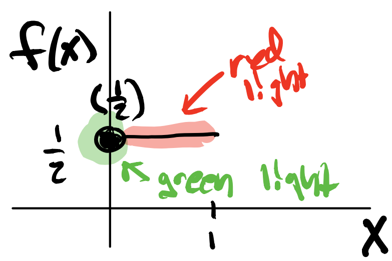
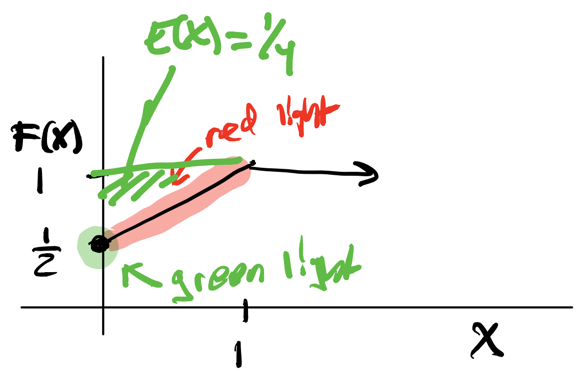
Stat 134 lec 27

warmup: 1:00–1:10

2. Suppose stop lights at an intersection alernately show green for one minute, and red for one minute (no yellow). Suppose a car arrives at the lights at a time distributed uniformly from 0 to 2 minutes. Let X be the delay of the car at the lights (assuming there is only one car on the road). Graph the density and the cdf of X. Also find $E(X)$

mixed density →





Last time

sec 4.5 Expectation of a nonnegative RV using CDF

$$E(X) = \int_0^\infty (1-F(x))\,dx$$

ex let $X \sim \text{Geom}\left(\frac{1}{2}\right)$

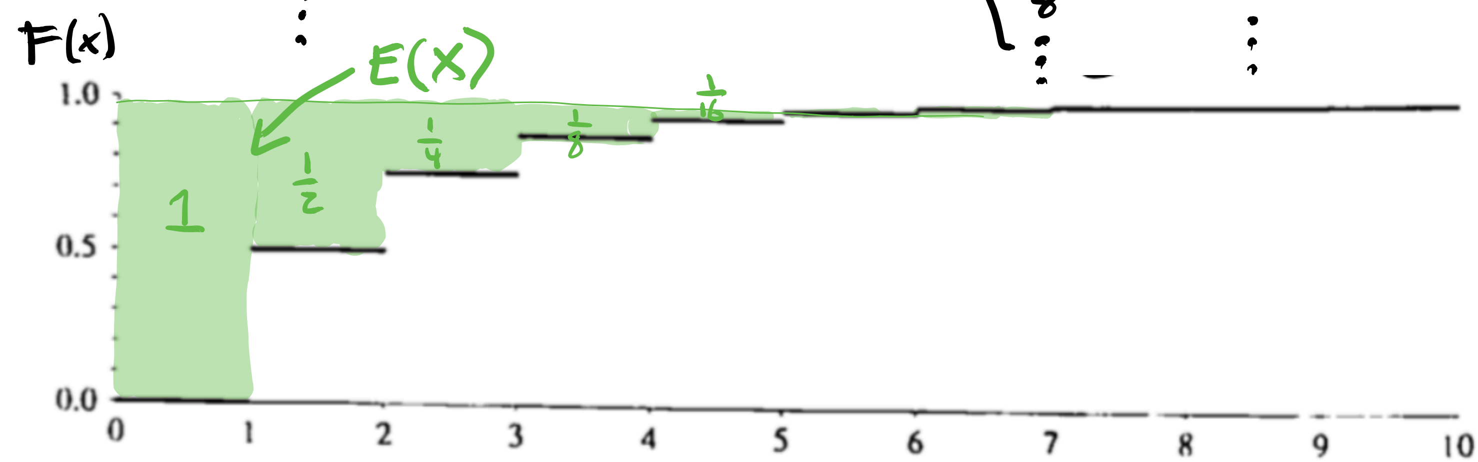
$P(X=1) = 1/2$

$P(X=2) = \frac{1}{2}\cdot\frac{1}{2} = 1/4$

Picture $P(X=3) = \left(\frac{1}{2}\right)^2\cdot\frac{1}{2} = 1/8$

$\vdots$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 1 \\ 1/2 & 1 \le x < 2 \\ \frac{3}{4} & 2 \le x < 3 \\ \frac{7}{8} & 3 \le x < 4 \\ \vdots & \vdots \end{cases}$$



$$E(X) = \int_0^\infty (1-F(x))\,dx$$

$$E(X) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \boxed{2}$$

p

$$= \sum_{j=0}^{\infty} P(X>j) = \sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^j \leftarrow \text{tail sum formula.}$$

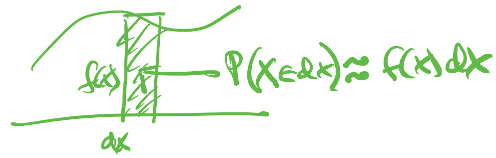
$\parallel$ q^j

Today

0. Overview of what we have learned since the midterm.
1. Review student explanations of MGF concept test.
2. sec 4.6 Order statistics
3. sec 5.1 continuous joint distributions.

0 Overview

Chap 4



density of distribution

change of variable formula for densities,

expectation

continuous distributions

- uniform
- exponential/gamma
- order statistics / beta

MGF - useful tool: calculate moments; identify a distribution by its MGF

CDF / mixed distributions

calculating expectation from cdf.

single variable unconditional prob

Chap 5

multiple variable unconditional prob: joint distributions

Chap 6

multiple variable conditional prob.: dependence

① student responses

Let X have density $f(x) = xe^{-x}$ for $x > 0$.
The MGF is?

a $M_X(t) = \frac{1}{1-t}$ for $t < 1$

b $M_X(t) = \frac{1}{(1-t)^2}$ for $t < 1$

c $M_X(t) = \frac{1}{(1+t)^2}$ for $t > -1$

d none of the above

a
b
c
d

c	We can see that X is the variable part of a gamma(2,-1). The MGF of a gamma is (lambda/(lambda-1))^r. Plugging in these values and dividing out the -1 yields answer C.

match xe^{-x} with $x^{r-1}e^{-\lambda x}$

$\Rightarrow X \sim$ Gamma$(2,1)$

Know $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r, t<$

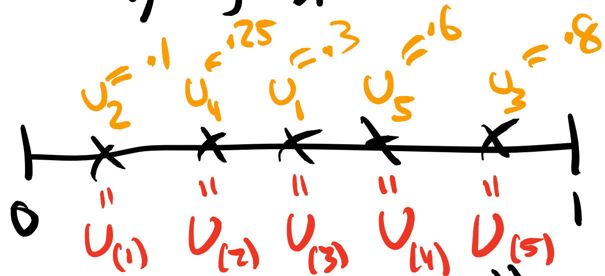
b	This is Gamma(2,1), so the MGF is (1/1-t)^2

b	Using wolfram alpha I know that the integral of x*e^(tx -x) dx from zero to infinity is 1/(1-t)^2 for t <1 thus b is the answer

Student didn't know xe^{-x} was Gamma and found $E(e^{tX}) = \int_0^\infty e^{tx} \cdot xe^{-x}\,dx = \frac{1}{(1-t)^2}$

② **Sec 4.6** <u>order statistic of $U(0,1)$</u>

let $U_1, \ldots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$



throw 5 darts at $[0,1]$.

let $U_{(k)}$ = called the k^{th} order statistic

= k^{th} largest value of $U_1, \ldots, U_n$

(assuming no ties)

ex

$$U_{(1)} = \min(U_1, \ldots, U_n)$$
$$U_{(n)} = \max(U_1, \ldots, U_n)$$

<u>Review counting</u>

You have 3 red, 2 green and 5 blue marbles,
How many orderings of these 10 marbles are there?

ex rrrggbbbbb
grrrgbbbbb
ggrrrbbbbb
⋮

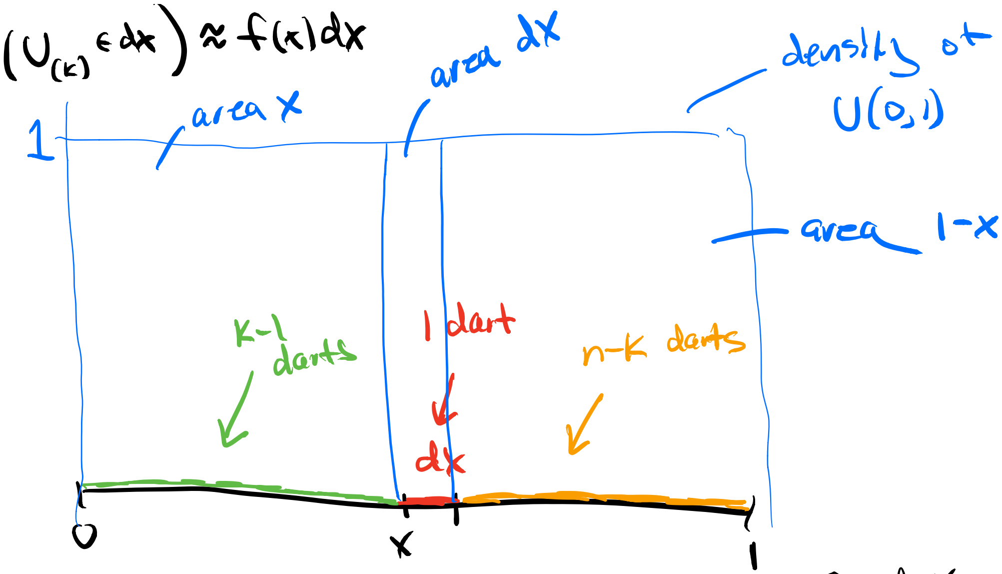
Answ

$$\binom{10}{3,2,5} = \binom{10}{3}\binom{7}{2}\binom{5}{5} = \frac{10!}{3!\,2!\,5!}$$

Next, find density of $U_{(k)}$

write $P(U_{(k)} \in dx) \approx f(x)dx$

<u>Picture</u>



$U_{(k)} \in dx$ means that $k-1$ darts are between 0 and x, and one is in dx, and $n-k$ darts are between x and 1

$$P(U_{(k)} \in dx) = P\left(k\text{-}1 \text{ darts} \in (0,x),\ 1 \text{ dart} \in dx,\ n\text{-}k \text{ darts} \in (x,1)\right)$$

$$= P\left(k\text{-}1 \text{ darts} \in (0,x)\right) \cdot P\left(1 \text{ dart} \in dx \mid k\text{-}1 \text{ darts} \in (0,x)\right)$$
$$\cdot P\left(n\text{-}k \text{ darts} \in (x,1) \mid 1 \text{ dart} \in dx,\ k\text{-}1 \text{ darts} \in (0,x)\right)$$

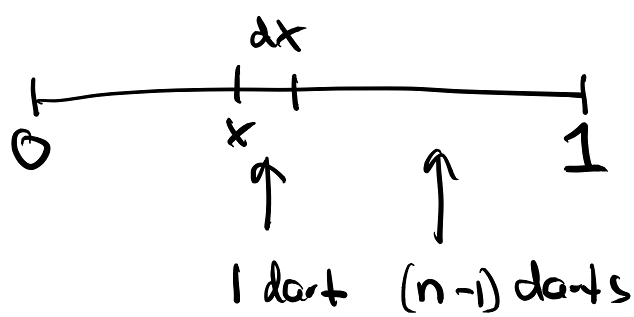
$$= \binom{n}{k-1} x^{k-1} \binom{n-k+1}{1} 1dx \binom{n-k}{n-k} (1-x)^{n-k}$$

$$= \underbrace{\binom{n}{k-1,1,n-k} x^{k-1} (1-x)^{(n-k+1)-1}}_{f_{U_{(k)}}(x)} dx$$

$$\Rightarrow f_{U_{(k)}}(x) = \binom{n}{k-1;\,1,\,n-k} x^{k-1} (1-x)^{(n-k+1)-1} \quad \text{for } 0<x<1$$

Ex Let $U_1, \ldots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$

Find the density of $U_{(1)}$

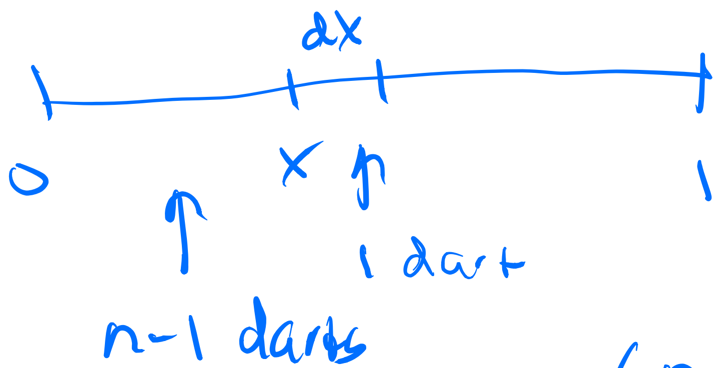


$$f_{U_{(1)}}(x)\,dx = \binom{n}{1,\,n-1} dx\,(1-x)^{n-1}$$

$$= n(1-x)^{n-1}\,dx$$

Ex Let $U_1, \dots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$

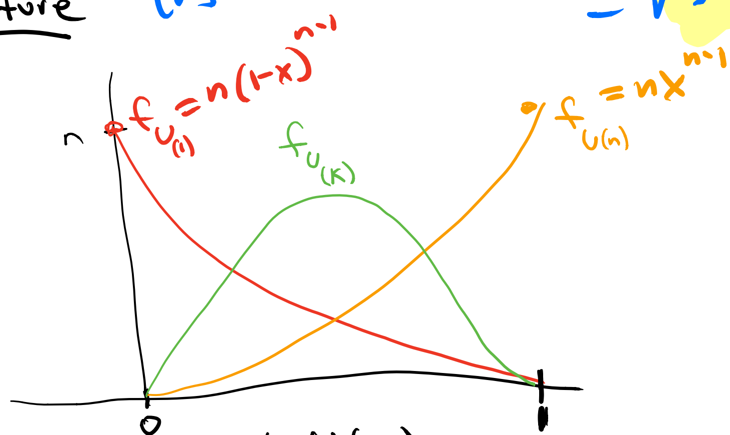
Find the density of $U_{(n)}$



$$f_{U_{(n)}}(x)\,dx \approx \binom{n}{n-1,\,1} x^{n-1}\,dx$$

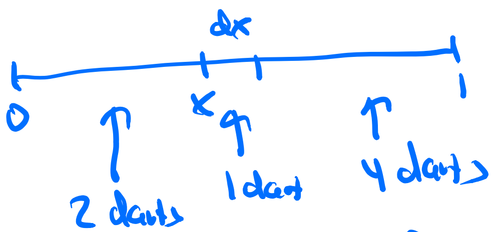
$$= n x^{n-1}\,dx$$

Picture



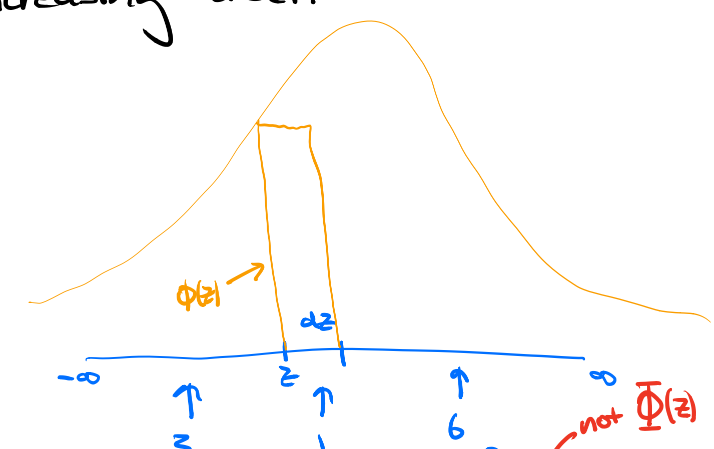
Order statistic of $U(0,1)$ provides a family of densities on the unit interval.

ex $x^2(1-x)^4$ for $0<x<1$ is the variable part of density of what RV? How many darts do you throw? $\rightarrow$ $\boxed{U_{(3)} \text{ of } n=7}$



$P(U_{(3)} \in dx) = f(x)dx$ $\quad f_{U_{(3)}}(x) = \binom{7}{2,1,4} x^2 (1-x)^4$

ex Let $Z_{(1)}, \dots, Z_{(10)}$ be the values of 10 independent standard normal variables arranged in increasing order. Find the density of $Z_{(4)}$



not $\Phi(z)$

$$f(z)\,dz = \binom{10}{3,1,6} (\Phi(z))^3 \, \phi(z)\,dz \, (1-\Phi(z))^6$$

$$\Rightarrow f(z) = \binom{10}{3,1,6} (\Phi(z))^3 \phi(z) (1-\Phi(z))^6$$