Stat 134 lec 40 (RRR review 1)

Friday — I am out of town.
Brian will give in class review

Today

① MGF review (lec 25)

② Conditional expectation. (lec 34)

③ Gamma/Poisson distribution. (lec 22)

$$M_X(t) = E(e^{tX})$$

Recall

$$M_X'(0) = E(X)$$

$$M_X''(0) = E(X^2)$$

$$\vdots$$

$$M_X^{(k)}(0) = E(X^k)$$

ex Consider a Pois(λ) RV X

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x = 0, 1, 2, \dots$$

a) Compute its MGF

b) Compute $E(X)$ by calculating the appropriate derivative of the MGF.

Recall $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

ex Consider a Pois(λ) RV X

$$P(X=x) = \frac{e^{-\lambda}\lambda^k}{k!} \quad k=0,1,2,\ldots$$

a) Compute its MGF

b) Compute $E(X)$ by calculating the appropriate derivative of the MGF.

Soln

a) $M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} \frac{e^{-\lambda}\lambda^k}{k!}$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda e^t - \lambda} = \boxed{e^{\lambda(e^t-1)}}$$

b) $M_X'(t) = e^{\lambda(e^t-1)} \cdot \underbrace{\frac{d}{dt}\lambda(e^t-1)}_{=\lambda e^t} = \lambda e^t \cdot e^{\lambda(e^t-1)}$

$$\Rightarrow M_X'(0) = \boxed{\lambda}$$

Review of conditional expectation Sec 6.4

Properties (from lec 34)

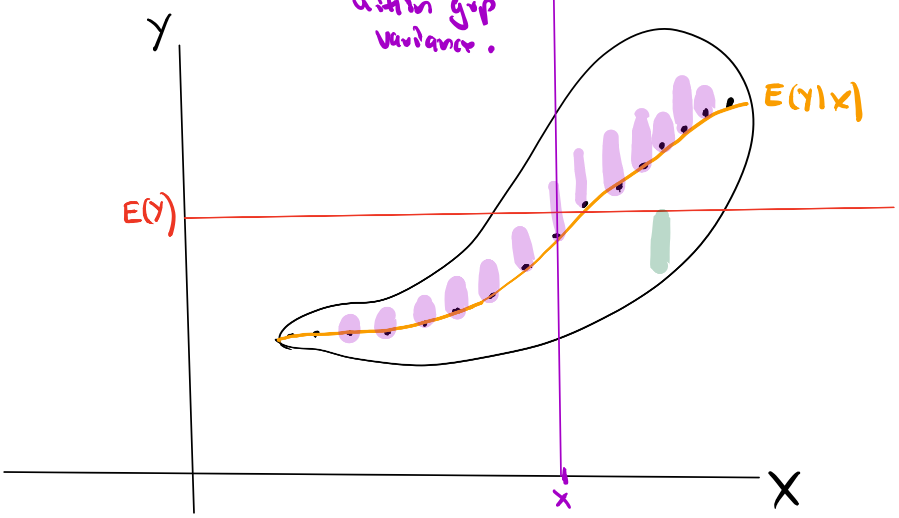
1. $E(Y) = E(E(Y|X))$ iterated expectation
2. $E(aY+b|X) = aE(Y|X)+b$
3. $E(Y+Z|X) = E(Y|X)+E(Z|X)$
4. $E(g(X)|X) = g(X)$
5. $E(g(X)Y|X) = g(X)E(Y|X)$
6. $Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$

$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$$

total variance decomposition (see 6.2.18)

tot var = avg( ) + — var of $E(Y|X)$ (between group variance)

within grp variance.



Note
$E(E(Y|X)) = E(Y)$

$X \mid U=u \sim \mathrm{Exp}\left(\frac{1}{u}\right)$

i.e. $E(X \mid U=u) = u$

ex Let $U \sim \mathrm{Unif}(0,1)$

Given $U=u$, X is exponential with mean u

Find $E(X)$, $E(UX)$ and $\mathrm{Var}(X)$.

Soln

Know $E(X \mid U=u) = u$

$E(X \mid U) = U$

$E(X) = E[E(X \mid U)] = E(U) = 1/2$

$E(UX) = E(E(UX \mid U)) = E[U\underbrace{E(X \mid U)}_{=U})$

$= \underbrace{E(U^2)}_{=\mathrm{Var}(U)+E(U)^2} = 1/3$, $1/12$, $1/4$

$\mathrm{Var}(X) = E\underbrace{(\mathrm{Var}(X \mid U))}_{=U^2} + \mathrm{Var}\underbrace{(E(X \mid U))}_{=U}$

$E(U^2) = 1/3$, $\mathrm{Var}(U) = 1/12$

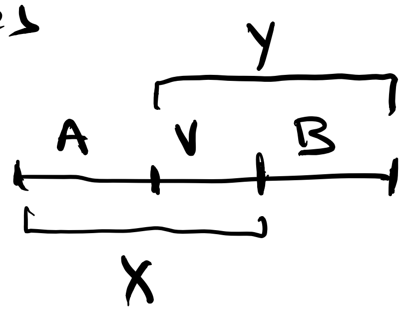
$= 1/3 + 1/12$

$= 5/12$

ex Toss a fair coin 30 times

X = # heads first 20

Y = # heads 2nd 20



$$\left.\begin{array}{l} A \sim Bin(10, \tfrac{1}{2}) \\ V \sim Bin(10, \tfrac{1}{2}) \\ B \sim Bin(10, \tfrac{1}{2}) \end{array}\right\} \text{ indep.}$$

$X = A + V$

$Y = V + B$

Find $Corr(X, Y)$?

$$Corr(X, Y) = \frac{Cov(X, Y)}{SD(X)SD(Y)}$$

$$= Cov(A+V, V+B) = Cov(V, V)$$

$$= Var(V) = 10\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)$$

$$= \frac{10}{4}$$

$$Corr(X, Y) = \frac{\frac{10}{4}}{20/4} = \frac{1}{2}$$

$20/4$ — $SD(X)SD(Y) = \sqrt{\tfrac{20}{4}} \cdot \sqrt{\tfrac{20}{4}}$

Find $E(Y|X) = E(V+B \mid X)$

$$= E(V|X) + E(B|X)$$

$$E(B|X) = E(B) = 10 \cdot \tfrac{1}{2} = 5$$

$V|X$ is hypergeom $(N = 20, n = 10, G = X)$

$$\boxed{\begin{array}{l} X \text{ heads} \\ N-X \text{ tails} \end{array}}$$

$\downarrow$ draw n w/o replacement

$V|X$ is number of good in sample size n

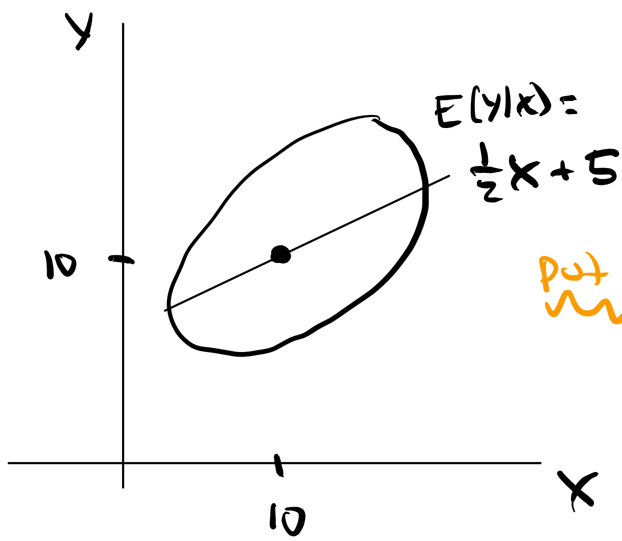
$$E(V|X) = n \cdot \frac{G}{N} = 10 \cdot \frac{X}{20} = \tfrac{1}{2}X$$

$$E(Y|X) = \tfrac{1}{2}X + 5$$

$X \sim N(10, 5)$
$Y \sim N(10, 5)$
$\mathrm{Corr}(X, Y) = 1/2$

binomial is approximately normal by CLT

<u>Picture</u>



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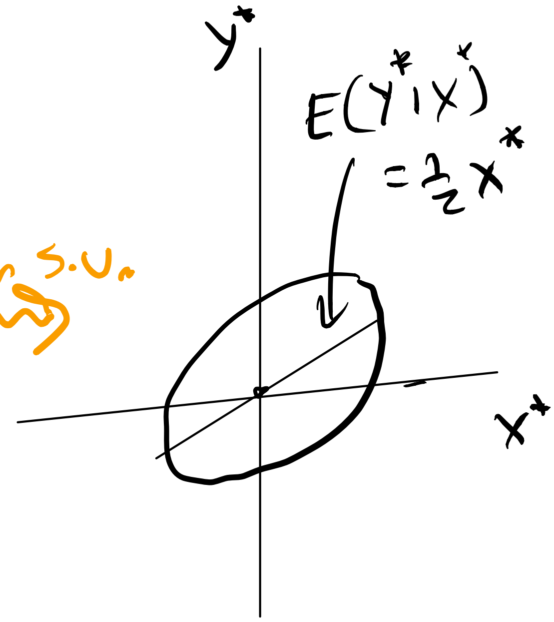


(X, Y) bivariate normal

(X*, Y*) std bivariate normal

Let N have the Poisson distribution with mean μ.
Let $U_1, U_2, \ldots \overset{iid}{\sim} U(0,1)$ independent of N.
Let $M = \min(U_1, \ldots, U_N)$. If $N=0$, define M to be 1.

a) Find $E(M \mid N)$

b) Find $E(M)$

Soln

a) if $n=0$, $M=1$ so $E(M \mid N=0) = 1$

else $M \mid N \sim \text{beta}(1, N)$

$\Rightarrow E(M \mid N) = \frac{1}{N+1}$

b) $E(M) = E(E(M \mid N)) = E\left(\frac{1}{N+1}\right)$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} e^{-\mu} \frac{\mu^n}{n!} = \frac{1}{\mu} \sum_{n=0}^{\infty} e^{-\mu} \frac{\mu^{n+1}}{(n+1)!}$$

$$= \frac{1}{\mu}\left[\frac{e^{-\mu}\mu}{1!} + \frac{e^{-\mu}\mu^2}{2!} + \cdots\right]$$

$$= \frac{1}{\mu}\left[\sum_{n=1}^{\infty} e^{-\mu}\frac{\mu^n}{n!}\right] = \frac{1-e^{-\mu}}{\mu}$$

$$\parallel$$

$$P(N \geq 1) = 1 - P(N=0)$$
$$= 1 - e^{-\mu}$$

Let N have the Poisson distribution with mean μ.
Let $U_1, U_2, \ldots \overset{iid}{\sim} N(0,1)$ independent of N.
Let $M = \min(U_1, \ldots, U_N)$. If $N=0$, define M to be 1.

c) Find the survival function of M
$P(M > m)$

soln

$$P(M>m) = \sum_{n=0}^{\infty} P(M>m \mid N=n) P(N=n)$$

For $n>0$, $P(M>m \mid N=n) = P(U_1 > m, U_2 > m, \ldots, U_n > m)$
$$= (1-m)^n$$

$$\Rightarrow P(M>m) = 1e^{-\mu} + \sum_{n=1}^{\infty} (1-m)^n e^{-\mu} \frac{\mu^n}{n!}$$

$$= \sum_{n=0}^{\infty} (1-m)^n e^{-\mu} \frac{\mu^n}{n!}$$

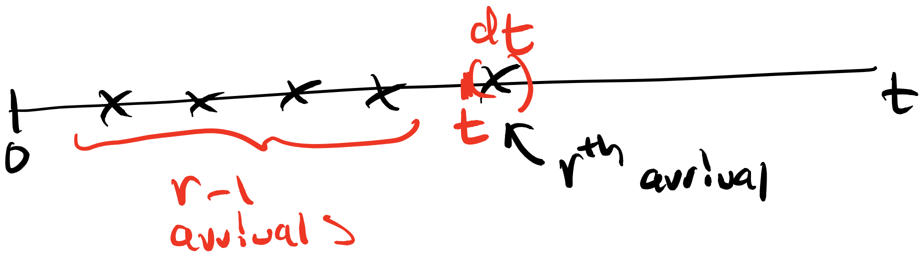
$$= e^{-\mu} \underbrace{\sum_{n=0}^{\infty} \frac{((1-m)\mu)^n}{n!}}_{= e^{(1-m)\mu}}$$

$$= e^{-\mu + \mu - m\mu} = e^{-m\mu}$$

So $P(M>m) = \begin{cases} 1 & m < 0 \\ e^{-m\mu} & 0 \le m < 1 \\ 0 & m \ge 1 \end{cases}$

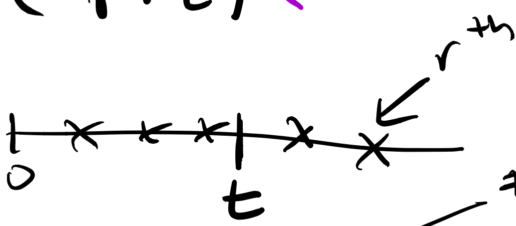
Sec 4.2 Gamma(r,λ) = distribution for r^{th} arrival time of a Poisson (λ) process.

T_r = arrival time of r^{th} call.



Let $T_r \sim \text{gamma}(r, \lambda)$

Find $P(T_r > t)$ (Right tail probability)



\# arrivals in time t

$X_t \sim \text{Pois}(\lambda t)$

$$P(T_r > t) = P(X_t \leq r-1)$$

$$= P(X_t=0) + P(X_t=1) + \dots + P(X_t=r-1)$$

$$= e^{-\lambda t} + e^{-\lambda t}\lambda t + \frac{e^{-\lambda t}(\lambda t)^2}{2!} + \dots + \frac{e^{-\lambda t}(\lambda t)^{r-1}}{(r-1)!}$$

ex

let $T \sim \text{Gamma}(r=4, \lambda=2)$

Find $P(T > 7)$

ex

let $T \sim \text{Gamma}(r=4, \lambda=2)$

Find $P(T>7)$

Soln

$$P(T>7) = P(N_7 \le \overset{r-1}{3})$$

where

$$N_7 \sim \text{Pois}(\overset{2\cdot 7}{14})$$

$$P(T>7) = e^{-14}\left(1 + \frac{14}{1} + \frac{14^2}{2!} + \frac{14^3}{3!}\right)$$