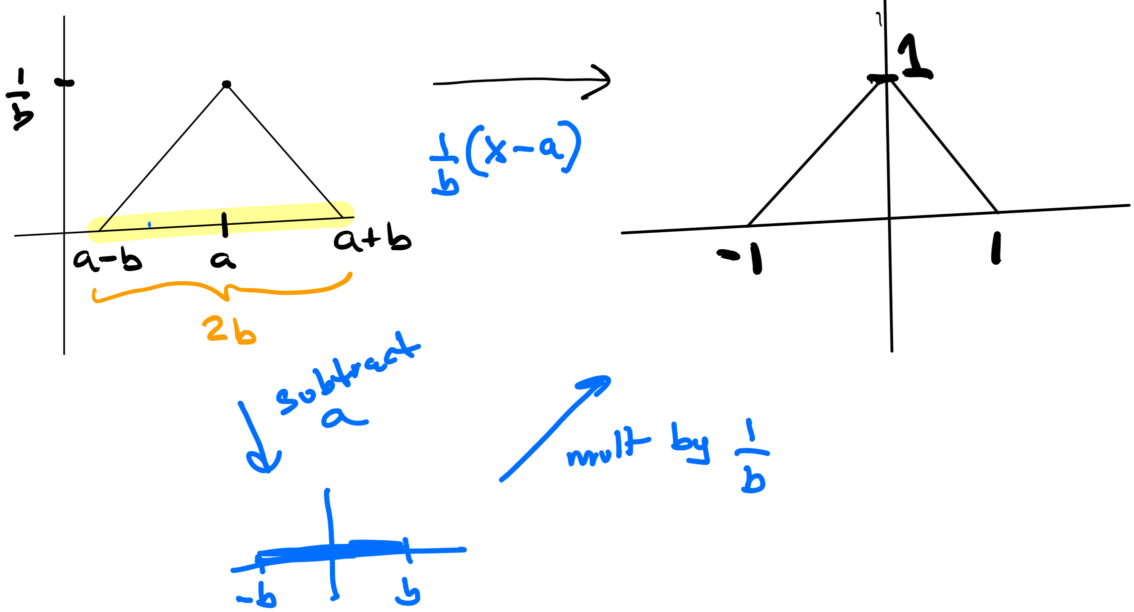
Stat 134 lec 21

Warmup 1:00-1:10

What change of scale do have here?

We are changing the values X takes. Because it is a linear change of scale a triangle shaped density maps to the same shape.



Last time sec 4.1 Continuous distributions

A continuous RV X, has a prob density function, $f(x)$, where $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

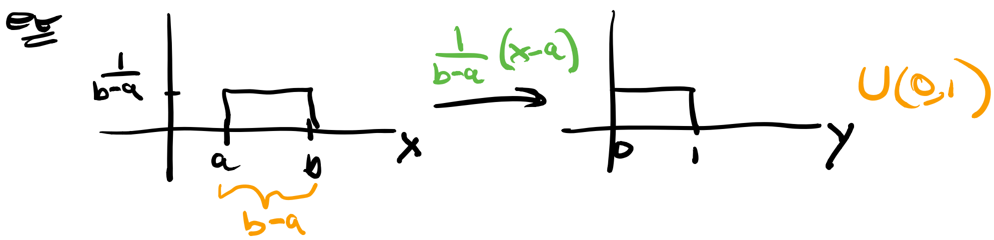
$P(X=a) = \int_a^a f(x)dx = 0$ so $P(X \geq a) = P(X > a)$.

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

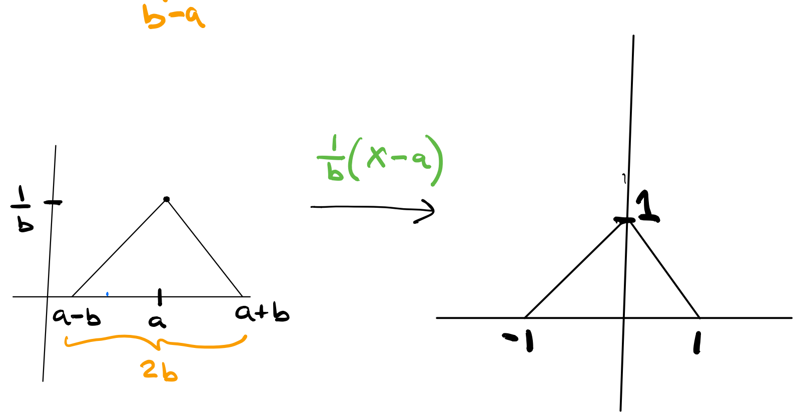
A change of scale is a transformation $Y = c + dX$, of X. The purpose is that it makes it easier to calculate $E(X)$ and $Var(X)$.

constants.

ex



ex



Today

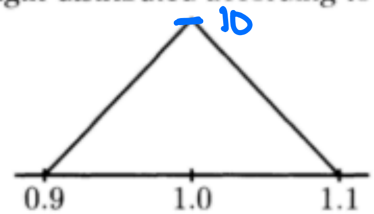
1. sec 4.1 change of scale calculations
2. briefly sec 4.5 Cummulative Distribution Function (CDF)
3. sec 4.2 Exponential Distribution.

① Sec 4.1 Change of scale calculation

tinyurl.com/oct12-pt1

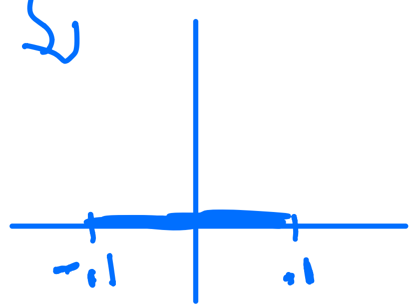
Suppose a manufacturing process designed to produce rods of length 1 inch exactly, in fact produces rods with length distributed according to the density graphed below.

$a = 1$
$b = \frac{1}{10}$
in previous example



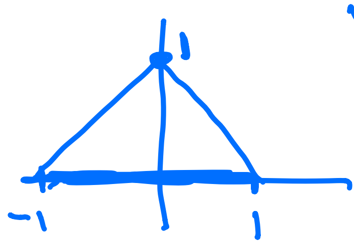
You should change the scale of X= the length of rods to:

subtract 1

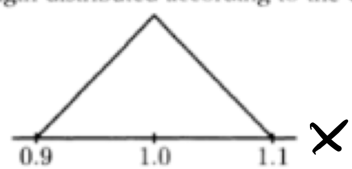


- ○ a: X-1
- ○ b: .1(X-1)
- ○ c: 10X-1
- ☒ d: none of the above

mult by 10



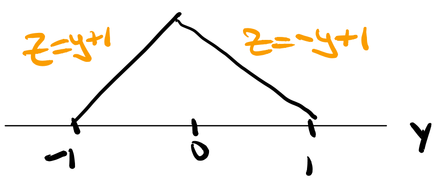
Suppose a manufacturing process designed to produce rods of length 1 inch exactly, in fact produces rods with length distributed according to the density graphed below.



Find the variance of the length of the rods.

$Y = 10(X-1)$ change of scale.

$\text{Var}(Y) = 100\,\text{Var}(X) \Rightarrow \text{Var}(X) = \dfrac{\text{Var}(Y)}{100}$ ← easier to find.



$$f(y) = \begin{cases} y+1 & -1 \le y \le 0 \\ -y+1 & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

Find Var(X)

$$\text{Var}(Y) = E(Y^2) - (E(Y))^2$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)\,dy = \int_{-1}^{0} y^2(y+1)\,dy + \int_{0}^{1} y^2(-y+1)\,dy$$

$$= \int_{-1}^{0}(y^3+y^2)\,dy + \int_{0}^{1}(-y^3+y^2)\,dy = \frac{y^4}{4} + \frac{y^3}{3}\Big|_{-1}^{0} + \frac{-y^4}{4} + \frac{y^3}{3}\Big|_{0}^{1} = \boxed{1/6}$$

$E(Y) = 0$

$\text{Var}(Y) = 1/6 \Rightarrow \text{Var}(X) = \dfrac{\text{Var}(Y)}{100} = \boxed{\dfrac{1}{600}}$

② briefly sec 4.5 The Cummulative Distribution Function (CDF)

Let X be a continuous RV

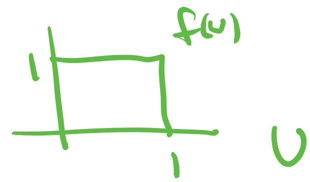
$F(x) = P(X \le x)$ — a number between 0 and 1

If $f(x)$ is a density of X,

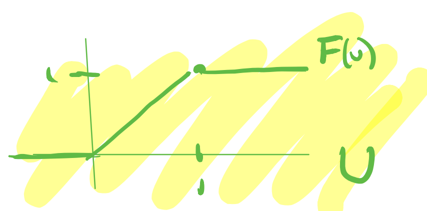
$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

ex $U \sim \text{Unif}(0,1)$

$$f(u) = \begin{cases} 1 & 0 < u < 1 \\ 0 & \text{else} \end{cases}$$



$$F(u) = \int_0^u 1\,dx = u$$



$$\boxed{F(u) = \begin{cases} 0 & -\infty < u \le 0 \\ u & 0 \le u \le 1 \\ 1 & u \ge 1 \end{cases}}$$

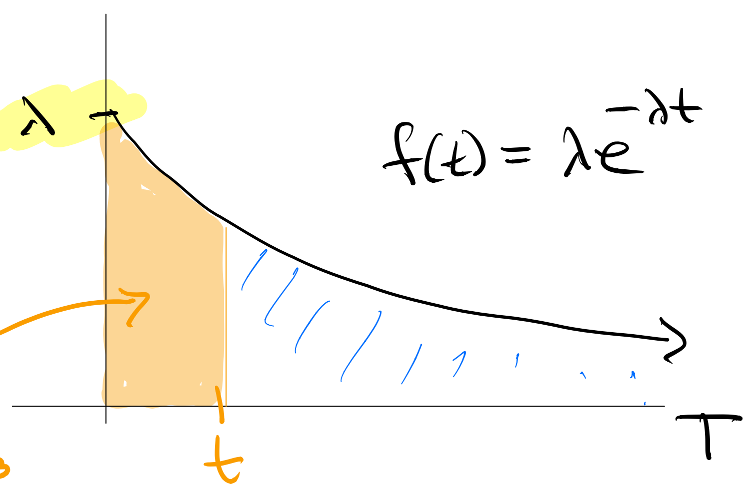
By FTC, $F'(x) = f(x)$

Consequently a density function and cdf are equivalent descriptions of a RV.

③ Sec 4.2 Exponential distribution

Def^n A random time T has exponential distribution with rate $\lambda > 0$.

$T \sim Exp(\lambda)$, if T has density $f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$



$F(t) = \int_0^t f(s)\,ds$

ex T = time until your first success where λ = rate of success, ex T = time until a lightbulb burns out

CDF and survival function

$T \sim \text{Exp}(\lambda)$ $\quad f(t) = \lambda e^{-\lambda t}$

Compute the CDF of T.

$$F(t) = P(T \leq t) = \int_{-\infty}^{t} f(s)\,ds = \underbrace{\int_{-\infty}^{0} f(s)\,ds}_{=0} + \int_{0}^{t} f(s)\,ds$$

$$= \int_0^t \lambda e^{-\lambda s}\,ds = \frac{\lambda e^{-\lambda s}}{-\lambda}\Big|_0^t$$

$$= -e^{-\lambda t} + 1 = \boxed{1 - e^{-\lambda t}}$$

$P(T > t) = e^{-\lambda t}$ is

called the <u>survival function</u>

$T \sim \text{Exp}(\lambda)$ iff $P(T > t) = e^{-\lambda t}$

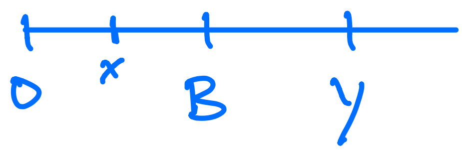
since $\underbrace{F(t)}_{=1-P(T>t)}$ and $f(t)$ both define distribution.

ex Brian and Yiming arrive at the beginning of a group advising session. The amount of time they will stay is exponentially distributed with rates λ_B and λ_Y, independent of each other.

(i.e. $\left.\begin{array}{l} B = \text{Brian's wait} \sim Exp(\lambda_B) \\ Y = \text{Yiming's wait} \sim Exp(\lambda_Y) \end{array}\right\}$ indep.)

What distribution is $X = \min(B, Y)$?

$P(X > x)$



$$= P(B > x, Y > x) = \underbrace{P(B > x)}_{= e^{-\lambda_B x}} \underbrace{P(Y > x)}_{= e^{-\lambda_Y x}} = e^{-(\lambda_B + \lambda_Y)x}$$

$$\underbrace{P(X \le x)}_{= F(x)} = 1 - e^{-(\lambda_B + \lambda_Y)x}$$

$$\boxed{X \sim Exp(\lambda_B + \lambda_Y)}$$

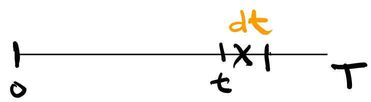
Memoryless property of Exponential

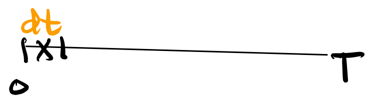
survival function

$T \sim \text{Exp}(\lambda)$ $\quad f(t) = \begin{cases} \lambda e^{-\lambda t} & t>0 \\ 0 & \text{else} \end{cases}$ $\quad$ or $\quad P(T>t) = e^{-\lambda t}$

T = time until 1^{st} success (arrival)

Memoryless Property : $P(T \in dt \mid T>t) = P(0<T<0+dt)$





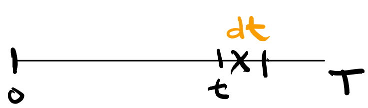
$$P(T \in dt \mid T>t) = \frac{P(T \in dt)}{P(T>t)} \approx \frac{\lambda \cancel{e^{-\lambda t}}\, dt}{\cancel{e^{-\lambda t}}} = \boxed{\lambda dt}$$

$$\boxed{\lambda dt} = P(0<T<0+dt)$$

conclusions :

① So long as a lightbulb is still functioning it is as good as new

② $\lambda = \dfrac{P(T \in dt \mid T>t)}{dt}$



instantaneous rate of success. This is always the same (not dependent on t).

Memoryless property of the geometric distribution

Let $X \sim \text{Geom}(1/2)$

$P(X > 3) = ?$

Question

Given it takes you more than $j=10$ coin tosses to get your first heads, what is the chance it will take you more than $k+j=13$ coin tosses to get your first heads?

proved in appendix of notes

Thm The Geometric distribution is the only discrete distribution with values $1, 2, 3, \ldots$ having the memoryless property

$$P(X > k+j \mid X > j) = P(X > k)$$

Only 2 distributions are memoryless:

For discrete $(X=1,2,3,\dots)$ – Geometric

For continuous $(T>0)$ – Exponential

Proof is similar to geom.

Appendix

$X \sim \text{Geom}(p)$

$X = \#$ trials until your first success.

$P(X=k) = q^{k-1} p$

recall $P(X > k) = P(X = k+1) + P(X = k+2) + \cdots$

$$= q^k p + q^{k+1} p + \cdots$$

$$= q^k p \left(\underbrace{1 + q + q^2 + \cdots}_{= \frac{1}{1-q}}\right)$$

$$= q^k.$$

Since $P(X=k) = \underbrace{P(X > k-1)}_{= q^{k-1}} - \underbrace{P(X > k)}_{= q^k}$

$$= q^{k-1}(1-q) = q^{k-1} p$$

$P(X > k) = q^k \Rightarrow X$ is $\text{Geom}(p)$.

$\Rightarrow X \sim \text{Geom}(p)$ iff $P(X > k) = q^k$

Thm The Geometric distribution is the only discrete distribution with values $1, 2, 3, \ldots$ having the

memoryless property

$$P(X > k+j \mid X > j) = P(X > k)$$

Proof/

Let $X \sim \text{Geom}(p)$, $X = 1, 2, 3, \ldots$

$$P(X > k+j \mid X > j) = \frac{P(X > k+j,\ X > j)}{P(X > j)}$$

$$= \frac{P(X > k+j)}{P(X > j)}$$

$$= \frac{q^{k+j}}{q^{j}} = q^{k}$$

$= P(X > k)$ ✓

Conversely,

For positive integers k, j

Suppose $P(X > k+j \mid X > j) = P(X > k)$

$$\parallel$$

$$\frac{P(X > k+j)}{P(X > j)}$$

$$\Rightarrow P(X > k+j) = P(X > j)P(X > k)$$

let $q = P(X > 1)$

Show that $P(X > j) = q^j$

for $j = 1, 2, 3, \ldots$ by induction.

Base case:

$P(X > 1) = q^1$ ✓

Assume $P(X > j-1) = q^{j-1}$

$$P(X > j) = P(X > j-1+1)$$

$$= \underbrace{P(X > j-1)}_{= q^{j-1}} \underbrace{P(X > 1)}_{= q} = q^j$$

$\Rightarrow X \sim \text{Geom}(p)$. $\square$