Stat 134 lec 32

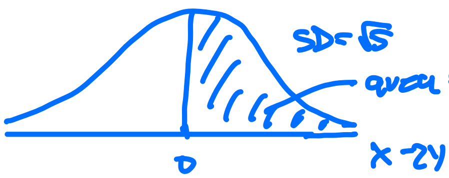
Warmup 11:00–11:10

Let $X, Y \overset{iid}{\sim} N(0,1)$

Find $P(X > 2Y) = \boxed{1/2}$

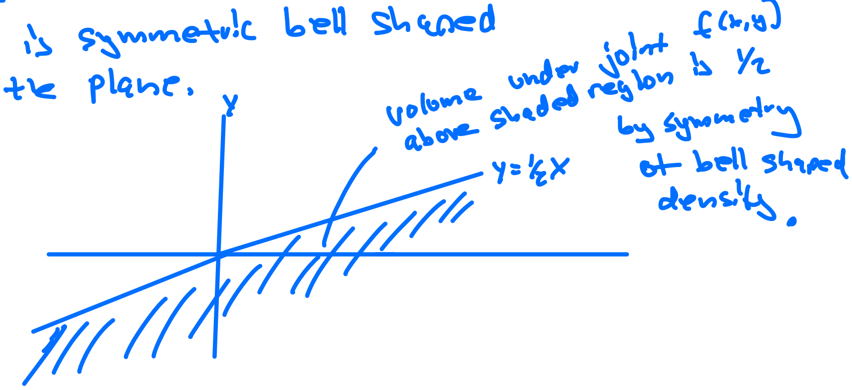
$= P(X - 2Y > 0)$

$X - 2Y \sim N(0, 5)$




Alternatively, $X < 2Y$ is the shaded region in the plane below.

$f(x,y)$ is symmetric bell shaped lying over the plane.



## Last time

### Sec 5.3

A linear combination of independent normals is normal.

**Thm** Let $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ } indep.

then $aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

**Note** In Chapter 6 we will generalize this result and show that $aX_1 + bX_2$ is normal iff $(X_1, X_2)$ are bivariate normal

### Sec 5.4 Convolution formula for density of sum

eg Let $X$ and $Y$ be discrete RVs

$$P(X+Y=z) = \sum_{\text{all } x \in X} P(X=x, Y=z-x)$$

eg $X, Y \overset{iid}{\sim} \text{Geom}(1/4)$ on $1, 2, 3, \ldots$

$X \sim \text{Geom}(p)$ on $1, 2, \ldots$

$P(X=n) = q^{n-1}p$

$$P(X+Y=4) = P(1,3) + P(2,2) + P(3,1)$$
$$= P(1)P(3) + P(2)P(2) + P(3)P(1)$$
$$= \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2\cdot\frac{1}{4} + \left(\frac{3}{4}\cdot\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2\frac{1}{4}\cdot\frac{1}{4}$$

Recall $Z \sim \text{Beta}(r, s) \Rightarrow f_Z(z) \propto z^{r-1}(1-z)^{s-1}$

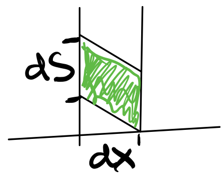
variable part

### Sec 5.4

1. Convolution formula for the density of $X+Y$
2. triangular density
3. Uniform spacing (see #13 p 355)

# (1) Sec 5.4 The Density Convolution Formula

A little geometry:




Area of parallelogram
$A = dx\,dS$

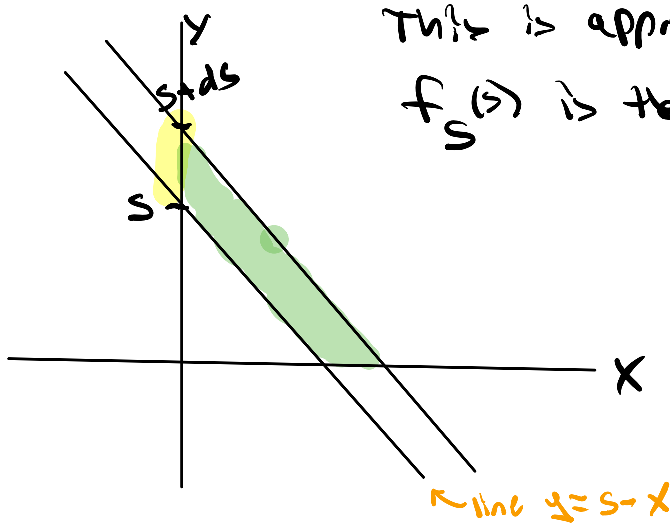
Let $X \geq 0$, $Y \geq 0$ be continuous RVs with joint density $f(x,y)$.

Let $S = X + Y$

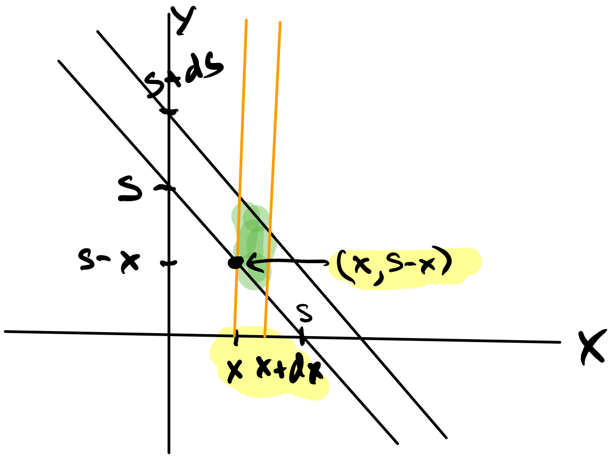
Find the density of $S$

$s = x + y$
$y = s - x$ ← y intercept

$P(S \in ds)$ is the volume under $f(x,y)$ over the green region. This is approx $f_S(s)\,ds$ where $f_S(s)$ is the density of $S$.



$P(X\in dX, S\in dS)$ is the volume under $f(x,y)$ over the green region.




$$\underbrace{P(S\in ds)}_{\substack{\approx \\ f_S(s)ds}} = \int_{x=0}^{x=s} \overbrace{P(X\in dx, S\in dS)}^{f(x,s-x)\,dx\,ds \;\approx}$$

$$= \underbrace{\int_{x=0}^{x=s} f(x,s-x)\,dx}_{f_S(s)}\,ds$$

$$\Rightarrow f_S(s) = \int_{x=0}^{x=s} f(x,s-x)\,dx$$

convolution formula for densities.

Compare with:

$$P(S=s) = \sum_{x=0}^{s} P(x, s-x)$$

convolution formula for p.m.f

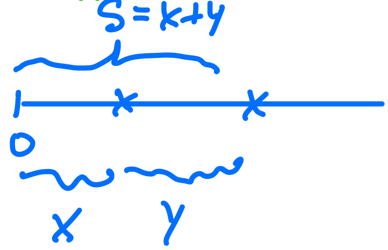
Ex $X, Y \overset{iid}{\sim} \text{expon}(\lambda)$ $\quad S = X+Y$

$X \sim \exp(\lambda)$

$f_X(x) = \lambda e^{-\lambda x}$

$$f_S(s) = \int_0^s f(x, s-x)\,dx$$

fixed




$$= \int_0^s f_X(x)\, f_Y(s-x)\,dx$$

$$= \int_0^s \lambda e^{-\lambda x} \lambda e^{-\lambda(s-x)}\,dx$$

$$= \lambda^2 e^{-\lambda s} \int_0^s dx = \lambda^2 e^{-\lambda x} x \Big|_0^s = \boxed{\lambda^2 s e^{-\lambda s}}$$

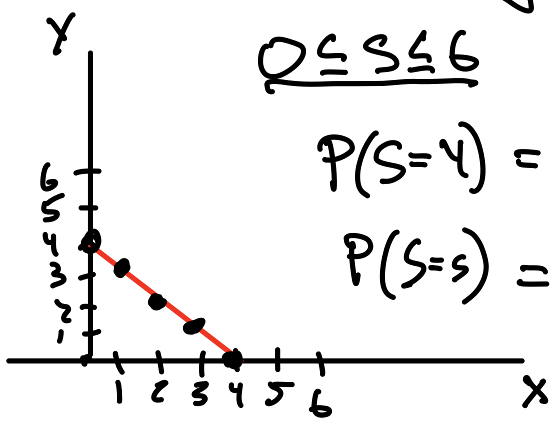
density of gamma$(2, \lambda)$

$$\Rightarrow S \sim \text{gamma}(2, \lambda)$$

② Sec 5.4 Triangular density

Let $X \sim \text{Unif}\{0,1,2,\dots,6\}$, $Y \sim \text{Unif}\{0,1,2,\dots,6\}$ } indep.

Find probability mass function of $S = X + Y$



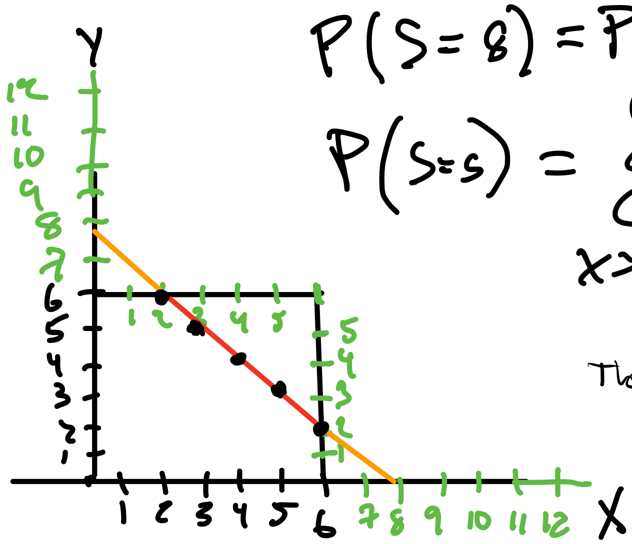
$\underline{0 \le S \le 6}$

$$P(S=4) = \underbrace{P(0,4)}_{=\left(\frac{1}{7}\right)^2} + \underbrace{P(1,3)}_{=\left(\frac{1}{7}\right)^2} + P(2,2) + P(3,1) + P(4,0)$$

$= 5/49$

$$P(S=s) = \sum_{x=0}^{s} P(X=x, Y=s-x)$$

$\underline{7 \le S \le 12}$

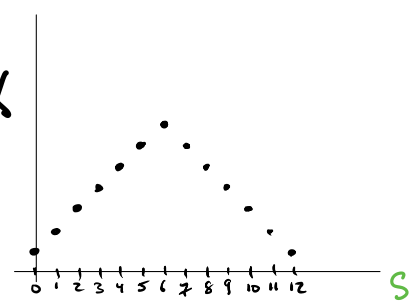


$$P(S=8) = P(2,6) + P(3,5) + \cdots + P(6,2)$$

$= 5/49$

$$P(S=s) = \sum_{x=s-6}^{6} P(x, s-x)$$

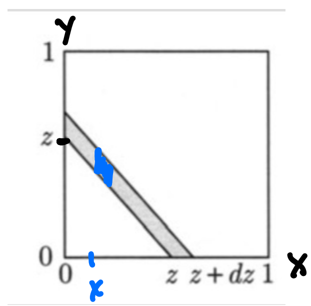
The distribution of $S = X + Y$ looks like



## Continuous case:

$$\left.\begin{array}{l} X \sim U(0,1) \\ Y \sim U(0,1) \end{array}\right\} \text{indep}$$

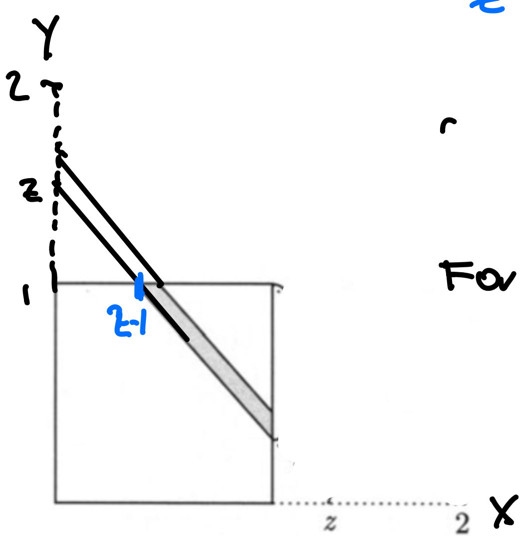
Find density of $Z = X + Y$




For $0 < z < 1$

$$P(Z \in dz) = \int_{x=0}^{x=z} P(X \in dx, Z \in dz) \approx \int_{x=0}^{x=z} f(x, z-x)\,dx\,dz$$

$$\Rightarrow f_Z(z) = \int_{x=0}^{x=z} \underbrace{f(x, z-x)}_{= f_X(x) f_Y(z-x)}\,dx = \int_0^z 1\,dx = \boxed{z}$$

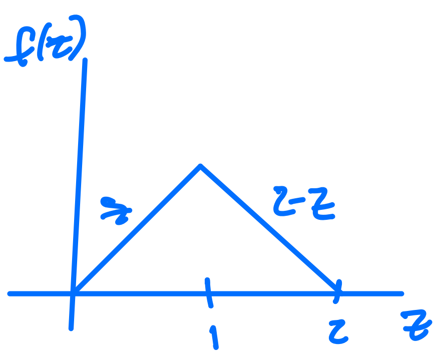



For $1 < z < 2$

$$P(Z \in dz) = \int_{x=z-1}^{x=1} f(x, z-x)\,dx\,dz$$

$$f_Z(z) = \int_{x=z-1}^{x=1} f_X(x) f_Y(z-x)\,dx = \int_{z-1}^{1} 1\,dx$$

$$= 1 - (z-1) = \boxed{2-z}$$



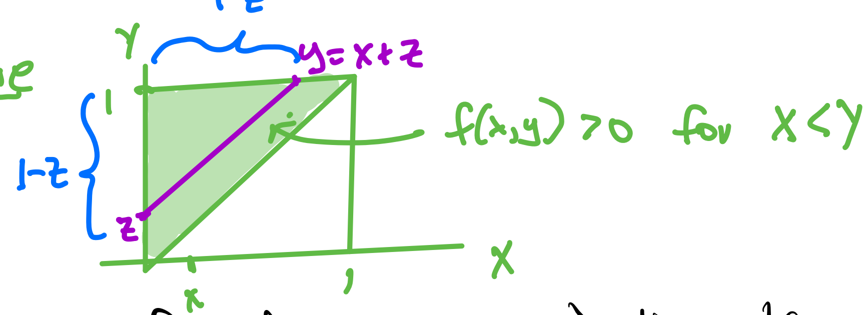
## ③ Uniform Spacing

ex: Let $X \sim U_{(7)}$, $Y \sim U_{(9)}$ for 10 iid $U(0,1)$.

The joint density $f_{X,Y}(x,y) = \binom{10}{6,1,1,1,1} x^6 (y-x)(1-y)$ for $0<x<y<1$.

Let $Z = Y - X$

Picture




For a fixed z, what is the largest value of X ?

$$\underset{\text{fixed}}{f(z)} = \int_{x=0}^{x=1-z} f(x, x+z)\,dx$$

what goes here? — remember that this must be a function of z since z is fixed.

$$f_Z(z) = \int_{x=0}^{x=1-z} f(x, x+z)\,dx$$

convolution formula