## Stat 134 lec 29

**Last time.**

① Sec 4.6 Beta Distribution

Let $r, s > 0$

$P \sim \text{Beta}(r,s)$ if

$$f(p) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} p^{r-1}(1-p)^{s-1} \quad \text{for } 0<p<1$$

**Applications**

ⓐ generalization of uniform ordered statistic

If throw $n$ darts at $[0,1]$

$U_{(k)} \sim \text{Beta}(k, n-k+1)$

Note $U(0,1) = \text{Beta}(1,1)$ — $n=1$, $k=1$

ⓑ $\text{Beta}(r,s)$ represents a distribution of probabilities (Bayesian statistics)

Recall from lecture 2 (Sec 1.2)

Posterior $\propto$ likelihood · Prior

$$f(p \mid x) \propto f(x \mid p)\, f(p)$$

An update of the distribution of $p$ given $X$. This will be a beta.

ex Bin form. $P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$

Geom form. $P(X=x) = p(1-p)^{x-1}$

ex $\text{Beta}(r,s)$

$f(p) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} p^{r-1}(1-p)^{s-1}$

**Today**

① Sec 5.2 Competing exponentials

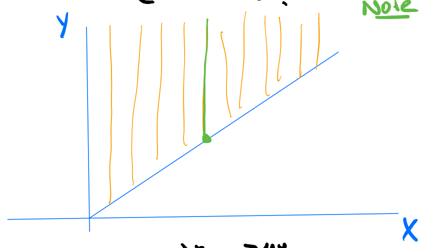
② Sec 5.2 Marginal density, expectation $E(g(X,Y))$

## ① Sec 5.2 Competing exponentials

Let $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$ be independent lifetimes of two bulbs.

Find $P(Y > X)$.

Note $\int_{y=x}^{y=\infty} e^{-\mu x} dy = \frac{e^{-\mu x}}{\mu}$.




$$f(x,y) = \lambda e^{-\lambda x} \mu e^{-\mu y}$$

$$P(X > Y) = \lambda\mu \int_{x=0}^{x=\infty} e^{-\lambda x} \underbrace{\int_{y=x}^{y=\infty} e^{-\mu y} dy}_{} dx$$

$$\underbrace{}_{} = -\frac{e^{-\mu y}}{\mu}\Bigg|_{y=x}^{y=\infty} = \frac{e^{-\mu x}}{\mu}$$

$$= \lambda \int_{x=0}^{\infty} e^{-(\lambda+\mu)x} dx = \boxed{\frac{\lambda}{\lambda+\mu}}$$

ex

(Exponential distributions) You are first in line to have your question answered by either of the 2 uGSIs, Brian and Yiming, who are busy with their current students. Your question will be answered as soon as the first uGSI finishes. The times spent with student by Brian and Yiming are independent and exponentially distributed with (positive) rates $\lambda_B$ and $\lambda_Y$ respectively, i.e. Brian's distribution is Exponential($\lambda_B$), and Yiming's is Exponential($\lambda_Y$).

(a) Find the probability that Yiming will be the one answering your questions.

$B$ = time Brian spent with his student $B \sim Exp(\lambda_B)$ } indep
$Y$ = time Yiming spent with her student $Y \sim Exp(\lambda_Y)$ }

$$P(Y < B) = \frac{\lambda_Y}{\lambda_Y + \lambda_B}$$

(b) What is the distribution of your wait time? Your answer should not include integrals.

$W = \min(Y, B)$ is waiting time

$$P(W>w) = P(\min(Y,B) > w) = P(Y>w, B>w)$$
$$= e^{-w\lambda_Y} \cdot e^{-w\lambda_B} = e^{-w(\lambda_Y+\lambda_B)}$$
$$\Rightarrow W \sim \text{Exp}(\lambda_Y + \lambda_B)$$

② Sec 5.2 Marginal densities

Recall marginal probability:

<u>discrete picture</u>

| Y \ X | 0 | 1 | P(y) |
|---|---|---|---|
| 2 | 0 | $\frac{1}{4}$ | $\frac{1}{4}$ |
| 1 | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{2}$ |
| 0 | $\frac{1}{4}$ | 0 | $\frac{1}{4}$ |
| P(x) | $\frac{1}{2}$ | $\frac{1}{2}$ | |

marginal probability of X

$P(x) = \sum_{y \in Y} p(x,y)$

Marginal prob of Y

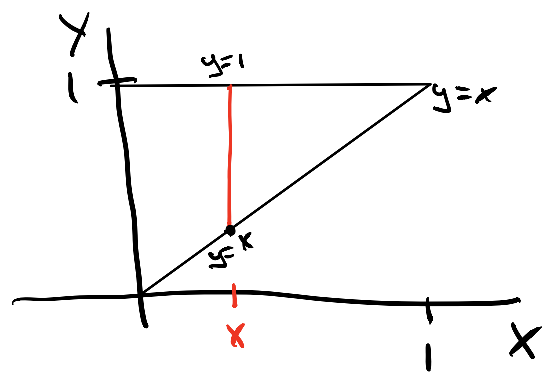
$P(y) = \sum_{x \in X} p(x,y)$

## marginal density

$$f(x,y) = \begin{cases} 30\,(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$

$Y = U_{(6)}$




$$\int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty} f(x,y)\,dy\,dx = 1$$

$$f_X(x) = \int_{y=-\infty}^{y=\infty} f(x,y)\,dy$$

marginal density of $X$

note
$f(x,y)$ is zero
outside of
$0<x<y<1$
so this is also

$$\int_{x=0}^{x=1}\int_{y=x}^{y=1} f(x,y)\,dy\,dx = 1$$

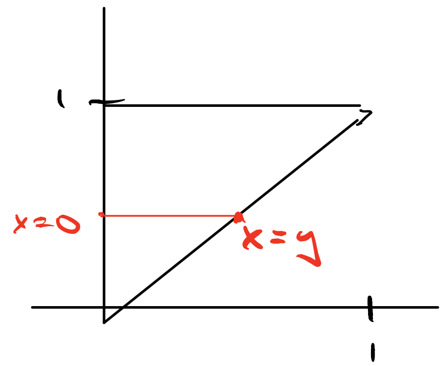
$$= \int_{y=x}^{y=1} 30\,(y-x)^4\,dy$$

$u = y-x$

$du = dy$

$$= \int_{u=0}^{u=1-x} 30u^4\,du = \frac{30u^5}{5}\Bigg|_0^{1-x} = \boxed{6(1-x)^5,\quad 0<x<1}$$

$$f_Y(y) = \int_{x=-\infty}^{x=\infty} 30\,(y-x)^4\,dx$$




$$= \int_{x=0}^{x=y} 30\,(y-x)^4\,dx$$

$u = y-x$
$du = -dx$

$$= -\int_{u=y}^{0} 30\,u^4\,du = \frac{30u^5}{5}\Big|_0^y = \boxed{6y^5,\ 0<y<1}$$

$\Rightarrow X = U_{(1)},\ Y = U_{(6)}$ aren't independent,

since $f(x,y) = 30\,(y-x)^4 \neq f(x)f(y)$

$$f(x)f(y) = 6(1-x)^5 \cdot 6(y)^5$$

## Expectation

$$E(g(x,y)) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} g(x,y)\,f(x,y)\,dx\,dy.$$

Check:

Find

$$E(Y) = \int_{y=0}^{y=1}\int_{x=0}^{x=y} y\,f(x,y)\,dx\,dy$$

$$= \int_{y=0}^{y=1} y \underbrace{\int_{x=0}^{x=y} f(x,y)\,dx}_{=\,f_Y(y)}\,dy$$

$$= \int_{y=0}^{y=1} y\,\underbrace{f_Y(y)}_{=\,6y^5}\,dy = \int_{y=0}^{y=1} 6y^6\,dy$$

$$= \frac{6y^7}{7}\Big|_0^1 = 6/7$$

<u>Note</u> $Y \sim U_{(6)} = \text{Beta}(\underset{k}{6},\underset{n-k+1\,=\,6-6+1}{1}) \Rightarrow E(Y) = \frac{6}{6+1}$ ✓

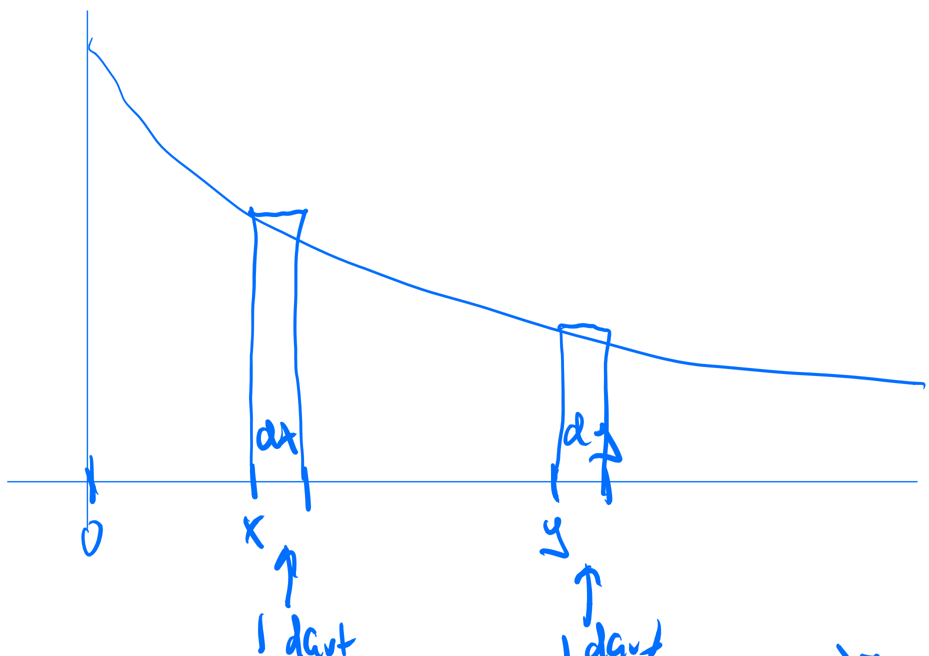
ex (5.2.9a)

$S, T \overset{iid}{\sim} \text{Exp}(\lambda)$

$X = \min(S,T) \leftarrow$ 1st ordered statistic of $\text{Exp}(\lambda)$

$Y = \max(S,T) \leftarrow$ 2nd ordered statistic of $\text{Exp}(\lambda)$

Find the joint density of $X$ and $Y$




$$P(X \in dx, Y \in dy) = \binom{2}{1} \lambda e^{-\lambda x} dx \binom{1}{1} \lambda e^{-\lambda y} dy$$

$$= 2\lambda^2 e^{-\lambda(x+y)} dx\,dy$$

$$\Rightarrow \boxed{f(x,y) = 2\lambda^2 e^{-\lambda(x+y)}}$$

Next, lets find the marginal densities for $X, Y$.

tinyurl

http://tinyurl.com/april8-pt1

http://tinyurl.com/april8-pt2

# Stat 134

## Monday April 8 2019

1. S and T are i.i.d. Exp($\lambda$). $X = \text{Min}(S, T)$ and $Y = \text{Max}(S, T)$. The joint density is $f(x, y) = 2\lambda^2 e^{\lambda(x+y)}$. The marginal density of X is:

   **a** $2\lambda e^{-2\lambda x}$ for $x > 0$

   **b** $2\lambda e^{-\lambda x}$ for $x > 0$

   **c** $\lambda e^{-\lambda x}$ for $x > 0$

   **d** none of the above

$X = \min(S,T)$

Method 1

$$f_X(x) = \int_{y=x}^{y=\infty} 2\lambda^2 e^{-\lambda(x+y)} dy$$

$$= 2\lambda^2 e^{-\lambda x} \int_x^\infty e^{-\lambda y} dy$$

$$\left. \frac{e^{-\lambda y}}{-\lambda} \right|_x^\infty = \frac{e^{-\lambda x}}{\lambda}$$

$$= \boxed{2\lambda e^{-2\lambda x}}$$

Method 2

$X = \min(S,T)$

$P(X > x) = P(S > x, T > x)$

$= P(S > x)^2 = (e^{-\lambda x})^2 = e^{-2\lambda x}$

$F(x) = 1 - e^{-2\lambda x}$

$f(x) = \frac{d}{dx} F(x) = \boxed{2\lambda e^{-2\lambda x}}$

2. S and T are i.i.d. Exp($\lambda$). $X = \text{Min(S, T)}$ and $Y = \text{Max(S, T)}$. The joint density is $f(x, y) = 2\lambda^2 e^{\lambda(x+y)}$. The marginal density of Y is:

**a** $\lambda(1 - e^{-\lambda y})e^{\lambda y}$ for $y > 0$

**b** $2\lambda(1 - e^{-\lambda y})e^{\lambda y}$ for $y > 0$

**c** $2\lambda(1 - e^{-\lambda y})$ for $y > 0$

**d** none of the above

method 1

$$f_Y(y) = 2\lambda^2 e^{-\lambda y} \int_0^y e^{-\lambda x}\, dx$$

$$= 2\lambda^2 e^{-\lambda y} \left( \frac{e^{-\lambda x}}{-\lambda} \right) \Big|_0^y$$

$$= \boxed{2\lambda (1 - e^{-\lambda \lambda})(e^{-\lambda y})}$$

method 2

$$F(y) = P(Y < y) = P(S < y, T < y)$$

$$= P(S < y)^2 = (1 - e^{-\lambda y})^2$$

$$f(y) = \frac{d}{dy} F(y) = 2(1 - e^{-\lambda y}) \cdot (e^{-\lambda y}) \cdot \lambda$$

$$= \boxed{2\lambda (1 - e^{-\lambda y})(e^{-\lambda y})}$$

For fun, check independence

$$f(x,y) \stackrel{?}{=} f(x) f(y) = 2\lambda e^{-2\lambda x} \cdot 2\lambda (1 - e^{-\lambda y})(e^{-\lambda y})$$

No!

$$f(x,y) = 2\lambda^2 e^{-\lambda(x+y)}$$

X, Y dependent.