Stat 134 lec 29

Quiz 6 wednesday 4.6, 5.1, 5.2

Last time.

① Sec 4.6 Beta Distribution

Let $r, s > 0$

$P \sim \text{Beta}(r,s)$ if

$$f(p) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} p^{r-1}(1-p)^{s-1} \quad \text{for } 0<p<1$$

Applications

ⓐ generalization of uniform ordered statistic

If throw n darts at $[0,1]$

$U_{(k)} \sim \text{Beta}(k, n-k+1)$

Note $U(0,1) = \text{Beta}(1,1)$ — $n=1$, $k=1$

ⓑ $\text{Beta}(r,s)$ represents a distribution of probabilities (Bayesian statistics)

Recall from lecture 2 (Sec 1.2)

Posterior $\propto$ likelihood $\cdot$ Prior

$$f(p|x) \propto f(x|p)\, f(p)$$

ex $f(p) = 1$ for $0<p<1$

An update of the distribution of P given that you have $X=5$ heads.

ex

Bin form. $P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$

Geom form. $P(X=x) = p(1-p)^{x-1}$

ex You have a prob p coin (p unknown). You flip coin n times and get x heads. What is posterior distribution of P?

prior: $P \sim \text{Beta}(r, s)$, $f(p) \propto p^{r-1}(1-p)^{s-1}$

likelihood: $X \mid P=p \sim \text{Bin}(n, p)$ $f(x \mid p) = \binom{n}{x} p^x (1-p)^{n-x}$

posterior:

$$f(x \mid p) f(p) \propto p^x (1-p)^{n-x} \cdot p^{r-1}(1-p)^{s-1}$$

$$= p^{x+r-1}(1-p)^{n-x+s-1}$$

$$\Rightarrow P \mid X=x \sim \text{Beta}(x+r, n-x+s)$$

② sec 5.1, 5.2 independent RVs

$f(x,y) = f(x) f(y)$

Today

① Review student explanations to concept test.

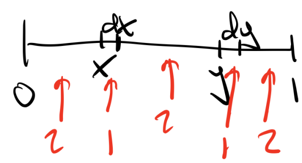
② sec 5.2 independence.
Marginal density
Expectation $E(g(x,y))$

③ sec 5.3 Independent normal variables.

Concept test

1. I throw down 8 darts on $(0,1)$. The variable part of the joint density of $X = U_{(3)}$ and $Y = U_{(6)}$ is:

   a $x(y-x)^5(1-y)^2$, $0<x<y<1$
   b $x^2(y-x)^2(1-y)^2$
   c $x^4(y-x)^2(1-y)^2$
   d none of the above



10/26/201	b	b	2 darts before x —> x^2 1 dart at x —> 1dx 2 darts between x and y —> (y-x)^2 1 dart at y —> dy 2 darts after y —> (1-y)^2 Don't you need to specify the ranges of x and y?	

10/26/201	b	b	Group 8 darts as the following: (1,2) (3) (4,5) (6) (7,8) corresponding to (before X) (dx) (between X and Y) (dy) (after Y). Then answer (b) follows.	

2. Is $f(x,y) = \binom{6}{1,4,1}(y-x)^4$ on $0 < x < y < 1$ a joint density function?

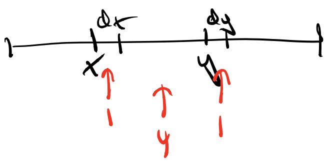
a yes

b no

c not enough info to decide

$X = U_{(1)}$

$Y = U_{(6)}$



Is X, Y independent? — find marginal densities

Recall marginal probability:

<u>discrete picture</u>

marginal probability of X

$P(x) = \sum_{y \in Y} P(x,y)$

$Y \backslash X$	0	1	$P(y)$
2	0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
0	$\frac{1}{4}$	0	$\frac{1}{4}$
$P(x)$	$\frac{1}{2}$	$\frac{1}{2}$	

Marginal prob of Y

$P(y) = \sum_{x \in X} P(x,y)$

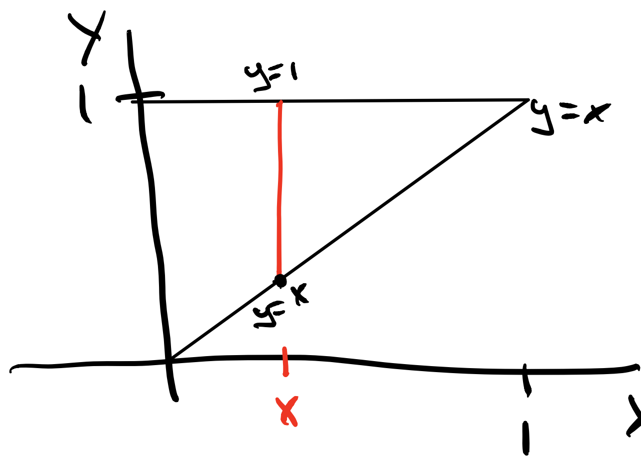
sec 5.2

marginal density

$$f(x,y) = \begin{cases} 30\,(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$

$Y = U_{(6)}$



$$\int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty} f(x,y)\,dy\,dx = 1$$

$$f_X(x) = \int_{y=-\infty}^{y=\infty} f(x,y)\,dy$$

marginal density of X

$$= \int_{y=x}^{y=1} 30\,(y-x)^4\,dy$$

$u = y - x$

$du = dy$

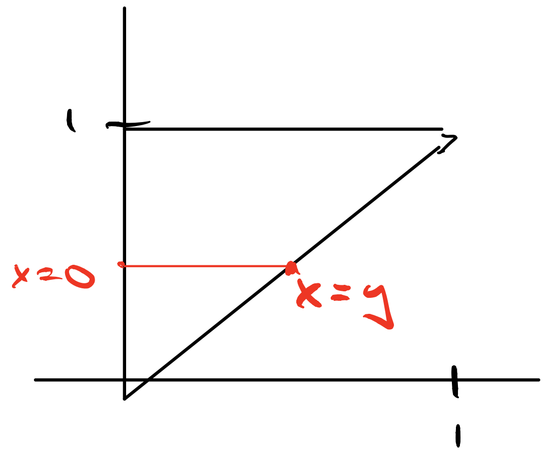
$$= \int_{u=0}^{u=1-x} 30u^4\,du = \left.\frac{30u^5}{5}\right|_0^{1-x} = \boxed{6(1-x)^5,\quad 0<x<1}$$

note

$f(x,y)$ is zero outside of $0<x<y<1$ so this is also

$$\int_{x=0}^{x=1}\int_{y=x}^{y=1} f(x,y)\,dy\,dx = 1$$

$$f_y(y) = \int_{x=-\infty}^{x=\infty} 30\,(y-x)^4\,dx$$



$$= \int_{x=0}^{x=y} 30\,(y-x)^4\,dt$$

$u = y-x$

$du = -dx$

$$= -\int_{u=y}^{0} 30\,u^4\,du = \frac{30u^5}{5}\Big|_0^y = 6y^5,\quad 0<y<1$$

$\Rightarrow X = U_{(1)},\ Y = U_{(6)}$ arent independent,

since $f(x,y) = 30\,(y-x)^4 \neq f(x)f(y) = 6(1-x)^5 \cdot 6(y)^5$

Expectation

$$E(g(x,y)) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} g(x,y) f(x,y)\,dx\,dy.$$

Check:

Find

$$E(Y) = \int_{y=0}^{y=1}\int_{x=0}^{x=y} y\, f(x,y)\,dx\,dy$$

$$= \int_{y=0}^{y=1} y \underbrace{\int_{x=0}^{x=y} f(x,y)\,dx}_{= f_Y(y)}\,dy$$

$$= \int_{y=0}^{y=1} y \underbrace{f_Y(y)}_{= 6y^5}\,dy = \int_{y=0}^{y=1} 6y^6\,dy$$

$$= \frac{6y^7}{7}\Big|_0^1 = 6/7$$

Note $Y \sim U_{(6)} = \text{Beta}(6,1) \Rightarrow E(Y) = \frac{6}{6+1}$ ✓

$k \quad n-k+1 = 6-6+1$

ex (5.2.9a)

$S, T \overset{iid}{\sim} \text{Exp}(\lambda)$

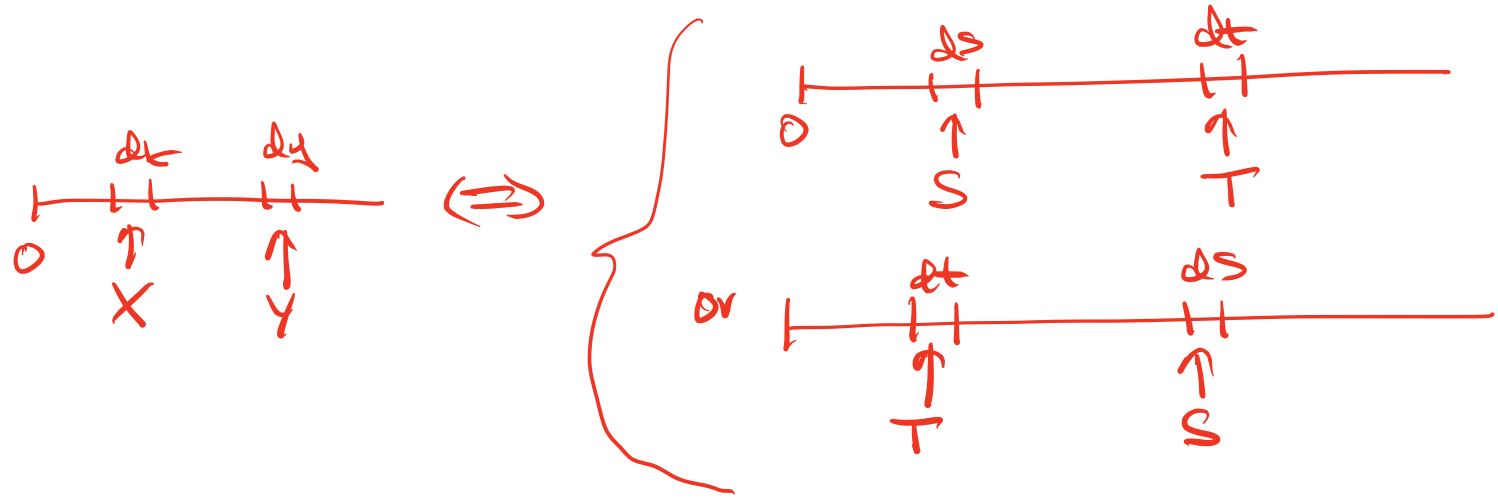
$X = \min(S,T)$

$Y = \max(S,T)$

Find the joint density of X and Y

Are X, Y indep.

Find joint density:



i.e $P(X \in dx, Y \in dy) = P(S \in ds, T \in dt) + P(T \in dt, S \in ds)$

$\Rightarrow P(X \in dx, Y \in dy) = 2P(S \in ds, T \in dt)$

$\Rightarrow f(x,y)\,dx\,dy = 2f(s,t)\,ds\,dt$

$= 2\lambda e^{-\lambda s} \cdot \lambda e^{-\lambda t}\,ds\,dt$

$= 2\lambda^2 e^{-\lambda(s+t)}\,ds\,dt$

$\Rightarrow \boxed{f(x,y) = 2\lambda^2 e^{-\lambda(x+y)}}$

Find $f_X(x), f_Y(y)$.

Is X, Y indep ?

Concept test.

Soln

Stat 134

Monday October 29 2018

1. S and T are i.i.d. Exp(λ). $X = \text{Min}(S, T)$ and $Y = \text{Max}(S, T)$. The joint density is $f(x, y) = 2\lambda^2 e^{\lambda(x+y)}$. The marginal density of X is:

   a $2\lambda e^{-2\lambda x}$ for $x > 0$

   b $2\lambda e^{-\lambda x}$ for $x > 0$

   c $\lambda e^{-\lambda x}$ for $x > 0$

   d none of the above

$X = \min(S,T)$

Method 1

Find marginal

$f(x) = \int_x^\infty 2\lambda^2 e^{-\lambda(x+y)}\, dy$

$= 2\lambda^2 e^{-\lambda x} \underbrace{\int_x^\infty e^{-\lambda y}\, dy}_{= \frac{e^{-\lambda y}}{-\lambda}\Big|_x^\infty = \frac{e^{-\lambda x}}{\lambda}}$

$= \boxed{2\lambda e^{-2\lambda x}}$

Method 2

$X = \min(S,T)$

$P(X > x) = P(S > x, T > x)$

$= P(S > x)^2 = (e^{-\lambda x})^2 = e^{-2\lambda x}$

$\Rightarrow F(x) = 1 - e^{-2\lambda x}$

$\boxed{f(x) = 2\lambda e^{-2\lambda x}}$

2. S and T are i.i.d. Exp(λ). $X = \text{Min(S, T)}$ and $Y = \text{Max(S, T)}$. The joint density is $f(x, y) = 2\lambda^2 e^{\lambda(x+y)}$. The marginal density of Y is:

a $\lambda(1 - e^{-\lambda x})e^{\lambda y}$ for $y > 0$

b $2\lambda(1 - e^{-\lambda x})e^{\lambda y}$ for $y > 0$

c $2\lambda(1 - e^{-\lambda x})$ for $y > 0$

d none of the above

Method 1

$$f_Y(y) = 2\lambda^2 e^{-\lambda y} \int_0^y e^{-\lambda x}\,dx$$

$$= 2\lambda^2 e^{-\lambda y} \left(\frac{e^{-\lambda x}}{-\lambda} \right) \Big|_0^y$$

$$= \boxed{2\lambda(1 - e^{-\lambda y})(e^{-\lambda y})}$$

method 2

$$F(y) = P(Y < y) = P(S < y, T < y)$$

$$= P(S < y)^2 = (1 - e^{-\lambda y})^2$$

$$f(y) = \frac{d}{dy} F(y) = 2(1 - e^{-\lambda y}) \cdot (e^{-\lambda y}) \cdot \lambda$$

$$= \boxed{2\lambda(1 - e^{-\lambda y})(e^{-\lambda y})}$$

$$f(x,y) \overset{?}{=} f(x)f(y) = 2\lambda e^{-2\lambda x} \cdot 2\lambda(1 - e^{-\lambda y})(e^{-\lambda y})$$

No!

$$\parallel$$

$$2\lambda^2 e^{-\lambda(x+y)}$$

X, Y dependent.

sec 5.3 Independent Normal variables.

In chapter 4 we introduced the standard normal RV but didnt prove much about it.

Properties of std normal	Proved it
$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$	no
$E(Z) = 0$	no
$SD(Z) = 1$	no

Even though Z is symmetric around zero it is possible E(x) is undefined

ex Cauchy distribution