Stat 134 lec 11

Last time sec 3.2 Expectation

$$E(X) = \sum_{x \in X} x P(X=x)$$

If X is a count, X can be written as a sum of indicators

$$X = I_1 + I_2 + \cdots + I_n, \quad I_j = \begin{cases} 1 & \text{prob } p \\ 0 & \text{prob } 1-p \end{cases} \quad 1 \le j \le n$$

$$E(I_j) = 1 \cdot p + 0 \cdot (1-p) = p.$$

Idea Even if indicators are dependent the expectation of each indicator is an unconditional probability.

Then $E(X) = n \cdot p$

we proved if $X \sim Bin(n,p) \Rightarrow E(X) = np$

if $X \sim Hyper(N, G, n) \Rightarrow E(X) = n\frac{G}{N}$

Today sec 3.2

1. review student concept test responses
2. More expectation with indicator examples
3. tail sum formula
4. Markov's inequality.

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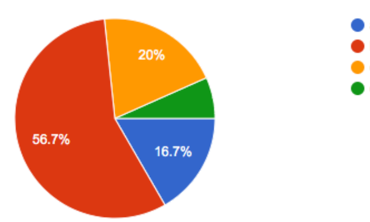
1. A forgetful valet is attempting to return n cars to their n rightful owners. For each driver, the valet remembers the car correctly 5% of the time; otherwise the valet retrieves a car at random (possibly the correct car). What is the probability that the second driver retrieves his own car.

a .05

b $.05 + .95/n$

c $.05 + .95/(n-1)$

d none of the above



c	There are n-1 cars left so the valet will either remember with a .05 probability or get him a random car out of n-1

b	P(he remembers) + P(he's randomly right) = .05 +.95(1/n) I think it's n instead of n-1 since it's like drawing from a deck where the prob of the second card being a specific card is the same as the first.

b	Consider the limiting case n=2. The probability that he gets his car should be slightly more than fifty percent, which is given by b.

simplify question :

A forgetful valet is attempting to return n cars to their n rightful owners. For each driver

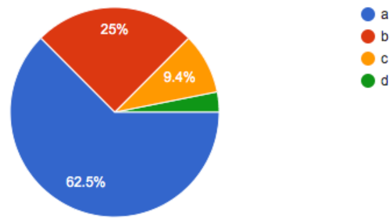
the valet retrieves a car at random (possibly the correct car). What is the probability that the second driver retrieves his own car.

C = 2nd driver gets his car
D = 1st driver gets second driver's car.

$$P(C) = P(C,D) + P(C, \overset{\text{not}}{\sim D})$$

$$= \underset{0}{P(C|D)}P(D) + \underset{\frac{1}{\cancel{n-1}}}{P(C|\sim D)}\,\underset{\frac{\cancel{n-1}}{n}}{P(\sim D)} = 1/n$$

A forgetful valet is attempting to return n cars to their n rightful owners. For each driver, the valet remembers the car correctly 5% of the time; otherwise the valet retrieves a car at random (possibly the correct car). Let N be the number of drivers who retrieve their own car. $E(N)$ is:



a $.05n + .95$

b $.05n$

c $.05n + 1$

d none of the above

a	Probability from part one times n

a	If I2 is the indicator the second driver gets their car, and there are n drivers (each with an identical indicator), then E(N) = n*E(I2) = n(0.05*0.95/n) = 0.05n + 0.95

(2) Sec 3.2 more expectation / indicator examples

ex Consider a 5 card deck consisting of 2, 2, 3, 4, 5.
Let X = number of cards before the first 2.

a) What are the range of values of X?

0, 1, 2, 3

b) Write X as a sum of indicators

$X = I_3 + I_4 + I_5$ (3, 4, 5 are the non 2 cards)

c) How is an indicator defined.

$$I_4 = \begin{cases} 1 & \text{if } 4 \text{ is before } 1^{st}\ 2 \\ 0 & \text{else.} \end{cases}$$

d) Find $E(I_4)$

$\frac{1}{3}$

The calculation of $E(I_4)$ only involves 3 cards 4, 2, 2. Take out all other cards. Now you have a 3 card deck.
Picture __ 2 __ 2 __ where each slot can be empty or have a 4 in it. It is equally likely you have a 4 in any of the 3 slots, hence $E(I_4) = 1/3$.

e) Find $E(X)$

$3 \cdot \frac{1}{3} = 1$

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Stat 134

Chapter 3 Friday February 15 2019

1. Consider a well shuffled deck of cards. The expected number of cards before the first ace is?

   a 52/5

   b 48/5

   c 48/4

   d none of the above

X = # cards before the 1st ace

0, 1, 2, …, 48

$4 \times 12 = 48$ nonaces

1 2 3 4 5 … 10 11 12 13 14 15 … 47 48

$\underbrace{2\ 3\ 4\ 5\ 6 \cdots J\ Q\ K}_{\text{hearts}}\ \underbrace{2\ 3\ 4 \cdots}_{\text{spades}}\ \underbrace{\quad}_{\text{clubs}}\ \underbrace{\cdots Q\ K}_{\text{diamonds}}$

$X = I_1 + I_2 + \cdots + I_{48}$

$I_j = \begin{cases} 1 & \text{if } j^{th} \text{ nonace before } 1^{st} \text{ ace} \\ 0 & \text{else} \end{cases}$ P

$\_\_\ A_1\ \_\_\ A_2\ \_\_\ A_3\ \_\_\ A_4\ \_\_$

$P = 1/5 \Rightarrow \boxed{E(X) = 48\left(\frac{1}{5}\right)}$

2. Consider a well shuffled deck of cards. The expected number of cards between the first and second ace (not counting either ace)?

a 52/5

b 48/5

c 48/4

d none of the above

X = # cards between 1^{st} and second ace

$X = I_1 + \dots + I_{48}$

$p = 1/5$

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ nonace is between } 1^{st} \text{ and } 2^{nd} \text{ ace} \\ 0 & \end{cases}$

Same as 1^{st} part of question

③ Sec 3.2 Tail Sum formula for expectation

Suppose X is a count $0, 1, 2, 3, \ldots$

$$E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) + \cdots$$

$$= 1 \cdot \underbrace{P(X=1)}_{p_1} + 2 \cdot \underbrace{P(X=2)}_{p_2} + \cdots$$

$$= \begin{array}{cccc} P(X\geq 1) & P(X \geq 2) & P(X \geq 3) & \\ p_1 & & & \\ p_2 & p_2 & & \\ p_3 & p_3 & p_3 & \\ p_4 & p_4 & p_4 & p_4 \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

$$= P(X \geq 1) + P(X \geq 2) + P(X \geq 3) + \cdots \quad \text{Tail Sum Formula}$$

This is useful when it is easy to find $P(X \geq k)$.

ex A fair die is rolled 10 times.
Let $X = \max(X_1, \ldots, X_{10})$.
Find $P(X \geq k)$

$$P(X \geq k) = 1 - P(X < k)$$
$$= 1 - P(X_1 < k, X_2 < k, \ldots, X_{10} < k)$$
$$= 1 - P(X_1 < k) P(X_2 < k) \cdots P(X_{10} < k)$$
$$= 1 - P(X_1 < k)^{10}$$
$$= 1 - \left(\frac{k-1}{6}\right)^{10}$$

so
$$E(X) = \underset{1}{\underset{\parallel}{P(X \geq 1)}} + \underset{1-\left(\frac{1}{6}\right)^{10}}{\underset{\parallel}{P(X \geq 2)}} + \cdots + \underset{1-\left(\frac{5}{6}\right)^{10}}{\underset{\parallel}{P(X \geq 6)}} + \underset{0}{\underset{\parallel}{P(X \geq 7)}} + \cdots$$

$$= 6 - \left(\frac{1}{6}\right)^{10} \left[1^{10} + 2^{10} + 3^{10} + 4^{10} + 5^{10}\right] = 5.82$$

ex A fair die is rolled 3 times, X_1, X_2, X_3.

Let Y be the sum of the largest 2 numbers.

Notice that $Y = X_1 + X_2 + X_3 - \min(X_1, X_2, X_3)$.

a) Find $P(\min(X_1, X_2, X_3) \geq 2)$

Picture

$$= P(X_1 \geq 2, X_2 \geq 2, X_3 \geq 2)$$

$$= P(X_1 \geq 2)^3 = \left(\frac{5}{6}\right)^3$$

(b) Find $E(\min(X_1, X_2, X_3))$

$$= P(\min \geq 1) + P(\min \geq 2) + \cdots + P(\min \geq 6)$$

$$= P(X_1 \geq 1)^3 + P(X_1 \geq 2)^3 + \cdots + P(X_1 \geq 6)^3$$

$$= \left(\frac{6}{6}\right)^3 + \left(\frac{5}{6}\right)^3 + \left(\frac{4}{6}\right)^3 + \cdots + \left(\frac{1}{6}\right)^3 = \frac{1}{6^3}\left[6^3+5^3+4^3+3^3+2^3+1^3\right]$$

(c) Find $E(Y) = E(X_1) + E(X_2) + E(X_3) - E(\min(X_1, X_2, X_3))$

$$E(X_1) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = \frac{1}{6}\left[1+2+3+4+5+6\right]$$

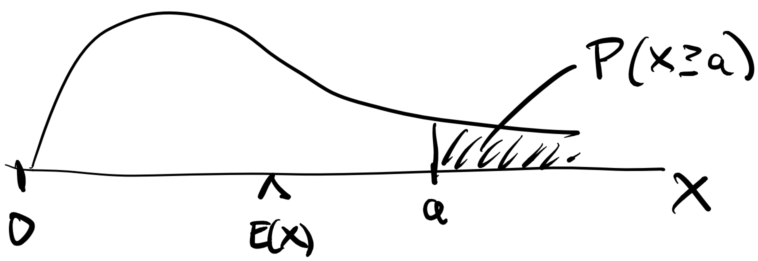
$$= \frac{21}{6} = \boxed{\frac{7}{2}}$$

$$\Rightarrow E(Y) = \boxed{3\left(\frac{7}{2}\right) + \frac{1}{6^3}\left[6^3+5^3+4^3+3^3+2^3+1^3\right]}$$

4 Sec 3.2 Markov's Inequality

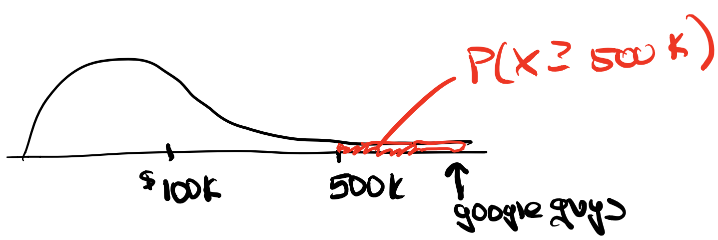
Let $X \ge 0$, $a > 0$

$E(X)$ is the center of the distribution



$$\boxed{P(X \ge a) \le \frac{E(X)}{a}}$$

ex Let X be the yearly income of Bay area residents. $E(X) = \$100K$. Find an upper bound for $P(X \ge 500K)$



By Markov ineq, $P(X \ge 500K) \le \frac{100}{500} = \frac{1}{5}$

Proof of Markov's inequality:

Property of expectation:

if $X \leq Y$ then $E(X) \leq E(Y)$

Markov's inequality

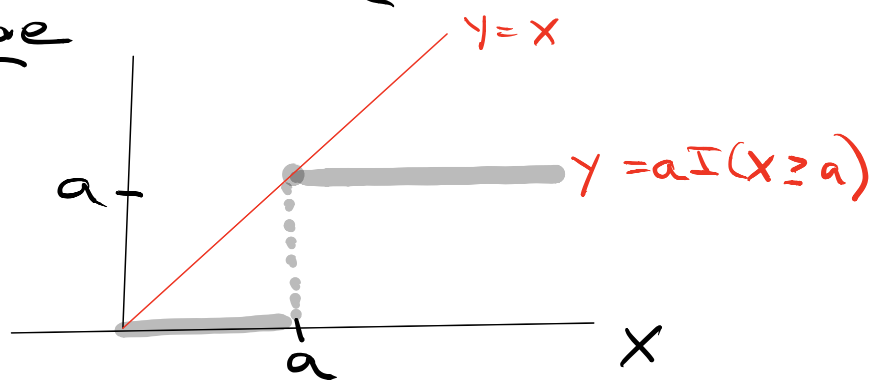
If $X \geq 0$, then $P(X \geq a) \leq \frac{E(X)}{a}$ for every $a > 0$.

Pf/

Let $I(X \geq a)$ be an indicator equal to 1 when $X \geq a$. Then

$$aI(X \geq a) = \begin{cases} a & \text{if } X \geq a \\ 0 & \text{else.} \end{cases}$$

Picture



$$aI(X \geq a) \leq X \Rightarrow E(aI(X \geq a)) \leq E(X)$$

$$E(aI(X \geq a)) = aE(I(X \geq a)) = aP(X \geq a)$$

$$\Rightarrow P(X \geq a) \leq \frac{E(X)}{a}$$

$\square$