

Stat 134 Lec 31

Warmup 1:00-1:10

Recall the Rayleigh distribution, $\text{Ray} \sim \sqrt{\text{Exp}(\frac{1}{2})}$
Let $R_1, R_2, R_3, R_4 \stackrel{iid}{\sim} \text{Rayleigh}$.

We showed $W = \min(R_1, R_2, R_3) \sim \frac{1}{\sqrt{3}} \text{Ray}$

Find $P(R_4 < W)$

Hint: Find $P(R_4^2 < W^2)$

$$R_4^2 \sim \text{Exp}(\frac{1}{2})$$

$$W^2 \sim \frac{1}{3} \text{Exp}(\frac{1}{2}) = \text{Exp}\left(\frac{1}{2} \cdot \frac{1}{3}\right) = \text{Exp}\left(\frac{1}{6}\right)$$

Competing exponentials

$$P(R_4 < W) = P(R_4^2 < W^2) = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{3}{2}} = \boxed{\frac{1}{4}}$$

Last time

Sec 5.3 independent normal variables

a) $T \sim \text{Exp}\left(\frac{1}{2}\right) \Rightarrow R = \sqrt{T}$ has density $f(r) = r e^{-\frac{1}{2}r^2}, r > 0$

↳ called Rayleigh RV

b) Using Rayleigh distribution we showed if $x, y \stackrel{iid}{\sim} N(0, 1)$

$$\textcircled{1} \quad \sqrt{x^2 + y^2} = R$$

$$\textcircled{2} \quad \Phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$\textcircled{3} \quad E(z) = 0$$

$$\textcircled{4} \quad \text{SD}(z) = 1$$

Today

(0) Sec 5.3 Sum of independent normals

(1) A linear combination of independent normals is normal

(2) Chi square distribution

Sec 5.4

(3) Convolution formula for the density of $X+Y$

① Sec 5.3 Sum of independent normals

Definition of the Normal (μ, σ^2) distribution:

We know the density of the std normal

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Make a change of

scale $X = \mu + \sigma z$ By the change of variable rule you can show

$$f_X(x) = \frac{1}{\sigma} \phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left(\frac{x-\mu}{\sigma}\right)^2}$$

$\uparrow z = \frac{x-\mu}{\sigma}$

MGF Review (lecture 24)

Recall the MGF of a RV X is $M(t) = E(e^{Xt})$.

MGF of std normal

$$X \sim N(0, 1), M(t) = e^{\frac{t^2}{2}} \text{ for all } t$$

Properties of MGF

If X, Y are independent $M_{X+Y}(t) = M_X(t)M_Y(t)$

$M_X(t) = M_Y(t)$ iff $X \stackrel{d}{=} Y$

$\rightarrow M_{b+ax}(t) = e^{bt} M_X(at)$

for t in an interval containing

some distributions

~~or~~ Use the properties of MGF to find the MGF of $X \sim N(\mu, \sigma^2)$

$$Z \sim N(0, 1) \quad \mu \in \mathbb{R}, \sigma > 0$$

$$X = \mu + \sigma Z$$

$$\begin{aligned} M_X(t) &= M_{\mu + \sigma Z}(t) = e^{\mu t + \frac{1}{2}(\sigma t)^2} \\ &= \boxed{e^{\mu t} \cdot e^{\frac{\sigma^2 t^2}{2}}} \quad || \\ &\qquad\qquad\qquad M_Z(\sigma t) \end{aligned}$$

$$X \sim N(0, 2)$$

$$M_X(t) = e^{t^2}$$

or Use MGF to prove that the sum of two independent std normals is $N(0, 2)$.

hint

$$X \sim N(\mu, \sigma^2) \Rightarrow M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$Z_1, Z_2 \stackrel{iid}{\sim} N(0, 1)$$

$$\begin{aligned} M_{Z_1 + Z_2}(t) &= M_{Z_1}(t)M_{Z_2}(t) \\ &= e^{t^2/2} \cdot e^{t^2/2} = e^{t^2} \end{aligned}$$



MGF of
 $N(0, 2)$

for all t

$$\Rightarrow Z_1 + Z_2 \sim N(0, 2)$$

Then Let $X_1 \sim N(\mu_1, \sigma_1^2)$ } indep.
 $X_2 \sim N(\mu_2, \sigma_2^2)$ }

then $\alpha X_1 + b X_2 \sim N(\alpha\mu_1 + b\mu_2, \alpha^2\sigma_1^2 + b^2\sigma_2^2)$

Pf) $M_{X_1}(t) = e^{\mu_1 t} e^{\frac{\sigma_1^2 t^2}{2}}$
 $M_{\alpha X_1 + b X_2}(t) = M_{X_1}(\alpha t) = e^{\mu_1 \alpha t} e^{\frac{\sigma_1^2 \alpha^2 t^2}{2}}$

then $M_{\alpha X_1 + b X_2}(t) = M_{\alpha X_1}(t) M_{b X_2}(t)$
 $= e^{\mu_1 \alpha t} e^{\frac{\sigma_1^2 \alpha^2 t^2}{2}} \cdot e^{\mu_2 b t} e^{\frac{\sigma_2^2 b^2 t^2}{2}}$
 $= e^{(\mu_1 \alpha + \mu_2 b)t} e^{(\sigma_1^2 \alpha^2 + \sigma_2^2 b^2) \frac{t^2}{2}}$

new mean

new variance

by uniqueness of MGF

$$\alpha X_1 + b X_2 \sim N(\mu_1 \alpha + \mu_2 b, \sigma_1^2 \alpha^2 + \sigma_2^2 b^2)$$

□



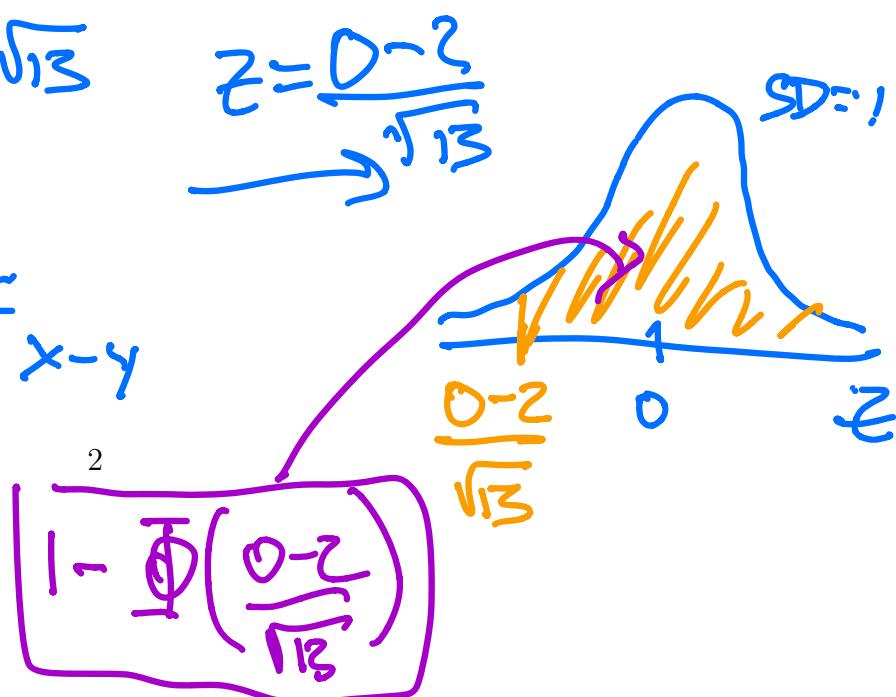
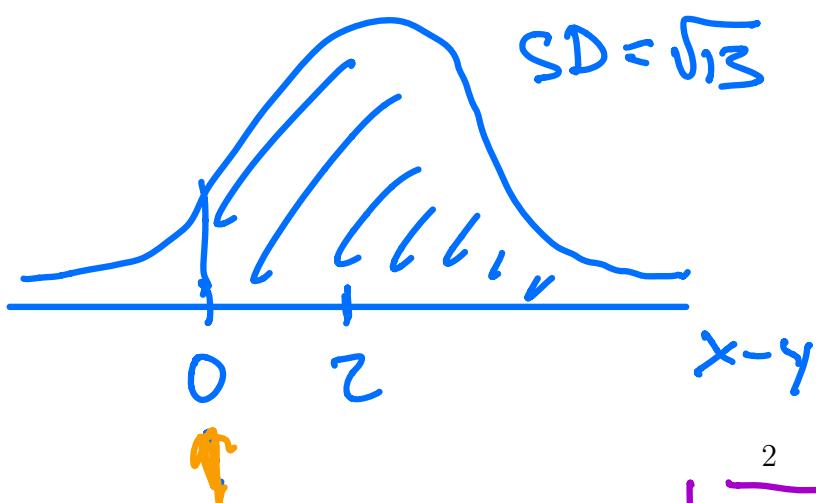
. Let $X \sim N(68, 3^2)$ and $Y \sim N(66, 2^2)$ be independent. $P(X > Y)$ equals

- a $1 - \Phi\left(\frac{0-2}{\sqrt{3^2+2^2}}\right)$
- b $1 - \Phi\left(\frac{0-2}{3^2+2^2}\right)$
- c $1 - \Phi\left(\frac{68-66}{\sqrt{3^2+2^2}}\right)$
- d none of the above

$$\begin{aligned}P(X > Y) \\ P(X - Y \geq 0)\end{aligned}$$

$$\begin{aligned}a &\approx 1 \\ b &\approx -1\end{aligned}$$

$$X - Y \sim N(68 - 66, 3^2 + 2^2) = N(2, 13)$$



② Sec 5.5 Chi-square distribution

see end of lecture notes for proof.

Fact If $Z \sim N(0,1)$ then

$$Z^2 \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$$

Recall that if $T \sim \text{Gamma}(r, \lambda)$,

$$M_T(t) = \left(\frac{\lambda}{\lambda-t}\right)^r, t < \lambda$$

Hence $M_{Z^2}(t) = \left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right)^{\frac{1}{2}}, t < \frac{1}{2}$

Let $Z_1, \dots, Z_n \stackrel{\text{iid}}{\sim} N(0,1)$

$$M_{\sum Z_i^2}(t) = \left(\frac{\frac{n}{2}}{\frac{1}{2}-t}\right)^{\frac{n}{2}}, t < \frac{1}{2}$$

By uniqueness of MGF

$$\sum Z_i^2 \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$$

\hookrightarrow called chi-square distribution with n degrees of freedom,
 χ_n^2 , with n degrees of freedom

ex Let $X, Y \stackrel{\text{iid}}{\sim} N(0,1)$

Let $R = \sqrt{x^2 + y^2}$ be Raleigh distribution,

$$\text{Then } R^2 = X^2 + Y^2 \sim \chi_2^2 = \text{Exp}\left(\frac{1}{2}\right) = \text{Gamma}\left(1, \frac{1}{2}\right)$$

(3) Sec 5.4 The Density Convolution Formula

Ex Let X and Y be discrete RVs taking values $\{0, 1, 2, 3, 4, 5, 6\}$.

$$\text{Let } S = X + Y.$$

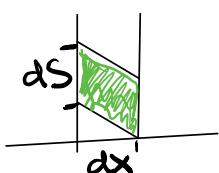
Find the probability mass function of S .

$$P(S=3) = P(X=0, Y=3-0) + P(X=1, Y=3-1) + P(X=2, Y=3-2) \\ + P(X=3, Y=3-3)$$

$$P(S=s) = \sum_{x=0}^s P(X=x, Y=s-x)$$

↗ convolution formula.

A little geometry:



Area of parallelogram
 $A = dx dS$

Let $X \geq 0, Y \geq 0$ be continuous RVs with joint density $f(x, y)$.

$$\text{let } S = X + Y$$

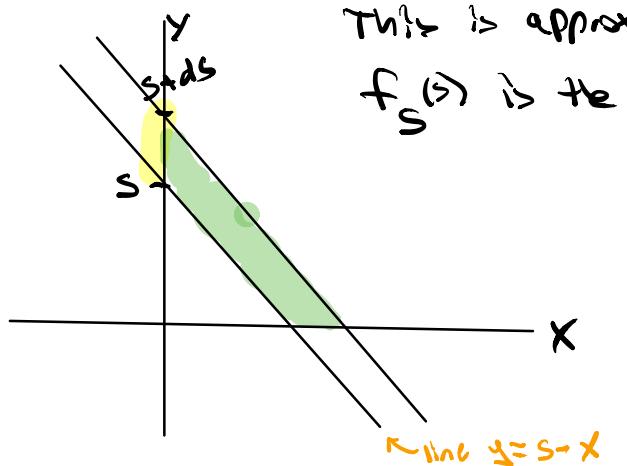
Find the density of S

$$s = x + y$$

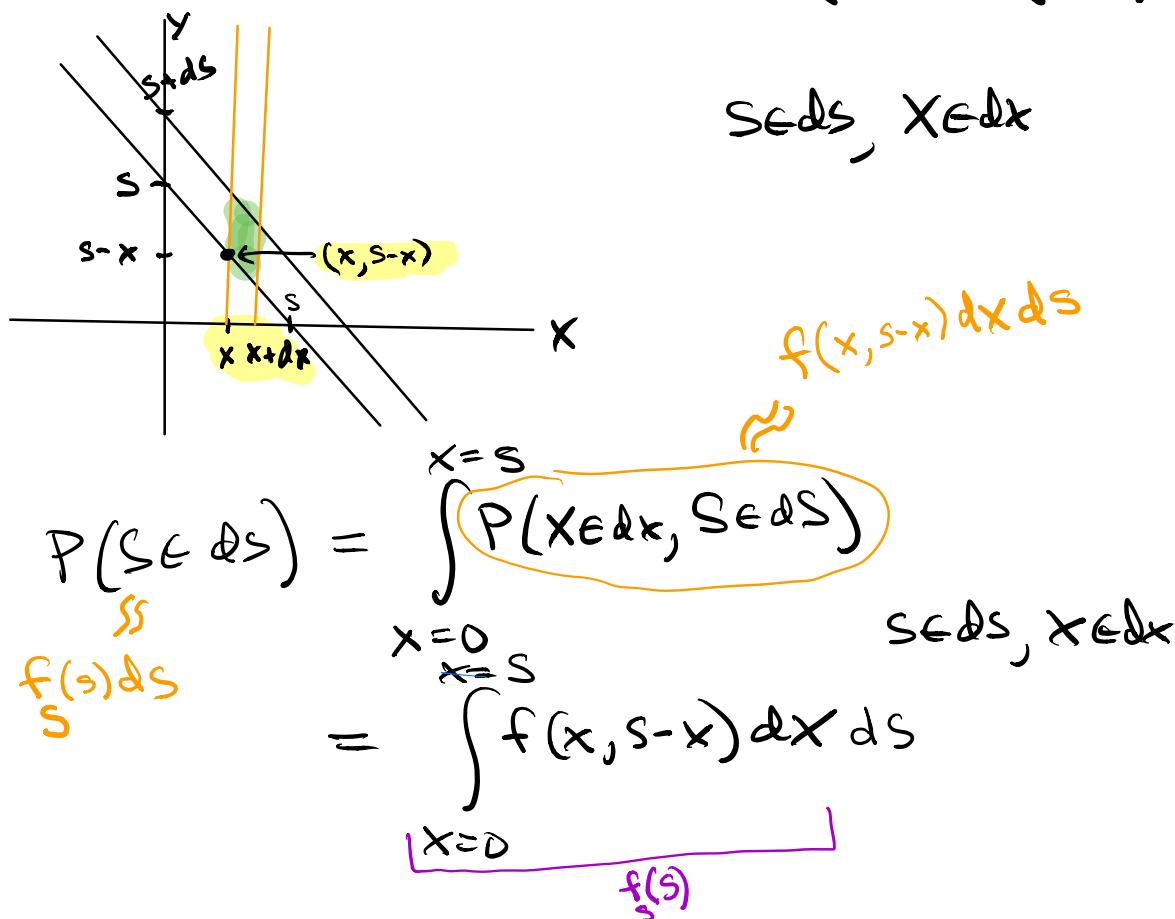
$$y = s - x$$

↗ of intercept.

$P(S \in ds)$ is the volume under $f(x, y)$ over the green region.
 This is approx $\int_S f(s) ds$ where $f_S(s)$ is the density of S .



$P(X \in dx, S \in ds)$ is the volume under $f(x, y)$ over the green region.



$$\Rightarrow f_s(s) = \int_{x=0}^{x=s} f(x, s-x) dx$$

Convolution formula for densities.

Compare with:

$$P(S=s) = \sum_{x=0}^s P(x, s-x)$$

Convolution formula for P.M.F

$$\stackrel{\text{def}}{=} X, Y \stackrel{iid}{\sim} \text{expon}(\lambda) \quad S = X + Y$$

$$\begin{aligned}
 f_s(s) &= \int_0^s f(x, s-x) dx \\
 &\stackrel{\text{independence}}{=} \int_0^s f_x(x) f_y(s-x) dx \\
 &= \int_0^s \lambda e^{-\lambda x} \lambda e^{-\lambda(s-x)} dx \\
 &= \int_0^s \lambda^2 e^{-\lambda s} dx = \lambda^2 e^{-\lambda s} \int_0^s dx = \lambda^2 e^{-\lambda s} s
 \end{aligned}$$

$$\boxed{\lambda^2 e^{-\lambda s}}$$

variable part of
gamma(2, λ)

$$\Rightarrow S \sim \text{gamma}(2, \lambda).$$

Appendix

If $Z \sim N(0, 1)$, then $Z^2 \sim \text{Gamma}(\frac{r}{2}, \frac{1}{2})$

Proof /

$\lambda > 0$, $r >$ integer r ,

gamma (r, λ) density

$$f(t) = \frac{\lambda^r}{\Gamma(r)} t^{r-1} e^{-\lambda t}, \quad t > 0$$

let $Z = \text{std normal}$
change of variable rule.

$$X = Z^2$$

Find $f_X(x) = \frac{1}{\sqrt{2\pi}} X^{-\frac{1}{2}} e^{-\frac{1}{2}x}, \quad x > 0$

$$= \frac{1}{\sqrt{2\pi}} X^{-\frac{1}{2}-1} e^{-\frac{1}{2}x}$$

 $\boxed{\sqrt{\pi}} \times \Gamma(\frac{1}{2})$

$$\Rightarrow X \sim \text{gamma}(\frac{r}{2}, \frac{1}{2})$$

□