Stat 134 lec 36

sec 6.3 Conditioning: density case.

Recall $f_{Y|X=x}(y) = \dfrac{f(x,y)}{f_X(x)}$ conditional density

$$P(A) = \int_{x \in X} P(A|X=x) f_X(x)\,dx$$ Integral Condition formula.

warm up $X \sim \text{Unif}(0,1)$, $X = p$

, $I_1, \ldots, I_n \overset{iid}{\sim} \text{Bern}(p)$

I have a p-coin but until I condition on $X=p$, p is a RV not a number

Find $P(2^{nd} \text{ toss } H)$

$$P(2^{nd} \text{ toss } H) = \int_0^1 \underbrace{P(2^{nd} H | X=p)}_{p} \cdot \underbrace{f_X(p)}_{1}\,dp$$

$$= \int_0^1 p\,dp = \frac{1}{2}$$

Bayesian Statistics

Last time we showed

$P(1^{st} H, 2^{nd} H) = 1/3$ and $P(2^{nd} H | 1^{st} H) = 2/3$

$\Rightarrow P(2^{nd} H | 1^{st} H) \neq P(2^{nd} H)$

so our coin tosses are not independent unless we condition on $X=p$.

Here is how we compute the posterior density:

$$\Rightarrow \underbrace{f_{X|1^{st} H}(p)}_{\text{posterior}} = \frac{\underbrace{P(1^{st} H | X=p)}_{\text{likelihood} = p} \underbrace{f_X(p)}_{\text{prior} = 1}}{\underbrace{P(1^{st} H)}_{\text{constant} = 1/2}} = 2p$$

Stat 134

Wednesday April 18 2018

1. Let $X \sim \text{Unif}(0,1)$. Given X=p, let $I_1, I_2, \ldots, I_n$ be iid Bernouilli(p) trials. Given that the first two tosses are heads, the posterior density $f_{X|\text{1st H,2nd H}}(p)$ is:

   a $2p^2$

   b $3p^2$

   c $6p^2$

   d none of the above

two ways to calculate:

$$f_{X|1^{st}H,2^{nd}H} = \frac{\overbrace{P(2^{nd}H|1^{st}H, X=p)}^{=p} \cdot \overbrace{f_{X|1^{st}H}(p)}^{=2p}}{\underbrace{P(2^{nd}H|1^{st}H)}_{=2/3}} = 3p^2$$

$$f_{X|1^{st}H,2^{nd}H} = \frac{\overbrace{P(2^{nd}H, 1^{st}H|X=p)}^{=p^2}\overbrace{f_X(x)}^{=1}}{\underbrace{P(1^{st}H, 2^{nd}H)}_{=1/3}} = 3p^2$$

ex Find $P(3^{rd}\ H \mid 1^{st}\ H, 2^{nd}\ H)$

$$= \int_0^1 \underbrace{P(3^{rd} H \mid 1^{st} H, 2^{nd} H, X=p)}_{= p} \cdot \underbrace{f_{X \mid 1^{st}, 2^{nd} H}(p)}_{= 3p^2} \, dp$$

$$= \int_0^1 3p^3 dp = \boxed{\frac{3}{4}}$$

$$P(1^{st} H, 2^{nd} H, 3^{rd} H) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \boxed{\frac{1}{4}}$$

what is

$$f_{X \mid 1^{st} H, 2^{nd} H, 3^{rd} H} \; ? \; \text{—} \; 4p^3$$

<u>Review of Beta</u>

$$\int_0^1 \underbrace{\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} p^{r-1}(1-p)^{s-1}}_{\text{beta}(r,s)} dp = 1$$

$$\Gamma(r) = (r-1)!$$

$$\text{beta}(1,1) = \text{Unif}(0,1)$$

ex $X \sim \text{beta}(1,1)$, $X = p$

S_n = the number of successes in n Bernoulli(p) trials.

Find $P(S_n = k)$:

$$P(S_n = k) = \int_0^1 P(S_n = k \mid X = p) f_X(p) dp$$

$$= \int_0^1 \binom{n}{k} p^{k+1-1} (1-p)^{n-k+1-1} \, 1 \, dp$$

$$= \binom{n}{k} \cdot \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)} = \frac{1}{n+1}$$

Generalize

$X \sim \text{beta}(r,s)$, $X = p$

$S_n = $ # successes in n Bern(p) trials

Find posterior density

$\binom{n}{k} p^k (1-p)^{n-k}$ $\quad \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} p^{r-1} (1-p)^{s-1}$

$$f_{X|S_n=k}(p) = \frac{P(S_n=k \mid X=p) f_X(x)}{P(S_n=k)}$$

constant

$$= \frac{\Gamma(r+n+s)}{\Gamma(k+r)\Gamma(n-k+s)} p^{k+r-1} (1-p)^{n-k+s-1}$$

beta$(k+r, n-k+s)$

$X \sim \text{beta}(r,s)$
our prior density is one of those,
which one is unbiased?
answ: $U_{(\frac{n}{2})}$ since $E(X) = \frac{1}{2}$
(i.e Prob$(1^{st} H) = \frac{1}{2}$).

Here is some extra material, I didnt have time to cover in lecture but you are responsible for it:

Please Read.

In the above example the prior is beta(r, s); the posterior is beta$(k+r, n-k+s)$ and the likelihood is binomial (n, p) where p is a value from the distribution of the prior.

When the prior and the posterior belong to the same family, we say that the prior and the likelihood are conjugate.

ex the beta and the binomial are conjugate

prior likelihood

ex the gamma and the Poisson are conjugate.

When this happens it is easy to update the prior with the posterior, as we did in the i-clicker question

The summary on p425 is useful.

Conditioning Formulae: Density Case

Multiplication rule: The joint density is the product of the marginal and the conditional

$$f(x,y) = f_X(x) f_Y(y \mid X = x)$$

Division rule: The conditional density of Y at y given $X = x$ is

$$f_Y(y \mid X = x) = \frac{f(x,y)}{f_X(x)}$$

Bayes' rule:

$$f_X(x \mid Y = y) = \frac{f_Y(y \mid X = x) f_X(x)}{f_Y(y)}$$

Conditional distribution of *Y* given *X* = *x*: Integrate the conditional density

$$P(Y \in B \mid X = x) = \int_B f_Y(y \mid X = x) dy$$

I wrote conditional density $f_{Y|X=x}(y)$

Conditional expectation of *g*(*Y*) given *X* = *x*: Integrate g against the conditional density:

$$E\left(g(Y) \mid X = x\right) = \int g(y) f_Y(y \mid X = x) dy$$

Average conditional probability:

$$P(B) = \int P(B \mid X = x) f_X(x)\, dx$$

$$f_Y(y) = \int f_Y(y \mid X = x) f_X(x)\, dx$$

Average conditional expectation:

$$E(Y) = \int E(Y \mid X = x) f_X(x)\, dx$$

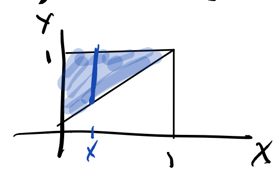
We use the conditional density to compute the conditional expectation as you would expect.

<u>Conditional Expectation</u>

$$E(g(Y) \mid X=x) = \int_{y \in Y} g(y) f_{Y|X=x}(y)\,dy$$

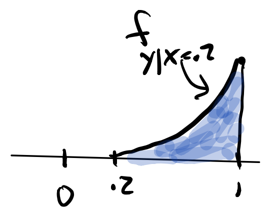
Ex Last time

$$f(x,y) = 90(y-x)^8, \quad 0<x<y<1$$



we found

$$f_{Y|X=x}(y) = \frac{9}{(1-x)^9}(y-x)^8$$



Find $E(Y \mid X=.2)$

<u>Soln</u>

$$E(Y \mid X=.2) = \int_{.2}^{1} y f_{Y|X=.2}(y)\,dy$$

$$= \int_{.2}^{1} y \frac{9}{(1-.2)^9}(y-.2)^8\,dy$$

substitute $u = y-.2$, $y = u+.2$
$du = dy$

$$= \frac{9}{(.8)^9}\int_{u=0}^{u=.8} (u+.2)u^8\,du$$

$$= \frac{9}{(.8)^9}\left[\int_0^{.8} u^9\,du + .2\int_0^{.8} u^8\,du\right]$$

$$= \frac{9}{(.8)^9}\left[\frac{(.8)^{10}}{10} + \frac{.2(.8)^9}{9}\right]$$

$$= 9\left[\frac{.8}{10} + \frac{.2}{9}\right] = .92$$