

Special lecture

Moment Generating Function of X

Not in book.

Next time Finish Sec 4.5, Start Sec 4.6.

Moment Generating Function (MGF) of X

(not in book).

The k^{th} moment of a RV X is the number

$$m_k = E(X^k) \quad \text{defined for } k=0, 1, 2, 3, \dots$$

$$m_0 = E(X^0) = E(1) = 1$$

$$m_1 = E(X)$$

$$m_2 = E(X^2)$$

⋮

$$\text{Note } \text{Var}(X) = E(X^2) - E(X)^2 = m_2 - m_1^2$$

} moments describe your distribution
ex 1st moment is mean
2nd moment relates to variance
3rd moment relates to how skewed the distribution is,

Computing the moments of a RV is sometimes hard, because it involves computing complicated integrals or sums

$$E(X) = \begin{cases} \int_{-\infty}^{\infty} x f(x) dx & \text{if } X \text{ continuous} \\ \sum_{x=1}^{\infty} x P(x) & \text{if } X \text{ discrete} \end{cases}$$

The MGF allows one to compute the moments by computing derivatives, which is easier.

Recall the Taylor series for e^y :

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots$$

let $t \in \mathbb{R}$, X RV.

You can do this for the RV tX

$$e^{tX} = 1 + tX + \frac{(tX)^2}{2!} + \dots$$

Take derivatives on both sides:

$$\frac{d}{dt} e^{tX} = 0 + X + \frac{2tX^2}{2!} + \frac{3t^2X^3}{3!} + \dots$$

$$\frac{d^2}{dt^2} e^{tX} = 0 + 0 + X^2 + \frac{6tX^3}{3!} + \dots$$

Evaluate derivatives at $t=0$:

$$\left. \frac{d}{dt} e^{tX} \right|_{t=0} = 0 + X + \left. \frac{2tX^2}{2!} \right|_{t=0} + \left. \frac{3t^2X^3}{3!} \right|_{t=0} + \dots$$

$$= X \quad 0$$

$$\left. \frac{d^2}{dt^2} e^{tX} \right|_{t=0} = 0 + 0 + X^2 + \left. \frac{6tX^3}{3!} \right|_{t=0} + \dots$$

$$= X^2 \quad 0$$

$$\rightarrow \left. \frac{d^K}{dt^K} e^{tX} \right|_{t=0} = X^K$$

Recall $E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) dx$

We define the MGF of X to be

$$M_X(t) = E(e^{tX}) = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x) dx & \text{if } X \text{ continuous} \\ \sum_{x=1}^{\infty} e^{tx} p(x) & \text{if } X \text{ discrete} \end{cases}$$

↑
Call $\Psi_X(t)$ in HW 9

Then $\left. \frac{d}{dt} M_X(t) \right|_{t=0} = \left. \frac{d}{dt} E(e^{tX}) \right|_{t=0}$

Let's not worry about why this step is true. Ask me for details if interested.

$$\begin{aligned} &= E\left(\left. \frac{d}{dt} e^{tX} \right|_{t=0}\right) \\ &= E(X^k) \end{aligned}$$

Note An MGF doesn't always exist in an interval around zero. (see end of lecture notes for an example).

Let $X \sim \text{Ber}(p)$ $X = \begin{cases} 1 & \text{if } X=1 \\ 0 & \text{if } X=0 \end{cases}$ Prob P

Find $M_X(t)$

$E(X)$

$\text{Var}(X)$

$$M_X(t) = E(e^{tX})$$

$$= e^{t \cdot 0} p(0) + e^{t \cdot 1} p(1)$$

$$= 1 \cdot (1-p) + e^t \cdot p$$

$$= \boxed{1 + p(e^t - 1)}$$

for all t

Next find

$$M'(t) = pe^t$$

$$M''(t) = pe^t$$

$$\Rightarrow E(X) = M'(t) \Big|_{t=0} = \boxed{p} \quad \checkmark$$

$$E(X^2) = M''(t) \Big|_{t=0} = p$$

$$\begin{aligned} \Rightarrow \text{Var}(X) &= E(X^2) - E(X)^2 \\ &= p - p^2 = \boxed{p(1-p)} \quad \checkmark \end{aligned}$$

let $X \sim \text{Gamma}(r, \lambda)$

$$f(x) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}, & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Find $M_X(t)$.

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{xt} \left(\frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \right) dx$$

$$= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^r e^{(t-\lambda)x} dx$$

which

converges
for

$$(t-\lambda) < 0$$

i.e. $t < \lambda$

Substitution

$$U = -(t-\lambda)x$$

> 0

$$\text{so } x = -\frac{U}{t-\lambda}, \quad dx = -\frac{1}{t-\lambda} du$$

$$= \frac{\lambda^r}{\Gamma(r)} \int_{U=0}^{U=\infty} \frac{U^{r-1}}{(\lambda-t)^{r-1}} e^{-U} \frac{1}{\lambda-t} du$$

$$\begin{aligned}
 &= \frac{\lambda^n}{\Gamma(n)} \frac{1}{(\lambda-t)^n} \int_0^\infty u^n e^{-\lambda u} du \\
 &= \frac{\lambda^n}{\Gamma(n)} \frac{P(n)}{(\lambda-t)^n} = \left(\frac{\lambda}{\lambda-t} \right)^n \quad \text{for } t < \lambda
 \end{aligned}$$

What makes MGF so helpful is :

Thm If a MGF exists in an interval

around zero, $M^{(x)}(t) \Big|_{t=0} = E(x^t)$

ex

Find $E(x)$ and $\text{Var}(x)$ if $X \sim \text{Gamma}(n, \lambda)$.

Step 1. Find $M'(t)$

Soln

$$M(t) = \left(\frac{\lambda}{\lambda-t}\right)^r = \lambda^r (\lambda-t)^{-r}$$

$$M'(t) = \lambda^r (-r)(\lambda-t)^{-r-1}(-1) = r \frac{\lambda^r}{(\lambda-t)^{r+1}}$$

$$\Rightarrow M'(0) = \frac{r}{\lambda} \Rightarrow \boxed{E(X) = \frac{r}{\lambda}}$$

$$M''(t) = r \lambda^r (-r-1) (\lambda-t)^{-r-2} (-1)$$

$$= r(r+1) \lambda^r \frac{1}{(\lambda-t)^{r+2}}$$

$$\Rightarrow M''(0) = r(r+1) \lambda^r \frac{1}{\lambda^{r+2}} = \frac{r(r+1)}{\lambda^2}$$

$$\Rightarrow E(X^2) = \frac{r(r+1)}{\lambda^2}$$

$$\Rightarrow \text{Var}(X) = E(X^2) - E(X)^2 = \frac{r(r+1)}{\lambda^2} - \frac{r^2}{\lambda^2}$$

$$= \frac{r^2}{\lambda^2} + \frac{r}{\lambda^2} - \frac{r^2}{\lambda^2} = \boxed{\frac{r}{\lambda^2}}$$

$\text{Ex } X \sim \text{Geom}\left(\frac{1}{3}\right)$

$$P(X=k) = \left(\frac{2}{3}\right)^{k-1} \left(\frac{1}{3}\right) \quad k=1, 2, 3, \dots$$

Find $M_X(t)$.

$$\begin{aligned} M_X(t) &= E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} \left(\frac{2}{3}\right)^{x-1} \left(\frac{1}{3}\right) \\ &= \sum_{t=1}^{\infty} e^t \left(e^t\left(\frac{2}{3}\right)\right)^{t-1} \left(\frac{1}{3}\right) \\ &= \frac{1}{e^t} e^t \sum_{x=1}^{\infty} \left(e^t \frac{2}{3}\right)^{x-1} \end{aligned}$$

$$\begin{aligned} \sum_{x=1}^{\infty} \left(e^t \frac{2}{3}\right)^{x-1} &= 1 + \left(e^t \frac{2}{3}\right) + \left(e^t \frac{2}{3}\right)^2 + \dots \\ &= \frac{1}{1 - \left(e^t \frac{2}{3}\right)} \quad \text{if } e^t \left(\frac{2}{3}\right) < 1 \\ &\quad \text{or} \end{aligned}$$

$$t < \log\left(\frac{3}{2}\right)$$

So

$$M_X(t) = \frac{1}{e^t} e^t \cdot \frac{1}{1 - \left(e^t \frac{2}{3}\right)} \quad \text{if } t < \log\left(\frac{3}{2}\right)$$

Appendix:

Claim the RV X with prob function,

$$P(x) = \begin{cases} \frac{6}{\pi^2 x^2} & x=1, 2, \dots \\ 0 & \text{else} \end{cases}$$

only exists at $x=1$, hence doesn't exist on an interval around zero.

Pf/

It is known that the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \text{ converges to } \frac{\pi^2}{6}.$$

Then $P(x) = \begin{cases} \frac{6}{\pi^2 x^2} & x=1, 2, 3, \dots \\ 0 & \text{else.} \end{cases}$

is the prob function of a RV X ,

$$M_X(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} P(x)$$

$$= \sum \frac{6e^{tx}}{\pi^2 x^2}$$

The ratio test can be used to show that it diverges if $t > 0$. Hence this RV only has an MGF at $x=0$.

□