Stat 134 lec 31

Last time

sec 5.3 independent normal variables

a) $T \sim Exp(\frac{1}{2}) \Rightarrow R = \sqrt{T}$ has density $f(r) = re^{-\frac{1}{2}r^2}$, $r > 0$

← called Rayleigh RV

b) Using Rayleigh distribution we showed if $X, Y \stackrel{iid}{\sim} N(0,1)$

① $\sqrt{X^2 + Y^2} = R$

② $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$

c)

If $Z \sim N(0,1)$ and $X = \mu + \sigma Z$ then

$X \sim N(\mu, \sigma^2)$ with density $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

d) $X \sim N(\mu, \sigma^2)$

$M_X(t) = e^{\mu t} e^{\frac{\sigma^2 t^2}{2}}$

e) Thm Let $\left.\begin{array}{l} X_1 \sim N(0,1) \\ X_2 \sim N(0,1) \end{array}\right\}$ indep.

then $X_1 + X_2 \sim N(0,2)$

Today sec 5.3

① A linear combination of independent normals is normal

② Chi square distribution

sec 5.4

③ Convolution formula for the density of $X+Y$

(1) A linear combination of independent normals is normal

Recall
1) $M_{aX}(t) = M_X(at)$
2) $M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t)$ for X_1, X_2 indep.
3) $M_X(t) = M_Y(t)$ in interval around 0
$\Rightarrow X \overset{d}{=} Y$.
4) $X \sim N(\mu, \sigma^2) \Rightarrow M_X(t) = e^{\mu t} e^{\sigma^2 \frac{t^2}{2}}$

Thm Let $\left.\begin{array}{l} X_1 \sim N(\mu_1, \sigma_1^2) \\ X_2 \sim N(\mu_2, \sigma_2^2) \end{array}\right\}$ indep.

then $aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

Pf/ $M_{X_1}(t) = e^{\mu_1 t} e^{\sigma_1^2 \frac{t^2}{2}}$

$M_{aX_1}(t) = M_{X_1}(at) = e^{\mu_1 a t} e^{\sigma_1^2 a^2 \frac{t^2}{2}}$

Then $M_{aX_1 + bX_2}(t) = M_{aX_1}(t) M_{bX_2}(t)$

$= e^{\mu_1 a t} e^{\sigma_1^2 a^2 \frac{t^2}{2}} \cdot e^{\mu_2 b t} e^{\sigma_2^2 b^2 \frac{t^2}{2}}$

$= e^{(\mu_1 a + \mu_2 b)t} e^{(\sigma_1^2 a^2 + \sigma^2 b^2)\frac{t^2}{2}}$

by uniqueness of MGF

$aX_1 + bX_2 \sim N(\mu_1 a + \mu_2 b, \sigma_1^2 a^2 + \sigma^2 b^2)$

$\square$

http://tinyurl.com/nov15-pt1

. Let $X \sim N(68, 3^2)$ and $Y \sim N(66, 2^2)$ be independent. $P(X > Y)$ equals

a $1 - \Phi(\frac{0-2}{\sqrt{3^2+2^2}})$

b $1 - \Phi(\frac{0-2}{3^2+2^2})$

c $1 - \Phi(\frac{68-66}{\sqrt{3^2+2^2}})$

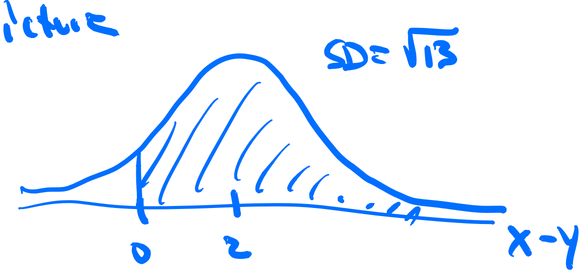
d none of the above

$\boxed{1 - \Phi\left(\frac{0-2}{\sqrt{13}}\right)}$

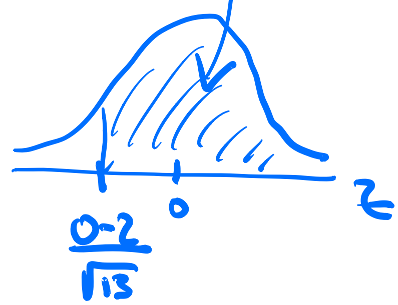
$P(X > Y) = P(X - Y > 0)$

$X - Y \sim N(68-66, 3^2+2^2) = N(2, 13)$

Picture



$z = \frac{0-2}{\sqrt{13}}$



Sec 5.3 Chi-square distribution

Fact see end of lecture notes for proof. if $Z \sim N(0,1)$ then

$$Z^2 \sim Gamma\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Recall that if $T \sim Gamma(r, \lambda)$,

$$M_T(t) = \left(\frac{\lambda}{\lambda - t}\right)^r, \quad t < \lambda$$

Hence $M_{Z^2}(t) = \left(\dfrac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{1/2}, \quad t < 1/2$

Let $Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$

$$M_{Z_1^2 + \dots + Z_n^2}(t) = \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{1/2 \cdot n}, \quad t < \frac{1}{2}$$

By uniqueness of MGF

$Z_1^2 + \dots + Z_n^2 \sim Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$

called chi square distribution with n degrees of freedom, χ^2_n, with n degrees of freedom

ex Let $X, Y \overset{iid}{\sim} N(0,1)$

Let $R = \sqrt{X^2 + Y^2}$ be Raleigh distribution.

Then $R^2 = X^2 + Y^2 \sim \chi^2_2 = Exp(1/2) = Gamma(1, 1/2)$

Sec 5.4 The Density Convolution Formula

ex Let X and Y be discrete RVs taking values $\{0,1,2,3,4,5,6\}$.

Let $S = X+Y$.

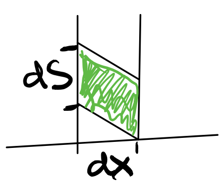
Find the probability mass function of S.

$$P(S=3) = P(X=0, Y=3-0) + P(X=1, Y=3-1) + P(X=2, Y=3-2) + P(X=3, Y=3-3)$$

$$P(S=s) = \sum_{x=0}^{s} P(X=x, Y=s-x)$$

convolution formula.

A little geometry:



Area of parallelogram
$A = dx\,dS$

Let $X \geq 0$, $Y \geq 0$ be continuous RVs with joint density $f(x,y)$.

Let $S = X+Y$

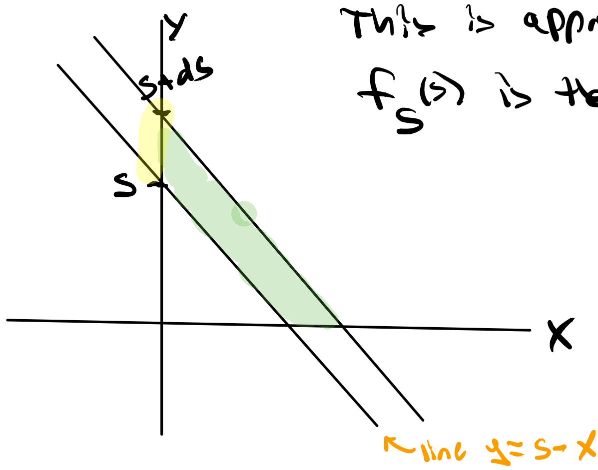
Find the density of S

$s = x+y$

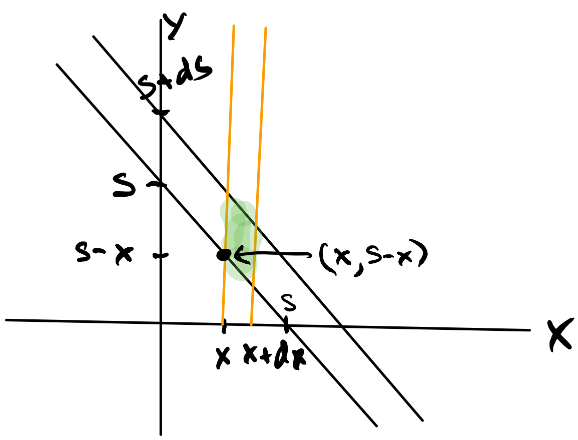
$y = s-x$

y intercept

$P(S \in ds)$ is the volume under $f(x,y)$ over the green region. This is approx $f_S(s)ds$ where $f_S(s)$ is the density of S.



$P(X \in dx, S \in dS)$ is the volume under $f(x,y)$ over the green region.



$S \in ds,\ X \in dx$

$$\underbrace{P(S \in ds)}_{f_S(s)ds} = \int_{x=0}^{x=s} \underbrace{P(X \in dx, S \in dS)}_{f(x,s-x)\,dx\,ds} \qquad S \in ds,\ X \in dx$$

$$= \underbrace{\int_{x=0}^{s} f(x,s-x)\,dx}_{f_S(s)}\ ds$$

$$\Rightarrow f_S(s) = \int_{x=0}^{x=s} f(x, s-x)\,dx$$

convolution formula for densities.

Compare with:

$$P(S=s) = \sum_{x=0}^{s} P(x, s-x)$$

convolution formula for p.m.f

ex $X, Y \overset{iid}{\sim} \text{expon}(\lambda) \quad S = X+Y$

$$f_S(s) = \int_0^s f(x, s-x)\,dx$$

fixed

independence

$$= \int_0^s f_X(x)\, f_Y(s-x)\,dx$$

$$= \int_0^s \lambda e^{-\lambda x} \lambda e^{-\lambda(s-x)}\,dx$$

$$= \int_0^s \lambda^2 e^{-\lambda s}\,dx = \lambda^2 e^{-\lambda s} \int_0^s dx = \lambda^2 e^{-\lambda s} x \Big|_0^s$$

$$= \boxed{\lambda^2 e^{-\lambda s} \cdot s}$$

variable part of gamma$(2, \lambda)$

$\Rightarrow S \sim \text{gamma}(2, \lambda)$.

ex Let $X \sim U_{(7)}$, $Y \sim U_{(9)}$ for 10 iid $U(0,1)$.

The joint density $f_{X,Y}(x,y) = \binom{10}{6,1,1,1,1} x^6 (y-x)(1-y)$

for $0<x<y<1$.

Let $Z = Y - X$

a) For a fixed z, what is the largest value of X?

$\leftarrow$ what goes here?

$$f(z) = \int_{x=0}^{x=} f(x, x+z)\,dx$$

$z = y - x$

$x = y - z$ where $y \in [0,1]$

largest x has $y = 1$

$$\boxed{x = 1 - z}$$

So our convolution formula is

$$f_Z(z) = \int_{x=0}^{x=1-z} f(x, x+z)\,dx$$

ex Let $X \sim U_{(7)}$, $Y \sim U_{(9)}$ for 10 iid $U(0,1)$.

The joint density $f_{X,Y}(x,y) = \binom{10}{6,1,1,1,1} x^6 (y-x)(1-y)$

for $0<x<y<1$.

Find the density of $Z = Y - X$

what distribution is Z?

$$f_Z(z) = \int_0^{1-z} f(x, x+z)\,dx$$

let $c = \binom{10}{6,1,1,1,1}$

$$f_Z(z) = \int_0^{1-z} c x^6 (x+z-x)(1-(x+z))\,dx$$

$$= cz \int_0^{1-z} \left((1-z)x^6 - x^7\right) dx$$

$$= cz \left[(1-z)\frac{x^7}{7} - \frac{x^8}{8}\right]\Bigg|_{x=0}^{x=1-z}$$

$$= cz\left(\frac{(1-z)^8}{7} - \frac{(1-z)^8}{8}\right) = \frac{c}{56} z(1-z)^8$$

$\Rightarrow Z \sim \text{Beta}(2, 9)$

Interpretation:

$U_{(9)} - U_{(7)} \sim \text{Beta}(2,9)$ means that the distribution of the distance between the 9^{th} largest and 7^{th} largest dart is Beta$(2,9)$.

Is there anything special about $U_{(9)} - U_{(7)}$. What about $U_{(8)} - U_{(6)}$ or $U_{(3)} - U_{(1)}$ or $U_{(2)} - 0$ for that matter? You can check that all of these have distribution Beta$(2,9)$.

The example $U_{(2)} - 0$ is particularly easy since $U_{(k)} \sim \text{Beta}(k, n-k+1)$ so

$$U_{(2)} \sim \text{Beta}(2, \underbrace{10-2+1}_{\substack{\shortparallel \\ 9}}).$$

Appendix

If $Z \sim N(0,1)$, then $Z^2 \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

Proof/

$\lambda > 0$, pos integer r,

gamma (r, λ) density

$$f(t) = \frac{\lambda^r}{\Gamma(r)} t^{r-1} e^{-\lambda t}, \quad t > 0$$

let Z = std normal

$X = Z^2$ change of variable rule.

$$\text{Find } f_X(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-\frac{1}{2}x}, \quad x > 0$$

$$= \frac{\left(\frac{1}{2}\right)^{1/2}}{\sqrt{\pi}} x^{\frac{1}{2}-1} e^{-\frac{1}{2}x}$$

$\Gamma(\frac{1}{2})$

$\Rightarrow X \sim \text{gamma}(\frac{1}{2}, \frac{1}{2})$

$\square$