Stat 134 lec 34

Quiz 6 Wednesday Sec 5.3 – 6.2

Last time

Sec 6.2 X discrete

Conditional expectation

$$E(T \mid S=s) = \sum_{t \in T} t \cdot P(T \mid S=s)$$

$E(T \mid S)$ is a RV (function of S)

Law of iterated expectation

$$E(T) = E(E(T \mid S))$$

$$E(T) = \sum_{\text{all } s} E(T \mid S=s) \cdot P(S=s)$$

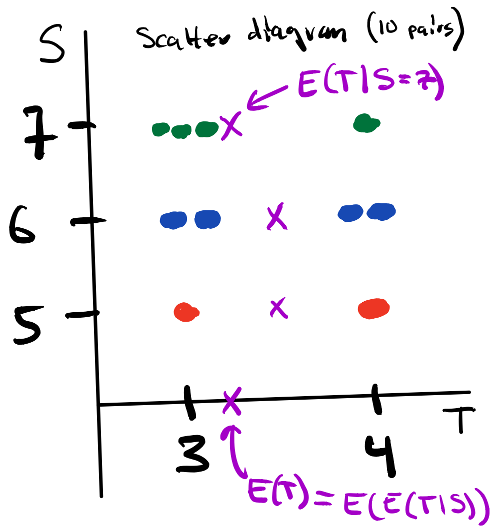
ex

Picture

joint distribution

	T=3	T=4	Sum
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1

$\longleftrightarrow$



Today

1. Sec 6.2 Law of iterated expectation
2. Sec 6.2 Properties of conditional expectation.
3. Sec 6.3
   a) Conditional densities.
   b) multiplication rule
   c) rule of average conditional probabilities

tinyurl: http://tinyurl.com/april19-pt1

http://tinyurl.com/april19-pt2

Stat 134

Friday April 19 2019

1. Let N have a Poisson(μ) distribution. Suppose that given $N = n$, random variable X follows a Binomial(n,p) distribution. E(X) is:

   a np

   b μ

   c $n\mu$

   d none of the above

$N \sim Pois(\mu)$

$X|N=n \sim Bin(n,p)$

$E(X|N=n) = np$

$E(X|N) = Np$

$E(X) = E(E(X|N)) = E(Np) = \underbrace{E(N)}_{\mu}p = \boxed{\mu p}$

2. Let N have a Geometric(p) distribution on $1, 2, 3 \ldots$. Suppose that given $N = n$, random variable X follows a Binomial(n,p) distribution. E(X) is:

a 1

b $1/p$

c p

d none of the above

$N \sim \text{Geom}(p)$

$X \mid N=n \sim \text{Bin}(n,p)$

$E(X \mid N=n) = np$

$E(X \mid N) = Np$

$E(X) = E(E(X \mid N)) = \underbrace{E(N)}_{=\frac{1}{p}}\, p = 1$

② <u>Sec 6.2</u> <u>Properties of conditional expectation</u>

$(Y+Z) \mid X=x \;=\; Y \mid X=x \;+\; Z \mid X=x$ so

$$E(Y+Z \mid X=x) = E(Y \mid X=x) + E(Z \mid X=x)$$

What is $E(X+Z \mid X=5) = ?$

$5 + E(Z \mid X=5)$

$$E(X+Y \mid X) = \underbrace{E(X \mid X)}_{X} + E(Y \mid X)$$

Properties

1. $E(X) = E(E(X \mid Y))$ — equality of numbers
2. $E(aY+b \mid X) = a\,E(Y \mid X) + b$
3. $E(Y+Z \mid X) = E(Y \mid X) + E(Z \mid X)$
4. $E(g(X) \mid X) = g(X)$
5. $E(g(X)Y \mid X) = g(X)\,E(Y \mid X)$

equality of RV

③

ⓐ

sec 6.3 Conditional density:

Let A be an event.

If X is a discrete RV

$$P(A \mid X=x) = \frac{P(A, X=x)}{P(X=x)}$$ by Bayes' rule.

If X is continuous,

$$P(A \mid X \in dx) = \frac{P(A, X \in dx)}{P(X \in dx)}$$

we define

$$P(A \mid X=x) = \lim_{dx \to 0} P(A \mid X \in dx)$$

If Y is continuous

$$P(Y \in dy \mid X=x) = \lim_{dx \to 0} \frac{P(Y \in dy, X \in dx)}{P(X \in dx)}$$

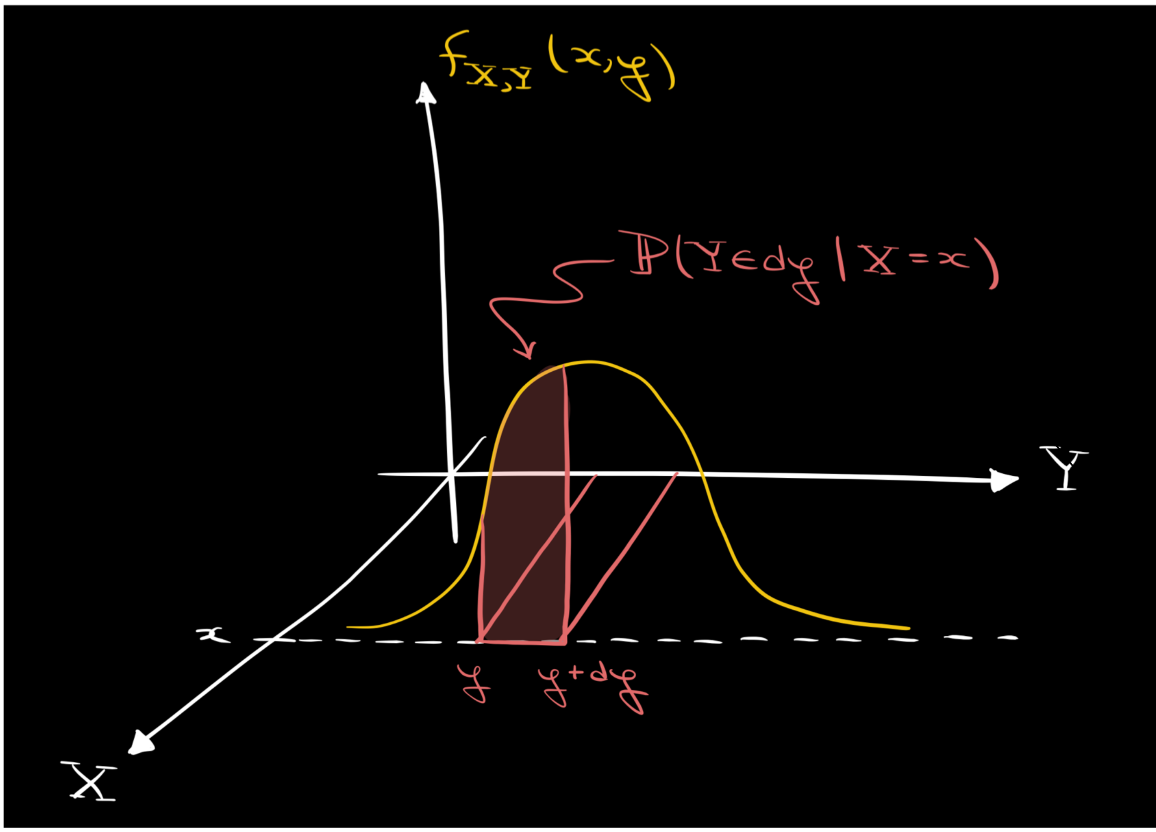
$$\parallel$$

$$f_{Y \mid X=x}(y)\, d\cancel{y} = \lim_{dx \to 0} \frac{f_{X,Y}(x,y)\, \cancel{dx}\, \cancel{dy}}{f_X(x)\, \cancel{dx}}$$

$$\Rightarrow \boxed{f_{Y \mid X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)}}$$

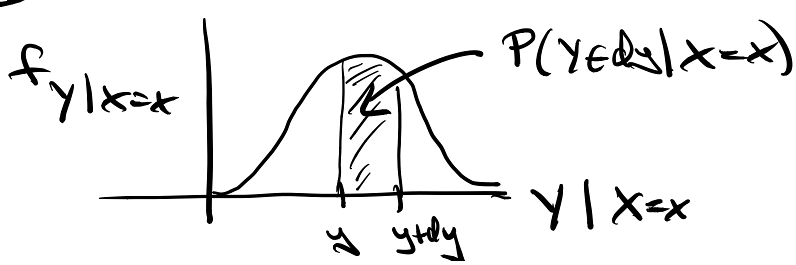
conditional density of Y given $X=x$

constant (not a function of y).



$f_{Y|X=x}(y)$ consists of a slice of $f(x,y)$ through $X=x$.

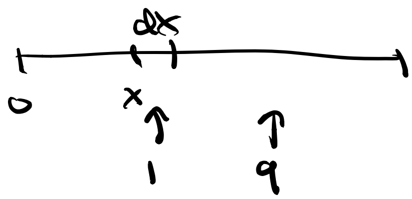
I will draw the distribution of $Y|X=x$ as



Ex Let $U_1, \ldots, U_{10} \overset{iid}{\sim} U(0,1)$

$X = U_{(1)}$, $Y = U_{(10)}$

$f(x,y) = 90\,(y-x)^8$

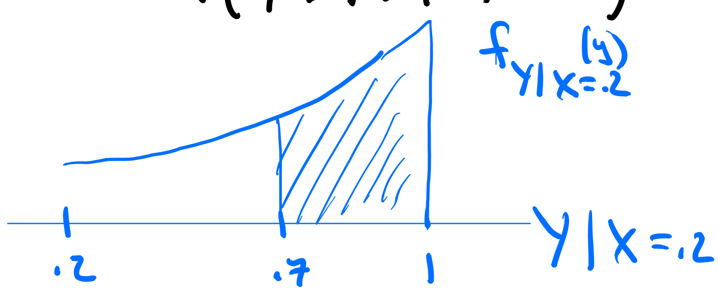


$$f_X(x) = \binom{10}{1,9}(1-x)^9$$
$$= 10\,(1-x)^9$$

a) Find $f_{Y|X=.2}(y)$

$$f_{Y|X=.2}(y) = \frac{f(.2,y)}{f_X(.2)} = \boxed{\frac{90\,(y-.2)^8}{10\,(.8)^9}}$$

b) Find $P(Y > .7 \mid X = .2) =$



$$\frac{9}{(.8)^9}\int_{.7}^{1}(y-.2)^8\,dy$$

let $u = y - .2$

$$= \frac{9}{(.8)^9}\int_{u=.5}^{u=.8} u^8\,du$$

$$= \frac{\cancel{9}}{(.8)^9}\frac{u^9}{\cancel{9}}\Big|_{.5}^{.8} = \boxed{1 - \left(\frac{.5}{.8}\right)^9}$$

Check

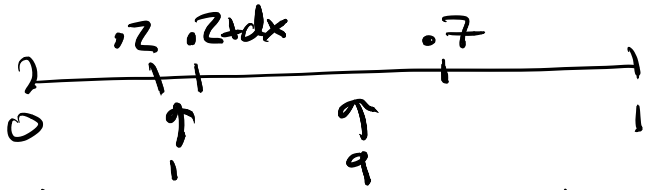
Method 2 (use fact that $X = U_{(1)}$ and $Y = U_{(10)}$)

$P(Y > .7 \mid X = .2) = 1 - P(Y < .7 \mid X = 0.2)$

By Baye's rule,

$$P(Y < .7 \mid X = 0.2) = \frac{P(Y < .7,\ X = .2)}{P(X = .2)}$$

Picture



$P(Y < .7, X = .2)$ is the chance that the remaining 9 darts land between .2 and .7. This is $(.7 - .2)^9 = (.5)^9$.

$P(X = .2)$ is the chance that the remaining 9 darts land between .2 and 1. This is $(1 - .2)^9 = (.8)^9$

Hence,

$$P(Y > .7 \mid X = .2) = \boxed{1 - \left(\frac{.5}{.8}\right)^9}.$$

(b) Multiplication rule

discrete case :

$$P(X=x, Y=y) = P(Y=y|X=x) \cdot P(X=x)$$

continuous case :

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} \Rightarrow \boxed{f(x,y) = f_{Y|X=x}(y)\, f_X(x)}$$

ex $X \sim \text{Gamma}(2,\lambda)$

$Y|X=x \sim \text{Unif}(0,x)$

Find $f(x,y)$

soln

$$f(x,y) = f_{Y|X=x}(y)\, f_X(x)$$

where

$$f_{Y|X=x}(y) = \frac{1}{x}, \quad 0<y<x<\infty$$

$$f_X(x) = \lambda^2 x e^{-\lambda x}$$

$$\Rightarrow f(x,y) = \frac{1}{x}\lambda^2 x e^{-\lambda x} = \boxed{\lambda^2 e^{-\lambda x}}$$

(c) The rule of average conditional probabilities

Let A be an event,

discrete case (X discrete):
$$P(A) = \sum_{x \in X} P(A \mid X=x)\, P(X=x)$$

continuous case (X continuous):

$$\boxed{P(A) = \int_{x \in X} P(A \mid X=x)\, f_X(x)\, dx}$$

integral conditioning formula

ex $X \sim \text{Unif}(0,1)$

given $X=x$, let $I_1, I_2 \overset{iid}{\sim} \text{Ber}(x)$

Let A be the event that the first toss is heads ($I_1 = 1$).

CAUTION X is continuous and I_1 is discrete.

We write $P(I_1 \mid X=x)$ for conditional probability mass function (pmf) of I_1 and $f_{X \mid I_1 = 1}(x)$ for the conditional density of X.

a) Find $P(I_1=1 \mid X=x) \sim x$

this is called a likelihood in Bayesian statistics. We dont need Bayes rule to find this.

b) Find $P(I_1=1)$

A is event $I_1=1$

$$P(A) = \int_{x=0}^{x=1} \overset{x}{\overset{\parallel}{P(A \mid X=x)}} \cdot \overset{1}{\overset{\parallel}{f_X(x)}}\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

interpretation: with no prior info about what is the chance of getting heads, $P(A)=1/2$.

c) Find $f_{X|I_1=1}(x)$ ← this is called a posterior in Bayesian statistics.

$$f_{X|I_1=1}(x)\, P(I_1=1) = f_{X,I_1}(x,1) \quad \text{mult rule}$$

$$P(I_1=1|X=x) f(X=x) = f_{X,I_1}(x,1) \quad \text{mult rule}$$

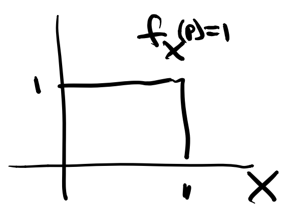
so

$$\underbrace{f_{X|I_1=1}(x)}_{\text{posterior}}\, P(I_1=1) = \underbrace{P(I_1=1|X=x)}_{\text{likelihood}}\, \underbrace{f(X=x)}_{\text{prior}}$$

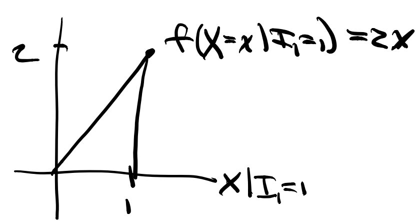
$$\Rightarrow f_{X|I_1=1}(x) = \frac{P(I_1=1|X=x)\, f(X=x)}{P(I_1=1)} \leftarrow \text{constant.}$$

$$= \frac{x \cdot 1}{1/2} = \boxed{2x}$$

Prior $X \sim \text{Unif}(0,1)$



Posterior



Interpretation: if we get a heads in 1st toss then it is more likely the prob of heads is closer to 1 than closer to 0.