Stat 134 lec 42

Final review RRR week pt 2

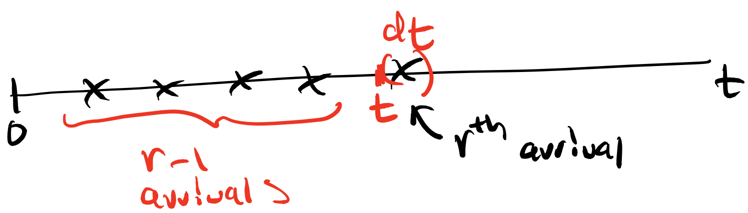
Today

1. sec 4.2 Gamma / Poisson Thinning (Rec 22)
2. CLT and MGF (Rec 24)
3. Expectation / Variance w/ indicators (Rec 14)

(1)

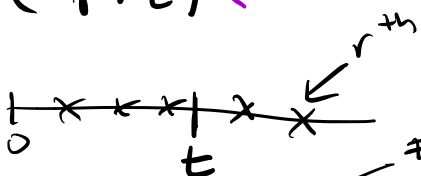
Sec 4.2 Gamma(r,λ) = distribution for r^{th} arrival time of a Poisson (λ) process.

T_r = arrival time of r^{th} call.



Let $T_r \sim$ gamma(r, λ)

Find $P(T_r > t)$ (Right tail probability)



\# arrivals in time t
$X_t \sim$ Pois(λt)

$$P(T_r > t) = P(X_t \le r-1)$$

$$= P(X_t = 0) + P(X_t = 1) + \dots + P(X_t = r-1)$$

$$= e^{-\lambda t} + e^{-\lambda t}\lambda t + \frac{e^{-\lambda t}(\lambda t)^2}{2!} + \dots + \frac{e^{-\lambda t}(\lambda t)^{r-1}}{(r-1)!}$$

ex

let $T \sim$ Gamma$(r=4, \lambda=2)$

Find $P(T > 7)$

$P(T>7) = P(N_7 \le 3)$ where $N_7 \sim$ Pois$(\underset{2\cdot 7}{14})$

$$\boxed{P(T>7) = e^{-14}\left(1 + \frac{14}{1} + \frac{14^2}{2!} + \frac{14^3}{3!}\right)}$$

5. Travelers arrive at an airport Information desk according to a Poisson process at the rate of 15 per hour. Assume that each traveler arriving at the desk has a 60% chance of being male and a 40% chance of being female, independent of all other travelers.

a) Fill in the blank with a number: The fifth male traveler is expected to arrive at the desk _______ minutes after the first male traveler.

$T \sim \text{Pois}\left(15 \cdot \frac{1}{60}\right)$

$M \sim \text{Pois}\left(15(.6)\frac{1}{60}\right) = \text{Pois}\left(9/60\right)$

$F \sim \text{Pois}\left(15(.4)\left(\frac{1}{60}\right)\right) = \text{Pois}\left(6/60\right)$

W_5 = wait time for 5^{th} male $\sim \text{Gamma}\left(5, 9/60\right)$

W_1 = wait time for 1^{st} male $\sim \text{Gamma}\left(1, 9/60\right)$

$$E(W_5 - W_1) = E(W_5) - E(W_1)$$

$$= \frac{5}{9/60} - \frac{1}{9/60} = \frac{4 \cdot 60}{9} = \boxed{\frac{80}{3} \text{ min}}$$

5. Travelers arrive at an airport Information desk according to a Poisson process at the rate of 15 per hour. Assume that each traveler arriving at the desk has a 60% chance of being male and a 40% chance of being female, independent of all other travelers.

b) Find the chance that the fifth male traveler arrives at the desk more than 30 minutes after the first male traveler.

$$P(W_5 - W_1 > 30) = P(W_4 > 30)$$

$$= P(N_{30} \leq 3) \quad \text{where } N_{30} \sim \text{Pois}\left(\frac{9}{60} \cdot 30\right) = \text{Pois}(4.5)$$

$$= \boxed{e^{-4.5}\left(1 + 4.5^1 + \frac{4.5^2}{2} + \frac{4.5^3}{3!}\right)}$$

5. Travelers arrive at an airport Information desk according to a Poisson process at the rate of 15 per hour. Assume that each traveler arriving at the desk has a 60% chance of being male and a 40% chance of being female, independent of all other travelers.

c) Find the expected number of female travelers who arrive at the desk before the fifth male traveler.

Recall $X \sim \text{NegBin}(r, p)$ on $0, 1, 2, \ldots$

is the number of failures before your r^{th} success.

$$E(X) = r \cdot \frac{1-p}{p}$$

Here let X = # female before 5^{th} male

$X \sim \text{NegBin}(5, .6)$ on $0, 1, 2, \ldots$

$$E(X) = 5\left(\frac{.4}{.6}\right) = 10/3$$

or

W_5 = waiting time (min) 5^{th} male

$\sim \text{Gamma}(5, 9/60)$

$E(W_5) = 300/9$ min.

N_{W_5} = # females in time W_5

$\sim \text{Pois}\left(\frac{6}{60} \cdot W_5\right)$

$$E(N_{W_5}) = E(\underbrace{E(N_{W_5} \mid W_5)}_{\frac{6}{60} W_5}) = \frac{6}{60} E(\overset{300/9}{W_5}) = 10/3$$

Review MGF $M_X(t) = E(e^{tX})$

Main properties

1. $M_X(0) = 1$

2. $M_{aX}(t) = M_X(at)$

3. $M_X'(0) = E(X)$

   $M_X''(0) = E^2(X)$

   $\vdots$

   $M_X^{(k)}(0) = E^{(k)}(X)$

4. If $X_1, \dots, X_n$ are independent then

   $M_{X_1 + \dots + X_n}(t) = M_{X_1}(t) \cdots M_{X_n}(t)$

5. $M_X(t)$ is unique for t in a neighborhood of 0. So if $M_X(t) = e^{t^2/2}$ for all t then $X \sim N(0,1)$.

ex

CLT

Let $X_1, \ldots, X_n \overset{iid}{\sim} F$, mean μ, SD σ ← any distribution

$S_n = \sum_{i=1}^{n} X_i$

$S_n \longrightarrow N(n\mu, n\sigma^2)$ as $n \to \infty$

Pf/ We show that

$$\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \longrightarrow Z \sim N(0,1) \quad \text{as } n \to \infty$$

Let $Y_i = \frac{X_i - \mu}{\sigma}$

$$\sum_{i=1}^{n} Y_i = \frac{\overbrace{\sum_{i=1}^{n} X_i}^{S_n} - n\mu}{\sigma}$$

So $\sum_{i=1}^{n} \frac{Y_i}{\sqrt{n}} = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$

We will show that for n large,

$\sum_{i=1}^{n} \frac{Y_i}{\sqrt{n}}$ and Z have the same MGF.

Note that

$E(Y_i) = E\left(\frac{X_i - \mu}{\sigma}\right) = \frac{1}{\sigma} E(X_i - \mu) = 0$

$Var(Y_i) = \frac{1}{\sigma^2} Var(X_i - \mu) = \frac{1}{\sigma^2} \cdot \sigma^2 = 1$

So $E(Y_i^2) = Var(Y_i) + E(Y_i)^2 = 1$

Make a Taylor series of $M_{\frac{Y_i}{\sqrt{n}}}(t)$ around 0:

$$M_{\frac{Y_i}{\sqrt{n}}}(t) = M_{Y_i}\left(\frac{t}{\sqrt{n}}\right) = M_{Y_i}(0) + M'_{Y_i}(0)\frac{t}{\sqrt{n}} + \frac{M''_{Y_i}(0)\frac{t^2}{n}}{2!} + \cdots$$

$$= 1 + \underbrace{E(Y_i)}_{=0}\frac{t}{\sqrt{n}} + \frac{\overbrace{E(Y_i^2)}^{=1}}{2!}\frac{t^2}{n} + \cdots$$

$$= 1 + \frac{1}{n}\left[\frac{t^2}{2} + \underbrace{\frac{t^3 M'''(0)}{3!\, n^{1/2}} + \cdots}_{\to 0}\right]$$

all terms $\to 0$ since have n in denom,

$$\to 1 + \frac{1}{n}\frac{t^2}{2} \quad \text{as } n \to \infty$$

So

$\frac{Y_i}{\sqrt{n}}$ are independent,

$$M_{\frac{S_n - n\mu}{\sqrt{n}\sigma}}(t) = M_{\frac{Y_1}{\sqrt{n}}}(t) \cdots M_{\frac{Y_n}{\sqrt{n}}}(t)$$

$$\to \left[1 + \frac{1}{n}\frac{t^2}{2}\right]^n \approx e^{\frac{t^2}{2}}$$

which is MGF of $N(0,1)$

Hence $\frac{S_n - n\mu}{\sqrt{n}\sigma} \to N(0,1)$ $\square$

③ Expectation / Variance with indicators.

6. A drawer contains s black socks and s white socks, where s is a positive integer. I pull two socks out at random without replacement and call that my first pair. Then I pull two socks out at random without replacement from the remaining socks in the drawer, and call that my second pair. I proceed in this way till I have s pairs and the drawer is empty.

Let D be the number of pairs in which the two socks are of different colors.

a) Find $E(D)$

any of 2s possible socks

$$D = I_1 + \cdots + I_s \qquad \frac{2s}{2s} \cdot \frac{s}{2s-1}$$

any of s possible socks of other color

$$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ pair diff color} \\ 0 & \text{else} \end{cases} \qquad = \frac{s}{2s-1}$$

$$E(D) = s \cdot \left(\frac{s}{2s-1}\right) = \boxed{\frac{s^2}{2s-1}}$$

6. A drawer contains s black socks and s white socks, where s is a positive integer. I pull two socks out at random without replacement and call that my first pair. Then I pull two socks out at random without replacement from the remaining socks in the drawer, and call that my second pair. I proceed in this way till I have s pairs and the drawer is empty.

Let D be the number of pairs in which the two socks are of different colors.

b) Find Var(D).

$$D = I_1 + \cdots + I_s$$

$$\text{Var}(D) = s\,\text{Var}(I_1) + s(s-1)\,\text{Cov}(I_1, I_2)$$

$$\text{Cov}(I_1, I_2) = E(\underset{I_1 I_2}{I_{12}}) - E(I_1)E(I_2)$$

$$\frac{2s}{2s}\frac{s}{2s-1}\frac{2s-2}{2s-2}\frac{s-1}{2s-3} = \frac{s}{2s-1}\cdot\frac{s-1}{2s-3}$$

$$I_{12} = \begin{cases} 1 & \text{if } 1^{st} \text{ and } 2^{nd} \text{ pair diff color.} \\ 0 & \text{else} \end{cases}$$

$$s(s-1)\,\text{Cov}(I_1, I_2) = s(s-1)\left[\frac{s}{2s-1}\cdot\frac{s-1}{2s-3} - \left(\frac{s}{2s-1}\right)^2\right]$$

$$s \cdot \text{Var}(I_1) = s \cdot \left[\left(\frac{s}{2s-1}\right)\left(1 - \frac{s}{2s-1}\right)\right]$$

$$\text{Var}(D) = s \cdot \text{Var}(I_1) + s(s-1)\,\text{Cov}(I_1, I_2)$$

$\square$