Stat 134 lec 7

Warmup: 1:00-1:10

Suppose you and I each have a box of 600 marbles. In my box, 4 of the marbles are black, while 3 of your marbles are black. We each draw 300 marbles **with replacement** from our own boxes. **Approximately**, what is the chance you and I draw the same number of black marbles?

X = # black I draw — $p_x = \frac{4}{600}$

Y = # black you draw — $p_y = \frac{3}{600}$

Def^n Poisson(μ)

$P(k) = \frac{e^{-\mu}\mu^k}{k!}$ for $k = 0, 1, 2, \dots$

$\mu_x = 300 \cdot \frac{4}{600} = 2$

$\mu_y = 300 \cdot \frac{3}{600} = 1.5$ ← draw with replacement

Find $P(X=Y) = \sum_{k=0}^{300} P(X=k, Y=k)$ by addition rule.

$= \sum_{k=0}^{300} P(X=k)P(Y=k)$ by independence (we draw from seperate boxes)

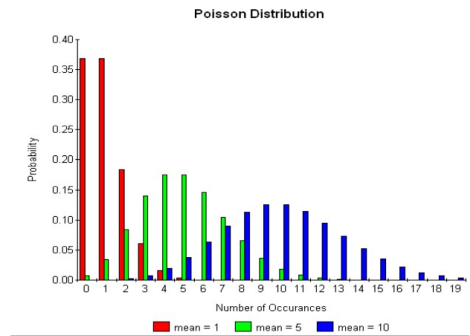
$\approx \boxed{\sum_{k=0}^{300} \frac{e^{-2}2^k}{k!} \cdot \frac{e^{-1.5}1.5^k}{k!}}$

quizz sec 2.1, 2.2, 2.4, 2.5 (fairly standard problem)

Last time

sec 2.4 Poisson Distribution

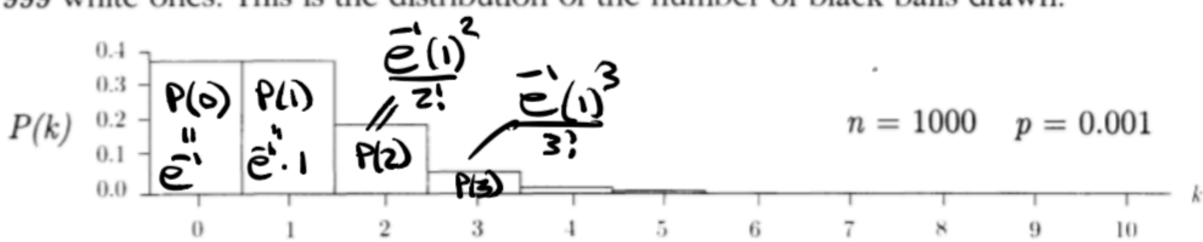
$$P(k) = \frac{e^{-\mu}\mu^{k}}{k!}, \quad k = 0, 1, 2, \dots$$



We saw that Pois(μ) is a limit of binomials for $n \to \infty$, $p \to 0$ and $np \to \mu$.

The binomial (1000, 1/1000) distribution.

Now take 1000 random draws with replacement from a box with 1 black ball and 999 white ones. This is the distribution of the number of black balls drawn:



mode of Bin(n,p):

$m = \lfloor np+p \rfloor$

$$\text{mode} = \begin{cases} m & \text{if } np+p \notin \mathbb{Z} \\ m-1, m & \text{if } np+p \in \mathbb{Z} \end{cases}$$

mode of Pois(μ):

$m = \lfloor \mu \rfloor$ since $np+p \to \mu$

$$\text{mode} = \begin{cases} m & \text{if } \mu \notin \mathbb{Z} \\ m-1, m & \text{if } \mu \in \mathbb{Z} \end{cases}$$

Today

1. sec 2.5 Random sampling

independent trials (draw w/ replacement)
- binomial distribution — 2 outcome trial
- multinomial distribution — k outcome trial

dependent trials (draw w/o replacement)
- hypergeometric distribution — 2 outcome trial
- multivariate hypergeometric distribution — k outcome trial

(1) Sec 2.5

Random sampling with replacement

ex Class 100 students

grade distribution:

A 50 students

B 30 students

C 15 students

D 5 students

You sample 10 students with replacement.

a) What is the chance you get

AAAABBBCCD ? AAABBBCCDA

$(.5)^4(.3)^3(.15)^2(.05)^1$

b) Find P(4 A's, 3B's, 2C's, 1D)

$\frac{10!}{4!3!2!1!}(.5)^4(.3)^3(.15)^2(.05)^1$

← # of Permutations of AAAABBBCCD

arrangements

$= \binom{10}{4,3,2,1} = \binom{10}{4}\binom{6}{3}\binom{3}{2}\binom{1}{1}$

$\underbrace{\binom{10}{4}(.5)^4(.5)^6}_{\text{Chance 4 As and 6 not As}} \cdot \underbrace{\qquad\qquad}_{\text{Chance 3 B's and 3 not Bs}}$

Draw 10

Find P(4 As, 6 Bs)

$$\binom{10}{4}(.5)^4(.5)^6 \cdot \binom{6}{6}(.3)^6$$

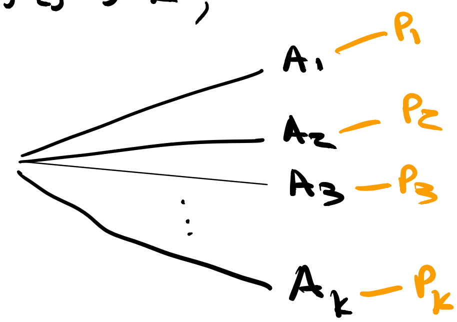
_ A _ A A A _ _ _ _ / B B _ _ _ B

P(4 As) $\qquad$ P(3 Bs)

Bin(10, .4) $\qquad$ Bin(6, .3)

Def^n Multinomial Distribution

If you have n independent trials, where each trial has K possible outcomes, $A_1, A_2, \ldots, A_K$ with probabilities $P_1, P_2, \ldots, P_K$,



then the probability you get n_1 outcome A_1, n_2 outcome A_2, ..., n_K outcome A_K is

$$P(n_1, n_2, \ldots, n_K) = \underbrace{\binom{n}{n_1, n_2, \ldots, n_K}}_{\frac{n!}{n_1!\,n_2!\cdots n_K!}} P_1^{n_1} P_2^{n_2} \cdots P_K^{n_K}$$

$n = n_1 + n_2 + \cdots + n_K$

Note: Binomial distribution is a special case with $K=2$.

independent trials (draw w/ replacement) $<$ binomial distribution — 2 outcome trial; multinomial distribution — K outcome trial random sample without replacement

ex In a very student friendly class with 100 students the grade distribution is:

A 70 students
B 30 students

You sample 5 students at random without replacement (called a simple random sample (SRS))

a) Find the chance you get

A A A B B

$$\frac{70}{100}\cdot\frac{69}{99}\frac{68}{98}\frac{30}{97}\frac{29}{96}$$

b) Find P(3A's, 2B's).

$$\binom{5}{2,3}\frac{70}{100}\cdot\frac{69}{99}\frac{68}{98}\frac{30}{97}\frac{29}{96}$$

$$\binom{5}{2,3} = \frac{5!}{2!3!}$$

(if drawn with replacement $\binom{5}{2,3}(.7)^3(.3)^2$)

$$\frac{5!}{2!3!}\frac{70}{100}\cdot\frac{69}{99}\frac{68}{98}\frac{30}{97}\frac{29}{96} = \frac{\frac{70\cdot 69\cdot 68}{3!}\cdot\frac{30\cdot 29}{2!}}{\frac{100\cdot 99\cdot 98\cdot 97\cdot 96}{5!}}$$

$$= \frac{\binom{70}{3}\binom{30}{2}}{\binom{100}{5}}$$ hypergeometric formula

Def^n hypergeometric distribution

Suppose a population of size N contains G good and B bad elements ($N=G+B$).
A sample, size n, with g good and b bad elements ($n=g+b$) is chosen at random without replacement.

$$P(g \text{ good and } b \text{ bad}) = \frac{\binom{G}{g}\binom{B}{b}}{\binom{N}{n}}$$

this generalizes to the multivariate hypergeometric distribution

Now instead of 2 types of elements we have k with sizes $G_1, G_2, \dots, G_k$ ($N=G_1+\dots+G_k$) and in our sample we have $n = g_1+\dots+g_k$.

$$P(g_1, g_2, \dots, g_k) = \frac{\binom{G_1}{g_1}\binom{G_2}{g_2}\cdots\binom{G_k}{g_k}}{\binom{N}{n}}$$

ex Class 100 students
grade distribution:

A 50 students
B 30 students
C 15 students
D 5 students

You sample 10 students with without replacement (SRS)

Find $P(4 A's, 3 B's, 2 C's, 1 D) = \dfrac{\binom{50}{4}\binom{30}{3}\binom{15}{2}\binom{5}{1}}{\binom{100}{10}}$

ex A 5 card <u>poker hand</u> consists of a SRS of 5 cards from a 52 card deck. There are $\binom{52}{5}$ poker hands.

a) Find P(poker hand has 4 aces and a king)

$$\frac{\binom{4}{4}\binom{4}{1}}{\binom{52}{5}} = \frac{\binom{4}{4}\binom{4}{1}\binom{44}{0}}{\binom{52}{5}}$$

b) Find P(poker hand has 4 aces)

$$\frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$$

c) Find P(poker hand has $\underbrace{\text{4 of a kind}}_{\text{ranks (aaaab)},\ a \neq b}$)

13 choices for a; a; not a

$$\frac{\binom{13}{1}\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$$

tinyurl.com/sept11-pt1

Stat 134

Chapter 2

1. The probability of being dealt a three of a kind poker hand (ranks $aaabc$ where $a \neq b \neq c$) is:

a $\binom{4}{3}\binom{4}{1}\binom{44}{1}/\binom{52}{5}$

b $\binom{13}{1}\binom{12}{2}\binom{4}{3}\binom{4}{1}\binom{4}{1}/\binom{52}{5}$

c $\binom{13}{1}\binom{12}{1}\binom{4}{3}\binom{4}{1}\binom{44}{1}/\binom{52}{5}$

d none of the above

Choose a rank for 3 of a kind, then choose 2 ranks for the singles.
We have $\binom{12}{2} = \frac{12 \cdot 11}{2}$ instead of $12 \cdot 11$
since aaabc = aaacb in a poker hand
so we double count.