Stat 134 lec 26

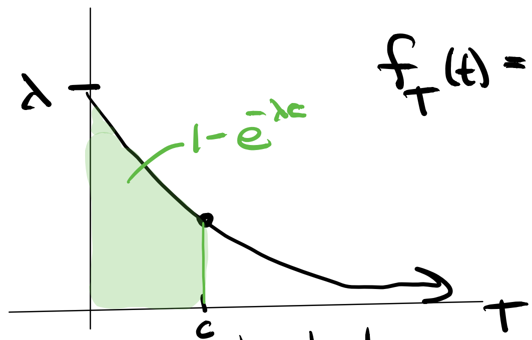
Last time

Sec 4.5 CDF $F_X(x) = P(X \le x)$

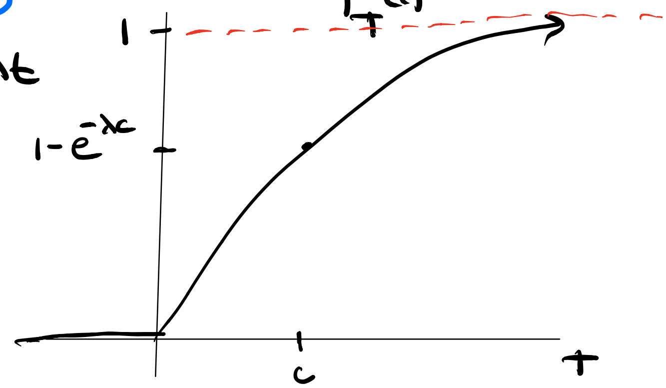
ex $T \sim Exp(\lambda)$ — values $[0, \infty)$

$c > 0$

$f_T(t) = \lambda e^{-\lambda t}$



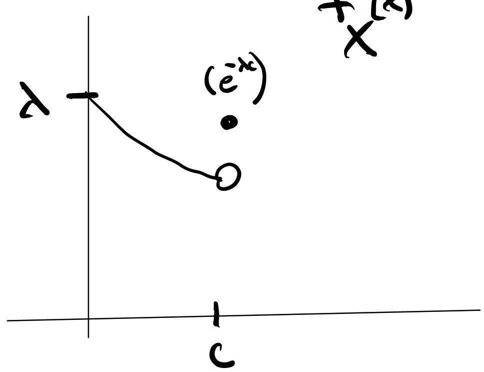
$F_T(t) = 1 - e^{-\lambda t}$



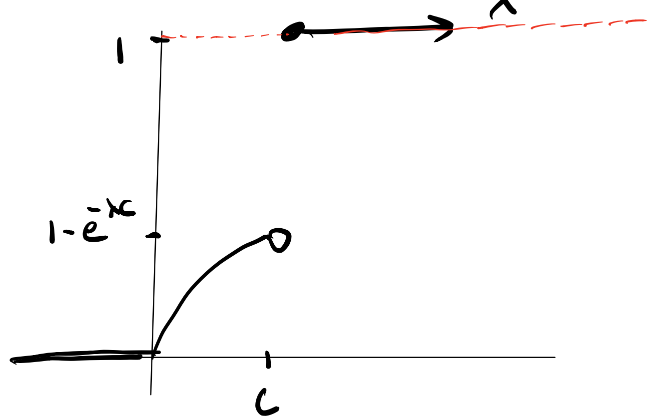
ex mixed distribution

$X = \min(T, c)$ — values $[0, c]$

$f_X(x)$



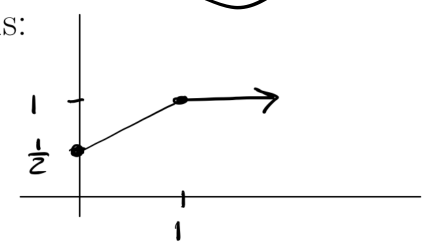
$F_X(x)$



Today

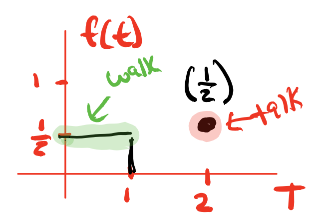
1. review student responses
2. review MGF
3. sec 4.5 Using CDF to find $E(X)$

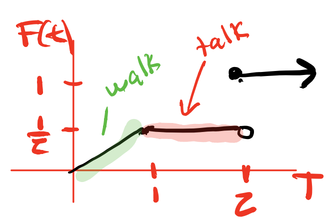
(1) . Suppose you are trying to discretly leave a party. Your time to leave is uniform from 0 to 2 minutes. However, if your walk to the exit takes more than 1 minute, you run into a friend at the door and and must spend the full 2 minutes to leave. Let T represent the time it takes you to leave. True or false, the graph of the cdf of T is:



FALSE	should start at 0 probability, increases to 1/2 and then jump to 1 when reaches 1

FALSE	the pdf for the distribution looks like a straight line at y=1/2 on x within (0, 1) then there is a point at x = 2 which takes the rest of the mass. So it is false, the cdf looks like a straight line from (0,0) to (1, 1/2) and then a flat, straight starting at (2, 1).

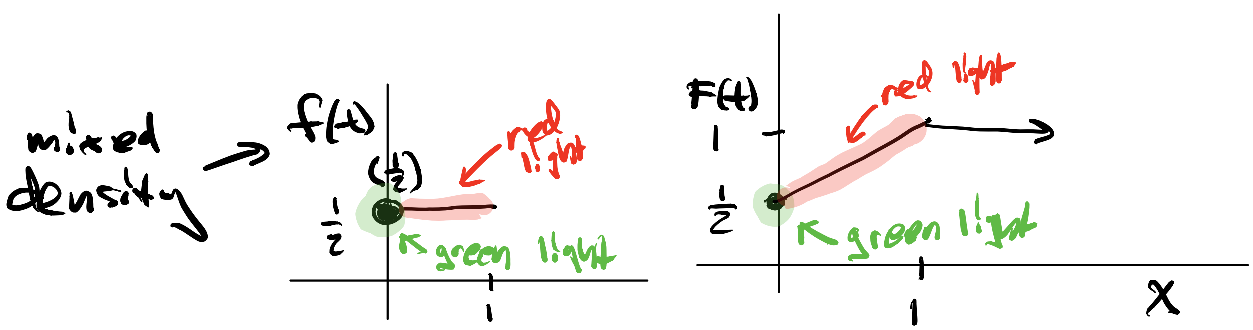




Another mixed distribution:

eg

Suppose stop lights at an intersection alernately show green for one minute, and red for one minute (no yellow). Suppose a car arrives at the lights at a time distributed uniformly from 0 to 2 minutes. Let X be the delay of the car at the lights (assuming there is only one car on the road). Graph the density and the cdf.



② Review MGF

X RV, $t \in \mathbb{R}$

$$M_X(t) = E(e^{tX})$$

ex $X \sim \text{Gamma}(r, \lambda)$

variable part

$$f_X(x) = \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}, \quad x > 0$$

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r, \quad t < \lambda$$

ex $X \sim \text{Unif}(0,1)$

$$f_X(x) = 1_{(x \in (0,1))}$$

Find $M_X(t)$

$$M_X(t) = E(e^{tX}) = \int_0^1 e^{tx} \cdot 1 \, dx = \frac{1}{t} e^{tx} \Big|_{x=0}^{x=1} = \boxed{\frac{1}{t}(e^t - 1)}$$

Property of MGF:

$M_X(t) = M_Y(t)$ for all t in a neighborhood of zero, iff $X \overset{d}{=} Y$ (i.e. X and Y have the same distribution)

ex $X \sim \text{Unif}(0,1)$

$Y = 1 - X$ ← also $\text{Unif}(0,1)$

by change of variable rule $(g(x) = 1 - x)$

$$f_Y(y) = \frac{1}{\underbrace{|g'(x)|}_{=-1}} f_X(x) \Big|_{x=1-y} = 1_{(y \in (0,1))}$$

Hence $X \overset{d}{=} Y$ (but $X \neq Y$)

Find $M_Y(t)$:

$$M_Y(t) = \int_0^1 e^{ty} 1 \, dy = \frac{1}{t} e^{ty} \Big|_{y=0}^{y=1} = \boxed{\frac{1}{t}(e^t - 1)}$$

Knowing the distribution uniquely specifies the MGF

And

Knowing the MGF uniquely specifies the distribution

Stat 134

Monday April 1 2019

1. Let X have density $f(x) = xe^{-x}$ for $x > 0$. The MGF is?

   a $\mathrm{M_X(t)} = \frac{1}{1-t}$ for $t < 1$

   b $\mathrm{M_X(t)} = \frac{1}{(1-t)^2}$ for $t < 1$

   c $\mathrm{M_X(t)} = \frac{1}{(1+t)^2}$ for $t > -1$

   d none of the above

variable part of Gamma $x^{r-1} e^{-\lambda x}$

$\Rightarrow X \sim \text{Gamma}(2, 1)$

know MGF of Gamma(r, λ) is $\left(\frac{\lambda}{\lambda - t}\right)^r$, $t < \lambda$

so $\boxed{M_X(t) = \frac{1}{(1-t)^2} \text{ for } t < 1}$

2. The MGF of X is $M_X(t) = \frac{1}{\sqrt{1-t}}$ for $t < 1$. The distribution of X is:

a Gamma(r $= 1/2, \lambda = 1$) and possibly another distribution.

b Gamma(r $= 2, \lambda = 1$)

c Gamma(r $= -1/2, \lambda = 1$)

d none of the above

$X \sim$ Gamma $(r = 1/2, \lambda = 1)$

Uniquely determined from

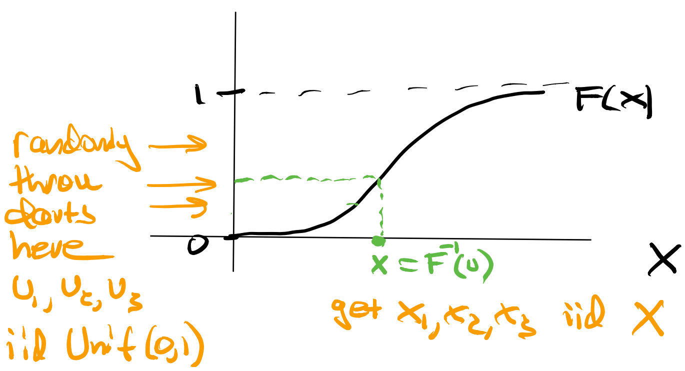
$M_X(t) = \frac{1}{\sqrt{1-t}}$ for $t < 1$.

see #9 p324

(3) Sec 4.5 Using CDF to find $E(X)$ for $X \geq 0$

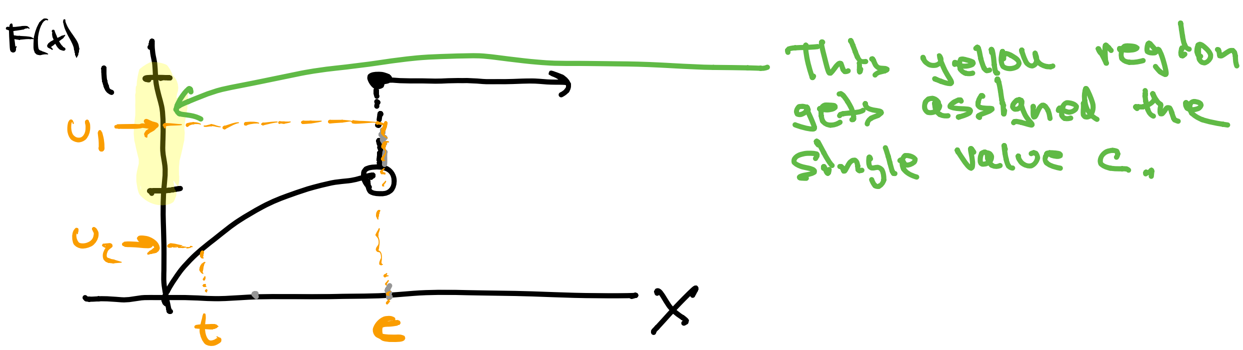
Inverse distribution function, $F^{-1}(u)$

Let X have CDF $F(x)$.



Note: doesn't have to be continuous RV.

eg $X = \min(T, c)$, $T \sim \exp(\lambda)$



Thm (p322) — Proof at end of lecture.

Let X have CDF F.

Then the RV $F^{-1}(U) = X$

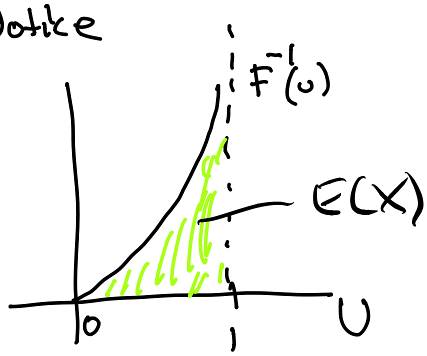
How is this useful to us finding $E(X)$?

It is sometimes easier to calculate $E(X)$ using the cdf (avoid doing integration by parts):

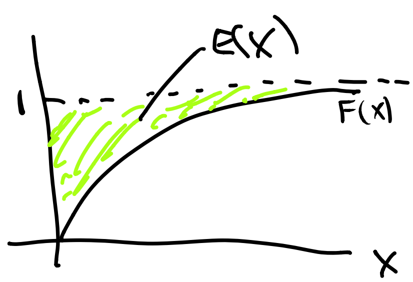
1 since $U \sim \text{Unif}(0,1)$

$$E(X) = E(F^{-1}(U)) = \int_0^1 F^{-1}(u) f_U(u)\, du$$

Notice



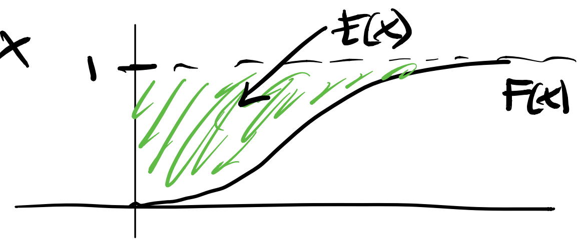
Now reflect the above graph about the diagonal $y = x$



We can find the shaded region by integrating $1 - F(x)$ with respect to x:

Thm Let X be a pos. random variable, with CDF F. (continuous, discrete, mixed),

$$E(X) = \int_0^\infty (1 - F(x))\,dx$$



ex

$T \sim \text{expon}(\lambda)$

$F_T(t) = 1 - e^{-\lambda t}$

Calculate $E(T)$.

$$E(T) = \int_0^\infty (1 - F(t))\, dt = \int_0^\infty e^{-\lambda t}\, dt$$

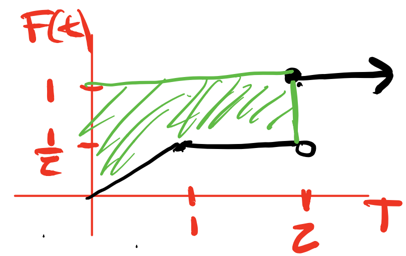
$$= \frac{1}{\lambda} e^{-\lambda t} \Big|_0^\infty = \frac{1}{\lambda}$$

wow that was easy!

ex

Suppose you are trying to discretly leave a party. Your time to leave is uniform from 0 to 2 minutes. However, if your walk to the exit takes more than 1 minute, you run into a friend at the door and and must spend the full 2 minutes to leave. Let T represent the time it takes you to leave. True or false, the graph of the cdf of T is:

$E(x) = \frac{5}{4}$



Appendix

See p 322 in book

Claim for any CDF F

$X = F^{-1}(U)$ is a RV with cdf F.

U ~ Unif(0,1)

Proof/ let $X = F^{-1}(U)$

$$F_X(x) = P(X \le x)$$

we will show $F_X = F$

$$= P(F^{-1}(U) \le x)$$

$$= P(F \cdot F^{-1}(U) \le F(x))$$

since F is increasing

$$= P(U \le F(x))$$

$$= F(x)$$

since $P(U \le u) = u$

$\square$