Stat 134 lec 16

last time sec 3.4
Geometric distribution
neg binomial distribution — sum of iid Geom(p)

today sec 3.5
Poisson distribution
Poisson random scatter (PRS) AKA Poisson Process

Sec 3.5

Poisson distribution

$X \sim Pois(\mu)$

$$P(X=k) = e^{-\mu}\frac{\mu^k}{k!} \quad k=0,1,2,\dots$$

Find $E(X)$, $Var(X)$.

Recall $e^{\mu} = 1+\mu+\frac{\mu^2}{2!}+\cdots$ Taylor series.

$$E(X) = \sum_{k=0}^{\infty} k e^{-\mu}\frac{\mu^k}{k!}$$

(note $0 \cdot e^{-\mu}\frac{\mu^0}{0!} = 0$)

$$= \sum_{k=1}^{\infty} \cancel{k} e^{-\mu}\frac{\mu^{k-1}\mu}{(k-1)!\cancel{k}}$$

$$= \mu e^{-\mu}\sum_{k=1}^{\infty}\frac{\mu^{k-1}}{(k-1)!}$$

$$= \mu e^{-\mu}\underbrace{\left(1+\mu+\frac{\mu^2}{2!}+\cdots\right)}_{e^{\mu}} = \boxed{\mu}$$

Makes sense since

$\text{bin}(n,p) \longrightarrow \text{Pois}(\mu)$ when
n large
p small
$np \to \mu$

and np is expectation of binomial.

$Var(X)$? We expect it to be μ
since $p \approx 0$, $q \approx 1$ and $npq \approx np \to \mu$.

$$Var(X) = E(X^2) - E(X)^2$$
$$= E(X^2) - E(X) + E(X) - E(X)^2$$
$$= E(X(X-1)) + E(X) - E(X)^2$$

$$E(X(X-1)) = \sum_{k=0}^{\infty} \cancel{k}(\cancel{k}-1) P(X=k)$$

$$P(X=k) = \frac{e^{-\mu}\mu^k}{\cancel{k}(\cancel{k-1})(k-2)!}$$

$$= e^{-\mu} \sum_{k=2}^{\infty} \frac{\mu^k}{(k-2)!}$$

$$= e^{-\mu}\mu^2 \underbrace{\sum_{k=2}^{\infty} \frac{\mu^{k-2}}{(k-2)!}}_{= e^{\mu}} = \mu^2$$

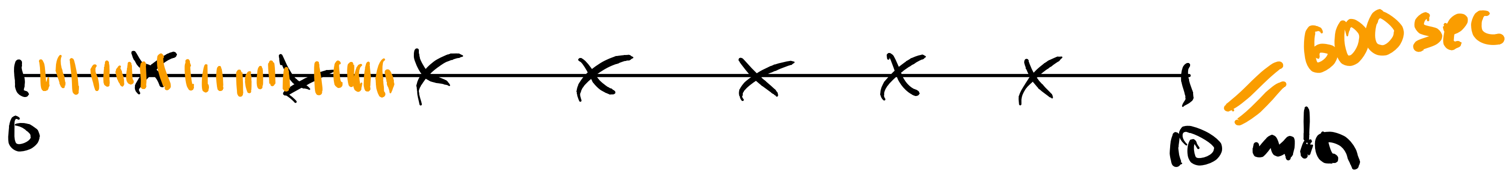
$$\Rightarrow Var(X) = \mu^2 + \mu - \mu^2 = \boxed{\mu}$$

Poisson Random Scatter (PRS)

Thinking of Poisson(μ) distribution as the limit of Bin(n,p) for $n \to \infty$, $p \to 0$, $np \to \mu$, we see that Pois(μ) can be used to model counts of low probability independent events.

ex The number of calls coming into a hotel reservation center in 10 minutes.

Say $\mu = 5$



distribution of calls should look random not clustered.

idea: divide 10 min into small intervals (say every second)

PRS assumptions

1) No time interval gets more than one call

2) Have iid Bernoulli p trials (i.e. all calls are independent of each other with same prob).

The mean number of calls in 10 minutes is $\underset{5}{\mu} = np$

let $\lambda = \frac{\mu}{10}$ be the intensity (rate) of calls/min for our PRS.
← or some unit of time.

or equivalently $\mu = 10 \cdot \lambda$ is the avg number of calls in 10 min.

Stat 134

Friday September 28 2018

1. Which of the following can be modeled as a Poisson Random Scatter with intensity $\lambda > 0$?

a The number of blueberries in a 3 cubic inch blueberry muffin

b The number of patients entering a doctor's office in a 24 hour period.

– no patients outside regular hours

c The number of times a day a person feels hungry

– not indep hits

d The number of air pulses counted every second from cars driving over an empty rubber hose lying across a highway between noon and 1pm.

– not indep hits

e more than one of the above

ex Blueberry muffins

PPS intensity $\lambda = 2$ blueberries per cubic inch

A muffin is 3 cubic inches.

On average how many blueberries per muffin?

$\mu = \lambda \cdot 3 = 6$ blueberries.

X_1 = # blueberries in first muffin

$X_1 \sim \text{Pois}(6)$

Another muffin (from same tub of batter)

size 4 cubic inches

let X_2 = # blueberries in second muffin

$X_2 \sim \text{Pois}(8)$

Find P(5 blueberries in each muffin)

$= P(X_1 = 5, X_2 = 5) = P(X_1 = 5)P(X_2 = 5)$

$$= \frac{e^{-6}\, 6^5}{5!} \cdot \frac{e^{-8}\, 8^5}{5!}$$

Find P(10 blueberries total in both months)

$$P(X_1 + X_2 = 10) = \frac{e^{-14}\, 14^{10}}{10!}$$

$$X_1 + X_2 \sim \text{Pois}(6+8)$$