stat 134 lec 30

Last time

sec 5.2 Marginal density $f_Y(y) = \int_{x=-\infty}^{x=\infty} f(x,y)\,dx$

Chap 4

Properties of std normal Z	Proved it
$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$	no
$E(Z) = 0$	no
$SD(Z) = 1$	no

Even though Z is symmetric around zero it is possible $E(X)$ is undefined
ex Cauchy distribution

Let $X, Y \overset{iid}{\sim} N(0,1)$ with density $\phi(x) = ce^{-\frac{1}{2}x^2}$, $c>0$

$f(x,y) = \phi(x)\phi(y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$ for $c>0$

we still need to show that $c = \frac{1}{\sqrt{2\pi}}$, $E(X)=0$, $SD(X)=1$.

Today

sec 5.3

1. Student explanation to concept test
2. Rayleigh distribution $f_R(r) = re^{-\frac{1}{2}r^2}$, $r>0$
3. Sums of independent normal RVs is normal.

① student explanations

1. S and T are i.i.d. Exp(λ). $X = \text{Min(S, T)}$ and $Y = \text{Max(S, T)}$. The joint density is $f(x,y) = 2\lambda^2 e^{\lambda(x+y)}$. The marginal density of X is:

a $2\lambda e^{-2\lambda x}$ for $x > 0$

b $2\lambda e^{-\lambda x}$ for $x > 0$

c $\lambda e^{-\lambda x}$ for $x > 0$

d none of the above

a	We know 0<x<y<inf so the density of x will be the integral of f(x,y) dy from y = x to y= inf thus a is the answer

$X = \min(S,T)$

$f_X(x) = \int_{y=x}^{y=\infty} 2\lambda^2 e^{-\lambda(x+y)} dy$

$= 2\lambda^2 e^{-\lambda x} \int_x^\infty e^{-\lambda y} dy$

$\frac{e^{-\lambda y}}{-\lambda}\Big|_x^\infty = \frac{e^{-\lambda x}}{\lambda}$

$= \boxed{2\lambda e^{-2\lambda x}}$

b	You can just integrate the function with respect to y we got with bounds (0,infinity). That will then give B as your answer

$0 < x < y < \infty$

not

$0 < y < \infty$

a	from the brian and yimingw problem

From last time,

$X = \min(B, Y) \sim \text{Exp}(\lambda_B + \lambda_Y)$

so

$f_X(x) = \lambda_B + \lambda_Y e^{-(\lambda_B+\lambda_Y)x}$

(2) Sec 5.3 Rayleigh Distribution

let $T \sim Exp(\frac{1}{2})$, $f(t) = \frac{1}{2}e^{-\frac{1}{2}t}$, $t > 0$

$R = \sqrt{T}$

Find $f_R(r)$.

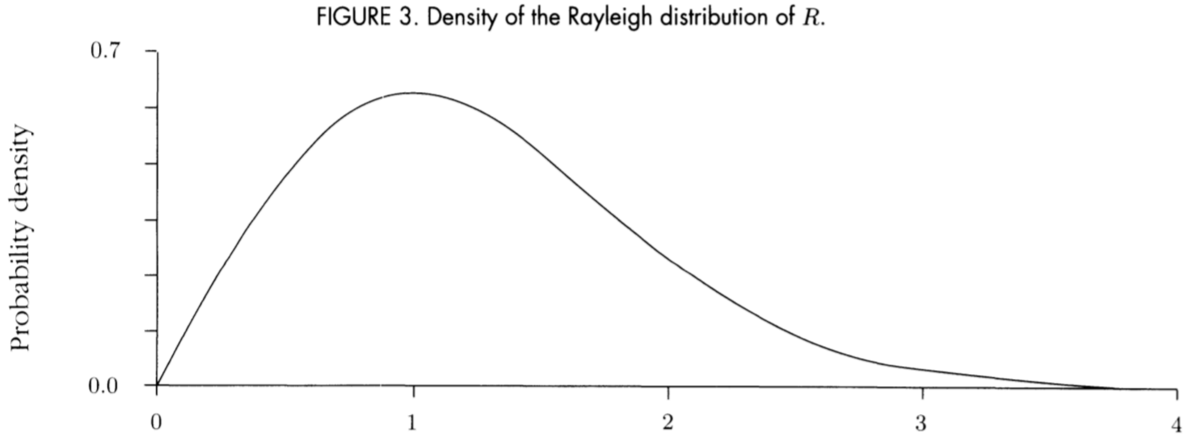
soln

$$f_R(r) = \frac{1}{|(\sqrt{t})'|} \cdot f_T(t) \Big|_{t=r^2}$$

$$f_R(r) = \frac{1}{\frac{1}{2\sqrt{t}}} \cdot \frac{1}{2} e^{-\frac{1}{2}t} \Big|_{t=r^2}$$

$$= \boxed{r e^{-\frac{1}{2}r^2}, \; r > 0}$$

FIGURE 3. Density of the Rayleigh distribution of R.



R is called the <u>Rayleigh distribution.</u>

Write $R \sim \text{Rayleigh}$.

Note:

$$P(R > r) = P(R^2 > r^2) = P(\underset{\sim \text{Exp}(\frac{1}{2})}{T} > r^2)$$

$$= e^{-\frac{1}{2}r^2}$$

So $\boxed{F_R(r) = 1 - e^{-\frac{1}{2}r^2}}$

and $\boxed{f_R(r) = r e^{-1/2 r^2}, \ r > 0}$

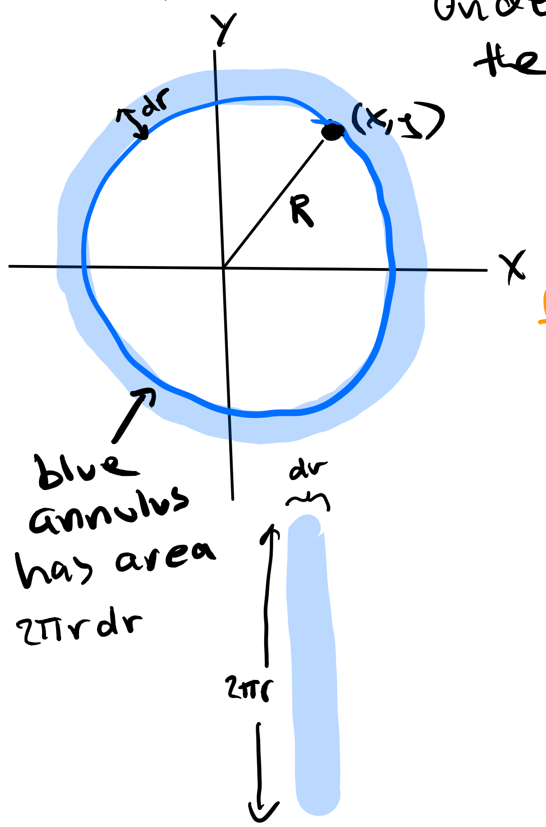
The Raleigh distribution will help us find the density of the standard normal:

For $X, Y \overset{iid}{\sim} N(0,1)$

Let $R = \sqrt{X^2 + Y^2}$

We will show that $R \sim \text{Raleigh}$

$P(R \in dr)$ is the volume of the cylinder under $f(x,y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$ over the shaded blue annulus.



$$P(R \in dr) = \text{height of } f(x,y) \text{ above blue annulus} \cdot \text{area of blue annulus}$$

$$P(R \in dr) = f_R(r)\,dr$$

$$= c^2 e^{-\frac{1}{2}r^2} \cdot (2\pi r\,dr)$$

$$= c^2 2\pi \, \underbrace{r e^{-\frac{1}{2}r^2}}_{\text{Rayleigh Density}} dr$$

$$1 = \int_{r=0}^{r=\infty} P(R \in dr) = c^2 2\pi \underbrace{\int_0^\infty r e^{-\frac{1}{2}r^2} dr}_{=1}$$

$$\Rightarrow c^2 \cdot 2\pi = 1$$

$$\Rightarrow \boxed{c = \frac{1}{\sqrt{2\pi}}}$$

Conclusions

① For $X, Y \overset{iid}{\sim} N(0,1)$ and $T \sim Exp(\frac{1}{2})$

$R = \sqrt{X^2+Y^2}$ and $R = \sqrt{T}$

are both the Rayleigh distribution

$\Rightarrow X^2+Y^2 \sim Exp(\frac{1}{2}) = gamma(1, \frac{1}{2})$

② $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ is density of the standard normal.

(i.e. $C = \frac{1}{\sqrt{2\pi}}$)

— see end of lecture notes

③ $E(X) = 0$ and $SD(X) = 1$

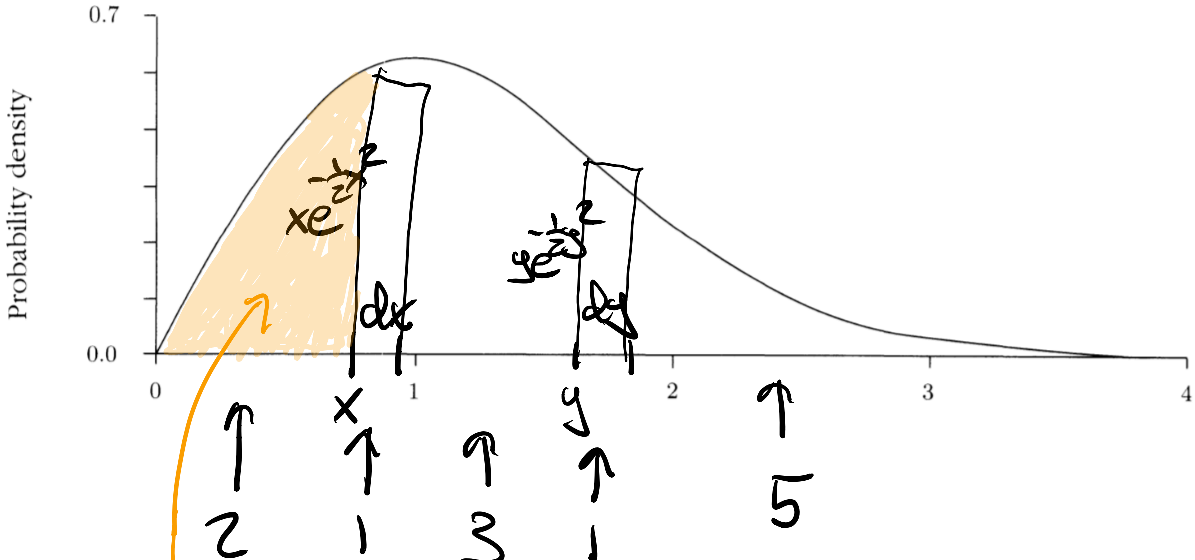
ex

Throw $n = 12$ darts at a target. Suppose the distance that each dart is away from the bull's eye follows a Rayleigh distribution and the distances are independent of each other. Let X be the 3^{rd} closest distance and Y be the 7^{th} closest distance among the 12 darts to the bull's eye.

Find $f_{X,Y}(x,y)$.

$$X \sim R_{(3)}$$
$$Y \sim R_{(7)}$$

FIGURE 3. Density of the Rayleigh distribution of R.



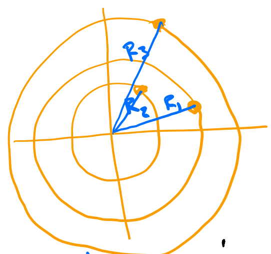
$F_X(x) = 1 - e^{-\frac{1}{2}x^2}$

$$f_{X,Y}(x,y) = \binom{12}{2,1,3,1,5} F_X(x)^2 f_X(x) \left(F_Y(y) - F_X(x)\right)^3 f_Y(y) \left(1 - F_Y(y)\right)^5$$

$$= \binom{12}{2,1,3,1,5} \left(1 - e^{-\frac{1}{2}x^2}\right)^2 x e^{-\frac{1}{2}x^2} \left(e^{-\frac{1}{2}x^2} - e^{-\frac{1}{2}y^2}\right)^3 y e^{-\frac{1}{2}y^2} \left(e^{-\frac{1}{2}y^2}\right)^5$$

ex Suppose 3 shots are fired at a target. Assume for each shot, both X and Y are standard normals. Let W be the closest distance among 3 shots to the bulls eye. Find $f_W(w)$

Picture



$W = \min(R_1, R_2, R_3)$

Where $R_1, R_2, R_3 \overset{iid}{\sim}$ Rayleigh.

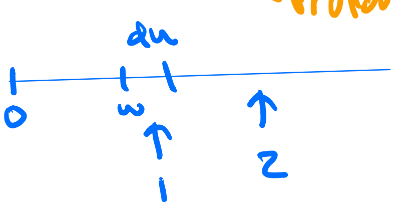
Method 1 (use CDF)

$$P(W > w) = P(R_1 > w,\ R_2 > w,\ R_3 > w)$$
$$= \left[P(R_1 > w)\right]^3$$
$$= \left(e^{-\frac{1}{2}w^2}\right)^3 = e^{-\frac{3}{2}w^2}$$
$$\Rightarrow F(w) = 1 - e^{-3/2\, w^2} \Rightarrow \boxed{f_W(w) = 3w e^{-\frac{3}{2}w^2}}$$

Method 2 (Rayleigh ordered statistic)

$$P(W \in dw) \propto P(R \in dw,\ R > w,\ R > w) = P(R \in dw) P(R > w)^2$$

proportional



$$P(W \in dw) = \binom{3}{1,2} w e^{-\frac{1}{2}w^2} dw \cdot \left(e^{-\frac{1}{2}w^2}\right)^2$$
$$= 3 w e^{-\frac{3}{2}w^2} dw$$
$$\Rightarrow \boxed{f_W(w) = 3w e^{-3/2\, w^2}}$$

Sec 5.3 Sum of independent normals

MGF Review (lecture 24)

Recall the MGF of a RV X is $M(t) = E(e^{tX})$ for some interval of t containing 0.

MGF of std normal

$X \sim N(0,1)$, $M(t) = e^{\frac{t^2}{2}}$ for all t

Properties of MGF

if X, Y are independent $M_{X+Y}(t) = M_X(t) M_Y(t)$

$M_X(t) = M_Y(t)$ iff $X \overset{d}{=} Y$ same distribution

$M_{b+aX}(t) = e^{bt} M_X(at)$

Definition of the Normal (μ, σ^2) distribution:

We know the density of the std normal

$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$. Make a change of scale $X = \mu + \sigma Z$. By the change of variable rule you can show

$$f_X(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

ex Use the properties of MGF to find the MGF of $X \sim N(\mu, \sigma^2)$

For $Z \sim N(0,1)$, $\mu \in \mathbb{R}$, $\sigma > 0$

$X = \mu + \sigma Z$

$b = \mu$ so $M_X(t) = M_{\underset{b}{\mu} + \underset{a}{\sigma} Z} = \boxed{e^{\mu t} e^{\frac{\sigma^2 t^2}{2}}}$

$a = \sigma$

ex Use MGF to prove that the sum of two independent std normals is $N(0,2)$.

$$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$$

$$M_{Z_1+Z_2}(t) = M_{Z_1}(t)\, M_{Z_2}(t)$$

$$= e^{\frac{t^2}{2}} \cdot e^{\frac{t^2}{2}} = e^{\frac{2t^2}{2}}$$

μ $\quad \sigma^2$

$$= e^{0 \cdot t}\, e^{\frac{2t^2}{2}}$$

by uniqueness

$$Z_1 + Z_2 \sim N(0,2)$$

<u>Thm</u> Let $\left.\begin{array}{l} X_1 \sim N(\mu_1, \sigma_1^2) \\ X_2 \sim N(\mu_2, \sigma_2^2) \end{array}\right\}$ indep.

then $aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

Pf/ $M_{X_1}(t) = e^{\mu_1 t} e^{\sigma_1^2 \frac{t^2}{2}}$

$M_{aX_1}(t) = M_{X_1}(at) = e^{\mu_1 a t} e^{\sigma_1^2 a^2 \frac{t^2}{2}}$

then $M_{aX_1 + bX_2}(t) = M_{aX_1}(t) M_{bX_2}(t)$

$= e^{\mu_1 a t} e^{\sigma_1^2 a^2 \frac{t^2}{2}} \cdot e^{\mu_2 b t} e^{\sigma_2^2 b^2 \frac{t^2}{2}}$

$= e^{(\mu_1 a + \mu_2 b)t} e^{(\sigma_1^2 a^2 + \sigma^2 b^2)\frac{t^2}{2}}$

by uniqueness of MGF

$aX_1 + bX_2 \sim N(\mu_1 a + \mu_2 b, \sigma_1^2 a^2 + \sigma^2 b^2)$

$\square$

Appendix

Claim Let $X \sim N(0,1)$. We show that $E(X) = 0$ and $SD(X) = 1$.

Let $X \sim N(0,1)$

Show $E(X) = 0$

$\phi(x)$ is symmetric about zero so all we have to show is that $E(X)$ converges absolutely.

(i.e $E(|X|) < \infty$).

$$E(|X|) = \int_{-\infty}^{\infty} |x| \phi(x) dx$$

$$= 2 \int_{0}^{\infty} x \phi(x) dx$$

$$= 2 \int_{0}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x e^{-\frac{x^2}{2}} dx$$

$\uparrow$ Rayleigh density

$= 1$

$$= \sqrt{\frac{2}{\pi}} < \infty.$$

$\Rightarrow E(X) = 0$ since $\phi(x)$ is semetric around zero

Next we show $SD(X) = 1$:

We know $X^2 + Y^2 \sim \text{expon}\left(\frac{1}{2}\right)$ from above

$\frac{1}{2} = \lambda$

so $E(X^2+Y^2) = \frac{1}{\frac{1}{2}} = 2$

$\frac{1}{2}$: λ rate of X^2+Y^2

$E(X^2) = E(Y^2)$

$\Rightarrow E(X^2) + E(Y^2) = 2$

$\Rightarrow 2E(X^2) = 2 \Rightarrow E(X^2) = 1$

but $SD(X) = \sqrt{E(X^2) - E(X)^2}$
$= \sqrt{1-0} = 1$ ✓