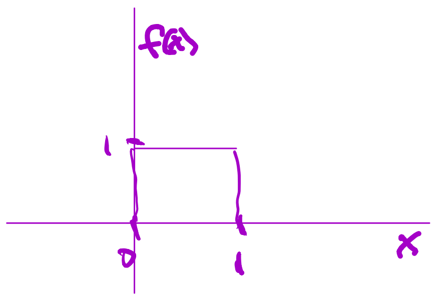
Stat 134 lec 20 (no lec 19)

warmup 1:00–1:10

Let $X \sim \text{Unif}(0,1)$ be the standard uniform distribution with histogram (density)

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \le 1 \\ 0 & \text{else} \end{cases}$$

Picture



Define

$$E(X) = \int_{x=-\infty}^{\infty} x f(x)\,dx$$

Find

$$E(X) = \int_0^1 x \cdot 1\,dx = \frac{x^2}{2}\Big|_0^1 = \boxed{\frac{1}{2}}$$

$$E(X^2) = \int_0^1 x^2 \cdot 1\,dx = \frac{x^3}{3}\Big|_0^1 = \boxed{1/3}$$

$$Var(X) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - 1/4 = \boxed{1/12}$$

Last time

Congratulations on finishing midterm 1 !

today

Sec 4.1 Continuous distributions

1. Probability density
2. expectation and variance.
3. Change of scale

① sec 4.1 Probability density.

let X be a continuous RV

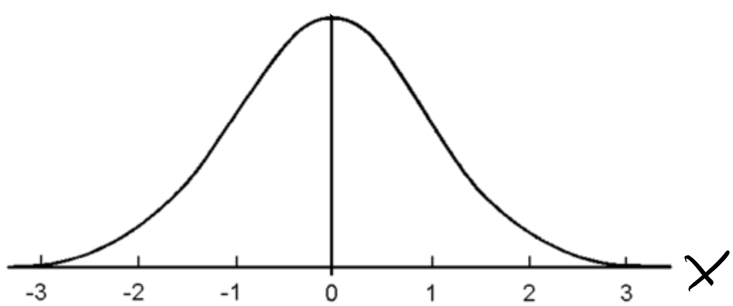
The probability density (histogram) of X is described by a prob density function

$$f(x) \geq 0 \quad \text{for } x \in X$$

and $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

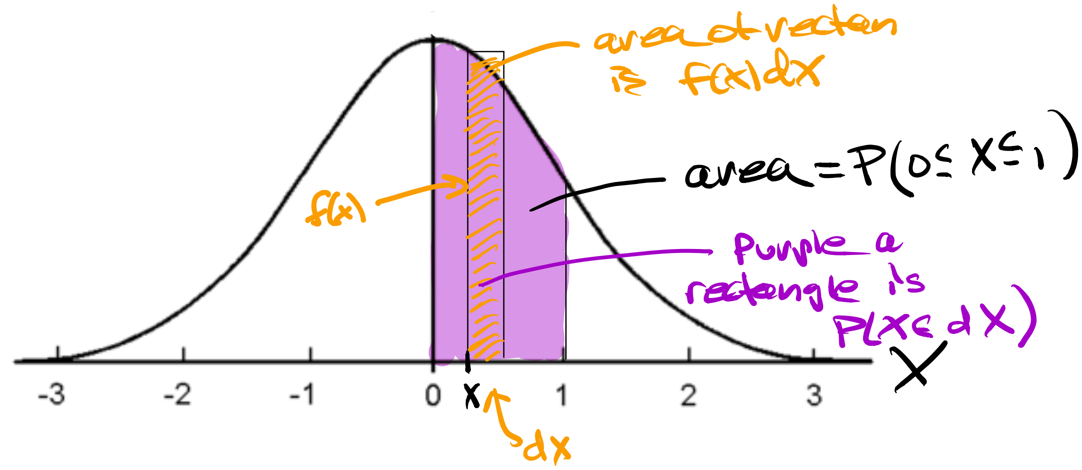
ex The standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$



Throwing a dart randomly at the histogram the x coordinate of your dart is most likely to be near zero.

The probability of getting an x coordinate of x is written $P(X \in dx)$.



we see from the rectangle in the picture,

$P(X \in dx) \approx f(x)dx$ (notice purple and orange area not same)

here dx = tiny interval around x and also the length of the interval

For all a, b $P(a \le X \le b) = \int_a^b P(X \in dx) = \int_a^b f(x)dx$

Note $f(x)$ is not a probability.

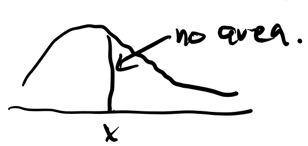
$f(x)dx$ is a probability

$f(x) \approx \dfrac{P(X \in dx)}{dx}$ ← probability

← length

units of f? probability / unit length ← probability density.

$P(X=x) = 0$

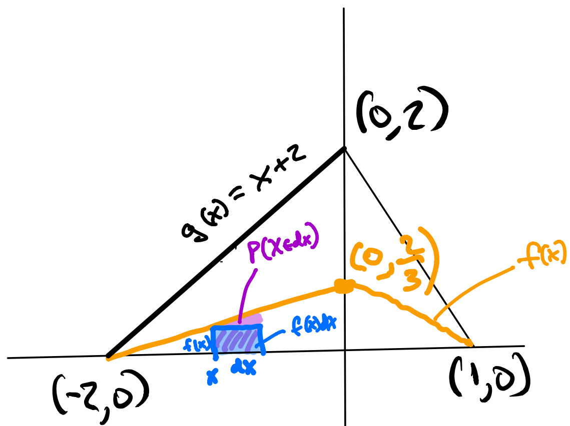


Hence $P(a \le X \le b) = P(a < X < b)$

(we don't have to worry about endpts).

ex 4.1.12 b

Consider a point picked uniformly at random from the area inside the following triangle



Find the density function of the x-coordinate f(x)

total area

$\frac{1}{2}(3)\cdot 2 = 3$

$$f(x) = \begin{cases} \frac{x+2}{3} & \text{for } -2 \leq x \leq 0 \\ \frac{-2x+2}{3} & \text{for } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

Procedure to find density:

ex area 3

① compute the area A of shape

② for a strip of width dx find

$P(x \in dx) = \frac{g(x)}{A} dx$ f(x)

ex $g(x) = x+2$ for $-2 \leq x \leq 0$

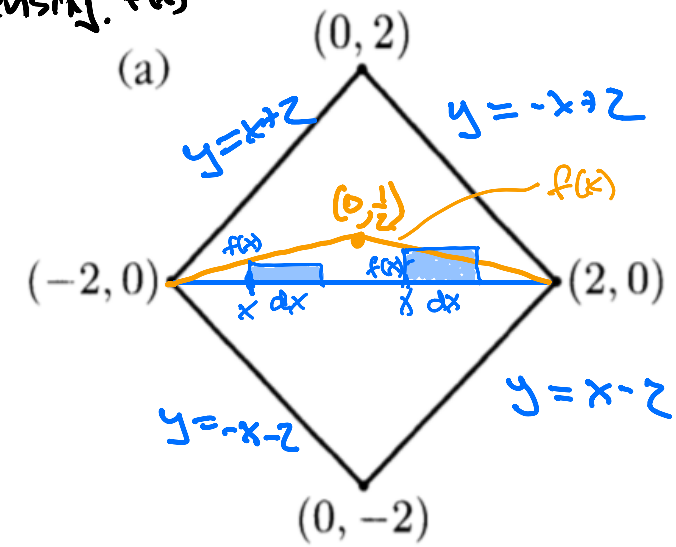
$A = 3$

③ repeat step 2 for other parts of shape

ex $g(x) = -2x+2$ for $0 \leq x \leq 1$

ex 4.1.12 a

Consider a point picked uniformly at random from the area inside the following shape. Find the density $f(x)$



total area

$$A = 2\left(\tfrac{1}{2}\cdot 4\cdot 2\right) = 8$$

top curve

$g(x) = (x+2) - (-x-2)$
$= 2(x+2)$

bottom curve

multiply by 2 since

$$f(x) = \begin{cases} \frac{2(x+2)}{8} = \frac{x+2}{4} & -2 \le x \le 0 \\ \frac{2(-x+2)}{8} = \frac{-x+2}{4} & 0 \le x \le 2 \\ 0 & \text{else} \end{cases}$$

(2)

Expectation and Variance

For discrete,

$$E(g(X)) = \sum_{x \in X} g(x) P(X=x)$$

For continuous,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) P(X \in dx) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$Var(X) = E(X^2) - E(X)^2$$

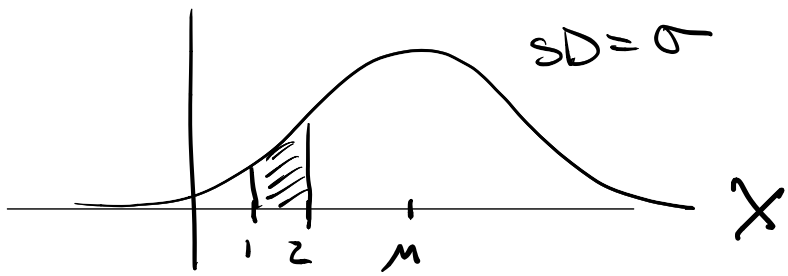
See Wampy for example.

③ Change of scale

To calculate $E(X)$, $Var(X)$, $P(X \in dx)$ we sometimes make a linear change of scale
$Y = c + dX$ where c, d are constants
Y hopefully has a simpler density function.
we can recover $E(X)$, $Var(X)$, $P(X \in dx)$
from $E(Y)$, $Var(Y)$, $P(Y \in dY)$.

$$Var(Y) = Var(c + dX) = d^2 Var(X)$$
$$\Rightarrow Var(X) = \frac{Var(Y)}{d^2}$$

ex Let $X \sim N(\mu, \sigma^2)$
Find $P(1 < X < 2)$

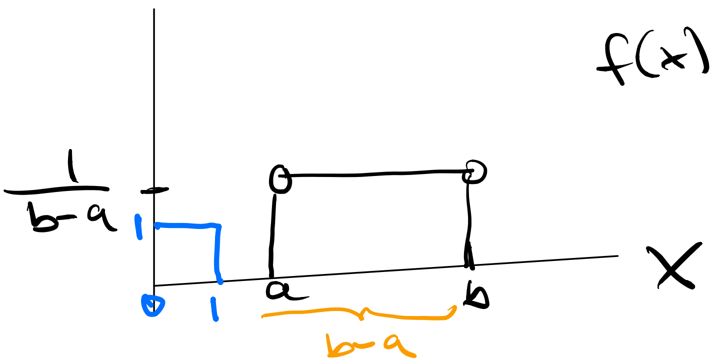


$$Z = \frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$$

Change of scale. $\quad d \quad c$

$$\boxed{P(1 < X < 2) = \Phi\left(\frac{2-\mu}{\sigma}\right) - \Phi\left(\frac{1-\mu}{\sigma}\right)}$$

ex: let $X \sim \text{Unif}(a,b)$

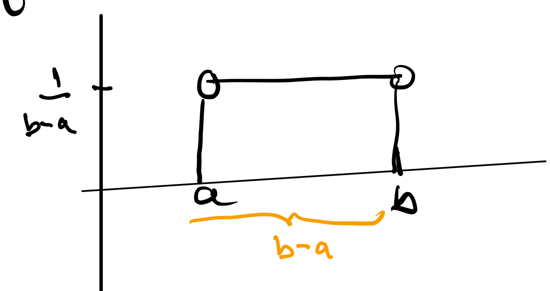


$$f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{else} \end{cases}$$

a) You should change the scale of X to?

$U \sim \text{Unif}(0,1)$

$U = \frac{1}{b-a}(X-a)$



b) Find $E(X)$

$b-a$

$$\frac{1}{2} = E(U) = E\left(\frac{1}{b-a}(x-a)\right) = \frac{1}{b-a}E(x-a) = \frac{E(x)-a}{b-a}$$

$$E(x) = \frac{1}{2}(b-a)+a = \boxed{\frac{a+b}{2}}$$

c) Find $Var(X)$.

$$\frac{1}{12} = Var(U) = Var\left(\frac{1}{b-a}(x-a)\right)$$

$$= \frac{1}{(b-a)^2}Var(x-a) = \frac{1}{(b-a)^2}Var(x)$$

$$\Rightarrow \boxed{Var(x) = \frac{(b-a)^2}{12}}$$