Stat 134, lecture 41 (RRR review #2)

Topics
① Minima & maxima via CDF
② Craps principle/conditioning on first trial
③ Conditional distributions examples

① Let X, Y be indep. RVs with cdfs F_X, F_Y.
Define $M = \min\{X, Y\}$, $W = \max\{X, Y\}$.

Find F_M, F_W in terms of F_X, F_Y.

Recall: $\mathbb{P}(M \le m) = \mathbb{P}(X \le m \cup Y \le m)$
$= \mathbb{P}(X \le m) + \mathbb{P}(Y \le m) - \mathbb{P}(X, Y \le m)$

$\mathbb{P}(W \le w) = \mathbb{P}(X \le w \cap Y \le w)$

Solution: $F_M(m) = F_X(m) + F_Y(m) - F_X(m) F_Y(m)$

$F_W(w) = F_X(w) F_Y(w)$

② Craps principle.

Suppose I am playing a game that takes place over multiple rounds. At each round, 3 events can occur (mut. excl.):

Win – prob. p

Lose – prob. q ← game ends on loss

Draw – prob. r

Find the chance I eventually win.

Method 1 – Infinite sum.

$$
\begin{aligned}
\mathbb{P}(\text{win}) &= \sum_{k=1}^{\infty} \mathbb{P}(\text{win on round } k) \\
&= \sum_{k=1}^{\infty} r^{k-1} p \quad \leftarrow \text{no losses, then win} \\
&= p \sum_{k=0}^{\infty} r^k = \frac{p}{1-r} = \frac{p}{p+q}.
\end{aligned}
$$

Comment: Either win or loss eventually happens. $\frac{p}{p+q}$ is the chance the win happens first.

Method 2 – Condition on first round.

$$
\begin{aligned}
\mathbb{P}(\text{win}) = \; & p \, \underbrace{\mathbb{P}(\text{win} \mid 1^{\text{st}} \text{ round win})}_{=1} + \\
& q \, \underbrace{\mathbb{P}(\text{win} \mid 1^{\text{st}} \text{ loss})}_{=0} + \\
& r \, \underbrace{\mathbb{P}(\text{win} \mid 1^{\text{st}} \text{ draw})}_{=\mathbb{P}(\text{win})}
\end{aligned}
$$

$$
\Rightarrow \mathbb{P}(\text{win}) = p + r(\mathbb{P}(\text{win})) \Rightarrow \mathbb{P}(\text{win}) = \frac{p}{1-r}.
$$

Example: 2 coins, $\mathbb{P}(H)=p$. All tails on table to begin. At each round, toss any coins showing T, leave H alone. Let $X :=$ # rounds to stop. Find $\mathbb{E}(X)$.

Remark: Note $X = \max\{G_1, G_2\}$ for $G_1, G_2 \overset{iid}{\sim} \text{Geom}(p)$ on $\{1,2,\dots\}$. (Not best way to proceed.)

Solution: Condition on first toss. Let $Y :=$ # H on toss 1.

$$\mu = \mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y))$$

$$= \sum_{k=0}^{2} \mathbb{E}(X|Y=k)\,\mathbb{P}(Y=k)$$

$$\mu = q^2 \underbrace{\mathbb{E}(X|Y=0)}_{1+\mathbb{E}(X^*)} + 2pq \underbrace{\mathbb{E}(X|Y=1)}_{1+\mathbb{E}(\text{Geom}(p))} + p^2 \underbrace{\mathbb{E}(X|Y=2)}_{1}$$

$$\mu = q^2(1+\mu) + 2pq\left(1+\frac{1}{p}\right) + p^2$$

$$(1-q^2)\mu = \underbrace{p^2+q^2+2pq}_{=(p+q)^2} + 2q$$

$$\Rightarrow \boxed{\mu = \frac{1+2q}{1-q^2}}$$

③ **Ex)** Suppose we have N light bulbs, where $N \sim Geom(p)$ on $\{0, 1, \ldots\}$. Each light bulb's lifetime is $Exp(\lambda)$, independent of other bulbs.
Find the chance that ≥ 1 light bulbs have died by some time t.

Solution: $\mathbb{P}(\geq 1 \text{ dead by } t)$

$$= 1 - \mathbb{P}(\text{all alive past } t)$$

$$= 1 - \sum_{k=0}^{\infty} \mathbb{P}(N=k)\underbrace{\mathbb{P}(\text{all alive past } t \mid N=k)}$$

Look at $M = \min\{X_1, X_2, \ldots X_k\}$ for $X \sim Exp(\lambda)$.
$M \sim Exp(k\lambda)$. Use $\mathbb{P}(M > t \mid N=k)$.

$$= 1 - \sum_{k=0}^{\infty} q^k p e^{-k\lambda}$$

$$= 1 - \sum_{k=0}^{\infty} (qe^{-\lambda})^k p = 1 - \frac{p}{1-qe^{-\lambda}}.$$

Ex| Let Y_t be the # of surviving bulbs at some time t. (Define $Y_t = 0$ if $N=0$). Find:

a) The conditional distribution of Y_t given $N=n$;

b) The distribution of Y_t.

ⓐ Note $Y_t|_{N=n} \in \{0, 1, 2, \ldots, n\}$.

$$\mathbb{P}(Y_t = k \mid N=n) = \binom{n}{k}(e^{-\lambda t})^k(1-e^{-\lambda t})^{n-k}$$

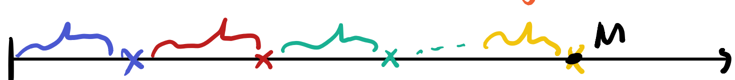
Observe $Y_t \mid N=n \sim Bin(n, e^{-\lambda t})$.

b) Recall: Poisson/binomial. If $X \sim \text{Bin}(N, p)$ for $N \sim \text{Pois}(\mu)$, $X \sim \text{Pois}(\mu p)$.
(Proof in appendix; see Pitman 6.1.7)

So $Y_t \sim \text{Pois}(\mu e^{-\lambda t})$.

Ex) $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Exp}(\lambda)$. Let $M = \max\{X_1, \ldots, X_n\}$.
Find $\mathbb{E}(M)$.

Solution: Look at times between failures.



$L_1 = 1^{st}$ failure $= \min\{X_1, \ldots, X_n\}$ $\quad L_n$

$L_2 =$ time bwn 1^{st} & 2^{nd} failures

So $M = L_1 + L_2 + \ldots + L_n$.
$L_1 \sim \text{Exp}(n\lambda)$. By memoryless property,

$$\mathbb{P}(L_2 > t + L_1 \mid L_2 > L_1) = \mathbb{P}(\underbrace{L_2}_{\text{Min of } n-1 \text{ exponentials.}} > t)$$

So $L_2 \sim \text{Exp}((n-1)\lambda)$, $L_1 \perp\!\!\!\perp L_2$.
Continuing, $L_i \sim \text{Exp}((n+1-i)\lambda)$.

$$\mathbb{E}(M) = \mathbb{E}\left(\sum_{i=1}^{n} L_i\right) = \sum_{i=1}^{n} \frac{1}{(n+1-i)\lambda}$$

Ex) Poisson process with rate λ. Find $Cov(T_1, T_3)$ using:
a) Bilinearity of covariance;
b) Computational formula.

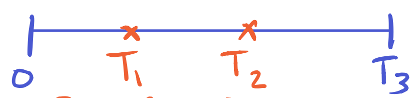
a) $$Cov(T_1,T_3) = Cov(W_1, W_1+W_2+W_3)$$
$$= Cov(W_1,W_1) + \cancel{Cov(W_1,W_2)} + \cancel{Cov(W_1,W_3)}$$
0 by indep.
$$= Var(W_1)$$
$$= \frac{1}{\lambda^2}$$

b) $$Cov(T_1,T_3) = \mathbb{E}(T_1T_3) - \mathbb{E}(T_1)\mathbb{E}(T_3)$$

$$\mathbb{E}(T_1T_3) = \mathbb{E}(\mathbb{E}(T_1T_3|T_3))$$
$$= \mathbb{E}(T_3\,\mathbb{E}(T_1|T_3))$$

Recall $T_1|T_3$ is equal in dist'n. to $\min\{U_1,U_2\}$ for $U \sim Unif(0,T_3)$.

Intuition:



Prob. of arrival in any small intvl.: λdt.
We know 2 must occur; take min.

So $\mathbb{E}(T_1|T_3) = \frac{T_3}{3}$ by spacings argument.

$$\Rightarrow \mathbb{E}(T_1T_3) = \mathbb{E}\left(\frac{T_3^2}{3}\right)$$

$$\mathbb{E}(T_3^2) = Var(T_3) + \mathbb{E}(T_3)^2 = \frac{3}{\lambda^2} + \frac{9}{\lambda^2}$$

$$= \frac{1}{3}\left(\frac{12}{\lambda^2}\right) = \frac{4}{\lambda^2}$$

$$\Rightarrow Cov(T_1,T_3) = \frac{4}{\lambda^2} - \frac{1}{\lambda}\cdot\frac{3}{\lambda} = \boxed{\frac{1}{\lambda^2}}.$$

Appendix

Let $X \sim Bin(N, p)$ for $N \sim Pois(\mu)$.

(I.e., given $N=n$, $X \sim Bin(n,p)$.)

Find marginal distribution of X.

Solution:

$$\mathbb{P}(X=k) = \sum_{n=k}^{\infty} \mathbb{P}(N=n)\,\mathbb{P}(X=k \mid N=n)$$

$$= \sum_{n=k}^{\infty} e^{-\mu}\frac{\mu^n}{n!}\cdot\binom{n}{k}p^k q^{n-k}$$

Break apart

$$= e^{-\mu}p^k \sum_{n=k}^{\infty} \frac{n!}{n!\,k!\,(n-k)!}\left(\mu^n \cdot q^{n-k}\right)$$

$$= \frac{e^{-\mu}\cdot(\mu p)^k}{k!} \underbrace{\sum_{n=k}^{\infty} \frac{(\mu q)^{n-k}}{(n-k)!}}_{= \sum_{n=0}^{\infty} \frac{\mu q^n}{n!} = e^{\mu q}}$$

re-index.

$$= e^{-\mu+\mu q}\frac{(\mu p)^k}{k!}$$

$$= e^{-\mu p}\cdot\frac{(\mu p)^k}{k!}, \quad k \in \{0,1,2,\dots\}$$

$\therefore X \sim Pois(\mu p)$.