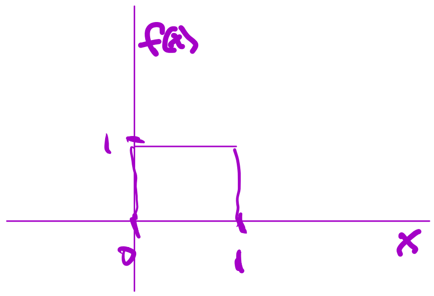
Stat 134 lec 20 (no lec 19)

warmup 1:00–1:10

Let $X \sim \text{Unif}(0,1)$ be the standard uniform distribution with histogram (density)

picture



$$f(x) = \begin{cases} 1 & \text{if } 0 < x \le 1 \\ 0 & \text{else} \end{cases}$$

Define

$$E(X) = \int_{x=-\infty}^{\infty} x f(x)\,dx$$

Find $E(X)$, $E(X^2)$, and $Var(X)$.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x \cdot 1\,dx = \frac{x^2}{2}\Big]_0^1 = \boxed{\frac{1}{2}}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big]_0^1 = \boxed{\frac{1}{3}}$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \boxed{\frac{1}{12}}$$

Last time

Congratulations on finishing midterm 1 !

today

Sec 4.1 Continuous Distributions

1. probability density
2. expectation and variance.
3. change of scale

① sec 4.1 Probability density.

let X be a continuous RV

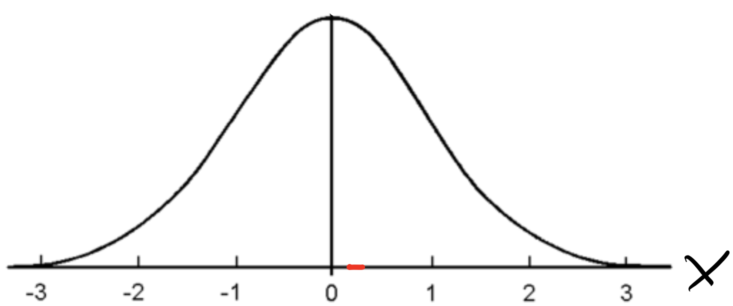
The probability density (histogram) of X is described by a prob density function

$$f(x) \geq 0 \text{ for } x \in X$$

and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

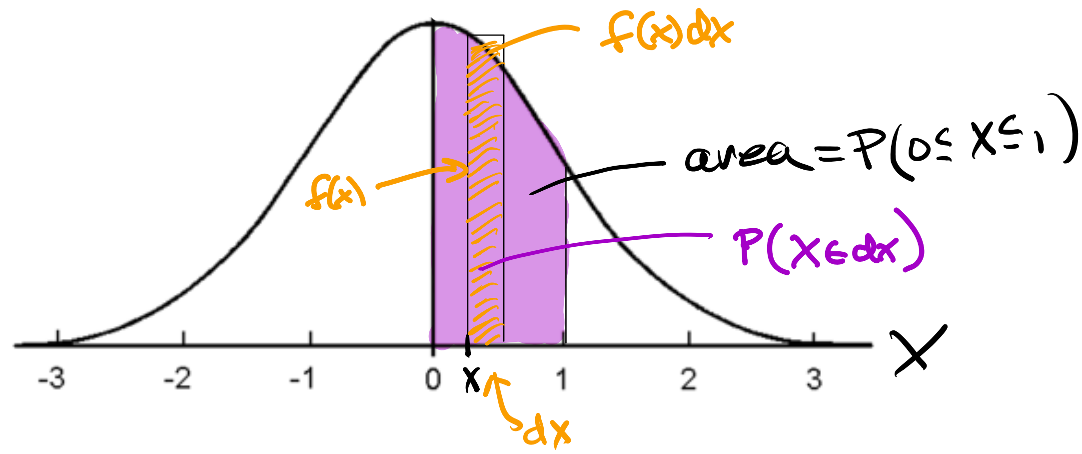
ex The standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$



Throwing a dart randomly at the histogram the x coordinate of your dart is most likely to be near zero.

The probability of getting an x coordinate in a small neighborhood of x is written $P(X \in dx)$.



we see from the rectangle in the picture,

$P(X \in dx) \approx f(x)dx$ (notice purple and orange area not same)

here dx = tiny interval around x and also the length of the interval

For all a, b $\quad P(a \leq X \leq b) = \int_a^b P(X \in dx) \approx \int_a^b f(x)dx$

Note $f(x)$ is not a probability.
$f(x)dx$ is a probability.

$$f(x) \approx \frac{P(X \in dx)}{dx} \begin{matrix} \leftarrow \text{probability} \\ \leftarrow \text{length} \end{matrix}$$

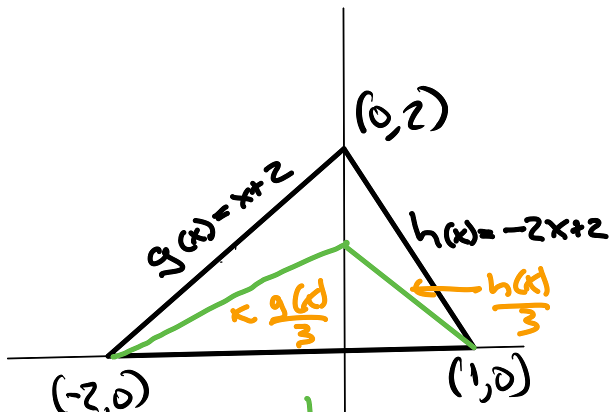
units of f? — probability / unit length ← probability density

$P(X = x) = 0$

Hence $P(a \leq X \leq b) = P(a < X < b)$
(we don't have to worry about endpts).

ex 4.1.12 b

Consider a point picked uniformly at random from the area inside the following triangle



Find the density function of the x-coordinate f(x)

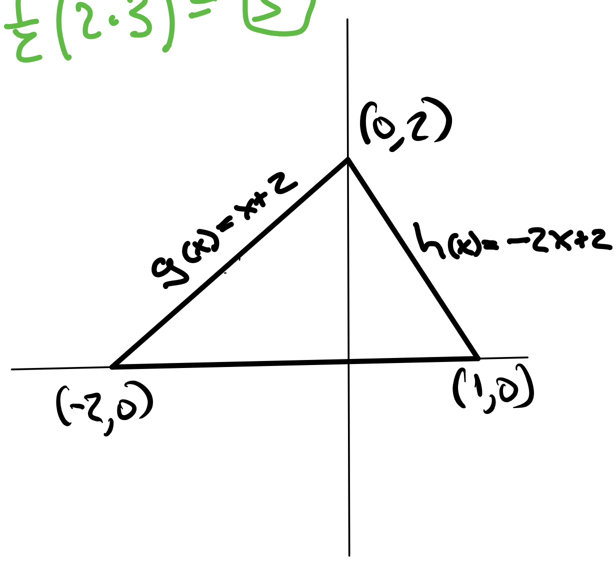
density

idea $\displaystyle\int_{-2}^{1} f(t)\,dt = \int_{-2}^{0} \frac{g(t)}{\text{area}} + \int_{0}^{1} \frac{h(t)}{\text{area}}\,dt$

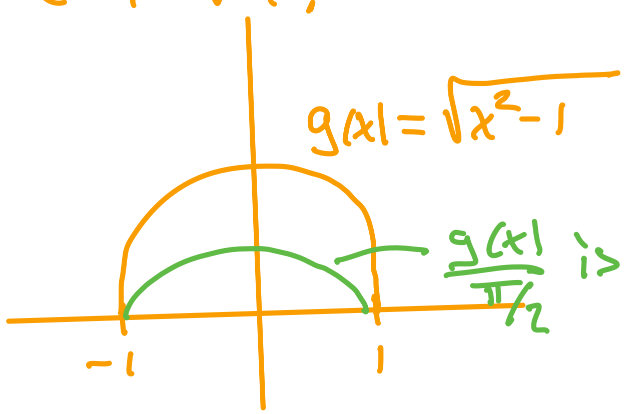
$$f(x) = \begin{cases} \dfrac{x+2}{3}, & -2 \le x \le 0 \\ \dfrac{-2x+2}{3}, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

$\frac{1}{2}(2 \cdot 3) = \boxed{3}$

Area of Δ = ?



Note there is nothing special about the shape being a triangle, It could be a half circle with radius 1 for example,



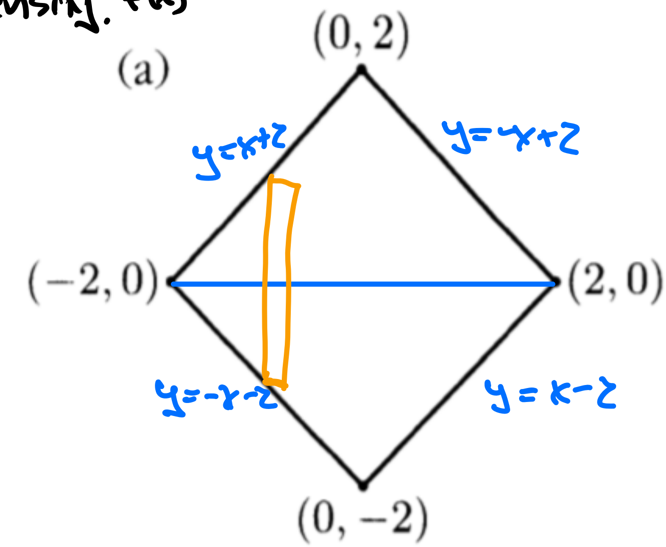
$\dfrac{g(x)}{\pi/2}$ is a density.

Here the area is $\frac{\pi}{2}$, To make g into a density divide it by $\pi/2$

$$f(x) = \frac{\sqrt{x^2-1}}{\pi/2}$$

ex 4.1.12 a

Consider a point picked uniformly at random from the area inside the following shape. Find the density. f(x)

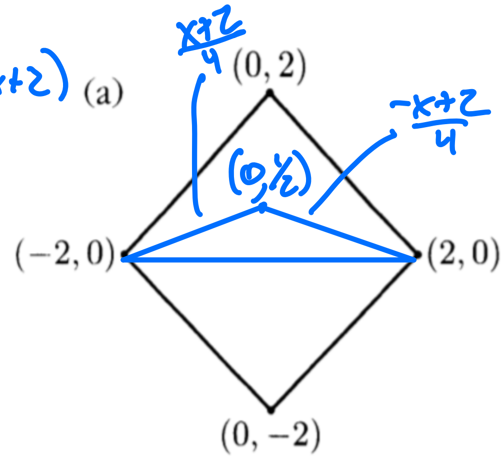


$$f(x) = \begin{cases} \frac{x+2}{4}, & -2 \le x \le 0 \\ \frac{-x+2}{4}, & 0 \le x \le 0 \\ 0 & \text{else} \end{cases}$$

$$\text{Area} = 2\left(\tfrac{1}{2}\cdot 4 \cdot 2\right) = 8$$

$$\int_{-2}^{2} f(x)\,dx = \frac{1}{\text{Area}} \cdot \int_{-2}^{0} 2(x+2)\,dx + \int_{0}^{2} 2(-x+2)$$

$$\text{Area} = 8$$



(2)

Expectation and Variance

For discrete,

$$E(g(X)) = \sum_{x \in X} g(x) P(X=x)$$

For continuous,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) P(X \in dx) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$Var(X) = E(X^2) - E(X)^2$$

See Wamby for example,

③ Change of scale

To calculate $E(X)$, $Var(X)$, $P(X \in dx)$ we sometimes make a linear change of scale
$Y = c + bX$ where c, b are constants

Y hopefully has a simpler density function.

We can recover $E(X)$, $Var(X)$, $P(X \in dx)$

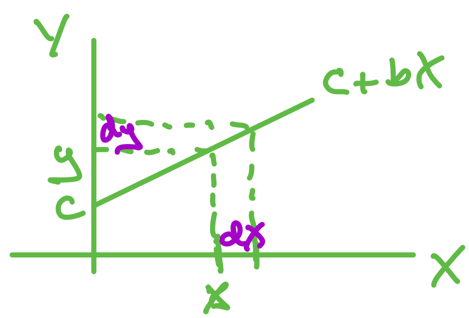
from $E(Y)$, $Var(Y)$, $P(Y \in dY)$.

$$E(Y) = E(bX + c) = b E(X) + c$$

$$\Rightarrow E(X) = \frac{E(Y) - c}{b}$$

$$Var(Y) = Var(bX + c) = b^2 Var(X)$$

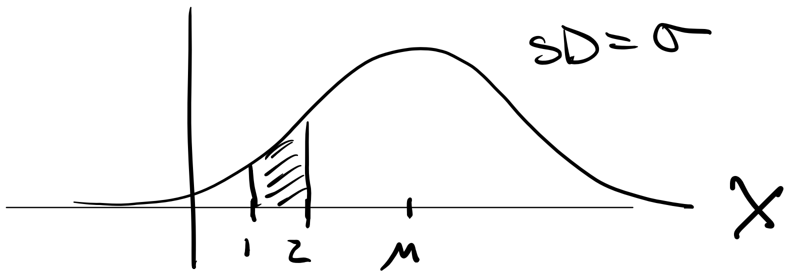
$$\Rightarrow Var(X) = \frac{Var(Y)}{b^2}$$



$$P(X \in dx) = P(Y \in dY)$$

ex Let $X \sim N(\mu, \sigma^2)$

Find $P(1 < X < 2)$



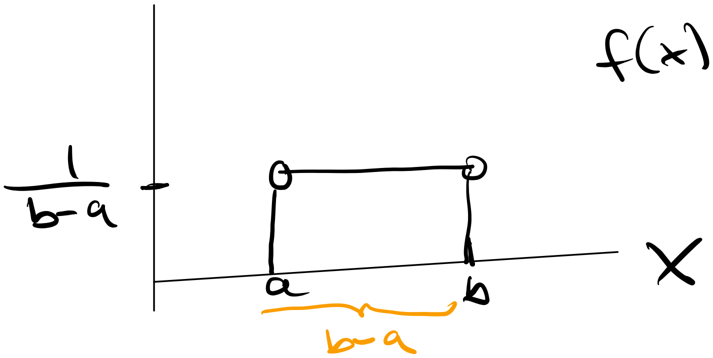
$$Z = \frac{X - \mu}{\sigma} = \frac{1}{\sigma} X - \frac{\mu}{\sigma}$$

$$\uparrow_{b} \qquad \uparrow_{c}$$

Change of scale

$$P(1 < X < 2) = \Phi\left(\frac{2-\mu}{\sigma}\right) - \Phi\left(\frac{1-\mu}{\sigma}\right)$$

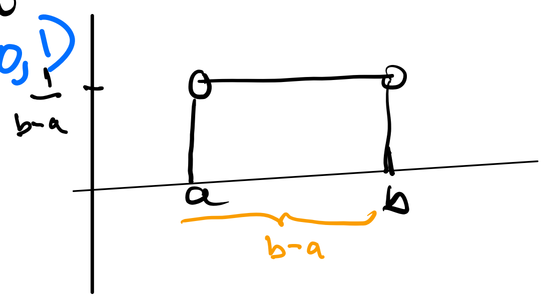
ex: let $X \sim \text{Unif}(a,b)$



$$f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{else} \end{cases}$$

a) You should change the scale of X to?

$$U = \frac{X-a}{b-a} \sim \text{Unif}(0,1)$$



b) Find $E(X)$

$$\tfrac{1}{2} = E(U) = E\left(\frac{1}{b-a}(x-a)\right) = \frac{1}{b-a}E(x-a)$$
$$= \frac{1}{b-a}\left(E(x)-a\right)$$

$$\boxed{E(x) = \frac{a+b}{2}}$$

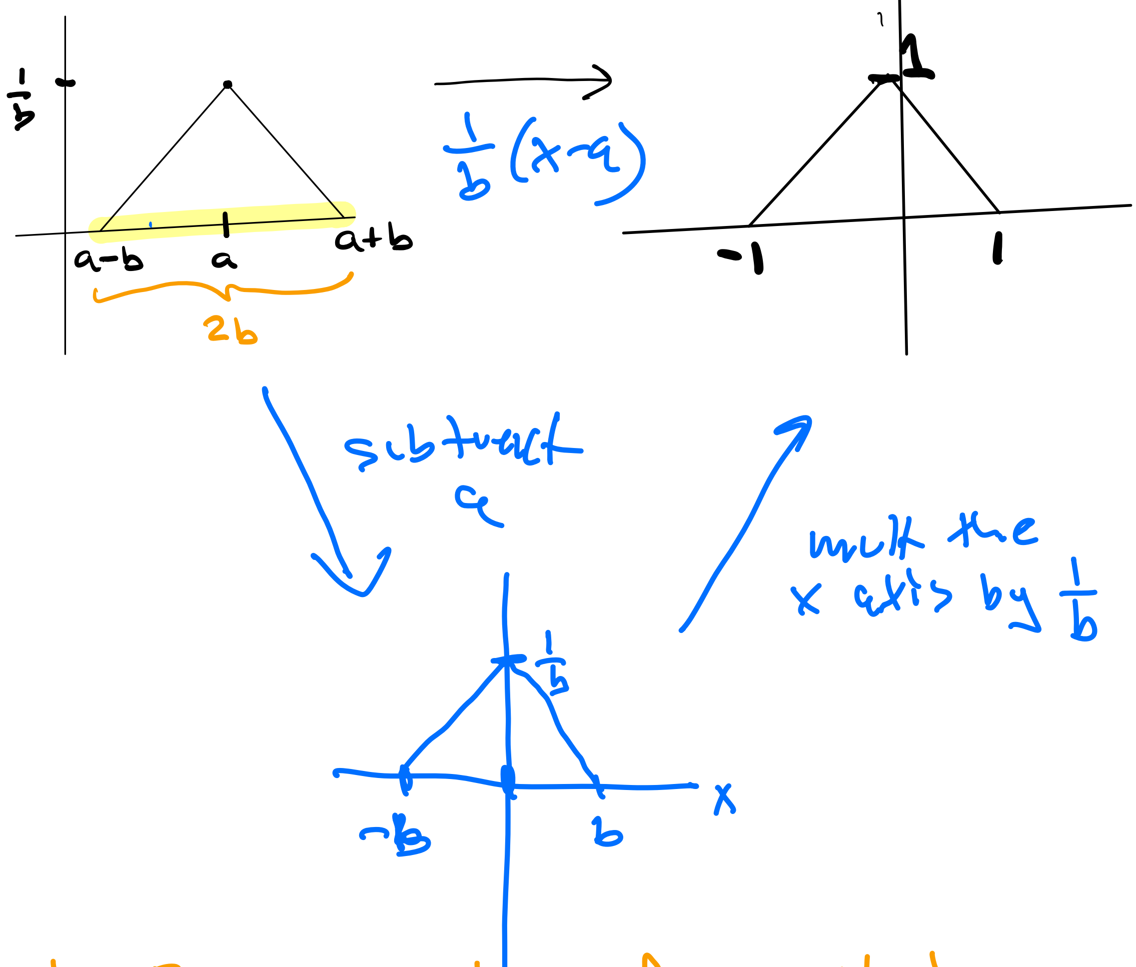
c) Find $Var(X)$.

$$\frac{1}{12} = Var(U) = \frac{1}{(b-a)^2} Var(x-a)$$

$$\Rightarrow \boxed{Var(x) = \frac{(b-a)^2}{12}}$$

What change of scale do have here?

we are changing the values X takes. Because it is a linear change of scale a triangle shaped density maps to the same shape.



<u>Note</u> Because the end result is a density with area 1 the height of the triangle is 1 not $\left(\frac{1}{b}\right)^2$.