Stat 134 lec 16

Warmup: 1:00-1:10

Matthew and two friends each have a p-coin and toss it independently at the same time.

a) What is the probability it takes Matthew more than n tosses to get heads?

$X = \#$ tosses till get heads

$P(X > n) = q^n$

b) What is the probability that the first person to get a head has to toss more than n times.

$P(\min(X_1, X_2, X_3) > n) = P(X_1 > n, X_2 > n, X_3 > n)$

$= (q^n)^3 = \boxed{q^{3n}}$

last time

Announcement:
next Friday! "in class" 50 minute midterm
review monday, wednesday

Sec 3.6 $X \sim Bin(n,p) \Rightarrow Var(X) = np(1-p)$

correction factor

$X \sim HG(n,N,G) \Rightarrow Var(X) = np(1-p)\left(\frac{N-n}{N-1}\right)$ where $p = \frac{G}{N}$

Sec 3.4 Geometric distribution $(Geom(p))$

X = # p-coin tosses until the first heads

takes values $1,2,3,\dots$

$P(X=k) = q^{k-1}p$

$E(X) = \frac{1}{p}$

$Var(X) = \frac{q}{p^2}$

Negative Binomial distribution $(NegBin(r,p))$

$T_r = W_1 + \dots + W_r$ where $W_1, W_2, \dots W_r \overset{iid}{\sim} Geom(p)$

$P(T_r = k) = \binom{k-1}{r-1} p^r q^{k-r}$

$E(T_r) = \frac{r}{p}$

$Var(T_r) = \frac{rq}{p^2}$

Today

1. Finish Sec 3.4 Minimum of independent geometrics
2. Poisson distribution
3. Poisson random scatter (PRS) AKA Poisson Process
4. Poisson thinning

① Sec 3.4 Minimum of independent geometrics

Adam, Beth and John independently flip a p_1, p_2, p_3 coin respectively.
let X = # trials until Adam, Beth or John get a heads.

eg A TTT $\quad X_1 \sim \text{Geom}(p_1)$

B TTT $\quad X_2 \sim \text{Geom}(p_2)$

J $\underbrace{TTH}_{X=3}$ $\quad X_3 \sim \text{Geom}(p_3)$

a) what is probability Adam, Beth or John get a head?

$P = \text{Prob}(A \text{ or } B \text{ or } J \text{ get heads})$

$= 1 - \text{Prob}(A, B, J \text{ dont get heads})$

$= \boxed{1 - q_1 q_2 q_3}$

b) what distribution is X?

$X = \min(X_1, X_2, X_3) \sim \boxed{\text{Geom}(1 - q_1 q_2 q_3)}$

② Sec 3.5 Poisson distribution ($Pois(\mu)$)

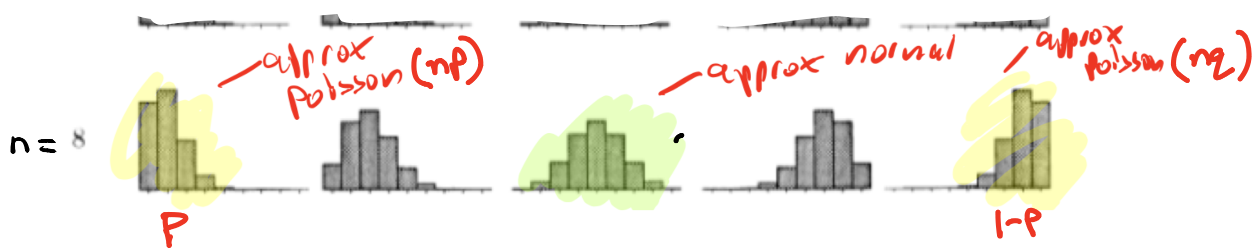
$X \sim Pois(\mu)$

$$P(X=k) = e^{-\mu}\frac{\mu^k}{k!} \quad k=0,1,2,\ldots$$

Intuitively, we know $E(X)=\mu$ and $Var(X)=\mu$

since,

$$Bin(n,p) \to Pois(\mu) \text{ when } \begin{array}{l} n\to\infty \\ p\to 0 \\ np\to\mu. \end{array}$$



Also we expect $npq \to \mu q \approx \mu$ so $var(X)$ should be μ. See appendix for a proof.

ex Let $X \sim Pois(\mu)$

Find $E(X(X+1)) = E(X^2+X) = E(X^2) + E(X)$

$E(X^2) = var(X) + (E(X))^2$

$= var(X) + (E(X))^2 + E(X)$

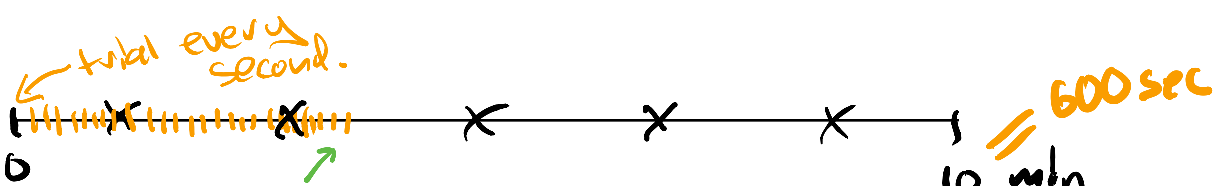
$= \mu + \mu^2 + \mu = \boxed{2\mu + \mu^2}$

③ Poisson Random Scatter (PRS)

A random scatter of points in a time line is an example of a Poisson random scatter,

ex X = number of calls coming into a hotel reservation center in 10 minutes.

Say on average there are $\mu = 5$ calls in 10 minutes



The distribution of calls should look random not clustered since we have independent trials w/ same p

PRS assumptions

1) No time interval gets more than one call

2) Have n iid Bernoulli p trials with $\mu = np$ large n, small p.
(i.e all calls are independent of each other with the same probability)

As $n \to \infty$ and $p \to 0$ and $np \to \mu$

let $X =$ # calls in $\underbrace{t \text{ seconds}}_{\text{time of } n \text{ trials}}$, ($t=600$ in this example.)

Then $X \sim Poi(\mu) \leftarrow$ limit of $Bin(n,p)$ as $n \to \infty$, $p \to 0$, $np \to \mu$.

Let λ be the rate (or intensity) of calls per second

ex $\lambda = 5/600$ calls/min in above example.

λ has units calls/min so λt has units calls in t min.

Since λ is the same every minute (PRS assumptions) $\mu = \lambda t$.

ex $\mu = \lambda t = \frac{5}{600} \cdot 600 = 5$ calls in 600 sec.

tinyurl.com/oct2-pt1a

Stat 134

1. Which of the following can be modeled as a Poisson Random Scatter with intensity $\lambda > 0$?

   a The number of blueberries in a 3 cubic inch blueberry muffin

   ~~**b**~~ The number of patients entering a doctor's office in a 24 hour period.

   different intensity λ different hours.

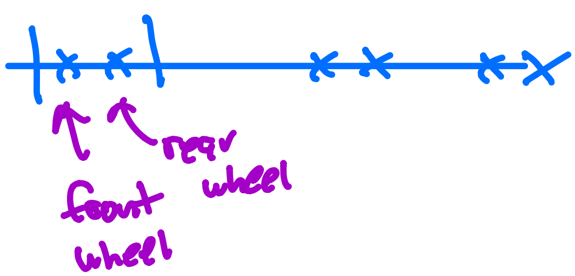
   ~~**c**~~ The number of times a day a person feels hungry

   trials not indep.

   ~~**d**~~ The number of air pulses counted every second from cars driving over an empty rubber hose lying across a highway between noon and 1pm.

   not indep trials.

   e more than one of the above



$\underline{\underline{ex}}$ (number of bb in a bb muffin)

A tub of bb muffin batter has

$\lambda = 2\ bb/in^3$.

A muffin is $3\ in^3$.

On average how many bb are there per muffin?

$\mu = 2\ bb/in^3 \cdot 3\ in^3 = 6\ bb$

Let $X_1 = \#$ bb in 1^{st} muffin

$X_1 \sim Pois(6)$

Another muffin is $4\ in^3$ (from the same batter)

let $X_2 = \#$ bb in 2^{nd} muffin.

a) Find P(5 bb in each muffin)

$X_1 \sim Pois(6)$

$X_2 \sim Pois(8)$

$$P(X_1=5, X_2=5) = \boxed{\frac{e^{-6}\,6^5}{5!} \cdot \frac{e^{-8}\,8^5}{5!}}$$

b) Find P(10 bb total in both muffin together)

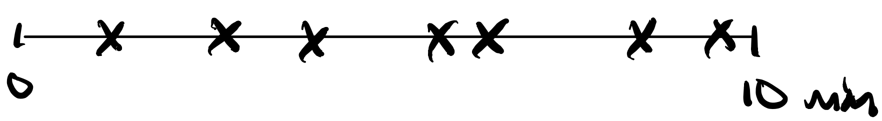
$X_1 + X_2 \sim Pois(14)$

$$P(X_1+X_2 = 10) = \boxed{\frac{e^{-14}\,14^{10}}{10!}}$$

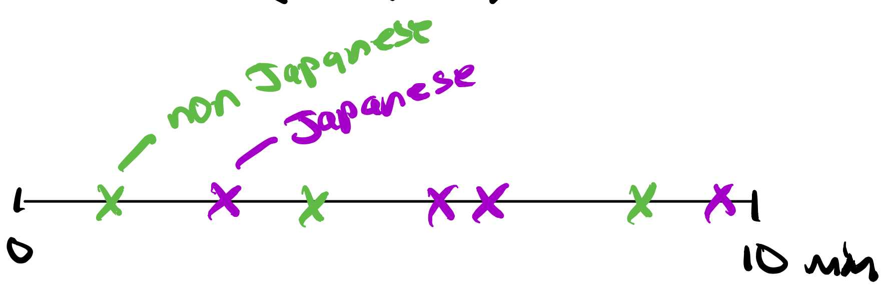
④ **Sec 3.5** **Poisson thinning**

Cars arrive at a toll booth according to a Poisson process at a rate $\lambda = 3$ arrivals/min

X = # cars arriving at a toll booth in 10 min. $X \sim \text{Pois}(\underbrace{\lambda \cdot 10}_{30})$



Of cars arriving, it is known, over the long term, that 60% are Japanese imports.



Call Japanese cars a success and non Japanese a failure.

cars $\sim \text{Pois}(\lambda \cdot 10) = \text{Pois}(30)$

The process of "success" hits in your PRS is a PRS with intensity λp

Japanese imports $\sim \text{Pois}(p\lambda \cdot 10) = \text{Pois}(18)$

non Japanese $\sim \text{Pois}(q\lambda \cdot 10) = \text{Pois}(12)$

Appendix

Let $X \sim \text{Pois}(\mu)$

Then $E(X) = \mu$ and

$Var(X) = \mu$

Pf/

Recall $e^{\mu} = 1 + \mu + \frac{\mu^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{\mu^k}{k!}$ Taylor series.

$$E(X) = \sum_{k=0}^{\infty} k \cdot P(X=k) = \sum_{k=0}^{\infty} k e^{-\mu} \frac{\mu^k}{k!}$$

$$= \sum_{k=1}^{\infty} \cancel{k} e^{-\mu} \frac{\mu^{k-1}\mu}{(k-1)!\cancel{k}}$$

(note $0 \cdot e^{-\mu} \frac{\mu^0}{0!} = 0$)

$$= \mu e^{-\mu} \sum_{k=1}^{\infty} \frac{\mu^{k-1}}{(k-1)!}$$

$$= \mu e^{-\mu} \underbrace{\left(1 + \mu + \frac{\mu^2}{2!} + \cdots\right)}_{e^{\mu}} = \boxed{\mu}$$

Next we show $Var(X) = \mu$:

$$Var(X) = E(X^2) - E(X)^2$$

$$= E(X^2) - E(X) + E(X) - E(X)^2$$

$$= E(X(X-1)) + E(X) - E(X)^2$$

$$E(X(X-1)) = \sum_{k=0}^{\infty} \cancel{k}(\cancel{k-1})\, P(X=k)$$

$$P(X=k) = \frac{e^{-\mu}\mu^{k}}{\cancel{k}(\cancel{k-1})(k-2)!}$$

$$= e^{-\mu} \sum_{k=2}^{\infty} \frac{\mu^{k}}{(k-2)!}$$

$$= e^{-\mu}\mu^{2} \underbrace{\sum_{k=2}^{\infty} \frac{\mu^{k-2}}{(k-2)!}}_{= e^{\mu}} = \mu^{2}$$

$$\Rightarrow \mathrm{Var}(X) = \mu^{2} + \mu - \mu^{2} = \boxed{\mu}$$

$\square$