Stat 134 lec 10

Warmup 11:00 – 11:10

(X_1, X_2) has joint distribution:

	$X_2=0$	$X_2=1$
$X_1=0$	$\frac{5}{36}$	$\frac{25}{36}$
$X_1=1$	$\frac{1}{36}$	$\frac{5}{36}$

Is X_1, X_2 independent?

Last time

Sec 3.1 Random Variables

The event $(X=x, Y=y)$ is the intersection of events $X=x$ and $Y=y$. sometimes written (x,y)

The probability X and Y satisfies some condition (i.e $P(X+Y=s)$) is the sum of $P(x,y)$ that satisfy that condition.

$$\text{ex } P(X+Y=s) = \sum_{(x,y):x+y=s} P(x,y) = \sum_{\text{all } x} P(x, s-x)$$

Independence of (X,Y,Z) means

$P(X=x, Y=y, Z=z) = P(X=x)P(Y=y)P(Z=z)$ for all $x \in X$, $y \in Y$, $z \in Z$.

Today

1. Sec 3.1 Sums of independent Poissons is Poisson
2. Sec 3.2 Expectation of a RV.

① Sum of independent Poisson is Poisson

informal arguement:

$X_1 \sim Bin(1000, \frac{1}{1000})$ $\approx Pois(1)$

$X_2 \sim Bin(2000, \frac{1}{1000})$ $\approx Pois(2)$

indep

$X_1 + X_2 \sim ?$ $Bin(3000, \frac{1}{1000}) \approx Pois(3)$

X_1+X_2 = # heads in 1000+2000 = 3000 $p = \frac{1}{1000}$ coin tosses.

Lets prove this rigorously:

Recall binomial theorem

$$(a+b)^3 = \binom{3}{3} a^3 b^0 + \binom{3}{2} a^2 b^1 + \binom{3}{1} a^1 b^2 + \binom{3}{0} a^0 b^3$$

$$= a^3 + 3a^2 b + 3ab^2 + b^3$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Recall $X \sim Pois(\mu)$

$$P(X=k) = \frac{e^{-\mu} \mu^k}{k!}$$

<u>Claim</u> If $X \sim \text{Pois}(\mu)$ and $Y \sim \text{Pois}(\lambda)$ are independent then

$S = X + Y \sim \text{Pois}(\mu + \lambda)$.

Pf/ $P(S=s) = P(X=0, Y=s) + P(X=1, Y=s-1) + \dots P(X=s, Y=0)$

$$= \sum_{k=0}^{s} P(X=k, Y=s-k)$$

$$= \sum_{k=0}^{s} P(X=k) P(Y=s-k)$$

$$= \sum_{k=0}^{s} \frac{e^{-\mu} \mu^k}{k!} \cdot \frac{e^{-\lambda} \lambda^{s-k}}{(s-k)!}$$

$$= e^{-(\lambda+\mu)} \frac{1}{s!} \sum_{k=0}^{s} \frac{s!}{k!(s-k)!} \mu^k \lambda^{s-k}$$

$$= e^{-(\lambda+\mu)} \frac{1}{s!} (\mu+\lambda)^s$$

$\Rightarrow S \sim \text{Pois}(\mu+\lambda)$.

$\square$

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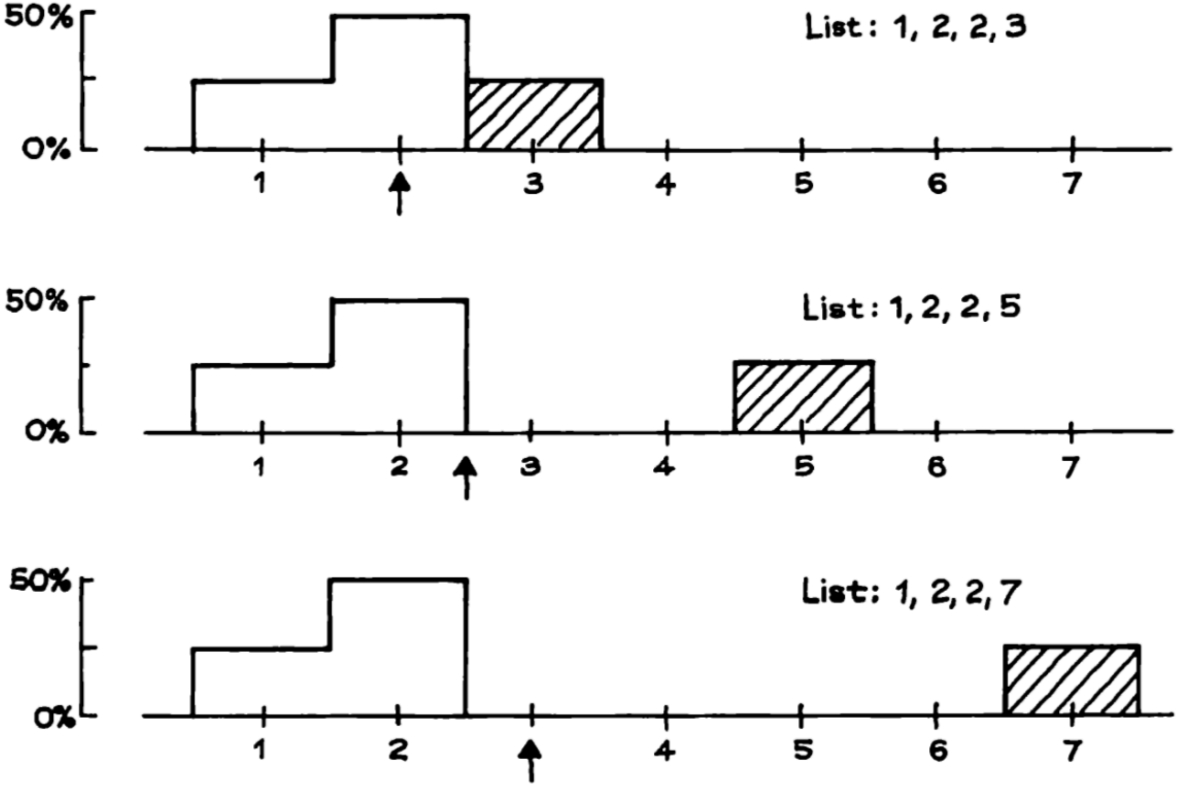
Sec 3.2 Expectation

$$E(X) = \sum_{x \in X} x \cdot P(X=x)$$

$$E(X) = 1 \cdot \tfrac{1}{4} + 2 \cdot \tfrac{1}{2} + 3 \cdot \tfrac{1}{4} = 2$$



Properties of Expectation — P167 Pitman

1. $E(c) = c$
2. $E(X+Y) = E(X) + E(Y)$ (X, Y don't need to be independent)
3. $E(aX+b) = aE(X) + b$

Indicators

An indicator is a RV that has only 2 values 1 (w/ prob p) and 0 (with prob 1-p).

$$I = \begin{cases} 1 & \text{prob } p \\ 0 & \text{prob } 1-p \end{cases}$$

— same as a Bernoulli p trial.

$$E(I) = 1 \cdot p + 0 \cdot (1-p) = p$$

RV that are counts can often be written as a sum of indicators.

ex $X \sim Bin(n,p)$

successes in n Bernoulli p trials.

ex $X = $ # heads in n flips of p coin

$$X = I_1 + I_2 + \dots + I_n$$

where $I_j = \begin{cases} 1 & \text{if } j^{th} \text{ trial success} \\ 0 & \text{else} \end{cases}$ p

$$E(X) = \underset{p}{E(I_1)} + \dots + \underset{p}{E(I_n)} = \boxed{np}$$

indicators are independent since trials are indep.

Ex X = # aces in a poker hand from a deck of cards

a) what are the range of values of X?

b) write X as a sum of indicators

c) How is I_2 defined?

d) Find $E(I_2)$

e) Find $E(X)$

$\underline{ex}$ Suppose a fair die is rolled 10 times.
Let $X =$ number of different faces that appear in 10 rolls.

$\underline{ex}$ if roll 2, 3, 4, 2, 3, 5, 2, 3, 3, 2 then $X = 4$

a) what are the range of values of X?

b) write X as a sum of indicators

c) How is I_2 defined?

d) Find $E(I_2)$

e) Find $E(X)$

1. n people with hats have had a bit too much to drink at a party. As they leave the party, each person randomly grabs a hat. A match occurs if a person gets his or her own hat.

a The expected number of matches depends on n

b The expected number of matches is 1

c The number of matches is hypergeometric

d more than one of the above

A drawer contains s black socks and s white socks (s> 0). I pull two socks out at random without replacement and call that my first pair. Then I pull out two socks at random without replacement and call that my second pair. I proceed in this way until I have s pairs and the drawer is empty. Find the expected number of pairs in which two socks are different colors.