

Stat 134 Lec 17

For Wednesday review, write down questions in discussion board on b-courses.

Last time sec 3.5 Poisson distribution

$$X \sim \text{Pois}(n)$$

$$P(X=k) = \frac{e^{-n} n^k}{k!}$$

$$E(X) = \text{Var}(X) = n.$$

Poisson Process or Poisson Random Scatter (PRS):

e.g. radioactive decay of Americium 241 in 10 seconds



Assumptions

- ① no two particles arrive at the same time.
(this allows us to divide 10 sec into n small time intervals each with at most one arrival.)
- ② X is a sum of n ^{large} iid Bernoulli (p) trials.

$n = 10$ is avg # of arrivals in 10 sec.

$\lambda = n/10$ is the arrival rate per second,

$X = \# \text{arrivals in 10 seconds.}$

Suppose $\lambda = 4$ arrivals/sec

then $n = \lambda \cdot 10 = 40 \Rightarrow X \sim \text{Pois}(40)$

Americium has a long half-life.

$Y = \# \text{arrivals in } 12070 \text{ sec.}$

$Y \sim \text{Pois}(\lambda \cdot 12070)$
^{K 4}

Today ① go over student comments of concept test

② finish sec 3.5 Poisson Theory

③ start midterm review. recognizing a distribution
binomial theorem

① Responses

1. Which of the following can be modeled as a Poisson Random Scatter with intensity $\lambda > 0$?
- (a) The number of blueberries in a 3 cubic inch blueberry muffin
 - (b) The number of patients entering a doctor's office in a 24 hour period.
 - (c) The number of times a day a person feels hungry
 - (d) The number of air pulses counted every second from cars driving over an empty rubber hose lying across a highway between noon and 1pm.
 - (e) more than one of the above

a	B isn't possible since the distribution would not be iid over a 24 hour period (who goes to the doctor at 4 am). C isn't possible since the trials are not independent. D isn't possible since you could feasibly have two or more cars run over the hose in one second A fits all of the criteria of PRS
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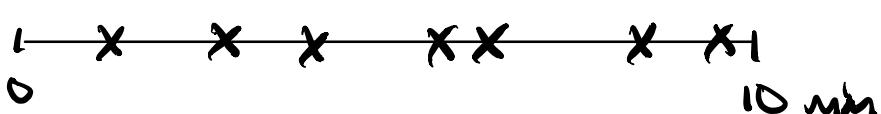
a	If the batter of your muffin mix is larger enough, the amount of blueberries per scoop of batter should be fixed with the blueberries being distributed uniformly
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a	The distribution is not fully random for patients entering a doctor's office over a 24 hour period and hunger pangs. Air pulses come in pairs which doesn't match poisson scatter unless you count pairs of hits.
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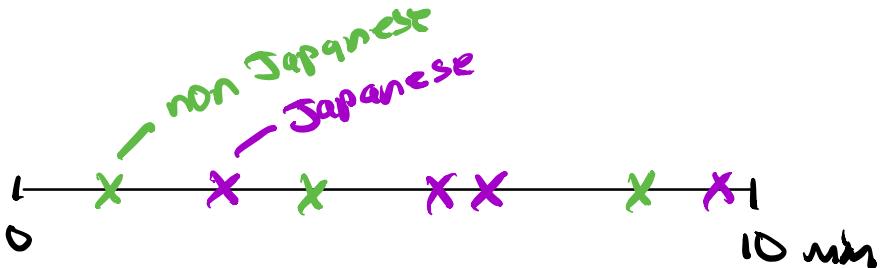
② Sec 3.5 Poisson thinning

Cars arrive at a toll booth according to a Poisson process at a rate $\lambda = 3$ arrivals/min

$X = \# \text{ cars arriving at a toll booth in 10 min. } X \sim \text{Pois}(\lambda \cdot 10)$



Of cars arriving, it is known, over the long term, that 60% are Japanese imports.



Call Japanese cars a success and non Japanese a failure.

$$\# \text{ cars} \sim \text{Pois}(\lambda \cdot 10) = \text{Pois}(30)$$

$$\# \text{ Japanese imports} \sim \text{Pois}(p\lambda \cdot 10) = \text{Pois}(18)$$

$$\# \text{ non Japanese} \sim \text{Pois}(q\lambda \cdot 10) = \text{Pois}(12)$$

Ex What is the probability that in a given 10 min interval, 15 cars arrive at the booth and 10 are Japanese imports?

$$X = \# \text{cars in 10 min} \sim \text{Pois}(30)$$

$$J = \# \text{Japanese cars in 10 min} \sim \text{Pois}(18)$$

$$nJ = \# \text{nonJapanese} \sim \text{Pois}(12)$$

We assume J and nJ are independent,

$$\begin{aligned} P(X=15, J=10) &= P(nJ=5, J=10) \\ &= P(nJ=5) P(J=10) \\ &= \frac{e^{-12}}{5!} \cdot \frac{e^{-18}}{10!} \end{aligned}$$

Which distributions involve drawing with replacement?

Discrete

name and range	$P(k) = P(X = k)$ for $k \in \text{range}$	mean	variance
uniform on $\{a, a+1, \dots, b\}$ $\{1, 2, \dots, n\}$	$\frac{1}{b-a+1}$	$\frac{a+b}{2}$ $\frac{n+1}{2}$	$\frac{(b-a+1)^2 - 1}{12}$ $\frac{n^2 - 1}{12}$
Bernoulli (p) on $\{0, 1\}$	$P(1) = p; P(0) = 1-p$	p	$p(1-p)$
binomial (n, p) on $\{0, 1, \dots, n\}$	$\binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
Poisson (μ) on $\{0, 1, 2, \dots\}$	$\frac{e^{-\mu} \mu^k}{k!}$	μ	μ
hypergeometric (n, N, G) on $\{0, \dots, n\}$	$\frac{\binom{G}{k} \binom{N-G}{n-k}}{\binom{N}{n}}$	$\frac{nG}{N}$	$n \left(\frac{G}{N}\right) \left(\frac{N-G}{N}\right) \left(\frac{N-n}{N-1}\right)$
geometric (p) on $\{1, 2, 3, \dots\}$	$(1-p)^{k-1} p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
geometric (p) on $\{0, 1, 2, \dots\}$	$(1-p)^k p$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$
negative binomial (r, p) on $\{0, 1, 2, \dots\}$	$\binom{k+r-1}{r-1} p^r (1-p)^k$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$

normal

$\phi(x)$

μ

σ

For Poisson and normal we are thinking of them as a limit of binomial.

8. Let X_1 and X_2 be independent random variables such that for $i = 1, 2$, the distribution of X_i is Poisson (μ_i). Let m be a fixed positive integer. Find the distribution of X_1 given that $X_1 + X_2 = m$. Recognize this distribution as one of the famous ones, and provide its name and parameters.

$$\begin{aligned}
 & \left. \begin{array}{l} X_1 \sim \text{Pois}(\mu_1) \\ X_2 \sim \text{Pois}(\mu_2) \end{array} \right\} \text{indep} \\
 & X_1 | X_1 + X_2 = m \text{ takes values } 0, 1, 2, \dots, m \\
 & P(X_1 = k | X_1 + X_2 = m) = \frac{P(X_1 = k, X_2 = m-k)}{P(X_1 + X_2 = m)} \\
 & = \frac{P(X_1 = k)P(X_2 = m-k)}{P(X_1 + X_2 = m)} \\
 & = \frac{\frac{e^{-\mu_1} \mu_1^k}{k!} \cdot \frac{e^{-\mu_2} \mu_2^{m-k}}{(m-k)!}}{\frac{e^{-(\mu_1 + \mu_2)} (\mu_1 + \mu_2)^m}{m!}} \\
 & = \binom{m}{k} \left(\frac{\mu_1}{\mu_1 + \mu_2} \right)^k \left(\frac{\mu_2}{\mu_1 + \mu_2} \right)^{m-k} \\
 \Rightarrow & X_1 | X_1 + X_2 = m \sim \text{Bin}\left(m, \frac{\mu_1}{\mu_1 + \mu_2}\right)
 \end{aligned}$$

Ex

8. Let m be a positive integer and let v and w be two probabilities strictly between 0 and 1. Let X have the binomial (m, v) distribution. Let Y be independent of X and suppose the distribution of Y is given by $P(Y = j) = (1 - w)^j w$ for $j \geq 0$. Fill in the blank with a formula in terms of v and w ; show your work below.

$$P(X = Y) = w \left(\text{_____} \right)^m$$

$$\begin{aligned} P(X = Y) &= P(X = 0, Y = 0) + P(X = 1, Y = 1) + \dots + P(X = m, Y = m) \\ &= P(X = 0)P(Y = 0) + P(X = 1)P(Y = 1) + \dots + P(X = m)P(Y = m) \end{aligned}$$

$$P(X = j) = \binom{m}{j} v^j (1-v)^{m-j}$$

$$P(Y = j) = (1-w)^j w$$

$$\begin{aligned} P(X = Y) &= \binom{m}{0} (1-v)^m w + \binom{m}{1} v (1-v)^{m-1} w + \binom{m}{2} v^2 (1-v)^{m-2} w \\ &\quad + \dots + \binom{m}{m} v^m (1-w)^m w \end{aligned}$$

$$\begin{aligned} &= w \left[\binom{m}{0} (1-v)^m + \binom{m}{1} v (1-v)^{m-1} + \binom{m}{2} v^2 (1-v)^{m-2} \right. \\ &\quad \left. + \dots + \binom{m}{m} v^m (1-w)^m \right] \end{aligned}$$

$$\begin{aligned} &= w \left[\binom{m}{0} \underset{a}{\cancel{(1-v)^m}} \underset{b}{\cancel{[(1-w)v]^0}} + \binom{m}{1} (1-v)^{\cancel{m-1}} \left[(1-w)v \right]^1 + \binom{m}{2} (1-v)^{\cancel{m-2}} \left[(1-w)v \right]^2 \right. \\ &\quad \left. + \dots + \binom{m}{m} (1-v)^0 \left[(1-w)v \right]^m \right] \end{aligned}$$

$$= w \left[(1-v) + (1-w)v \right]^m \text{ by binomial theorem}$$

recall

$$(a+b)^m = \binom{m}{0} a^m b^0 + \binom{m}{1} a^{m-1} b^1 + \dots + \binom{m}{m} a^0 b^m$$