Stat 134: Section 23

Brett Kolesnik

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Conceptual Review

a. Given the joint density $f_{X,Y}$ of (X, Y) how to compute the density of X?

b. Given the joint density $f_{X,Y}$ of (X, Y) and marginal densities f_X, f_Y and a point y such that $f_Y(y) > 0$ what's the expression for the conditional distribution of X given $Y = y$ (give the density $f_X(x|Y = y)$ as a function of x).

c. Give a relation between $f_{X,Y}(x,y)$, $f_Y(y)$ and $f_X(x|Y = y)$.

a) $f_X(x) = \int_y f_{X,Y}(x,y)\, dy$

$\mathbb{P}(X=x) = \sum_y \mathbb{P}(X=x, Y=y)$

b) $f_X(x|Y=y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

$\mathbb{P}(X=x|Y=y) = \dfrac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)}$

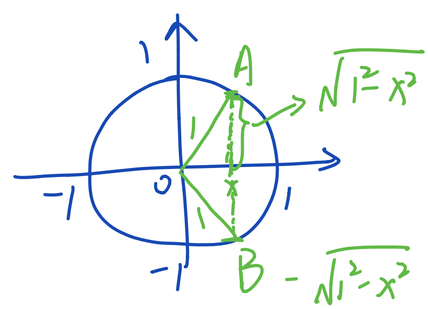
c) $f_X(x|Y=y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

$f_{X,Y}(x,y) = f_X(x|Y=y) f_Y(y).$

Problem 1

Suppose that X, Y is distributed uniformly on the disk of center $(0, 0)$ and of radius 1. Find the conditional distribution of Y given $X = 0$ by carefully compting the density $f_Y(y|X = 0)$.

$f_Y(y|X=0) = \dfrac{f_{X,Y}(0,y)}{f_X(0)}$



$f_{X,Y}(x,y) = \dfrac{1}{\pi \cdot 1^2} = \dfrac{1}{\pi}$

$f_X(x) = \int_y f_{X,Y}(x,y)\, dy$

$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy = \dfrac{2\sqrt{1-x^2}}{\pi}$

$f_Y(y|X=0) = \dfrac{\frac{1}{\pi}}{\frac{2\sqrt{1-0^2}}{\pi}}$

$= \dfrac{1}{2}$

Problem 2

Suppose (X, Y) are random variables with joint density

$$f(x,y) = \begin{cases} \lambda^3 x e^{-\lambda y} & \text{for } 0 < x < y \\ 0 & \text{otherwise} \end{cases}$$

a. Find the density f_Y of Y and compute $E[Y]$

b. Find the conditional distribution of X given $Y = 1$.

c. Deduce $E[X|Y = 1]$

Ex 6.3.4 in Pitman's Probability

a) $f_Y(y) = \int_0^y \lambda^3 x e^{-\lambda y}\, dx \quad \frac{1}{\Gamma(3)}$

$= \lambda^3 \frac{1}{2} x^2 e^{-\lambda y} \Big|_0^y = \underbrace{\boxed{\tfrac{1}{2}} \lambda^3 y^2 e^{-\lambda y}}$

$Y \sim \text{Gamma}(3, \lambda) \Rightarrow E(Y) = \frac{3}{\lambda}$

b) $f_X(x|Y=1) = \frac{f_{X,Y}(x,1)}{f_Y(1)}$

$= \frac{\cancel{\lambda^3}\, x\, \cancel{e^{-\lambda}}}{\frac{1}{2}\cancel{\lambda^3}\, 1^2\, \cancel{e^{-\lambda}}} = 2x \quad 0 < x < 1$

Beta$(2,1)$

c) $E(X|Y=1) = \frac{2}{2+1} = \frac{2}{3}.$

Problem 3

Let X, Y be independent random variables, X is uniform on $(0, 3)$ and Y is Poisson(λ) for some $\lambda > 0$. Find

a. Find $P(X < Y)$ in terms of λ

b. Find the conditional density of X given $X < Y$ (and try sketching its graph for $\lambda = 1, 2, 3$)

c. Compute $E[X|X < Y]$

Ex 6.3.13 in Pitman's Probability

$f_X(x) = \frac{1}{3}$ $\quad P(X > x) = \frac{3-x}{3}$

$\perp\!\!\!\perp$ $\quad P(Y = y) = e^{-\lambda} \frac{\lambda^y}{y!}$

$P(X < Y) = 1 - P(X > Y)$

$P(X > Y) = E[P(X > Y)|Y] = \sum_{y=0}^{2} \underbrace{P(X > Y | Y = y)}_{P(X > y)} P(Y = y)$

$= \sum_{y=0}^{2} \frac{3-y}{3} \cdot e^{-\lambda} \frac{\lambda^y}{y!} = e^{-\lambda}\left[1 + \frac{2}{3}\lambda + \frac{1}{6}\lambda^2\right]$

$P(X < Y) = 1 -$

b) $f_X(x | X < Y) = \frac{P(X \in dx | X < Y)}{dx}$

$P(X \in dx | X < Y) = \frac{P(X \in dx, X < Y)}{P(X < Y)} = \frac{P(X \in dx) P(X < Y | X \in dx)}{P(X < Y)}.$

$f_X(x)dx \quad P(Y > x)$ R.V. constant

$= \frac{f_X(x) P(Y > x)}{P(X < Y)} dx \Rightarrow f_X(x | X < Y) = \frac{\overset{\frac{1}{3}}{f_X(x)}\, \boxed{P(Y > x)}\, ?}{P(X < Y) = (a)}$

$f_X(x | X < Y) = \begin{cases} \boxed{\dfrac{\frac{1}{3}\left[1 - e^{-\lambda}\right]}{1 - e^{-\lambda}\left(1 + \frac{2}{3}\lambda + \frac{1}{6}\lambda^2\right)}} & 0 \le X < 1 \end{cases}$

$= P(Y > 1)$

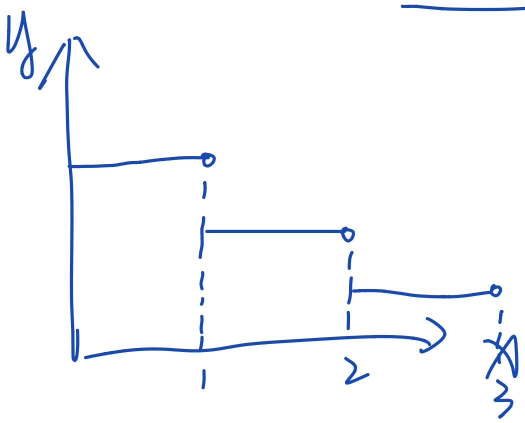
Prepared by Yassine El Maazouz

$$\frac{1}{3}\left[1-e^{-\lambda}-e^{-\lambda}\cdot\lambda\right] \quad 1\le x<2$$

$$''$$

$$\frac{1}{3}\Big[\underbrace{1-e^{-\lambda}-e^{-\lambda}\cdot\lambda-e^{-\lambda}\frac{\lambda^2}{2!}}_{P(Y>x)}\Big] \quad 2\le x<3$$

$$''$$



(c) $E[X \mid X<Y]$

$$= \int_0^3 x\, \underbrace{f_X(x \mid X<Y)}_{''}\, dx$$

$$= \int_0^1 '' \, dx + \int_1^2 '' \, dx + \int_2^3 \, dx$$

$$= \frac{9-e^{-\lambda}\left(9+8\lambda+\frac{5}{2}\lambda^2\right)}{6-e^{-\lambda}\left(6+4\lambda+\lambda^2\right)}$$