Stat 134 <u>lec 11</u>

Warmup 11:00 – 11:10

n people with hats have had a bit too much to drink at a party. As they leave the party, each person randomly grabs a hat. A match occurs if a person gets his or her own hat.

a The expected number of matches depends on n

b The expected number of matches is 1

c The number of matches is hypergeometric

d more than one of the above

X = # people (out of n) who get a match,

$X = I_1 + \dots + I_n$

$I_2 = \begin{cases} 1 & \text{if 2nd person gets match} \\ 0 & \text{else} \end{cases}$

$P = \frac{1}{n}$

$E(X) = n\left(\frac{1}{n}\right) = \boxed{1}$

indicators dependent but X not HG

For HG you must be able to tell if element in pop is good before you draw. Here you can only tell after you draw.

Note: A student in class asked about making Adam's hat good and the rest bad, and if now X is HG?

The answer is no.

In this case $G=1$ in the population.

If you define

$$I_1 = \begin{cases} 1 & \text{if Adam gets his hat} \\ 0 & \text{else} \end{cases}$$

then I_1 is HG (with $N=n$, $G=1/n$, sample size $=1$)

However,

$$X = \underbrace{I_1}_{\substack{\text{1 if}\\ \text{Adam gets}\\ \text{hat back}}} + \underbrace{I_2}_{\substack{\text{1 if}\\ \text{Talia}\\ \text{gets}\\ \text{hat back}}} + \cdots + \underbrace{I_n}_{\substack{\text{1 if}\\ \text{Mitchel}\\ \text{gets}\\ \text{hat}\\ \text{back}}}$$

isn't hypergeometric since sum of dependent hypergeometrics isn't HG.

Last time sec 3.2 Expectation

$$E(X) = \sum_{x \in X} x P(X=x)$$

If X is a count, X can be written as a sum of indicators

$$X = I_1 + I_2 + \cdots + I_n \quad , \quad I_j = \begin{cases} 1 & \text{Prob } p \\ 0 & \text{Prob } 1-p \end{cases}$$

$1 \le j \le n$

$$E(I_j) = 1 \cdot p + 0 \cdot (1-p) = p.$$

Idea Even if indicators are dependent the expectation of each indicator is an unconditional probability.

Try choosing indicators such that all indicators have the same expectation p.

Then $E(X) = n \cdot p$

we proved if $X \sim Bin(n,p) \Rightarrow E(X) = np$

if $X \sim HG(n,N,G) \Rightarrow E(X) = n\frac{G}{N}$

ex

ex X = # aces in a poker hand from a deck of cards

$X \sim HG(n,N,G)$ $N = 52$, $G = 4$, $n = 5$

$X = I_1 + I_2 + I_3 + I_4 + I_5$

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ card is an ace} \\ 0 & \text{else} \end{cases}$ — $p = 4/52$

$E(X) = 5 \cdot E(I_1) = \boxed{5 \cdot \left(\frac{4}{52}\right)}$

Today

1. Sec 3.2 More expectation with indicator examples
2. Sec 3.3 tail sum formula
3. Sec 3.3 Markov's inequality.

1 Sec 3.2 More expectation / indicator examples

ex Consider a 5 card deck consisting of 2, 2, 3, 4, 5. Shuffle the cards.
Let X = number of cards before the first 2.

a) What are the range of values of X?

$0, 1, 2, 3$

b) Write X as a sum of indicators

$X = I_3 + I_4 + I_5$ (3, 4, 5 are the non 2 cards)

c) How is an indicator defined.

$$I_4 = \begin{cases} 1 & \text{if } 4 \text{ is before } 1^{st}\ 2 \\ 0 & \text{else.} \end{cases}$$

d) Find $E(I_4)$

$\frac{1}{3}$

The calculation of $E(I_4)$ only involves 3 cards 4, 2, 2. Take out all other cards. Now you have a 3 card deck.

Picture __ 2 __ 2 __ where each slot can be empty or have a 4 in it. It is equally likely you have a 4 in any of the 3 slots, hence $E(I_4) = 1/3$.

e) Find $E(X)$

$3 \cdot \frac{1}{3} = 1$

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Stat 134

1. Consider a well shuffled deck of cards. The expected number of cards before the first ace is?

a 52/5

b 48/5

c 48/4

d none of the above

Intuitively spread out the 48 nonaces among 4 aces gives # cards before 1^{st} ace = # cards between 1^{st} and 2^{nd} ace etc $\Rightarrow$ 48/5.

X = # cards before the 1^{st} ace

0, 1, 2, ..., 48

4 × 12 = 48 nonaces

$$\underbrace{\overset{1}{2}\ \overset{2}{3}\ \overset{3}{4}\ \overset{4}{5}\ \overset{5}{6} \cdots \overset{10}{J}\ \overset{11}{Q}\ \overset{12}{K}}_{\text{hearts}}\ \underbrace{\overset{13}{2}\ \overset{14}{3}\ \overset{15}{4} \cdots}_{\text{spades}}\ \underbrace{\quad}_{\text{clubs}}\ \underbrace{\cdots \overset{47}{Q}\ \overset{48}{K}}_{\text{diamonds}}$$

$X = I_1 + I_2 + \cdots + I_{48}$

$I_j = \begin{cases} 1 & \text{if } j^{th} \text{ nonace before } 1^{st} \text{ ace} \\ 0 & \text{else} \end{cases}$ P

$$\_\_\ A_1\ \_\_\ A_2\ \_\_\ A_3\ \_\_\ A_4\ \_\_$$

$P = 1/5 \Rightarrow \boxed{E(X) = 48\left(\tfrac{1}{5}\right)}$

③ <u>Sec 3.2</u> <u>Tail Sum formula for expectation</u>

Suppose X is a count $0, 1, 2, 3, \ldots$

$$E(X) = 0\cdot P(X=0) + 1\cdot P(X=1) + 2\cdot P(X=2) + \cdots$$

$$= 1\cdot \underbrace{P(X=1)}_{p_1} + 2\cdot \underbrace{P(X=2)}_{p_2} + \cdots$$

$$= \begin{array}{cccc} P(X\ge 1) & P(X \ge 2) & P(X\ge 3) & \\ p_1 & & & \\ p_2 & p_2 & & \\ p_3 & p_3 & p_3 & \\ p_4 & p_4 & p_4 & p_4 \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

$$= P(X\ge 1) + P(X \ge 2) + P(X \ge 3) + \cdots$$ Tail Sum Formula

This is useful when it is easy to find $P(X \ge k)$.

ex A fair die is rolled 10 times.
Let $X = \max(X_1, \ldots, X_{10})$.
Find $P(X \ge k)$

$$P(X\ge k) = 1 - P(X<k)$$
$$= 1 - P(X_1<k, X_2<k, \ldots, X_{10}<k)$$
$$= 1 - P(X_1<k)P(X_2<k)\cdots P(X_{10}<k)$$
$$= 1 - P(X_1<k)^{10}$$
$$= 1 - \left(\frac{k-1}{6}\right)^{10}$$

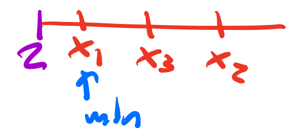
so
$$E(X) = \underset{1}{\underset{\parallel}{P(X\ge 1)}} + \underset{1-\left(\frac{1}{6}\right)^{10}}{\underset{\parallel}{P(X\ge 2)}} + \cdots + \underset{1-\left(\frac{5}{6}\right)^{10}}{\underset{\parallel}{P(X\ge 6)}} + \underset{0}{\underset{\parallel}{P(X\ge 7)}} + \cdots$$

$$= 6 - \left(\frac{1}{6}\right)^{10}\left[1^{10} + 2^{10} + 3^{10} + 4^{10} + 5^{10}\right] = 5.82$$

ex A fair die is rolled 3 times, X_1, X_2, X_3.
Let Y be the sum of the largest 2 numbers.
Notice that $Y = X_1 + X_2 + X_3 - \min(X_1, X_2, X_3)$.

a) Find $P(\min(X_1, X_2, X_3) \geq 2)$

Picture



$= P(X_1 \geq 2, X_2 \geq 2, X_3 \geq 2)$

$= P(X_1 \geq 2)^3 \quad = \left(\frac{5}{6}\right)^3$

(b) Find $E(\min(X_1, X_2, X_3))$

$= P(\min \geq 1) + P(\min \geq 2) + \cdots + P(\min \geq 6)$

$= P(X_1 \geq 1)^3 + P(X_1 \geq 2)^3 + \cdots + P(X_1 \geq 6)^3$

$= \left(\frac{6}{6}\right)^3 + \left(\frac{5}{6}\right)^3 + \left(\frac{4}{6}\right)^3 + \cdots + \left(\frac{1}{6}\right)^3 = \frac{1}{6^3}\left[6^3 + 5^3 + 4^3 + 3^3 + 2^3 + 1^3\right]$

(c) Find $E(Y) = E(X_1) + E(X_2) + E(X_3) - E(\min(X_1, X_2, X_3))$

$E(X_1) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = \frac{1}{6}\left[1+2+3+4+5+6\right]$

$= \frac{21}{6} = \boxed{\frac{7}{2}}$

$\Rightarrow E(Y) = \boxed{3\left(\frac{7}{2}\right) + \frac{1}{6^3}\left[6^3 + 5^3 + 4^3 + 3^3 + 2^3 + 1^3\right]}$