Stat 134 lec 29

sec 5.2

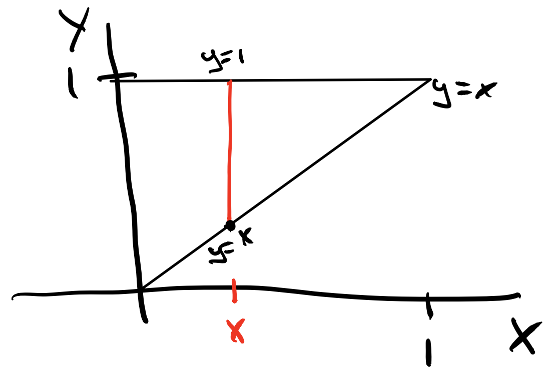
marginal density

$$f(x,y) = \begin{cases} 30(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$

$Y = U_{(6)}$

$$\int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty} f(x,y)\,dy\,dx = 1$$



note
$f(x,y)$ is zero outside of $0<x<y<1$
so this is also

$$\int_{x=0}^{x=1}\int_{y=x}^{y=1} f(x,y)\,dy\,dx = 1$$

$$f_X(x) = \int_{y=-\infty}^{y=\infty} f(x,y)\,dy$$

marginal density of X

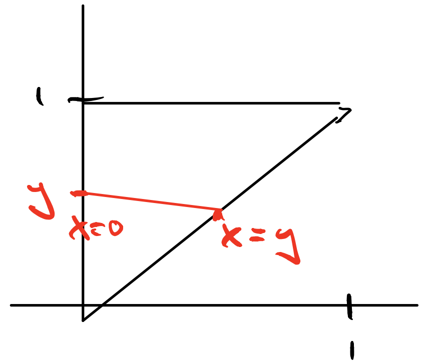
$$= \int_{y=x}^{y=1} 30(y-x)^4\,dy$$

$u = y-x$

$du = dy$

$$= \int_{u=0}^{u=1-x} 30u^4\,du = \left.\frac{30u^5}{5}\right|_0^{1-x} = \boxed{6(1-x)^5, \quad 0<x<1}$$

$$f_Y(y) = \int_{x=-\infty}^{x=\infty} 30\,(y-x)^4\,dx$$



$$= \int_{x=0}^{x=y} 30\,(y-x)^4\,dt$$

$u = y-x$
$du = -dx$

$$= -\int_{u=y}^{0} 30\,u^4\,du = \frac{30u^5}{5}\Big|_0^y = \boxed{6y^5,\ 0<y<1}$$

$\Rightarrow X = U_{(1)},\ Y = U_{(6)}$ aren't independent.

<u>Expectation</u>

$$E(g(X,Y)) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} g(x,y)\,f(x,y)\,dx\,dy.$$

Find

$$E(Y) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} y\,f(x,y)\,dx\,dy$$

$$= \int_{y=-\infty}^{y=\infty} y \left(\int_{x=-\infty}^{x=\infty} f(x,y)\,dx \right) dy$$

$$\left(\int_{x=-\infty}^{x=\infty} f(x,y)\,dx \right) = f_Y(y)$$

$$= \int_{-\infty}^{\infty} y f_Y(y) = \int_{-\infty}^{\infty} y \cdot 6y^5\,dy$$

$$= \frac{6y^7}{7}\Big|_0^1 = 6/7$$

<u>Note</u> $Y \sim \text{Beta}(6,1)$

$$E(Y) = \frac{6}{6+1} = 6/7.$$

Stat 134

Monday April 2 2018

1. Let $T \sim expon(1/2)$ and $R = \sqrt{T}$. The density of R is:

   a $f(r) = e^{-\frac{1}{2}r^2}, r > 0$

   b $f(r) = \frac{1}{4r}e^{-\frac{1}{2}r^2}, r > 0$

   c $f(r) = re^{-\frac{1}{2}r^2}, -\infty < r < \infty$

   d none of the above

$T \sim expon(1/2)$

$f_T(t) = \frac{1}{2}e^{-\frac{1}{2}t}$

$R = \sqrt{T}$ — $r > 0$

$$f_R(r) = \left|\frac{1}{\frac{1}{2\sqrt{t}}}\right| \cdot f_T(t) \qquad t = r^2$$

change of variable rule

$$= \cancel{2}r \cdot \frac{1}{\cancel{2}}e^{-\frac{1}{2}r^2} = \boxed{re^{-\frac{1}{2}r^2}, r > 0}$$

Rayleigh Dist.

sec 5.3 Independent Normal variables.

In chapter 4 we introduced the standard normal RV but didnt prove much about it.

Properties of std normal	Proved it
$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$	no
$E(Z) = 0$	no
$SD(Z) = 1$	no

Even though Z is symmetric around zero it is possible $E(x)$ is undefined

ex Cauchy distribution

Let $X, Y \overset{iid}{\sim} N(0,1)$ w/ density $c e^{-\frac{1}{2}z^2}$ $-\infty < z < \infty$

joint dist:

$$f(x,y) = f_X(x) f_Y(y)$$

$$= c e^{-\frac{1}{2}x^2} \cdot c e^{-\frac{1}{2}y^2}$$

$$= c^2 e^{-\frac{1}{2}(x^2+y^2)} \qquad x^2+y^2 = r^2$$

constant on circle radius r.

Compare this with the Rayleigh distribution:

$$R=\sqrt{T}\ , \quad T \sim \text{Expon}\left(\tfrac{1}{2}\right)$$

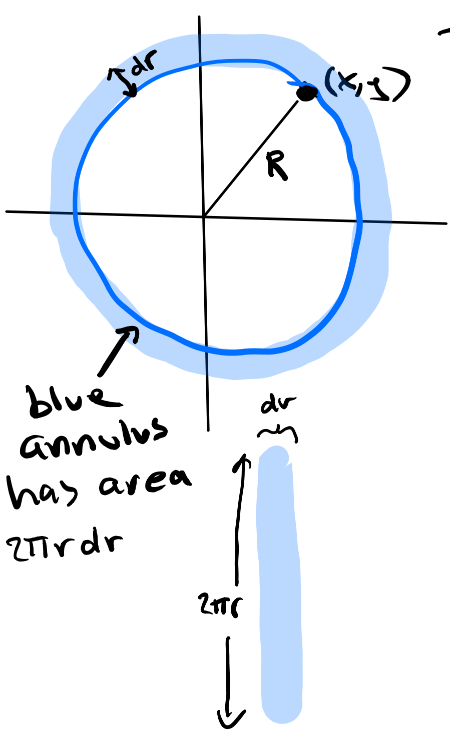
$$f_R(r) = re^{-\frac{1}{2}r^2}\ , \quad r>0$$

it looks very similar.

let $R=\sqrt{X^2+Y^2}$ be the distance from (x,y) to the origin in the plane.

Lets find $f_R(r)$ with this definition of R:

$P(R\in dr)$ is the volume of the cylinder under $f(x,y)=c^2 e^{-\frac{1}{2}(x^2+y^2)}$ over the shaded blue annulus.



$P(R\in dr)$ = height of $f(x,y)$ above blue annulus · area of blue annulus

$\parallel$

$f_R(r)\,dr$

$$= c^2 e^{-\frac{1}{2}r^2} \cdot (2\pi r\,dr)$$

$$\Rightarrow \quad f_R(r)\,dr = c^2 2\pi\, r e^{-\frac{1}{2}r^2}\,dr$$

Rayleigh Density

$\Rightarrow$ $c^2 2\pi$ must be 1 $\Rightarrow c = \sqrt{\frac{1}{2\pi}}$

Two conclusions:

1. $X^2 + Y^2 \sim \text{expon}\left(\frac{1}{2}\right)$ since density of $f_{\sqrt{T}}(r)$ and $f_{\sqrt{X^2+Y^2}}(r)$ is the same.

2. $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ is density of the standard normal.
(i.e $c = \frac{1}{\sqrt{2\pi}}$)

Next Let $X \sim N(0,1)$

Show $E(X) = 0$

$\phi(x)$ is symmetric about zero so all we have to show is that

$E(X)$ converges absolutely.

(i.e $E(|X|) < \infty$).

$$E(|X|) = \int_{-\infty}^{\infty} |x| \phi(x) dx$$

$$= 2 \int_0^{\infty} x \phi(x) dx$$

$$= 2 \int_0^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}} \underbrace{\int_0^{\infty} x e^{-\frac{x^2}{2}} dx}_{= 1}$$

Rayleigh density

$$= \sqrt{\frac{2}{\pi}} < \infty .$$

$\Rightarrow E(X) = 0$ since $\phi(x)$ is semetric around zero

Next we show $SD(X) = 1$:

We know $X^2 + Y^2 \sim \text{expon}\left(\frac{1}{2}\right)$ from above

$\frac{1}{2} = \lambda$

so $E(X^2+Y^2) = \frac{1}{\frac{1}{2}} = 2$

λ rate of X^2+Y^2

$$\Rightarrow E(X^2) + \underbrace{E(Y^2)}_{= E(X^2)} = 2$$

$$\Rightarrow 2E(X^2) = 2 \Rightarrow E(X^2) = 1$$

but $SD(X) = \sqrt{E(X^2) - E(X)^2}$

$= \sqrt{1 - 0} = 1$ ✓