Stat 134 lec 24

Special lecture

Moment Generating Function of X

Not in book.

Next time Finish Sec 4.5, Start Sec 4.6.

Moment Generating Function (MGF) of X

The k^{th} moment of a RV X is the number

$\mu_k = E(X^k)$ defined for $k = 0, 1, 2, 3, \ldots$

$\mu_0 = E(X^0) = E(1) = 1$

$\mu_1 = E(X)$

$\mu_2 = E(X^2)$

$\vdots$

moments describe your distribution

ex 1st moment is mean

2nd moment relates to variance

3rd moment relates to how skewed the distribution is.

Note $Var(X) = E(X^2) - E(X)^2 = \mu_2 - \mu_1^2$

Computing the moments of a RV is sometimes hard, because it involves computing complicated integrals or sums

$$E(X) = \begin{cases} \int_{-\infty}^{\infty} x f(x)\,dx & \text{if } X \text{ continuous} \\ \sum_{-\infty}^{\infty} x P(x) & \text{if } X \text{ discrete} \end{cases}$$

The MGF allows one to compute the moments by computing derivatives, which is easier.

ex $X \sim Ber(p)$

we know $E(X) = 1 \cdot p + 0 \cdot q = p$

$E(X^2) = 1^2 \cdot p + 0^2 \cdot q = p$

The moment generating function for X is

$M_X(t) = 1 + p(e^t - 1)$ for all t. — see next page.

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = pe^t \Big|_{t=0} = p \leftarrow E(X)$$

$$\frac{d}{dt^2} M_X(t)\Big|_{t=0} = \frac{d}{dt} pe^t \Big|_{t=0} = p \leftarrow E(X^2)$$

You can easily see that $E(X^k) = p$.

To find moments we take derivatives of the MGF and evaluate at $t=0$.

What is the MGF?

Recall $E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$

We define the MGF of X to be

$$M_X(t) = E(e^{tX}) = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{if } X \text{ continuous} \\ \sum_{x=-\infty}^{\infty} e^{tx} p(x) & \text{if } X \text{ discrete} \end{cases}$$

↑ Call $\psi_X(t)$ in HW 9

ex $X \sim Ber(p)$

Find $M_X(t)$

$X = \begin{cases} 1 & \text{if } X=1 \\ 0 & \text{if } X=0 \end{cases}$ prob p

so $tX = \begin{cases} t & \text{if } X=1 \\ 0 & \text{if } X=0 \end{cases}$ prob p

$$M_X(t) = E(e^{tX})$$
$$= e^{t\cdot 0} p(0) + e^{t\cdot 1} p(1)$$
$$= 1\cdot(1-p) + e^t \cdot p$$
$$= \boxed{1 + p(e^t - 1) \text{ for all } t}$$

Why is $M_X^{(k)}(t)\Big|_{t=0} = E(X^k)$?

Recall the Taylor series for e^y:

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

let $t \in \mathbb{R}$, X RV.

You can do this ~~for~~ ~~the~~ RV tX

$$e^{tX} = 1 + tX + \frac{(tX)^2}{2!} +$$

$$\underbrace{E(e^{tX})}_{M_X(t)} = E(1) + E(tX) + E\left(\frac{(tX)^2}{2!}\right) + \cdots$$
$$= E(1) + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots$$

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = E(X)$$

$$\frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = E(X^2)$$

more generally,
$$\frac{d^k}{dt^k} M_X(t)\Big|_{t=0} = \frac{d^k}{dt^k} E(e^{tX})\Big|_{t=0}$$

Lets not worry about why this step is true. Ask me for details if interested. —!!!

$$\overset{!!}{=} E\left(\frac{d^k}{dt^k} e^{tX}\Big|_{t=0}\right)$$

$$= E(X^k)$$

Summary, the MGF $M_X(t) = E(e^{tX})$ contains info about all of the moments of X. By taking the k^{th} derivative and evaluating at 0 you get the k^{th} moment.

<u>Note</u> An MGF doesn't always exist in an interval around zero. (see end of lecture notes for an example).

What makes MGF so helpful is:

<u>Thm</u> If a MGF exists in an interval around zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

ex let $X \sim \text{Gamma}(r, \lambda)$

$$f(x) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}, & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Recall $\Gamma(r) = \int_0^\infty u^{r-1} e^{-u} \, du$

Find $M_X(t)$.

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{xt} \left(\frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \right) dx$$

$$= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^{r-1} e^{(t-\lambda)x} \, dx$$

which converges for $(t-\lambda) < 0$ i.e. $\boxed{t < \lambda}$

Substitution

$$u = \underbrace{-(t-\lambda)}_{>0} x$$

so $x = -\frac{u}{t-\lambda}$, $dx = -\frac{1}{t-\lambda} du$

$$= \frac{\lambda^r}{\Gamma(r)} \int_{u=0}^{u=\infty} \frac{u^{r-1}}{(\lambda - t)^{r-1}} e^{-u} \frac{1}{\lambda - t} \, du$$

$$= \frac{\lambda^r}{\Gamma(r)} \frac{1}{(\lambda-t)^r} \underbrace{\int_0^\infty u^{r-1} e^{-u}\, du}_{\Gamma(r)}$$

$$= \frac{\lambda^r}{\cancel{\Gamma(r)}} \frac{\cancel{\Gamma(r)}}{(\lambda-t)^r} = \left(\frac{\lambda}{\lambda-t}\right)^r \text{ for } t < \lambda$$

Recall: If a MGF exists in an interval around zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

ex

Let $X \sim \text{Gamma}(r, \lambda)$

Find $\frac{d}{dt} M_X(t)\Big|_{t=0}$

and

$\frac{d^2}{dt^2} M_X(t)\Big|_{t=0}$

$$M(t) = \left(\frac{\lambda}{\lambda - t}\right)^r = \lambda^r (\lambda - t)^{-r}$$

$$M'(t) = \lambda^r (-r)(\lambda - t)^{-r-1}(-1) = \frac{r\lambda^r}{(\lambda - t)^{r+1}}$$

$$M'(0) = \frac{r}{\lambda} \Rightarrow \boxed{E(X) = \frac{r}{\lambda}}$$

$$M''(t) = r\lambda^r(-r-1)(\lambda - t)^{-r-2}(-1)$$

$$= r(r+1)\lambda^r \frac{1}{(\lambda - t)^{r+2}}$$

$$\Rightarrow M''(0) = r(r+1)\lambda^r \frac{1}{\lambda^{r+2}} = \frac{r(r+1)}{\lambda^2}$$

$$\Rightarrow \boxed{E(X^2) = \frac{r(r+1)}{\lambda^2}}$$

$$\Rightarrow Var(X) = E(X^2) - E(X)^2 = \frac{r(r+1)}{\lambda^2} - \frac{r^2}{\lambda^2}$$

$$= \frac{r^2}{\lambda^2} + \frac{r}{\lambda^2} - \frac{r^2}{\lambda^2} = \boxed{\frac{r}{\lambda^2}}$$

$\underline{\underline{ex}}$ A RV X ~~takes~~ takes values 1, 2, 3 with prob $1/2$, $1/3$, $1/6$.

Find $M_X(t)$. $= \displaystyle\sum_{x=1}^{3} e^{tx} p(x)$

$$= \boxed{e^{t} \cdot \frac{1}{2} + e^{2t} \cdot \frac{1}{3} + e^{3t} \cdot \frac{1}{6} \quad \text{for all } t}$$

ex $X \sim \text{Geom}\left(\frac{1}{3}\right)$

$P(X=x) = \left(\frac{2}{3}\right)^{x-1}\left(\frac{1}{3}\right) \qquad x = 1, 2, 3, \ldots$

Find $M_X(t) = E(e^{tX})$

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx}\left(\frac{2}{3}\right)^{x-1}\left(\frac{1}{3}\right)$$

$$= \sum_{x=1}^{\infty} e^{t}\left(e^{t}\right)^{x-1}\left(\frac{2}{3}\right)^{x-1}\left(\frac{1}{3}\right)$$

$$= \frac{1}{3}e^{t}\sum_{x=1}^{\infty}\left(e^{t}\frac{2}{3}\right)^{x-1}$$

$$\sum_{x=1}^{\infty}\left(e^{t}\frac{2}{3}\right)^{x-1} = 1 + \left(e^{t}\frac{2}{3}\right) + \left(e^{t}\frac{2}{3}\right)^{2} + \cdots$$

$$= \frac{1}{1-\left(e^{t}\left(\frac{2}{3}\right)\right)} \quad \text{if } e^{t}\frac{2}{3} < 1$$

or

$t < \log\left(\frac{3}{2}\right)$

so $\boxed{M_X(t) = \frac{1}{3}e^{t}\cdot\frac{1}{1-\left(e^{t}\left(\frac{2}{3}\right)\right)} \quad \text{if } t < \log\left(\frac{3}{2}\right)}$

Appendix:

Let X be a discrete RV with probability mass function $p(x) = \begin{cases} \frac{6}{\pi^2 x^2} & x = 1, 2, \ldots \\ 0 & \text{else} \end{cases}$

The MGF, $M_X(t)$, only exists at $t=0$, and hence doesn't exist on an interval around zero.

Pf/

It is known that the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$ converges to $\pi^2/6$.

Then $$p(x) = \begin{cases} \frac{6}{\pi^2 x^2} & x = 1, 2, 3, \ldots \\ 0 & \text{else.} \end{cases}$$

is the prob function of a RV X,

$$M_X(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} \frac{6e^{tx}}{\pi^2 x^2}$$

The ratio test can be used to show that diverges if $t > 0$. Hence this RV only has an MGF at $t=0$.

$\square$