Stat 134 lec 39

last time sec 6.5 Bivariate normal distribution.

std bivariate normal construction

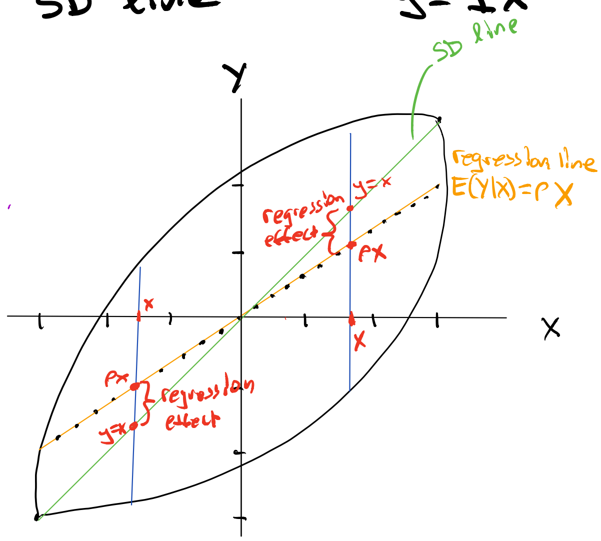
$X, Z \overset{iid}{\sim} N(0,1), \quad -1 \le \rho \le 1$

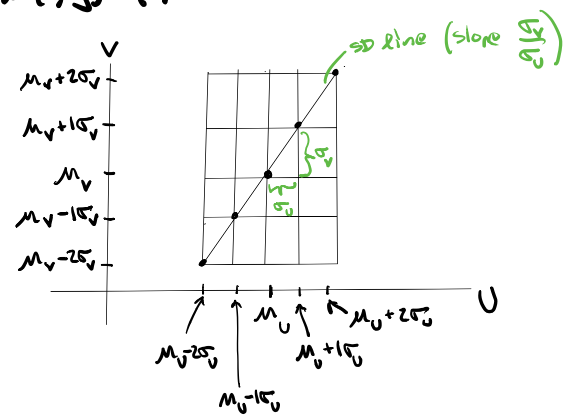
$Y = \rho X + \sqrt{1-\rho^2}\, Z \sim N(0,1)$

(X,Y) is std bivariate normal with $\mathrm{Cov}(X,Y) = \rho$.

Regression line $E(Y|X) = \rho X$

SD line $Y = 1X$





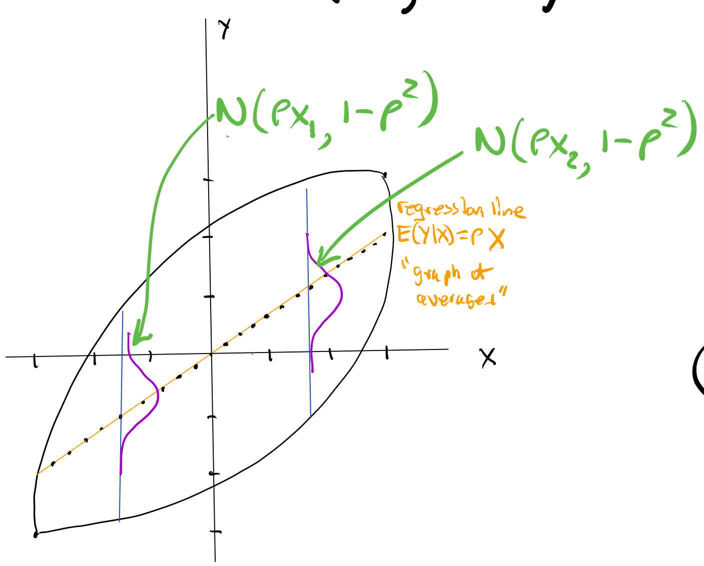
Regression effect,

Corr (test 1, test 2) = .6

If 1 SD above mean on test 1 then on average you will be less than 1 SD above average on test 2.

(regression line is less steep than SD line).

$Y|X \sim N(\rho X, 1-\rho^2)$



Today

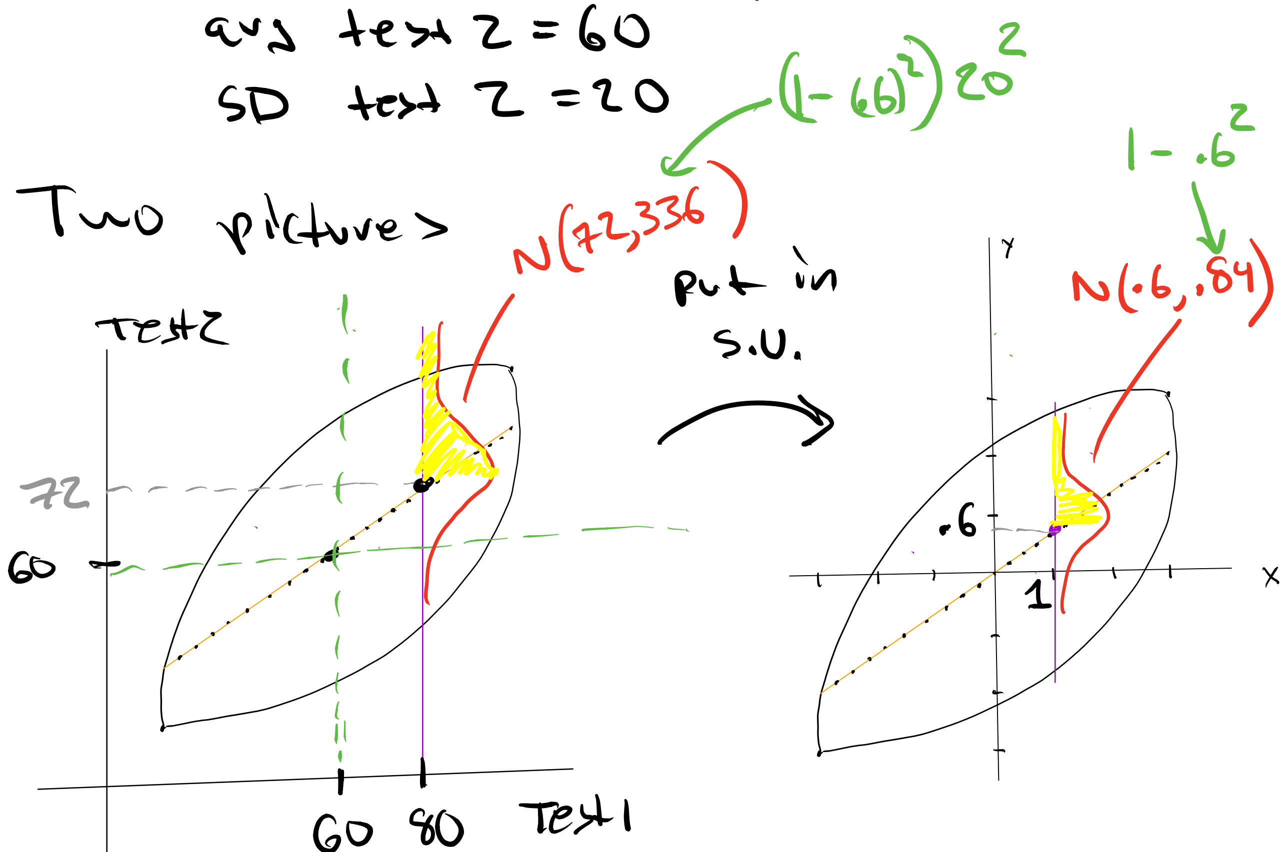
1. Examples with Bivariate normal
2. properties of Bivariate normal

① practice with bivariate normal

ex avg test 1 = 60
SD test 1 = 20
avg test 2 = 60
SD test 2 = 20

$\rho = .6$

Two pictures



You can answer the questions below using either of the 2 pictures above!

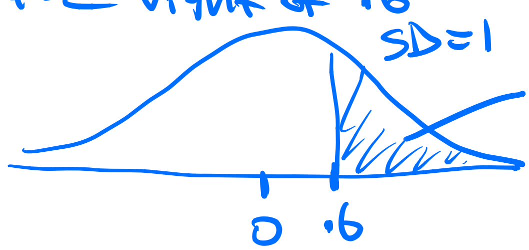
a) Among students who get an 80 on test 1, what is the chance you get greater than 72 if you get an 80 on Test 1?

First lets calculate what Test 2 score we expect if we get an 80 on test 1. 80 in s.u. is $X=1$ and so we expect $\hat{Y} = .6(1) = .6$ in s.u.
The corresponding test 2 score is
Test 2 $= 60 + (.6)(20) = 72$. From the picture above we can see we want the area of $N(72, 16.8)$ greater than 72 which is $.5$.

b) Among all students in the class what is the chance you get greater than 72 if you get 80 on test 1?

Soln

The distribution of test 2 scores in the whole class is $N(60, 400)$.
In s.u. $\frac{72-60}{20} = .6$ and we want area to the right of .6 under std normal curve,



$1 - \Phi(.6) = .27$

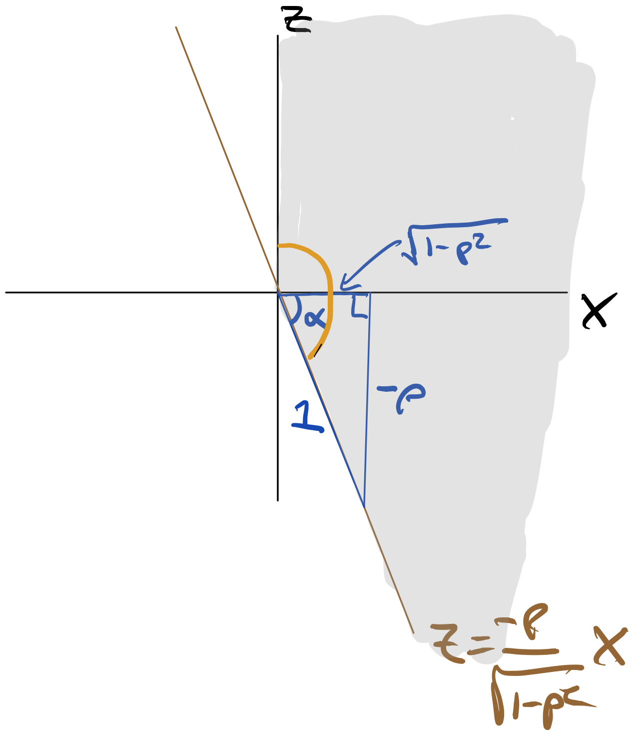
You are part of a more elite grp.

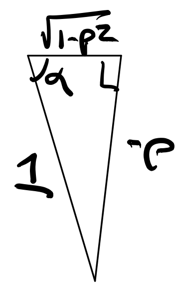
eg: X, Y std bivariate normal, $\rho > 0$

Find $P(X>0, Y>0)$

$$P(X>0, Y>0) = P(X>0, \rho X + \sqrt{1-\rho^2}\, Z > 0)$$

$$= P\left(X>0, Z > \frac{-\rho}{\sqrt{1-\rho^2}} X\right)$$





$$\tan\alpha = \frac{-\rho}{\sqrt{1-\rho^2}}$$

$$\alpha = \tan^{-1}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)$$

$$P\left(X>0, Z > \frac{-\rho}{\sqrt{1-\rho^2}} X\right) = \frac{90 + |\alpha|}{360} = \boxed{\frac{90 + \left|\tan^{-1}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)\right|}{360}}$$

tinyurl:

http://tinyurl.com/april29-pt1

http://tinyurl.com/april29-pt2

Stat 134

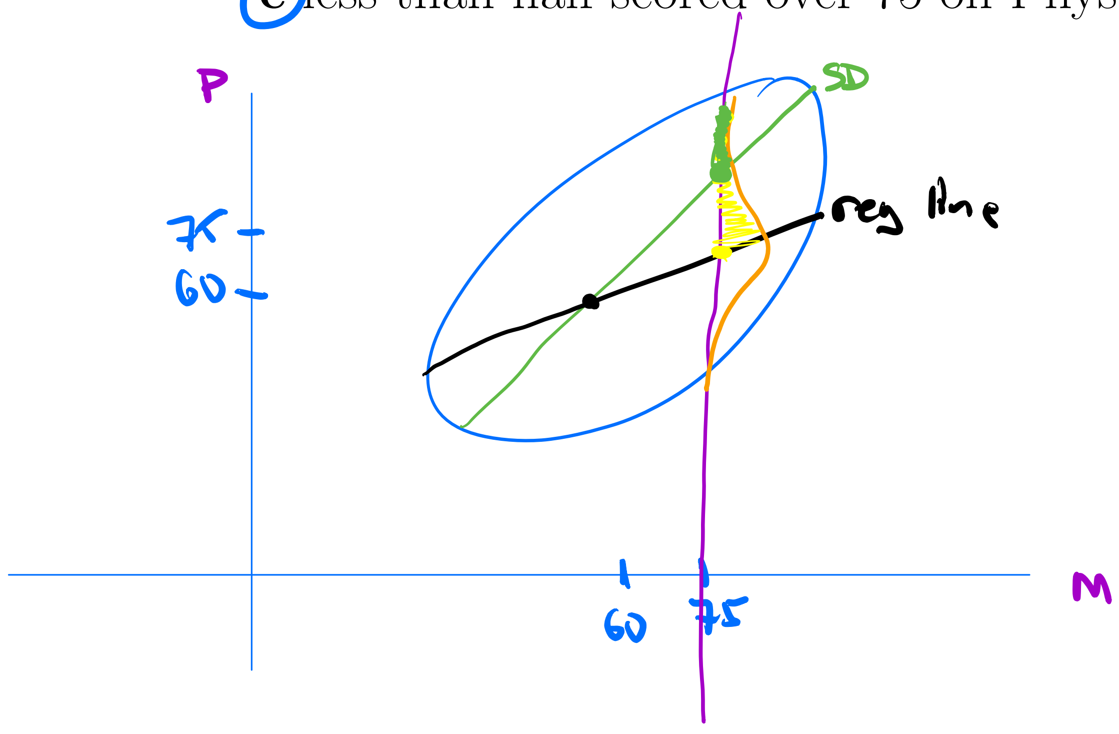
Monday April 29 2019

1. A test score in Math and Physics is bivariate normal, $\rho > 0$. The average is 60 on both tests and the SDs are the same. Of students scoring 75 on the Math test:

   a about half scored over 75 on Physics

   b more than half scored over 75 on Physics

   c less than half scored over 75 on Physics

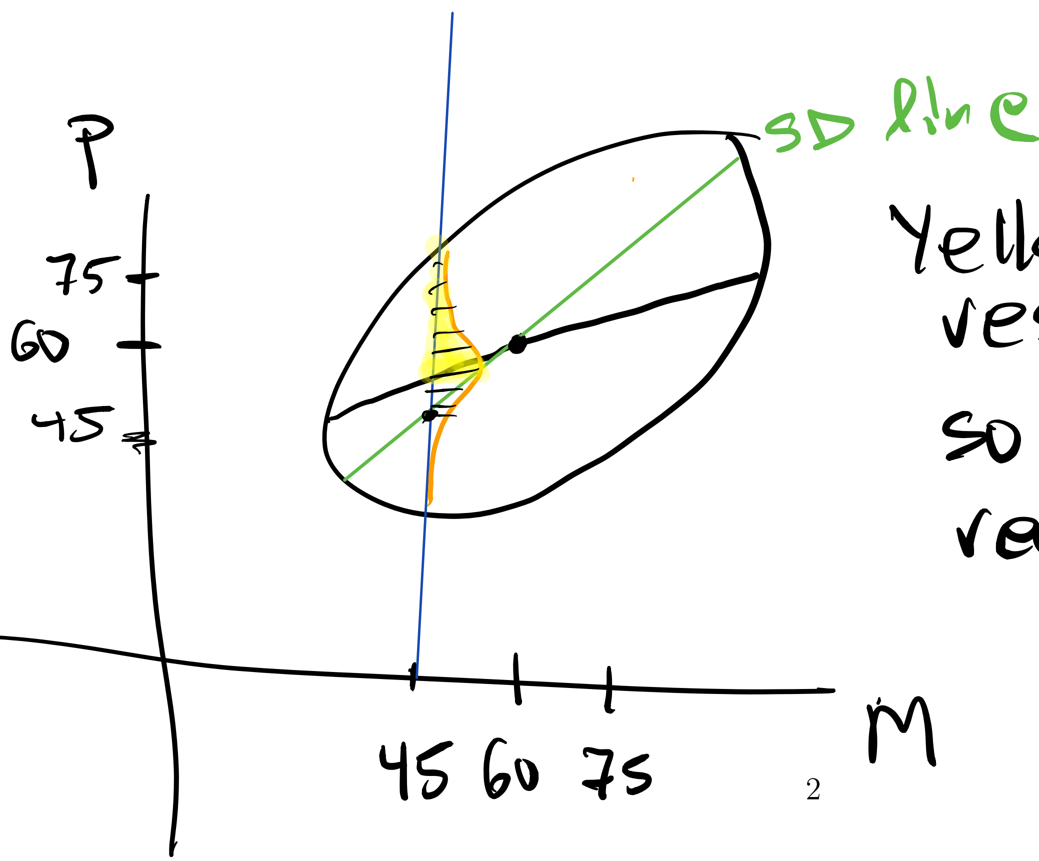


2. A test score in Math and Physics is bivariate normal, $\rho > 0$. The average is 60 on both tests and the SDs are the same. Of students scoring 45 on the Math test:

a about half scored over 45 on Physics

b more than half scored over 45 on Physics

c less than half scored over 45 on Physics



Yellow shaded region is 50%

so black shaded region > 50%

② Properties of bivariate normal

The single and multivariate MGF is defined as:

$$M_Y(t) = E(e^{tY})$$

$$M_{(X,Y)}(s,t) = E(e^{sX+tY})$$ multivariate MGF

Note that

$$M_Y(t) = E(e^{tY}) = M_{tY}(1)$$ $// \; E(e^{1 \cdot tY})$

$$M_{(X,Y)}(s,t) = E(e^{sX+tY}) = M_{sX+tY}(1)$$

recall,

$$Z \sim N(\mu, \sigma^2) \text{ iff } M_Z(a) = e^{\mu a} e^{\sigma^2 \frac{a^2}{2}}$$

Thm Let (X, Y) be standard bivariate normal.

The MGF of (X, Y) is

$$M_{(X,Y)}(s,t) = e^{\frac{s^2}{2} + \frac{t^2}{2} + st\rho}$$

Pf/

Recall that $X \sim N(0,1)$ so $M_X(s) = e^{\frac{s^2}{2}}$

$$M_{X,Y}(s,t) = E\left[e^{sX+tY}\right]$$

$$= E\left[e^{sX+t(\rho X+\sqrt{1-\rho^2}Z)}\right]$$

$$= E\left[e^{(s+t\rho)X} \cdot e^{t\sqrt{1-\rho^2}Z}\right]$$

$$\overset{\text{indenpendence}}{=} E\left[e^{(s+t\rho)X}\right] E\left[e^{t\sqrt{1-\rho^2}Z}\right]$$

$$= M_X(s+t\rho) \cdot M_Z(t\sqrt{1-\rho^2})$$

Finish proof.

$$= e^{\frac{(s+t\rho)^2}{2}} \cdot e^{\frac{(t\sqrt{1-\rho^2})^2}{2}}$$

$$= e^{\frac{s^2}{2} + st\rho + \frac{t^2\rho^2}{2}} \cdot e^{\frac{t^2}{2} - \frac{t^2\rho^2}{2}}$$

$$= \boxed{e^{s^2/2 + t^2/2 + st\rho}}$$

$\square$