Problem 2: Competing Exponentials

Suppose $X \sim \text{Exp}(\lambda_X)$, $Y \sim \text{Exp}(\lambda_Y)$, and X, Y are independent.

a. Find $P(X < Y)$.

b. Now suppose $\lambda_X = \lambda_Y = \lambda$. Using part (a), find the density of $Z = X/Y$. (Hint: look at the CDF of Z.)

c. By a similar process as in (b), find the density of $W = \frac{X}{X+Y}$.

a) See p.352 in Pitman. Answer: $\frac{\lambda_X}{\lambda_X + \lambda_Y}$.

b) Key fact: for $X \sim \text{Exp}(\lambda)$, $aX \sim \text{Exp}\left(\frac{\lambda}{a}\right)$ (for $a > 0$.)

Note $Z \in (0, \infty)$. $F_Z(z) = \mathbb{P}(Z < z) = \mathbb{P}\left(\frac{X}{Y} < z\right)$

$$= \mathbb{P}(\underbrace{X}_{\sim \text{Exp}(\lambda)} < \underbrace{zY}_{\sim \text{Exp}\left(\frac{\lambda}{z}\right)}) \overset{\text{by (a)}}{=} \frac{\lambda}{\lambda + \frac{\lambda}{z}} = \frac{1}{1 + \frac{1}{z}} = \frac{z}{z+1}.$$

$$\Rightarrow \boxed{f_Z(z) = \frac{1}{(z+1)^2}, \quad z > 0.}$$

c) Note $W \in (0,1)$. Similarly to (b), for $w \in (0,1)$,

$$\mathbb{P}(W < w) = \mathbb{P}\left(\frac{X}{X+Y} < w\right) = \mathbb{P}(X < w(X+Y))$$

$$= \mathbb{P}(\underbrace{(1-w)}_{>0} X < wY) = \frac{\frac{\lambda}{1-w}}{\frac{\lambda}{1-w} + \frac{\lambda}{w}} = \frac{\frac{1}{1-w}}{\frac{1}{1-w} + \frac{1}{w}} = \frac{1}{1 + \frac{(1-w)}{w}} \ldots$$

$$\ldots = \frac{1}{1 + \frac{1}{w} - 1} = w. \quad \boxed{\therefore\ W \sim \text{Unif}(0,1)}.$$