Stat 134 lec 41

Warmup 8:00 – 8:10 AM

Let (X,Y) be bivariate normal. Then (2X+3Y+4, 6X-Y-4) is bivariate normal.

a true

b false

c not enough info to decide

$a(2X+3Y+4) + b(6X-Y-4)$

$= \underbrace{(2a+6b)X + (3a-4b)Y}_{\text{normal since } (X,Y) \text{ bivariate normal}} + 4a-4b$ $\Rightarrow$ $(2X+3Y+4, 6X-Y-4)$ is bivariate normal

Final review RRR ~~week~~

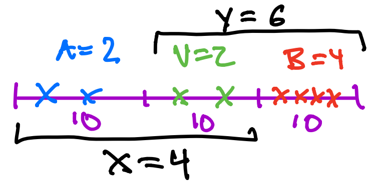
Today

1. Sec 6.5 Regression and bivariate normal (Rec 39)
2. Sec 4.2 Gamma / Poisson Thinning (Rec 22)
3. Expectation / Variance w/ indicators (Rec 14)

ex Toss a fair coin 30 times

$X = \#$ heads first 20

$Y = \#$ heads 2nd 20



$$\left.\begin{array}{l} A \sim Bin(10, 1/2) \\ V \sim Bin(10, 1/2) \\ B \sim Bin(10, 1/2) \end{array}\right\} \text{ indep.}$$

$$X = A + V$$
$$Y = V + B$$

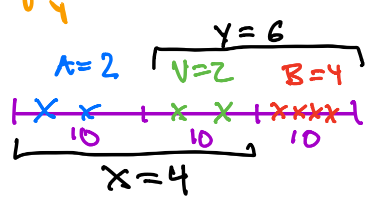
a) Find $Corr(X, Y)$?

$$Corr(X, Y) = \frac{Cov(X, Y)}{SD(X)SD(Y)}$$

$$Cov(X, Y) = Cov(A+V, V+B) = Cov(V, V)$$
$$= Var(V) = 10\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)$$
$$= \frac{10}{4}$$

$$Corr(X, Y) = \frac{\frac{10}{4}}{20/4} = 1/2$$

$20/4$ — $SD(X)SD(Y) = \sqrt{\frac{20}{4}} \cdot \sqrt{\frac{20}{4}}$



b) Find $E(Y \mid X) = E(V + B \mid X)$

$$= E(V \mid X) + E(B \mid X)$$

$$E(B \mid X) = E(B) = 10 \cdot 1/2 = 5$$

What distribution is $V|X$?

hint use Bayes rule to find

$$P(V=v|X=x) = \frac{P(V=v, X=x)}{P(X=x)}$$

$$= \frac{P(V=v, A=x-v)}{P(X=x)}$$

$$P(V=v, A=x-v) = P(V=v)P(A=x-v)$$

$$= \frac{\binom{10}{v}\cancel{\left(\frac{1}{2}\right)^{10}}\binom{10}{x-v}\cancel{\left(\frac{1}{2}\right)^{10}}}{\binom{20}{x}\cancel{\left(\frac{1}{2}\right)^{20}}}$$

$$\Rightarrow V|X \sim \text{Hyper}(N=20, G=10, n=X)$$

Interpretation: Out of 20 coin tosses, the good ones are the last 10 (tosses 11–20).

we sample the X heads out of the 20 and find the probability that v are good (i.e. in tosses 11–20).

$$E(V|X) = n \cdot \frac{G}{N} = 10 \cdot \frac{X}{20} = \frac{1}{2}X$$

$$E(Y|X) = E(V|X) + \underbrace{E(B|X)}_{=5} = \frac{1}{2}X + 5$$

Note that (X,Y) is approximately Bivariate normal:

$$X \sim Bin(20, 1/2) \approx N(\overset{20\cdot 1/2}{10}, \overset{20\cdot 1/2 \cdot 1/2}{5})$$

$$Y \sim Bin(20, 1/2) \approx N(10, 5)$$

$$Corr(X,Y) = 1/2$$

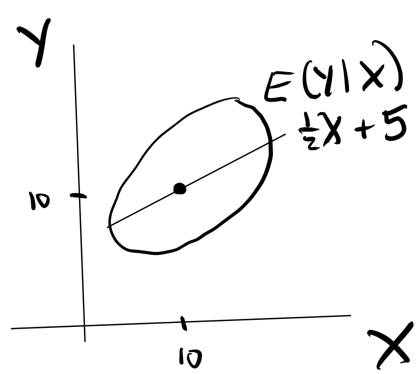
$(X,Y) \sim BV(10, 10, 5, 5, 1/2)$ since

$$aX + bY = a(A+V) + b(V+B)$$
$$= aA + (a+b)V + bB$$

is normal since A, V, B are indep normals.

eqn of regression line $E(Y|X)$:

$$\frac{\hat{Y} - \mu_y}{\sigma_y} = \frac{1}{2}\,\frac{X - \mu_x}{\sigma_x}$$



$$\frac{\hat{y} - 10}{\sqrt{5}} = \frac{1}{2}\left(\frac{X - 10}{\sqrt{5}}\right)$$

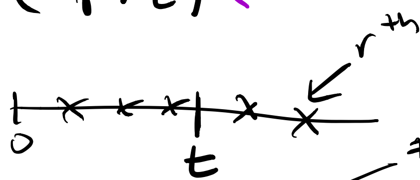
$$\boxed{\hat{y} = \tfrac{1}{2}X + 5}$$

(2)

<u>Sec 4.2</u> Gamma(r,λ) = distribution for r^{th} arrival time of a Poisson (λ) process.

T_r = arrival time of r^{th} call. $\sim$ Gamma(r,λ)

Find $P(T_r > t)$ (Right tail probability)



\# arrivals in time t

$N_t \sim$ Pois(λt)

$$P(T_r > t) = P(N_t \leq r-1)$$

$$= P(N_t = 0) + P(N_t = 1) + \cdots + P(N_t = r-1)$$

$$= e^{-\lambda t} + e^{-\lambda t}\lambda t + \frac{e^{-\lambda t}(\lambda t)^2}{2!} + \cdots + \frac{e^{-\lambda t}(\lambda t)^{r-1}}{(r-1)!}$$

<u>ex</u>

let $T \sim$ Gamma$(r=4, \lambda=2)$

Find $P(T > 7)$

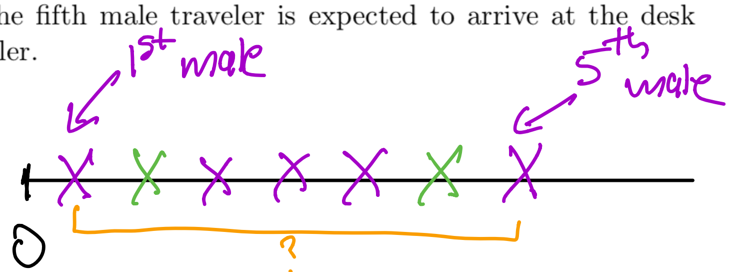
$$P(T > 7) = P(N_7 \leq \underset{r-1}{3})$$

where $N_7 \sim$ Pois$(\underset{2\cdot 7}{14})$

$$P(T > 7) = \boxed{e^{-14}\left(1 + \frac{14}{1} + \frac{14^2}{2!} + \frac{14^3}{3!}\right)}$$

5. Travelers arrive at an airport Information desk according to a Poisson process at the rate of 15 per hour. Assume that each traveler arriving at the desk has a 60% chance of being male and a 40% chance of being female, independent of all other travelers.

a) Fill in the blank with a number: The fifth male traveler is expected to arrive at the desk ________ minutes after the first male traveler.



$N \sim \text{Pois}\left(15 \cdot \frac{1}{60}\right)$ male and female

$M \sim \text{Pois}\left(15(.6)\left(\frac{1}{60}\right)\right) = \text{Pois}\left(9/60\right)$ male

$F \sim \text{Pois}\left(15(.4)\left(\frac{1}{60}\right)\right) = \text{Pois}\left(6/60\right)$ female

T_5 = wait time for 5^{th} male $\sim \text{Gamma}(5, 9/60)$

T_1 = " " " 1^{st} male $\sim \text{Gamma}(1, 9/60)$

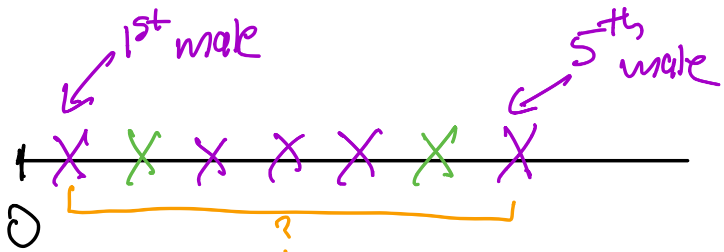
$$E(T_5 - T_1) = E(W_4) = 4/\tfrac{9}{60} = \frac{4 \cdot 60}{9} = \boxed{\frac{80}{3} \text{ min}}$$

$T_5 - T_1 = W_4$

$W_4 \sim \text{Gamma}(4, 9/60)$

5. Travelers arrive at an airport Information desk according to a Poisson process at the rate of 15 per hour. Assume that each traveler arriving at the desk has a 60% chance of being male and a 40% chance of being female, independent of all other travelers.

b) Find the chance that the fifth male traveler arrives at the desk more than 30 minutes after the first male traveler.



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let T_5 = arrival time for 5th male

T_1 = arrival time 1st male

$$P(T_5 - T_1 > 30) = P(T_4 > 30)$$

$$= P(N_{30} \leq 3)$$

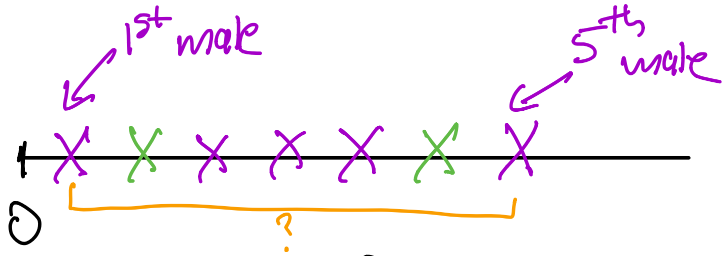
$$= P(N_{30}=0) + P(N_{30}=1) + P(N_{30}=2) + P(N_{30}=3)$$

$$N_{30} \sim \text{Pois}\left(\frac{9}{60} \cdot 30\right) = \text{Pois}(4.5)$$

$$\Rightarrow P(N_{30} \leq 3) = e^{-4.5}\left(1 + 4.5^1 + \frac{4.5^2}{2!} + \frac{4.5^3}{3!}\right)$$

5. Travelers arrive at an airport Information desk according to a Poisson process at the rate of 15 per hour. Assume that each traveler arriving at the desk has a 60% chance of being male and a 40% chance of being female, independent of all other travelers.

c) Find the expected number of female travelers who arrive at the desk before the fifth male traveler.



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Recall $X \sim \text{NegBin}(r, p)$ on $0, 1, 2, \ldots$ is the number of failures before the r^{th} success

$E(X) = r \cdot \frac{1-p}{p}$

Here X = # female before the 5^{th} male

$X \sim \text{NegBin}(5, .6)$ on $0, 1, 2, \ldots$

$E(X) = 5\left(\frac{.4}{.6}\right) = 10/3$

3 Expectation / Variance with indicators.

6. A drawer contains s black socks and s white socks, where s is a positive integer. I pull two socks out at random without replacement and call that my first pair. Then I pull two socks out at random without replacement from the remaining socks in the drawer, and call that my second pair. I proceed in this way till I have s pairs and the drawer is empty.

Let D be the number of pairs in which the two socks are of different colors.

a) Find $E(D)$

any of 2s poss. socks

$\frac{2s}{2s} \cdot \frac{s}{2s-1}$ any of s poss socks remaining

$= \frac{s}{2s-1}$

$D = I_1 + \cdots + I_s$

$I_2 = \{ 1$ if 2^{nd} pair diff color

$E(D) = s \cdot \left(\frac{s}{2s-1}\right) = \boxed{\frac{s^2}{2s-1}}$

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Let D be the number of pairs in which the two socks are of different colors.

$I_1, \ldots, I_s$ exchangeable, since draw socks w/o replacement from drawer.

b) Find Var(D).

$$D = I_1 + \cdots + I_s$$

$$\text{Var}(D) = s\,\text{Var}(I_1) + s(s-1)\,\text{Cov}(I_1, I_2)$$

$$\text{Cov}(I_1, I_2) = E(\underbrace{I_{12}}_{I_1 I_2}) - E(I_1)E(I_2)$$

$$\frac{2s}{2s}\,\frac{s}{2s-1}\,\frac{2s-2}{2s-2}\,\frac{s-1}{2s-3} = \frac{s}{2s-1}\cdot\frac{s-1}{2s-3}$$

$$I_{12} = \begin{cases} 1 & \text{if } 1^{st} \text{ and } 2^{nd} \text{ pair diff color} \\ 0 & \text{else} \end{cases}$$

$$s(s-1)\,\text{Cov}(I_1, I_2) = s(s-1)\left[\frac{s}{2s-1}\cdot\frac{s-1}{2s-3} - \left(\frac{s}{2s-1}\right)^2\right]$$

$$s\cdot\text{Var}(I_1) = s\cdot\left[\left(\frac{s}{2s-1}\right)\left(1 - \frac{s}{2s-1}\right)\right]$$

$$\text{Var}(D) = s\cdot\text{Var}(I_1) + s(s-1)\,\text{Cov}(I_1, I_2)$$

$\square$