Stat 134 lec 31

warmup 11:00-11:10

Rayleigh

If $R_1 \sim Ray$ (density $f_R(r) = re^{-\frac{1}{2}r^2}$)

find the density of $W = \frac{1}{\sqrt{3}} R_1$

$$f_W(w) = \frac{1}{1/\sqrt{3}} r e^{-\frac{1}{2}r^2} \Big|_{r=\sqrt{3}w}$$

$$= \sqrt{3} \cdot \sqrt{3} w e^{-\frac{3}{2}w^2} = \boxed{3we^{-\frac{3}{2}w^2}}$$

so $W = \frac{1}{\sqrt{3}} Ray$ is also the min of R_1, R_2, R_3.

i.e $\min(R_1, R_2, R_3) \sim \frac{1}{\sqrt{3}} Ray$

Last time

Sec 5.3 independent normal variables

a) $T \sim Exp(\frac{1}{2}) \Rightarrow R = \sqrt{T}$ has density $f(r) = re^{-\frac{1}{2}r^2}$, $r > 0$

$\nwarrow$ called Rayleigh RV

b) Using Rayleigh distribution we showed if $X, Y \overset{iid}{\sim} N(0,1)$

(1) $\sqrt{X^2 + Y^2} = R$

(2) $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$

(3) $E(Z) = 0$

(4) $SD(Z) = 1$

c) for $R_1, R_2, R_3 \overset{iid}{\sim} Ray \Rightarrow \min(R_1, R_2, R_3) \sim \frac{1}{\sqrt{3}} Ray$

ex Let $R_1, R_2, R_3, R_4 \overset{iid}{\sim} Ray$

let $W = \min(R_1, R_2, R_3)$

Find $P(R_4 < W)$

Hint: Find $P(\underbrace{R_4^2}_{Exp(\frac{1}{2})} < W^2)$

$c\,Exp(\lambda) = Exp(\frac{\lambda}{c})$

$W^2 \sim \left(\frac{1}{\sqrt{3}} Ray\right)^2 = \frac{1}{3} Exp(\frac{1}{2}) = Exp\left(\frac{\frac{1}{2}}{\frac{1}{3}}\right) = Exp(\frac{3}{2})$

competing exponentials

$P(R_4 < W) = P(R_4^2 < W^2) = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{3}{2}} = \boxed{\frac{1}{4}}$

Today

(1) Sec 5.3 Sum of independent normals

(2) Chi square distribution

Sec 5.4

(3) Convolution formula for the density of $X+Y$

① Sec 5.3 Sum of independent normals

Definition of the Normal (μ,σ^2) distribution:

We know the density of the std normal $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ Make a change of scale $X = \mu + \sigma Z$. By the change of variable rule you can show

$$f_X(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

MGF Review (lecture 24)

Recall the MGF of a RV X is $M(t) = E(e^{Xt})$.

MGF of std normal

$X \sim N(0,1)$, $M(t) = e^{\frac{t^2}{2}}$ for all t

Properties of MGF

X, Y are independent if $M_{X+Y}(t) = M_X(t)M_Y(t)$

$M_X(t) = M_Y(t)$ for t in an interval containg zero iff $X \overset{d}{=} Y$ same distribution

$M_{b+aX}(t) = e^{bt} M_X(at)$

ex Use the properties of MGF to find the MGF of $X \sim N(\mu, \sigma^2)$

Hint let $Z \sim N(0,1)$, $\mu \in \mathbb{R}$, $\sigma > 0$

and $X = \mu + \sigma Z$

$$M_X(t) = M_{\mu+\sigma Z}(t) = e^{\mu t} M_Z(\sigma t) = e^{\mu t} \cdot e^{\frac{1}{2}(\sigma t)^2}$$

$$= \boxed{e^{\mu t} \cdot e^{\frac{\sigma^2 t^2}{2}} \quad \text{for all } t \in \mathbb{R}}$$

Note if $X \sim N(0,2)$

$$M_X(t) = e^{t^2}$$

ex Use MGF to prove that the sum of two independent std normals is $N(0,2)$.

hint

$$X \sim N(\mu,\sigma^2) \Rightarrow M_X(t) = e^{\mu t} e^{\frac{\sigma^2 t^2}{2}}$$

$$M_{Z_1+Z_2}(t) = M_{Z_1}(t)\, M_{Z_2}(t)$$

$$= e^{t^2/2} \cdot e^{t^2/2} = e^{t^2} \quad \text{for all } t$$

MGF of $N(0,2)$ for all t

$$\Rightarrow Z_1 + Z_2 \sim N(0,2)$$

Thm Let $X_1 \sim N(\mu_1, \sigma_1^2)$ } indep.
$X_2 \sim N(\mu_2, \sigma_2^2)$

then $aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

Pf/ $M_{X_1}(t) = e^{\mu_1 t} e^{\sigma_1^2 \frac{t^2}{2}}$

$M_{aX_1}(t) = M_{X_1}(at) = e^{\mu_1 a t} e^{\sigma_1^2 a^2 \frac{t^2}{2}}$

Then $M_{aX_1 + bX_2}(t) = M_{aX_1}(t) M_{bX_2}(t)$

$= e^{\mu_1 a t} e^{\sigma_1^2 a^2 \frac{t^2}{2}} \cdot e^{\mu_2 b t} e^{\sigma_2^2 b^2 \frac{t^2}{2}}$

$= e^{(\mu_1 a + \mu_2 b)t} e^{(\sigma_1^2 a^2 + \sigma^2 b^2)\frac{t^2}{2}}$

by uniqueness of MGF

$aX_1 + bX_2 \sim N(\mu_1 a + \mu_2 b, \sigma_1^2 a^2 + \sigma^2 b^2)$

$\square$

. Let $X \sim N(68, 3^2)$ and $Y \sim N(66, 2^2)$ be independent. $P(X > Y)$ equals

a $1 - \Phi(\frac{0-2}{\sqrt{3^2+2^2}})$

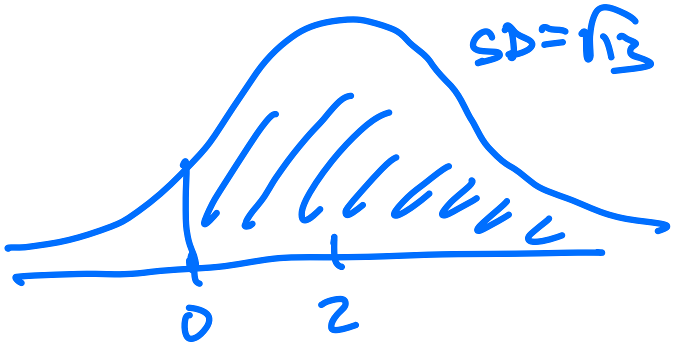
b $1 - \Phi(\frac{0-2}{3^2+2^2})$

c $1 - \Phi(\frac{68-66}{\sqrt{3^2+2^2}})$

d none of the above

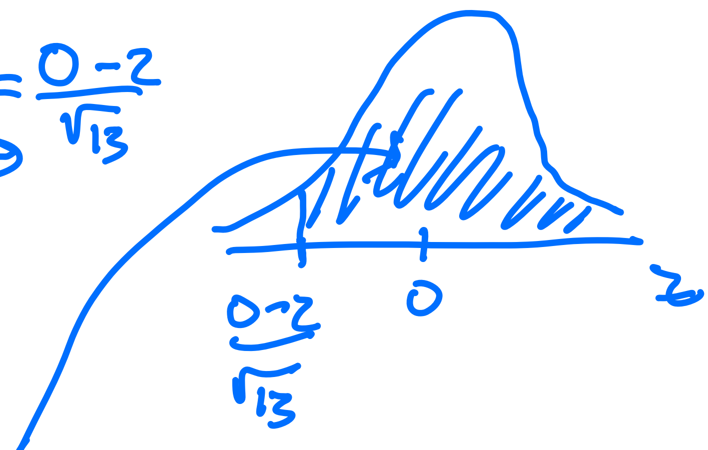
$P(x > y) = P(x - y > 0)$

$x - y \sim N(68-66, 3^2+2^2) = N(2, 13)$



$SD = \sqrt{13}$

$z = \frac{0-2}{\sqrt{13}}$



area $1 - \Phi(\frac{0-2}{\sqrt{13}})$

② Sec 5.3 Chi-square distribution

Fact — see end of lecture notes — if $Z \sim N(0,1)$ then

$$Z^2 \sim Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$$

Recall that if $T \sim Gamma(r, \lambda)$,

$$M_T(t) = \left(\frac{\lambda}{\lambda - t}\right)^r, \quad t < \lambda$$

Hence $M_{Z^2}(t) = \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{1/2}, \quad t < 1/2$

Let $Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$

$$M_{Z_1^2 + \dots + Z_n^2}(t) = \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{1/2 \cdot n}, \quad t < \frac{1}{2}$$

By uniqueness of MGF

$$Z_1^2 + \dots + Z_n^2 \sim Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$$

called chi square distribution with n degrees of freedom, χ^2_n, with n degrees of freedom

ex Let $X, Y \overset{iid}{\sim} N(0,1)$

Let $R = \sqrt{X^2 + Y^2}$ be Raleigh distribution.

Then $R^2 = X^2 + Y^2 \sim \chi^2_2 = Exp(1/2) = Gamma(1, 1/2)$

(3) Sec 5.4 The Density Convolution Formula

eg Let X and Y be discrete RVs taking values $\{0, 1, 2, 3, 4, 5, 6\}$.

Let $S = X + Y$.

Find the probability mass function of S.

$$P(S=3) = P(X=0, Y=3-0) + P(X=1, Y=3-1) + P(X=2, Y=3-2) + P(X=3, Y=3-3)$$

$$P(S=s) = \sum_{x=0}^{s} P(X=x, Y=s-x)$$

convolution formula.

Appendix

If $Z \sim N(0,1)$, then $Z^2 \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

Proof/

$\lambda > 0$, pos integer r,

gamma (r, λ) density

$$f(t) = \frac{\lambda^r}{\Gamma(r)} t^{r-1} e^{-\lambda t}, \quad t > 0$$

let Z = std normal

$X = Z^2$ change of variable rule.

Find $$f_X(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-\frac{1}{2}x}, \quad x > 0$$

$$= \frac{\left(\frac{1}{2}\right)^{1/2}}{\sqrt{\pi}} x^{\frac{1}{2}-1} e^{-\frac{1}{2}x}$$

$\Gamma(\frac{1}{2})$

$\Rightarrow X \sim \text{gamma}(\frac{1}{2}, \frac{1}{2})$

$\square$