Stat 134 lec 22

next Wednesday Q4 covers 4.1, 4.2

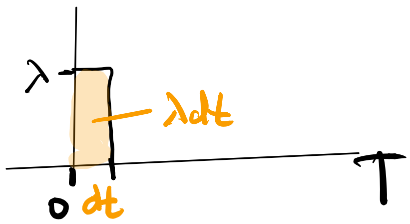
__last time__ __sec 4.2__ __exponential distribution.__

$T \sim Exp(\lambda)$ $\quad f(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$

T = time until 1^{st} success (arrival)

$$P(0 < T < 0 + dt) = f(0)dt = \lambda e^{-\lambda 0} dt = \boxed{\lambda dt}$$

$$\Rightarrow \lambda = \frac{P(0 < T < 0 + dt)}{dt} \text{ is the}$$



instantaneous rate of success,

__Memoryless property__: $P(T \in dt \mid T > t) = P(0 < T < 0 + dt)$

$$P(T \in dt \mid T > t) = \frac{P(T \in dt)}{P(T > t)} = \frac{\lambda e^{-\lambda t} dt}{e^{-\lambda t}} = \boxed{\lambda dt}$$

interpretation:

So long as a lightbulb is still functioning it is as good as new

__Relationship w/ $X \sim Geom(p)$__

$pX \to Exp(1)$ as $p \to 0$

__Today__ __sec 4.2__

1. Expectation and Variance of $Exp(\lambda)$.
2. Gamma distribution.

Sec 4.2 Expectation and Variance of Exp(λ).

$T \sim \text{Exp}(\lambda)$

see end of lec notes.

$$E(T) = \int_0^\infty t f(t)\,dt = \lambda \int_0^\infty t e^{-\lambda t}\,dt = \boxed{\frac{1}{\lambda}}$$

see end of lec notes.

$$E(T^2) = \int_0^\infty t^2 f(t)\,dt = \lambda \int_0^\infty t^2 e^{-\lambda t}\,dt = \frac{2}{\lambda^2}$$

$$\text{var}(T) = E(T^2) - E(T)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

check for $\lambda = 1$

Recall that for $X \sim \text{Geom}(p)$

$E(X) = \frac{1}{p}$

$\text{var}(X) = \frac{q}{p^2}$

$E(pX) = pE(X) = \frac{p}{p} = 1 \longrightarrow 1$ as $p \to 0$

$pX \to T \sim \text{Exp}(1)$ as $p \to 0$

$E(pX) \to E(T)$ as $p \to 0$

$\Rightarrow E(T) = 1$ ✓

$\text{var}(pX) = p^2 \text{var}(X) = \frac{p^2 q}{p^2} = q \longrightarrow 1$ as $p \to 0$.

$\text{var}(pX) \to \text{var}(T)$ as $p \to 0$

$\Rightarrow \text{var}(T) = 1$ ✓

Interpretation of λ:

$$\lambda = \frac{1}{E(T)} \quad \text{(one over average arrival time)}$$

ex Let T = waiting time (min) until a blue Honda enters a toll gate starting at noon. The waiting time is on average 2 minutes. Find $P(T>3)$.

$E(T) = 2\text{ min}$

$\lambda = \frac{1}{2}\text{ min}^{-1}$

$\Rightarrow T \sim \text{Exp}(1/2\text{ min}^{-1})$

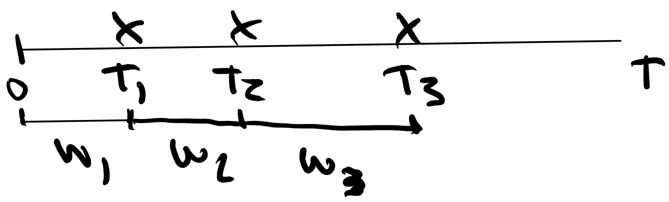
$P(T>3) = e^{-\frac{1}{2}\cdot 3} = \boxed{e^{-3/2}}$

ex Gamma Distribution

A gamma distribution $T \sim \text{Gamma}(r, \lambda)$, $\lambda > 0$ is a sum of r iid $\text{Exp}(\lambda)$

$T = W_1 + \dots + W_r \qquad W_i \sim \text{Exp}(\lambda)$

Picture



dep. $\begin{cases} T_1 \sim \text{Gamma}(1,\lambda) \\ T_2 \sim \text{Gamma}(2,\lambda) \\ T_3 \sim \text{Gamma}(3,\lambda) \end{cases}$ but W_1, W_2, W_3 indep.

Describe $\text{Gamma}(r,\lambda)$ as time of r^{th} arrival in a Poisson Process.

Review Poisson Process

$N_t \sim \text{Poisson}(\lambda t)$

N_t = # arrivals in time t where rate of arrivals is t.

$$P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

ex Let T = waiting time (min) until a blue Honda enters a toll gate starting at noon. The waiting time is on average 2 minutes. Find the probability 3 blue Hondas arrive in 8 minutes.

soln $\lambda = \frac{1}{2}$ blue Honda/minute.

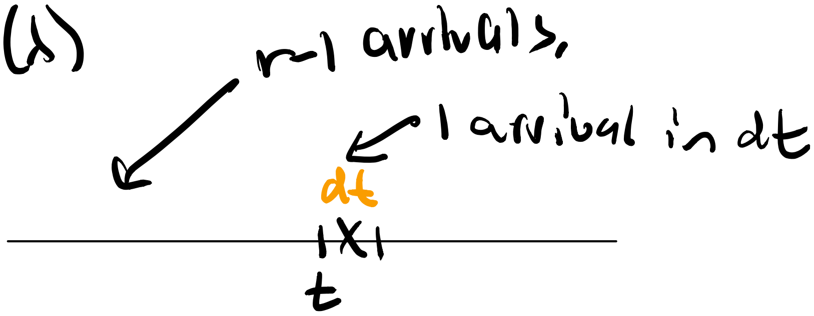
$$N_8 \sim \text{Poisson}(\overset{\frac{1}{2}\cdot 8}{4})$$

$$P(N_8 = 3) = \boxed{\frac{e^{-4}\,4^3}{3!}}$$

Back to gamma distribution:

Let $N_t \sim \text{Pois}(\lambda t)$

$W \sim \text{Exp}(\lambda)$

$P(T \in dt)$ means 

so $P(T \in dt) = P(N_t = r-1)\,P(W \in dt \mid W > t)$

$$= \frac{e^{-\lambda t}(\lambda t)^{r-1}}{(r-1)!} \cdot \lambda\, dt$$

$$= \underbrace{\frac{1}{(r-1)!} \lambda^r t^{r-1} e^{-\lambda t}}_{f(t)} \, dt$$

The Gamma distribution can be defined for $r > 0$ real number not just for $r \in \mathbb{Z}^+$.

If $r \in \mathbb{Z}^+$ define $\Gamma(r) = (r-1)!$ where $\Gamma(r)$ is called the gamma function

Then $f(t) = \underbrace{\frac{1}{\Gamma(r)} \lambda^r}_{\text{const}} \underbrace{t^{r-1} e^{-\lambda t}}_{\text{variable}}$

and $\int_0^\infty f(t)\,dt = 1 \Rightarrow$

$$\frac{1}{\Gamma(r)} \int_0^\infty \lambda^r t^{r-1} e^{-\lambda t}\,dt = 1$$

$$\Rightarrow \Gamma(r) = \int_0^\infty \lambda^r t^{r-1} e^{-\lambda t}\,dt.$$

This is actually the definition of the gamma function and using induction you can show that $\Gamma(r) = (r-1)!$ for $r \in \mathbb{Z}^+$.

This allows us to define $T \sim \text{Gamma}(r, \lambda)$ for $r > 0$ as having density $f(t) = \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}$.

Ex Suppose $T \sim \text{Gamma}(r=4, \lambda = 2s^{-1})$

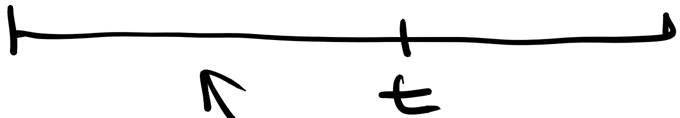
Find $f(t)$

$$= \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-t} = \frac{1}{\Gamma(4)} 2^4 t^3 e^{-2t}$$

$$= \boxed{\frac{8}{3} t^3 e^{-2t}}$$

Let $T \sim \text{Gamma}(r, \lambda)$

$$P(T > t) = \int_t^\infty \frac{1}{\Gamma(r)} \lambda^r s^{r-1} e^{-\lambda s}\, ds$$



$\leq r-1$ arrivals here.

$$P(T > t) = P(N_t \leq r-1)$$

$$= \sum_{i=0}^{r-1} P(N_t = i)$$

$$= \boxed{\sum_{i=0}^{r-1} \frac{e^{-\lambda t} (\lambda t)^i}{i!}}$$

~~Ex~~ $T \sim \text{Gamma}(r=4, \lambda = 2\,s^{-1})$

Find $P(T>7)$, $P(T<7)$

$$P(T>7) = P(N_7 \leq 3)$$

where $N_7 \sim \text{Pois}(\overset{2\cdot 7}{14})$

$$= \boxed{e^{-14}\left(1 + \frac{14}{1!} + \frac{14^2}{2!} + \frac{14^3}{3!}\right)}$$

$$P(T<7) = 1 - P(T>7)$$

$\downarrow$

Suppose customers are arriving at a ticket booth at rate of five per minute, according to a Poisson arrival process. Find the probability that:

(a) Starting from time 0, the $9th$ customer doesn't arrive within 5 minutes;

(b) At least one customer arrives within 40 seconds after the arrival of the $13th$ customer.

$N_5 \sim \text{Pois}(5 \cdot 5)$

soln

(a) $P(T_9 > 5) = P(N_5 < 9) = \sum_{k=0}^{8} e^{-25} \frac{25^k}{k!}$

(b) $P(W_{14} < \frac{2}{3}) = 1 - e^{-5 \cdot 2/3}$

Appendix

Expectation and Variance of exponential

let $T \sim Exp(\lambda)$

$f(t) = \lambda e^{-\lambda t}$ density

$$E(T) = \lambda \int_0^\infty t e^{-\lambda t} dt$$

I recommend using the tabular method for integration by parts. This works well when the function you are integrating is the product of two expressions, where the n^{th} derivative of one expression is zero.

ex $t^4 e^{3t}$ ✓ good

ex $\sin t \, e^{3t}$ ✗ bad.

To find $\lambda \int_0^\infty t e^{-\lambda t}$:

$\frac{d}{dt}$		$\int$
t	$+$	$e^{-\lambda t}$
1	$-$	$\frac{-e^{-\lambda t}}{\lambda}$
0		$\frac{e^{-\lambda t}}{\lambda^2}$

$$E(T) = \lambda \int_0^\infty t e^{-\lambda t} dt = \lambda\left(t\left(-\frac{e^{-\lambda t}}{\lambda}\right) - 1 \cdot \frac{e^{-\lambda t}}{\lambda^2}\right)\Bigg|_0^\infty$$

$$= 0 - \left(0 - \frac{1}{\lambda}\right)$$

$$= \boxed{\frac{1}{\lambda}}$$

Next find $Var(T) = E(T^2) - E(T)^2$:

$$E(T^2) = \lambda \int_0^\infty t^2 e^{-\lambda t} dt$$

$\frac{d}{dt}$		$\int$
t^2	$+$	$e^{-\lambda t}$
$2t$	$-$	$-\frac{1}{\lambda} e^{-\lambda t}$
2	$+$	$\frac{1}{\lambda^2} e^{-\lambda t}$
0		$-\frac{1}{\lambda^3} e^{-\lambda t}$

$$E(T^2) = \lambda\left(t^2\left(-\frac{1}{\lambda}e^{-\lambda t}\right) - 2t\left(\frac{1}{\lambda^2}e^{-\lambda t}\right) + 2\left(-\frac{1}{\lambda^3}e^{-\lambda t}\right)\right)\Big|_0^\infty$$

$$= \frac{2}{\lambda^2}$$

$$\Rightarrow \mathrm{Var}(T) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \boxed{\frac{1}{\lambda^2}}$$