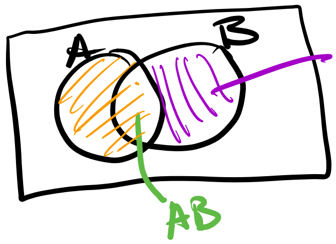
Stat 134 lec 2

Warm up

Prove the inclusion exclusion rule :



$B(AB)^c$ ← complement

$B(AB)^c$

$$P(A\cup B) = P(A) + P(B) - P(AB)$$

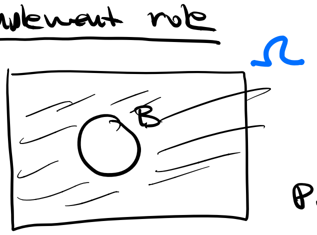
$A \cup B = A \cup B(AB)^c$ disjoint union

misprint in lecture (I wrote A^cB)

$$P(A\cup B) = P(A) + P(B(AB)^c)$$

$$= P(A) + \underbrace{P(B) - P(AB)}_{\text{difference rule.}}$$

Complement rule



$P(B^c) = 1 - P(B).$

Prove the complement rule

trick: $A = \Omega$

$B^c = \Omega B^c$

$$P(B^c) = P(\Omega B^c) = \underset{\overset{\shortparallel}{1}}{P(\Omega)} - P(B) = 1 - P(B).$$

Last time

(OR)

Addition rule if A, B mutually exclusive sets

$$P(A \text{ or } B) = P(A) + P(B).$$

(OR)

Inclusion exclusion $P(A \text{ or } B) = P(A) + P(B) - P(AB)$

Today

0. A little set theory
1. Sec 1.3 Distributions
2. Sec 1.4 Conditional Probability

0. A little set theory

Some easy facts about sets:

$$A_1 \cup A_2 \cup A_3 = (A_1 \cup A_2) \cup A_3 \quad \Rightarrow \quad \bigcup_{i=1}^{m+1} A_i = \left(\bigcup_{i=1}^{m} A_i\right) \cup A_{m+1}$$

$$A_1A_3 \cup A_2A_3 = (A_1 \cup A_2)A_3 \quad \Rightarrow \quad \bigcup_{i=1}^{m} (A_i A_{m+1}) = \left(\bigcup_{i=1}^{m} A_i\right) A_{m+1}$$

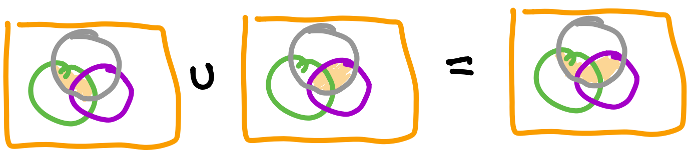
These are proven by induction. I sketch such a proof for the second one below:

Show

$$\bigcup_{i=1}^{m}(A_i A_{m+1}) = \left(\bigcup_{i=1}^{m} A_i\right)\overbrace{A_{m+1}}^{B}$$

base case $m=2$

$$A_1A_3 \cup A_2A_3 = (A_1 \cup A_2)A_3$$



assume true for $m \le 4$

$$\bigcup_{i=1}^{4}(A_iB) = \left(\bigcup_{i=1}^{4} A_i\right)\overbrace{B}^{A_{m+1}}$$

Show true for $m=5$

$$\bigcup_{i=1}^{5}(A_iB) = \underbrace{\bigcup_{i=1}^{4}(A_iB)}_{} \cup A_5B$$

$\parallel$ ← by induction $(m=4)$

$$\left(\bigcup_{i=1}^{4} A_i\right)B$$

$$= \left(\bigcup_{i=1}^{4} A_i \cup A_5\right)B$$

↑ by induction $(m=2)$

$$= \left(\bigcup_{i=1}^{5} A_i\right)B \quad \checkmark$$

Sec 1.3 Distributions

generalized inclusion exclusion:

see #12 p31

ex

Let $A_1, \dots, A_{n+1}$ be events.

Show $P\left(\bigcup_{i=1}^{n+1} A_i\right) = P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1}) - P\left(\bigcup_{i=1}^{n} (A_i A_{n+1})\right)$

Pf/ we prove by induction.

① base case is true, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 A_2)$ by inclusion exclusion rule.

② Assume true for union of $\leq n$ events and show true for union of $n+1$ events:

$$P\left(\bigcup_{i=1}^{n+1} A_i\right) = P\left(\underbrace{\bigcup_{i=1}^{n} A_i}\ \cup\ \underbrace{A_{n+1}}\right)$$

$$= P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1}) - P\left(\left(\bigcup_{i=1}^{n} A_i\right) A_{n+1}\right)$$

by inclusion exclusion rule

$$= P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1}) - P\left(\bigcup_{i=1}^{n} (A_i A_{n+1})\right)$$

by proof above

By the principle of mathematical induction we have shown the claim for all n. □

Named distribution

Uniform distribution on a finite set $\{x_1, \dots, x_n\}$:

Imagine you have numbers $x_1, \dots, x_n$ in a hat.

ex $\{1, 1, 2\}$ with x_1, x_2, x_3

Let X be a random draw of one of these numbers (i.e $P(X = x_i) = \frac{1}{n}$ for all i)

ex $P(X=1) = P(X = x_1 \text{ or } x_2) = \frac{2}{3}$, $P(X=2) = \frac{1}{3}$

We write that $X \sim \text{Unif}(\{x_1, \dots, x_n\})$

ex Suppose a word is randomly picked from this sentance.

7 1 4 2 8 6 4 4 8

What is the distribution of the length of the word picked?

$$\text{Unif}(\{7, 1, 4, 2, 8, 6, 4, 4, 8\})$$

$P(X=7) = 1/9$

$P(X=1) = 1/9$

$P(X=4) = 3/9$

$\vdots$

tinyurl.com/august28-pt1

Stat 134

1. A deck of cards is shuffled. What is the chance that the top card is the **king** of spades **or** the bottom card is the **king** of spades

   a $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \times \frac{1}{52}$
   b $\frac{1}{52} + \frac{1}{51}$
   c $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \times \frac{1}{51}$
   d none of the above

King of spades on top and bottom are mutually exclusive so have addition rule.

$\underbrace{P\begin{pmatrix}KS\\top\end{pmatrix}}$ + $P\begin{pmatrix}KS\\bottom\end{pmatrix}$

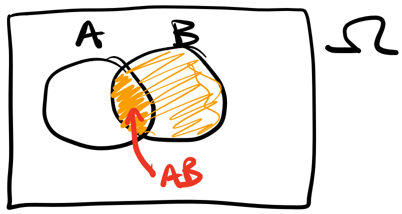
unconditional probability.

$= \boxed{\frac{1}{52} + \frac{1}{52}}$

You know to use unconditional probability because the addition rule always always involves unconditional probability.

Sec 1.4 Conditional Probability and Independence

Let A, B be subsets of Ω (i.e events).



Baye's rule says $P(A|B) = \frac{P(AB)}{P(B)}$ (given)

$\Leftrightarrow$ $\boxed{P(AB) = P(A|B)\,P(B)}$ multiplication rule. (AND)

A and B

We say A and B are <u>independent</u> if

$P(A|B) = P(A)$

or equivalently if $P(AB) = P(A)P(B)$

ex
A = last card is queen of spades
B = 1st card is king of spades

A and B are dependent

$P(AB) = P(B)P(A|B)$ or $P(A|B)P(B)$ $= \boxed{\frac{1}{52}\cdot\frac{1}{51}}$

$P(B) = \frac{1}{52}$, $P(A|B) = \frac{1}{51}$

$\neq P(B)\cdot P(A) = \frac{1}{52}\cdot\frac{1}{52}$

different hence A, B are dependent (i.e not independent)