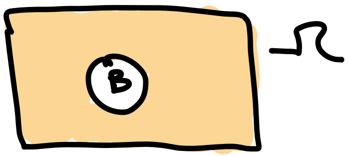
Stat 134 lec 2

Warm up 9:00 - 9:10

Prove the complement rule

$P(B^c) = 1 - P(B)$

Picture



$\Omega = B \cup B^c$ disjoint union

$P(\Omega) = P(B) + P(B^c)$ addition rule

$\Vert$

1

$P(B^c) = 1 - P(B)$

Difference rule

$P(AB^c) = P(A) - P(AB)$

Last time

(OR)

Addition rule if A, B mutually exclusive sets
$P(A \text{ or } B) = P(A) + P(B)$.

(OR)

Inclusion exclusion $P(A \text{ or } B) = P(A) + P(B) - P(AB)$

Today

(0) Mathematical Induction
(1) Sec 1.3 Distributions
(2) Sec 1.4 Conditional Probability

(0) Mathematical Induction

A **proof by induction** consists of two cases. The first, the **base case** (or **basis**), proves the statement for $n = 0$ (=1) without assuming any knowledge of other cases. The second case, the **induction step**, proves that *if* the statement holds for any given case $n = k$, *then* it must also hold for the next case $n = k + 1$. These two steps establish that the statement holds for every natural number n.

ex (1.3.12 in HW#1)

12. Inclusion–exclusion formula for n events. Derive the inclusion–exclusion formula for n events

$$P(\bigcup_{i=1}^{n} A_i) = \sum_i P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k) - \cdots + (-1)^{n+1} P(A_1 \ldots A_n)$$

ex
n=3

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1A_2) - P(A_1A_3) - P(A_2A_3) + P(A_1A_2A_3)$$

Assume the following fact from set theory:

$$\left(\bigcup_{i=1}^{k} A_i\right) A_{k+1} = \bigcup_{i=1}^{k} (A_i A_{k+1}) \quad (*)$$

ex $(A_1 \cup A_2) A_3 = A_1 A_3 \cup A_2 A_3$ $\quad k=2$

To prove generalized inclusion exclusion for $n=3$ we show by induction.

True for $n=1$ $\quad P(A_1) = P(A_1)$ ✓

Assume true for $n=2$.

Show true for $n=3$:

$P(A_1 \cup A_2 \cup A_3) =$

$$P\left(\bigcup_{i=1}^{2} A_i \cup A_3\right) = P\left(\bigcup_{i=1}^{2} A_i\right) + P(A_3) - P\left(\left(\bigcup_{i=1}^{2} A_i\right) A_3\right)$$

$$\left(\bigcup_{i=1}^{2} A_i\right) A_3 = \bigcup_{i=1}^{2} A_i A_3 \text{ by } (*)$$

$$= \underbrace{P\left(\bigcup_{i=1}^{2} A_i\right)}_{= P(A_1)+P(A_2)-P(A_1A_2)} + P(A_3) - \underbrace{P\left(\bigcup_{i=1}^{2} A_i A_3\right)}_{= P(A_1A_3) + P(A_2A_3) - P(A_1A_3A_2A_3)}$$

$$P(A_1)+P(A_2)-P(A_1A_2) \quad - P(A_1A_3) - P(A_2A_3) + P(A_1A_3A_2A_3)$$

$$A_1A_3A_2A_3 = A_1A_2A_3$$

For HW, try $n=4$ and generalize.

① Sec 1.3 Distributions

Uniform distribution

Let $\{x_1, x_2, \ldots, x_n\}$ be a finite set.

Suppose the probability of drawing each element is equally likely (i.e each has prob $\frac{1}{n}$)

We say $\{x_1, \ldots, x_n\}$ has the uniform distribution.

We write $\text{Unif}(\{x_1, \ldots, x_n\})$.

ex $\{1, 1, 2\}$ is a finite set.

$\text{Unif}(\{1,1,2\})$ means 1 has probability $\frac{2}{3}$ and 2 has probability $\frac{1}{3}$.

ex $\overset{7}{\text{Suppose}}$ $\overset{1}{\text{a}}$ $\overset{4}{\text{word}}$ $\overset{2}{\text{is}}$ $\overset{8}{\text{randomly}}$ $\overset{6}{\text{picked}}$ $\overset{4}{\text{from}}$ $\overset{4}{\text{this}}$ $\overset{8}{\text{Sentance}}$.

Name the distribution of the length of the word picked?

$\text{Unif}(\{7,1,4,2,8,6,4,4,8\})$

tinyurl.com/aug25-2022

Stat 134

1. A deck of cards is shuffled. What is the chance that the top card is the **king** of spades **or** the bottom card is the **king** of spades

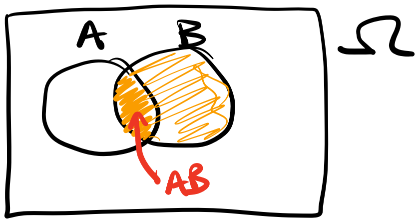
   a $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \times \frac{1}{52}$
   b $\frac{1}{52} + \frac{1}{51}$
   c $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \times \frac{1}{51}$
   d none of the above

$\frac{1}{52} + \frac{1}{52} \leftarrow P(\text{last card is king of spades})$

This is an unconditional probability.

② Sec 1.4 Conditional Probability and Independence

Let A, B be subsets of Ω (i.e events).



Baye's rule says $P(A|B) = \frac{P(AB)}{P(B)}$ (given)

$\Leftrightarrow$ $\boxed{P(\underset{A \text{ and } B}{AB}) = P(A|B)P(B)}$ multiplication rule. (AND)

We say A and B are <u>independent</u> iff

$P(A|B) = P(A)$

or equivalently if $P(AB) = P(A)P(B)$

ex

A = last card is queen of spades

B = 1st card is king of spades

Is A and B independent?

$$P(AB) = \underset{1/52}{P(B)}\,\underset{1/51}{P(A|B)} \overset{?}{\neq} \underset{1/52}{P(B)}\,\underset{1/52}{P(A)}$$

$\Rightarrow A, B$ are dependent.

ex

(10 pts) An airport bus drops off 35 passengers at ~~7~~ 2 ← 3 stops. Each passenger is equally likely to get off at any stop, and passengers act independently of one another. The bus makes a stop only if someone wants to get off. Find the probability that the bus drops off passengers at every stop.

Let B_i = event drop off at least one person at stop i.

$$P(B_1B_2) = 1 - P((B_1B_2)^c)$$

$$= 1 - P(B_1^c \cup B_2^c)$$

$$= 1 - \underbrace{P(B_1^c)}_{=\left(\frac{1}{2}\right)^{35}} - \underbrace{P(B_2^c)}_{=\left(\frac{1}{2}\right)^{35}} + \underbrace{P(B_1^cB_2^c)}_{=0}$$

$(B_1B_2)^c = B_1^c \cup B_2^c$

$(B_1 \cup B_2)^c = B_1^cB_2^c$

De Morgan's Law rule

If the bus has 3 stops:

$$P(B_1B_2B_3) = 1 - P((B_1B_2B_3)^c)$$

$$= 1 - P(B_1^c \cup B_2^c \cup B_3^c)$$

$$= 1 - \underbrace{\sum_{i=1}^{3} P(B_i^c)}_{=\binom{3}{1}\left(\frac{2}{3}\right)^{35}} + \underbrace{\sum_{i<j} P(B_i^cB_j^c)}_{\binom{3}{2}\left(\frac{1}{3}\right)^{35}} - \underbrace{P(B_1^cB_2^cB_3^c)}_{=0}$$

3 choose 1 $\binom{3}{1} = 3$

3 choose 2 $\binom{3}{2} = 3$

If the bus has 7 stops:

$$P(B_1B_2\cdots B_7) = 1 - \binom{7}{1}\left(\frac{6}{7}\right)^{35} + \binom{7}{2}\left(\frac{5}{7}\right)^{35} - \cdots - \binom{7}{7}\left(\frac{7-7}{7}\right)^{35}$$

stops

$$= \boxed{\sum_{j=0}^{7} (-1)^j \binom{7}{j}\left(\frac{7-j}{7}\right)^{35}}$$

Inclusion–exclusion formula for n events. Derive the inclusion–exclusion formula for n events

$$P(\bigcup_{i=1}^{n} A_i) = \sum_i P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) - \cdots + (-1)^{n+1} P(A_1 \ldots A_n)$$