Stat 134 lec 38

Warmup 1:00-1:10

rule of average conditional probability:

Suppose $Y|X=x \sim Pois(x)$ where $X \sim Exp(\lambda)$.

Show $Y \sim Geom\left(\frac{\lambda}{\lambda+1}\right)$ on $0,1,2,\dots$

$$P(Y=y) = \int_0^\infty P(Y=y|X=x)f_X(x)\,dx$$

$$= \int_0^\infty \frac{e^{-x}x^y}{y!}\lambda e^{-\lambda x}\,dx$$

$$= \frac{\lambda}{(\lambda+1)^{y+1}}\underbrace{\boxed{\int_0^\infty \frac{(\lambda+1)^{y+1}}{y!}x^{y+1-1}e^{-x(\lambda+1)}\,dx}}_{=1}$$

$$= \left(\frac{1}{\lambda+1}\right)^y \frac{\lambda}{\lambda+1}$$

$$\Rightarrow \boxed{Y \sim Geom\left(\frac{\lambda}{\lambda+1}\right)}.$$

recall if $Y \sim Geom\left(\frac{\lambda}{\lambda+1}\right)$ on $0,1,2,\dots$

$$P(Y=y) = \left(\frac{1}{\lambda+1}\right)^y\left(\frac{\lambda}{\lambda+1}\right)$$

$$P(Y=y|X=x) = \frac{e^{-x}x^y}{y!}$$

$$f_X(x) = \lambda e^{-\lambda x}$$

recall

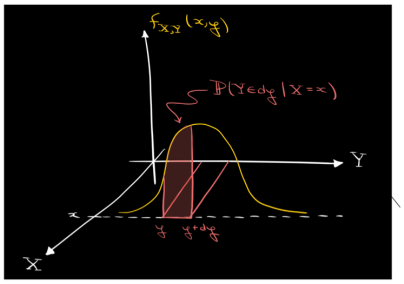
$X|_{Y=y} \sim Gamma(y+1, \lambda+1)$

$$f_{X|Y=y}(x) = \frac{(\lambda+1)^{y+1}}{\Gamma(y+1)}x^{y+1-1}e^{-x(\lambda+1)}$$

Quiz 6: Wednesday sec 5.4 through sec 6.3

Last time

sec 6.3 Conditional density



X, Y continuous RVs.

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} \quad \text{Bayes' rule}$$

$$f(x,y) = f_{Y|X=x}(y)\, f_X(x) \quad \text{mult rule.}$$

The analog in discrete probability is conditional probability, e.g. what is chance second card is an ace? We can find this by conditioning it on what the 1st card is:

$$P(2^{nd}\text{ ace}) = \sum_x P(2^{nd}\text{ ace} \mid 1^{st} = x) \cdot P(1^{st} = x) \quad \text{where } x \text{ is a suit.}$$

Average conditional probabilities

X discrete, Y discrete

$$P(Y=y) = \sum_{x \in X} P(Y=y \mid X=x)\, P(X=x)$$

X continuous Y continuous

$$P(Y \in dy) = \int_{x \in X} P(Y \in dy \mid X=x)\, f_X(x)\, dx$$

X discrete Y continuous

$$P(Y \in dy) = \sum_{x \in X} P(Y \in dy \mid X=x)\, P(X=x)$$

X continuous Y discrete

$$P(Y=y) = \int_{x \in X} P(Y=y \mid X=x)\, f_X(x)\, dx$$

Today ① sec 6.3 more practice

② covariance and the variance of sum

① More practice

Let $Y|X=x \sim \text{Pois}(x)$ where $X \sim \text{Exp}(\lambda)$

We saw that $Y \sim \text{Geom}\left(\frac{\lambda}{\lambda+1}\right)$ on $0, 1, 2, \ldots$

Show that $X|Y=y$ is Gamma and find parameters.

recall if $Y \sim \text{Geom}\left(\frac{\lambda}{\lambda+1}\right)$ on $0, 1, 2, \ldots$

$$P(Y=y) = \left(\frac{1}{\lambda+1}\right)^y \left(\frac{\lambda}{\lambda+1}\right)$$

$$P(Y=y|X=x) = \frac{e^{-x} x^y}{y!}$$

$$f_X(x) = \lambda e^{-\lambda x}$$

$$P(X \in dx | Y=y) = \frac{P(X \in dx, Y=y)}{P(Y=y)}$$ Bayes rule

$$= \frac{P(Y=y|X=x)\, P(X \in dx)}{P(Y=y)}$$ mult rule

$$= \frac{\frac{e^{-x} x^y}{y!} \cdot \lambda e^{-\lambda x} dx}{\left(\frac{1}{\lambda+1}\right)^y \frac{\lambda}{\lambda+1}}$$ $\approx P(X \in dx)$

$$\propto x^y e^{-(\lambda+1)x} dx$$

$$\boxed{X|Y=y \sim \text{Gamma}(y+1, \lambda+1)}$$

② Sec 6.4 Covariance and variance of a sum

$X, Y, \quad S = X + Y$

mean $\mu_X, \mu_Y, \mu_S = \mu_X + \mu_Y$

$$D_S = S - \mu_S \quad \text{D_S deviation from mean}$$
$$= X + Y - (\mu_X + \mu_Y)$$
$$= D_X + D_Y$$

$$\mathrm{Var}(S) = E\left((D_X + D_Y)^2\right)$$
$$= E\left(D_X^2 + D_Y^2 + 2D_X D_Y\right)$$
$$= \underbrace{E(D_X^2)}_{= \mathrm{Var}(X)} + \underbrace{E(D_Y^2)}_{= \mathrm{Var}(Y)} + 2\underbrace{E(D_X D_Y)}_{= \mathrm{Cov}(X,Y)}$$

Defn The Covariance of X and Y is

$$\mathrm{Cov}(X,Y) = E((X - \mu_X)(Y - \mu_Y))$$

Bilinearity Properties

Proved end of lecture.

(a) $\text{Cov}(X+Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$

(b) $\text{Cov}(aX, bY) = ab\,\text{Cov}(X, Y)$

$$\boxed{\begin{array}{l}\underline{\text{More generally}} \\ \text{Cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) \\ \qquad = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \text{Cov}(X_i, Y_j)\end{array}}$$

Proved end of lecture.

Thm $\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$

__Easy facts__

$\text{Cov}(X, X) = E(X^2) - E(X)^2 = \text{Var}(X)$

$\text{Cov}(X, X) = \text{Var}(X)$

$\text{Cov}(X, Y) = \text{Cov}(Y, X)$

$\text{Cov}(X, c) = 0$

constant

ex

Simplify

$$\mathrm{Cov}(X-5Y,\ 3X+Y-Z+10)$$

$$= 3\mathrm{Var}(X) + \mathrm{Cov}(X,Y) - \mathrm{Cov}(X,Z) + 0$$

$$- 15\mathrm{Cov}(X,Y) - 5\mathrm{Var}(Y) + 5\mathrm{Cov}(Y,Z) + 0$$

Recall X, Y independent

$$\Rightarrow \quad E(XY) = E(X)E(Y)$$

$\mathrm{Cov}(X,Y) = 0$ if X, Y independent.

Hence if X, Y indep,

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\overset{=0}{\mathrm{Cov}(X,Y)}$$

$$= \mathrm{Var}(X) + \mathrm{Var}(Y)$$

http://tinyurl.com/nov30-pt1

Stat 134

Wednesday April 24 2019

1. Consider a Poisson(λ) process. Let $T_r \sim$ gamma(r, λ) be the rth arrival time. Cov(T_1, T_3) equals:

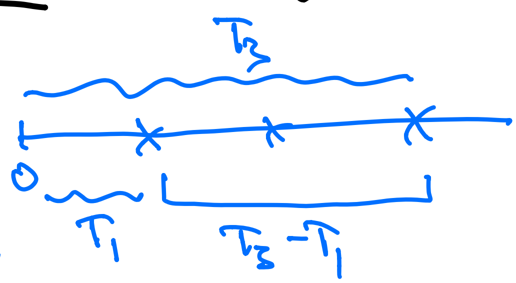
   a λ

   b λ^2

   c $1/\lambda^2$

   d none of the above

Recall $\text{Var}(T_r) = \frac{r}{\lambda^2}$



$\text{Cov}(T_1, T_3 - T_1) \overset{\text{indep}}{=} 0$

$\text{Cov}(T_1, T_3) - \text{Var}(T_1) = 0$

$\text{Cov}(T_1, T_3) = \text{Var}(T_1) = \boxed{\frac{1}{\lambda^2}}$

<u>ex</u>

Let $X_1, \dots, X_n$ be identically distributed

$$\operatorname{Var}\left(\sum_{i=1}^n X_i\right) = \operatorname{Cov}\left(\sum_{i=1}^n X_i, \sum_{j=1}^n X_j\right)$$

<u>Defn</u>

The <u>variance-covariance matrix</u> has all n^2 terms

$$\begin{array}{c|c}
 & \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} & \begin{bmatrix} \operatorname{Cov}(x_1,x_1) & \operatorname{Cov}(x_1,x_2) & & \\ & \operatorname{Cov}(x_2,x_2) & & \\ & & \ddots & \\ & & & \operatorname{Cov}(x_n,x_n) \end{bmatrix}_{n \times n}
\end{array}$$

$$\begin{aligned}
\operatorname{Var}\left(\sum_{i=1}^n x_i\right) &= \operatorname{Cov}\left(\sum_{i=1}^n x_i, \sum_{i=1}^n x_i\right) \\
&= \underbrace{n \operatorname{Var}(x_1)}_{\text{diagonal}} + \underbrace{n(n-1)\operatorname{Cov}(x_1, x_2)}_{\text{off diagonal}}
\end{aligned}$$

<u>Note</u> $\operatorname{Cov}(x_1, x_2) = \operatorname{Cov}(x_i, x_j)$ since $x_1, \dots, x_n$ are identically distributed.

Appendix

Bilinearity Properties

Thm

ⓐ $\mathrm{Cov}(X+Y, Z) = \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Z)$

ⓑ $\mathrm{Cov}(aX, bY) = ab\,\mathrm{Cov}(X,Y)$

Pf/

a)

$$\mathrm{Cov}(X+Y,Z) = E\big((X+Y-\mu_{X+Y})(Z-\mu_Z)\big)$$

$$= E\big([(X-\mu_X)+(Y-\mu_Y)](Z-\mu_Z)\big)$$

$$= E\big((X-\mu_X)(Z-\mu_Z) + (Y-\mu_Y)(Z-\mu_Z)\big)$$

$$= E\big((X-\mu_X)(Z-\mu_Z)\big) + E\big((Y-\mu_Y)(Z-\mu_Z)\big)$$

$$= \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Z). \quad \square$$

b) $$\mathrm{Cov}(aX,bY) = E\big((aX-\mu_{aX})(bY-\mu_{bY})\big)$$

$$= E\big((aX-a\mu_X)(bY-b\mu_Y)\big)$$

$$= E\big(ab(X-\mu_X)(Y-\mu_Y)\big)$$

$$= ab\,E\big((X-\mu_X)(Y-\mu_Y)\big)$$

$$= ab\,\mathrm{Cov}(X,Y) \quad \square$$

Appendix

Thm $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

pf/ $\mathrm{Cov}(X,Y) = E(D_X D_Y) = E\big((X-\mu_X)(Y-\mu_Y)\big)$

$$= E\left(XY - \mu_X Y - X\mu_Y + \mu_X \mu_Y\right)$$

$$= E(XY) - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_Y)$$

$$= E(XY) - E[X]E(Y)$$

$\square$