Last time  Stat 134  lec 27

sec 4.5 Expectation of a nonnegative RV using CDF

$$E(X) = \int_0^\infty (1-F(x))\,dx$$

ex let $X \sim \text{Geom}\left(\frac{1}{2}\right)$

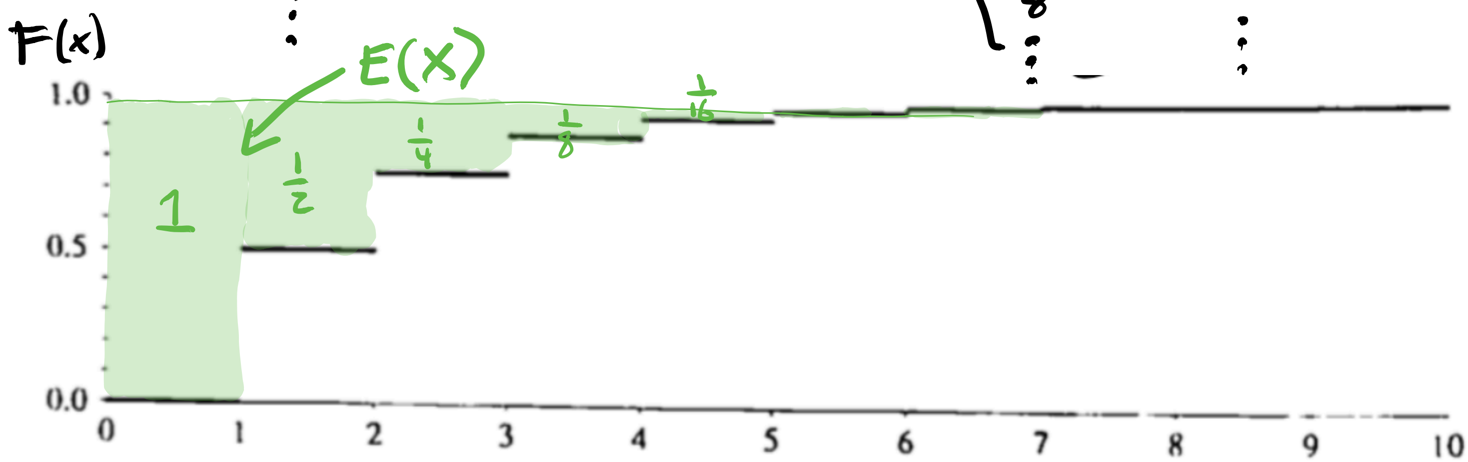
$P(X=1) = 1/2$

$P(X=2) = \frac{1}{2}\cdot\frac{1}{2} = 1/4$

Picture $P(X=3) = (1/2)^2 \cdot \frac{1}{2} = 1/8$

$\vdots$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 1 \\ 1/2 & 1 \le x < 2 \\ \frac{3}{4} & 2 \le x < 3 \\ \frac{7}{8} & 3 \le x < 4 \\ \vdots & \vdots \end{cases}$$




$$E(X) = \int_0^\infty (1-F(x))\,dx$$

$$E(X) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} \leftarrow p = \boxed{2}$$

$$= \sum_{j=0}^{\infty} P(X>j) = \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j \leftarrow \text{tail sum formula.}$$

Today

1. Review student explanations of MGF concept test.
2. sec 4.6 order statistics
3. sec 5.1 continuous joint distributions.

① Review of student explanations of MGF concept test,

**Stat 134**

**Monday October 21 2018**

1. Let $X$ have density $f(x) = xe^{-x}$ for $x > 0$. The MGF is?

   **a** $\mathrm{M_X(t)} = \frac{1}{1-\mathrm{t}}$ for $t < 1$

   **b** $\mathrm{M_X(t)} = \frac{1}{(1-\mathrm{t})^2}$ for $t < 1$

   **c** $\mathrm{M_X(t)} = \frac{1}{(1+\mathrm{t})^2}$ for $t > -1$

   **d** none of the above

Soln

variable part of Gamma $x^{r-1}e^{-\lambda x}$

$\Rightarrow X \sim \text{Gamma}(2,1)$

know MGF of Gamma$(r,\lambda)$ is $\left(\frac{\lambda}{\lambda-t}\right)^r, t<\lambda$

so $\boxed{M_X(t) = \frac{1}{(1-t)^2} \text{ for } t<1}$

| | | | | |
|---|---|---|---|---|
| 10/22/2018 10:25:4 | b | b | b) | |
| 10/22/201 | b | b | This is the probability density function for Gamma(r=2,lambda=1). MGF of gamma distributions is (lambda/lambda - t)^r when t<lambda. Plug in r=2 and lambda=1 to get answer b | |
| 10/22/2018 12:15:5 | b | b | through integration by parts, the constant outside of the integral is 1/(t-1)^2 | |
| 10/22/201 | b | b | Integrate by part | |
| 10/22/2018 13:41:2 | b | b | | |
| 10/22/2018 13:41:5 | b | b | cX~gamma(2,1) | |
| 10/22/2018 14:13:5 | a | b | | |
| 10/22/2018 14:41:4 | b | b | | |
| 10/22/2018 15:08:5 | b | b | | |
| 10/22/2018 15:17:0 | b | b | | |
| 10/22/2018 16:52:2 | b | b | This is the density of a gamma distribution with r=2 and lambda=1. The formula for the MGF of a gamma distributed RV is (lambda/lambda - t) ^ r for values of t that are less than lambda. Plugging in the values you get b. | |
| 10/22/2018 16:55:3 | b | b | calculate the integral 0—∞ : xe^ (t-1) x of course t<1 | |
| $M_X(t) = E(e^{tX})$ $= \int_0^\infty e^{tx} x e^{-x} dx$ $= \int_0^\infty x e^{x(t-1)} dx$ $= \frac{1}{1-t}\int_0^\infty x(1-t)e^{-(1-t)x} dx$ ($\lambda$) $= \frac{1}{1-t}\int_0^\infty x\lambda e^{-\lambda x}$ expectation of Exp(1-t) $= \frac{1}{1-t}\cdot\frac{1}{1-t}$ <br>10/23/201 | a | b | Computing the moment generating function: M_{X}(t) = E(e^{tX}) = \int_{\mathbb{R}^+} e^{tX} \cdot xe^{-x}dx = \int_{\mathbb{R}^+} xe^{x(t-1)}dx. Rewriting the integral, we produce the following: \frac{1}{1-t} \int_{\mathbb{R}^+} x(1-t)e^{-(1-t)x}dx. Looking closely at the above expression, we can clearly see that it is the expectation of an Exponential Distribution with \lambda = 1-t (or equivalently Gamma Distribution with r = 1 and \lambda = 1-t ), multiplied by a factor \frac{1}{1-t}. Using this knowledge, we can conclude that M_{X}(t) = (\frac{1}{1-t})^2 with t < 1. The constraint of t < 1 is enforced to preserve convergence. | |
| 10/22/201 | b | b | similar to the lecture example. converges for t<1, use U substitution...get Mx (t)=1/(1-t)^2 (t<1) | |
| | | | The density is the variable part of the gamma density, where r = | |

2. The $MGF$ of $X$ is $M_X(t) = \frac{1}{\sqrt{1-t}}$ for $t < 1$. The distribution of $X$ is:

**a** Gamma(r $= 1/2, \lambda = 1$) and possibly another distribution.

**b** Gamma(r $= 2, \lambda = 1$)

**c** Gamma(r $= -1/2, \lambda = 1$)

**d** none of the above

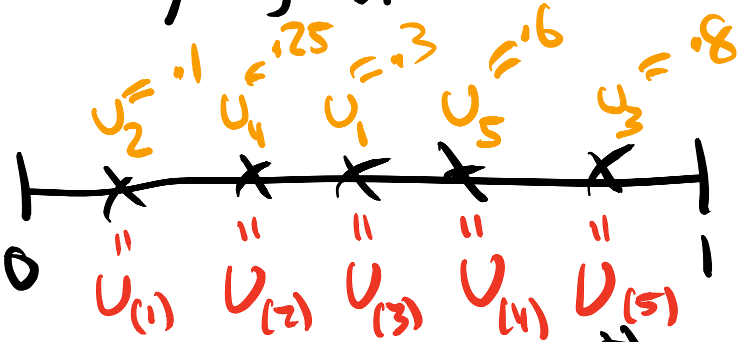
soln The MGF uniquely determines the distribution.

$M_X(t)$ for Gamma$(r, \lambda)$ is $\left(\frac{\lambda}{\lambda - t}\right)^r$ for $t < \lambda$

$$\Rightarrow \frac{1}{\sqrt{1-t}} = \left(\frac{1}{1-t}\right)^{1/2} \Rightarrow \boxed{X \sim \text{Gamma}(r = 1/2, \lambda = 1)}$$

② Sec 4.6 Order statistics of Unif(0,1)

let $U_1, \dots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$




throw 5 darts at [0, 1].

let $U_{(k)}$ = called the $k^{th}$ order statistic

$= k^{th}$ largest value of $U_1, \dots, U_n$

(assuming no ties)

ex

$U_{(1)} = \min(U_1, \dots, U_n)$

$U_{(n)} = \max(U_1, \dots, U_n)$

Review counting

You have 3 red, 2 green and 5 blue marbles,
How many orderings of these 10 marbles are there?

ex r r r g g b b b b b
g r r r g b b b b b
g g r r r b b b b b
⋮

Answer

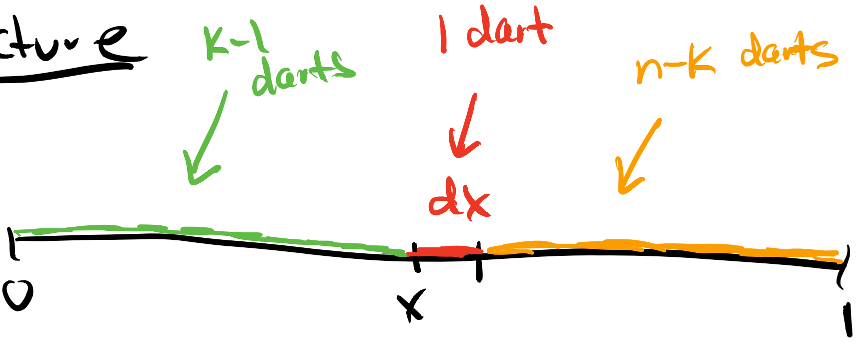
$$\binom{10}{3,2,5} = \binom{10}{3}\binom{7}{2}\binom{5}{5}$$

$$= \frac{10!}{3!\,2!\,5!}$$

Next, find density of $U_{(k)}$

write $P(U_{(k)} \in dx) = f(x)dx$

Picture




$U_{(k)} \in dx$ means that k-1 of

that k-1 **darts** are between 0 and $x$,

and one is in $dx$, and n-k **darts** are between $x$ and 1

$$P(U_{(k)} \in dx) = P\left(k\text{-}1 \text{ darts} \in (0,x),\ 1 \text{ dart} \in dx,\ n\text{-}k \text{ darts} \in (x,1)\right)$$

$$= P\left(k\text{-}1 \text{ darts} \in (0,x)\right) \cdot P\left(1 \text{ dart} \in dx \,\middle|\, k\text{-}1 \text{ darts} \in (0,x)\right)$$

$$\cdot P\left(n\text{-}k \text{ darts} \in (x,1) \,\middle|\, 1 \text{ dart} \in dx,\ k\text{-}1 \text{ darts} \in (0,x)\right)$$

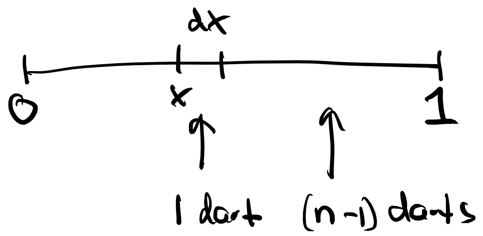
$$= \binom{n}{k-1} x^{k-1} \binom{n-k+1}{1} 1dx \binom{n-k}{n-k} (1-x)^{n-k}$$

$$= \underbrace{\binom{n}{k-1,\, 1,\, n-k} x^{k-1} (1-x)^{(n-k+1)-1}}_{f_{U_{(k)}}(x)} dx$$

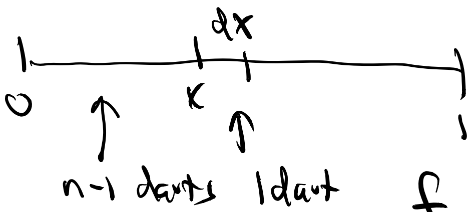
$$\Rightarrow f_{U_{(k)}}(x) = \binom{n}{k-1,\ 1,\ n-k} x^{k-1} (1-x)^{(n-k+1)-1} \quad \text{for } 0<x<1$$

Ex: Let $U_1, \ldots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$

Find the density of $U_{(1)}$ and $U_{(n)}$

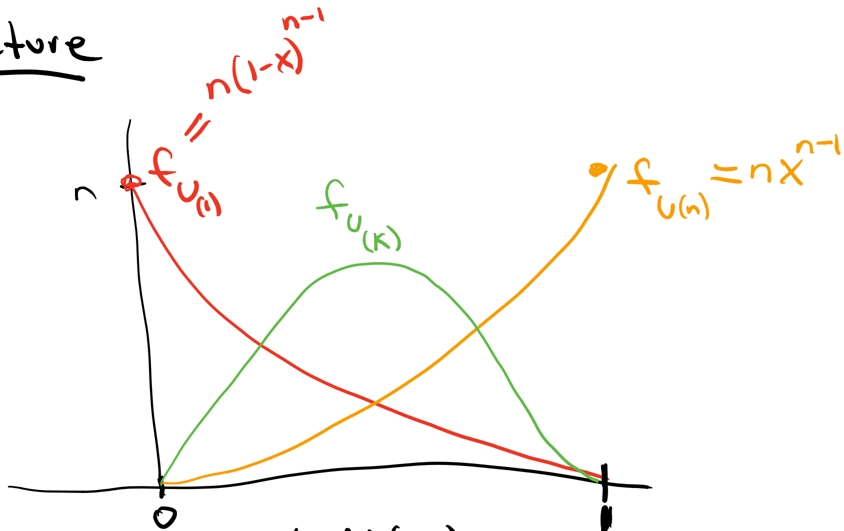



$$f_{U_{(1)}}(x)dx = \binom{n}{1,\ n-1} dx\,(1-x)^{n-1} = n(1-x)^{n-1}\,dx$$



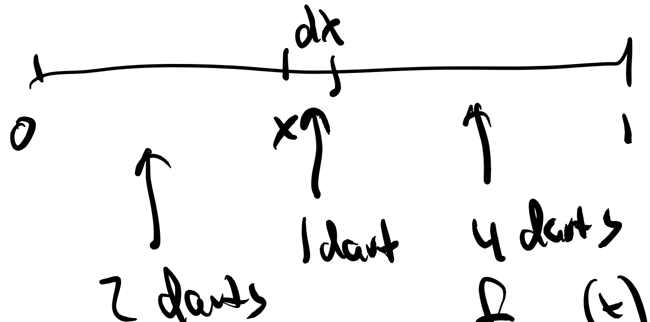

$$f_{U_{(n)}}(x)dx = \binom{n}{n-1,\ 1} x^{n-1}\,dx = n x^{n-1}\,dx$$

**Picture**




Order statistic of $U(0,1)$ provides a family of densities on the unit interval.

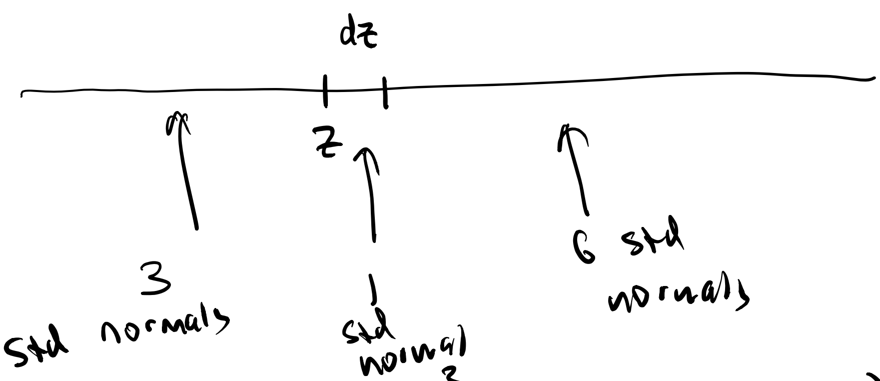
ex $x^2(1-x)^4$ for $0<x<1$ is the variable part of what RV? How many darts do you throw?




$$f_{U_{(3)}}(x) = \binom{7}{2,1,4} x^2 (1-x)^4$$

$P(U_{(3)} \in dx) = f(x)dx$

ex Let $Z_{(1)}, \dots, Z_{(10)}$ be the values of 10 independent standard normal variables arranged in increasing order. Find the density of $Z_{(4)}$




$$f(z)\,dz = \binom{10}{3,1,6} \left(\Phi(z)\right)^3 \phi(z)\,dz \left(1-\Phi(z)\right)^6$$

$$f(z) = \binom{10}{3,1,6} \Phi(z)\, \phi(z) \left(1-\Phi(z)\right)^6$$

# Chap 5 Continuous Joint Dist

## sec 5.1, 5.2

$X, Y$ have **joint density** $f(x,y)$ means $f$ must satisfy

$f(x,y) \geq 0$ ← think of this as a surface over the plane

$$\int_y \int_x f(x,y)\,dx\,dy = \int_x \int_y f(x,y)\,dy\,dx = 1$$

the total volume under the surface is 1

Let $A$ be a subset of the plane

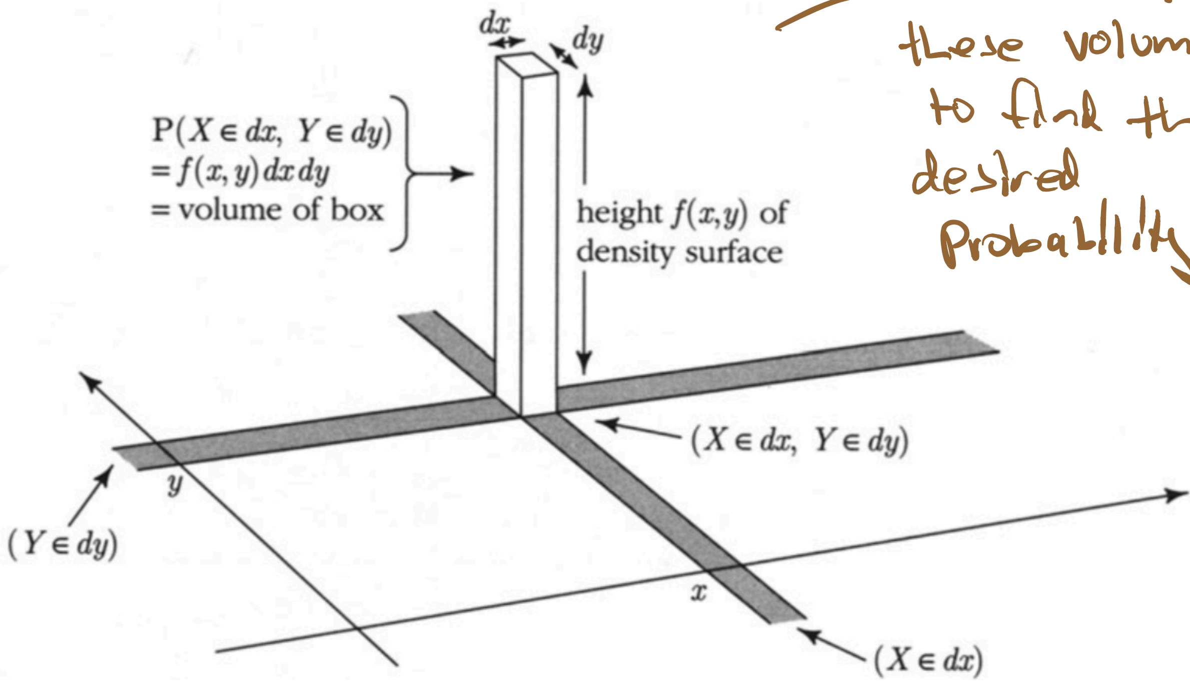
$$P((X,Y) \in A) = \iint_A f(x,y)\,dx\,dy$$

the volume of the surface over region $A$. this is a number between 0 and 1.

$$P(X \in dx, Y \in dy) = f(x,y)\,dx\,dy$$

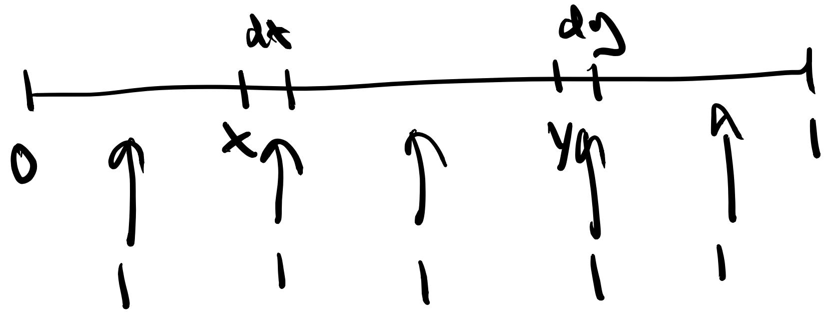
the volume of the surface over a little rectangle in the plane.

# Picture




— we add up the these volumes to find the desired probability.

ex Throw down 5 darts on $(0,1)$.
Find the joint density of
$X = U_{(2)}$ and $Y = U_{(4)}$.




$$f(x,y)\,dx\,dy = \binom{5}{1,1,1,1,1} x\,dx\,(y-x)\,dy\,(1-y)$$

$$= \underbrace{\binom{5}{1,1,1,1,1}}_{=5} x(y-x)(1-y)\,dx\,dy \qquad 0<x<y<1$$

$$\Rightarrow \boxed{f(x,y) = 5x(y-x)(1-y) \quad \text{for } 0<x<y<1}$$