

Stat 134 Lec 11

Last time sec 3.2 Expectation

$$E(X) = \sum_{x \in X} x P(X=x)$$

If X is a count, X can be written as
a sum of indicators

$$X = I_1 + I_2 + \dots + I_n, \quad I_j = \begin{cases} 1 & \text{Prob } p \\ 0 & \text{Prob } 1-p \end{cases}$$

$$E(I_j) = 1 \cdot p + 0 \cdot (1-p) = p.$$

Idea Even if indicators are dependent, the expectation of each indicator is an unconditional probability.

then $E(X) = n \cdot p$

We proved if $X \sim \text{Bin}(n, p) \Rightarrow E(X) = np$

if $X \sim \text{Hyper}(N, G, n) \Rightarrow E(X) = n \frac{G}{N}$

Today sec 3.2

- ① review student concept test responses
- ② More expectation with indicator examples
- ③ tail sum formula
- ④ Markov's inequality.

①

1. A forgetful valet is attempting to return n cars to their n rightful owners. For each driver, the valet remembers the car correctly 5% of the time; otherwise the valet retrieves a car at random (possibly the correct car). What is the probability that the second driver retrieves his own car.

a .05

b $.05 + .95/n$

c $.05 + .95/(n - 1)$

d none of the above

b Everyone should have the same probability. This should be symmetric across the board.

b $P(\text{he remembers}) + P(\text{he's randomly right}) = .05 + .95(1/n)$
I think it's n instead of $n-1$ since it's like drawing from a deck where the prob of the second card being a specific card is the same as the first.

c there are 2 options: the valet remembering the car (0.05) and the valet randomly retrieving a car (0.95). for this 95%, there are $n-1$ cars left because the first driver already drove away a car.

b Consider the limiting case $n=2$. The probability that he gets his car should be slightly more than fifty percent, which is given by b.

now solve pt 2:

2. A forgetful valet is attempting to return n cars to their n rightful owners. For each driver, the valet remembers the car correctly 5% of the time; otherwise the valet retrieves a car at random (possibly the correct car). Let N be the number of drivers who retrieve their own car. $E(N)$ is:

- (a) $0.05n + .95$
- b $.05n$
- c $.05n + 1$
- d none of the above

$$N = \# \text{ drivers who retrieve their own car}$$

$$\hookrightarrow_{\{1, 2, \dots, n\}}$$

$$N = I_1 + I_2 + \dots + I_n, \quad I_i = \begin{cases} 1 & \text{if } j \text{ driver retrieves their own car,} \\ 0 & \text{else,} \end{cases}$$

$$E(N) = n \left(.05 + \frac{.95}{n} \right) = \underline{\underline{(.05n + .95)}}$$

$.05 + .95/n$

(2) Sec 3.2 more expectation / indicator examples

ex Consider a 5 card deck consisting of 2, 2, 3, 4, 5.
Let X = number of cards before the first 2.

a) What are the range of values of X ?

$$0, 1, 2, 3$$

b) Write X as a sum of indicator(s).

$$X = I_3 + I_4 + I_5 \quad (3, 4, 5 \text{ are the non } 2 \text{ cards})$$

c) How is an indicator defined.

$$I_4 = \begin{cases} 1 & \text{if 4 is before 1^{st} 2} \\ 0 & \text{else.} \end{cases}$$

d) Find $E(I_4)$

$$\frac{1}{3}$$

The calculation of $E(I_4)$ only involves 3 cards 4, 2, 2. Take out all other cards. Now you have a 3 card deck.

Picture $\underline{2} \underline{2} \underline{2}$ where each slot can be empty or have a 4 in it.

e) Find $E(X)$ It is equally likely you have a 4 in any of the 3 slots, hence $E(I_4) = \frac{1}{3}$.

$$3 \cdot \frac{1}{3} = 1$$

Stat 134

Chapter 3 Friday February 15 2019

1. Consider a well shuffled deck of cards. The expected number of cards before the first ace is?
- a $52/5$
 - b $48/5$
 - c $48/4$
 - d none of the above

$X = \# \text{ cards before the } 1^{\text{st}} \text{ ace}$

$0, 1, 2, \dots, 48$

$4 \times 12 = 48 \text{ nonaces}$

hearts spades clubs diamonds

$$X = I_1 + I_2 + \dots + I_{48} \quad I_j = \begin{cases} 1 & \text{if } j^{\text{th}} \text{ nonace} \\ & \text{before } 1^{\text{st}} \text{ ace} \\ 0 & \text{else} \end{cases}$$

$P = Y_5 \Rightarrow \boxed{E(X) = 48\left(\frac{1}{5}\right)}$

2. Consider a well shuffled deck of cards. The expected number of cards between the first and second ace (not counting either ace)?

a $52/5$

b $48/5$

c $48/4$

d none of the above

$X = \# \text{ cards between } 1^{\text{st}} \text{ and second ace}$

$X = I_1 + \dots + I_{48}$

$I_i = \begin{cases} 1 & \text{if } 2^{\text{nd}} \text{ nonace is between } i^{\text{th}} \text{ and } 2^{\text{nd}} \text{ ace} \\ 0 & \end{cases}$

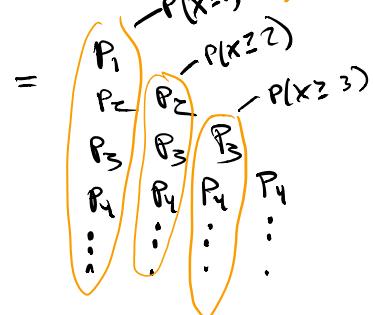
Same as 1st part of question

(3) Sec 3.2 Tail Sum formula for expectation

Suppose X is a count 0, 1, 2, 3, ...

$$E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) + \dots$$

$$= 1 \cdot \underbrace{P(X=1)}_{P_1} + 2 \cdot \underbrace{P(X=2)}_{P_2} + \dots$$



$$= P(X \geq 1) + P(X \geq 2) + P(X \geq 3) + \dots \quad \text{Tail Sum Formula}$$

This is useful when it is easy to find $P(X \geq k)$.

\Leftrightarrow A fair die is rolled 10 times.

Let $X = \max(X_1, \dots, X_{10})$.

Find $P(X \geq k)$

$$\begin{aligned} P(X \geq k) &= 1 - P(X < k) \xrightarrow{\substack{1 \text{ event} \\ X_1, X_2, \dots, X_{10} \text{ max}}} \frac{1}{k} \\ &= 1 - P(X_1 < k, X_2 < k, \dots, X_{10} < k) \\ &= 1 - P(X_1 < k) P(X_2 < k) \cdots P(X_{10} < k) \\ &= 1 - P(X_1 < k)^{10} \\ &= 1 - \left(\frac{k-1}{6}\right)^{10} \end{aligned}$$

$$\begin{aligned} \therefore E(X) &= P(X \geq 1) + P(X \geq 2) + \dots + P(X \geq 6) + P(X \geq 7) + \dots \\ &\quad \stackrel{\text{"}}{=} \stackrel{\text{"}}{1} - \left(\frac{1}{6}\right)^{10} \quad \stackrel{\text{"}}{=} \stackrel{\text{"}}{1} - \left(\frac{5}{6}\right)^{10} \quad \stackrel{\text{"}}{=} \stackrel{\text{"}}{0} \\ &= 6 - \left(\frac{1}{6}\right)^{10} [1^{10} + 2^{10} + 3^{10} + 4^{10} + 5^{10}] = (5.82) \end{aligned}$$

ex A fair die is rolled 3 times, X_1, X_2, X_3 .

Let Y be the sum of the largest 2 numbers.

Notice that $Y = X_1 + X_2 + X_3 - \min(X_1, X_2, X_3)$

a) Find $P(\min(X_1, X_2, X_3) \geq 2)$ Picture

$$= P(X_1 \geq 2, X_2 \geq 2, X_3 \geq 2)$$

$$= P(X_1 \geq 2)^3 = \left(\frac{5}{6}\right)^3$$

(b) Find $E(\min(X_1, X_2, X_3))$

$$= P(\min \geq 1) + P(\min \geq 2) + \dots + P(\min \geq 6)$$

$$= P(X_1 \geq 1)^3 + P(X_1 \geq 2)^3 + \dots + P(X_1 \geq 6)^3$$

$$= \left(\frac{5}{6}\right)^3 + \left(\frac{4}{6}\right)^3 + \left(\frac{3}{6}\right)^3 + \dots + \left(\frac{1}{6}\right)^3 = \boxed{\frac{1}{6^3} [6^3 + 5^3 + 4^3 + 3^3 + 2^3 + 1^3]}$$

(c) Find $E(Y) = E(X_1) + E(X_2) + E(X_3) - E(\min(X_1, X_2, X_3))$

$$E(X_1) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \dots + 6 \cdot \frac{1}{6} = \frac{1}{6} [1 + 2 + 3 + 4 + 5 + 6]$$

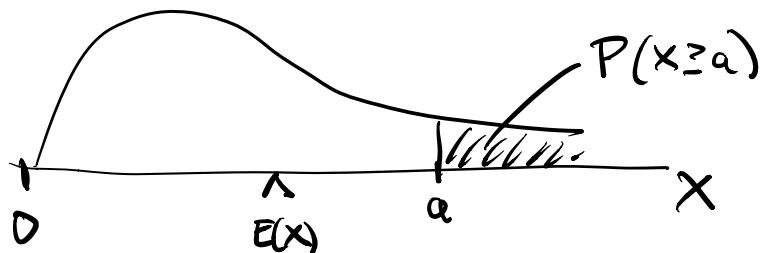
$$= \frac{21}{6} = \boxed{\frac{7}{2}}$$

$$\Rightarrow E(Y) = \boxed{3 \left(\frac{7}{2}\right) + \frac{1}{6^3} [6^3 + 5^3 + 4^3 + 3^3 + 2^3 + 1^3]}$$

④ Sec 3.2 Markov's Inequality

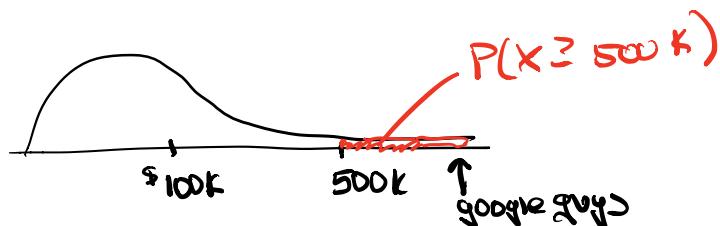
Let $X \geq 0$, $a > 0$

$E(X)$ is the center of the distribution



$$P(X \geq a) \leq \frac{E(X)}{a}$$

ex let X be the yearly income of Bay area residents.
 $E(X) = \$100K$. Find an upper bound for $P(X \geq 500K)$



By Markov Ineq, $P(X \geq 500K) \leq \frac{100}{500} = \frac{1}{5}$

Ex Let X_1, X_2, \dots, X_{100} be independent and identically distributed (iid) $\text{Pois}(.01)$.

$$\text{Let } S = X_1 + X_2 + \dots + X_{100}$$

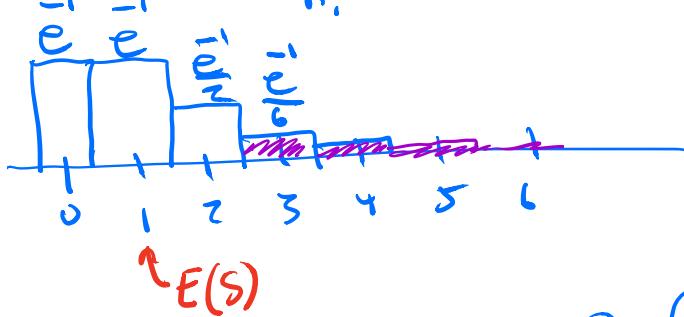
Draw the histogram of S and approximate an upperbound for $P(S \geq 3)$ using Markov's inequality.

Soln $S = X_1 + X_2 + \dots + X_{100} \sim \text{Pois}(100 \cdot .01) = \text{Pois}(1)$,

$$S \sim \text{Pois}(1)$$

$$E(S) = 1$$

$$P(S=k) = \frac{\bar{e}^k}{k!} = \frac{\bar{e}^1}{1!}$$



$$\text{By Markov's inequality, } P(S \geq 3) \leq \left(\frac{1}{3}\right)$$

Note Exact: $P(S \geq 3) = 1 - P(0) - P(1) - P(2) = 1 - \bar{e}^{(1+1+\frac{1}{2})}$

$$\bar{e}^{-1} \quad \bar{e}^{-1} \quad \frac{\bar{e}^{-\frac{1}{2}}}{2!} = 0.08$$

Proof of Markov's inequality:

Property of expectation:

if $X \leq Y$ then $E(X) \leq E(Y)$

Markov's inequality

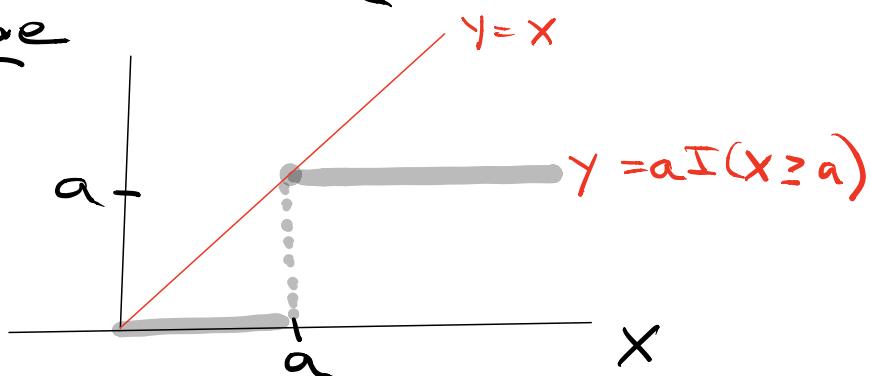
If $X \geq 0$, then $P(X \geq a) \leq \frac{E(X)}{a}$ for every $a > 0$.

Pf/

Let $I(X \geq a)$ be an indicator
equal to 1 when $X \geq a$. Then

$$aI(X \geq a) = \begin{cases} a & \text{if } X \geq a \\ 0 & \text{else.} \end{cases}$$

Picture



$$aI(X \geq a) \leq X \Rightarrow E(aI(X \geq a)) \leq E(X)$$

$$\stackrel{\text{"}}{=} aE(I(X \geq a))$$

$$\stackrel{\text{"}}{=} aP(X \geq a)$$

$$\Rightarrow P(X \geq a) \leq \frac{E(X)}{a}$$

□