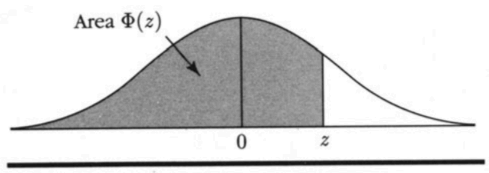
Stat 134 lec 20

Warmup 9:00 – 9:10

**11.** A large lot of marbles have diameters which are approximately normally distributed with a mean of 1 cm. One third have diameters greater than 1.1 cm. Find:

a) the standard deviation of the distribution;

You might need this:




# Appendix 5
# Normal Table

Table shows values of $\Phi(z)$ for $z$ from 0 to 3.59 by steps of .01. Example: to find $\Phi(1.23)$, look in row 1.2 and column .03 to find $\Phi(1.2 + .03) = \Phi(1.23) = .8907$. Use $\Phi(z) = 1 - \Phi(-z)$ for negative $z$.

| | .0 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| 0.1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| 0.2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| 0.3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| 0.4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| 0.5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| 0.6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| 0.7 | .7580 | .7611 | .7642 | .7673 | .7703 | .7734 | .7764 | .7794 | .7823 | .7852 |

Change of scale

$$z = \frac{X - E(X)}{SD(X)}$$

$\Phi(.44) = .67$ from table

so $z = .44$

$$.44 = \frac{1.1 - 1}{SD(X)}$$

$$SD(X) = \frac{.1}{.44} = \boxed{.23}$$

## Last time  sec 4.1  Continuous distributions

A continuous RV $X$, has a prob density function, $f(x)$, where $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

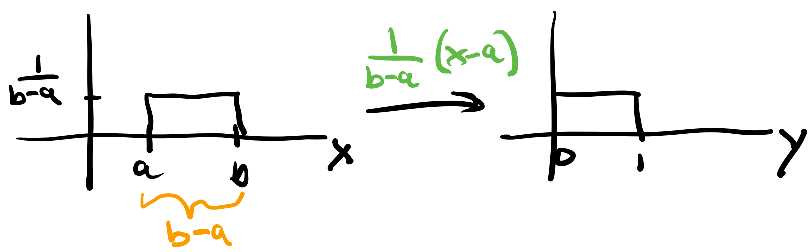
$$P(X=a) = \int_a^a f(x)dx = 0 \quad \text{so} \quad P(X \geq a) = P(X > a).$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

A change of scale is a transformation $Y = m + nX$, of $X$. The purpose is that it makes it easier to calculate $E(X)$ and $Var(X)$. It maps one density to another.

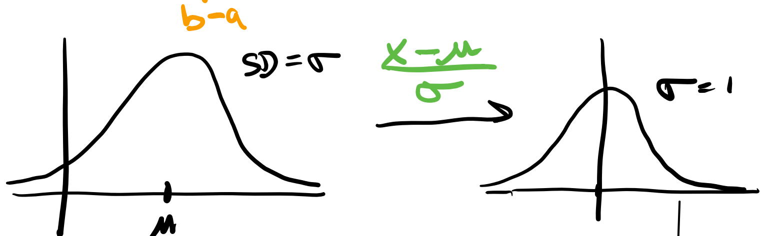
constants.

ex



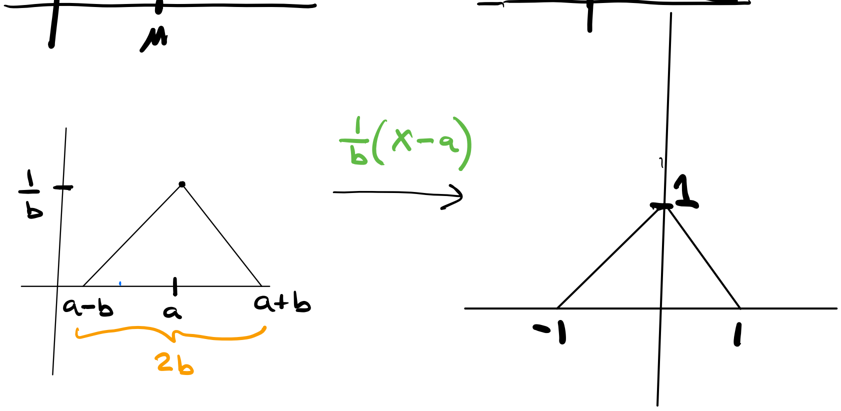

$U(0,1)$

ex



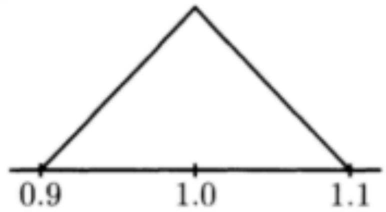

std normal

ex




Concept test

Suppose a manufacturing process designed to produce rods of length 1 inch exactly, in fact produces rods with length distributed according to the density graphed below.




You should change the scale of X= the length of rods to:

- a: X-1
- b: .1(X-1)
- c: 10X-1
- d: none of the above

| | |
|---|---|
| a: X-1 | The area is already 1 |

| | |
|---|---|
| d: none of the above | You need to move the center of the rod from x = 1 to x = 0 so first we need to subtract 1. Now we have the end point at x = .1 and to normalize this so the end point is at x = 1 we need to divide by .1 which is equivalent to multiplying by 10. So our final equation is 10(x - 1) |

Today

1. briefly sec 4.5 Cummulative Distribution Function (CDF)
2. Sec 4.2 Exponential Distribution.

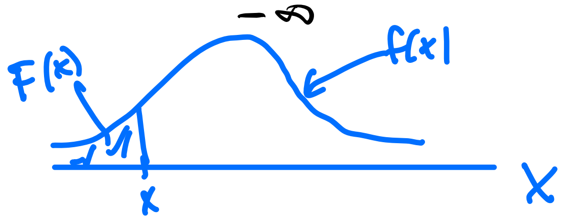
① briefly sec 4.5 The Cumulative Distribution Function (CDF)

Let $X$ be a continuous RV

$F(x) = P(X \le x)$ — a number between 0 and 1

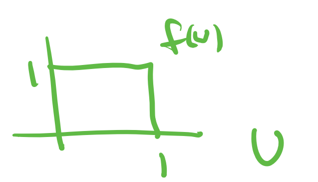
If $f(x)$ is a density of $X$,

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

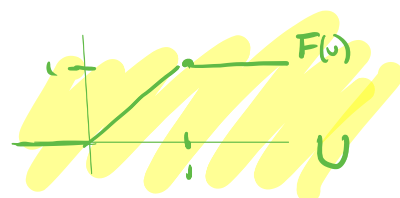



ex $U \sim \text{Unif}(0,1)$

$$f(u) = \begin{cases} 1 & 0 < u < 1 \\ 0 & \text{else} \end{cases}$$




$$F(u) = \int_0^u 1\,dx = u$$




$$\boxed{F(u) = \begin{cases} 0 & -\infty < u \le 0 \\ u & 0 \le u \le 1 \\ 1 & u \ge 1 \end{cases}}$$

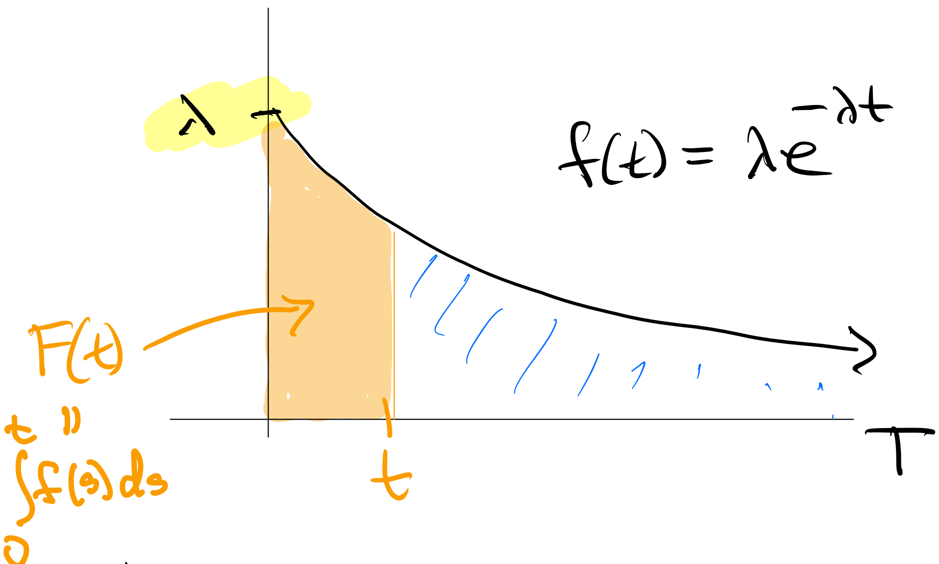
By FTC, $F'(x) = f(x)$

Consequently a density function and cdf are equivalent descriptions of a RV.

② Sec 4.2 <u>Exponential distribution</u>

<u>Def^n</u> A random time $T$ has exponential distribution with rate $\lambda > 0$.

$T \sim \mathrm{Exp}(\lambda)$, if $T$ has density
$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$$




ex $T$ = time until your first success where $\lambda$ = rate of success, ex $T$ = time until a lightbulb burns out

## CDF and survival function

$T \sim \text{Exp}(\lambda)$ $\quad f(t) = \lambda e^{-\lambda t}$

Compute the CDF of $T$.

$$F(t) = P(T \leq t) = \int_{-\infty}^{t} f(s)\,ds$$

$$= \int_{-\infty}^{0} \underbrace{f(s)}_{0}\,ds + \int_{0}^{t} f(s)\,ds$$

$$= \int_0^t \lambda e^{-\lambda s}\,ds = \frac{\lambda e^{-\lambda s}}{-\lambda}\Bigg|_0^t = -e^{-\lambda t} + 1 = \boxed{1 - e^{-\lambda t}}$$

$P(T > t) = e^{-\lambda t}$ is

called the survival function

$T \sim \text{Exp}(\lambda)$ iff $P(T > t) = e^{-\lambda t}$

since $\underbrace{F(t)}_{= 1 - P(T > t)}$ and $f(t)$ both

define distribution.

tinyurl.com/oct10-2022

**Stat 134**

**Monday October 10 2022**

1. GSI Brian and Yiming are each helping a student. Brian and Yiming see students at a rate of $\lambda_B$ and $\lambda_Y$ students per hour respectively.
   Let

   $$B = \text{wait time for Brian } \sim Exp(\lambda_B)$$
   $$Y = \text{wait time for Yiming } \sim Exp(\lambda_Y)$$

   What distribution is $T = \min(B, Y)$?
   Hint: compute $P(T > t)$

   **a** $Exp(\max(\lambda_B, \lambda_Y))$

   **b** $Exp(\lambda_B - \lambda_Y)$

   **c** $Exp(\lambda_B + \lambda_Y)$

   **d** none of the above

$$P(T > t) = \underbrace{P(B > t)}_{= e^{-\lambda_B t}} \underbrace{P(Y > t)}_{= e^{-\lambda_Y t}} = \boxed{e^{-(\lambda_B + \lambda_Y)t}}$$

## The memoryless property

This property relates to the geometric and exponential distributions.

In words, it says if you haven't had success yet then you can reset the clock to zero.

More formally,

if $T \sim \mathrm{Exp}(\lambda)$, $T$ = time until your first success or arrival.

the memoryless property says:

$$P(T \in dt \mid T > t) = P(0 < T < 0 + dt)$$

you have success in small interval after time $t$

given you haven't had success before time $t$

you have success in small time interval after time 0.


By the graph of $T \sim \mathrm{Exp}(\lambda)$, we see $P(0 < T < dt) \approx \lambda dt$

By Baye's rule $P(T \in dt \mid T > t) = \dfrac{P(T \in dt, T > t)}{P(T > t)}$

$$= \frac{P(T \in dt)}{P(T > t)}$$

$$\approx \frac{f(t)\,dt}{e^{-\lambda t}} = \frac{\lambda \cancel{e^{-\lambda t}}\,dt}{\cancel{e^{-\lambda t}}}$$

$$= \lambda\,dt$$

$$\approx P(0 \leq T < dt)$$

Proving the memoryless property of exponential.

Interestingly,

Only 2 distributions are memoryless:

For discrete $(X = 1, 2, 3, \ldots)$ — Geometric (*) ← Proved in appendix below

For continuous $(T > 0)$ — Exponential ← Proof is similar to geom

We prove below in the appendix that the only memoryless discrete distribution is geometric.

## Appendix

The def$^n$ for memoryless in the discrete case is a little different. It says, given that $X > j$ then the chance $X > j + k$ is the same as the unconditional probability that $X > k$.

**Definition**

$X \sim Geom(p)$ is memoryless if

$$P(X > j+k \mid X > j) = P(X > k)$$

**Thm** A discrete distribution, $X$, taking values $1, 2, 3, \ldots$ is memoryless iff it is $Geom(p)$ where $p = P(X=1)$.

Pf/ First we show that if $X$ is memoryless with values $X = 1, 2, 3, \ldots$ then $X$ is $Geom(p)$ where $p = P(X=1)$.

For positive integers $k, j$

Suppose $P(X > k+j \mid X > j) = P(X > k)$

$$\parallel$$

$$\frac{P(X > k+j)}{P(X > j)}$$

$$\Leftrightarrow P(X > k+j) = P(X > j) P(X > k)$$

It follows from this that

$$P(X>j) = P(X>1)^j \quad \text{for } j=1,2,\ldots$$

This can be shown by induction

base case $j=1$ ✓

assume true for $j-1$

$$P(X>j) = P(X>j-1+1) = P(X>j-1)\,P(X>1)$$

$$P(X>j-1) = P(X>1)^{j-1}$$

$$= P(X>1)^j \checkmark$$

let $q = P(X>1)$

then $P(X>j) = q^j \Rightarrow X \sim \text{Geom}(1-q)$

where $p = 1-q = P(X=1)$.

Next, we show the converse is true:

If $X \sim \text{Geom}(p)$ where $p = P(X=1)$ then $X$ is memoryless.

Let $X \sim \text{Geom}(p)$, $X = 1,2,3,\ldots$

$$P(X>k+j \mid X>j) = \frac{P(X>k+j,\, X>j)}{P(X>j)}$$

$$= \frac{P(X>k+j)}{P(X>j)}$$

$$= \frac{q^{k+j}}{q^j} = q^k$$

$$= P(X > k) \checkmark$$

$$\square$$