iff $\rho = 0$

since $M_X(s) = e^{s^2/2}$

$M_Y(t) = e^{t^2/2}$

$\square$

$\square$

Recall from lec 30 that the sum of independent normal random variables is normal.

What about dependent normal random variables?

Thm Let $X, Y \sim N(0,1)$ and $\text{corr}(X,Y) = \rho$.
(X,Y) is std bivariate normal iff
$sX + tY$ is normal for all constants s, t.

Pf/

$\Rightarrow$: Suppose X, Y std bivariate normal.
(i.e. $Y = \rho X + \sqrt{1-\rho^2}\, Z$ for X, Z iid $N(0,1)$)

$$sX + tY = sX + t(\rho X + \sqrt{1-\rho^2}\, Z)$$
$$= (s + t\rho)X + t\sqrt{1-\rho^2}\, Z$$

is normal since X, Z are independent normals.

$\Leftarrow$: Suppose $sX + tY$ is normal.

note $E(sX+tY) = sE(X) + tE(Y) = 0$

$$Var(sX+tY) = Var(sX) + Var(tY) + \underbrace{2Cov(sX,tY)}_{= 2st\,Cov(X,Y) = 2st\,corr(X,Y) = 2st\rho}$$
$$= s^2 + t^2 + 2st\rho$$

hence, $sX + tY \sim N(0, s^2 + t^2 + 2st\rho)$

Recall,
For $Z \sim N(\mu, \sigma^2) \Rightarrow M_Z(a) = e^{\mu a} e^{\sigma^2 \frac{a^2}{2}}$ σ^2

Then $M_{(X,Y)}(s,t) = M_{sX+tY}(1) = e^{(s^2+t^2+2st\rho)\frac{1^2}{2}}$

normal

$$= e^{\frac{s^2}{2} + \frac{t^2}{2} + st\rho}$$ MGF of (X,Y)

$\Rightarrow (X,Y)$ is standard bivariate normal $\square$

We dont need to restrict ourselves to $X, Y \sim N(0,1)$

Corollary — Proof at end of lecture

Let $U \sim N(\mu_U, \sigma_U^2)$
$V \sim N(\mu_V, \sigma_V^2)$
and $\text{Corr}(U,V) = \rho$

$(U,V) \sim BV(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$ iff
$sU + tV$ is normal for all constants s, t.

tinyurl: http://tinyurl.com/may3-part1

Stat 134

Friday May 3 2019

1. Let (X,Y) be bivariate normal. Then (2X+3Y+4, 6X-Y-4) is bivariate normal.

   a true

   b false

   c not enough info to decide

$a(2x+3y+4) + b(6x-y-4)$

$= (2a+6b)x + (3a-4b)y + 4a-4b$

is normal since (x,y) bivariate normal using corollary above.

ex Let M = a students score on the midterm
F = a students score on the final

$$(M,F) \sim BV(70, 65, 8^2, 10^2, .6)$$

Find the chance that a student scores higher on the final than the midterm?

Hint $P(F>M) = P(F-M>0)$

what distribution is $F-M$?

normal since (M,F) is bivariate normal!

$$P\left(\frac{F-M - E(F-M)}{SD(F-M)} > \frac{0 - E(F-M)}{SD(F-M)}\right)$$

$$E(F-M) = 65 - 70 = \boxed{-5}$$

$$Var(F-M) = Var(F) + Var(M) - 2\underbrace{Cov(F,M)}_{= Corr(F,M)SD(M)SD(F)}$$

$$= 10^2 + 8^2 - 2(.6)(8)(10)$$

$$SD(F-M) = \boxed{\sqrt{68}}$$

$$\Rightarrow \frac{0 - E(F-M)}{SD(F-M)} = \frac{0-(-5)}{\sqrt{66}} = .61$$

$$\Rightarrow P(F>M) = 1 - \Phi(.61) = .27$$

Appendix

Corollary

Let $U \sim N(\mu_U, \sigma_U^2)$
$V \sim N(\mu_V, \sigma_V^2)$
and $\mathrm{Corr}(U,V) = \rho$
$(U,V) \sim BV(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$ iff
$sU + tV$ is normal for all constants s,t.

Pf/ let $X = \dfrac{U-\mu_U}{\sigma_U}$ $(U = \mu_U + X\sigma_U)$

$Y = \dfrac{V-\mu_V}{\sigma_V}$ $(V = \mu_V + Y\sigma_V)$

$(U,V) \sim BV(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$
iff $(X,Y) \sim BV(0,0,1,1,\rho)$

$$sU + tY = s(\mu_U + X\sigma_U) + t(\mu_V + Y\sigma_V)$$
$$= \underbrace{s\sigma_U}_{=a} X + \underbrace{t\sigma_V}_{=b} Y + \underbrace{sx\sigma_U + ty\sigma_V}_{\text{constant}}$$

this is normal for all constants s,t
iff $aX + bY$ is normal for all constants a,b.

$\square$