Solutions to Stat 134 Old Finals

Final#1

1) a) $\frac{13}{4}$ by using 13 indicators, 1 for each card in the hand.

b) The number of hearts in the hand is hypergeometric, so $\frac{\binom{13}{3}\binom{39}{10}}{\binom{52}{13}}$

2) Using indicators I_i, one for each card, E(number of records) is

$$E(\sum_{i=1}^{10} I_i) = \sum_{i=1}^{10} E(I_i) = \sum_{i=1}^{10} \frac{1}{i}$$

where $\frac{1}{i}$ is the probability that the ith card is the largest of the cards so far, in other words, the probability that the ith card is a record.

3) a) binomial(100,.4)

b) $E(X) = \mu = 40$, $SD(X) = \sigma = \sqrt{npq} = 4.9$ Since $\sigma > 3$, normal approximation works well, and

$$\begin{aligned} P(X = 45) &= P(44.5 \le X \le 45.5) \\ &= P(\frac{44.5 - 40}{4.9} \le \frac{X - 40}{4.9} \le \frac{45.5 - 40}{4.9}) \\ &= \Phi(1.12) - \Phi(.92) \end{aligned}$$

c) negative binomial(4,.4) This is waiting until the 4th head.

d) The negative binomial(4,.4) can be thought of as a sum of 4 geometric(.4) random variables, each of which has expected value $\frac{1}{.4} = 2.5$, so $E(Z) = 10$.

4) a)

$$\int_0^1 cx^2 dx = c\frac{1}{3}x^3|_0^1 = \frac{c}{3} = 1$$

since the integral of the density over entire range must equal 1. Thus $c = 3$.

b) Using the density function,

$$E(X) = \int_0^1 x(3x^2)dx = \frac{3}{4}x^4|_0^1 = \frac{3}{4}$$

c) $SD(X) = \sqrt{var(X)} = \sqrt{E(X^2) - [E(X)]^2}$ so we need only find $E(X^2)$.

$$E(X^2) = \int_0^1 x^2(3x^2)dx = \frac{3}{5}x^5|_0^1 = \frac{3}{5}$$

and so $SD(X) = \sqrt{\frac{3}{5} - \frac{9}{16}} = \sqrt{\frac{3}{80}} = .194$.

d) The cdf of X is

$$F(x) = P(X \le x) = \int_0^x f(x)dx = \int_0^x 3x^2 dx = x^3|_0^x = x^3$$

e) Since $g(x)$ is strictly increasing,

$$f_Y(y) = \frac{3x^2}{.5x^{-.5}} = 6y^5 \text{ for } 0 \le y \le 1, 0 \text{ else.}$$

5) a)

$$\int_0^\infty \int_0^x \lambda_1 \lambda_2 e^{-\lambda_1 x} e^{-\lambda_2 y} dy dx = \int_0^\infty \lambda_1 e^{-\lambda_1 x}(1 - e^{-\lambda_2 x}) dx = 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

b) $P(\frac{1}{2} < \frac{X}{Y} < 2) = P(.5Y < X < 2Y)$. Figuring out what region this is in the plane, integrate the jt density over this region.

$$\int_0^\infty \int_{.5y}^{2y} \lambda^2 e^{-\lambda x - \lambda y} dx dy = \int_0^\infty \lambda e^{-\lambda y}(e^{-\lambda .5y} - e^{-\lambda 2y}) dy = -\frac{2}{3} e^{-\lambda 1.5y} + \frac{1}{3} e^{-\lambda 3y} |_0^\infty = \frac{1}{3}$$

c) $P(T < 3) = P(2 \text{ or more hits by time } 3)$. Using the Poisson, this is $1 - e^{-6} - 6e^{-6} = 1 - 7e^{-6}$.
d) $1 - e^{-4x}$ for $x > 0$.
e) $f_X(x|T = t) = \frac{1}{t}$; $f_T(t) = \lambda^2 t e^{-\lambda t}$ so $f(x,t) = 4e^{-2t}$.
6) The number of dice landing any particular number is a thinned Poisson process, thus the number of 6s, for example, is Poisson (1), and each of the six Poissons thus generated is independent. Thus $X = \sum_{i=1}^6 iY_i$ where the Y_i are the number of dice showing i. a) $E(X) = 21$
b) $Var(X) = \sum_{i=1}^6 i^2 var(Y_i) = \sum_{i=1}^6 i^2 = 91$, so $SD(X) = \sqrt{91} = 9.54$

Final#2

1) a) $\frac{5}{13}$ b) $\frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}$ c) $5\left(\frac{1}{13}\right)\left(\frac{12}{13}\right)\left(\frac{52-5}{52-1}\right)$

2) a) Let W_i be the number of times die i is rolled. Then $\mathrm{P}(N \leq 3) = \mathrm{P}(W_i \leq 3)^{10} = \left[1 - \left(\frac{5}{6}\right)^3\right]^{10}$

b) $T = \sum_{i=1}^{10} W_i$ so $E(T) = 10E(W_i) = 60$.

3) a) $\int_{-\infty}^{\infty} f(x)dx = 2\int_0^{\infty} ce^{-x}dx = 2(-ce^{-x})|_0^{\infty} = 2c = 1$ so $c = \frac{1}{2}$.

b) $Y = X^2$, range of Y is $[0, \infty)$, $X = \pm\sqrt{Y}$ is not one-to-one, so must consider two values of x for each y. $\frac{dy}{dx} = 2x$.

$$f(y) = \frac{\frac{1}{2}e^{-|x|}}{|2x|} + \frac{\frac{1}{2}e^{-|x|}}{|2x|} = \frac{e^{-x}}{2x}$$

for $x > 0$. And finally

$$f(y) = \frac{e^{-\sqrt{y}}}{2\sqrt{y}} \text{ for } y \geq 0$$

c) E(X)=0, var(X)=2

d)

$$g(x) = F(x) = \begin{cases} \frac{1}{2}e^{x} & \text{for } x < 0 \\ 1 - \frac{1}{2}e^{-x} & \text{for } x \geq 0 \end{cases} \tag{1}$$

4) a) $\frac{1}{3}$

b) Let N_t be the number of particles in t minutes.

$$\begin{aligned} P(1 < T_3 < 2) &= P(T_3 \geq 1) - P(T_3 \geq 2) \\ &= P(N_1 \leq 2) - P(N_2 \leq 2) \\ &= \left(e^{-3} + 3e^{-3} + \frac{9}{2}e^{-3}\right) - \left(e^{-6} + 6e^{-6} + \frac{36}{2}e^{-6}\right) \end{aligned}$$

5) a) Let R_A = Annie's distance from center. R_A is Rayleigh, and P($R \leq 1$)=$1 - e^{-\frac{1}{2}}$.

b) Let R_B be Butch's distance from the center. We want P($R_A \leq R_B$). Could use joint density or recall that R_A^2 is exp($\frac{1}{2}$), notice that $R_B^2 = 4R^2$, where R^2 is exp($\frac{1}{2}$) so R_B^2 is exp($\frac{1}{8}$). P($R_A \leq R_B$) = P($R_A^2 \leq R_B^2$) $= \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{8}} = \frac{4}{5}$. (like 4.rev 13c, or chance that first email is spam done in lecture.)

6) a) $\binom{4}{2}\left(\frac{1}{2}\right)^4$

b) $\frac{24}{99}$ see pg 397-398 of text.

Final#3

1) a) P($T > t$)=e^{-6t}

b) P($D > t$)=P((no B buses) and (0 or 1 A buses)) $= e^{-4t}\left(e^{-6t} + 6te^{-6t}\right) = e^{-10t} + 6te^{-10t}$

c)

$$\begin{aligned} E(D) &= \int_0^\infty \left(e^{-10t} + 6te^{-10t}\right) dt \\ &= \frac{1}{10} + \frac{6}{10}\int_0^\infty t10e^{-10t}dt \\ &= \frac{1}{10} + \frac{6}{100} = \frac{4}{25} \end{aligned}$$

d) P($T_3 \geq \frac{1}{2}$)=P(0,1, or 2 buses in half an hour) $= e^{-5} + 5e^{-5} + \frac{25}{2}e^{-5}$

2) a)

x	0	1	2	3	4
P($S_2 = x$)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

b) E(S_{50})=50, var(S_{50})=25, so $\Phi\left(\frac{50+0.5-50}{5}\right) - \Phi\left(\frac{50-0.5-50}{5}\right)$

c) This is perhaps a bit mean, but S_1 has the same distribution as the number of heads in 2 coin tosses, so S_n has the same distribution as the number of heads in $2n$ coin tosses, so the answer is $\binom{2n}{k}\left(\frac{1}{2}\right)^{2n}$.

3) a) $f_X(x) = \int_0^\infty f(x,y)dy = 2\lambda e^{-2\lambda x}$, similarly $f_Y(y) = \lambda e^{-y}$.

b) Yes, because $f(x,y) = f_X(x)f_Y(y)$.

c) If you're lucky, you notice that $X + Y$ has the same distribution as the maximum of two exponentials with parameter λ. Think of X as the time of the minimum of two such exponentials and then Y is the waiting time for the other one. Look at pages 316-317. If you see this, then P($X + Y > 2$)= $1-(1-e^{-2\lambda})^2$ =$2e^{-2\lambda}-e^{-4\lambda}$. If not, you can always do $1-\int_0^2\int_0^{2-x} f(x,y)dydx$. That's a messy integral, but it works.

d) Again, you can do an integral, or think about X and Y coming from Poisson processes where $\frac{2}{3}$ of the hits from from the X process, so the chance that the first hit comes from that process is $\frac{2}{3}$.

e) $1 - e^{-3\lambda z}$, look at p317.

4) a) X is binomial$(4, \frac{1}{6})$ and T is geometric$(\frac{1}{6})$.
b) Four equally likely ways to have $X = 1$, each corresponding to a different

t	1	2	3	4
$\mathrm{P}(T = t \mid X = 1)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

value of T.
c)

t	1	2	3
$\mathrm{P}(T = t \mid X = 1)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

5) a) $\binom{80}{1} \frac{1}{100} \left(\frac{99}{100}\right)^{79}$
b) Poisson approx isn't *great*, but you can't do normal since $\sqrt{npq} < 3$. $\mathrm{P}(X \geq 2) = 1 - \left(e^{-\frac{4}{5}} + \frac{4}{5} e^{-\frac{4}{5}}\right)$
6) a) $80 \times \frac{120}{199} = 48.24$
b) $80 \frac{120}{199} \times \frac{79}{199} + (80)(79) \left[\frac{120}{199} \frac{119}{197} - \left(\frac{120}{199}\right)^2\right] = 23.14$