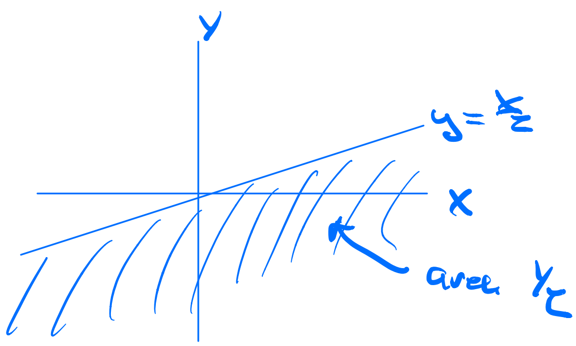
Stat 134 lec 32

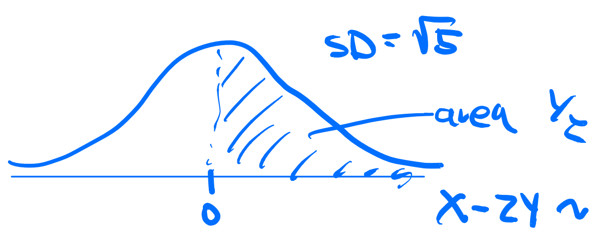
Warmup 11:00-11:10

Let $X, Y \overset{iid}{\sim} N(0,1)$

Find $P(X > 2Y)$

$\frac{1}{2}$





$X - 2Y \sim N(0, 5)$

__Last time__

__Sec 5.3__

A linear combination of independent normals is normal.

__Thm__ Let $\left.\begin{array}{l} X_1 \sim N(\mu_1, \sigma_1^2) \\ X_2 \sim N(\mu_2, \sigma_2^2) \end{array}\right\}$ indep.

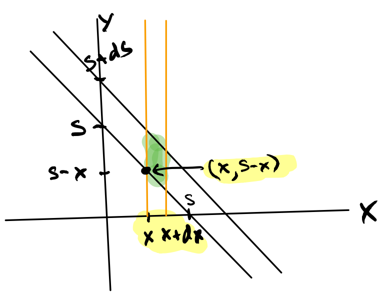
then $aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

__Note__ In Chapter 6 we will generalize this result and show that $aX_1 + bX_2$ is normal iff (X, Y) are bivariate normal

__Sec 5.4__ __Convolution formula for density of sum__

Let X, Y be RVs and $S = X + Y$.

if $X > 0, Y > 0$



$$P(S \in ds) = \int_{x=0}^{x=s} P(X \in dx, S \in ds)$$

$f_S(s)\,ds$; $f(x, s-x)\,dx\,ds$

if X, Y independent

$$f_S(s) = \int_{x=0}^{x=s} f(x, s-x)\,dx = \int_{x=0}^{x=s} f_X(x) f_Y(s-x)\,dx$$

Recall $Z \sim Beta(r, s) \Rightarrow f_Z(z) \propto z^{r-1}(1-z)^{s-1}$

Variable Part

__Today__

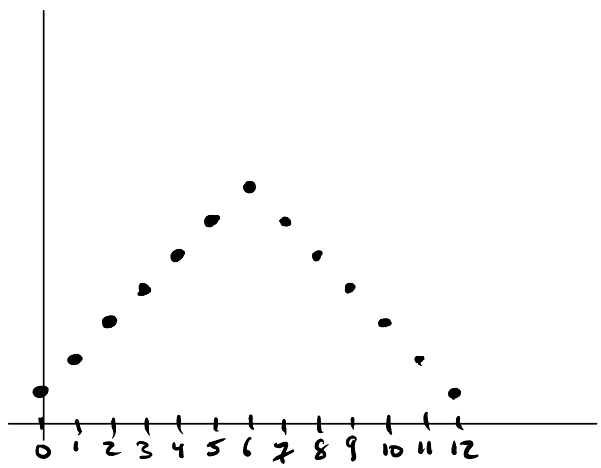
__Sec 5.4__

1. triangular density
2. Uniform spacing (see #13 p 355)
3. Convolution formula of ratio Y/X

① Sec 5.4 Triangular density

Let
$$\left.\begin{array}{l} X \sim \text{Unif}\{0,1,2,\ldots,6\} \\ Y \sim \text{Unif}\{0,1,2,\ldots,6\} \end{array}\right\} \text{indep.}$$

The distribution of $S = X + Y$ looks like



True ?

False ?

$$P(S=0) = P(0,0) = P(0)P(0) = \left(\tfrac{1}{7}\right)^2$$

$$P(S=6) = \underset{\substack{\parallel \\ P(0)P(6)}}{P(0,6)} + P(1,5) + \cdots + P(6,0) = \frac{7}{49} = 1/7$$

$$P(S=s) = \sum_{x=0}^{x=s} P(x)P(s-x) \quad , \quad 0 \le s \le 6$$

$$P(S=7) = P(1,6) + P(2,5) + \cdots + P(6,1) = 6/49$$

$$P(S=s) = \sum_{x=s-6}^{x=6} P(x)P(s-x) \qquad 6 \le s \le 12$$

Continuous case:

$$\left.\begin{array}{l} X \sim U(0,1) \\ Y \sim U(0,1) \end{array}\right\} \text{ indep}$$

Find density of $S = X + Y$

if X, Y independent

$$f_S(s) = \int_{x=0}^{x=s} f(x, s-x)\,dx = \int_{x=0}^{x=s} f_X(x) f_Y(s-x)\,dx$$

for $0 < s < 1$

$$f_S(s) = \int_{x=0}^{x=s} f_X(x) f_Y(s-x)\,dx = \int_0^s 1 \cdot 1\,dx = \boxed{s}$$

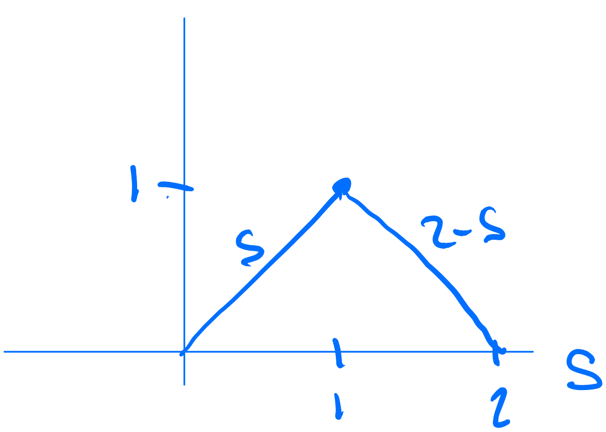
for $1 < s < 2$

$$f_S(s) = ?$$

$$f_S(s) = \int_{x=s-1}^{x=1} f_X(x) f_Y(s-x)\,dx = \int_{s-1}^{1} 1 \cdot 1\,dx$$

$$= 1 - (s-1)$$

$$= \boxed{2-s}$$



② Uniform Spacing

ex Let $X \sim U_{(7)}$, $Y \sim U_{(9)}$ for 10 iid $U(0,1)$.
The joint density $f_{X,Y}(x,y) = \binom{10}{6,1,1,1,1} x^6 (y-x)(1-y)$
for $0<x<y<1$.

Let $Z = Y - X$

$$f_S(s) = \int_{x=0}^{x=s} f(x, s-x)\,dx$$

a) For a fixed z, what is the largest value of X?

$\leftarrow$ what goes here?

$$f(z) = \int_{x=0}^{x=} f(x, x+z)\,dx$$

$z = y - x$

$x = y - z$ where $y \in [0,1]$

largest x has $y = 1$

$$\boxed{x = 1 - z}$$

So our convolution formula is

$$f_Z(z) = \int_{x=0}^{x=1-z} f(x, x+z)\,dx$$

ex Let $X \sim U_{(7)}$, $Y \sim U_{(9)}$ for 10 iid $U(0,1)$.

The joint density $f_{X,Y}(x,y) = \binom{10}{6,1,1,1,1} x^6 (y-x)(1-y)$

for $0<x<y<1$.

Find the density of $Z = Y - X$

what distribution is Z?

$$f_Z(z) = \int_0^{1-z} f(x, x+z)\,dx$$

let $c = \binom{10}{6,1,1,1,1}$

$$f_Z(z) = \int_0^{1-z} c x^6 (x+z-x)(1-(x+z))\,dx$$

$$= cz \int_0^{1-z} \left((1-z)x^6 - x^7\right) dx$$

$$= cz \left[(1-z)\frac{x^7}{7} - \frac{x^8}{8}\right]\Bigg|_{x=0}^{x=1-z}$$

$$= cz \left(\frac{(1-z)^8}{7} - \frac{(1-z)^8}{8}\right) = \frac{c}{56} z (1-z)^8$$

$\Rightarrow Z \sim Beta(2, 9)$

Interpretation:

$U_{(9)} - U_{(7)} \sim \text{Beta}(2,9)$ means that the distribution of the distance between the 9^{th} largest and 7^{th} largest dart out of 10 darts is Beta $(2,9)$.

Is there anything special about $U_{(9)} - U_{(7)}$.

What about $U_{(8)} - U_{(6)}$ or $U_{(3)} - U_{(1)}$ or $U_{(2)} - 0$ for that matter? You can check that all of these have distribution Beta$(2,9)$.

The example $U_{(2)} - 0$ is particularly easy since $U_{(k)} \sim \text{Beta}(k, n-k+1)$ so

$$U_{(2)} \sim \text{Beta}(2, \underbrace{10-2+1}_{=9}).$$

<u>More generally</u> (Uniform Spacing)

You randomly throw n dart at $[0,1]$.

For $0 < a < a+k \leq n$, $U_{(a+k)} - U_{(a)}$ is? $U_{(k)}$ out of n

$\sim \boxed{\text{Beta}(k, n-k+1)}$

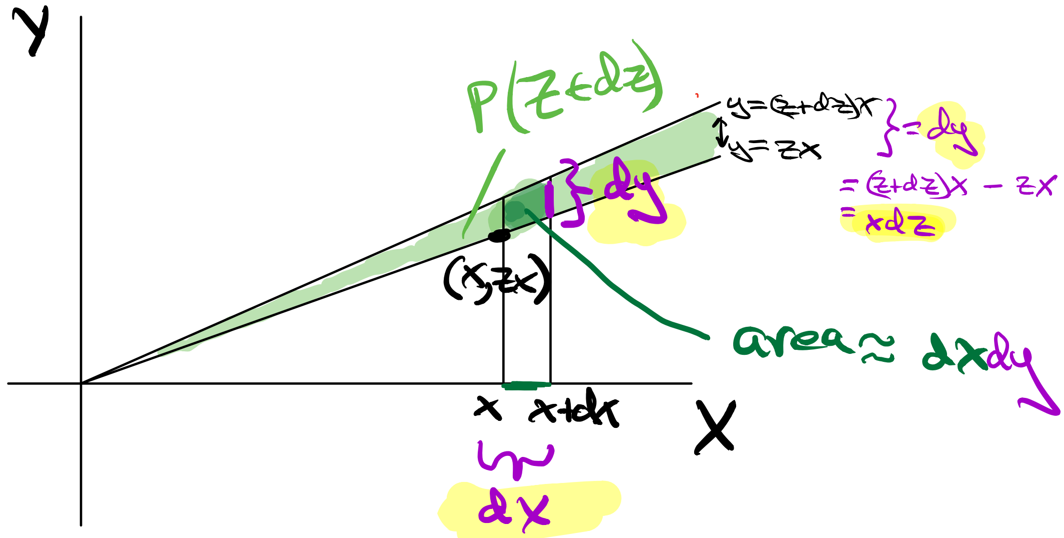
③ Convolution formula for density of ratio Y/X

$X > 0,\ Y > 0$

let $Z = \dfrac{Y}{X}$.

Find $f_Z(z)$.

Picture $\quad Y = \underbrace{Z}_{\text{slope}} X$



$$\underbrace{P(Z \in dz)}_{= f_Z(z)\,dz} = \int_{x=0}^{x=\infty} P(Z \in dz, X \in dx)$$

$$= \int_{x=0}^{x=\infty} f(x, zx)\, \underbrace{dy}_{dy = x\,dz}\, dx$$

$$\Rightarrow f_Z(z) = \int_{x=0}^{x=\infty} f(x, zx)\, x\, dx \overset{\text{if } X, Y \text{ indep.}}{=} \int_{x=0}^{x=\infty} f_X(x) f_Y(zx)\, x\, dx$$

Convolution formula.