Stat 134 lec 34

Warmup 8:00 - 8:10 AM

Iterated expectation

Let $N \sim Geom(p)$ on $1, 2, 3, \ldots$

Suppose $X \mid N = n \sim Unif(0, 1, 2, \ldots, n)$

Find $E(X)$

$N \sim Pois(\mu)$

$X \mid N = n \sim Unif(0, 1, 2, \ldots, n)$

$E(X \mid N = n) = \frac{n}{2}$

$E(X \mid N) = \frac{N}{2}$

$E(X) = E(E(X \mid N)) = E\left(\frac{N}{2}\right) = \frac{1}{2p}$

Stat 134 lec 34

Quiz 6 Wednesday Dec 4 Sec 5.3 – 6.3

Last time 3 discrete

Rule of average conditional probabilities (discrete case)

marginal prob of X

$$P(X=x) = \sum_{s} P(X=x, S=s)$$
$$= \sum_{s} P(X=x \mid S=s)P(S=s)$$

Conditional expectation

$$E(T \mid S=s) = \sum_{t \in T} t \cdot P(T \mid S=s)$$

$E(T \mid S)$ is a RV (function of S)

Law of Iterated Expectation

$$E(T) = E(E(T \mid S))$$
$$E(T) = \sum_{\text{all } s} E(T \mid S=s) \cdot P(S=s)$$

Today

1. Sec 5.4 General convolution formula
2. Sec 6.2 properties of conditional expectation.
3. Sec 6.3
   a) Conditional densities,
   b) multiplication rule
   c) rule of average conditional probabilities

① sec 5.4 General convolution formula

We have different convolution formulas for sums and quotients.

We can write a general convolution formula for any operation.

1 dimensional change of variables

RV	transformed RV
Y	$Z(Y)$ — a 1-1 differentiable function
f_Y	$f_Z = \left\lvert \frac{\partial y}{\partial z} \right\rvert f_Y$ ← $\frac{1}{\left\lvert \frac{\partial z}{\partial y} \right\rvert}$

2 dimensional change of variables

RV	transformed RV
(X, Y)	$(X, Z(X,Y))$ — a 1-1 differentiable function
$f_{X,Y}$	$f_{X,Z} = \left\lvert \det \frac{\partial(X,Y)}{\partial(X,Z)} \right\rvert f_{X,Y}$

$$= \left\lvert \det \begin{bmatrix} \underbrace{\frac{\partial X}{\partial X}}_{=1} & \frac{\partial X}{\partial Z} \\ \underbrace{\frac{\partial Y}{\partial X}}_{=0} & \frac{\partial Y}{\partial Z} \end{bmatrix} \right\rvert f_{X,Y}$$

$$= \left\lvert \frac{dy}{dz} \right\rvert f_{X,Y}$$

Convolution formula

let $Z(X,Y)$ be a 1-1 differentiable function of X, Y

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Z}(x,z)\, dx = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y) \left\lvert \frac{\partial y}{\partial z} \right\rvert dx$$

ex Let $Z = \frac{Y}{X}$. Find the convolution formula for Z.

$\Rightarrow Y = XZ \Rightarrow \frac{\partial y}{\partial z} = X$

$$\Rightarrow f_Z(z) = \int_{x=-\infty}^{x=\infty} f_{X,Y}(x, xz)\,|x|\,dx$$

convolution formula for quotient.

ex Let $Z = \frac{X}{X+Y}$. Find the convolution formula for Z.

$ZX + ZY = X \Rightarrow ZY = X - ZX$

$\Rightarrow Y = \frac{X(1-Z)}{Z}$

$$\Rightarrow \frac{\partial Y}{\partial Z} = X\left[\frac{(1-Z)'Z - (1-Z)(Z)'}{Z^2}\right]$$

$$= X\left[\frac{-Z - 1 + Z}{Z^2}\right] = \frac{-X}{Z^2}$$

$$\Rightarrow f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}\left(x, \frac{x(1-z)}{z}\right)\frac{|x|}{z^2}\,dx$$

② Sec 6.2 Properties of Conditional expectation

$(Y+Z)|X=x = Y|X=x + Z|X=x$ so

$E(Y+Z|X=x) = E(Y|X=x) + E(Z|X=x)$

What is $E(X+Z|X=5) = ?$

$5 + E(Z|X=5)$

$E(X+Y|X) = \underbrace{E(X|X)}_{X} + E(Y|X)$

Properties

① $E(X) = E(E(X|Y))$ equality of numbers

② $E(aY+b|X) = aE(Y|X)+b$

③ $E(Y+Z|X) = E(Y|X)+E(Z|X)$

④ $E(g(X)|X) = g(X)$

⑤ $E(g(X)Y|X) = g(X)E(Y|X)$

equality of RV

(3) sec 6.3 Conditional Density:

(a)

Let A be an event.

If X is a discrete RV

$$P(A \mid X=x) = \frac{P(A, X=x)}{P(X=x)}$$ by Bayes' rule.

If X is continuous,

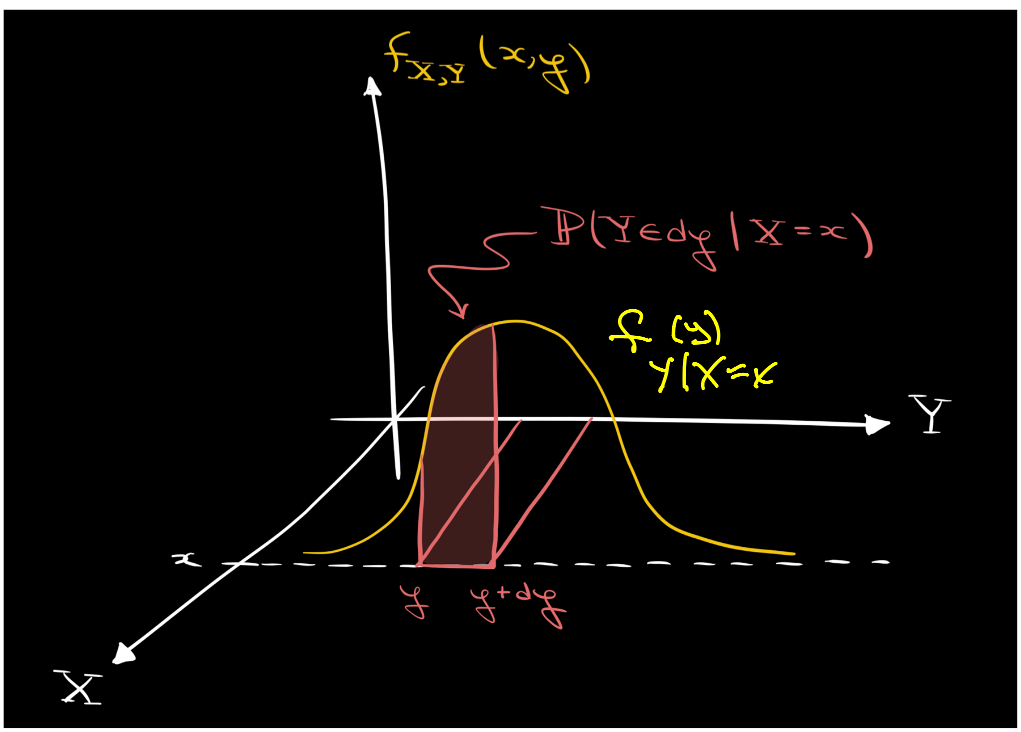
$$P(A \mid X\in dx) = \frac{P(A, X\in dx)}{P(X\in dx)}$$

we define

$$P(A \mid X=x) = \lim_{dx\to 0} P(A \mid X\in dx)$$

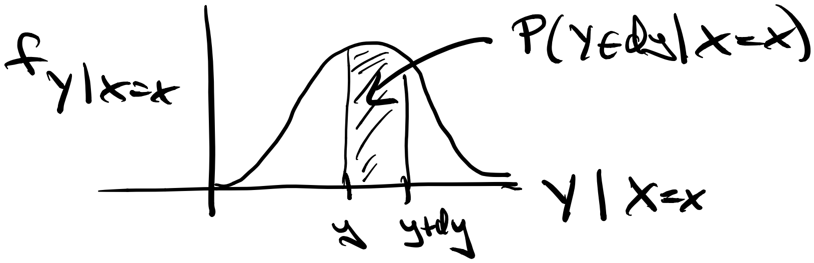
If Y is continuous

$$P(Y\in dy \mid X=x) = \lim_{dx\to 0} \frac{P(Y\in dy, X\in dx)}{P(X\in dx)}$$



$f_{Y|X=x}(y)$ is a slice of $f(x,y)$ through $X=x$.

We will draw the distribution of $Y|X=x$ as



$$P(Y\in dy \mid X=x) = \lim_{dx\to 0} \frac{\overbrace{P(Y\in dy, X\in dx)}^{= f(x,y)\,dx\,dy}}{\underbrace{P(X\in dx)}_{= f_X(x)\,dx}}$$

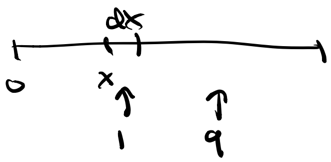
$$\| \quad f_{Y|X=x}(y)\,dy$$

$$\boxed{f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)}}$$ Conditional density of Y given $X=x$

ex Let $U_1, \dots, U_{10} \overset{iid}{\sim} U(0,1)$

$X = U_{(1)}$, $Y = U_{(10)}$

$f(x,y) = 90(y-x)^8$



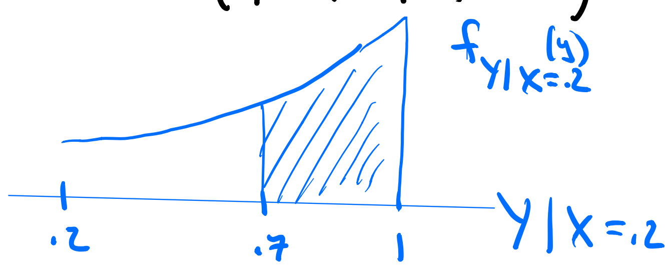
$$f_X(x) = \binom{10}{1,9}(1-x)^9 = 10(1-x)^9$$

$$\boxed{f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)}}$$

a) Find $f_{Y|X=.2}(y)$

$$f_{Y|X=.2}(y) = \frac{f(.2,y)}{f_X(.2)} = \boxed{\frac{90(y-.2)^8}{10(.8)^9}}$$

b) Find $P(Y>.7 \mid X=.2) = \frac{9}{(.8)^9}\int_{.7}^{1}(y-.2)^8\,dy$



let $u = y - .2$

$$= \frac{9}{(.8)^9}\int_{u=.5}^{u=.8} u^8\,du$$

$$= \frac{\cancel{9}}{(.8)^9}\frac{u^9}{\cancel{9}}\Big|_{.5}^{.8} = \boxed{1 - \left(\frac{.5}{.8}\right)^9}$$

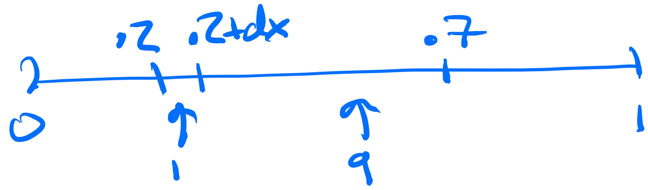
Method 2 (use fact that $X = U_{(1)}$ and $Y = U_{(10)}$)

$$P(Y > .7 \mid X = .2) = 1 - P(Y < .7 \mid X = 0.2)$$

By Bayes' rule,

$$P(Y < .7 \mid X = 0.2) = \lim_{dx \to 0} \frac{P(Y < .7,\ X \in .2 + dx)}{P(X \in .2 + dx)}$$

Picture



$P(Y < .7, X \in .2 + dx)$ is the chance that the $U_{(1)}$ is in dx and the remaining 9 darts land between .2 and .7

this is $1 dx (.7 - .2)^9 = dx(.5)^9$.

$P(X \in .2 + dx)$ is the chance that the $U_{(1)}$ is in dx and the remaining 9 darts land between .2 and 1.

this is $1 dx (1 - .2)^9 = dx(.8)^9$.

Hence
$$P(Y > .7 \mid X = .2) = \lim_{dx \to 0} 1 - \frac{\cancel{dx}(.5)^9}{\cancel{dx}(.8)^9}$$

$$= \boxed{1 - \left(\frac{.5}{.8}\right)^9}$$

(b) Multiplication rule

discrete case:

$$P(X=x, Y=y) = P(Y=y|X=x) \cdot P(X=x)$$

continuous case:

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} \Rightarrow \boxed{f(x,y) = f_{Y|X=x}(y)\, f_X(x)}$$

ex $X \sim \text{Gamma}(2, \lambda)$

$Y|X=x \sim \text{Unif}(0, x)$

Find $f(x,y)$

soln

$$f(x,y) = f_{Y|X=x}(y)\, f_X(x)$$

where

$$f_{Y|X=x}(y) = \frac{1}{x}, \quad 0<y<x<\infty$$

$$f_X(x) = \lambda^2 x e^{-\lambda x}$$

$$\Rightarrow f(x,y) = \frac{1}{x} \lambda^2 x e^{-\lambda x} = \boxed{\lambda^2 e^{-\lambda x}}$$

(c) The rule of average conditional probabilities

Let A be an event,

discrete case (X discrete) :

$$P(A) = \sum_{x \in X} P(A \mid X=x) P(X=x)$$

continuous case (X continuous) :

$$P(A) = \int_{x \in X} P(A \mid X=x) P(X \in dx)$$

$$\boxed{P(A) = \int_{x \in X} P(A \mid X=x) f_X(x)\, dx}$$

integral conditioning formula

ex $X \sim \text{Unif}(0,1)$

given $X=x$, let $I_1, I_2 \overset{iid}{\sim} \text{Ber}(x)$

Let A be the event that the first toss is heads $(I_1 = 1)$,

CAUTION X is continuous and I_1 is discrete

We write $P(I_1 \mid X=x)$ for conditional probability mass function (pmf) of I_1 and $f_{X \mid I_1 = 1}(x)$ for the conditional density of X.

a) Find $P(I_1=1 \mid X=x)$

x

this is called a likelihood in Bayesian statistics.

We dont need Baye's rule to find this.

b) Find $P(I_1=1)$

A is event $I_1=1$

$$P(A) = \int_{x=0}^{x=1} \overbrace{P(A \mid X=x)}^{x} \cdot \overbrace{f_X(x)}^{1}\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

interpretation: with no prior info about what is the chance of getting heads $P(A) = 1/2$.

c) Find $f_{X|I_1=1}(x)$ ← this is called a posterior in Bayesian statistics.

$$f_{X|I_1=1}(x)\, P(I_1=1) = f_{X,I_1}(x,1) \quad \text{mult rule}$$

$$P(I_1=1 \mid X=x) f(X=x) = f_{X,I_1}(x,1) \quad \text{mult rule}$$

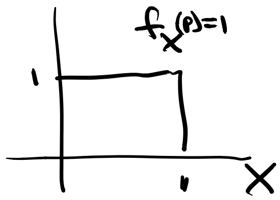
so

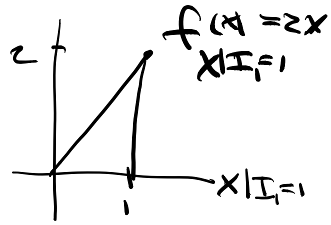
$$\underbrace{f_{X|I_1=1}(x)}_{\text{posterior}}\, P(I_1=1) = \underbrace{P(I_1=1 \mid X=x)}_{\text{likelihood}}\, \underbrace{f(X=x)}_{\text{prior}}$$

$$\Rightarrow f_{X|I_1=1}(x) = \frac{P(I_1=1 \mid X=x)\, f(X=x)}{P(I_1=1)} \leftarrow \text{constant.}$$

$$= \frac{x \cdot 1}{1/2} = \boxed{2x}$$

Prior $X \sim Unif(0,1)$ **Posterior**





Interpretation: if we get a heads in 1^{st} toss then it is more likely the prob of heads is closer to 1 than closer to 0.