Stat 134 lec 39

Warmup 8:00 - 8:10 AM

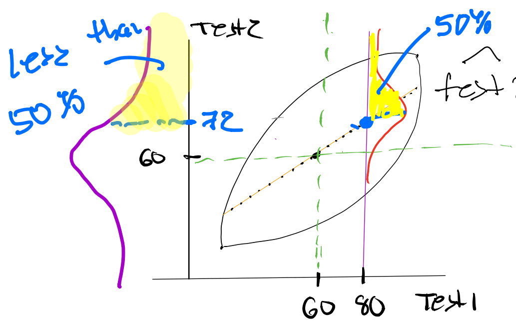
$(\text{Test 1}, \text{Test 2}) \sim BV(60, 60, 20^2, 20^2, 0.6)$

mean variance

What is greater?

a) The chance you get greater than 72 on test 2 among students who got 80 on test 1 — 50%

b) The chance you get greater than 72 on test 2 among all students. — less than 50%



$\text{test 2} = 0.6(\text{test 1}) + 24$ ← $\mu_V - \frac{\sigma_V}{\sigma_U}\rho\mu_U$ — V intercept

$60 - (.6)60 = 24$

$0.6(80) + 24 = 72$

This is why you sit next to the successful kids in class. You have a higher chance of doing well on the next test!

<u>last time</u> sec 6.5 Bivariate normal distribution.

std bivariate normal construction

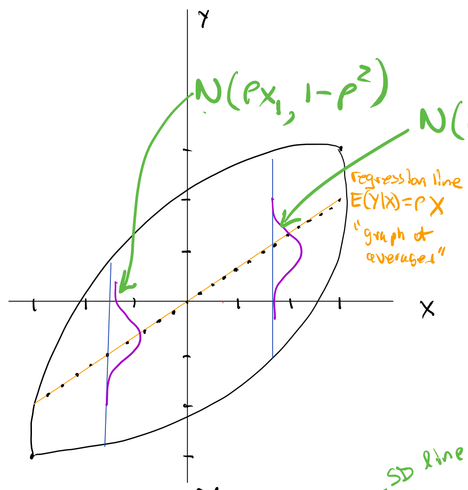
$X, Z \overset{iid}{\sim} N(0,1)$, $-1 \le \rho \le 1$

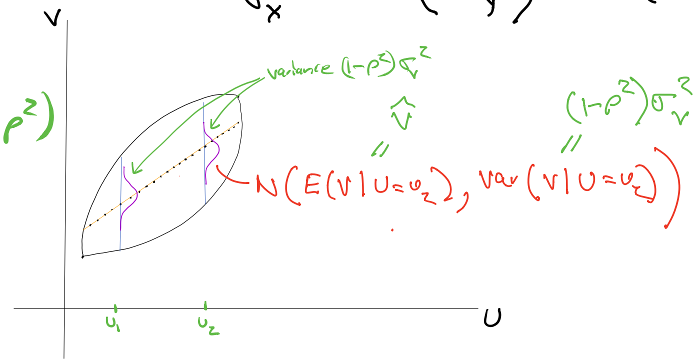
$Y = \rho X + \sqrt{1-\rho^2}\, Z \sim N(0,1)$

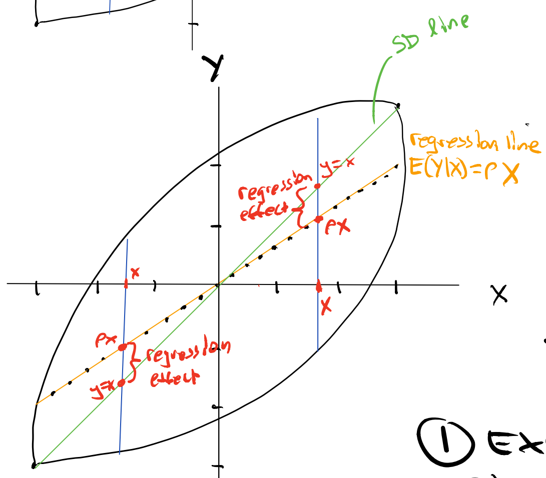
(X,Y) is std bivariate normal with $Cov(X,Y) = \rho$.

$$Y \mid X \sim N(\rho X, 1-\rho^2)$$

$$\frac{\hat{V} - \mu_x}{\sigma_x} = \rho\left(\frac{U - \mu_x}{\sigma_y}\right) \text{ regression line}$$







Picture of regression effect.

<u>Today</u>

1. Examples with Bivariate normal (Sec 6.5)
2. MGF of Bivariate normal
3. properties of Bivariate normal
4. Review of Conditional Expectation (Sec 6.4)

① Examples with bivariate normal,

tinyurl: http://tinyurl.com/dec6-pt1

Stat 134

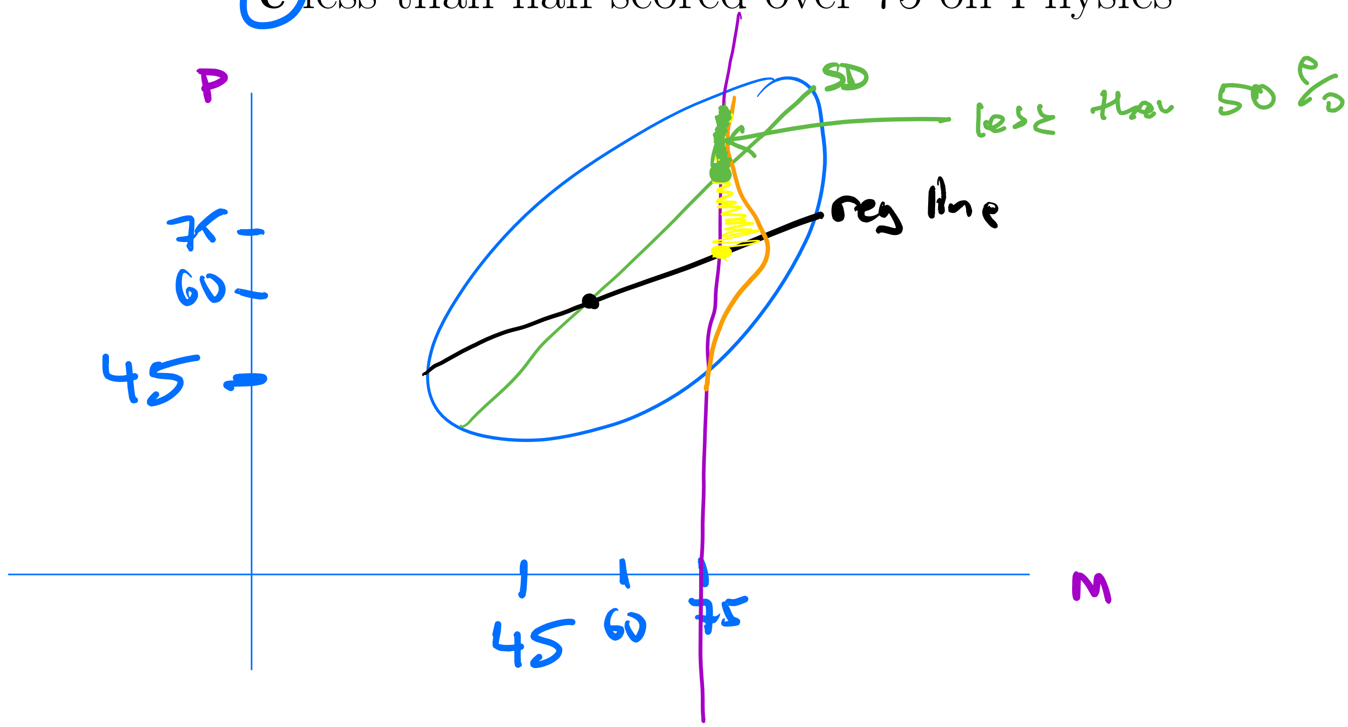
Monday April 29 2019

1. A test score in Math and Physics is bivariate normal, $\rho > 0$. The average is 60 on both tests and the SDs are the same. Of students scoring 75 on the Math test:

   a about half scored over 75 on Physics

   b more than half scored over 75 on Physics

   c less than half scored over 75 on Physics



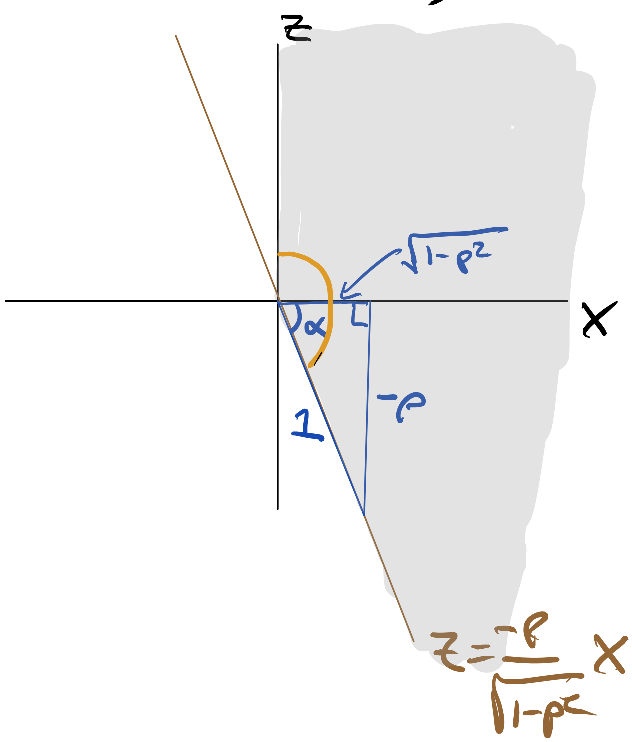
eg: X, Y std bivariate normal, $\rho > 0$

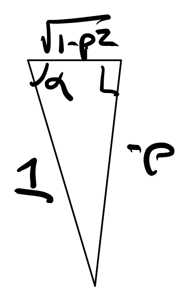
Find $P(X>0, Y>0)$

$$P(X>0, Y>0) = P\left(X>0,\ \rho X + \sqrt{1-\rho^2}\, Z > 0\right)$$

$$= P\left(X>0,\ Z > \frac{-\rho}{\sqrt{1-\rho^2}} X\right)$$

→ note X and Z are uncorrelated so joint (X, Z) is a symmetric bell over X, Z plane,





$$\tan\alpha = \frac{-\rho}{\sqrt{1-\rho^2}}$$

$$\alpha = \tan^{-1}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)$$

$$P\left(X>0,\ Z > \frac{-\rho}{\sqrt{1-\rho^2}} X\right) = \frac{90 + |\alpha|}{360} = \boxed{\frac{90 + \left|\tan^{-1}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)\right|}{360}}$$

② MGF of Bivariate normal

The single and multivariate MGF is defined as:

$$M_Y(t) = E(e^{tY})$$

$$M_{(X,Y)}(s,t) = E(e^{sX+tY})$$ multivariate MGF

Note that

$$E(e^{1 \cdot tY})$$
$$\parallel$$
$$M_Y(t) = E(e^{tY}) = M_{tY}(1)$$

$$M_{(X,Y)}(s,t) = E(e^{sX+tY}) = M_{sX+tY}(1)$$

recall,

$$Z \sim N(\mu, \sigma^2) \text{ iff } M_Z(a) = e^{\mu a} e^{\frac{\sigma^2 a^2}{2}}$$

Thm $M_{(X,Y)}(s,t) = M_X(s)\, M_Y(t)$ iff

X, Y are independent.

Pf/

$$M_{(X,Y)}(s,t) = E\left(e^{sX+tY}\right)$$

multivariate MGF

$$= E\left(e^{sX} \cdot e^{tY}\right)$$

iff X, Y indep.

$$= E\left(e^{sX}\right) E\left(S^{tY}\right) = M_X(s)\, M_Y(t)$$

$\square$

Thm Let (X,Y) be standard bivariate normal.

The MGF of (X,Y) is

$$M_{(X,Y)}(s,t) = e^{\frac{s^2}{2}+\frac{t^2}{2}+st\rho}$$

Pf/

Recall that $X \sim N(0,1)$ so $M_X(s) = e^{\frac{s^2}{2}}$

$$\begin{aligned}
M_{X,Y}(s,t) &= E\left[e^{sX+tY}\right] \\
&= E\left[e^{sX+t(\rho X+\sqrt{1-\rho^2}Z)}\right] \\
&= E\left[e^{(s+t\rho)X} \cdot e^{t\sqrt{1-\rho^2}Z}\right] \\
&\overset{\text{independence}}{=} E\left[e^{(s+t\rho)X}\right] E\left[e^{t\sqrt{1-\rho^2}Z}\right] \\
&= M_X(s+t\rho) \cdot M_Z(t\sqrt{1-\rho^2})
\end{aligned}$$

Finish proof.

$$\begin{aligned}
&= e^{\frac{(s+t\rho)^2}{2}} \cdot e^{\frac{(t\sqrt{1-\rho^2})^2}{2}} \\
&= e^{\frac{s^2}{2}+st\rho+\frac{t^2\rho^2}{2}} \cdot e^{\frac{t^2}{2}-\frac{t^2\rho^2}{2}} \\
&= \boxed{e^{s^2/2+t^2/2+st\rho}}
\end{aligned}$$

$\square$

③ Properties of Bivariate Normal

Recall that if 2 RVs X, Y are independent, $Cov(X,Y)=0$

$\Rightarrow Cov(X,Y)=0$

However the converse isn't true in general. ($Corr(X,Y)=0 \not\Rightarrow X,Y$ indep.)

Let $X \sim N(0,1)$
$Y = X^2$

X and Y are dependent

We show X, Y are uncorrelated.

$$Cov(X,Y) = \underset{\underset{E(X^3)=0}{\parallel}}{E(XY)} - \underset{\underset{0}{\parallel}}{E(X)}E(Y)$$

Find $E(X^3)$

$X \sim N(0,1)$

$M_X(t) = e^{t^2/2}$

$$M_X'''(t)\Big|_{t=0} = te^{t^2/2} + 2te^{t^2/2} + t^3 e^{t^2/2}\Big|_{t=0} = 0$$

$\Rightarrow E(X^3) = 0$

$\Rightarrow Corr(X,Y) = 0$

Thm If (X,Y) is std bivariate normal then $\rho = \mathrm{Corr}(X,Y) = 0$ iff X, Y are independent.

Pf/

$$M_{(X,Y)}(s,t) = M_{sX+tY}(1) = M_{sX}(1) \cdot M_{tY}(1) = M_X(s) M_Y(t)$$

iff X and Y are independent.

Since (X,Y) is std bivariate normal,

$$M_{(X,Y)}(s,t) = e^{s^2/2 + t^2/2 + st\rho}$$

Finish proof.

$= M_X(s) M_Y(t)$ iff $\rho = 0$

since $M_X(s) = e^{s^2/2}$

$M_Y(t) = e^{t^2/2}$

$\square$

$\square$

Recall from lec 30 that the sum of independent normal random variables is normal.

What about dependent normal random variables?

Thm Let $X, Y \sim N(0,1)$ and $\mathrm{corr}(X,Y) = \rho$.

(X,Y) is std bivariate normal iff

$sX + tY$ is normal for all constants s, t.

Pf/

$\Rightarrow$: Suppose X, Y std bivariate normal.

(i.e. $Y = \rho X + \sqrt{1-\rho^2}\, Z$ for X, Z iid $N(0,1)$)

$$sX + tY = sX + t\left(\rho X + \sqrt{1-\rho^2}\, Z\right)$$
$$= (s + t\rho)X + t\sqrt{1-\rho^2}\, Z$$

is normal since X, Z are independent normals.

$\Leftarrow$: Suppose $sX + tY$ is normal.

note $E(sX+tY) = sE(X) + tE(Y) = 0$

$$\mathrm{Var}(sX+tY) = \mathrm{Var}(sX) + \mathrm{Var}(tY) + \underbrace{2\,\mathrm{Cov}(sX, tY)}_{= 2st\,\mathrm{Cov}(X,Y) = 2st\,\mathrm{corr}(X,Y) = 2st\rho}$$
$$= s^2 + t^2 + 2st\rho$$

hence, $sX + tY \sim N(0, s^2 + t^2 + 2st\rho)$

Recall,

For $Z \sim N(\mu, \sigma^2) \Rightarrow M_Z(a) = e^{\mu a} e^{\sigma^2 \frac{a^2}{2}}$

Then $M_{(X,Y)}(s,t) = M_{sX+tY}(1) = e^{(s^2+t^2+2st\rho)\frac{1^2}{2}}$

normal

$$= e^{\frac{s^2}{2} + \frac{t^2}{2} + st\rho} \quad \leftarrow \text{MGF of } (X,Y)$$

$\Rightarrow (X,Y)$ is standard bivariate normal $\square$

We dont need to restrict ourselves to $X, Y \sim N(0,1)$

Corollary — Proof at end of lecture

Let $U \sim N(\mu_U, \sigma_U^2)$
$V \sim N(\mu_V, \sigma_V^2)$
and $\text{Corr}(U,V) = \rho$
$(U,V) \sim BV(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$ iff
$sU + tV$ is normal for all constants s, t.

ex

Let M be a students score on the midterm of a class and F the student's score on the final of the same class.

Suppose $(M, F) \sim BV(\underbrace{70}_{\mu_M}, \underbrace{65}_{\mu_F}, \underbrace{8^2}_{\sigma_M^2}, \underbrace{10^2}_{\sigma_F^2}, \underbrace{0.6}_{\rho})$.

Find the chance that the student scores higher on the final than the midterm.

Hint $P(F > M) = P(F - M > 0)$

What distribution is $F - M$? — Normal! (Since (M,F) is Bivariate Normal)

$E(F-M) = 65 - 70 = -5$

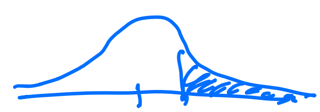
$$Var(F-M) = Var(F) + Var(M) - 2\underbrace{Cov(F,M)}_{= Corr(F,M)\,SD(M)\,SD(F)}$$

$$= 10^2 + 8^2 - 2(.6)(8)(10) = 68$$

$SD(F-M) = \sqrt{68}$

$\Rightarrow F - M \sim N(-5, 68)$

$$\Rightarrow \frac{0 - E(F-M)}{SD(F-M)} = \frac{0-(-5)}{\sqrt{68}} = \boxed{.61}$$



$P(F > M) = P(F - M > 0) = 1 - \Phi(.61) = .27$

Appendix

Corollary

Let $U \sim N(\mu_U, \sigma_U^2)$
$V \sim N(\mu_V, \sigma_V^2)$
and $\mathrm{Corr}(U,V) = \rho$
$(U,V) \sim BV(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$ iff
$sU + tV$ is normal for all constants s,t.

Pf/ let $X = \dfrac{U-\mu_U}{\sigma_U}$ $\quad (U = \mu_U + X\sigma_U)$

$Y = \dfrac{V-\mu_V}{\sigma_V}$ $\quad (V = \mu_V + Y\sigma_V)$

$(U,V) \sim BV(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$
iff $(X,Y) \sim BV(0,0,1,1,\rho)$

$$sU + tY = s(\mu_U + X\sigma_U) + t(\mu_V + Y\sigma_V)$$
$$= \underbrace{s\sigma_U}_{=a} X + \underbrace{t\sigma_V}_{=b} Y + \underbrace{sx\sigma_U + ty\sigma_V}_{\text{constant}}$$

this is normal for all constants s,t
iff $aX + bY$ is normal for all constants a,b.

$\square$

4 Review of conditional expectation Sec 6.4

Properties (from lec 34)

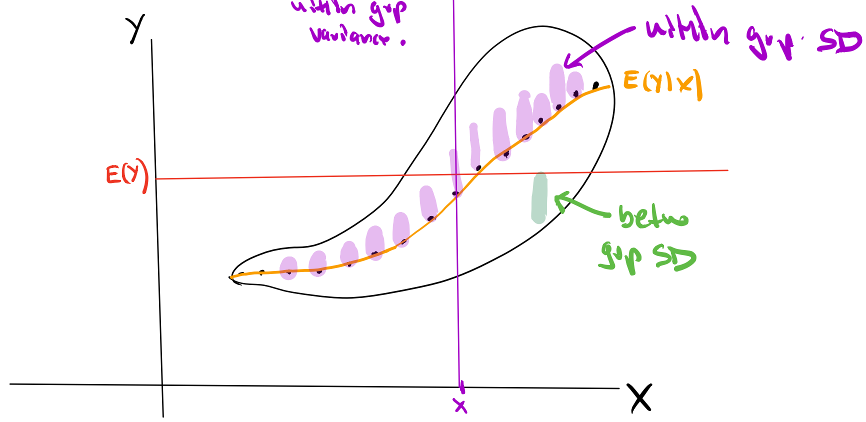
1. $E(Y) = E(E(Y|X))$ iterated expectation
2. $E(aY+b|X) = aE(Y|X)+b$
3. $E(Y+Z|X) = E(Y|X)+E(Z|X)$
4. $E(g(X)|X) = g(X)$
5. $E(g(X)Y|X) = g(X)E(Y|X)$
6. $Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$ total variance decomposition

$Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$ (sec 6.2.18)

tot var = avg($\square^2$) + $\square^2$

within grp variance.

var of $E(Y|X)$ (between group variance)



$\underline{\text{ex}}$ Let $U \sim \text{Unif}(0,1)$

Given $U=u$, X is exponential with mean u

$\underline{\text{i.e}}$ $X|U=u \sim \text{Exp}\left(\frac{1}{u}\right)$

$E(X|U) = U$

$\text{Var}(X|U) = U^2$

Find $E(X)$, $E(UX)$ and $\text{Var}(X)$.

$E(X) = E(E(X|U)) = E(U) = \frac{1}{2}$

$E(UX) = E(E(UX|U)) = E(U\underbrace{E(X|U)}_{=U})$

$= E(U^2) = \underbrace{\text{Var}(U)}_{=1/12} + \underbrace{E(U)^2}_{=(1/2)^2} = \frac{1}{3}$

$\text{Var}(X) = \underbrace{E(\underbrace{\text{Var}(X|U)}_{=U^2})}_{=1/3} + \underbrace{\text{Var}(\underbrace{E(X|U)}_{=U})}_{=1/12} = \frac{5}{12}$