Stat 134: Densities, CDFs, and Order Statistics Review

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Problem 1

Let $X_1, X_2, ..., X_n, \sim Beta(r, s)$ and $Y = X_{(k)}$ be the k^{th} order statistic of the *iid* $Beta(r, s)$ distributions. Find the CDF of Y.

The CDF of X_i is $\sum_{i=1}^{n} \binom{r+s-1}{i} x^i (1-x)^{r+s-1-i}$

$P(Y \leq y) = P(\text{at least k } X_i \leq y)$

$P(Y \leq dy) = \sum_{j=k}^{n} P(\text{exactly j } X_i \leq y)$

$P(Y \leq dy) = \sum_{j=k}^{n} \binom{n}{j} (F_X(y))^{j-1} (1 - F_X(y))^{n-j}$ where F_X denotes the CDF of X_i

Problem 2

Let $X_1, X_2, ..., X_n$ be independent Poisson Processes, all with identical rate λ and $Y = X_{(k)}$, where $X_{(k)}$ denotes the time t when exactly k of these Poisson Processes have had at least one arrival, while the remaining $n - k$ do not have any arrivals in the time interval $(0, t)$. Find the density of Y.

Since we only care about the Poisson Process having at least one arrival, we only need to look at the time of the first arrival for all n Poisson processes. Y then beecomes the k^{th} order statistic of n *iid* $Exp(\lambda)$

$P(Y \in dy) = P(k-1 \text{ exponentials} < y, \text{ one} \in dy, n-k \text{ exponentials} > y)$

$P(Y \in dy) = n\binom{n-1}{k-1}(1 - e^{-\lambda y})^{k-1} \lambda e^{-\lambda y} (e^{-\lambda y})^{n-k}$

Problem 3

1. Let Θ be uniformly distributed on $[0, \pi]$. Find the distribution of $cos(\Theta)$.
2. Let X be uniformly distributed on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Find the distribution of sin(X + $\frac{\pi}{2}$) without change of variable calculations.
3. Let Y now be distributed uniformly on $[0, 1]$. Find the distribution of $sin^{-1}(Y)$.
4. Let B be the minimum of n i.i.d uniform(0,1) random variables. Find the distribution of $cos^{-1}(\sqrt{B})$. We dare you to do this with the change of variables formula.

1. Note that cos^{-1} is decreasing.

   $P(cos(\Theta) < s) = P(\Theta > cos^{-1}(s))$

   $= \frac{\pi - cos^{-1}(s)}{\pi}$
2. From trigonometry, $cos(\theta) = sin(\theta + \frac{\pi}{2})$.

   X + $\frac{\pi}{2}$ is identical in distribution to Θ.

   Thus the distribution is exactly the same as cos(Θ)
3. sin^{-1} is increasing on this interval. Possible values $[0, \frac{\pi}{2}]$

   $P(sin^{-1}(Y) < s)$

   $= P(Y < sin(s))$

   $= sin(s)$
4. cos^{-1} is decreasing on this interval. The survival function of B is $(1 - x)^n$

   $P(cos^{-1}(\sqrt{B} < s)$

   $= P(\sqrt{B} > cos(s))$

   $= P(B > cos^2(s))$

   $= (1 - cos^2(s))^n$

   $= sin^{2n}(s)$

Problem 4

Boxes arrive according to a Poisson process with rate λ.

Inside each box is a random number of messages. The number of messages in each box follows a Poisson(μ) distribution, independently of when it arrives and independent of other boxes.

Find the distribution of the waiting time for the first message. Does it look familiar?

Solution:

Easy way: Notice that the probability that any given box has a message is 1 - $e^{-\mu}$). Apply Poisson thinning.

Hard way: Let T denote the waiting time for the first message. Let A_i denote the number of messages in box i.

$P(T > t) = P(\sum_{i=1}^{K} A_i = 0)$ where K has Poisson(λt) distribution.

$= \sum_{k=0}^{\infty} P(\sum_{i=1}^{K} A_i = 0 | K = k) P(K = k)$

But the sum of the A_i's conditional on K = k is Poisson($k\mu$).

So the above expression equals

$\sum_{k=0}^{\infty} e^{-k\mu} P(K = k)$

$= \sum_{k=0}^{\infty} e^{-k\mu} \frac{e^{-t\lambda}(t\lambda)^k}{k!}$

$= e^{-t\lambda} \sum_{k=0}^{\infty} \frac{(t\lambda e^{-\mu})^k}{k!}$

$= e^{-t\lambda} e^{t\lambda e^{-\mu}}$

$= e^{-t(\lambda - e^{-\mu}\lambda)}$

This is the survival function of the exponential distribution with rate $\lambda(1 - e^{-\mu})$, consistent with Poisson thinning.