Section 25: Solutions

Conceptual review:

ⓐ Let X, Y be r.v.:

$$\text{Cov}(X,Y) = \mathbb{E}\big((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\big)$$
$$= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

ⓑ $X, Y \sim$ iid $N(0,1)$ $\quad \alpha, \beta \in \mathbb{R}$

$$\alpha X + \beta Y \sim \; ?$$

$$\alpha X \sim N(0, \alpha^2) \perp \beta Y \sim N(0, \beta^2)$$

$$\alpha X + \beta Y \sim N(0, \alpha^2 + \beta^2)$$

ⓒ We say that X, Y have standard bivariate normal distribution with correlation $-1 \leq \rho \leq 1$ if:

$$X \sim N(0,1) \quad , \quad Y = \rho X + \sqrt{1-\rho^2}\, Z$$

and $Z \sim N(0,1)$ independent of X

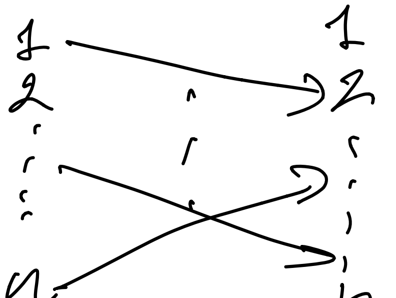
if $\rho = 0$ then this is just saying that X, Y are iid $N(0,1)$.

Problem 1 :

$X_1, \ldots, X_n$ random variables. We say that they are exchangeable if the joint distribution of $(X_1, \ldots, X_n)$ does not depend on the order of variables.

$(X_1, \cdots, X_n)$ has the same distribution as $(X_{\pi(1)}, \ldots, X_{\pi(n)})$ for any permutation π of $\{1, 2, \ldots, n\}$.

π :



when $n = 2$: (X_1, X_2) has same distrib as (X_2, X_1)

If $X_1, \dots, X_n$ are exchangeable then all the X_i's have the same distribution

Explanation, fix $i \neq 1$:

$(X_1, \dots, X_n)$ has same distrib as

$(X_i, X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$

X_1 and X_i have same distrib.

(X_1, X_2) has the same distribution as (X_i, X_j) for any $i \neq j$

$S_n = X_1 + \dots + X_n$

$$\mathrm{Var}(S_n) = \mathrm{Cov}(S_n, S_n)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} \mathrm{Var}(X_i) + \sum_{1 \le i \neq j \le n} \mathrm{Cov}(X_i, X_j)$$

since all X_i's have the same dist

$$Var(X_i) = Var(X_1) \quad \text{for any } i$$

and $$Cov(X_i, X_j) = Cov(X_1, X_2) \quad \text{for } i \neq j$$

$$= n Var(X_1) + n(n-1) Cov(X_1, X_2).$$

Problem2:

$X_1, X_2 \overset{iid}{\sim} N(0,1)$

$Y_1 = X_1 + X_2$

$Y_2 = \alpha X_1 + 2X_2$

such that $Cov(Y_1, Y_2) = 0$ $\leftarrow$

$\alpha \in \mathbb{R}$ unknown.

Compute/find α:

$Cov(Y_1, Y_2) = 0$ then we get

$$Cov(X_1 + X_2, \alpha X_1 + 2X_2) = 0$$

$$\alpha Var(X_1) + 2 Var(X_2) = 0$$

Then $\alpha + 2 = 0 \implies \alpha = -2$

<u>Density of Y_2:</u>

$$Y_2 = -2X_1 + 2X_2 \sim N(0, 2^2 + (-2)^2)$$
$$= N(0, 8)$$

$$f_{Y_2}(x) = \frac{1}{\sqrt{2\pi 8}} \exp\left(-\frac{x^2}{2 \times 8}\right)$$
$$= \frac{1}{4\sqrt{\pi}} \exp\left(-\frac{x^2}{16}\right)$$

<u>$Cov(X_2, Y_2)$:</u> Bilinearity of Cov

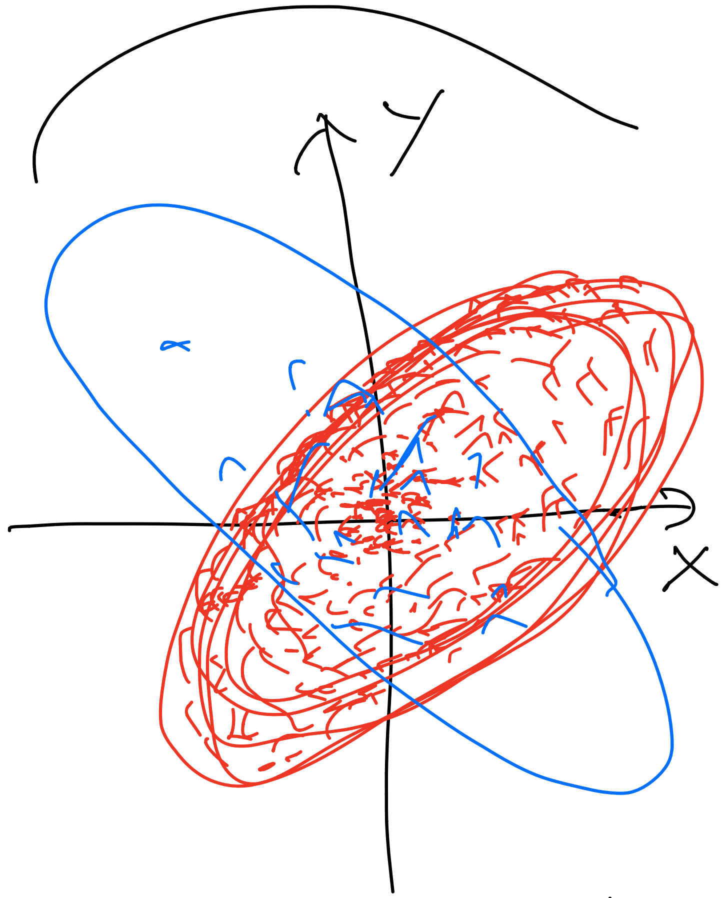
$$Cov(X_2, Y_2) = Cov(X_2, -2X_1 + 2X_2)$$
$$= 2\,Cov(X_2, X_2) = \underline{2}$$

Problem 3:

to show (X, Y) have bivariate normal, it's not enough to say $X \sim N(0,1)$, $Y \sim N(0,1)$

X, Y iid $N(0,1)$

$\rho = 0$



X, Y are standard bivariate with $\rho > 0$

$\rho < 0$

$X \sim N(0,1)$ $\underline{Y \perp X}$ $Y \begin{cases} \to +1 \\ \to -1 \end{cases}$

$$\mathbb{P}(Y = \pm 1) = 1/2$$

$Z = YX$ $\underline{\mathrm{Cov}(X,Z) = 0}$

$X, Z \sim N(0,1)$ $\underline{X \sim N(0,1)}$ ✓

$\underline{Z \sim N(0,1)}$: $z \in \mathbb{R}$

$$\mathbb{P}(Z \le z) = \mathbb{P}(YX \le z)$$

$$= \mathbb{P}(YX \le z \mid Y = 1)\, \mathbb{P}(Y=1)$$

$$+ \mathbb{P}(\underline{Y}X \le z \mid Y = -1)\, \mathbb{P}(Y = 1)$$

$$= \mathbb{P}(X \le z \mid \underline{Y = 1}) \times 1/2$$

$$+ \mathbb{P}(X \ge -z \mid \underline{Y = -1}) \times 1/2$$

So since $X \perp Y$

$$\mathbb{P}(Z \le z) = \frac{1}{2}\,\mathbb{P}(X \le z) + \frac{1}{2}\,\mathbb{P}(X \ge -z)$$

$$= \frac{1}{2}\big(\underbrace{\Phi(z) + 1 - \Phi(-z)}_{2\Phi(z)}\big)$$

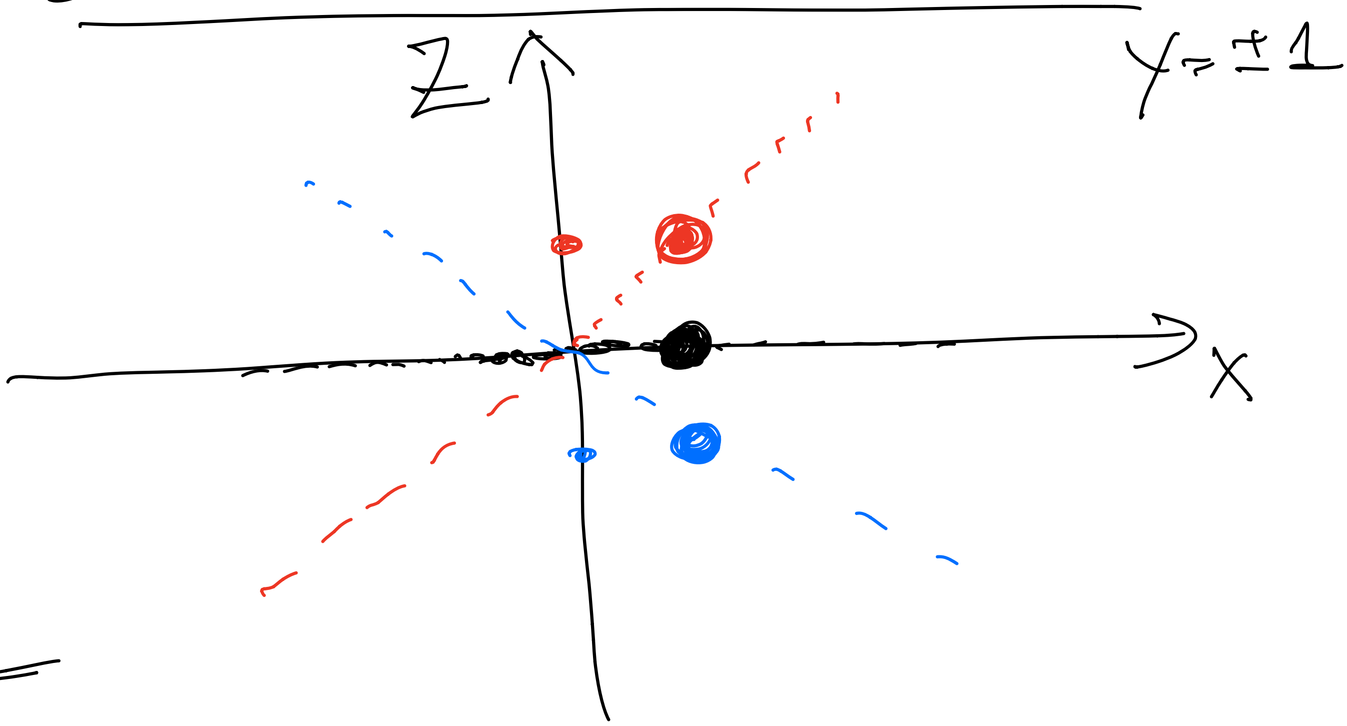
$$= \Phi(z)$$

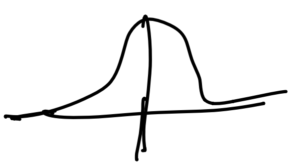
$$= \mathbb{P}(X \le z)$$

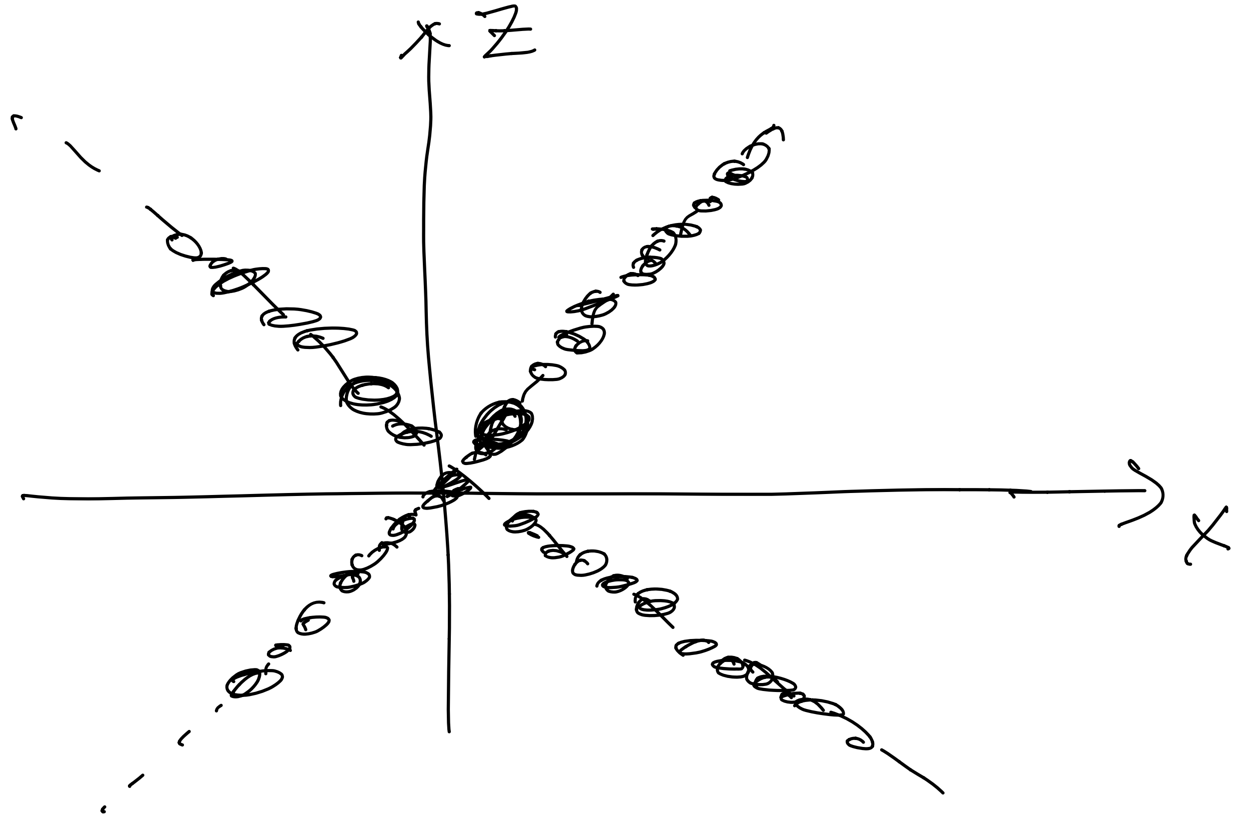
So X, Z have same distribution $N(0,1)$ and are uncorrelated

$$\mathrm{Cov}(X,Z) = 0$$

<u>BUT X, Z NOT Bivariate</u>







Z
X