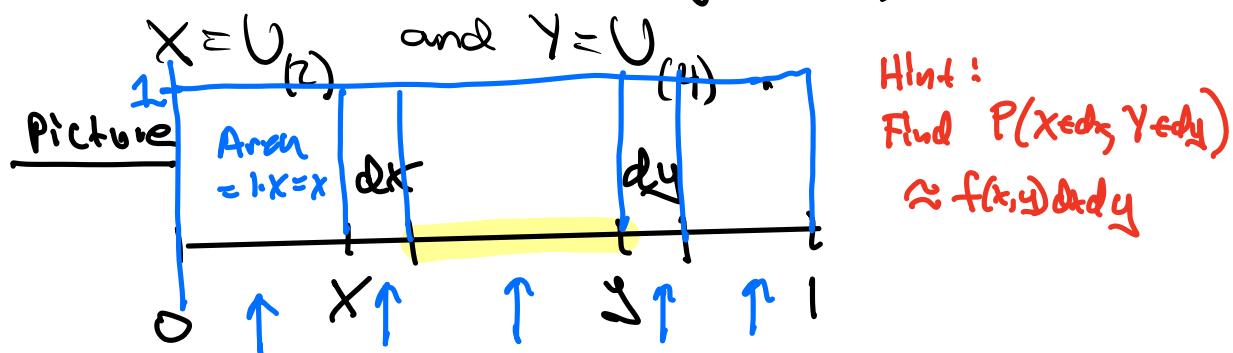


Stat 134 Sec 28

warmup 11:00-11:10

Ex Throw down 5 darts on $(0,1)$.

Find the joint density, $f(x,y)$, of



$$P(X \in dx, Y \in dy) = \binom{5}{1} \cdot x \cdot \binom{4}{1} dx \binom{3}{1} (y-x) \binom{2}{1} dy \binom{1}{1} (1-y)$$

$$\underbrace{f(x,y)dx dy}_{\substack{\approx \\ f(x,y)dx dy}} = \underbrace{\binom{5}{1,1,1,1,1}}_{f(x,y)} \times (y-x)(1-y)dx dy$$

$f(x,y)$ for $0 < x, y < 1$

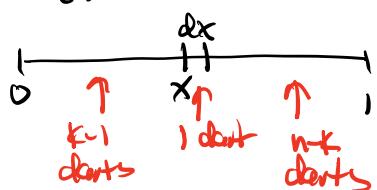
Last time

Sec 4.6 Uniform order statistic

$U_1, \dots, U_n \stackrel{iid}{\sim} U(0, 1)$

$U_{(1)}, \dots, U_{(n)}$ order statistics

$$P(U_{(k)} \in dx) = f(x)dx$$



$$\begin{aligned} \text{Note} \\ \binom{n}{a,b,c} &= \frac{n!}{a!b!c!} \\ &= \binom{n}{a} \binom{n-a}{b} \binom{n-a-b}{c} \end{aligned}$$

$$f_{U_{(k)}}(x) = \binom{n}{k-1, 1, n-k} x^{k-1} (1-x)^{(n-k+1)-1} \quad \text{on } 0 < x < 1$$

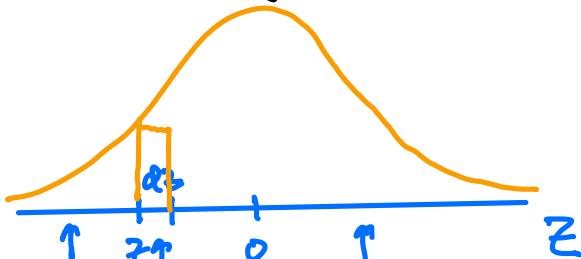
Normal order statistic

ex Let $Z_{(1)}, \dots, Z_{(10)}$ be the values of 10

independent standard normal variables arranged in increasing order. Find the density of $Z_{(4)}$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$\Phi(z) = P(Z \leq z)$$



$$P(Z_{(4)} \leq z) = 1$$

$$f(z)dz = \binom{10}{3,1,6} (\Phi(z))^3 \phi(z)dz \quad \Phi(z)^6$$

$$\Rightarrow f(z) = \binom{10}{3,1,6} (\Phi(z))^3 \phi(z) / (1 - \Phi(z))$$

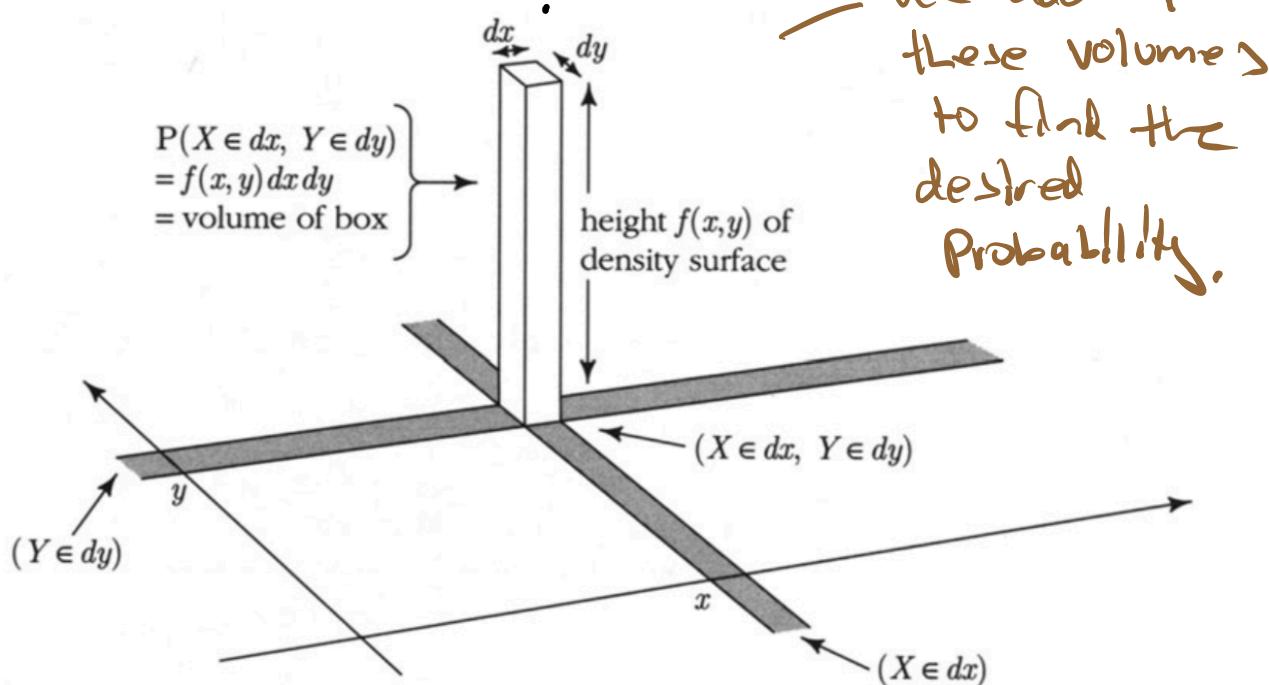
today ① Sec 5.1, 5.2 Continuous Joint Distribution

② Sec 4.6 Beta Distribution

③ Sec 5.1, 5.2 Calculate probabilities with $f(x, y)$.

① Sec 5.1, 5.2 Joint Density

$$P(X \in dx, Y \in dy) \approx f(x, y) dx dy$$

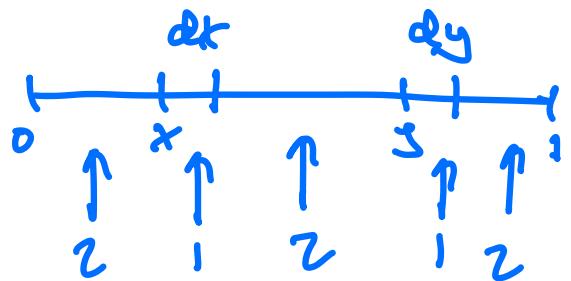


$$\iint_{\text{xy}} f(x, y) dx dy = \iint_{\text{xy}} f(x, y) dy dx = 1$$

Stat 134

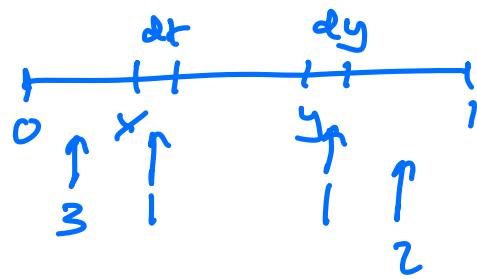
1. I throw down 8 darts on $(0, 1)$. The variable part of the joint density of $X = U_{(3)}$ and $Y = U_{(6)}$ is:

- a $x(y - x)^5(1 - y)^2$
- b $x^2(y - x)^2(1 - y)^2$
- c $x^4(y - x)^2(1 - y)^2$
- d none of the above



Let (X, Y) have joint density $f_{X,Y}(x, y) = 420x^3(1-y)^2$ for $0 < x < y < 1$.

Fill in the blanks: X and Y represent the 4th smallest and 5th smallest of 7 i.i.d. Unif (0,1) random variables, respectively.



② Sec 4.6 Beta distribution.

$X \sim \text{Beta}(r, s)$ for $r > 0, s > 0$ is a distribution often used to model physical processes that take values between 0 and 1,
ex: the proportion of defective items in a shipment.

Defⁿ Let $r, s > 0$
 $X \sim \text{Beta}(r, s)$ if

$$f(x) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} x^{r-1} (1-x)^{s-1} \quad \text{for } 0 < x < 1,$$

where $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$ Gamma function for $r > 0$
or $\Gamma(r) = (r-1)! \quad r \in \mathbb{Z}^+$

Notice if $r=1, s=1$, $f(x) = 1_{x \in (0,1)}$
 $\Rightarrow \text{Beta}(1, 1) = \text{Unif}(0,1)$.

Ex Let $U_1, \dots, U_n \stackrel{iid}{\sim} U(0,1)$
 $f_{U_k}(x) = \binom{n}{k-1, n-k} x^{k-1} (1-x)^{(n-k+1)-1}$ on $0 < x < 1$

Compare with,

$$f_{\text{Beta}(r,s)}(x) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} x^{r-1} (1-x)^{s-1} \quad \text{for } 0 < x < 1.$$

Notice that $f_{U_{(k)}}(x)$ and $f_{\text{Beta}(s,s)}(x)$ have the same variable part of their density when $r = k$

$$s = n - k + 1$$

then $\Gamma(s+r) = \Gamma(n-k+1+k) = \Gamma(n+1) = n!$

$$\Gamma(r) = (k-1)!$$

$$\Gamma(s) = (n-k)!$$

$$\Rightarrow \frac{\Gamma(s+r)}{\Gamma(r)\Gamma(s)} = \binom{n}{k-1, 1, n-k}$$

\Rightarrow Standard uniform ordered statistics are beta!

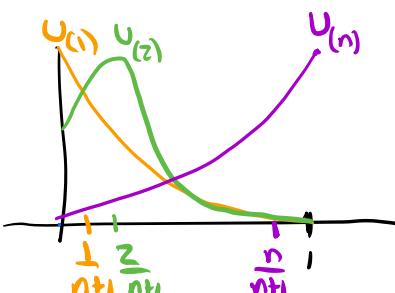
Thm *see appendicit notes* $X \sim \text{Beta}(r, s)$

$$E(X) = \frac{r}{r+s}$$

Hence if $X \sim U_{(k)}$

$$E(X) = \frac{k}{n-k+1+k} = \boxed{\frac{k}{n+1}}$$

$$\begin{aligned} E(U_{(1)}) &= \frac{1}{n+1} \\ E(U_{(2)}) &= \frac{2}{n+1} \\ &\vdots \\ E(U_{(n)}) &= \frac{n}{n+1} \end{aligned}$$



Ex (Bayesian Statistics)

Let P be the chance a coin lands head. Suppose the prior distribution of P is

$$f_p(p) = \begin{cases} C(1-p)^4 & \text{for } 0 \leq p \leq 1 \\ 0 & \text{else} \end{cases}$$

a) Is this a beta distribution? If so,

what are the parameters?

$$\text{Yes, compare } (1-p)^4 \text{ and } p^{r-1}(1-p)^{s-1}$$

$$\Rightarrow P \sim \text{Beta}(1, 5)$$

b) Calculate the constant C

$$C = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} = \frac{\Gamma(6)}{\Gamma(1)\Gamma(5)} = \frac{5!}{0!4!} = 5$$

c) What is the mean of P ?

$$E(P) = \frac{r}{r+s} = \frac{1}{1+4} = \frac{1}{5}$$

If $X \sim \text{Beta}(r, s)$

$$f(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1} (1-x)^{s-1}$$

Since $\int_0^1 f(x) dx = 1 \Rightarrow \int_0^1 x^{r-1} (1-x)^{s-1} dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$

ex Let $X \sim \text{Beta}(3, 4)$

Compute $E(7x - 5x^6)$

$$E(7x - 5x^6) = 7E(x) - 5E(x^6)$$

$$E(x^6) = \int_0^1 x^6 \underbrace{\frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} x^2 (1-x)^3}_{f(x)} dx$$

$$= \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} \int_0^1 x^8 (1-x)^3 dx$$

$$= \frac{\Gamma(9)\Gamma(4)}{\Gamma(13)} = \frac{\Gamma(9)\Gamma(7)}{\Gamma(13)}$$

$$= \frac{\Gamma(9)\Gamma(7)}{\Gamma(3)\Gamma(13)} \Rightarrow E(7x - 5x^6) = 3 - 5 \frac{\Gamma(9)\Gamma(7)}{\Gamma(3)\Gamma(13)}$$

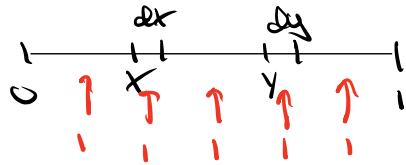
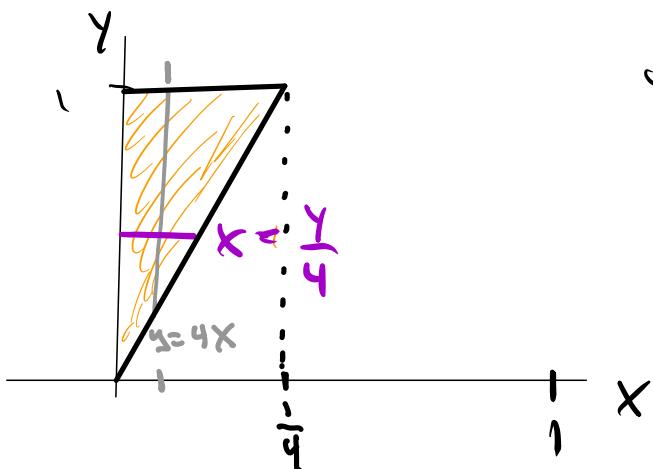
③ See 5.1, 5.2 Calculate probabilities with $f(x, y)$.

Throw down 5 darts on $(0, 1)$.

$$\text{ex } X = U(2) \quad Y = U(4)$$

$$\text{Find } P(Y > 4X)$$

recall,



$$f(x, y) = \binom{5}{1, 1, 1, 1} \times (y-x)(1-y)$$

$$\Leftarrow 5! \times (y-x)(1-y)$$

$$P(Y > 4X) = \int_{y=0}^{y=1} \int_{x=\frac{y}{4}}^{x=y} 5! \times (y-x)(1-y) dx dy$$

(or)

$$P(Y > 4X) = \int_{x=0}^{x=\frac{y}{4}} \int_{y=x}^{y=1} 5! \times (y-x)(1-y) dy dx$$

details:

$$\begin{aligned} P(Y > 4x) &= \int_{y=0}^{y=1} \int_{x=0}^{x=y/4} 120 \times (y-x)(1-y) dx dy \\ &= \int_{y=0}^{y=1} 120(1-y) \int_{x=0}^{x=y/4} (xy - x^2) dx dy \\ &= \int_{y=0}^{y=1} 120(1-y) \left[\frac{x^2 y}{2} - \frac{x^3}{3} \right] \Big|_{x=0}^{x=y/4} dy \\ &= \int_{y=0}^{y=1} 120(1-y) \left(\frac{y^3}{32} - \frac{y^3}{3 \cdot 64} \right) dy \\ &\quad \frac{6y^3 - y^3}{3 \cdot 64} = \frac{5y^3}{192} \\ &= \frac{5}{192} \cdot 120 \int_{y=0}^{y=1} y^3 - y^4 dy \\ &= \frac{5}{192} \cdot 120 \int_0^1 y^3 - y^4 dy = \frac{5 \cdot 120 \left(\frac{y^4}{4} - \frac{y^5}{5} \right)}{192} \Big|_0^1 \\ &= \frac{5 \cdot 120}{192} \left(\frac{1}{4} - \frac{1}{5} \right) = \frac{30}{192} = \textcircled{156} \end{aligned}$$

Appendix

Let $X \sim \text{Beta}(r, s)$

then $E(X) = \frac{r}{r+s}$,

Pf/ Note that $\int_0^1 f(x) dx = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 x^{r-1} (1-x)^{s-1} dx = 1$

$$\Rightarrow \int_0^1 x^{r-1} (1-x)^{s-1} dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$$

$$E(X) = \int_0^1 x f(x) dx = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 x x^{r-1} (1-x)^{s-1} dx$$

$$\frac{\cancel{\Gamma(s)\Gamma(r+1)}}{\Gamma(s+r+1)}$$

$$= \frac{(r+s-1)! \cdot r!}{(s+r)!} = \boxed{\frac{r}{r+s}}$$

□