Stat 134 Lec 21

Last time sec 4.1 Continuous distributions

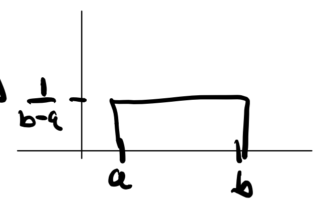
A continuous RV X, has a prob density function, $f(x)$, where $f(x) \ge 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

$P(X=a) = \int_a^a f(x)dx = 0$ so $P(X \ge a) = P(X > a)$.

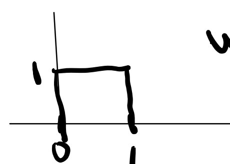
A change of scale is a transformation $Y = c + dX$, of X. The density of X gets transformed into the density of Y.

constants.

ex $X \sim \text{Unif}(a,b)$ has prob density



The change of scale $U = \frac{X-a}{b-a}$ has prob density



which is easier to work with.

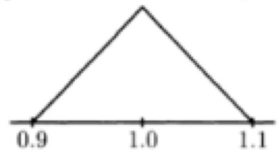
Expectation $E(X) = \int_{-\infty}^{\infty} x f(x) dx$

Variance $Var(X) = E(X^2) - E(X)^2$

Today
1. review concept test responses from last time.
2. sec 4.1 change of scale calculations
3. briefly sec 4.5 Cummulative Distribution Function (CDF)
4. Sec 4.2 Exponential Distribution.

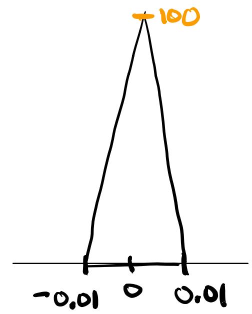
① student responses.

Suppose a manufacturing process designed to produce rods of length 1 inch exactly, in fact produces rods with length distributed according to the density graphed below.

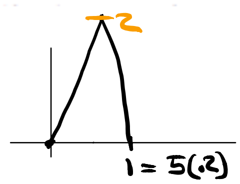


You should change the scale of X= the length of rods to:

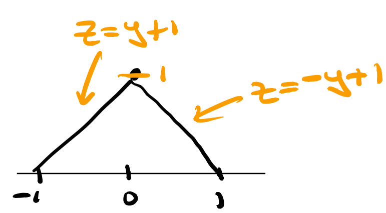
- a: X-1
- b: .1(X-1)
- c: 10X-1
- d: none of the above



b: .1(X-1)	To make the triangle into standard normal, translate it so that 1.0 becomes 0, and 0.9 becomes -0.1 and 1.1 becomes 0.1. This is the translation of X-1. To make the area equal to 1, divide height by 10, which is b.



d: none of the above	a in this case is 0.9 and b is 1.1, so using the structure of the last example, it should be (x-0.9)/0.2 to make it centered around 0



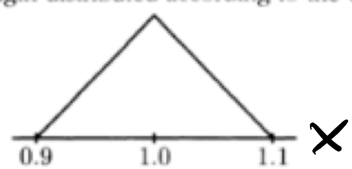
d: none of the above	10(X-1) = 10X-10 to center it at 0 and make the range -1 to 1

$$f(y) = \begin{cases} y+1 & -1<y<0 \\ -y+1 & 0<y<1 \\ 0 & \text{else} \end{cases}$$

② Sec 4.1 Change of scale calculation

ex

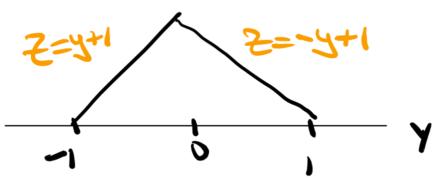
Suppose a manufacturing process designed to produce rods of length 1 inch exactly, in fact produces rods with length distributed according to the density graphed below.



Find the variance of the length of the rods.

$Y = 10(X-1)$ change of scale.

$Var(Y) = 100\, Var(X) \Rightarrow Var(X) = \frac{Var(Y)}{100}$ ← easier to find.



Find Var(X)

$Var(Y) = E(Y^2) - E(Y)^2$

$E(Y^2) = 2\int_0^1 y^2(-y+1)\,dy = 2\left(\frac{y^3}{3} - \frac{y^4}{4}\right)\Big|_0^1$

$= \frac{2}{3} - \frac{2}{4} = \boxed{1/6}$

$Var(X) = \frac{Var(Y)}{100} = \boxed{1/600}$

③ briefly sec 4.5 The Cummulative Distribution Function (CDF)

Let X be a continuous RV

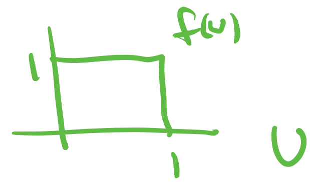
$F(x) = P(X \le x)$ — a number between 0 and 1

If $f(x)$ is a density of X,

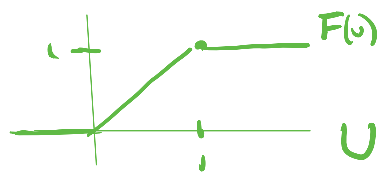
$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

ex $U \sim \text{Unif}(0,1)$

$$f(u) = \begin{cases} 1 & 0 < u < 1 \\ 0 & \text{else} \end{cases}$$



$$F(u) = \int_0^u 1\,dx = u$$



$$\boxed{F(u) = \begin{cases} 0 & -\infty < u \le 0 \\ u & 0 \le u \le 1 \\ 1 & u \ge 1 \end{cases}}$$

By FTC, $F'(x) = f(x)$

Consequently a density function and cdf are equivalent descriptions of a RV.

(4) Sec 4.2 Exponential and Gamma Distribution

Memoryless property of the geometric distribution

$X \sim Geom(p)$

$X = \#$ trials until your first success.

$P(X=k) = q^{k-1}p$

recall $P(X>k) = P(X=k+1) + P(X=k+2) + \cdots$

$= q^{k}p + q^{k+1}p + \cdots$

$= q^{k}p\,\underbrace{(1+q+q^2+\cdots)}_{=\frac{1}{1-q}}$

$= q^{k}$

Since $P(X=k) = \underbrace{P(X>k-1)}_{=q^{k-1}} - \underbrace{P(X>k)}_{=q^{k}}$

$= q^{k-1}(1-q) = q^{k-1}p$

$P(X>k) = q^{k} \Rightarrow X$ is $Geom(p)$.

$\Rightarrow$ $X \sim Geom(p)$ iff $P(X>k) = q^{k}$

Question

If it takes you more than $j=10$ p-coin tosses to get your first heads, what is the chance it will take you more than $k+j=13$ coin tosses to get your first heads?

Answer

$X \sim \text{Geom}(p)$

Think of starting on your $j+1 = 11^{th}$ toss.

So it should be $P(X>3) = q^3$

i.e.

$$P(X > k+j \mid X > j) = P(X>k)$$

Thm The Geometric distribution is the only discrete distribution with values $1, 2, 3, \ldots$ having the

memoryless property

$$P(X > \underset{\underset{3}{\shortparallel}}{k} + \underset{\underset{10}{\shortparallel}}{j} \mid X > \underset{\underset{10}{\shortparallel}}{j}) = P(X > \underset{\underset{3}{\shortparallel}}{k})$$

Pf/ Let $X \sim \text{Geom}(p)$, $X = 1, 2, 3, \ldots$

$$P(X > k+j \mid X > j) = \frac{P(X > k+j,\ X > j)}{P(X > j)}$$

$$= \frac{P(X > k+j)}{P(X > j)}$$

$$= \frac{q^{k+j}}{q^{j}} = q^{k}$$

$= P(X > k)$ ✓

Conversely,

For positive integers k, j

Suppose $P(X > k+j \mid X > j) = P(X > k)$

$$\begin{gathered} \shortparallel \\ \frac{P(X > k+j)}{P(X > j)} \end{gathered}$$

$\Rightarrow P(X > k+j) = P(X > j)P(X > k)$

let $q = P(X > 1)$

Show that $P(X > j) = q^j$

for $j = 1, 2, 3, \ldots$ by induction.

<u>Base case</u>:

$P(X>1)=q^1$ ✓

Assume $P(X>j-1)=q^{j-1}$

Finish the proof.

$$P(X>j) = P(X>j-1+1)$$

$$= \underbrace{P(X>j-1)}_{=\,q^{j-1}}\underbrace{P(X>1)}_{=\,q}$$

$$= q^j$$

$$\Rightarrow X \sim \text{Geom}(p).$$

$\square$

Only 2 distributions are memoryless:

For discrete $(X=1,2,3,\ldots)$ — Geometric

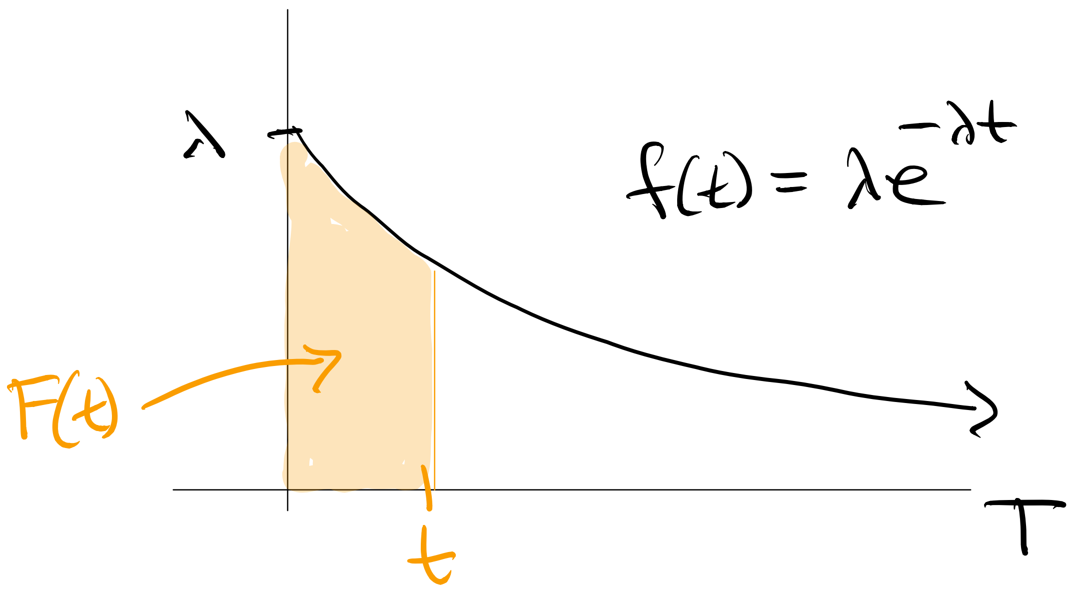
For continuous $(T>0)$ — Exponential

Proof is similar to geom.

Exponential distribution

Def^n A random time T has exponential distribution with rate $\lambda>0$.

$T \sim Exp(\lambda)$, if T has density $f(t)=\lambda e^{-\lambda t}$ $\quad t \geq 0$



ex T = time until your first success where λ = rate of success,

ex T = time untill a lightbulb burns out

CDF and survival function

$T \sim Exp(\lambda) \quad f(t) = \lambda e^{-\lambda t}$

Compute the CDF of T.

$$F(t) = P(T \leq t) = \int_0^t f(s)\,ds$$

$$= \int_0^t \lambda e^{-\lambda s}\,ds = \frac{\lambda e^{-\lambda s}}{-\lambda}\Big|_0^t$$

$$= -e^{-\lambda t} + 1 = \boxed{1 - e^{-\lambda t}}$$

$P(T>t) = e^{-\lambda t}$ is

called the survival function

$T \sim Exp(\lambda)$ iff $P(T>t) = e^{-\lambda t}$

since $F(t)$ $= 1 - P(T>t)$ and $f(t)$ both

define distribution.

We can think of Geom(p)

as a discrete version of Exp(λ)

here is how:

$X \sim Geom(p)$ $X = 1, 2, 3, \ldots$

let pX be a change of scale

of X.

We will show that $pX \to \text{Exp}(1)$ as $p \to 0$.

pX takes values $p, 2p, 3p, \ldots$

$$P(pX = p) = P(X=1) = p$$
$$P(pX = 2p) = P(X=2) = qp$$
$$P(pX = 3p) = P(X=3) = q^2 p$$

Prob mass function for pX: