

Solutions

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Problem 1

Let $X_1 \sim \text{Geom}(p_1)$, $X_2 \sim \text{Geom}(p_2)$, $X_1 \perp X_2$, both on $\{1, 2, \dots\}$.
Find:

- a. $P(X_1 \leq X_2)$;
- b. $P(X_1 = x \mid X_1 \leq X_2)$. Recognize $X_1 \mid X_1 \leq X_2$ as a named distribution, and state the parameter(s).

$$\begin{aligned}
 a) P(X_1 \leq X_2) &= \sum_{k=1}^{\infty} P(X_1 = k, X_2 \geq k) \\
 &= \sum_{k=1}^{\infty} q_1^{k-1} p_1 q_2^{k-1} \\
 &= p_1 \sum_{k=1}^{\infty} (q_1 q_2)^{k-1} = \boxed{\frac{p_1}{1-q_1 q_2}}
 \end{aligned}$$

b) We proceed using the conditional prob. rule:

For $x \in \mathbb{N}$,

$$\begin{aligned}
 P(X_1 = x \mid X_1 \leq X_2) &= \frac{P(X_1 = x, X_1 \leq X_2)}{P(X_1 \leq X_2)} \\
 &= \frac{P(X_1 = x) P(X_2 \geq x)}{P(X_1 \leq X_2)} \\
 &= \frac{(q_1^{x-1} p_1)(q_2^{x-1})}{\cancel{(p_1)}} = (q_1 q_2)^{x-1} (1-q_1 q_2)
 \end{aligned}$$

$\therefore, X_1 \mid X_1 \leq X_2 \sim \text{Geom}(1-q_1 q_2)$. (on $\{1, 2, \dots\}$)

Problem 2

Let $Y \sim \text{Beta}(r, s)$. Conditioned on $Y = y$, let $X \sim \text{Geometric}(y)$ on $\{0, 1, 2, \dots\}$. For simplicity, assume $r, s > 1$.

- What is $E(X | Y = y)$?
- Find $E(X)$.
- Find $P(X = x)$, for $x \in \{0, 1, 2, \dots\}$.

$$a) E(X | Y = y) = \frac{1-y}{y} = \frac{1}{y} - 1$$

$$b) E(X) = E(E(X | Y)) \quad (\text{law of iterated expectations}) \\ = E\left(\frac{1}{Y} - 1\right) = E\left(\frac{1}{Y}\right) - 1, \text{ where}$$

$$\begin{aligned} E\left(\frac{1}{Y}\right) &= \int_0^1 \frac{1}{y} \cdot \frac{1}{B(r,s)} y^{r-1} (1-y)^{s-1} dy \\ &= \frac{1}{B(r,s)} \cdot \underbrace{\frac{B(r-1,s)}{B(r-1,s)}}_{=1} \int_0^1 \frac{1}{y} y^{r-2} (1-y)^{s-1} dy \\ &= \frac{B(r-1,s)}{B(r,s)}. \end{aligned}$$

$$\therefore, \boxed{E(X) = \frac{B(r-1,s)}{B(r,s)} - 1}, \text{ where}$$

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$$

c) To find $P(X=x)$, we use the law of total prob.:

$$\begin{aligned}
 P(X=x) &= \int_0^1 P(Y \in dy) P(X=x | Y=y) \\
 &= \int_0^1 f_Y(y) (1-y)^x (y) dy \\
 &= \int_0^1 \frac{1}{B(r,s)} y^{r-1} (1-y)^{s-1} (1-y)^x (y) dy \\
 &= \frac{1}{B(r,s)} B(r-1, s+x) \int_0^1 \frac{1}{B(r-1, s+x)} y^r (1-y)^{s+x-1} dy \\
 &= \boxed{\frac{B(r-1, s+x)}{B(r, s)}}, \text{ where } B(r, s) \text{ is defined as above.}
 \end{aligned}$$

Feel free to simplify this result if you like; for the intrepid, you may find that the result has a special form if r, s are integers. It will resemble the negative hypergeometric distribution, though the underlying reason is unclear.

Problem 3

Suppose a proportion p of a population has a gene m that makes them predisposed to migraines. Of these people, the number of migraines they experience in a year follows a Poisson process with rate μ per year, whereas the rest of the population experiences migraines according to a Poisson process with rate λ .

- What is the probability that a randomly selected individual experiences no migraines in a given year?
- Let N_t denote the number of migraines a randomly selected individual experiences in t years. Find $\mathbb{E}(N_t)$.
- Find $\text{Var}(N_t)$.

Let \mathbb{I}_m be the indicator that an individual has gene m . The process here is to condition on \mathbb{I}_m .

Observe given $\mathbb{I}_m = 1$, $N_t | \mathbb{I}_m = 1 \sim \text{Pois}(\mu t)$,
 & given $\mathbb{I}_m = 0$, $N_t | \mathbb{I}_m = 0 \sim \text{Pois}(\lambda t)$.

$$\begin{aligned} a) \mathbb{P}(N_t = 0) &= \sum_{k=0}^1 \mathbb{P}(\mathbb{I}_m = k) \mathbb{P}(N_t = 0 | \mathbb{I}_m = k) \\ &= p(e^{-\mu \cdot 1}) + q(e^{-\lambda \cdot 1}) = \boxed{pe^{-\mu} + qe^{-\lambda}}. \end{aligned}$$

$$\begin{aligned} b) \mathbb{E}(N_t) &= \mathbb{E}(\mathbb{E}(N_t | \mathbb{I}_m)) \\ &= \sum_{k=0}^1 \mathbb{E}(N_t | \mathbb{I}_m = k) \mathbb{P}(\mathbb{I}_m = k) \\ &= \boxed{p(\mu t) + q(\lambda t)}. \end{aligned}$$

c) We use our rule for variance by conditioning:

$$\text{Var}(N_t) = \mathbb{E}(\text{Var}(N_t | I_m)) + \text{Var}(\mathbb{E}(N_t | I_m))$$

Examine each term individually:

- $\mathbb{E}(\text{Var}(N_t | I_m)) = \sum_{k=0}^1 \text{Var}(N_t | I_m = k) P(I_m = k)$

$$= p(\mu t) + q(\lambda t) \quad \begin{matrix} \text{For } X \sim \text{Pois}(\mu), \\ \text{Var}(X) = \mu. \end{matrix}$$

- $\text{Var}(\mathbb{E}(N_t | I_m))$: $\mathbb{E}(N_t | I_m)$ is a RV that is a function of I_m . It has two poss. values:

$$\mathbb{E}(N_t | I_m) = \begin{cases} \mu t & \text{with prob. } p \\ \lambda t & \text{with prob. } q \end{cases}$$

We use our definition of $\text{Var}(x) = \mathbb{E}((x - \mathbb{E}(x))^2)$:

$$\begin{aligned} \text{Var}(\mathbb{E}(N_t | I_m)) &= \mathbb{E}((\mathbb{E}(N_t | I_m) - \mathbb{E}(\mathbb{E}(N_t | I_m)))^2) \\ &= \mathbb{E}(N_t) \\ &= p(\mu t - \mathbb{E}(N_t))^2 + q(\lambda t - \mathbb{E}(N_t))^2, \\ &\quad \text{where } \mathbb{E}(N_t) \text{ is given in (b).} \end{aligned}$$

$\therefore \boxed{\text{Var}(N_t) = p(\mu t) + q(\lambda t) + p(\mu t - \mathbb{E}(N_t))^2 + q(\lambda t - \mathbb{E}(N_t))^2}$

Problem 4

Let X, Y have joint density $f_{X,Y}(x,y) = 2\lambda^2 e^{-\lambda(x+y)}$, $0 < x < y$. It can be shown that $f_X(x) = 2\lambda e^{-2\lambda x}$, $x > 0$. Find:

- The conditional density of Y , given $X = x$;
- $\mathbb{E}(Y|X = x)$.

$$\begin{aligned} a) f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2\lambda^2 e^{-\lambda x} e^{-\lambda y}}{2\lambda e^{-2\lambda x}} \\ &= \lambda \frac{e^{-\lambda y}}{e^{-\lambda x}}, \quad 0 < x < y. \end{aligned}$$

A good check: does this cond'l. density integrate to 1?
Try for yourself.

$$\begin{aligned} b) \mathbb{E}(Y|X=x) &= \int_x^\infty y f_{Y|X}(y|x) dy \\ &= \int_x^\infty y \lambda \frac{e^{-\lambda y}}{e^{-\lambda x}} dy \quad u\text{-sub: } u=y-x \\ &= \frac{1}{e^{-\lambda x}} \int_x^\infty \lambda y e^{-\lambda y} dy \\ &= \frac{1}{e^{-\lambda x}} \int_0^\infty \lambda(u+x) e^{-\lambda(u+x)} du \\ &= \frac{\cancel{e^{-\lambda x}}}{\cancel{e^{-\lambda x}}} \int_0^\infty \lambda(u+x) e^{-\lambda u} du \\ &= \mathbb{E}(Z+x) \text{ for } Z \sim \text{Exp}(1) \\ &= \boxed{\lambda + x.} \end{aligned}$$