Stat 134 lec 4

warmup: 11:00 - 11:10

Suppose you draw a number from a bag, with equal probabilities across the choices $\{1, 2, 3\}$.

Once you draw a number, you toss a coin until you get that many number of heads followed by a tails—so if you draw a 3, you keep tossing until you encounter the sequence {Heads, Heads, Heads, Tails}.

What is the probability of tossing a coin seven times given that you draw the number 2?

forward conditional

Find $P(7 \mid \text{draw } 2)$

$\underbrace{* \; * \; * \; *}$ H H T

16 possibilities but can't have

* HHT or HHT *

so 12 possibilities

each with probability $\left(\frac{1}{2}\right)^7$

$\Rightarrow \boxed{12\left(\frac{1}{2}\right)^7}$

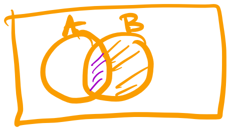
Last time

sec 1.4 Independence

Note that if $P(AB)=P(A)P(B)$ then $P(A^cB)=P(A^c)P(B)$

since,

difference rule



$P(A^cB) = P((AB)^cB) = P(B) - P(AB)$

independence of A,B

$= P(B) - P(A)P(B) = (1-P(A))P(B) = P(A^c)P(B)$

If A and B are indep then so to is A, B^c, and A^c, B and A^c, B^c.

sec 1.5 Bayes rule

There are two types of conditional probabilities:

ex From todays warmup question

see warmup

$P(7 \mid \text{draw } 2)$ is forward conditional (likelihood conditional)
DON'T NEED BAYES TO COMPUTE

$P(\text{draw } 2 \mid 7)$ is backwards conditional (posterior conditional)
NEED BAYES TO COMPUTE

Today

1. sec 1.6 independence of 3 or more events
2. sec 2.1 Binomial Distribution

sec 1.6 Independence of 3 events

Def^n (pairwise independence of 3 events)

A, B, C are pairwise independent if

$P(AB) = P(A)P(B)$ and $P(AC) = P(A)P(C)$ and $P(BC) = P(B)P(C)$

ex

> One ball is drawn randomly from a bowl containing four balls numbered 1, 2, 3, and 4. Define the following three events:
>
> - Let A be the event that a 1 or 2 is drawn. That is, $A = \{1, 2\}$.
> - Let B be the event that a 1 or 3 is drawn. That is, $B = \{1, 3\}$.
> - Let C be the event that a 1 or 4 is drawn. That is, $C = \{1, 4\}$.

Is A, B, C pairwise independent?

$P(AB) = \frac{1}{4} = \underbrace{P(A)}_{= 1/2}\underbrace{P(B)}_{= 1/2}$ ✓ Yes

Similarly $P(AC) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$ ✓

$P(BC) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$ ✓

Is $P(ABC) = P(A)P(B)P(C)$

No $P(ABC) = \frac{1}{4} \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

Def^n (mutual independence of 3 events)

A, B, C are mutually independent if

$P(ABC) = P(A)P(B)P(C)$, (and the same for any of the events replaced by its complement) we require showing 8 equations is true for mutual independence. This is a strong condition.

Thus Suppose A, B, C are mutually independent. Then they are also pairwise independent.

Pf/
we can write

$$P(AB) = \underbrace{P(ABC) + P(ABC^c)}_{\text{add}^n \text{ rule}}$$

$$= P(A)P(B)P(C) + P(A)P(B)P(C^c)$$

$$= P(A)P(B)\big[\underbrace{P(C) + P(C^c)}_{=1}\big]$$

$$= P(A)P(B).$$

similarly for other cases.

$\square$

Note that $P(ABC) = P(A)P(B)P(C)$ by itself doesn't imply pairwise independence:

ex let $\Omega = \{1,2,3,4,5,6,7,8\}$

$A = B = \{1,2,3,4\}$

$C = \{1,5,6,7\}$

Is $P(ABC) = P(A)P(B)P(C)$?

$P(A) = P(B) = P(C) = 1/2$

$P(ABC) = P(\{1\}) = 1/8$

$\Rightarrow P(ABC) = P(A)P(B)P(C)$ ✓

Is A, B, C pairwise indep?

No $\underset{1/2}{\underset{\parallel}{P(AB)}} \neq \underset{1/2}{\underset{\parallel}{P(A)}}\,\underset{1/2}{\underset{\parallel}{P(B)}}$

Thm

A, B, C are mutually independent iff

1) A, B, C are pairwise indep

2) $P(ABC) = P(A)P(B)P(C)$.

Sketch

Suppose (1) and (2) hold,

lets show $P(ABC^c) = P(A)P(B)P(C^c)$

write

$$P(ABC^c) = \underbrace{P(ABC^c) + P(ABC)}_{= P(AB)} - P(ABC)$$

$\Rightarrow$

$$P(ABC^c) = P(A)P(B) - P(A)P(B)P(C)$$

$$= P(A)P(B)[\underbrace{1 - P(C)}_{= P(C^c)}]$$

Similarly for other cases

$\square$

(2) Sec 2.1 Binomial distribution.

Bernoulli(p) trial (distribution)

two outcomes — success p / failure $1-p$

ex roll a die.
success is getting a 6 — $1/6$
failure is not getting a 6 — $5/6$

Binomial (n,p) distribution (Bin(n,p))

We have n independent Bernoulli(p) trials

n: fixed; p: fixed (unconditional probability)

ex we roll a die n times.

What are the possible number of successes? — $0, 1, 2, \dots, n$

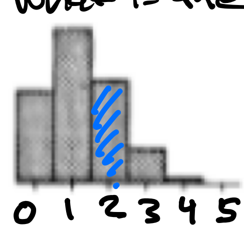
In Bin(n,p) the chance of having k successes $(0 \le k \le n)$ is given by the Binomial formula:

$$P(k) = \frac{n!}{k!\,(n-k)!}\, p^{k} (1-p)^{n-k}$$

n: # trials; k: number of successes; p: chance of success.

ex You roll a die 5 times. What is the chance of getting 2 sixes?

$n = ?$ 5
$k = ?$ 2
$p = ?$ $1/6$



$$P(2) = \frac{5!}{2!3!}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 = \frac{5\cdot4\cdot\cancel{3\cdot2\cdot1}}{2\cdot1\cdot\cancel{3\cdot2\cdot1}}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$$

$$= \boxed{10\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3}$$

What is chance of getting

success (6)

failure (not 6)

1 1 0 0 0 ?

0 1 1 0 0 ?

⋮

How many of these are there?

$$\boxed{\frac{5!}{2!3!}}$$

5 positions for 1st 1

4 positions for 2nd 1

3 positions for 1st 0

2 " " 2nd 0

1 " " 3rd 0

We write $\frac{5!}{2!3!}$ as $\binom{5}{2}$ or $\binom{5}{3}$ or $\binom{5}{2,3}$

$$\binom{5}{3} = \frac{5!}{3!2!}$$

[redacted]. A well shuffled deck is cut in half so there are 7 aces in the first half deck and 6 aces in the second half deck. Five cards are dealt off the top of one half deck and five cards are dealt off the top of the other half deck. The binominal formula doesn't apply to find the chance of getting exactly three ~~diamonds~~ aces total because:

get ace

n=10

a The probability of a trial being successful changes – Prob of drawing an ace is $7/26$ in 1^{st} half deck and $6/26$ in second half deck

we draw without replacement

b The trials aren't independent

c There isn't a fixed number of trials

n=10 is fixed

d more than one of the above