Stat 134 lec 33

Last time:

Sec 5.4 Convolution formula for density of Y/X

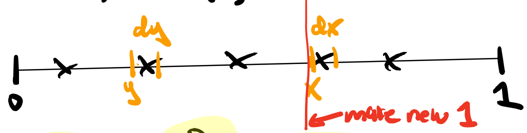
$Z = Y/X \quad X>0,\ Y>0$

$$f(z) = \int_{x=0}^{x=\infty} f(x, zx)\, x\, dx = \int_{x=0}^{x=\infty} f_X(x) f_Y(zx)\, x\, dx$$

- if X, Y indep.

ex $U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$

$Y = U_{(k)},\ X = U_{(m)}$ with $k<m$



$$\frac{U_{(k)} \text{ in } n}{U_{(m)} \text{ in } n} \overset{D}{=} U_{(k)} \text{ in } m-1 \quad (\text{both are Beta}(k, n-k))$$

Today

① Sec 5.4 Review student explanation from concept test.

② Sec 6.1 Conditional Distribution: Discrete case.

③ Sec 6.2

ⓐ Conditional expectation: $E(T \mid X=x)$

ⓑ Rule of average conditional expectations,

$$E(T) = E(E(T \mid X))$$

Let $X \sim U(0,1)$ and $Y \sim U(0,1)$ be independent. The density of $Z = Y/X$ for $0 < z < 1$ is:

a $1/(2z)$

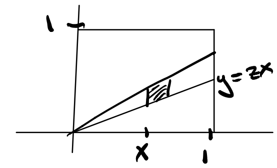
b $1/2$

c $1/(2z^2)$

d none of the above

Discuss with your neighbor for 1 minute how you did this.

b	Integrate from 0 to 1 because the slope is between 0 to 1 so it goes to the end of where x can be in the joint



b	I used the previous formula and integrated (fx=1)*(fy=1)*x from 0 to 1; intuitively, however, I have no idea what's going on

$$f_Z(z) = \int_{x=0}^{x=1} f_X(x) f_Y(zx)\, x\, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

TRUE

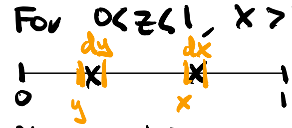
$0<z<1 \iff x>y$

$P(0<z<1) = 1/2$

For $0<z<1$, $x>y$

d	y/x is uniform over 0 to 1

Problem is that $P(0<z<1) = 1/2$ Since $0<z<1$ iff $x>y$



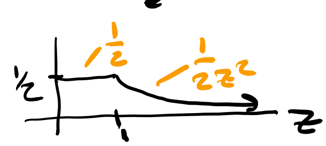
$$\frac{Y}{X} = \frac{U_{(1)} \text{ in } 2}{U_{(2)} \text{ in } 2} \stackrel{D}{=} U_{(1)} \text{ in } 1 = U(0,1)$$

$Z \sim U(0,1)$ half the time so $f(z) = 1/2$.

b	For 0<z<1, then X>Y. Since they're both uniform and by symmetry P(X>Y)=p(Y>X) So the density should be 1/2 since it'll happen half the time uniformly

Let $T = \frac{X}{Y}$ i.e. $T = \frac{1}{Z}$

$$f_T(t) = \frac{1}{\left|-\frac{1}{z^2}\right|} \cdot f_Z(z)\Big|_{z=\frac{1}{t}} = \frac{z^2}{2}\Big|_{z=\frac{1}{t}} = \frac{1}{2t^2}$$



② Sec 6.1 Conditional Distribution: Discrete case.

let X, N discrete RVs w/ joint distribution $P(X=x, N=n)$.

Bayes rule

$$P(X=x \mid N=n) = \frac{P(X=x, N=n)}{P(N=n)}$$

or $$P(X=x, N=n) = P(X=x \mid N=n)P(N=n)$$

Rule of average conditional probabilities

marginal prob of X

$$P(X=x) = \sum_n P(X=x, N=n)$$

$$= \sum_n P(X=x \mid N=n)P(N=n)$$

ex

Let N have Poisson (λ) distribution. Let X be a random variable with the following property: for every n, the conditional distribution of X given $(N = n)$ is binomial (n, p). Find the unconditional distribution of X and state its parameter(s). Show all your work for full credit.

$$P(N=n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

$$P(X=x \mid N=n) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

Find $P(X=x)$

$$P(X=x) = \sum_{n=x}^{\infty} P(X=x \mid N=n)P(N=n)$$

$$= \sum_{n=x}^{\infty} \frac{\cancel{n!}}{x!(n-x)!} p^x q^{n-x} \frac{e^{-\lambda}\lambda^n}{\cancel{n!}} \quad \lambda^x \cdot \lambda^{n-x}$$

$$= \frac{e^{-\lambda}\lambda^x p^x}{x!} \sum_{n=x}^{\infty} \frac{\lambda^{n-x} q^{n-x}}{(n-x)!}$$

Finish

$$= \underbrace{\frac{e^{-\lambda}\lambda^x p^x}{x!}}\ \underbrace{\left(1+\lambda q+\frac{(\lambda q)^2}{2!}+\cdots\right)}_{\substack{\shortparallel \\ e^{\lambda q}}}$$

$$= \frac{e^{-\lambda \overbrace{(1-q)}^{p}} (\lambda p)^x}{x!}$$

$$\boxed{X \sim \mathrm{Pois}(\lambda p)}$$

<u>explanation</u> (Poisson thinning)

N = # cars in 1 minute $\quad N \sim \mathrm{Pois}(\lambda \cdot 1)$

p = Prob of car being red

$X \mid N=n$ = # of red cars given $N=n$

$X \mid N=n \sim \mathrm{Bin}(n,p)$

X = # red cars in 1 minute

$X \sim \mathrm{Pois}(\lambda p)$ by Poisson thinning.

Sec 6.2 Conditional Expectation (discrete case)

recall Bayes rule: $P(T=t \mid S=s) = \dfrac{P(T=t, S=s)}{P(S=s)}$

ex (T,S) is joint distribution below.

Find $P(T=3 \mid S=7)$

$$= \frac{P(T=3, S=7)}{P(S=7)} = \frac{.3}{.4} = \boxed{.75}$$

	T=3	T=4	Sum ← marginal of S
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1.0

marginal of T

Find $P(T=4 \mid S=7)$

$$\frac{P(T=4, S=7)}{P(S=7)} = \frac{.1}{.4} = .25$$

Find $E(T)$ $= \sum_{t \in T} t P(T=t) = 3(.6) + 4(.4) = 3.4$

Find $E(T \mid S=7)$

$$= \sum_{t \in T} t \cdot P(T=t \mid S=7) = 3 \cdot P(T=3 \mid S=7) + 4 \cdot P(T=4 \mid S=7)$$

$$= 3(.75) + 4(.25) = 3.25$$

Find $E(T \mid S=6)$

$= 3\,P(T=3 \mid S=6) + 4 \cdot P(T=4 \mid S=6)$

$= 3\left(\frac{.2}{.4}\right) + 4\left(\frac{.2}{.4}\right) = 3.5$

	T=3	T=4	Sum ← marginal of S
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1.0

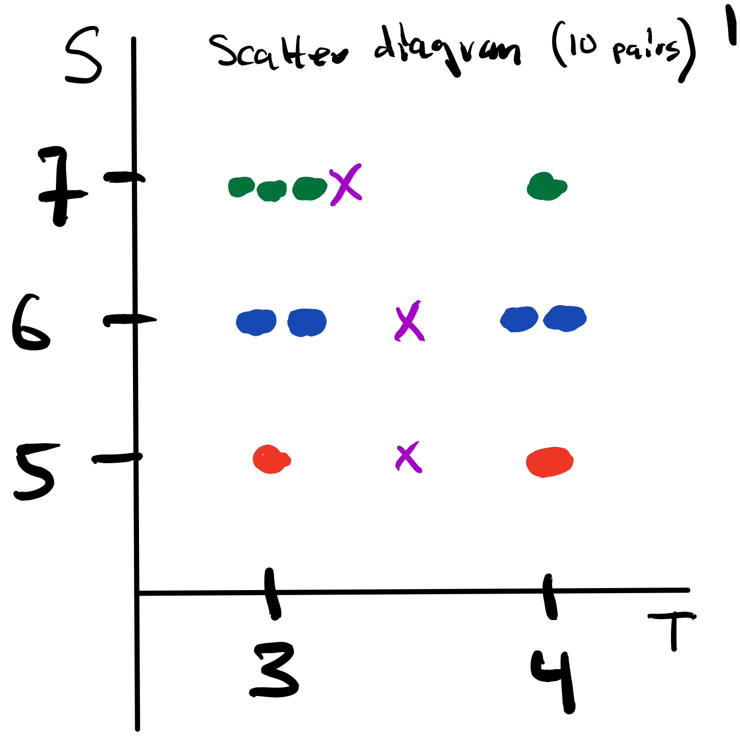
marginal of T

$$\left.\begin{aligned} E(T \mid S=7) &= 3.25 \\ E(T \mid S=6) &= 3.5 \\ E(T \mid S=5) &= 3.5 \end{aligned}\right\} \text{function of } S$$

Picture

joint distribution

	T=3	T=4	Sum
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1



Two main points:

(1) $E(T|S)$ is a function of S.

(2) $E(T|S)$ is a RV so it has an expectation.

Next we explore the expectation of $E(T|S)$.

Let $g(s) = E(T|S)$

$g(7) = E(T|S=7) = 3.25$

$g(6) = 3.5$

$g(5) = 3.5$

$$E(g(S)) = \sum_{s \in S} g(s) P(S=s)$$

$$= 3.25(.4) + 3.5(.4) + 3.5(.2)$$

$$= 3.4$$

← this is $E(T)$.

it is the weighted average of all of the group averages.

In other words,

$$E(E(T|S)) = E(T)$$

This is called the property of <u>iterated expectations</u>.

Intuitively,

If you have a class that is $2/3$ girls and $1/3$ boys and the girls weigh on average 100 lbs and boys weigh 200 lbs then the average weight of the class should be $\frac{2}{3}(100) + \frac{1}{3}(200)$. i.e. we take the weighted average of the averages.

Rule of average conditional expectations

For any random variable Y with finite expectation and any discrete RV X,

$$E(T) = \sum_{\text{all } s} E(T \mid S = s) \cdot P(S = s)$$

(see end of this lecture for a formal proof) ex

8 transistors (type 1) are distributed Exp$\left(\frac{1}{100}\right)$ and 4 transistors (type 2) are Exp$\left(\frac{1}{200}\right)$.

Let T be the lifetime of a randomly picked transistor.

Find $E(T)$.

soln

Let X = type of transistor

$$E(T) = E(E(T|X))$$

$$= E(T|X=1)\cdot P(X=1) + E(T|X=2)P(X=2)$$

$$= 100 \cdot \frac{8}{12} + 200 \cdot \frac{4}{12} = 133.3$$

ex

(2 pts) Let $X \sim$ Geom (p) on $\{1, 2, \ldots\}$. Suppose Y is a variable such that given $X = x$, Y is uniformly distributed on $\{1, 2, \ldots, x\}$. Find $\mathbb{E}(Y)$. (Hint: for $U \sim$ Unif $\{a, a+1, \ldots, b\}$, $\mathbb{E}(U) = \frac{a+b}{2}$.)

$E(Y) = E(E(Y|X))$

$Y|X \sim \text{Unif}(1, 2, \ldots, x)$

$E(Y|X) = \frac{1+X}{2}$

$E(X) = 1/p$

$$E(Y) = E(E(Y|X)) = E\left(\frac{1+X}{2}\right) = \underbrace{E\left(\tfrac{1}{2}\right)}_{=\frac{1}{2}} + \underbrace{E\left(\tfrac{X}{2}\right)}_{=\frac{1}{2p}}$$

$$= \boxed{\frac{1}{2} + \frac{1}{2p}}$$

Appendix

Iterated Expectation

We show $E(Y) = E(E(Y|X))$:

$$E(Y) = \sum_{\text{all } y} y P(Y=y)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} P(X=x, Y=y)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} \frac{P(X=x, Y=y)}{P(X=x)} P(X=x)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} P(Y=y \mid X=x) P(X=x)$$

$$= \sum_{\text{all } x} \left(\sum_{\text{all } y} y P(Y=y \mid X=x) \right) \cdot P(X=x)$$

$$= \sum_{\text{all } x} E(Y \mid X=x) \cdot P(X=x)$$

$$= E(E(Y|X))$$

$\square$