Stat 134 lec 5

last time

sec 2.1

binomial distribution — n independent Bernoulli (p) trials

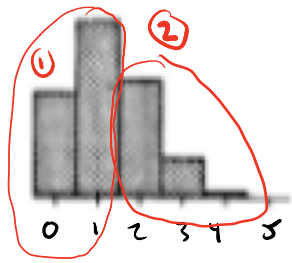
binomial formula $P(k) = \binom{n}{k} p^k q^{n-k}$ where $q = 1-p$

The consecutive odds ratio is

$$\frac{P(k)}{P(k-1)} = \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k-1} p^{k-1} (1-p)^{n-k+1}} = \boxed{\frac{n-k+1}{k} \cdot \frac{p}{1-p}}$$

helps show* — see end of lecture notes

ex $n=5$, $p=\frac{1}{4}$



$np+p = 1.5$

$\lfloor np+p \rfloor = 1$

① $k < np+p$ iff $P(k-1) < P(k)$

② $k > np+p$ iff $P(k-1) > P(k)$

③ $k = np+p$ iff $P(k-1) = P(k)$

This helps us find the mode (most likely outcome)

$$\text{mode} = \begin{cases} m & \text{if } np+p \notin \mathbb{Z} \quad \text{where } m = \lfloor np+p \rfloor \\ m-1, m & \text{if } np+p \in \mathbb{Z} \end{cases}$$

Today ① review student responses to concept test

② Finish sec 2.1 Binomial distribution

③ Start sec 2.2 Normal approximation to the binomial.

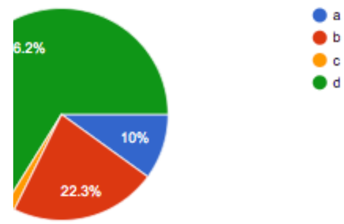
(1) Ten cards are dealt off the top of a well shuffled deck. The binominal formula doesn't apply to find the chance of getting exactly three diamonds because:

a The probability of a trial being successful changes

b The trials aren't independent

c There isn't a fixed number of trials

d more than one of the above



d	After taking any card, the probability of getting a diamond changes. That also means the trials aren't independent, since P(diamond) will depend on how many diamonds are left in the deck

b	Probabilities are not independent because cards are not replaced; however, the probability of a card being a diamond is unconditional (same Bernoulli p for each card, not conditioning on events occurring before or after a draw) so the probability of a trial succeeding does not change.

A well shuffled deck is cut in half. Five cards are dealt off the top of one half deck and five cards are dealt off the top of the other half deck. The binominal formula doesn't apply to find the chance of getting exactly three diamonds total because:

if say "2D in 1st half and 11D in 2nd half". Then answer (d).

a The probability of a trial being successful changes

b The trials aren't independent

c There isn't a fixed number of trials

d more than one of the above

This isn't different than first problem except now draw cards 1-5 and 27-31

$$\text{Prob}(2^{n}\text{ card D}) = P\left(2^{nd}\text{ card D}, \begin{matrix}0\text{D in }1^{st}\text{ half}\\ 13\text{D in }2^{nd}\text{ half}\end{matrix}\right) + P\left(2^{nd}\text{ card D}, \begin{matrix}1\text{D in }1^{st}\text{ half}\\ 12\text{D in }2^{nd}\text{ half}\end{matrix}\right)$$

$$+ \cdots + P\left(2^{nd}\text{ card D}, \begin{matrix}13\text{D in }1^{st}\text{ half}\\ 0\text{D in }2^{nd}\text{ half}\end{matrix}\right)$$

$P\left(2^{nd}\text{ card D} \,\middle|\, \begin{matrix}0\text{D }1^{st}\text{ half}\\ 13\text{D }2^{nd}\text{ half}\end{matrix}\right)$

1^{st} half deck

2^{nd} half deck

$$= \frac{0}{26}\,\frac{\binom{26}{0}\binom{26}{13}}{\binom{52}{13}} + \frac{1}{26}\,\frac{\binom{26}{1}\binom{26}{12}}{\binom{52}{13}} + \frac{2}{26}\,\frac{\binom{26}{2}\binom{26}{11}}{\binom{52}{13}} + \cdots + \frac{13}{26}\,\frac{\binom{26}{13}\binom{26}{0}}{\binom{52}{13}}$$

choose 13 diamonds in full deck

$P\left(\begin{matrix}0\text{D }1^{st}\text{ half}\\ 13\text{D }2^{nd}\text{ half}\end{matrix}\right)$

$$= \underbrace{\frac{1}{26\binom{52}{13}}\left[\sum_{i=1}^{13} i\binom{26}{i}\binom{26}{13-i}\right]}_{\text{done in R}} = .25$$

```
> (choose(26,1)*choose(26,12) +2*choose(26,2)*choose(26,11) + 3*choose(26,3)*choose(26,10) + 4*choose(26,4)*choose(26,9) +5*choose(26,5)*choose(26,8) + 6*choose(26,6)*choose(26,7) + 7*choose(26,7)*choose(26,6) + 8*choose (26,8)*choose(26,5) +9*choose(26,9)*choose(26,4) + 10*choose(26,10)*choose(26,3) + 11*choose(26,11)*choose(26,2) +12*choose(26,12)*choose(26,1) + 13*choose(26,13)*choose(26,0))/(26*choose(52,13))
[1] 0.25
```

② Sec 2.1 Binomial distribution

Recall the binomial distribution $\text{Bin}(n,p)$ has

$$\text{mode} = \begin{cases} m & \text{if } np+p \notin \mathbb{Z} \\ m-1, m & \text{if } np+p \in \mathbb{Z} \end{cases} \quad \text{where } m = \lfloor np+p \rfloor$$

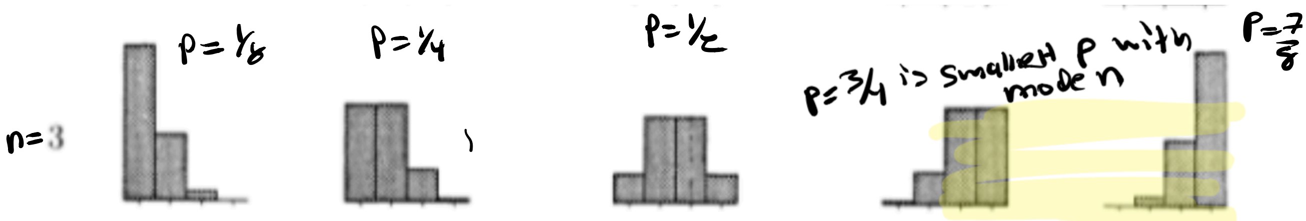
ex $\text{Bin}(3, \frac{3}{4})$

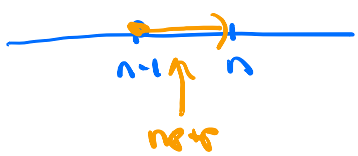
$$np+p = 3\left(\tfrac{3}{4}\right) + \left(\tfrac{3}{4}\right) = \tfrac{9}{4} + \tfrac{3}{4} = 3 \leftarrow \text{integer}$$

so mode 2, 3

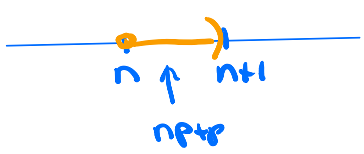
ex Fix n. What is the smallest p such that n is part of the mode?

For example





if $np+p<n$ then mode isn't n



make p small so

$np+p=n$

$n = p(n+1)$

$\Rightarrow \boxed{p = \frac{n}{n+1}}$

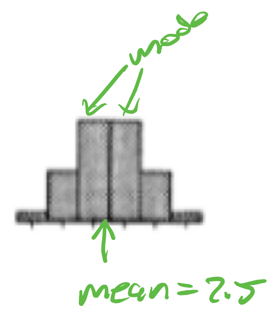
The mode is a measure of the center of your data.

The true center of your data is the <u>expectation</u> or <u>mean</u>

shown in chap 3

<u>Fact</u> The expected (mean) number of successes is $\boxed{\mu = np}$

This isn't usually an integer



<u>ex</u> $n=5$, $p=1/2$ $\quad \mu = np = 5/2$

If the mean is an integer is it the mode?

<u>Yes</u> $np \in \mathbb{Z}$ $\quad np+p \in \mathbb{Z}$

so have one mode and $m = \lfloor np+p \rfloor = np$

shown in Chap 3

<u>Fact</u> the average spread around the mean (standard deviation) is $\boxed{\sigma = \sqrt{npq}}$ where $q = 1-p$

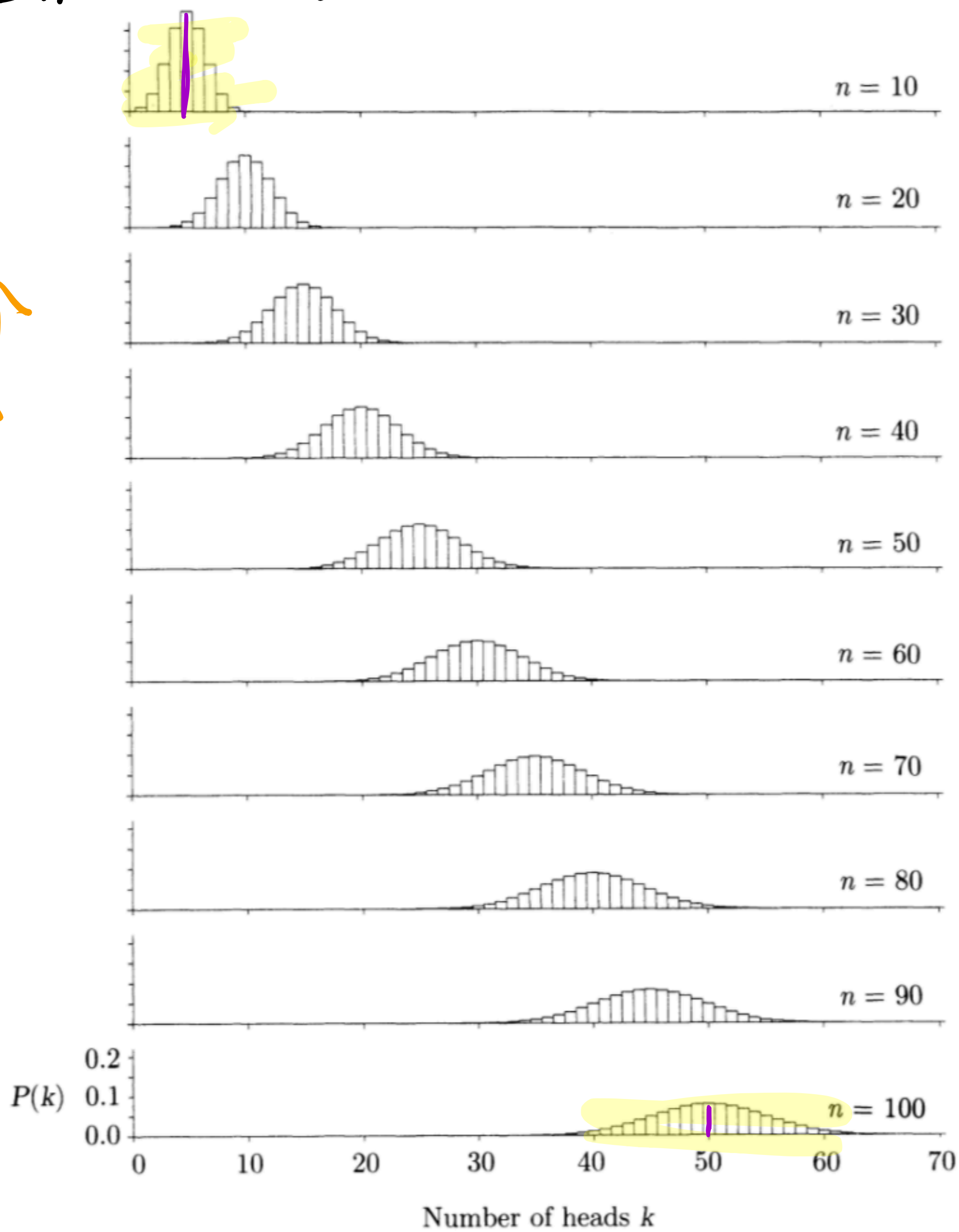
Notice that the spread around the mean gets larger as n gets bigger.

P88

Binomial $(n, \frac{1}{2})$

$\sigma = \sqrt{npq} = \sqrt{n \frac{1}{2} \cdot \frac{1}{2}} = \frac{\sqrt{n}}{2}$

Notice spread ↑ as n ↑



tinyurl.com/sept9-pt1

Stat 134

Chapter 2 Friday February 1 2019

1. A fair coin is tossed, and you win a dollar if there are exactly 50% heads. Which is better?

   a 10 tosses

   b 100 tosses

As n increases the spread increases, and the probability of getting any particular value goes down

The binomial formula gives

$P(5) = \binom{10}{5}\left(\frac{1}{2}\right)^{10} = .246$ ← more likely with 10 tosses.

$P(50) = \binom{100}{50}\left(\frac{1}{2}\right)^{100} = .08$

Stat 134

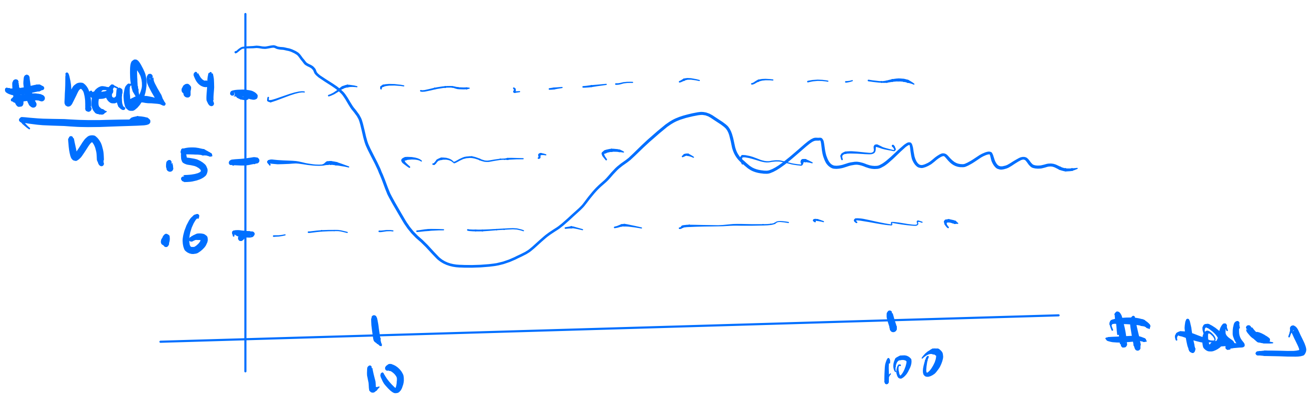
Chapter 2 Friday February 3 2019

1. A fair coin is tossed, which is more likely:

   a between 4 and 6 heads in 10 tosses

   b between 40 and 60 heads in 100 tosses

You saw in sec 1.2 that relative frequency of heads gets closer to the average.



sec 2.2 The normal distribution

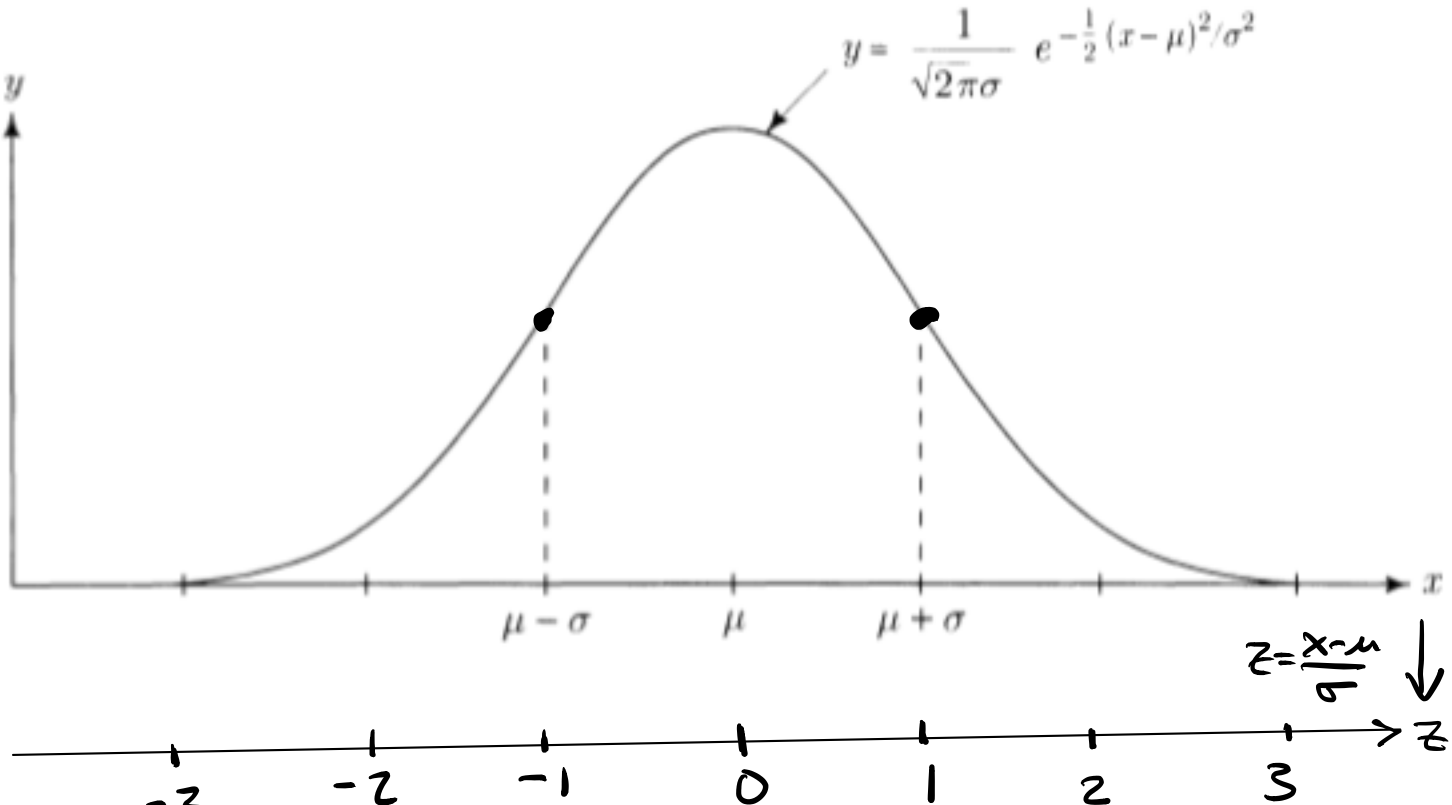
The normal curve is
$$y = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Notice :

1. two param μ = mean
   σ = std dev
2. inflection pts $\mu \pm \sigma$
3. almost all data between $\mu \pm 3\sigma$

P93

FIGURE 1. The normal curve.



To find the area under the curve it is convenient to make a change of coordinates
$$z = \frac{x-\mu}{\sigma}$$
This makes $\mu = 0$ and $\sigma = 1$

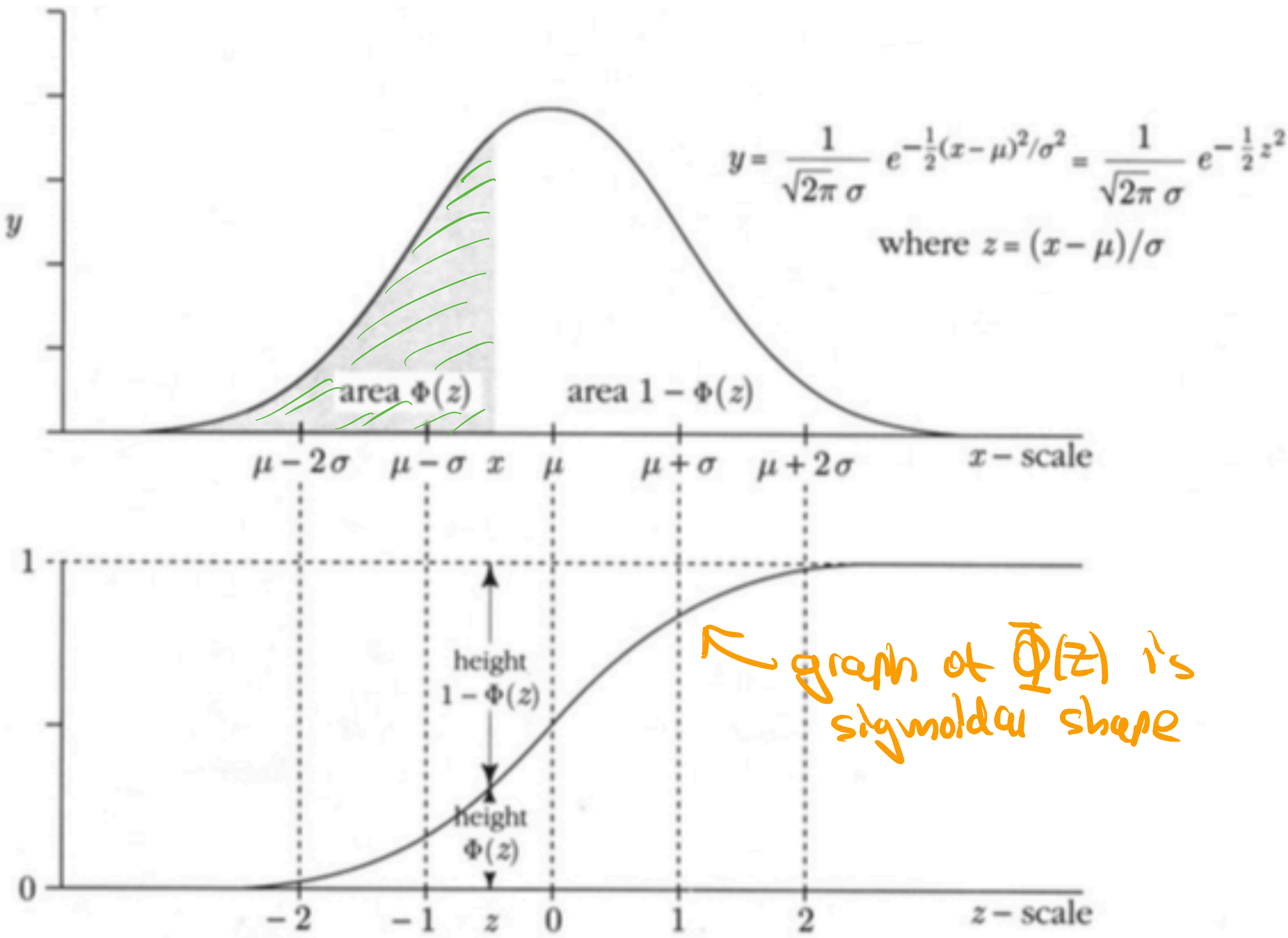
density function

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$ std normal curve

Define cumulative distribution function (cdf)

as $$\Phi(z) = \int_{-\infty}^{z} \phi(t)\,dt$$ area between $-\infty$ and z

P95

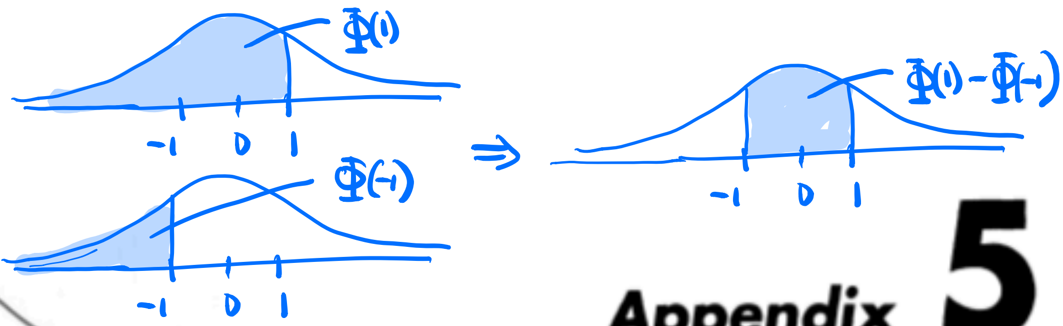


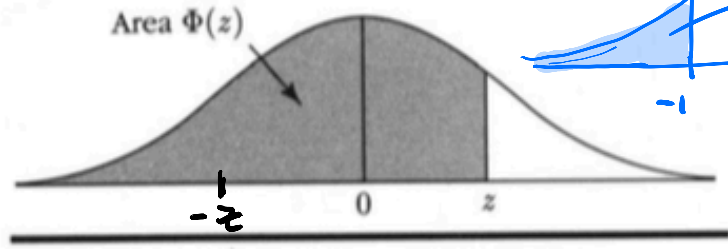
graph of $\Phi(z)$ is sigmoidal shape

we can't solve integral $\Phi(z)$ but instead use look up table.

Notice

$\Phi(-z) = 1 - \Phi(z)$





Appendix 5

Normal Table

Table shows values of $\Phi(z)$ for z from 0 to 3.59 by steps of .01. Example: to find $\Phi(1.23)$, look in row 1.2 and column .03 to find $\Phi(1.2 + .03) = \Phi(1.23) = .8907$. Use $\Phi(z) = 1 - \Phi(-z)$ for negative z.

	.0	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998

Find area between 1 and -1 in std normal curve:

$\Phi(1) - \Phi(-1)$

$= \Phi(1) - (1 - \Phi(1))$

$= 2\Phi(1) - 1$

$= 2(.8413) - 1$

$= .68$

Find area between 2 and -2.

$2\Phi(2) - 1$

$2(.9772) - 1$

$= .95$

Find area between 3 and -3.

$= .997$

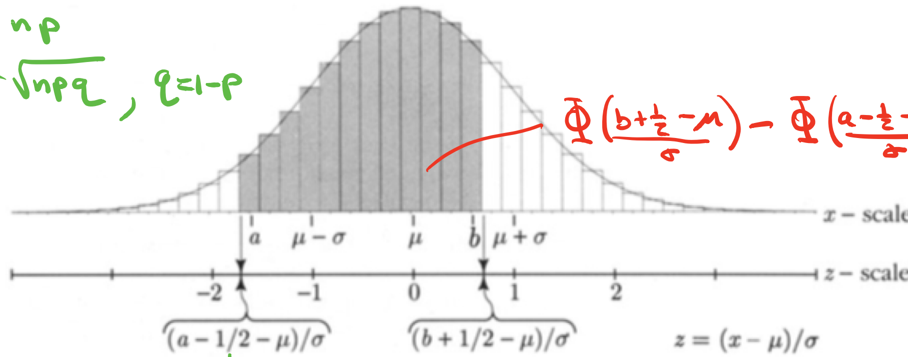
Empirical rule.

Normal Approx to binomial

Motivation: It is difficult to calculate exact probabilities with the binomial formula. It is easier to calculate the area under the normal curve.

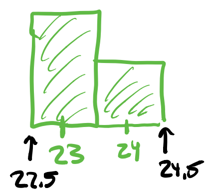
$\mu = np$

$\sigma = \sqrt{npq}$, $q = 1-p$



$$\Phi\left(\frac{b+\frac{1}{2}-\mu}{\sigma}\right) - \Phi\left(\frac{a-\frac{1}{2}-\mu}{\sigma}\right)$$

why $\frac{1}{2}$?



Continuity correction

We are approximating a discrete distribution (binomial) by a continuous one (normal)

ex Find the approximate chance of getting 75 sixes in 600 rolls of a fair die.

$$\mu = np = 600\left(\tfrac{1}{6}\right) = 100$$

$$\sigma = \sqrt{npq} = \sqrt{600\left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right)} = 9.1$$

$$a = b = 75$$

$$\Phi\left(\frac{75.5 - 100}{9.1}\right) - \Phi\left(\frac{74.5 - 100}{9.1}\right) = .00101$$

Exact value

$$\binom{600}{75}\left(\frac{1}{6}\right)^{75}\left(\frac{5}{6}\right)^{525} = .00067$$

Appendix

Fact *

① $K < np + p$ iff $P(k-1) < P(k)$

② $K > np + p$ iff $P(k-1) > P(k)$

③ $K = np + p$ iff $P(k-1) = P(k)$

Proof of Fact * above

First note that
$$\frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{\frac{\cancel{n!}}{k!(n-k)!}}{\frac{\cancel{n!}}{(k-1)!(n-k+1)!}} = \boxed{\frac{n-k+1}{k}}$$

binomial formula

$$\frac{P(k)}{P(k-1)} = \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k-1} p^{k-1} (1-p)^{n-k+1}} = \boxed{\frac{n-k+1}{k} \cdot \frac{p}{1-p}}$$

$$P(k-1) < P(k)$$

$$\Leftrightarrow \quad 1 < \frac{P(k)}{P(k-1)}$$

$$\Leftrightarrow \quad 1 < \frac{n-k+1}{k} \cdot \frac{p}{1-p}$$

$$\Leftrightarrow \quad k(1-p) < (n-k+1)p$$

$$\Leftrightarrow \quad k - \cancel{kp} < np - \cancel{pk} + p$$

$$\Leftrightarrow \quad k < np + p$$

so $P(k-1) < P(k) \Leftrightarrow k < np + p$

similarly for $>$ or $=$ instead of $<$