Section 25 : Solutions

Conceptual Review :

ⓐ X, Y

* $\text{Cov}(X,Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(X)))$

$$= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}}$$

* X, Y have finite second moment. ($\mathbb{E}(|X^2|)$, $\mathbb{E}(|Y^2|)$ are finite)

ⓑ $X, Y \sim \text{iid } N(0,1)$ $\quad \alpha, \beta \in \mathbb{R}$

$\alpha X + \beta Y \sim$?

$\alpha X \sim \underline{N(0, \alpha^2)}$, $\beta Y \sim \underline{N(0, \beta^2)}$

$$\alpha X + \beta Y \sim N(0, \alpha^2 + \beta^2)$$

ⓒ $\boxed{X, Y}$ are standard bivariate normal with correlation $\rho \geq 0$ if:

$(\rho \geq 0)$

$$Y = \rho X + \sqrt{1-\rho^2}\, Z$$

where X, Y are iid $N(0,1)$

Problem 1 :

$X_1, \dots, X_n$ are exchangeable

$(X_1, \dots, X_n)$ has the same distribution as $\boxed{(X_{\pi(1)}, \dots, X_{\pi(n)})}$ (π is shuffling (permutation))

$$(X_1, \dots, X_n) \stackrel{d}{=} (X_1, X_3, X_2, X_4, \dots, X_n)$$

$\stackrel{d}{=}$

X, Y iid $N(0,1)$ $\qquad X \neq Y$

$(X, Y) \stackrel{d}{=} (Y, X)$

If $X_1, -, X_n$ are exchangeable

$S_n = \sum_{k=1}^{n} X_k$.

$Var(S_n) = Cov(S_n, S_n)$ (bilin. of Cov)

$= \sum_{j=1}^{n} \sum_{i=1}^{n} Cov(X_i, X_j)$

$= \underbrace{\sum_{i=1}^{n} Cov(X_i, X_i)}_{} + \underbrace{\sum_{i \neq j} Cov(X_i, X_j)}_{}$

$= \underbrace{\sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)}_{}$

Since $X_1, -, X_n$ are exchangeable then: $X_1, -, X_n$ all have the same distribution and also $(i \neq j)$ (X_i, X_j) has the same distribution as (X_1, X_2)

$\Rightarrow$ $Var(X_i) = Var(X_1)$ for all i

and $Cov(X_i, X_j) = Cov(X_1, X_2)$.

Then

$$Var(S_n) = \sum_{k=1}^{n} Var(X_1) + \sum_{i \neq j} Cov(X_1, X_2)$$

$$\boxed{Var(S_n) = n\,Var(X_1) + n(n-1)\,Cov(X_1, X_2)}$$

Problem 2 :

$X_1, X_2 \sim$ iid $N(0,1)$.

$Y_1 = X_1 + X_2$

$Y_2 = \alpha X_1 + 2X_2$

with $\alpha \in \mathbb{R}$ s.t

$$Cov(Y_1, Y_2) = 0$$

$$Cov(Y_1, Y_2) = \underbrace{Cov(X_1 + X_2,\ \alpha X_1 + 2X_2)}_{} = 0$$

using bilinearity of Cov.

$$\alpha \, Cov(X_1, X_1) + \cancel{2\,Cov(X_1, X_2)}^{0} + \cancel{\alpha\,Cov(X_1, X_2)}^{0}$$
$$+ 2\,Cov(X_2, X_2) = 0$$

$$\alpha \, Var(X_1) + 2\,Var(X_2) = 0$$
$$\alpha + 2 = 0 \qquad \text{then} \qquad \alpha = -2$$

$$Y_2 = \alpha X_1 + 2X_2 = \boxed{2(X_2 - X_1)}$$
$$Y_1 = X_1 + X_2$$

$$\underline{X_2 - X_1} \sim N(0, 1^2 + (-1)^2) = N(0, 2)$$
$$Y_2 \sim N(0, 2 \times 2^2) = N(0, 8)$$

$$f_{Y_2}(x) = \frac{1}{\sqrt{16\pi}} \exp\left(-\frac{x^2}{16}\right)$$

$$\mathrm{Cov}(X_2, Y_2) = \mathrm{Cov}(X_2, 2X_2 - 2X_1)$$

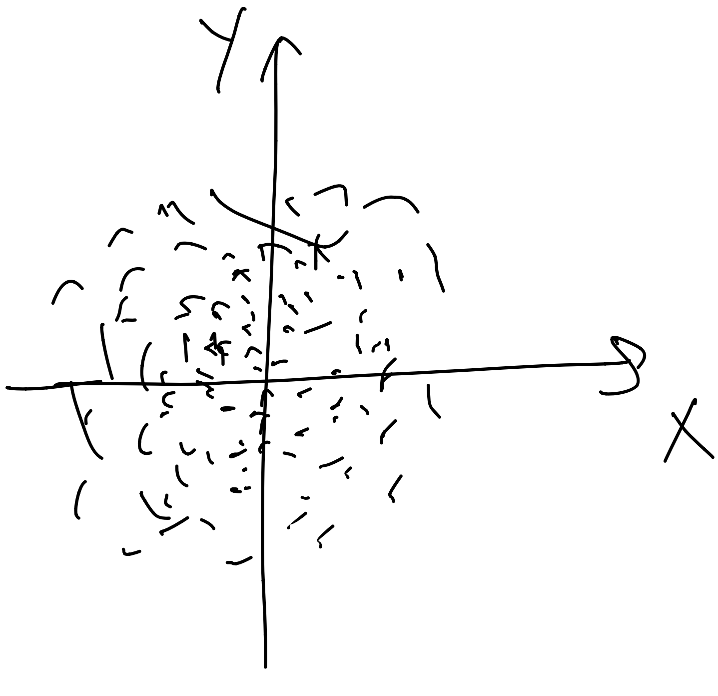
$$= 2\,\mathrm{Cov}(X_2, X_2) = 2$$

__Problem 3:__

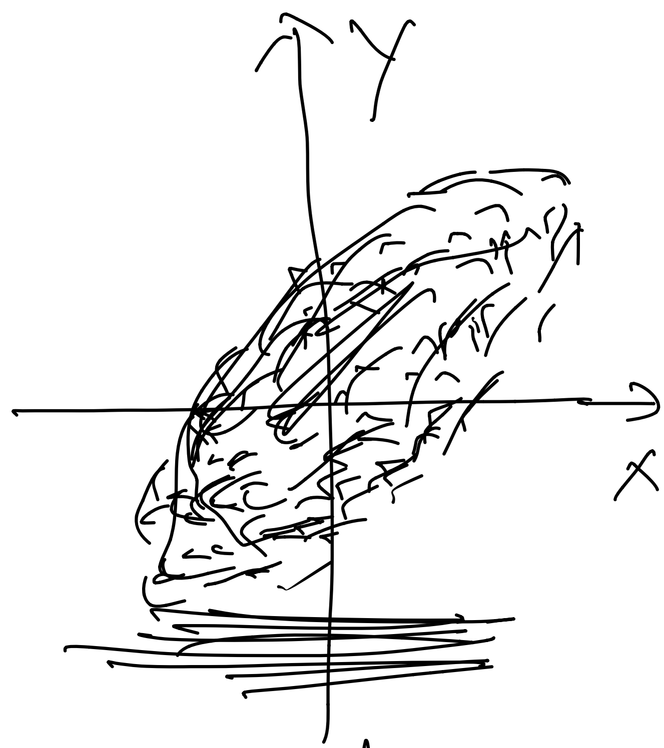
$X \sim N(0,1)$ $\quad$ Y indep of X

$$P(Y = \pm 1) = 1/2$$

$Z = YX$ $\quad$ $\underline{\mathrm{Cov}(X, Z) = 0}$



X, Y iid



$X, Y \sim$ bivariate stand with cor ρ

$\checkmark\ X \sim N(0,1)$

$\checkmark\ Z \sim N(0,\ )$

but X, Z not <u>bivariate</u>

$Z \sim N(0,\ )$:

$$\mathbb{P}(Z \leq x) = \mathbb{P}(Z \leq x \mid Y = 1)\ \tfrac{1}{2}$$
$$+\ \mathbb{P}(Z \leq x \mid Y = -1)\ \tfrac{1}{2}$$

$$= \frac{1}{2}\mathbb{P}(X \leq x \mid Y = 1) + \frac{1}{2}\mathbb{P}(X \geq -x \mid Y = -1)$$

$$= \frac{1}{2}\big(\mathbb{P}(X \leq x) + \mathbb{P}(X \geq -x)\big)$$

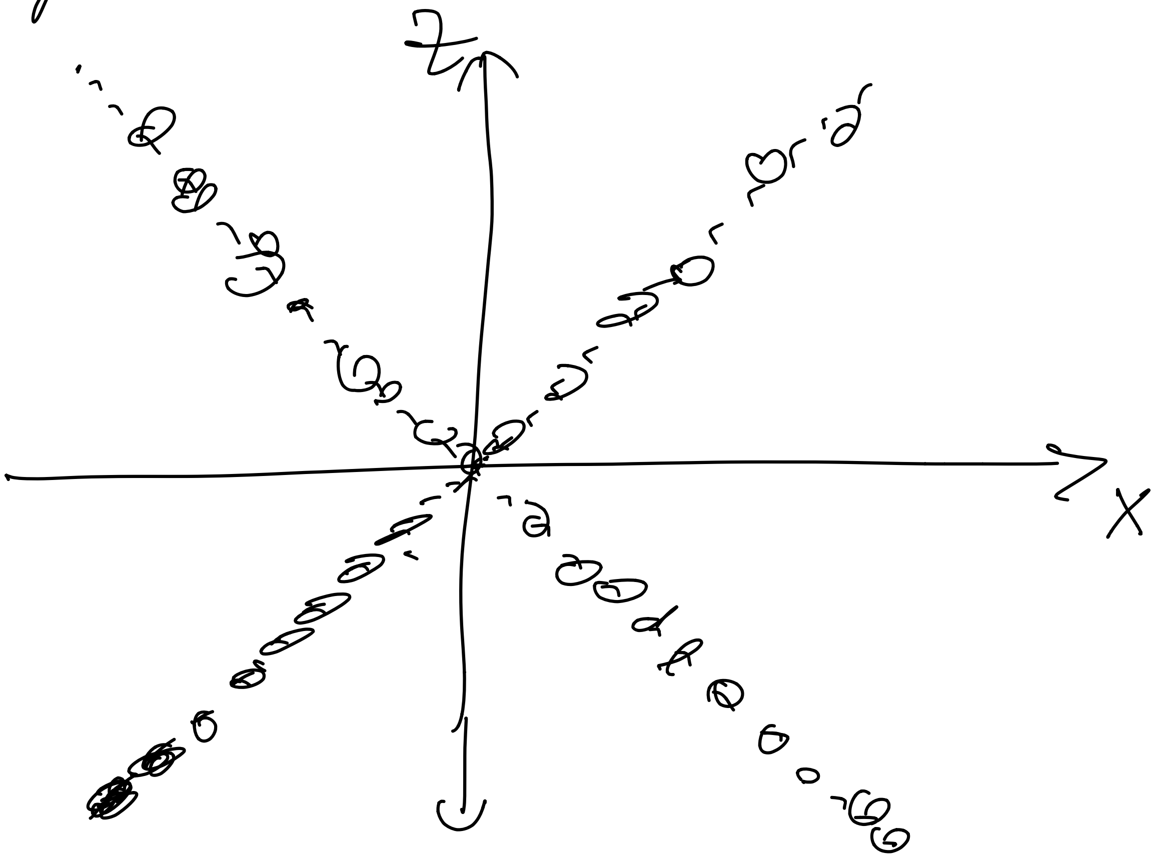
$$= \frac{1}{2}\big(\Phi(x) + 1 - \Phi(-x)\big)$$

$$= \frac{1}{2}\big(2\Phi(x)\big) = \Phi(x)$$

$$\mathbb{P}(Z \leq x) = \Phi(x) = \mathbb{P}(X \leq x)$$

meaning $Z \sim N(0,1)$

(X,Z)



$Z = X$

$Z = -X$

(X,Z)

$\Big((X,Z)$ are <u>not</u> bivariate normal $\Big)$