

Last time

Stat 134 lec 27

sec 4.5 Expectation of a non-negative RV using CDF

$$E(X) = \int_0^{\infty} (1 - F(x)) dx$$

e.g. let  $X \sim \text{Geom}\left(\frac{1}{2}\right)$

$$P(X=1) = \frac{1}{2}$$

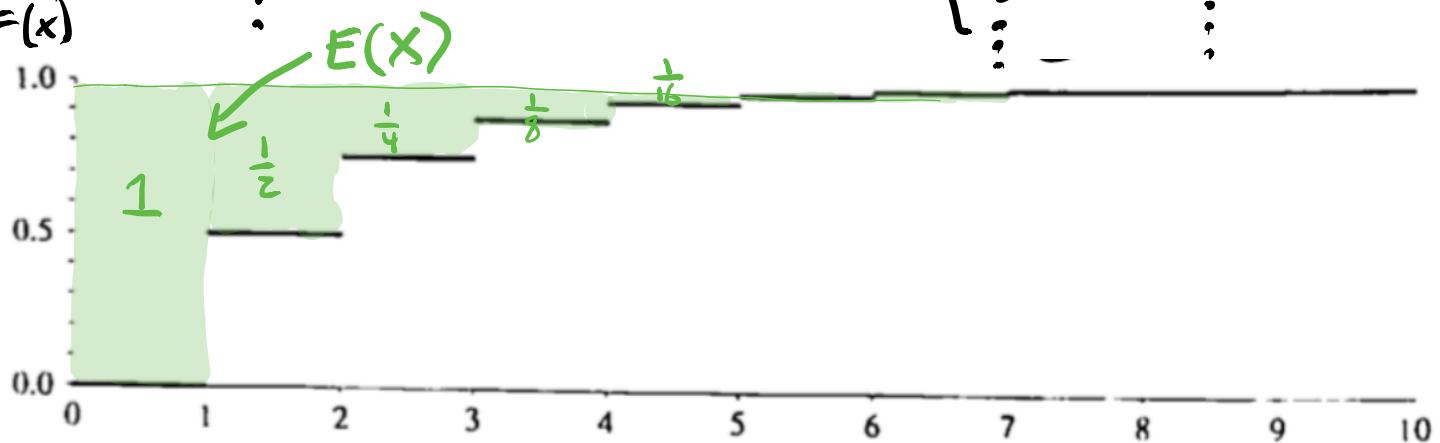
$$P(X=2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P(X=3) = \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} = \frac{1}{8}$$

Picture

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2} & 1 \leq x < 2 \\ \frac{3}{4} & 2 \leq x < 3 \\ \frac{7}{8} & 3 \leq x < 4 \\ \vdots & \vdots \end{cases}$$

$F(x)$



$$E(X) = \int_0^{\infty} (1 - F(x)) dx$$

$$E(X) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

$$= \sum_{j=0}^{\infty} P(X > j) = \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j \quad \leftarrow \text{tail sum formula, } q_j$$

Today

- ① Overview of what we have learned since the midterm.
- ② Review student explanations of MGF concept test,
- ③ sec 4.6 Order statistics
- ④ sec 5.1 continuous joint distributions,

## ① Overview

### Chap 4

Single  
variable  
unconditional  
Prob

density of distribution  
change of variable formula for densities,

expectation

continuous distributions >

- uniform
- exponential / gamma
- order statistics / beta

MGF - useful tool calculate moments identify a distribution by its MGF

CDF / mixed distributions

calculating expectation from Cdf.

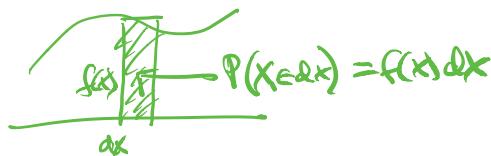
### Chap 5

multiple  
variable  
unconditional  
Prob

joint distributions >

### Chap 6

multiple  
variable  
conditional  
Prob.



① Student response

Let  $X$  have density  $f(x) = xe^{-x}$  for  $x > 0$ .

The MGF is?

- a  $M_X(t) = \frac{1}{1-t}$  for  $t < 1$
- b  $M_X(t) = \frac{1}{(1-t)^2}$  for  $t < 1$
- c  $M_X(t) = \frac{1}{(1+t)^2}$  for  $t > -1$
- d none of the above

a  
b  
c  
d

c

We can see that  $X$  is the variable part of a gamma(2,-1). The MGF of a gamma is  $(\lambda/\lambda-1)^r$ . Plugging in these values and dividing out the -1 yields answer C.

b

This is Gamma(2,1), so the MGF is  $(1/(1-t))^2$

match  $x e^{-x}$  with  
 $x^{r-1} e^{-tx}$

$\Rightarrow X \sim \text{Gamma}(2,1)$

Know  $M_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^r, t < 1$

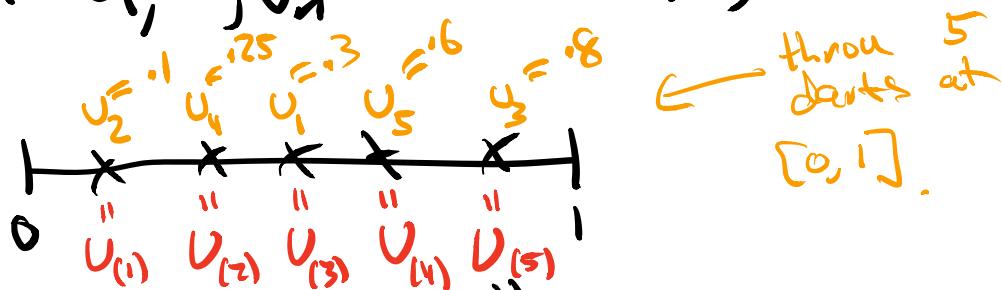
b

Using wolfram alpha I know that the integral of  $x^r e^{-(t-x)}$  dx from zero to infinity is  $1/(1-t)^{r+1}$  for  $t < 1$  thus b is the answer

cool !

② Sec 4.6 order statistic at  $U(0,1)$

let  $U_1, \dots, U_n \sim \text{Unit}(0,1)$  iid



let  $U_{(k)}$  = called the  $k^{\text{th}}$  order statistic  
 $= k^{\text{th}}$  largest value of  $U_1, \dots, U_n$   
 (assuming no ties)

$$U_{(1)} = \min(U_1, \dots, U_n)$$

$$U_{(n)} = \max(U_1, \dots, U_n)$$

Review counting  
 You have 3 red, 2 green and 5 blue marbles,  
 How many orderings of these 10 marbles are there?

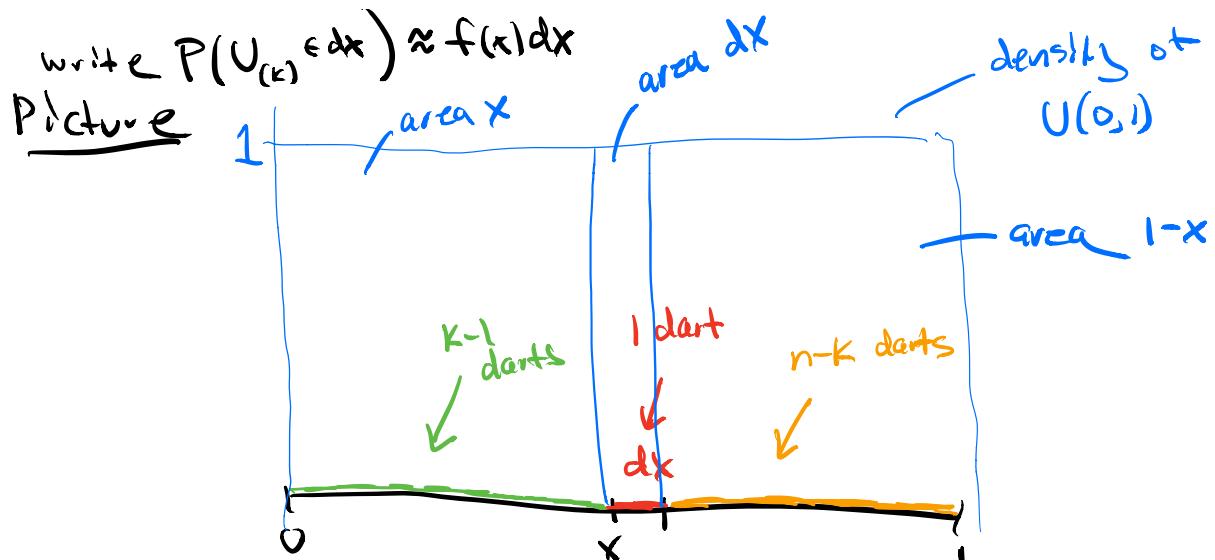
ex rrr ggg bbb bb  
 grrr g bb b bb  
 ggrrr bb b bb

Answe

$$\binom{10}{3,2,5} = \binom{10}{3} \binom{7}{2} \binom{5}{5}$$

$$\frac{10!}{3!2!5!}$$

Next, find density of  $U_{(k)}$



$U_{(k)} \in dx$  means that  $k-1$  darts are between  $0$  and  $x$ ,  
 and one is in  $dx$ , and  $n-k$  darts are between  $x$  and  $1$

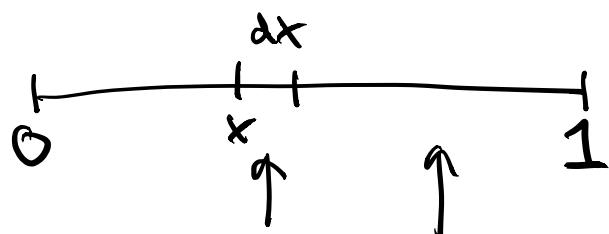
$$\begin{aligned}
 P(U_{(k)} \in dx) &= P(k-1 \text{ darts} \in (0, x), 1 \text{ dart} \in dx, n-k \text{ darts} \in (x, 1)) \\
 &= P(k-1 \text{ darts} \in (0, x)) \cdot P(1 \text{ dart} \in dx \mid k-1 \text{ darts} \in (0, x)) \\
 &\quad \cdot P(n-k \text{ darts} \in (x, 1) \mid 1 \text{ dart} \in dx, k-1 \text{ darts} \in (0, x)) \\
 &= \binom{n}{k-1} x^{k-1} \binom{n-k+1}{1} dx \binom{n-k}{n-k} (1-x)^{n-k} \\
 &= \binom{n}{k-1, 1, n-k} x^{k-1} (1-x)^{(n-k+1)-1} dx \\
 &\qquad\qquad\qquad f_{U_{(k)}}(x)
 \end{aligned}$$

$$\Rightarrow f_{U_{(k)}}(x) = \binom{n}{k-1, 1, n-k} x^{k-1} (1-x)^{(n-k+1)-1}$$

for  
 $0 < x < 1$

$\Leftarrow$  Let  $U_1, \dots, U_n \stackrel{\text{iid}}{\sim} \text{Unif}(0, 1)$

Find the density of  $U_{(1)}$



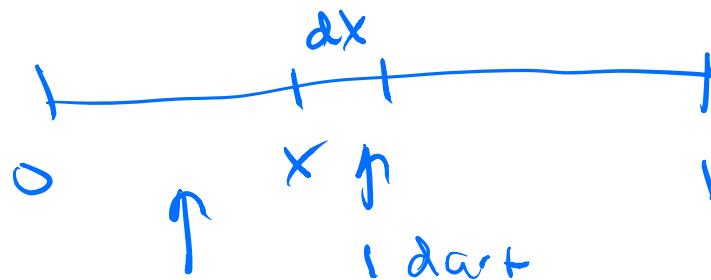
1 dart  $(n-1)$  darts

$$f_{U_{(1)}}(x)dx = \binom{n}{1, n-1} dx (1-x)^{n-1}$$

$$= n (1-x)^{n-1} dx$$

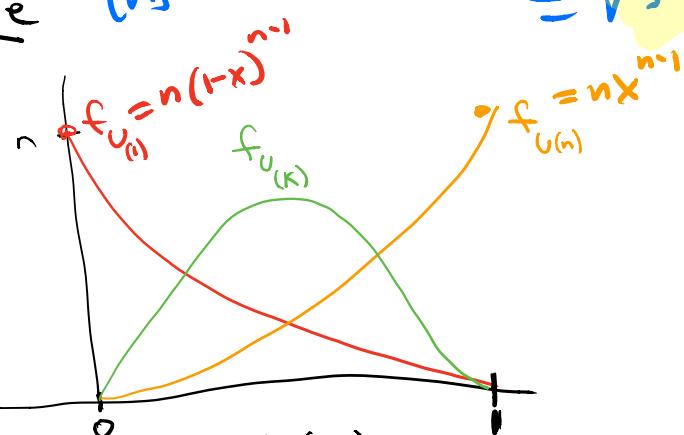
$\Leftarrow$  Let  $U_1, \dots, U_n \stackrel{iid}{\sim} \text{Unif}(0, 1)$

Find the density of  $U_{(n)}$



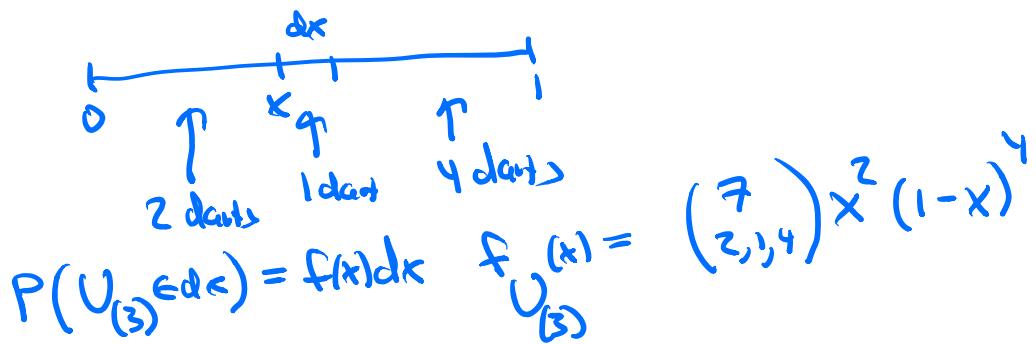
Picture

$$f_{U_{(n)}}(x) dx \approx \binom{n}{n-1} x^{n-1} dx = nx^{n-1} dx$$

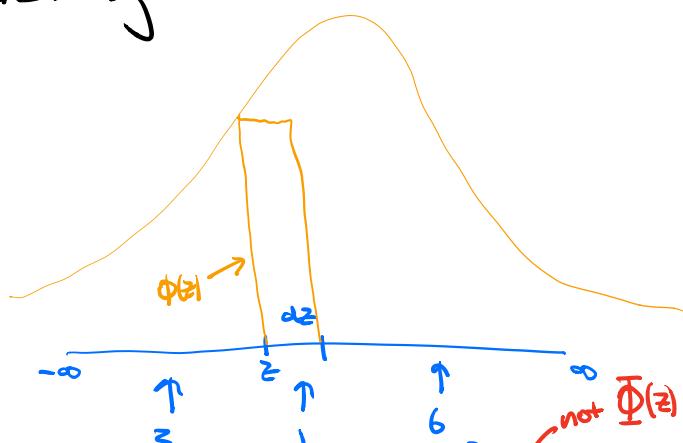


Order statistic of  $U(0, 1)$  provides a family of densities on the Unit Interval.

$\hat{x} = x^2(1-x)^4$  for  $0 < x < 1$  is the  
variable part of what RV? How many  
darts do you throw?



$\hat{x}$  Let  $Z_{(1)}, \dots, Z_{(10)}$  be the values of 10  
independent standard normal variables arranged  
in increasing order. Find the density of  $Z_{(4)}$



$$f(z)dz = \binom{10}{3,1,6} (\Phi(z))^3 \phi(z)dz (1 - \Phi(z))^6$$

$$\Rightarrow f(z) = \binom{10}{3,1,6} (\Phi(z))^3 \phi(z) (1 - \Phi(z))^6$$

## Chap 5 Continuous Joint Dist

sec 5.1, 5.2

$x, y$  have joint density  $f(x, y)$   
means  $f$  must satisfy  
 $f(x, y) \geq 0$  ← think of this as a surface over the plane

$$\iint_{\mathbb{R}^2} f(x, y) dx dy = \iint_{\mathbb{R}^2} f(x, y) dy dx = 1$$

the total volume under the surface is 1

Let  $A$  be a subset of the plane

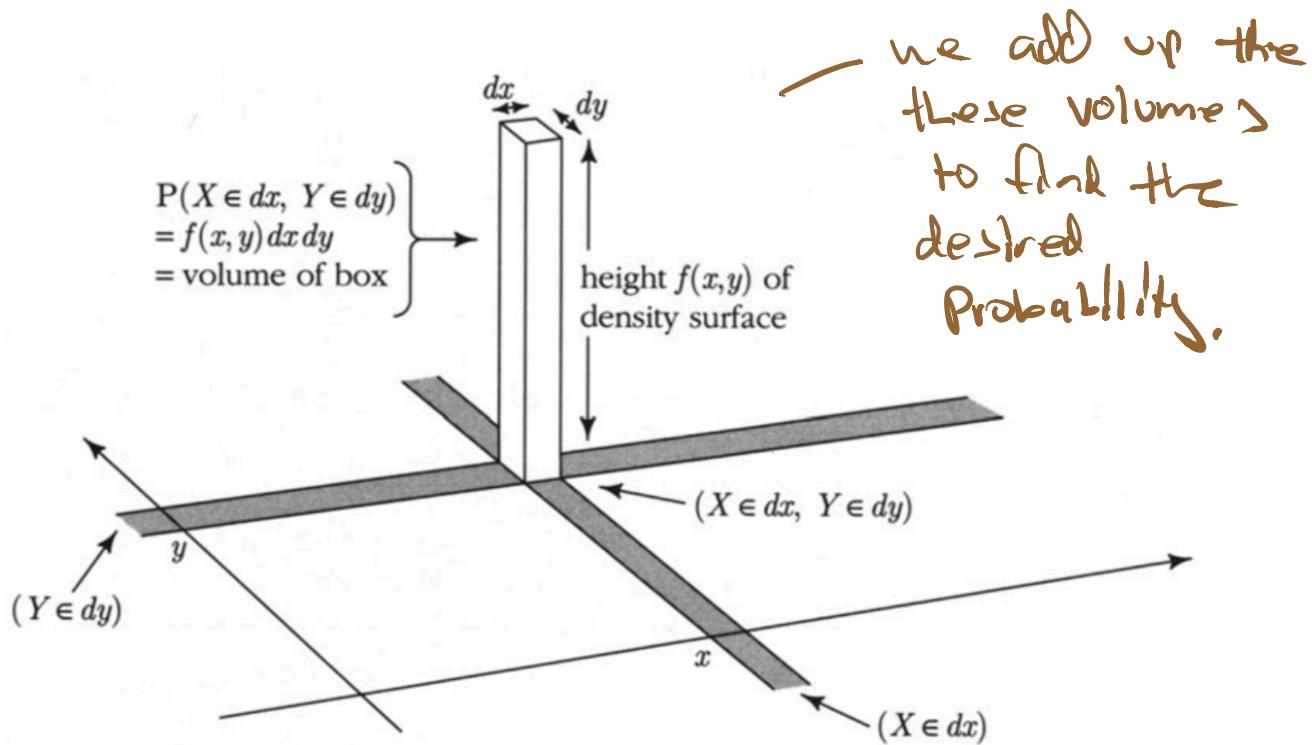
$$P(A) = P((x, y) \in A) = \iint_A f(x, y) dx dy$$

the volume of the surface over region  $A$ . This is a number between 0 and 1.

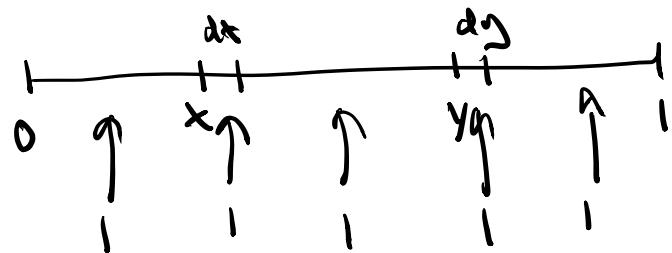
$$P(X \in dx, Y \in dy) = f(x, y) dx dy$$

the volume of the surface over a little rectangle in the plane.

## Picture



Ex Throw down 5 darts on  $(0,1)$ .  
 Find the joint density of  
 $X \in U_{(2)}$  and  $Y \in U_{(4)}$ .



$$f(x,y) = \binom{5}{1,1,1,1,1} \times dx \times (y-x) \times dy \times (1-y)$$

$$= \binom{5}{1,1,1,1,1} \times (y-x)(1-y)$$

"5!"       $0 < x < y < 1$

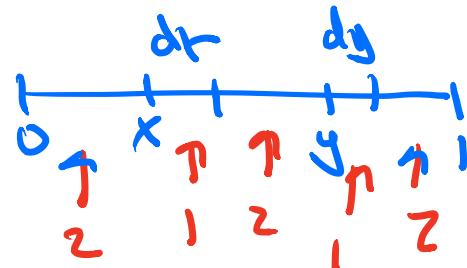
$$\Rightarrow \boxed{f(x,y) = 5! \times (y-x)(1-y) \quad \text{for } 0 < x < y < 1}$$

Stat 134

Wednesday April 3 2019

1. I throw down 8 darts on  $(0, 1)$ . The variable part of the joint density of  $X = U_{(3)}$  and  $Y = U_{(6)}$  is:

- a  $x(y - x)^5(1 - y)^2$
- b  $x^2(y - x)^2(1 - y)^2$
- c  $x^4(y - x)^2(1 - y)^2$
- d none of the above



2. Is  $f(x, y) = \binom{6}{1,4,1}(y - x)^4$  on  $0 < x < y < 1$  a joint density function?

**a** yes

**b** no

**c** not enough info to decide

