Stat 134: Section

Brett Kolesnik

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Conceptual Review

Please discuss these short questions with those around you in section. These problems are intended to highlight concepts that will be relevant for today's problems.

a. For a normal variable with mean μ and variance σ^2, verify that its mean and variance are μ and σ^2.

b. What are the mean and variance of the sum of independent variables?

c. What is the Chi-squared variable? What is its degree of freedom?

Problem 1

Let X, Y be independent normal variables, X with mean 0 and variance 1, Y with mean 1. Suppose $P(X > Y) = 1/3$. Find the standard deviation of Y.
Ex 5.3.5 in Pitman's Probability

Note that $X - Y$ is also a normal distribution with mean -1 and variance $1 + \sigma^2$, where σ^2 is the variance of Y.

$$P(X > Y) = \frac{1}{3} \implies P(X - Y - (-1) > 1) = \frac{1}{3} \tag{1}$$

Looking up the standard normal table, one finds that the corresponding Z-score is about 0.43. Hence, we conclude that $0.43 * \sqrt{1+\sigma^2} = 1$. Thus, $\sigma \approx 2.1$.

Problem 2

Let X, Y be independent standard normals. Find:
(a) $P(|\min(X,Y)| < 1)$; (b) $P(\min(X,Y) > \max(X,Y) - 1)$.
Ex 5.3.6 in Pitman's Probability

For part (a), $|\min(X,Y)| < 1$ implies $\min(X,Y) \in (-1,1)$. Hence, we can compute:

$$P(|\min(X,Y)| < 1) = P(\min(X,Y) > -1) - P(\min(X,Y) > 1).$$

We can compute both term using the normal table to be 0.84134^2 and 0.15866^2, so the result is about 0.69.

For part (b), note that $P(\min(X,Y) > \max(X,Y) - 1) = P(|X - Y| < 1)$, and $X - Y$ is a normal variable with mean 0 and variance 2. Then, $P(|X-Y| < 1) = P(|X-Y|/\sqrt{2} < 1/\sqrt{2})$ is about 0.76.

Problem 3

If X has normal $(0, \sigma^2)$ distribution, then X^2 has gamma $\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right)$.
Ex 5.3.15 in Pitman's Probability

First we consider the case of $\sigma^2 = 1$. Then, we may derive the PDF of X^2 by:

$$f_{X^2}(x) = \frac{dF_{X^2}(x)}{dx} = \frac{d(F_X(\sqrt{x}) - F_X(-\sqrt{x}))}{dx}$$

We can derive that $f_{X^2}(x)$ is proportional to $x^{-1/2}e^{-x/2}$, and since X^2 shares the same domain of the Gamma(1/2,1/2) distribution, whose pdf is also proportional to $x^{-1/2}e^{-x/2}$, we conclude that X^2 has a Gamma(1/2,1/2) distribution. Extending this by choosing $\sigma^2 X^2$ and showing a similar result yields the final answer.