Stat 134 lec 41 (RRR review 1)

Correction Bivariate normal notation

$(U,V) \sim BV(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$ (corrected in lec 40 notes)

Today

(1) Convolution formula (lec 32)

(2) Conditional expectation. (lec 34)

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(1) Sec 5.4

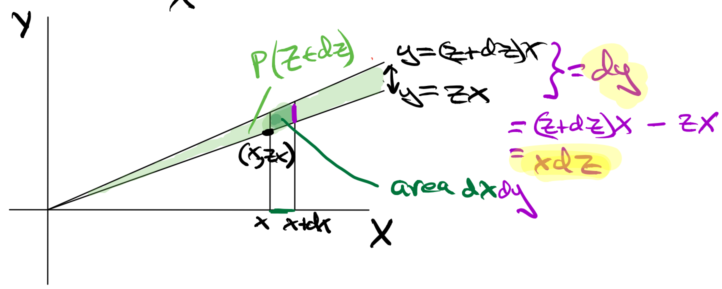
Convolution formula for density of ratio Y/X

$X > 0$, $Y > 0$

let $Z = \frac{Y}{X}$.

Find $f_Z(z)$.

Picture $\quad Z = \frac{Y}{X} \Rightarrow Y = \underset{\text{slope.}}{Z}X$



$$P(Z \in dz) = \int_{x=0}^{x=\infty} P(Z \in dz, X \in dx)$$

$$\| $$

$$f(z)dz = \int_{x=0}^{x=\infty} f_{(X,Y)}(x, zx)\, \underbrace{x}_{\left|\frac{dy}{dz}\right|}\, dz\, dx$$

$\left|\frac{dy}{dz}\right|$ ← absolute value since $x > 0$.

$$\Rightarrow f(z) = \int_{x=0}^{x=\infty} f(x, zx)\left|\frac{dy}{dz}\right| dx \overset{\text{if } X,Y \text{ indep.}}{=} \int_{x=0}^{x=\infty} f_X(x) f_Y(zx)\left|\frac{dy}{dz}\right| dx$$

Convolution formula.

2. (4 pts) Let X and Y be independent variables with $Gamma(r,\lambda)$ and $Gamma(s,\lambda)$ distributions, respectively. Using convolution, show that $Z = \frac{X}{X+Y}$ follows a $Beta(r,s)$ distribution. Recall $f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$ for $Gamma(r,\lambda)$, $f_Z(z) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} z^{r-1}(1-z)^{s-1}$ for $Z \sim Beta(r,s)$. Hint: rewrite integrand to be one of the known distributions after taking out constants.

$$Z = \frac{X}{X+Y} \Rightarrow Z(X+Y) = X \Rightarrow Y = \frac{1-Z}{Z}X \Rightarrow \underbrace{\frac{1-Z}{Z}}_{W} = \frac{Y}{X}$$

$$f_W(w) = \int_0^\infty f_{X,Y}(x, wx)\left|\frac{dy}{dw}\right| dx$$

$$\| \qquad \left|\frac{dy}{dz}\right|$$

$$\frac{1}{\left|\frac{dw}{dz}\right|} f_Z(z) \Rightarrow f_Z(z) = \int_0^\infty f_{X,Y}\left(x, \frac{1-z}{z}x\right)\left|\frac{dw}{dz}\cdot\frac{dy}{dw}\right| dx$$

$$= \int_0^\infty f_X(x) f_Y\left(\frac{1-z}{z}x\right)\left|\frac{dy}{dz}\right| dx$$

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$$

$$f_Y\left(\frac{1-z}{z}x\right) = \frac{\lambda^s}{\Gamma(s)}\left(\frac{1-z}{z}x\right)^{s-1} e^{-\lambda\left(\frac{1-z}{z}x\right)}$$

$$\left|\frac{dy}{dz}\right| = \frac{x}{z^2}$$

?

$$\Rightarrow f_Z(z) = \int_0^\infty \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \frac{\lambda^s}{\Gamma(s)}\left(\frac{1-z}{z}x\right)^{s-1} e^{-\lambda\left(\frac{1-z}{z}x\right)} \frac{x}{z^2}\, dx$$

$$= \frac{\lambda^{r+s}}{\Gamma(r)\Gamma(s)} \frac{(1-z)^{s-1}}{z^{s+1}} \int_0^\infty x^{r+s-1} e^{-\frac{\lambda}{z}x}\, dx$$

$$= \frac{\lambda^{r+s}}{\Gamma(r)\Gamma(s)} \frac{(1-z)^{s-1}}{z^{s+1}} \frac{\Gamma(r+s)}{\left(\frac{\lambda}{z}\right)^{r+s}} \underbrace{\int_0^\infty \frac{\left(\frac{\lambda}{z}\right)^{r+s}}{\Gamma(r+s)} x^{r+s-1} e^{-\frac{\lambda}{z}x}\, dx}_{=1}$$

$$= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \cdot \underbrace{\frac{\lambda^{r+s}}{\lambda^{r+s}}}_{=1} \cdot \underbrace{\frac{z^{r+s}}{z^{s+1}}}_{=z^{r-1}} \cdot (1-z)^{s-1}$$

$$= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} z^{r-1}(1-z)^{s-1} \Rightarrow \boxed{Z \sim Beta(r,s)}$$

(2) Review of conditional expectation   Sec 6.4

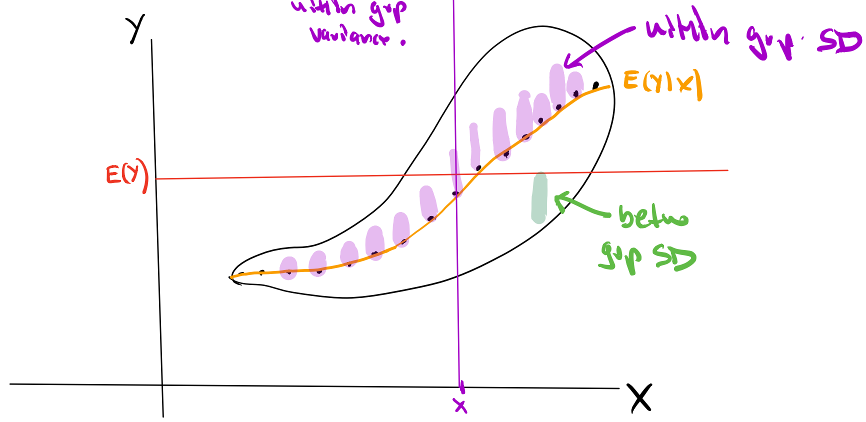
Properties (from lec 34)

1. $E(Y) = E(E(Y|X))$ iterated expectation
2. $E(aY+b|X) = aE(Y|X)+b$
3. $E(Y+Z|X) = E(Y|X)+E(Z|X)$
4. $E(g(X)|X) = g(X)$
5. $E(g(X)Y|X) = g(X)E(Y|X)$
6. $Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$ total variance decomposition

$Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$ (sec 6.2.18)

tot var = avg($\;^2$) + $\;^2$ var of E(Y|X) (between group variance)

within grp variance.



ex Let $U \sim \text{Unif}(0,1)$

Given $U=u$, X is exponential with mean u

i.e $X|U=u \sim \text{Exp}\left(\frac{1}{u}\right)$

$E(X|U)=U$

$\text{Var}(X|U)=U^2$

Find $E(X)$, $E(UX)$ and $\text{Var}(X)$.

$E(X)=E(E(X|U))=E(U)=\frac{1}{2}$

$E(UX)=E(E(UX|U))=E(U\underbrace{E(X|U)}_{=U})$

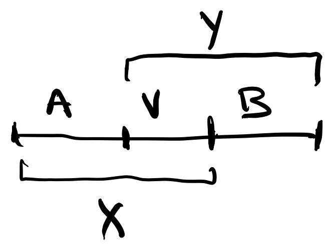
$=E(U^2)=\underbrace{\text{Var}(U)}_{=1/12}+\underbrace{E(U)^2}_{=(1/2)^2}=\frac{1}{3}$

$\text{Var}(X)=\underbrace{E(\underbrace{\text{Var}(X|U)}_{=U^2})}_{=1/3}+\underbrace{\text{Var}(\underbrace{E(X|U)}_{=U})}_{=1/12}=\frac{5}{12}$

ex Toss a fair coin 30 times

X = # heads first 20

Y = # heads 2nd 20



$$\left.\begin{array}{l} A \sim Bin(10, \tfrac{1}{2}) \\ V \sim Bin(10, \tfrac{1}{2}) \\ B \sim Bin(10, \tfrac{1}{2}) \end{array}\right\} \text{ indep.} \qquad \begin{array}{l} X = A + V \\ Y = V + B \end{array}$$

a) Find $Corr(X,Y)$?

$$Corr(X,Y) = \frac{Cov(X,Y)}{SD(X)SD(Y)}$$

$$Cov(X,Y) = Cov(A+V, V+B) = Cov(V,V)$$
$$= Var(V) = 10\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)$$
$$= \frac{10}{4}$$

$$Corr(X,Y) = \frac{\frac{10}{4}}{20/4} = 1/2$$

$20/4$ — $SD(X)SD(Y) = \sqrt{\frac{20}{4}} \cdot \sqrt{\frac{20}{4}}$

b) Find $E(Y \mid X) = E(V + B \mid X)$

$$= E(V \mid X) + E(B \mid X)$$

$$E(B \mid X) = E(B) = 10 \cdot \tfrac{1}{2} = 5$$

What distribution is $V|X$?

hint use Bayes rule to find $P(V=v|X=x)$

$$P(V=v)P(A=20-v)$$

$$= \frac{P(V=v, X=x)}{P(X=x)} = \frac{P(V=v, A=\cancel{20}-v)}{P(X=x)}$$

$$= \frac{\binom{10}{v}\cancel{\left(\frac{1}{2}\right)^{10}}\binom{10}{\cancel{20}-v}\cancel{\left(\frac{1}{2}\right)^{10}}}{\binom{20}{x}\cancel{\left(\frac{1}{2}\right)^{20}}}$$

$$= \frac{\binom{10}{v}\binom{10}{x-v}}{\binom{20}{x}}$$

$$\Rightarrow V|X \sim \text{Hyper}(N=20, G=x, n=10)$$

$$E(Y|X) = n\cdot\frac{6}{N} = 10\cdot\frac{X}{20} = \tfrac{1}{2}X$$

$$E(Y|X) = \tfrac{1}{2}X + 5$$

Note that (X,Y) is approximately Bivariate normal:

$20\cdot\frac{1}{2}$ $20\cdot\frac{1}{2}\cdot\frac{1}{2}$

$$X \sim \text{Bin}(20, \tfrac{1}{2}) \approx N(10, 5)$$
$$Y \sim \text{Bin}(20, \tfrac{1}{2}) \approx N(10, 5)$$
$$\text{Corr}(X,Y) = \tfrac{1}{2}$$

$(X,Y) \sim BV(10, 10, 5, 5, \frac{1}{2})$ since

$$aX + bY = a(A+V) + b(V+B)$$
$$= aA + (a+b)V + bB$$

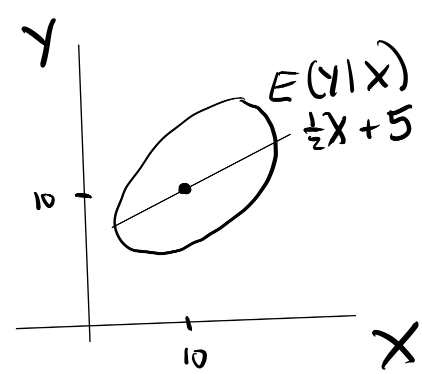
is normal since A, V, B are indep normals.

eqn of regression line $E(Y|X)$:

$$\frac{Y-\mu_y}{\sigma_y} = \frac{1}{2}\,\frac{X-\mu_x}{\sigma_x}$$

$$\frac{y-10}{\sqrt{5}} = \frac{1}{2}\left(\frac{x-10}{\sqrt{5}}\right)$$

$$\boxed{\hat{y} = \tfrac{1}{2}X + 5}$$



ex

Let N have the Poisson distribution with mean μ.
Let $U_1, U_2, \ldots \overset{iid}{\sim} U(0,1)$ independent of N.
Let $M = \min(U_1, \ldots, U_N)$. If $N=0$, define M to be 1.

a) Find $E(M|N)$

If $N=0$, $M=1$ so $E(M|N=0)=1$

Otherwise,

$M|N \sim \text{beta}(1, N)$ since $M|N = U_{(1)}$ ← 1st order statistic of $U(0,1)$

$E(M|N) = \frac{1}{N+1}$

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Let $M = \min(U_1, \ldots, U_N)$. If $N=0$, define M to be 1.

b) Find $E(M)$

$$E(M) = E\left(E(M|N)\right) = E\left(\frac{1}{N+1}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} e^{-\mu} \frac{\mu^n}{n!} = \frac{1}{\mu} \sum_{n=0}^{\infty} e^{-\mu} \frac{\mu^{n+1}}{(n+1)!}$$

$$= \frac{1}{\mu}\left[\sum_{n=1}^{\infty} e^{-\mu} \frac{\mu^n}{n!}\right] = \boxed{\frac{1-e^{-\mu}}{\mu}}$$

$$\sum_{n=1}^{\infty} e^{-\mu} \frac{\mu^n}{n!} = P(N \geq 1) = 1 - P(N=0)$$

$$= 1 - e^{-\mu}$$