Stat 134 lec 32

Last time

sec 5.3 independent normal variables

- we used MGF to prove

① a linear combination of independent normals is normal

② If $X_1, \ldots, X_n \overset{iid}{\sim} N(0,1)$ then
$X_1^2 + X_2^2 + \cdots + X_n^2 \sim \text{Gamma}(r = \frac{n}{2}, \lambda = \frac{1}{2})$

called chisquare distribution with n degrees of freedom written χ^2_n.

sec 5.4 Convolution formula for density of sum

Let X, Y be RVs and $S = X+Y$,

if $X>0, Y>0$

if X,Y independent

$$f_S(s) = \int_{x=0}^{x=s} f_{(X,Y)}(x, s-x)\,dx = \int_{x=0}^{x=s} f_X(x) f_Y(s-x)\,dx$$

more generally

$$f_S(s) = \int_{x=-\infty}^{x=\infty} f(x, s-x)\,dx = \int_{x=-\infty}^{x=\infty} f_X(x) f_Y(s-x)\,dx$$

Today

① Go over student comments from concept test last time,

Finish sec 5.4

② example: $X, Y \overset{iid}{\sim} U(0,1)$. $S = X+Y$. Find $f_S(s)$.

③ Convolution formula of ratio Y/X

④ Proof that $f_p(p) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} p^{r-1}(1-p)^{s-1}$, $0<p<1$ is a density for $r>0, s>0$.

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Stat 134

Friday November 2 2018

1. Let $X \sim N(0,1)$ and $Y \sim N(0,1)$ be independent. $P(X > 2Y)$ equals

a $1 - \Phi(0)$ =

b $1 - \Phi(\frac{1}{\sqrt{3}})$

c $1 - \Phi(\sqrt{3})$

d none of the above

Discuss with neighbor your solution for 1 min.

Soln

$Y \sim N(0,1) \Rightarrow 2Y \sim N(0,4)$

and $X - 2Y \sim N(0, 1^2 + 2^2) = N(0,5)$

$P(X > 2Y) = P(X - 2Y > 0)$

normalize $= P\left(\frac{(X-2Y)-0}{\sqrt{5}} > \frac{0-0}{\sqrt{5}}\right)$

$= P(Z > 0) = 1 - \Phi(0) = \frac{1}{2}$

11/2/2018	c	a	cuz the sum 2y-x is normal(0,3) which is still symmetric about 0
11/2/2018 9:37:09	a	a	evaluating the integral of the joint density over the plane gives 1/2 which is equal to 1-phi(0) = 1/2
11/2/2018	a	a	Wait, the image is symmetry on the plane!
11/2/2018 9:37:18	a	a	I think it's a), as X - 2Y > 0 has X - 2Y has N 0,5. Therefore, cdf is PHI(0) for P (X-2Y < 0).
11/2/2018 9:38:40	a	a	We have P(X>2Y). Transfer the Y to the left to get P(X-2Y>0). X-2Y ~ N(0,5) (just square and add the SDs). Now we can find the probability. Since mu is 0 and our x is 0, x-mu/sigma is 0 so we get 1-Phi(0)
11/2/2018 9:39:26	a	a	
11/4/2018	a	a	I have a question to the answer. Since ax1 + bx2 ~ N(au1 + bu2, a^2 sig^2 + b^2 sig^2) why x-2y does not follow N(0, 1^2 *1 + (-2)^2 * 1 = 5) instead of the answer N(0,3)?

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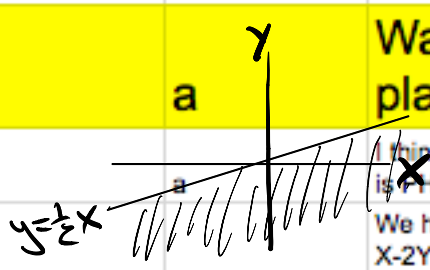


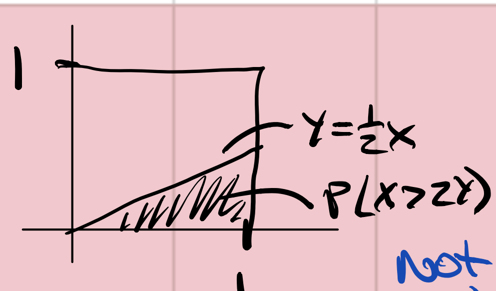
thanks!

11/2/2018	a	a	Z=X-2Y~Normal(0,5), P(Z>0)=1-fi(0)

Screen Shot 2018-11-04 at 9.31.12 PM

Search

11/2/2018	c	b	half of all Y values will be between 1/2 and 1, which means x cannot be greater for half of the time. thus the probability must be less than 1/2. We can compare x~N(0,1) and y~N(0,2) to get our answer



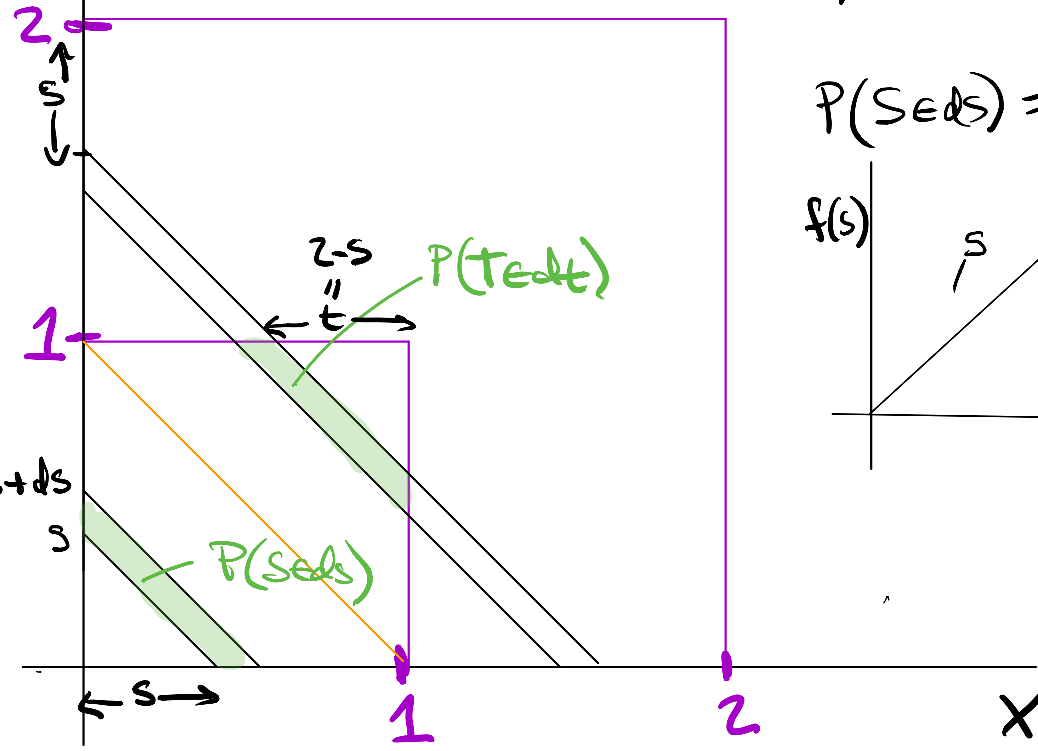
Not U(0,1)

② Example Convolution formula of Sum:

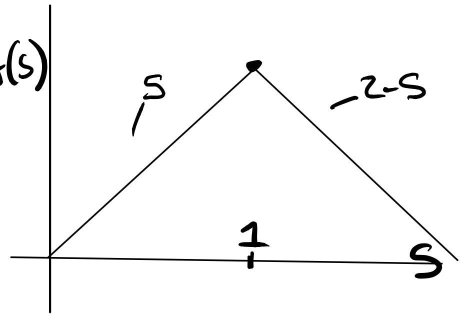
eg $X, Y \overset{iid}{\sim} \text{Unif}(0,1)$, $S = X + Y$

Find $f_S(s)$.

$y = -x + s$



$P(S \in ds) = f(s)\,ds$



For $0 < s < 1$

$$f_S(s) = \int_{x=0}^{x=s} f_X(x) f_Y(s-x)\,dx = \int_{x=0}^{x=s} 1 \cdot 1\,dx = s$$

For $1 < s < 2$

Note if $X, Y \overset{iid}{\sim} U(0,1)$ then $1-X, 1-Y \overset{iid}{\sim} U(0,1)$

Let $T = 2 - S = (1-X) + (1-Y)$

then $0 < t < 1$

By convolution formula for T, $f_T(t) = t$, $0 < t < 1$

2-S

so $\boxed{f_S(s) = \begin{cases} s & \text{for } 0 < s < 1 \\ 2-s & \text{for } 1 < s < 2 \end{cases}}$

Knowing $f_T(t) = t$ we find $f_S(s) = 2-s$ by chg of variable formula

③

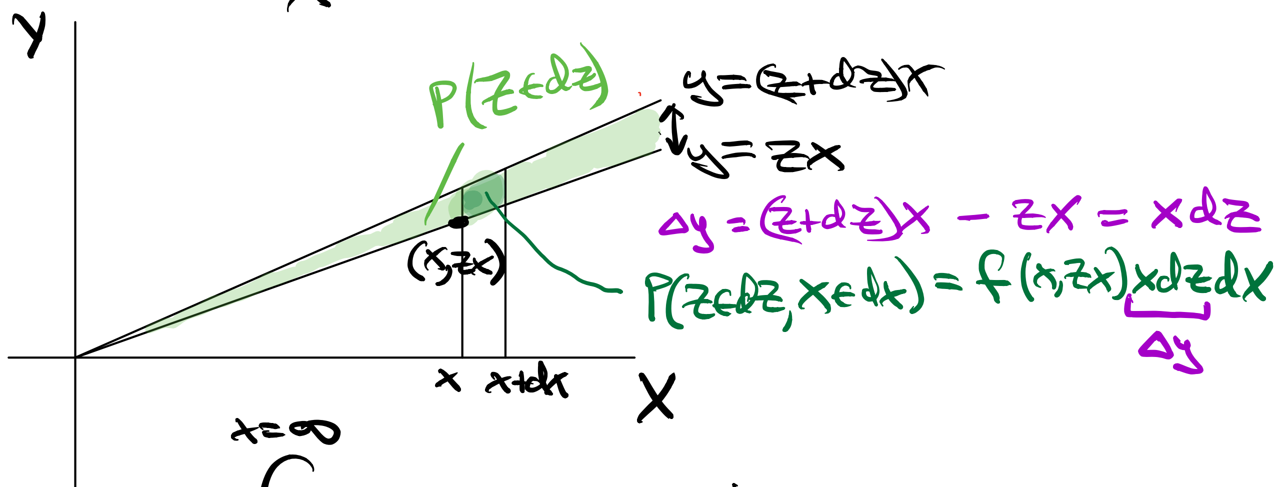
Convolution formula for density of ratio Y/X

$X>0,\ Y>0$

let $Z = \dfrac{Y}{X}$.

Find $f_Z(z)$.

Picture $\quad Z = \dfrac{Y}{X} \Rightarrow Y = ZX$ ← slope.



$$P(Z\in dz) = \int_{x=0}^{x=\infty} P(Z\in dz, X\in dx)$$

$$\parallel$$

$$f(z)\,dz = \int_{x=0}^{x=\infty} f_{(X,Y)}(x,zx)\,x\,dz\,dx$$

$$\Rightarrow f(z) = \int_{x=0}^{x=\infty} f(x,zx)\,x\,dx = \int_{x=0}^{x=\infty} f_X(x) f_Y(zx)\,x\,dx$$

– if X, Y indep.

Convolution formula.

Recall gamma density

$X \sim \text{Gamma}(r, \lambda)$

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$$

variable part

so

$$\boxed{\int_0^\infty x^{r-1} e^{-\lambda x} = \frac{\Gamma(r)}{\lambda^r}}$$

ex Let $X, Y \overset{iid}{\sim} \text{Exp}(1)$. $Z = \frac{Y}{X}$

Find $f_Z(z)$.

soln $f_X(x) = e^{-x}$, $x > 0$.

$$f_Z(z) = \int_{x=0}^{\infty} e^{-x} e^{-zx} x \, dx$$

$$= \int_{x=0}^{\infty} x \, e^{-(1+z)x} \, dx$$

gamma$(r=2, \lambda = 1+z)$

$$= \frac{\Gamma(2)}{(1+z)^2} = \boxed{\frac{1}{(1+z)^2} \text{ for } 0 < z < \infty.}$$

Concept test

Stat 134

Monday November 5 2018

1. Let $U \sim U(0,1)$ and $V \sim U(0,1)$ be independent. The density of $Y = U/V$ for $0 < y < 1$ is:

   a $1/(2y)$

   b $1/2$

   c $1/(2y^2)$

   d none of the above

Soln

$Y = U/V$

since

need $v < 1$ need $yv < 1$ ✓

$$f_Y(y) = \int_{v=0}^{v=1} f_V(v) f_U(yv)\, v\, dv$$

$$= \int_0^1 1 \cdot 1\, v\, dv = \frac{v^2}{2}\Big|_0^1 = \boxed{\frac{1}{2} \quad \text{for } 0<y<1}$$

2. Let $U \sim U(0,1)$ and $V \sim U(0,1)$ be independent. The density of $Y = U/V$ for $y > 1$ is:

a $1/(2y)$

b $1/2$

c $1/(2y^2)$

d none of the above

for $y>1$

$Y = U/V$

$f_Y(y) = \int_{v=0}^{v=\frac{1}{y}} f_V(v) f_U(yv)\, v\, dv$

need $v<1$ true if $v < \frac{1}{y}$

need $yv < 1 \Rightarrow v < \frac{1}{y}$

$= \int_0^{\frac{1}{y}} 1 \cdot 1\, v\, dv = \frac{v^2}{2}\Big|_0^{\frac{1}{y}} = \boxed{\frac{1}{2y^2} \text{ for } y>1}$

④ In sec 4.6 we defined the Beta distribution $P \sim \text{Beta}(r,s)$ for $r>0$, $s>0$ to have density

$$f_P(p) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} p^{r-1}(1-p)^{s-1}, \quad 0<p<1$$

We still need to show that this is a density!

we know:

① Gamma $r>0, \lambda>0$

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, \quad x>0$$

where $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}, \quad r>0$

② Convolution formula for sum, $Z = X+Y$ where $X>0$, $Y>0$ independent.

$$f_Z(z) = \int_{x=0}^{x=z} f_X(x) f_Y(z-x)\,dx$$

← This is a density

Let $X \sim \text{Gamma}(r,\lambda)$, $Y \sim \text{Gamma}(s,\lambda)$ } indep.

Let $Z = X+Y$

By the convolution formula for $Z = X + Y$ we know $f_Z(z)$ below is a density:

$$f_Z(z) = \int_0^z f_X(x) f_Y(z-x)\,dx \quad , z > 0$$

$$= \int_0^z \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \cdot \frac{\lambda^s}{\Gamma(s)} (z-x)^{s-1} e^{-\lambda(z-x)}\,dx$$

$$= \int_0^z \frac{\lambda^{r+s}}{\Gamma(r)\Gamma(s)} x^{r-1} (z-x)^{s-1} e^{-\lambda z}\,dx$$

$$= \frac{\lambda^{r+s}}{\Gamma(r)\Gamma(s)} e^{-\lambda z} \int_0^z x^{r-1} (z-x)^{s-1}\,dx$$

let $p = \frac{x}{z}$ to change limit of integration to 1,

$x = pz$, $dx = z\,dp$

$$z^{r+s-1} p^{r-1} (1-p)^{s-1}$$

$$f_Z(z) = \frac{\lambda^{r+s}}{\Gamma(r)\Gamma(s)} e^{-\lambda z} \int_{p=0}^{p=1} (pz)^{r-1} (z-pz)^{s-1} z\,dp$$

$$f_Z(z) = \underbrace{\frac{\lambda^{r+s}}{\Gamma(r)\Gamma(s)} \left(\int_{p=0}^{p=1} p^{r-1}(1-p)^{s-1}\,dp \right)}_{\text{constant part no } z.} \cdot \underbrace{z^{r+s-1} \cdot e^{-\lambda z}}_{\text{variable part } (z)}$$

variable part (z)

gamma $(r+s, \lambda)$

$$= \frac{\cancel{\lambda^{r+s}}}{\Gamma(r)\Gamma(s)} \left(\int_{p=0}^{p=1} p^{r-1}(1-p)^{s-1}\,dp \right) \frac{\Gamma(r+s)}{\cancel{\lambda^{r+s}}} \cdot \frac{\lambda^{r+s}}{\Gamma(r+s)} z^{r+s-1} e^{-\lambda z}$$

$$f_Z(z) = \left[\int_{p=0}^{p=1} \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} p^{r-1}(1-p)^{s-1}\,dp \right] \frac{\lambda^{r+s}}{\Gamma(r+s)} z^{r+s-1} e^{-\lambda z}$$

density of Gamma $(r+s, \lambda)$

must be 1 since $f_Z(z)$ is a density, This proves that

$$\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} p^{r-1}(1-p)^{s-1}$$ is a density.

$\square$