

lec 40 stat 134

Last time sec 6.5

we write  $(U, V)$  bivariate normal w/  $\text{corr}(U, V) = \rho$  as

$$(U, V) \sim BV(M_U, \sigma_U, M_V, \sigma_V, \rho)$$

↑  
not  
 $\sigma_U^2$

std bivariate  
normal.

Thm Let  $(\zeta, \gamma) \sim BV(0, 1, 0, 1, \rho)$

normal.

The MGF of  $(\zeta, \gamma)$  is

$$M_{(\zeta, \gamma)}(s, t) = e^{\frac{s^2}{2} + \frac{t^2}{2} + st\rho}$$

- Today
- ① review concept test from last time
  - ② Review MGF
  - ③ Sec 6.5 Properties of Bivariate normal,

① A test score in Math and Physics is bivariate normal,  $\rho > 0$ . The average is 60 on both tests and the SDs are the same. Of students scoring 75 on the Math test:

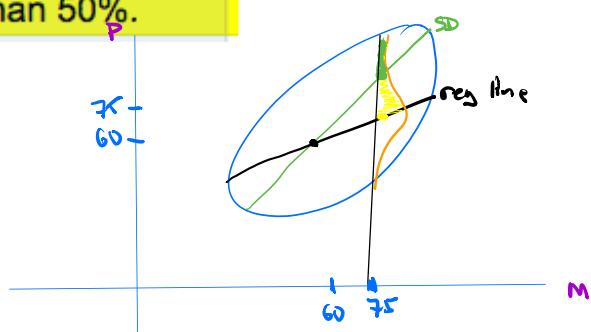
- a about half scored over 75 on Physics
- b more than half scored over 75 on Physics
- c less than half scored over 75 on Physics

*discuss how you did this for 1 minute.*

c If  $\rho=1$ , it would be exactly half

c Due to the regression effect, those who scored 75 on math should expect to regress to the mean for physics since both tests are independent

c The scores of students who scored 75 on math is normal centered on the regression line. 75 is above the regression line, so the proportion of students above 75 is less than 50%.



② Review MGF  $M_X(t) = E(e^{tX})$

Main properties

a)  $M_{aX}(t) = M_X(at)$

b)  $X, Y$  indep iff  $M_{X+Y}(t) = M_X(t)M_Y(t)$

c)  $M'_X(0) = E(X)$

$$M''_X(0) = E(X^2)$$

$$\vdots$$

$$M^{(k)}_X(0) = E(X^k)$$

d)  $M_X(t)$  is unique for  $t$  in a neighbourhood of 0. So if  $M_X(t) = e^{tx/\sigma^2}$  for all  $t$   
then  $X \sim N(0, 1)$

## Multivariate MGF

$$M_{(XY)}(s,t) = E(e^{sx+ty})$$

Multivariate MGF

Note that

$$M_{(XY)}(s,t) = E(e^{sx+ty}) = M_{sX+tY}^{(1)}$$

$X, Y$  indep iff  $M_{(XY)}(s,t) = M_X(s)M_Y(t)$

### ③ Properties of Bivariate Normal

Recall that if 2 RVs  $X, Y$  are independent,  $\text{Cov}(X, Y) = 0$   
 $\Rightarrow \text{Corr}(X, Y) = 0$

However the converse is not true  
 in general. ( $\text{Corr}(X, Y) = 0 \not\Rightarrow X, Y \text{ indep.}$ )

Let  $X \sim N(0, 1)$

$$Y = X^2$$

$X$  and  $Y$  are dependent

we show  $X, Y$  are uncorrelated.

$$\begin{aligned} \text{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\ &\stackrel{\substack{= \\ "}}{=} E(X^3) = 0 \end{aligned}$$

Find  $E(X^3)$

$$X \sim N(0, 1) \quad Z \sim N(0, 1)$$

$$M_X(t) = e^{t^2/2}$$

$$M_X'''(t) = t^3 e^{t^2/2} + 3t e^{t^2/2} + t^2 e^{t^2/2} \Big|_{t=0} = 0$$

$$\Rightarrow E(X^3) = 0$$

$$\Rightarrow \text{Corr}(X, Y) = 0$$

Then If  $(X, Y)$  is <sup>std</sup> bivariate normal then

$\rho = \text{Corr}(X, Y) = 0$  iff  $X, Y$  are independent.

Pf

$$M_{(X,Y)}^{(s,t)} = M_{sX+tY}^{(1)} = M_{sX}^{(1)} \cdot M_{tY}^{(1)} = M_X^{(s)} M_Y^{(t)}$$

iff  $X$  and  $Y$  are independent.

Since  $(X, Y)$  is <sup>std</sup> bivariate normal,

$$M_{(X,Y)}^{(s,t)} = e^{\frac{s^2}{2} + \frac{t^2}{2} + st\rho}$$

Finish proof.

$$= M_X^{(s)} M_Y^{(t)} \text{ iff } \rho = 0$$

$$\text{since } M_X^{(s)} = e^{\frac{s^2}{2}}$$

$$M_Y^{(t)} = e^{\frac{t^2}{2}}$$

□

□

Recall from Lec 30 that the sum of independent normal random variables is normal.

What about dependent normal random variables?

Then Let  $X, Y \sim N(0, 1)$  and  $\text{Corr}(X, Y) = \rho$ .

$(X, Y)$  is std bivariate normal iff

$sX + tY$  is normal for all constants  $s, t$ .

Pf/

$\Rightarrow$ : Suppose  $X, Y$  std bivariate normal.  
 (i.e.  $Y = \rho X + \sqrt{1-\rho^2} Z$  for  $X, Z \sim N(0, 1)$ )

$$\begin{aligned}sX + tY &= sX + t(\rho X + \sqrt{1-\rho^2} Z) \\ &= (s+t\rho)X + t\sqrt{1-\rho^2} Z\end{aligned}$$

is normal since  $X, Z$  are independent normals,

$\Leftarrow$ : Suppose  $sX + tY$  is normal,

note  $E(sX + tY) = sE(X) + tE(Y) = 0$

$$\begin{aligned}\text{Var}(sX + tY) &= \text{Var}(sX) + \text{Var}(tY) + \text{Cov}(sX, tY) \\ &= s^2 + t^2 + 2st\rho\end{aligned}$$

hence,  $sX + tY \sim N(0, s^2 + t^2 + 2st\rho)$

"  
 $2st\rho$   
 "  $\text{Cov}(X, Y)$   
 "  $\text{Corr}(X, Y)$

Recall,

$$\text{For } Z \sim N(\mu, \sigma^2) \Rightarrow M_Z^{(n)} = e^{\mu n} e^{\frac{\sigma^2 n^2}{2}}$$

$$\begin{aligned}\text{Then } M_{(X, Y)}^{(s, t)} &= M_{sX+tY}^{(1)} = e^{(s^2 + t^2 + 2st\rho)\frac{1}{2}} \\ &= e^{s^2 + t^2 + st\rho} \quad \text{normal} \quad \text{M6F of } (X, Y)\end{aligned}$$

$\Rightarrow (X, Y)$  is standard bivariate normal  $\square$

we dont need to restrict  
ourselves to  $X, Y \sim N(0, 1)$

Corollary → proof at end of lecture

Let  $U \sim N(\mu_U, \sigma_U^2)$

$V \sim N(\mu_V, \sigma_V^2)$

and  $\text{Corr}(U, V) = \rho$

$(U, V) \sim BV(\mu_U, \sigma_U, \mu_V, \sigma_V, \rho)$  iff

$sU + tV$  is normal for all constants  $s, t$ .

tinyurl:

<http://tinyurl.com/may3-part1>

## Stat 134

Friday May 3 2019

1. Let  $(X, Y)$  be bivariate normal. Then  $(2X+3Y+4, 6X-Y-4)$  is bivariate normal.

**a** true

**b** false

**c** not enough info to decide

$$\begin{aligned} & a(2x+3y+4) + b(6x-y-4) \\ &= (2a+6b)x + (3a-4b)y + 4a-4b \\ & \text{is normal since } (x,y) \text{ bivariate normal} \\ & \text{using corollary above,} \end{aligned}$$

e.g. Let  $M$  = a student's score on the midterm  
 $F$  = a student's score on the final

$$(M, F) \sim BV(70, 8, 65, 10, .6)$$

Find the chance that a student scores higher on the final than the midterm?

Hint  $P(F > M) = P(F - M > 0)$

What distribution is  $F - M$ ?

$$P\left(\frac{F-M - E(F-M)}{SD(F-M)} > \frac{0 - E(F-M)}{SD(F-M)}\right)$$

↑ normal since  
 $(M, F)$  is bivariate normal!

$$E(F-M) = 65 - 70 = -5$$

$$\begin{aligned} \text{Var}(F-M) &= \text{Var}(F) + \text{Var}(M) - 2 \underbrace{\text{Corr}(F, M)}_{\text{Corr}(F, M)SD(M)SD(F)} \\ &= 10^2 + 8^2 - 2(.6)(8)(10) \end{aligned}$$

$$SD(F-M) = \sqrt{168}$$

$$\Rightarrow \frac{0 - E(F-M)}{SD(F-M)} = \frac{0 - (-5)}{\sqrt{168}} = .61$$

$$\Rightarrow P(F > M) = 1 - \Phi(.61) = .27$$

## Appendix

### Corollary

Let  $U \sim N(\mu_U, \sigma_U^2)$   
 $V \sim N(\mu_V, \sigma_V^2)$

and  $\text{Corr}(U, V) = \rho$

$(U, V) \sim BV(\mu_U, \sigma_U, \mu_V, \sigma_V, \rho)$  iff  
 $sU + tV$  is normal for all constants  $s, t$ .

Pf/ let  $X = \frac{U - \mu_U}{\sigma_U}$  ( $U = \mu_U + X\sigma_U$ )

$$Y = \frac{V - \mu_V}{\sigma_V} \quad (V = \mu_V + Y\sigma_V)$$

$$(U, V) \sim BV(\mu_U, \sigma_U, \mu_V, \sigma_V, \rho)$$

$$\text{iff } (X, Y) \sim BV(0, 1, 0, 1, \rho)$$

$$\begin{aligned} sU + tV &= s(\mu_U + X\sigma_U) + t(\mu_V + Y\sigma_V) \\ &= \cancel{s\sigma_U} \underset{a}{X} + \cancel{t\sigma_V} \underset{b}{Y} + \cancel{s\mu_U + t\mu_V} \underset{\text{constant}}{\text{constant}} \end{aligned}$$

this is normal for all constants  $s, t$

if  $ax + by$  is normal for all constants  $a, b$ ,

□

