Stat 134 Lec 25

Warmup 1:00-1:10

(3 pts) Suppose the random variable X, which measures the magnitude of an earthquake (on the Richter scale) in the Bay Area, follows the Exponential (λ) distribution. Since the Richter scale is logarithmic, we want to study the distribution of the total energy of earthquakes. Find the distribution of $Y = e^X$.

Change of variable formula:

$$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \frac{f_X(x)}{|g'(x)|} \quad \text{evaluated at } x = g^{-1}(y).$$

$X \sim Exp(\lambda)$ $\quad f_X(x) = \lambda e^{-\lambda x}, \; x \geq 0$

$Y = g(X) = e^X$ one to one

$g'(x) = e^x \Rightarrow x = \log y$

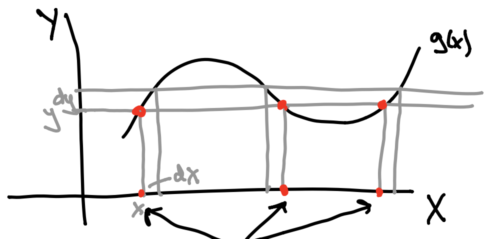
$$f_Y(y) = \frac{\lambda e^{-\lambda x}}{e^x}\Bigg|_{x=\log y} = \lambda e^{(-\lambda-1)x}\Bigg|_{x=\log y} = \lambda e^{(-\lambda-1)\log y}$$

$$= \boxed{\lambda y^{(-\lambda-1)}, \quad y \geq 1}$$

Last 2 times

① sec 4.4 Change of Variable rule

many to one g:



$g^{-1}(y) = \{x : g(x) = y\}$

$f_Y(y)\,dy = f_X(x_1)dx_1 + f_X(x_2)dx_2 + f_X(x_3)dx_3$

$f_Y(y) = f_X(x_1)\frac{dx_1}{dy} + f_X(x_2)\frac{dx_2}{dy} + f_X(x_3)\frac{dx_3}{dy}$

$= \frac{f_X(x_1)}{g'(x_1)} + \frac{f_X(x_2)}{|g'(x_2)|} + \frac{f_X(x_3)}{g'(x_3)}$

$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{g'(x)}$

$P(x \in dx_2) \geq 0$

② MGF (not in book)

$M_X(t) = E(e^{tX})$

Thm If a MGF exists in an interval around zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ Taylor for all $x \in \mathbb{R}$

ex $X \sim \text{Pois}(\mu)$

$M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} \frac{\mu^k e^{-\mu}}{k!} = e^{-\mu} \sum_{k=0}^{\infty} \frac{(\mu e^t)^k}{k!}$ $= e^{\mu e^t}$ for all t

$= e^{-\mu} e^{\mu e^t} = \boxed{e^{\mu(e^t - 1)} \text{ for all } t}$

then $E(X) = \frac{d}{dt} M_X(t)\Big|_{t=0} = e^{\mu(e^t-1)} \cdot \mu e^t \Big|_{t=0} = \mu$ ✓

Today

① Key properties of MGF

② Recognizing a distribution from the variable part of its density.

(1) Key Properties of MGF

(a) If an MGF exists in an interval containing zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

last time

(b) If X and Y are independent RVs,

$M_{X+Y}(t) = M_X(t)M_Y(t)$.

Proved in MGF HW.

(c) $M_X(t) = M_Y(t)$ for all t in an interval around 0 then $F_X(z) = F_Y(z)$ (i.e X and Y have the same distribution).

Skip proof — we can invert a MGF to get the CDF.

$= E(e^{tX})$

ex If $M_X(t) = \frac{1}{2}e^{1t} + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}$,

e^{xt} tells us the value of X and the associated coefficients tell us the probabilities (i.e $X = 1, 2, 3$ w/ prob $1/2, 1/3, 1/6$.)

so MGF ⇒ distribution of X when X has finite # values.

Property (a) is useful to find $E(X)$, $Var(X)$,

Property (b) and (c) allow us to prove for example that sum of independent Poisson is Poisson.

ex

$X_1 \sim Pois(\mu_1)$
$X_2 \sim Pois(\mu_2)$ } independent.

show that $X_1 + X_2 \sim Pois(\mu_1 + \mu_2)$.

$$M_{X_1}(t) = e^{\mu_1(e^t - 1)} \quad \text{for all } t$$

$$M_{X_2}(t) = e^{\mu_2(e^t - 1)} \quad \text{for all } t$$

$$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t) = \boxed{e^{(\mu_1+\mu_2)(e^t-1)}}$$

MGF of Pois $(\mu_1 + \mu_2)$ for all t.

$\Rightarrow X_1 + X_2 \sim Pois(\mu_1 + \mu_2)$

ex Let X be a RV and a a constant.

Show that $M_{aX}(t) = M_X(at) \leftarrow E(e^{Xat})$

hint
$$M_{aX}(t) = E(e^{aXt})$$
$$= E(e^{Xat})$$
$$= M_X(at).$$

For $X \sim \text{Gamma}(r, \lambda)$

recall $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$ for $t < \lambda$

ex Let $X \sim \text{Exp}(\lambda)$ $= \text{Gamma}(1, \lambda)$ and $a > 0$.

Show that $Y = aX$ is also exponential, and specify the new parameter.

$Y = aX$

$$M_Y(t) = M_{aX}(t) = M_X(at) = \left(\frac{\lambda}{\lambda - at}\right)^1 \text{ for } at < \lambda$$

$$= \left(\frac{\frac{\lambda}{a}}{\frac{\lambda}{a} - t}\right)^1 \text{ for } t < \frac{\lambda}{a}$$

> 0

$\Rightarrow Y \sim \text{Exp}\left(\frac{\lambda}{a}\right)$

since is MGF of $\text{Exp}\left(\frac{\lambda}{a}\right)$.

② **Recognizing a distribution from the variable part of its density.**

A density can be written as

$$f(t) = c\, h(t)$$

c ← constant

$h(t)$ ← variable part.

$$1 = \int_{-\infty}^{\infty} f(t)\,dt = c \int_{-\infty}^{\infty} h(t)\,dt \Rightarrow c = \frac{1}{\int_{-\infty}^{\infty} h(t)\,dt}$$

So you can figure out the density from its variable part.

List of densities. Please circle their variable parts:

ex

$T \sim \text{Gamma}(r, \lambda)$ $\quad f(t) = \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}, \; t > 0$

$T \sim \text{Normal}(\mu, \sigma^2)$ $\quad f(t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$

$T \sim \text{Unif}(a, b)$ $\quad f(t) = \frac{1}{b-a} \mathbb{1}_{(t \in (a,b))}$

$T_r \sim \text{Gamma}(r, \lambda)$, $r, \lambda > 0$ $\quad f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$

$T \sim \text{Exp}(\lambda)$, $\lambda > 0$ $\quad f(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$ ← variable part

ex Name the distribution with the following variable part **ex** $\text{Gamma}(r = \frac{1}{2}, \lambda = 3)$

a) $h(t) = t^3 e^{-\frac{1}{2}t}$ $\quad \text{Gamma}(r = 4, \lambda = \frac{1}{2})$

b) $h(t) = e^{-\frac{1}{2}t^2}$ $\quad \text{Normal}(\mu = 0, \sigma = 1)$

c) $h(t) = e^{-3t}$ $\quad \text{Gamma}(r = 1, \lambda = 3) = \text{Exp}(\lambda = 3)$

d) $h(t) = t^{-\frac{1}{2}} e^{-t}$ $\quad \text{Gamma}(r = \frac{1}{2}, \lambda = 1)$

e) $h(t) = \mathbb{1}_{(t \in (0,1))}$ $\quad \text{Unif}(0, 1)$

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Stat 134

1. Let Z be a standard normal RV (with variable part $e^{-\frac{z^2}{2}}$). The variable part of the distribution $X = Z^2$ is?

   a Gamma $x^{-\frac{1}{2}}e^{-\frac{x}{2}}$

   b Gamma $x^{\frac{1}{2}}e^{-\frac{x}{2}}$

   c Exponential $e^{-\frac{x}{2}}$

   d Normal $e^{-\frac{x^2}{2}}$

   e none of the above

$X = g(z) = z^2$ 2 to 1 function

$g'(z) = 2z$

$f_X(x) \propto \frac{e^{-\frac{z^2}{2}}}{2z}$ $z = \pm\sqrt{x}$

$= \frac{e^{-\frac{x}{2}}}{2\sqrt{x}} + \frac{e^{-\frac{x}{2}}}{2\sqrt{x}}$

$= \frac{1}{\sqrt{x}} e^{-\frac{x}{2}}$

$= x^{-1/2} e^{-\frac{x}{2}}$