Stat 134 lec 22

Warmup: 11:00–11:10

Recall $P(T>k) = e^{-\lambda k}$

Let $T \sim Exp(\lambda)$

a) Find $P(T>5)$ — $e^{-5\lambda}$

b) Find $P(T>\underset{8+5}{13} \mid T>8)$ — $\dfrac{P(T>13)}{P(T>8)} = \dfrac{e^{-13\lambda}}{e^{-8\lambda}} = e^{-5\lambda}$

This is the memoryless property of the exponential distribution.

last time

Sec 4.5 Cummulative Distribution Function (CDF)

The CDF of a RV X is $P(X \le x)$. The survival function is $P(X > x)$.

ex If $X \sim Geom(p)$, what is $P(X > x)$? $-q^x$

The CDF or survival function uniquely determine a distribution.

sec 4.2 exponential distribution.

$T \sim Exp(\lambda)$ $\quad f(t) = \begin{cases} \lambda e^{-\lambda t} & t \ge 0 \\ 0 & \text{else} \end{cases}$ or $P(T > t) = e^{-\lambda t}$ survival function

T = time until 1^{st} success (arrival)

ex

A family is getting ready for their trip to Yosemite. Each person is in their room, packing their bags. For each person, the time it takes them to pack their bag is exponentially distributed and independent of the time it takes any other person. On average, it takes each parent 1 hour and each child 2 hours to get ready. In a family with 2 parents and 4 children, what is the probability that it takes the family more than 2 hours to get ready?

$P_1, P_2, C_3, C_4, C_5, C_6$

$P_1, P_2 \sim Exp(1)$

$C_3, C_4, C_5, C_6 \sim Exp\left(\frac{1}{2}\right)$

$E(C) = 2 \Rightarrow C \sim Exp\left(\frac{1}{2}\right)$

$M = \max(P_1, P_2, C_3, C_4, C_5, C_6)$

$P(M > 2) = 1 - P(M \le 2)$

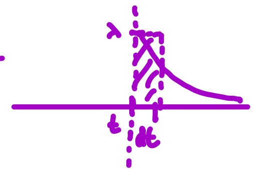
$$P(M \le 2) = P(P_1 \le 2) P(P_2 \le 2) P(C_3 \le 2) \cdots P(C_6 \le 2)$$

$$= \left(1 - e^{-1 \cdot 2}\right)^2 \cdot \left(1 - e^{-\frac{1}{2} \cdot 2}\right)^4$$

$$P(M > 2) = 1 - \left(1 - e^{-1 \cdot 2}\right)^2 \cdot \left(1 - e^{-\frac{1}{2} \cdot 2}\right)^4$$

Memoryless Property : $P(T \in dt \mid T > t) = P(0 < T < 0 + dt)$

Picture



$$P(T \in dt \mid T > t) = \frac{P(T \in dt)}{P(T > t)} \approx \frac{\lambda \cancel{e^{-\lambda t}}\, dt}{\cancel{e^{-\lambda t}}} = \boxed{\lambda\, dt}$$

Only 2 distributions are memoryless :

For discrete $(X = 1, 2, 3, \dots)$ – Geometric

For continuous $(T > 0)$ – Exponential

Today sec 4.2

1. Expectation and Variance of $\text{Exp}(\lambda)$.
2. Gamma distribution .

1

Sec 4.2 Expectation and Variance of Exp(λ)

$T \sim \text{Exp}(\lambda)$

$$E(T) = \int_0^\infty t f(t)\,dt = \lambda \int_0^\infty t e^{-\lambda t}\,dt = \boxed{\frac{1}{\lambda}}$$ see end of lec notes.

$$E(T^2) = \int_0^\infty t^2 f(t)\,dt = \lambda \int_0^\infty t^2 e^{-\lambda t}\,dt = \frac{2}{\lambda^2}$$ see end of lec notes.

$$\text{var}(T) = E(T^2) - E(T)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

Interpretation of λ:

$\lambda = \frac{1}{E(T)}$ (one over average arrival time)

This makes sense since the instantaneous rate of success is constant by the memoryless property of Exp(λ).

So if it takes on avg 5 min to have a success then $\lambda = 1/5$.

ex Let T = waiting time (min) until a blue Honda enters a toll gate starting at noon. The waiting time is on average 2 minutes. Find $P(T > 3)$

$E(T) = 2 \Rightarrow \lambda = 1/2$

$T \sim \text{Exp}(1/2)$

$$P(T > 3) = e^{-\frac{1}{2}\cdot 3} = \boxed{e^{-3/2}}$$

② Gamma Distribution

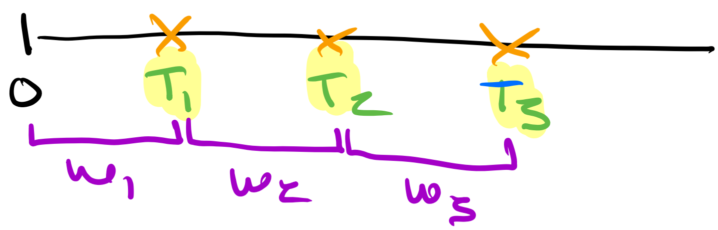
A gamma distribution, $T_r \sim \text{Gamma}(r, \lambda)$, $\lambda > 0$, is a sum of r iid $\text{Exp}(\lambda)$.

$$T_r = W_1 + W_2 + \dots + W_r, \quad W_i \sim \text{Exp}(\lambda)$$

ex T_r = time of the r^{th} arrival of a Poisson Process $(\text{Pois}(\lambda t))$

ex T_r = time when the r^{th} lightbulb dies.

Picture



$$\left.\begin{aligned} T_1 &\sim \text{Gamma}(1, \lambda) \\ T_2 &\sim \text{Gamma}(2, \lambda) \\ T_3 &\sim \text{Gamma}(3, \lambda) \end{aligned}\right\}$$

dependent but W_1, W_2, W_3 are independent, $\text{Exp}(\lambda)$

$$T_1 = W_1$$
$$T_2 = W_1 + W_2$$
$$T_3 = W_1 + W_2 + W_3$$

Review Poisson Process (a.k.a PRS)

$N_t \sim Pois(\lambda t)$

N_t = # arrivals in time t where the rate of arrivals is λ.

$$P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

ex Let T_1 = waiting time (min) until a blue Honda enters a toll gate starting at noon. The waiting time is on average 2 minutes. Find the probability 3 blue Hondas arrive in 8 minutes.

$N_8 \sim Pois\left(\frac{1}{2} \cdot 8\right) = Pois(4)$

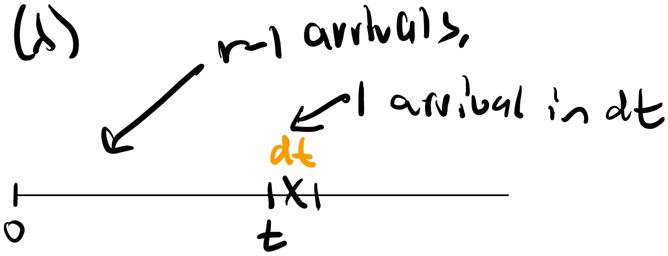
$$P(N_8 = 3) = \boxed{\frac{e^{-4} 4^3}{3!}}$$

Back to gamma distribution: $T_r \sim \text{Gamma}(r, \lambda)$

Let $N_t \sim \text{Pois}(\lambda t)$

$W \sim \text{Exp}(\lambda)$

$P(T_r \in dt)$ means



$= f(t)\,dt$ where $f(t)$ is the density of T_r

so $P(T_r \in dt) = P(N_t = r-1)\,P(W \in dt \mid W > t)$

$$\approx \frac{e^{-\lambda t} (\lambda t)^{r-1}}{(r-1)!} \cdot \lambda\, dt$$

$$= \underbrace{\frac{1}{(r-1)!} \lambda^r t^{r-1} e^{-\lambda t}}_{f(t)}\, dt$$

Gamma function $\Gamma(r)$, $r > 0$

If $r \in \mathbb{Z}^+$ define $\Gamma(r) = (r-1)!$ where $\Gamma(r)$ is called the **gamma function**

For $\lambda = 1$, $\quad f(t) = \underbrace{\frac{1}{\Gamma(r)}}_{\text{const}} \underbrace{t^{r-1} e^{-t}}_{\text{variable}}$

and $\int_0^\infty f(t)\,dt = 1 \Rightarrow$

$$\frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-t}\, dt = 1$$

$$\Rightarrow \Gamma(r) = \int_0^\infty t^{r-1} e^{-t}\, dt$$

This allows us to define

$T \sim \text{Gamma}(r, \lambda)$ for $r > 0$ as

having density $f(t) = \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}$

$t \geq 0$

ex Let $T_r \sim \text{Gamma}(r=4, \lambda=2)$

Find $f(t)$

$$f(t) = \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}$$

$$= \frac{1}{3!} 2^4 t^3 e^{-2t} = \boxed{\frac{8}{3} t^3 e^{-2t}}$$

The formula $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} \, dt$.

is the definition of the gamma function.

You can show that $\Gamma(r) = (r-1)!$

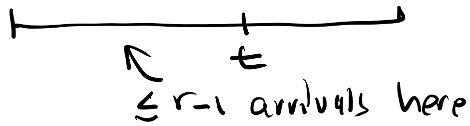
for $r \in \mathbb{Z}^+$

Now, $T_r \sim \text{Gamma}(r, \lambda)$ has density

$$f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}, & t \geq 0 \\ 0, & \text{else} \end{cases}$$

Let $T_r \sim \text{Gamma}(r, \lambda)$

$$P(T_r > t) = \int_t^\infty \frac{1}{\Gamma(r)} \lambda^r s^{r-1} e^{-\lambda s}\, ds$$



$N_t \sim \text{Pois}(\lambda t)$

$$P(T_r > t) = P(N_t < r)$$

$$= \sum_{i=0}^{r-1} P(N_t = i)$$

$$= \boxed{\sum_{i=0}^{r-1} \frac{e^{-\lambda t}(\lambda t)^i}{i!}}$$

ex let $T_4 \sim \text{Gamma}(r=4, \lambda=2)$

Find $P(T_4 > 7)$

$$P(T_4 > 7) = P(N_7 < 4) = P(N_7 \leq 3)$$

$$N_7 \sim \text{Pois}(\underbrace{2 \cdot 7}_{14})$$

$$= \boxed{e^{-14}\left(1 + \frac{14}{1!} + \frac{14^2}{2!} + \frac{14^3}{3!}\right)}$$

ex

Suppose customers are arriving at a ticket booth at rate of five per minute, according to a Poisson arrival process. Find the probability that:

λ

(a) Starting from time 0, the $9th$ customer doesn't arrive within 5 minutes;

t

$$P(T_9 > 5) = P(N_5 \le 9) \quad \le 8$$

$$N_5 \sim \text{Pois}(\lambda t) \quad 25$$

$$\boxed{\sum_{k=0}^{8} \frac{e^{-25} 25^k}{k!}}$$

Appendix

Expectation and Variance of exponential

let $T \sim Exp(\lambda)$

$f(t) = \lambda e^{-\lambda t}$ density

$$E(T) = \lambda \int_0^\infty t e^{-\lambda t} dt$$

I recommend using the tabular method for integration by parts. This works well when the function you are integrating is the product of two expressions, where the n^{th} derivative of one expression is zero.

ex $t^4 e^{3t}$ ✓ good

ex $\sin t \, e^{3t}$ ✗ bad.

To find $\lambda \int_0^\infty t e^{-3t}$:

$\frac{d}{dt}$		$\int$
t	$+$	$e^{-\lambda t}$
1	$-$	$\frac{-e^{-\lambda t}}{\lambda}$
0		$\frac{e^{-\lambda t}}{\lambda^2}$

$$E(T) = \lambda \int_0^\infty t e^{-\lambda t} dt = \lambda \left(t \left(-\frac{e^{-\lambda t}}{\lambda} \right) - 1 \cdot \frac{e^{-\lambda t}}{\lambda^2} \right) \Bigg|_0^\infty$$

$$= 0 - \left(0 - \frac{1}{\lambda} \right)$$

$$= \boxed{\frac{1}{\lambda}}$$

Next find $Var(T) = E(T^2) - E(T)^2$:

$$E(T^2) = \lambda \int_0^\infty t^2 e^{-\lambda t} dt$$

$\frac{d}{dt}$		$\int$
t^2	$+$	$e^{-\lambda t}$
$2t$	$-$	$-\frac{1}{\lambda} e^{-\lambda t}$
2	$+$	$\frac{1}{\lambda^2} e^{-\lambda t}$
0		$-\frac{1}{\lambda^3} e^{-\lambda t}$

$$E(T^2) = \lambda \left(t^2 \left(-\frac{1}{\lambda} e^{-\lambda t} \right) - 2t \left(\frac{1}{\lambda^2} e^{-\lambda t} \right) + 2 \left(-\frac{1}{\lambda^3} e^{-\lambda t} \right) \right) \Bigg|_0^\infty$$

$$= \frac{2}{\lambda^2}$$

$$\Rightarrow Var(T) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda} \right)^2 = \boxed{\frac{1}{\lambda^2}}$$