# Stat 134 lec 25

Q5 next wednesday covers change of variable formula, CDF, MGF.

## Special lecture

### Moment Generating Function of X

Not in book.

Next time Finish Sec 4.5, start Sec 4.6.

### Moment Generating Function (MGF) of X

The $k^{th}$ moment of a RV X is the number

$\mu_k = E(X^k)$ defined for $k = 0, 1, 2, 3, \ldots$

$\mu_0 = E(X^0) = E(1) = 1$

$\mu_1 = E(X)$

$\mu_2 = E(X^2)$

$\vdots$

moments describe your distribution
ex $1^{st}$ moment is mean
$2^{nd}$ moment relates to variance
$3^{rd}$ moment relates to how skewed the distribution is.

Note $Var(X) = E(X^2) - E(X)^2 = \mu_2 - \mu_1^2$

Computing the moments of a RV is sometimes hard, because it involves computing complicated integrals or sums

$$E(X) = \begin{cases} \int_{-\infty}^{\infty} x f(x)\,dx & \text{if } X \text{ continuous} \\ \sum_{x=1}^{\infty} x P(x) & \text{if } X \text{ discrete} \end{cases}$$

The MGF allows one to compute the moments by computing derivatives, which is easier.

Recall the Taylor series for $e^y$:

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots$$

let $t \in \mathbb{R}$, $X$ RV.

You can do this ~~for~~ the RV $tX$

$$e^{tX} = 1 + tX + \frac{(tX)^2}{2!} + \dots$$

Take derivatives on both sides:

$$\frac{d}{dt} e^{tX} = 0 + X + \frac{2tX^2}{2!} + \frac{3t^2X^3}{3!} + \dots$$

$$\frac{d^2}{dt^2} e^{tX} = 0 + 0 + X^2 + \frac{6tX^3}{3!} + \dots$$

$\vdots$

Evaluate derivatives at $t=0$:

$$\frac{d}{dt} e^{tX}\Big|_{t=0} = \underbrace{0 + X}_{= X} + \underbrace{\frac{2tX^2}{2!}\Big|_{t=0}}_{= 0} + \underbrace{\frac{3t^2X^3}{3!}\Big|_{t=0}}_{= 0} + \dots$$

$$\frac{d^2}{dt^2} e^{tX}\Big|_{t=0} = \underbrace{0 + 0 + X^2}_{= X^2} + \underbrace{\frac{6tX^3}{3!}\Big|_{t=0}}_{= 0} + \dots$$

$\vdots$

$\rightarrow$ $$\frac{d^k}{dt^k} e^{tX}\Big|_{t=0} = X^k$$

Recall $$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

We define the MGF of $X$ to be

$$M_X(t) = E(e^{tX}) = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{if } X \text{ continuous} \\ \sum_{x=1}^{\infty} e^{tx} p(x) & \text{if } X \text{ discrete} \end{cases}$$

Call $\psi_X(t)$ in HW 9

Then $$\left.\frac{d^k}{dt^k} M_X(t)\right|_{t=0} = \left.\frac{d^k}{dt^k} E(e^{tX})\right|_{t=0}$$

Lets not worry about why this step is true. Ask me for details if interested.

$$\overset{!!!}{=} E\left(\left.\frac{d^k}{dt^k} e^{tX}\right|_{t=0}\right)$$

$$= E(X^k)$$

Note An MGF doesn't always exist in an interval around zero. (see end of lecture notes for an example).

ex $X \sim \text{Ber}(p)$ $\quad X = \begin{cases} 1 & \text{if } X=1 \\ 0 & \text{if } X=0 \end{cases}$ Prob $p$

Find $M_X(t)$

$E(X)$

$Var(X)$

$$M_X(t) = E(e^{tX})$$
$$= e^{t\cdot 0} p(0) + e^{t\cdot 1} p(1)$$
$$= 1\cdot(1-p) + e^t \cdot p$$
$$= \boxed{1 + p(e^t - 1) \text{ for all } t}$$

Next find

$M'(t) = pe^t$

$M''(t) = pe^t$

$\Rightarrow E(X) = M'(t)\big|_{t=0} = \boxed{p}$ ✓

$E(X^2) = M''(t)\big|_{t=0} = p$

$\Rightarrow Var(X) = E(X^2) - E(X)^2$
$= p - p^2 = \boxed{p(1-p)}$ ✓

ex let $X \sim \text{Gamma}(r, \lambda)$

$$f(x) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}, & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Find $M_X(t)$.

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{xt} \left( \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \right) dx$$

$$= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^{r-1} e^{(t-\lambda)x} \, dx$$

which converges for $(t-\lambda) < 0$ i.e. $\boxed{t < \lambda}$

Substitution

$$U = \underbrace{-(t-\lambda)}_{>0} x$$

so $X = -\frac{U}{t-\lambda}$, $dx = -\frac{1}{t-\lambda} dU$

$$= \frac{\lambda^r}{\Gamma(r)} \int_{U=0}^{U=\infty} \frac{U^{r-1}}{(\lambda - t)^{r-1}} e^{-U} \frac{1}{\lambda - t} \, dU$$

$$= \frac{\lambda^r}{\Gamma(r)} \frac{1}{(\lambda-t)^r} \underbrace{\int_0^\infty u^{r-1} e^{-u}\, du}_{\Gamma(r)}$$

$$= \frac{\lambda^r}{\cancel{\Gamma(r)}} \frac{\cancel{\Gamma(r)}}{(\lambda-t)^r} = \left(\frac{\lambda}{\lambda-t}\right)^r \text{ for } t < \lambda$$

What makes MGF so helpful is :

__Thm__ If a MGF exists in an interval around zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

ex

Find $E(X)$ and $Var(X)$ of $X \sim Gamma(r, \lambda)$.

Step 1. Find $M'(t)$

Soln

$$M(t) = \left(\frac{\lambda}{\lambda - t}\right)^r = \lambda^r (\lambda - t)^{-r}$$

$$M'(t) = \lambda^r (-r)(\lambda - t)^{-r-1}(-1) = \frac{r\lambda^r}{(\lambda - t)^{r+1}}$$

$$\Rightarrow M'(0) = \frac{r}{\lambda} \Rightarrow \boxed{E(X) = \frac{r}{\lambda}}$$

$$M''(t) = r\lambda^r(-r-1)(\lambda - t)^{-r-2}(-1)$$

$$= r(r+1)\lambda^r \frac{1}{(\lambda - t)^{r+2}}$$

$$\Rightarrow M''(0) = r(r+1)\lambda^r \frac{1}{\lambda^{r+2}} = \frac{r(r+1)}{\lambda^2}$$

$$\Rightarrow E(X^2) = \frac{r(r+1)}{\lambda^2}$$

$$\Rightarrow Var(X) = E(X^2) - E(X)^2 = \frac{r(r+1)}{\lambda^2} - \frac{r^2}{\lambda^2}$$

$$= \frac{r^2}{\lambda^2} + \frac{r}{\lambda^2} - \frac{r^2}{\lambda^2} = \boxed{\frac{r}{\lambda^2}}$$

ex $X \sim \text{Geom}\left(\frac{1}{3}\right)$

$$P(X=x) = \left(\frac{2}{3}\right)^{x-1}\left(\frac{1}{3}\right) \qquad x = 1, 2, 3, \ldots$$

Find $M_X(t)$.

$$M_X(t) = E\left(e^{tX}\right) = \sum_{x=1}^{\infty} e^{tx}\left(\frac{2}{3}\right)^{x-1}\left(\frac{1}{3}\right)$$

$$= \sum_{x=1}^{\infty} e^{t}\left(e^{t}\right)^{x-1}\left(\frac{2}{3}\right)^{x-1}\left(\frac{1}{3}\right)$$

$$= \frac{1}{3} e^{t} \sum_{x=1}^{\infty}\left(e^{t}\frac{2}{3}\right)^{x-1}$$

$$\sum_{x=1}^{\infty}\left(e^{t}\frac{2}{3}\right)^{x-1} = 1 + \left(e^{t}\frac{2}{3}\right) + \left(e^{t}\frac{2}{3}\right)^{2} + \cdots$$

$$= \frac{1}{1-\left(e^{t}\left(\frac{2}{3}\right)\right)} \quad \text{if } e^{t}\left(\frac{2}{3}\right) < 1$$

or

$$t < \log\left(\frac{3}{2}\right)$$

So $\boxed{M_X(t) = \frac{1}{3}e^{t} \cdot \frac{1}{1-\left(e^{t}\left(\frac{3}{3}\right)\right)} \quad \text{if } t < \log\left(\frac{3}{2}\right)}$

Appendix:

Let $X$ be a discrete RV with probability mass function

$$p(x) = \begin{cases} \frac{6}{\pi^2 x^2} & x = 1, 2, \ldots \\ 0 & \text{else} \end{cases}$$

The MGF, $M_X(t)$, only exists at $t=0$, and hence doesn't exist on an interval around zero.

pf/

It is known that the series

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ converges to $\pi^2/6$.

Then $$p(x) = \begin{cases} \frac{6}{\pi^2 x^2} & x = 1, 2, 3, \ldots \\ 0 & \text{else.} \end{cases}$$

is the prob function of a RV $X$,

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} \frac{6e^{tx}}{\pi^2 x^2}$$

The ratio test can be used to show this diverges if $t>0$. Hence this RV only has an MGF at $t=0$.

$\square$