Stat 134 lec 21

Warmup: 9:00-9:10

Recall $P(T>k) = e^{-\lambda k}$

Let $T \sim Exp(\lambda)$

a) Find $P(T>5) = e^{-5\lambda}$

b) Find $P(T>\underbrace{13}_{=5+8} \mid T>8) = e^{-5\lambda}$

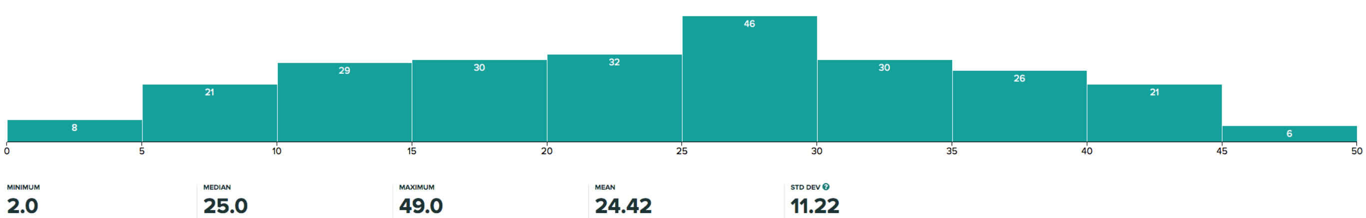
memoryless property of exponential.

or Bayes rule gives

$$\frac{P(T>13)}{P(T>8)} = e^{-5\lambda} \checkmark$$

Congratulations on a challenging midterm!

You can clobber the midterm if you do better on the final.



last time

Sec 4.5 Cummulative Distribution Function (CDF)

The CDF of a RV X is $P(X \le x)$. The survival function is $P(X > x)$.

The CDF or survival function uniquely determine a distribution.

Sec 4.2 exponential distribution.

survival function

$T \sim Exp(\lambda)$ $\quad f(t) = \begin{cases} \lambda e^{-\lambda t} & t \ge 0 \\ 0 & \text{else} \end{cases}$ or $P(T > t) = e^{-\lambda t}$

T = time until 1st success (arrival)

Memoryless property of the geometric distribution and exponential distributions

memoryless property

$P(X > k + j \mid X > j) = P(X > k)$

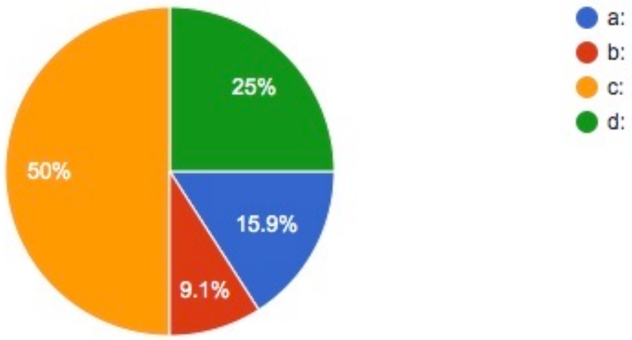
proved in appendix of notes

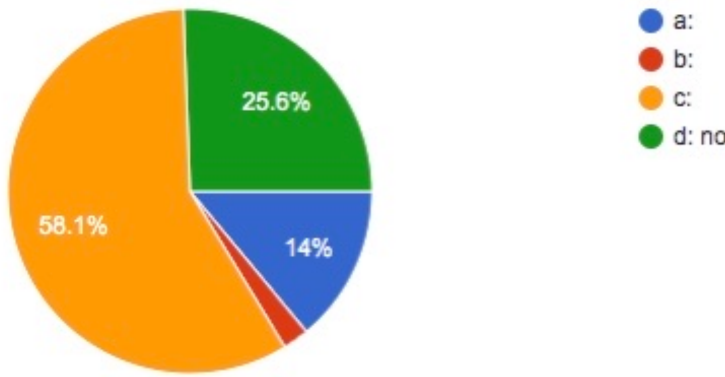
Thm. The Geometric distribution is the only discrete distribution with values $1, 2, 3, \ldots$ having the **memoryless property**

Only 2 distributions are memoryless:

For discrete $(X = 1, 2, 3, \ldots)$ — Geometric

For continuous $(T > 0)$ — Exponential ← Proof is similar to geom.





1. GSI Brian and Yiming are each helping a student. Brian and Yiming see students at a rate of λ_B and λ_Y students per hour respectively.
   Let

   $$B = \text{wait time for Brian } \sim Exp(\lambda_B)$$
   $$Y = \text{wait time for Yiming } \sim Exp(\lambda_Y)$$

   What distribution is $T = \min(B, Y)$?
   Hint: compute $P(T > t)$

   a $Exp(\max(\lambda_B, \lambda_Y))$

   b $Exp(\lambda_B - \lambda_Y)$

   c $Exp(\lambda_B + \lambda_Y)$

   d none of the above

a:	The greater the lambda, the smaller the density for T > t. Therefore, we want the greater lambda of the two.

c:	They are independent and since we are working with the minimum. Both of them are greater than t. So we get the probability of Brian's rate greater than t AND Yiming's rate greater than t. Which uding the survival function. We get e^(-lambda(b)-lambda(y)) then you get exp(lambda(b)+lambda(y)).

Today sec 4.2

① Expectation and Variance of $Exp(\lambda)$.

② Gamma distribution.

1

Sec 4.2 Expectation and Variance of Exp(λ)

$T \sim \text{Exp}(\lambda)$

see end of lec notes.

$$E(T) = \int_0^\infty t f(t)\,dt = \lambda \int_0^\infty t e^{-\lambda t}\,dt = \boxed{\frac{1}{\lambda}}$$

see end of lec notes.

$$E(T^2) = \int_0^\infty t^2 f(t)\,dt = \lambda \int_0^\infty t^2 e^{-\lambda t}\,dt = \frac{2}{\lambda^2}$$

$$\text{var}(T) = E(T^2) - E(T)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

Interpretation of λ:

$\lambda = \frac{1}{E(T)}$ (one over average arrival time)

This makes sense since the instantaneous rate of success is constant by the memoryless property of Exp(λ).

So if it takes on avg 5 min to have a success then $\lambda = 1/5$.

ex Let T = waiting time (min) until a blue Honda enters a toll gate starting at noon. The waiting time is on average 2 minutes. Find $P(T > 3)$

$E(T) = 2 \Rightarrow \lambda = \frac{1}{2}$

$T \sim \text{Exp}(1/2)$

$P(T > 3) = e^{-\frac{1}{2} \cdot 3} = \boxed{e^{-3/2}}$

② Gamma Distribution

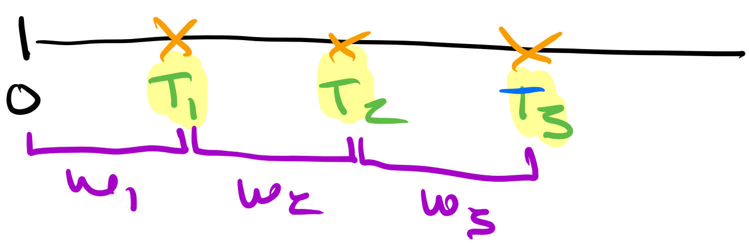
A gamma distribution, $T_r \sim \text{Gamma}(r, \lambda)$, $\lambda > 0$, is a sum of r iid $\text{Exp}(\lambda)$.

$$T_r = W_1 + W_2 + \dots + W_r \quad , \quad W_i \sim \text{Exp}(\lambda)$$

ex T_r = time of the r^{th} arrival of a Poisson Process $(\text{Pois}(\lambda t))$

ex T_r = time when the r^{th} lightbulb dies.

Picture



$$\left.\begin{aligned} T_1 &\sim \text{Gamma}(1, \lambda) \\ T_2 &\sim \text{Gamma}(2, \lambda) \\ T_3 &\sim \text{Gamma}(3, \lambda) \end{aligned}\right\}$$

dependent but W_1, W_2, W_3 are independent, $\text{Exp}(\lambda)$

$$T_1 = W_1$$
$$T_2 = W_1 + W_2$$
$$T_3 = W_1 + W_2 + W_3$$

Review Poisson Process (a.k.a. PRS)

$N_t \sim Pois(\lambda t)$

N_t = # arrivals in time t where the rate of arrivals is λ.

$$P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

ex Let T_1 = waiting time (min) until a blue Honda enters a toll gate starting at noon. The waiting time is on average 2 minutes. Find the probability 3 blue Hondas arrive in 8 minutes.

$t = 8$

$\lambda = 1/2$

$N_8 \sim Pois(\underbrace{\tfrac{1}{2}\cdot 8}_{\mu = 4})$

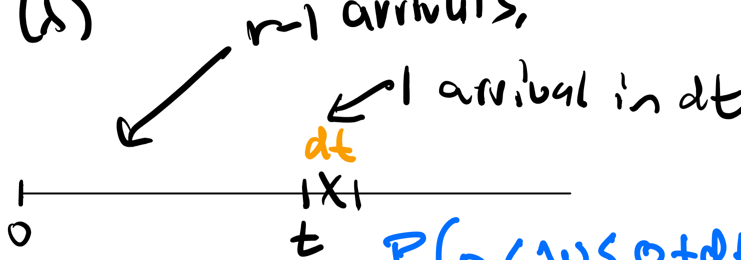
$$P(N_8 = 3) = \boxed{\frac{e^{-4}\,4^3}{3!}}$$

Back to gamma distribution: $T_r \sim \text{Gamma}(r, \lambda)$

Let $N_t \sim \text{Pois}(\lambda t)$

$W \sim \text{Exp}(\lambda)$

$\overset{\approx f(t)dt}{P(T_r \in dt)}$ means



$P(0 < W < 0 + dt) \approx \lambda dt$

so $P(T_r \in dt) = P(N_t = r-1)\, P(W \in dt \mid W > t)$

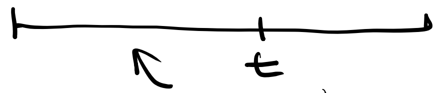
$$\approx \frac{e^{-\lambda t} (\lambda t)^{r-1}}{(r-1)!} \cdot \lambda dt$$

$$= \underbrace{\frac{1}{(r-1)!} \lambda^r t^{r-1} e^{-\lambda t}}_{f(t)} \, dt$$

Now, $T_r \sim \text{Gamma}(r, \lambda)$ for $r \in \mathbb{Z}^+$ has density

$$f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}, & t \geq 0 \\ 0, & \text{else} \end{cases}$$

$N_t \sim \text{Pois}(\lambda t)$



$\le r-1$ arrivals here.

$$P(T_r > t) = P(N_t < r)$$

$$= \sum_{i=0}^{r-1} P(N_t = i)$$

$$= \boxed{\sum_{i=0}^{r-1} \frac{e^{-\lambda t}(\lambda t)^i}{i!}}$$

Ex let $T_4 \sim \text{Gamma}(r=4, \lambda=2)$

Find $P(T_4 > 7)$

$$P(T_4 > 7) = P(N_7 < 4) = P(N_7 \le 3)$$

$$N_7 \sim \text{Pois}(\underbrace{2 \cdot 7}_{=14})$$

$$= e^{-14}\left(1 + \frac{14}{1!} + \frac{14^2}{2!} + \frac{14^3}{3!}\right)$$

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Stat 134

Monday October 10 2022

1. Suppose customers arrive at a ticket booth at a rate of λ five per minute, according to a Poisson arrival process. Find the probability that starting from time 0, the $9th$ customer doesn't arrive within 5 minutes:

k

t

a $\sum_{k=0}^{9} \frac{e^{-25}25^k}{k!}$

b $\sum_{k=0}^{8} \frac{e^{-25}25^k}{k!}$

c $\sum_{k=0}^{8} \frac{e^{-5}5^k}{k!}$

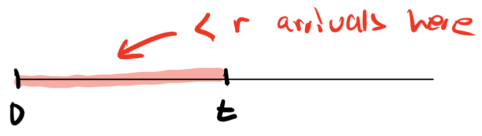
d none of the above

$P(T_9 > 5) = P(N_5 < 9)$

$N_5 \sim Pois(\underbrace{\lambda t}_{25})$

$\sum_{k=0}^{8} \frac{e^{-25} 25^k}{k!}$

$P(T_r > t) = P(N_t < r)$ where $N_t \sim Pois(\lambda t)$



Gamma function $\Gamma(r)$, $r>0$

If $r \in \mathbb{Z}^+$ define $\Gamma(r) = (r-1)!$ where $\Gamma(r)$ is called the gamma function

For $\lambda = 1$, $f(t) = \underbrace{\frac{1}{\Gamma(r)}}_{\text{const}} \underbrace{t^{r-1} e^{-t}}_{\text{variable}}$

and $\int_0^\infty f(t)dt = 1 \Rightarrow$

$$\frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-t} dt = 1$$

$$\Rightarrow \Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$$

The formula $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t}\, dt$.

is the definition of the gamma function.

You can show that $\Gamma(r) = (r-1)!$

for $r \in \mathbb{Z}^+$

Now, $T_r \sim \text{Gamma}(r, \lambda)$ has density

$$f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} & , t \geq 0 \\ 0 & , \text{else} \end{cases}$$

ex Let $T_r \sim \text{Gamma}(r=4, \lambda=2)$

Find $f(t)$

$$f(t) = \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}$$

$$= \frac{1}{3!} 2^4 t^3 e^{-2t} = \boxed{\frac{8}{3} t^3 e^{-2t}}$$

Appendix

Expectation and Variance of exponential

let $T \sim \text{Exp}(\lambda)$

$f(t) = \lambda e^{-\lambda t}$ density

$$E(T) = \lambda \int_0^\infty t e^{-\lambda t} dt$$

I recommend using the <u>tabular method</u> for integration by parts. This works well when the function you are integrating is the product of two expressions, where the n^{th} derivative of one expression is zero.

ex $t^4 e^{3t}$ ✓ good

ex $\sin t \, e^{3t}$ ✗ bad.

To find $\lambda \int_0^\infty t e^{-3t}$:

$\frac{d}{dt}$		$\int$
t	$+$	$e^{-\lambda t}$
1	$-$	$-\frac{e^{-\lambda t}}{\lambda}$
0		$\frac{e^{-\lambda t}}{\lambda^2}$

$$E(T) = \lambda \int_0^\infty t e^{-\lambda t}\,dt = \lambda\left(t\left(-\frac{e^{-\lambda t}}{\lambda}\right) - 1\cdot\frac{e^{-\lambda t}}{\lambda^2}\right)\Bigg|_0^\infty$$

$$= 0 - \left(0 - \frac{1}{\lambda}\right)$$

$$= \boxed{\frac{1}{\lambda}}$$

Next find $Var(T) = E(T^2) - E(T)^2$:

$$E(T^2) = \lambda \int_0^\infty t^2 e^{-\lambda t}\,dt$$

$\frac{d}{dt}$		$\int$
t^2	$+$	$e^{-\lambda t}$
$2t$	$-$	$-\frac{1}{\lambda}e^{-\lambda t}$
2	$+$	$\frac{1}{\lambda^2}e^{-\lambda t}$
0		$-\frac{1}{\lambda^3}e^{-\lambda t}$

$$E(T^2) = \lambda\left(t^2\left(-\frac{1}{\lambda}e^{-\lambda t}\right) - 2t\left(\frac{1}{\lambda^2}e^{-\lambda t}\right) + 2\left(-\frac{1}{\lambda^3}e^{-\lambda t}\right)\right)\Bigg|_0^\infty$$

$$= \frac{2}{\lambda^2}$$

$$\Rightarrow Var(T) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \boxed{\frac{1}{\lambda^2}}$$