Stat 134 lec 29

warmup 11:00–11:10

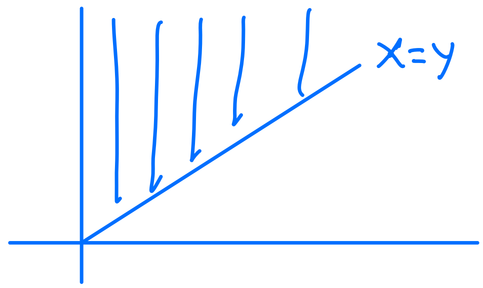
Let $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$

(recall, $f_X(x) = \lambda e^{-\lambda x}$)

be independent lifetimes of two bulbs.

Find $P(X<Y)$.

Hint: use $f(x,y) = f_X(x) f_Y(y)$



$f(x,y) = \lambda e^{-\lambda x} \mu e^{-\mu y}$

If choose $dx\,dy$:

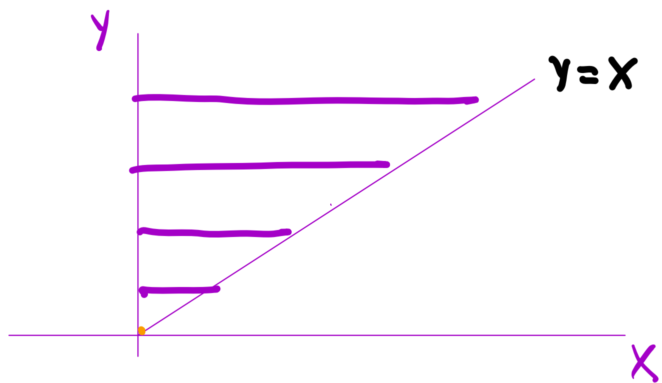
$$P(X<Y) = \lambda \cancel{\mu} \int_{x=0}^{x=\infty} e^{-\lambda x}\, dx \int_{y=x}^{y=\infty} e^{-\mu y}\, dy$$

$$\| \qquad \frac{e^{-\lambda x}}{\mu}$$

easier

$$= \lambda \int_{x=0}^{x=\infty} e^{-(\mu+\lambda)x}\, dx = \boxed{\frac{\lambda}{\mu+\lambda}}$$

If you choose dydx you get



harder

$$P(X<Y) = \cancel{\lambda}\mu \int_{y=0}^{y=\infty} e^{-\mu y} \underbrace{\int_{x=0}^{x=y} e^{-\lambda x}\, dx}\, dy$$

$$-\frac{e^{-\lambda x}}{\lambda}\Bigg|_{x=0}^{x=y} = \frac{1}{\cancel{\lambda}}\left(1-e^{-\lambda y}\right)$$

$$= \mu \int_{y=0}^{y=\infty} \left(e^{-\mu y} - e^{-(\mu+\lambda)y}\right) dy$$

$$= \mu \left[\int_{y=0}^{y=\infty} e^{-\mu y}\, dy - \int_{y=0}^{y=\infty} e^{-(\mu+\lambda)y}\, dy \right] = \mu\left[\frac{1}{\mu} - \frac{1}{\mu+\lambda}\right] = 1 - \frac{\mu}{\mu+\lambda}$$

$$= \boxed{\frac{\lambda}{\mu+\lambda}}$$

Last time.

Sec 4.6 Beta Distribution

Let $r, s > 0$

$P \sim \text{Beta}(r, s)$ if

$$f(p) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} p^{r-1}(1-p)^{s-1} \quad \text{for } 0 < p < 1$$

Applications

a) Beta(r, s) takes values between 0 and 1 and commonly models the prior distribution of a probability in Bayesian statistics.

b) generalization of standard uniform ordered statistic

If throw n darts at [0,1]

$U_{(k)} \sim \text{Beta}(k, n-k+1)$

Today

1. Sec 5.1, 5.2 Independent RVs
2. Sec 5.2 Competing exponentials
3. Sec 5.2 Marginal density

① Sec 5.1, 5.2

Independent RVs

defn X and Y are independent if

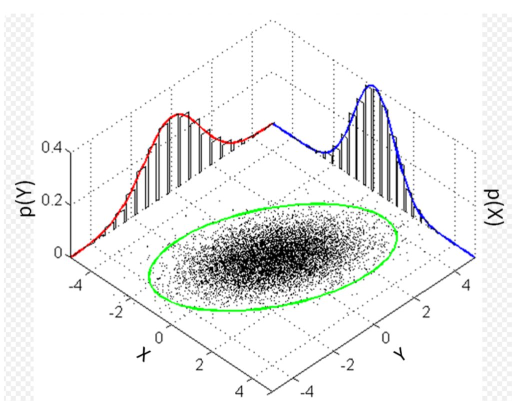
$$\underset{\substack{\parallel \\ f(x,y)\,dx\,dy}}{P(X\in dx, Y\in dy)} = \underset{\substack{\parallel \\ f(x)\,dx}}{P(X\in dx)}\ \underset{\substack{\parallel \\ f(y)\,dy}}{P(Y\in dy)}$$

$$\iff \boxed{f(x,y) = f(x)f(y)}$$

ex $X, Y \overset{iid}{\sim} N(0,1)$

$$f(x,y) = \phi(x)\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$



Not a great picture because the oval in green should be a circle. This is the picture of a correlated bivariate normal from Chapter 6 instead of an uncorrelated bivariate normal.

ex

If $X, Y \overset{iid}{\sim} U(0,1)$

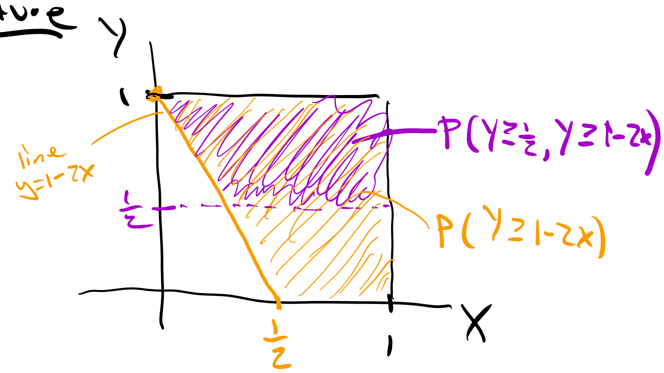
Find $P(Y \geq \frac{1}{2} \mid Y \geq 1-2X)$

Soln

$f(x,y) = \underset{1}{f(x)}\,\underset{1}{f(y)} = 1$ for $0<x,y<1$, 0 else.

$P(Y \geq \frac{1}{2} \mid Y \geq 1-2X) = \dfrac{P(Y \geq \frac{1}{2}, Y \geq 1-2X)}{P(Y \geq 1-2X)}$ Baye's rule

Picture



So,

$$\frac{P(Y \geq \frac{1}{2}, Y \geq 1-2X)}{P(Y \geq 1-2X)} = \frac{\frac{8}{16} - \frac{1}{16} = \frac{7}{16}}{\frac{1}{2} + \frac{1}{4} = \frac{3}{4}} = \boxed{7/12}$$

(2) sec 5.2 Competing exponentials

superposition of Poisson random scatters:

let $N_t \sim \text{Pois}(\mu_1 = \lambda_1 t)$ and $M_t \sim \text{Pois}(\mu_2 = \lambda_2 t)$ be independent PRS corresponding to the number of arrivals of red and purple cars in time t.

Then $N_t + M_t \sim \text{Pois}(\mu = (\lambda_1 + \lambda_2)t)$

What is the chance that the first car is red?

$$\frac{\mu_1}{\mu_1 + \mu_2} = \frac{\lambda_1 t}{\lambda_1 t + \lambda_2 t} = \boxed{\frac{\lambda_1}{\lambda_1 + \lambda_2}}$$

$$\frac{\mu_1}{\mu_1+\mu_2} = \frac{np_1}{np_1 + np_2} = \frac{p_1}{p_1 + p_2}$$

competing exponentials:

Let X = time until the first red car

Y = time until the first purple car

What is the chance the first car is red? $\longrightarrow \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$

by warmup

ex

(Exponential distributions) You are first in line to have your question answered by either of the 2 uGSIs, Brian and Yiming, who are busy with their current students. Your question will be answered as soon as the first uGSI finishes. The times spent with student by Brian and Yiming are independent and exponentially distributed with (positive) rates λ_B and λ_Y respectively, i.e. Brian's distribution is Exponential(λ_B), and Yiming's is Exponential(λ_Y).

(a) Find the probability that Yiming will be the one answering your questions.

$$P(Y < B) = \boxed{\frac{\lambda_y}{\lambda_y + \lambda_B}}$$

(b) What is the distribution of your wait time? Your answer should not include integrals.

$W = \min(Y, B)$ is the shortest wait time,

$$P(W > w) = P(\min(Y, B) > w)$$
$$= P(Y > w, B > w) = P(Y > w)P(B > w)$$
$$= e^{-w\lambda_y} \cdot e^{-w\lambda_B} = e^{-w(\lambda_y + \lambda_B)}$$

$\Rightarrow W \sim Exp(\lambda_y + \lambda_B)$

Think of $\min(Y,B)$ as a superposition of Poisson processes with rate $\lambda_y + \lambda_B$.

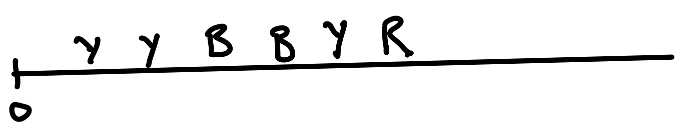
So $P(Y = \min(Y,B)) = P(Y<B) = \dfrac{\lambda_Y}{\lambda_Y + \lambda_B}$

If $X_1, \ldots, X_n$ are indep exponentials with rates $\lambda_1, \ldots, \lambda_n$

$P(X_i = \min(X_1, \ldots, X_n)) = \ ? \quad \dfrac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$

Now have 3 GSI, Yiming, Bilan and Rowan. What is chance Yiming done first, then Bilan and then Rowan (indep exponentials with rates $\lambda_Y, \lambda_B, \lambda_R$)?

i.e. $P(Y<B<R)$



$= P(Y = \min(Y,B,R),\ B = \min(B,R))$

$= P(\underline{Y = \min(Y,B,R)}) \cdot P(B = \min(B,R) \mid Y = \min(Y,B,R))$

$= \boxed{\dfrac{\lambda_Y}{\lambda_Y + \lambda_B + \lambda_R} \cdot \dfrac{\lambda_B}{\lambda_B + \lambda_R}}$

(3) Sec 5.2 Marginal densities

Recall marginal probability:

discrete picture

X \ Y	0	1	2	P(x)
0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$
1	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
P(y)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

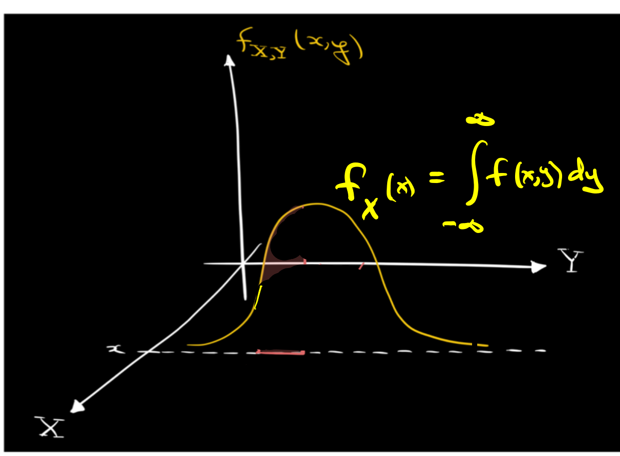
Marginal probability of X

$$P(x) = \sum_{y \in Y} p(x,y)$$

Marginal prob of Y

$$P(y) = \sum_{x \in X} p(x,y)$$

Continuous Picture: marginal density



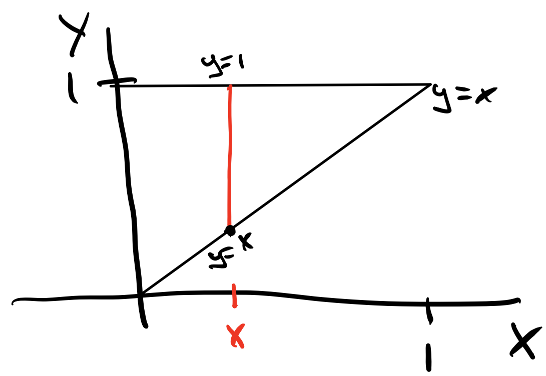
ex

joint density

$$f(x,y) = \begin{cases} 30\,(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$

$Y = U_{(6)}$



marginal density

$$f_X(x) = \int_{y=-\infty}^{y=\infty} f(x,y)\,dy$$

$$= \int_{y=x}^{y=1} 30\,(y-x)^4\,dy$$

$u = y-x$

$du = dy$

$$= \int_{u=0}^{u=1-x} 30u^4\,du = \frac{30u^5}{5}\Big|_0^{1-x} = \boxed{6(1-x)^5,\quad 0<x<1}$$

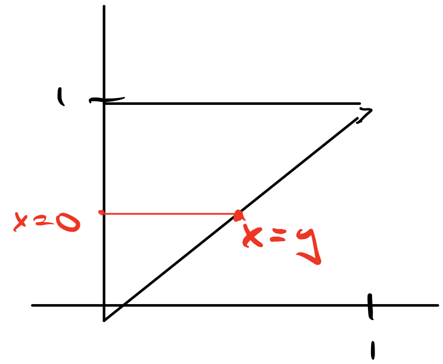
Find $f_Y(y)$

$$= \int_{x=0}^{x=y} 30(y-x)^4 \, dx$$

$u = y - x$

$du = - dx$

$$= -\int_{u=y}^{u=0} 30u^4 \, du = \frac{30u^5}{5}\Big|_0^y = \boxed{6y^5} \quad 0<y<1$$



$$f(x,y) = 30(y-x)^4 \neq \underset{\underset{6(1-x)^5}{\shortparallel}}{f_X(x)} \; \underset{\underset{6y^5}{\shortparallel}}{f_Y(y)}$$

so $X = U_{(1)}$ and $Y = U_{(6)}$ are dependent.