Stat 134 lec 2

Warm up 11:00 - 11:10

Prove the complement rule

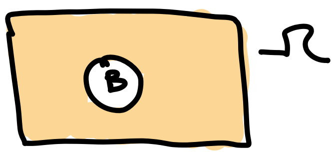
$P(B^c) = 1 - P(B)$

Difference rule

if $B \subseteq A$

$P(AB^c) = P(A) - P(B)$

Picture



trick $A = \Omega$

$B^c = \Omega B^c$

$P(B^c) = P(\Omega B^c) \overset{\text{diff rule.}}{=} P(\Omega) - P(B) = 1 - P(B)$

$P(\Omega) = 1$

or

$B \cup B^c = \Omega$ disjoint union

By add^n rule

$P(B \cup B^c) = P(B) + P(B^c)$

$B \cup B^c = \Omega$, so $P(B \cup B^c) = 1$

$\Rightarrow P(B) = 1 - P(B^c)$ ✓

Last time

Addition rule (OR) if A, B mutually exclusive sets

$$P(A \text{ or } B) = P(A) + P(B).$$

Inclusion exclusion (OR)

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

Today

(0) Mathematical Induction

(1) Sec 1.3 Distributions

(2) Sec 1.4 Conditional Probability

(0) Mathematical Induction

$n=1$

A **proof by induction** consists of two cases. The first, the **base case** (or **basis**), proves the statement for $n = 0$ without assuming any knowledge of other cases. The second case, the **induction step**, proves that *if* the statement holds for any given case $n = k$, *then* it must also hold for the next case $n = k + 1$. These two steps establish that the statement holds for every natural number n.

ex (1.3.12 in HW#1)

12. Inclusion–exclusion formula for n events. Derive the inclusion–exclusion formula for n events

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) - \cdots + (-1)^{n+1} P(A_1 \ldots A_n)$$

Lets assume the following fact from set theory:

$$\bigcup_{i=1}^{K}(A_i A_{K+1}) = \left(\bigcup_{i=1}^{K} A_i\right) A_{K+1} \quad (*)$$

ex $\quad A_1A_3 \cup A_2A_3 = (A_1 \cup A_2)A_3 \qquad K=2$

To prove generalized inclusion exclusion.

Show base case $n=1$
then assume true for $n=K$ and show true for $n=K+1$.

To get going let $K=2$

$$P\left(\bigcup_{i=1}^{2} A_i \cup A_3\right) = P\left(\bigcup_{i=1}^{2} A_i\right) + P(A_3) - P\left(\left(\bigcup_{i=1}^{2} A_i\right) A_3\right)$$

$$\left(\bigcup_{i=1}^{2} A_i\right) A_3 = \bigcup_{i=1}^{2} A_i A_3 \text{ by } (*)$$

$A_1A_3 \cup A_2A_3$

$$= P\left(\bigcup_{i=1}^{2} A_i\right) + P(A_3) - P\left(\bigcup_{i=1}^{2} A_i A_3\right)$$

$P\left(\bigcup_{i=1}^{2} A_i\right) = P(A_1)+P(A_2)-P(A_1A_2)$

$P\left(\bigcup_{i=1}^{2} A_i A_3\right) = P(A_1A_3)+P(A_2A_3) - P(A_1A_3A_2A_3)$, $\quad A_1A_3A_2A_3 = A_1A_2A_3$

$$= \sum_{i=1}^{3} P(A_i) - \sum_{i<j} P(A_iA_j) + P(A_1A_2A_3)$$

try $K=3$ and generalize,

$$P\left(\bigcup_{i=1}^{3} A_i \cup A_4\right) = P\left(\bigcup_{i=1}^{3} A_i\right) + P(A_4) - P\left(\bigcup_{i=1}^{3} A_i A_4\right)$$

① <u>Sec 1.3</u> <u>Distributions</u>

Uniform distribution

Let $\{x_1, x_2, \ldots, x_n\}$ be a finite set.

Suppose the probability of drawing each element is equally likely (i.e each has prob $\frac{1}{n}$)

We say $\{x_1, \ldots, x_n\}$ has the uniform distribution.

We write $\text{Unif}(\{x_1, \ldots, x_n\})$.

$\underline{\text{ex}}$ $\{1, 1, 2\}$ is a finite set.

$\text{Unif}(\{1,1,2\})$ means 1 has probability $\frac{2}{3}$ and 2 has probability $\frac{1}{3}$.

$\underline{\text{ex}}$ $\overset{7}{\text{Suppose}}$ $\overset{1}{\text{a}}$ $\overset{4}{\text{word}}$ $\overset{2}{\text{is}}$ $\overset{8}{\text{randomly}}$ $\overset{6}{\text{picked}}$ $\overset{4}{\text{from}}$ $\overset{4}{\text{this}}$ $\overset{8}{\text{Sentance}}$.

What is the distribution of the length of the word picked?

$$\text{Unif}(\{7, 1, 4, 2, 8, 6, 4, 4, 8\})$$

$$P(X=7) = 1/9$$

$$P(X=4) = 3/9$$

$$\vdots$$

Stat 134

1. A deck of cards is shuffled. What is the chance that the top card is the **king** of spades **or** the bottom card is the ~~**king**~~ Queen of spades

   a $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \times \frac{1}{52}$

   b $\frac{1}{52} + \frac{1}{51}$

   c $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \times \frac{1}{51}$

   d none of the above

$P(A)+P(B)-P(AB)$

$\frac{1}{52} + \frac{1}{52}$ by addition rule.

$A \cap B = \emptyset$

$P(A \cup B) = P(A) + P(B)$

unconditional probabilities.

If replace card after you draw a card the two events are no longer mutually exclusive so use inclusion exclusion formula $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \cdot \frac{1}{52}$

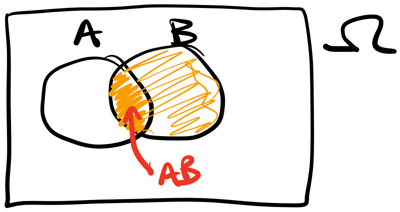
52 not 51 since replace card.

If drawn without replacement but bottom card QS then

$\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \cdot \frac{1}{51}$

② Sec 1.4 Conditional Probability and Independence

Let A, B be subsets of Ω (i.e events).



Baye's rule says $P(A|B) = \frac{P(AB)}{P(B)}$ given

$\Leftrightarrow$ $\boxed{P(AB) = P(A|B)P(B)}$ multiplication rule. (AND)

A and B

We say A and B are <u>independent</u> iff

$P(A|B) = P(A)$

or equivalently if $P(AB) = P(A)P(B)$

ex

A = last card is queen of spades

B = 1st card is king of spades

Is A and B independent ?

$P(AB) = P(A|B)P(B)$
$\quad\quad\quad\quad \overset{\parallel}{1/51} \quad \overset{\parallel}{1/52}$

different hence A and B are dependent.

$P(A)P(B) = \frac{1}{52} \cdot \frac{1}{52}$

ex

(10 pts) An airport bus drops off 35 passengers at 7 stops. Each passenger is equally likely to get off at any stop, and passengers act independently of one another. The bus makes a stop only if someone wants to get off. Find the probability that the bus drops off passengers at every stop.

Let B_i = event drop off at least one person at stop i.

Find $P(B_1) = 1 - P(B_1^c) = 1 - (6/7)^{35}$

$P(B_1 B_2) = 1 - P((B_1 B_2)^c) = 1 - P(B_1^c \cup B_2^c)$

De Morgan's law

$= 1 - \left(P(B_1^c) + P(B_2^c) - P(B_1^c B_2^c)\right)$

$P(B_1^c) = (6/7)^{35}$, $P(B_2^c) = (6/7)^{35}$, $P(B_1^c B_2^c) = (5/7)^{35}$

Find a formula for $P(B_1 B_2 \cdots B_7)$ by next class.

Inclusion–exclusion formula for n events. Derive the inclusion–exclusion formula for n events

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) - \cdots + (-1)^{n+1} P(A_1 \dots A_n)$$