STAT 134 - Instructor: Adam Lucas

Midterm

Friday, March 8, 2019

SOLUTIONS

Exam Information and Instructions:

- You will have 45 minutes to take this exam. Closed book/notes/etc. No calculator or computer.
- We will be using Gradescope to grade this exam. Write any work you want graded on the front of each page, in the space below each question. Additionally, write your SID number in the top right corner on every page.
- Please use a dark pencil (mechanical or #2), and bring an eraser. *If you use a pen and make mistakes, you might run out of space to write in your answer.*
- Provide calculations or brief reasoning in every answer.
- Unless stated otherwise, you may leave answers as unsimplified numerical and algebraic expressions, and in terms of the Normal c.d.f. Φ. Finite sums are fine, but simplify any infinite sums.
- Do your own unaided work. Answer the questions on your own. The students around you have different exams.

1. (5 pts)

a Suppose that there is a machine that gives out a random number Y between 0 and 80. You are also told that $\mathbb{E}[Y] = 20$. Now someone proposes you a game where you win if the number that shows up is strictly smaller than 40. Assume that you always play games when you have a chance of at least $\frac{1}{2}$ of winning. Given the information you have, can you determine whether you should agree to play the game?

b Suppose that there is a machine that gives out a random number X between 20 and 100. You are also told that $\mathbb{E}[X] = 40$. Now someone proposes you a game where you win if the number that shows up is strictly smaller than 60. Assume that you always play games when you have a chance of at least $\frac{1}{2}$ of winning. Given the information you have, can you determine whether you should agree to play the game?

a Using Markov's inequality we get

$$\mathbb{P}(Y \geq 40) \leq \frac{\mathbb{E}[Y]}{40} = \frac{20}{40} = \frac{1}{2}.$$

This implies that

$$\mathbb{P}(Y < 40) = 1 - \mathbb{P}(Y \geq 40) \geq 1 - \frac{1}{2} = \frac{1}{2}$$

hence you should agree to play the game.

b Substitute $Y = X - 20$ into your conclusion from part a and you get

$$\mathbb{P}(X < 60) = \mathbb{P}(X - 20 < 40) = \mathbb{P}(Y < 40) \geq \frac{1}{2}$$

hence you should agree to play the game

2. (5 pts) An airport bus drops off 35 passengers at 7 stops. Each passenger is equally likely to get off at any stop, and passengers act independently of one another. The bus makes a stop only if someone wants to get off. Find the probability that the bus drops off passengers at every stop.

Let A_i be the event that nobody gets off at the i_{th} stop. By De Morgan's law, the desired probability is $1 - P(A_1 \cup A_2 \cup \cdots \cup A_7)$. The probability that nobody gets off at n of the 7 stops is $(\frac{7-n}{7})^{35}$. By the inclusion-exclusion formula,

$$\begin{aligned} P(A_1 \cup A_2 \cup \cdots \cup A_7) &= \sum_{i=1}^{7} P(A_i) - \sum_{i<j\leq 7} P(A_iA_j) + \cdots + (-1)^{7+1} P(A_1A_2A_3 \ldots A_7) \\ &= \sum_{n=1}^{7} (-1)^{n+1} \binom{7}{n} (\frac{7-n}{7})^{35} \end{aligned}$$

The final answer is $1 - P(A_1 \cup A_2 \cup \cdots \cup A_7) = 1 - \sum_{n=1}^{7} (-1)^{n+1} \binom{7}{n} (\frac{7-n}{7})^{35}$

3. (5 pts) Consider a grid of n^2 cups, with n rows of n cups. Toss n ping pong balls at random into these cups, where at most one ball can occupy a particular cup. Let X be the number of unoccupied rows (i.e., where every cup in that row contains no balls). Find:

   (a) $P(X \geq 1)$;

   (b) $E(X)$;

   (c) $Var(X)$.

   Solutions

   (a) We think of this as selecting n cups (without replacement) from the n^2 total. We want to find the chance that all n cups selected come from different rows. (This is similar to the chance of all ranks appearing in 13 cards, from Quiz 2.)

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - \frac{\binom{n}{1}^n}{\binom{n^2}{n}} \end{aligned}$$

   Or, an alternate method for finding $P(X = 0)$: we think about the slots the balls are allowed to go in; i.e., the first ball can go in any row, then the second ball must go in a row different from the first one, and so on. Note that after k balls have been placed, $n(n-k)$ possible slots will yield the $k+1$th ball being in a new row, while $n^2 - k$ slots remain. This yields

$$\begin{aligned} P(X = 0) &= \frac{n^2}{n^2} \cdot \frac{n(n-1)}{n^2-1} \cdot \frac{n(n-2)}{n^2-2} \cdot \ldots \cdot \frac{n(1)}{n^2-(n-1)} \\ &= \frac{n^n \cdot n!}{(n^2)_n} \end{aligned}$$

   These approaches are equivalent; expanding out the binomial coefficients above will demonstrate this result. This second approach could then be used for (b) and (c) as well.

(b) We proceed here using indicators. Let I_i be the indicator that row i is unoccupied. This is the same as the chance that all n balls fall in the n^2-n remaining cups.

$$\begin{aligned}
E(X) &= E(\sum_{i=1}^{n} I_i) \\
&= \sum_{i=1}^{n} E(I_i) \\
&= nP(I_i = 1) \\
&= n\frac{\binom{n}{0}\binom{n^2-n}{n}}{\binom{n^2}{n}}
\end{aligned}$$

(c) Again, we use indicators. Following the method discussed in Lecture 14, $Var(X) = E(X^2) - E(X)^2$, where $E(X)$ is given in part (b), and

$$\begin{aligned}
E(X^2) &= E\left[\left(\sum_{i=1}^{n} I_i\right)^2\right] \\
&= E\left(\sum_{i=1}^{n} I_i^2 + \sum_{j\neq k} I_j I_k\right) \\
&= nE(I_1^2) + n(n-1)E(I_1 I_2)
\end{aligned}$$

Here, $E(I_1^2) = E(I_1)$, and

$$\begin{aligned}
E(I_1 I_2) &= P(\text{rows 1, 2 unoccupied}) \\
&= \frac{\binom{n^2-2n}{n}}{\binom{n^2}{n}}
\end{aligned}$$

So our final answer is

$$Var(X) = n\frac{\binom{n}{0}\binom{n^2-n}{n}}{\binom{n^2}{n}} + n(n-1)\cdot\frac{\binom{n^2-2n}{n}}{\binom{n^2}{n}} - \left(n\frac{\binom{n}{0}\binom{n^2-n}{n}}{\binom{n^2}{n}}\right)^2$$

4. (5 pts) Consider a box containing 1000 balls, of which m are gold and the remaining $1000 - m$ are blue (Go Bears!). You draw 1000 balls from the box, randomly with replacement. Let X be the number of gold balls you get.

   (a) For any $1 \leq m \leq 1000$, what is $P(X = m)$? *Your answer should be in terms of m.*

   (b) For $m = 500$, what is $P(X = 500)$, approximately?

   (c) For $m = 2$, what is $P(X = 2)$, approximately?

   (a) This is a $\text{Bin}(n, p)$ probability, with $n = 1000$ and $p = m/1000$. Accordingly,

   $$P(X = m) = \binom{1000}{m} p^m (1-p)^{1000-m}.$$

   (b) This is a $\text{Bin}(1000, 1/2)$ probability, which is best approximated by the Normal distribution with mean $np = 500$ and variance $np(1-p) = 250$. Let $X \sim \text{Bin}(1000, 1/2)$ be the number of gold balls produced from the draws. Then we approximate $P(X = m)$ as

   $$P(X = 500) = P\left(500 \leq X \leq 500\right) \approx \Phi\left(\frac{500 + 0.5 - 500}{\sqrt{250}}\right) - \Phi\left(\frac{500 - 0.5 - 500}{\sqrt{250}}\right).$$

   (c) This is a $\text{Bin}(1000, 2/1000)$ probability, which is best approximated by the Poisson distribution, with mean $\mu = np = 2$. Let $X \sim \text{Bin}(1000, 2/1000)$ be the number of gold balls produced by the draws. Then we approximate $P(X = 2)$ as

   $$P(X = 2) \approx e^{-2}\frac{2^2}{2!} = 2e^{-2}.$$

5. (5 pts) Suppose that Player A and Player B take turns rolling a pair of balanced dice and that the winner is the first player who obtains the sum of 7 on a given roll of the two dice. Assume that the rolls are independent. What is the probability that Player A wins the game if he rolls first?

The probability that one of the players gets a 7 on any given roll is $\frac{6}{36} = \frac{1}{6}$ since there are 6 outcomes that lead to a sum of 7 ($(1,6),(2,5),(3,4),(4,3),(5,2)$ and $(6,1)$) and 36 is the total number of possible outcomes.

Let W be the event that Player A wins. Then $W = \cup_{n=1}^{\infty} W_n$, where W_n is the event that Player A wins at the nth roll of the dice (we count the total number of rolls of both Player A and B). First note that Player A can only win at odd times, since they only rolls the dice at odd times. So $\mathbb{P}(W_{2n}) = 0$ for all n. Player A wins at time $2n+1$ if the sum of the two dice is always different from 7 until time $2n$ and it is 7 at roll $2n+1$. So $\mathbb{P}(W_{2n+1}) = \left(\frac{5}{6}\right)^{2n} \frac{1}{6}$. Using the infinite sum rule this gives

$$\begin{aligned}
\mathbb{P}(W) = \mathbb{P}(\cup_{n=1}^{\infty} W_n) &= \sum_{n=1}^{\infty} \mathbb{P}(W_n) = \sum_{n=0}^{\infty} \mathbb{P}(W_{2n+1}) \\
&= \sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^{2n} \frac{1}{6} = \frac{1}{6} \frac{1}{1 - \left(\frac{5}{6}\right)^2} = \frac{1}{6 - \frac{25}{6}}.
\end{aligned}$$