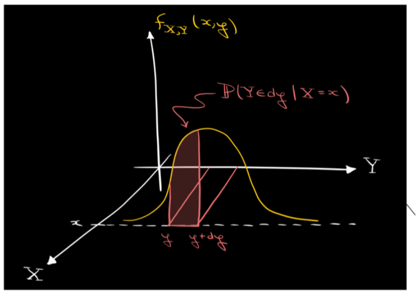
Stat 134 lec 36

Last time

sec 6.3 Conditional density



$$f_{Y|X=x}(y) = \frac{f_{(X,Y)}(x,y)}{f_X(x)}$$

Average conditional probabilities

X discrete, Y discrete

$$P(Y=y) = \sum_{x \in X} P(Y=y \mid X=x) P(X=x)$$

X continuous Y continuous

$$P(Y \in dy) = \int_{x \in X} P(Y \in dy \mid X=x) f_X(x)\,dx$$

X discrete Y continuous

$$P(Y \in dy) = \sum_{x \in X} P(Y \in dy \mid X=x) P(X=x)$$

X continuous Y discrete

$$P(Y=y) = \int_{x \in X} P(Y=y \mid X=x) f_X(x)\,dx$$

Today ① sec 6.3 more practice

② covariance and the variance of sum

④ **More practice**

3. (6 pts) Suppose $Y \sim$ Pois (X), where $X \sim$ Exp (λ). That is, given $X = x$, $Y \sim$ Pois (x).

   (a) (3 pts) Show that the unconditional distribution of Y is Geometric $(\frac{\lambda}{\lambda+1})$ on $\{0, 1, 2, \ldots\}$.

$$P(Y=y) = \int_0^\infty P(Y=y \mid X=x) f_X(x)\,dx$$

$$= \int_0^\infty \frac{e^{-x} x^y}{y!} \lambda e^{-\lambda x}\,dx$$

$$= \frac{\lambda}{(\lambda+1)^{y+1}} \underbrace{\int_0^\infty \frac{(\lambda+1)^{y+1}}{y!} x^{y+1-1} e^{-x(\lambda+1)}\,dx}_{=1}$$

recall

$X \sim \text{Gamma}(y+1, \lambda+1)$

$$f_X(x) = \frac{(\lambda+1)^{y+1}}{\Gamma(y+1)} x^{y+1-1} e^{-x(\lambda+1)}$$

$$= \frac{\lambda}{(\lambda+1)^{y+1}} = \left(\frac{1}{\lambda+1}\right)^y \frac{\lambda}{\lambda+1}$$

$\Rightarrow Y \sim \text{Geom}\left(\frac{\lambda}{\lambda+1}\right)$.

recall if $Z \sim \text{Geom}(p)$ on $0, 1, 2, \ldots$

$P(Z=k) = q^k p$

so $Z \sim \text{Geom}\left(\frac{\lambda}{\lambda+1}\right)$ on $0, 1, 2, \ldots$

$$P(Z=k) = \left(\frac{1}{\lambda+1}\right)^k \left(\frac{\lambda}{\lambda+1}\right)$$

Suppose $Y \sim$ Pois (X), where $X \sim$ Exp (λ). That is, given $X = x$, $Y \sim$ Pois (x).

(b) (3 pts) Show that given $Y = k$, the conditional distribution of X is a gamma distribution, and provide the parameters. (Hint: use the result of part (a), even if you weren't able to prove it.)

$$P(X \in dx \mid Y=k) = f_{X|Y=k}(x)\, dx = \frac{f_{X,Y}(x,k)\, dx}{P(Y=k)}$$

$$= \frac{P(Y=k \mid X=x) f_X(x)\, dx}{P(Y=k)}$$

$$= \frac{\frac{e^{-x} x^{k}}{k!} \lambda e^{-\lambda x} dx}{\frac{\lambda}{\lambda+1}\left(\frac{1}{\lambda+1}\right)^{k}}$$

$$\propto x^{k} e^{-(\lambda+1)x}$$

$$\Rightarrow \boxed{X \sim \text{Gamma}(k+1, \lambda+1)}$$

(2) Sec 6.4 Covariance and variance of a sum

$X, Y, \quad S = X + Y$

mean $\mu_X, \mu_Y, \mu_S = \mu_X + \mu_Y$

$$\mathrm{Var}(S) = E\left(\left(\underbrace{S - \mu_S}_{D_S}\right)^2\right) = E(D_S^2)$$

D_S deviation from mean

$$\begin{aligned} D_S &= S - \mu_S \\ &= X + Y - (\mu_X + \mu_Y) \\ &= D_X + D_Y \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(S) &= E\left((D_X + D_Y)^2\right) \\ &= E\left(D_X^2 + D_Y^2 + 2D_X D_Y\right) \\ &= \underbrace{E(D_X^2)}_{\mathrm{Var}(X)} + \underbrace{E(D_Y^2)}_{\mathrm{Var}(Y)} + 2\underbrace{E(D_X D_Y)}_{\mathrm{Cov}(X,Y)} \end{aligned}$$

Defn The covariance of X and Y is

$$\mathrm{Cov}(X,Y) = E\left((X - \mu_X)(Y - \mu_Y)\right).$$

Proved end of lecture.

Bilinearity Properties

ⓐ $\mathrm{Cov}(X+Y, Z) = \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Z)$

ⓑ $\mathrm{Cov}(aX, bY) = ab\,\mathrm{Cov}(X,Y)$

<u>More generally</u>

$$\boxed{\mathrm{Cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \,\mathrm{Cov}(X_i, Y_j)}$$

Proved end of lecture.

Thm $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

<u>Easy facts</u>

$\mathrm{Cov}(X,X) = E(X^2) - E(X)^2 = \mathrm{Var}(X)$

$\mathrm{Cov}(X,X) = \mathrm{Var}(X)$

$\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$

$\mathrm{Cov}(X,c) = 0$

constant

ex

Simplify

$Cov(X-5Y,\ 3X+Y-Z+10)$

$3Var(X)+Cov(X,Y)-Cov(X,Z)+0$

$-15Cov(X,Y)-5Var(Y)+5Cov(Y,Z)$
$+0$

Recall X, Y independent

$\Rightarrow \quad E(XY)=E(X)E(Y)$

$Cov(X,Y)=0$ if X, Y independent,

Hence if X, Y indep,

$Var(X+Y) = Var(X)+Var(Y)+2\underbrace{Cov(X,Y)}_{=0}$

$= Var(X)+Var(Y)$

tinyurl: http://tinyurl.com/april24-pt1

Stat 134

Wednesday April 24 2019

1. Consider a Poisson(λ) process. Let $T_r \sim$ gamma(r, λ) be the rth arrival time. Cov(T_1, T_3) equals:

   Recall $Var(T_r) = \frac{r}{\lambda^2}$

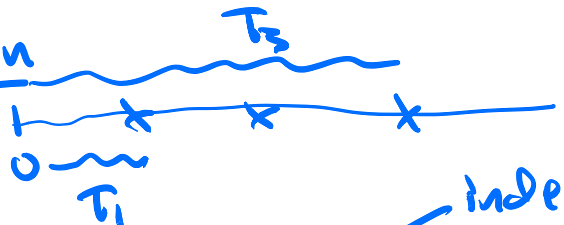
   a λ

   b λ^2

   c $1/\lambda^2$

   d none of the above

Soln



$Cov(T_1, T_3 - T_1) = 0$ indep

$Cov(T_1, T_3) - var(T_1) = 0$

$Cov(T_1, T_3) = var(T_1) = \frac{1}{\lambda^2}$

ex

Let $X_1, \dots, X_n$ be identically distributed

$$\mathrm{var}\left(\sum_{i=1}^{n} X_i\right) = \mathrm{Cov}\left(\sum_{i=1}^{n} X_i , \sum_{j=1}^{n} X_j\right)$$

Defn

The variance-covariance matrix has all n^2 terms

$$\begin{array}{c|c} & \begin{array}{ccccc} x_1 & x_2 & \cdots & & x_n \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ \\ x_n \end{array} & \begin{bmatrix} \mathrm{Cov}(x_1,x_1) & \mathrm{Cov}(x_1,x_2) & & \\ & \mathrm{Cov}(x_2,x_2) & & \\ & & \ddots & \\ & & & \mathrm{Cov}(x_n,x_n) \end{bmatrix}_{n \times n} \end{array}$$

$$\mathrm{var}\left(\sum_{i=1}^{n} X_i\right) = \underbrace{n\,\mathrm{var}(X_1)}_{\text{diagonal}} + \underbrace{n(n-1)\,\mathrm{Cov}(X_1, X_2)}_{\text{off diagonal}}$$

ex

$x_1, \dots, x_N$ — pop mean μ
pop SD σ

draw n w/o replacement
(called a simple random sample (SRS))

$X_1, \dots, X_n$

let $S_n = X_1 + \dots + X_n$ be the sample sum

Find $E(S_n)$
$\quad Var(S_n)$.

$$E(S_n) = E(X_1 + \dots + X_n) = nE(X_1) = n\mu$$

$$Var(S_n) = n\underbrace{Var(X_1)}_{= \sigma^2} + n(n-1)Cov(X_1, X_2)$$

trick to find $Cov(X_1, X_2)$
take a census $n = N$

$$Var(S_n) = Var(S_N) = 0$$

$$\Rightarrow 0 = N\sigma^2 + N(N-1)Cov(X_1, X_2)$$

$$\Rightarrow \operatorname{Cov}(X_1, X_2) = -\frac{N\sigma^2}{N(N-1)} = \boxed{-\frac{\sigma^2}{N-1}}$$

So $\operatorname{Var}(S_n) = n\sigma^2 + n(n-1)\left(-\frac{\sigma^2}{N-1}\right)$

$$= n\sigma^2\left[\frac{N-n}{N-1}\right]$$

correction factor < 1

(note if draw w/ replacement

$\operatorname{Var}(S_n) = n\sigma^2$).

Recall, we have seen this correction factor before (p241)

$$X \sim \operatorname{Bin}(n,p) \Rightarrow \operatorname{Var}(X) = (npq)$$

$$X \sim \text{hyper-geom}(n, N, G) \Rightarrow \operatorname{Var}(X) = n\left(\frac{G}{N}\right)\left(\frac{N-G}{N}\right)\left(\frac{N-n}{N-1}\right).$$

$\uparrow$ p $\quad$ $\uparrow$ q $\quad$ $\uparrow$ correction factor

Appendix

Bilinearity Properties

Thm

(a) $\text{Cov}(X+Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$

(b) $\text{Cov}(aX, bY) = ab\,\text{Cov}(X, Y)$

Pf/

a)

$$\text{Cov}(X+Y, Z) = E\left((X+Y-\mu_{X+Y})(Z-\mu_Z)\right)$$

$$= E\left(\left[(X-\mu_X)+(Y-\mu_Y)\right](Z-\mu_Z)\right)$$

$$= E\left((X-\mu_X)(Z-\mu_Z) + (Y-\mu_Y)(Z-\mu_Z)\right)$$

$$= E\left((X-\mu_X)(Z-\mu_Z)\right) + E\left((Y-\mu_Y)(Z-\mu_Z)\right)$$

$$= \text{Cov}(X, Z) + \text{Cov}(Y, Z). \quad \square$$

b)

$$\text{Cov}(aX, bY) = E\left((aX-\mu_{aX})(bY-\mu_{bY})\right)$$

$$= E\left((aX-a\mu_X)(bY-b\mu_Y)\right)$$

$$= E\left(ab(X-\mu_X)(Y-\mu_Y)\right)$$

$$= ab\,E\left((X-\mu_X)(Y-\mu_Y)\right)$$

$$= ab\,\text{Cov}(X, Y) \quad \square$$

Appendix

Thm $Cov(X,Y) = E(XY) - E(X)E(Y)$

pf/ $Cov(X,Y) = E(D_X D_Y) = E\big((X-\mu_X)(Y-\mu_Y)\big)$

$$= E\left(XY - \mu_X Y - X\mu_Y + \mu_X\mu_Y\right)$$

$$= E(XY) - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_Y)$$

$$= E(XY) - E(X)E(Y)$$

$\square$