Stat 134 lec 27

Sec 4.6

last time

$U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$

$U_{(1)}, \dots, U_{(n)}$ order statistics

learned

constant

$$f_{U_{(k)}}(x) = \frac{n!}{(k-1)!\,((n-k+1)-1)!} x^{k-1} (1-x)^{(n-k+1)-1} \text{ on } 0<x<1$$

let r, s be pos integers

define Beta(r,s) density as

$$f(x) = \text{constant} \cdot x^{r-1}(1-x)^{s-1} \text{ on } 0<x<1$$

Compare with density of $U_{(k)}$ and find the constant

$r = k$

$s = n-k+1 \implies n = s+r-1$

$$\text{constant} = \frac{(s+r-1)!}{(r-1)!\,(s-1)!} = \frac{\Gamma(s+r)}{\Gamma(r)\Gamma(s)}$$

so $\boxed{f(x) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} x^{r-1}(1-x)^{s-1} \text{ for } 0<x<1}$ Beta(r,s) density

By principle of ignoring constants

$$\boxed{\int_0^1 x^{r-1}(1-x)^{s-1}\,dx = \frac{1}{\text{constant}} = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}}$$

let $X \sim \text{Beta}(r, s)$

$$E(X) = \int_0^1 x f_X(x)\,dx$$

$$= \frac{\Gamma(r+s)}{\Gamma(r)\cancel{\Gamma(s)}} \underbrace{\int_0^1 x\,x^{r-1}(1-x)^{s-1}\,dx}_{= \frac{\cancel{\Gamma(s)}\Gamma(r+1)}{\Gamma(s+r+1)}}$$

$x\,x^{r-1} = x^{r+1-1}$

$$= \boxed{\frac{r}{r+s}}$$

Conclusions

- The order statistics $U_{(1)}, \dots, U_{(k)}$ of the standard uniform give a family of densities with a single mode along $(0,1)$.

- $U_{(k)} \sim \text{Beta}(k, n-k+1)$

- $$\int_0^1 x^r (1-x)^{r+s-1} = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} = \frac{(r-1)!\,(s-1)!}{(r+s-1)!}$$

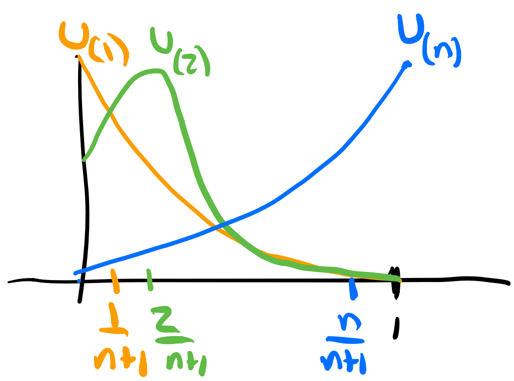
  the principle of ignoring constants.

- $X \sim \text{Beta}(r,s)$

  $$E(X) = \frac{r}{r+s}$$ — see p 329

  $r = k$, $s = n-k+1$ $\Rightarrow$ $$E(U_{(k)}) = \frac{k}{n+1}$$

  $$\Rightarrow E(U_{(1)}) = \frac{1}{n+1}$$
  $$E(U_{(2)}) = \frac{2}{n+1}$$
  $$\vdots$$
  $$E(U_{(n)}) = \frac{n}{n+1}$$



Chap 5 Continuous Joint Dist

sec 5.1, 5.2

X, Y have joint density $f(x,y)$ means f must satisfy

$$f(x,y) \geq 0$$

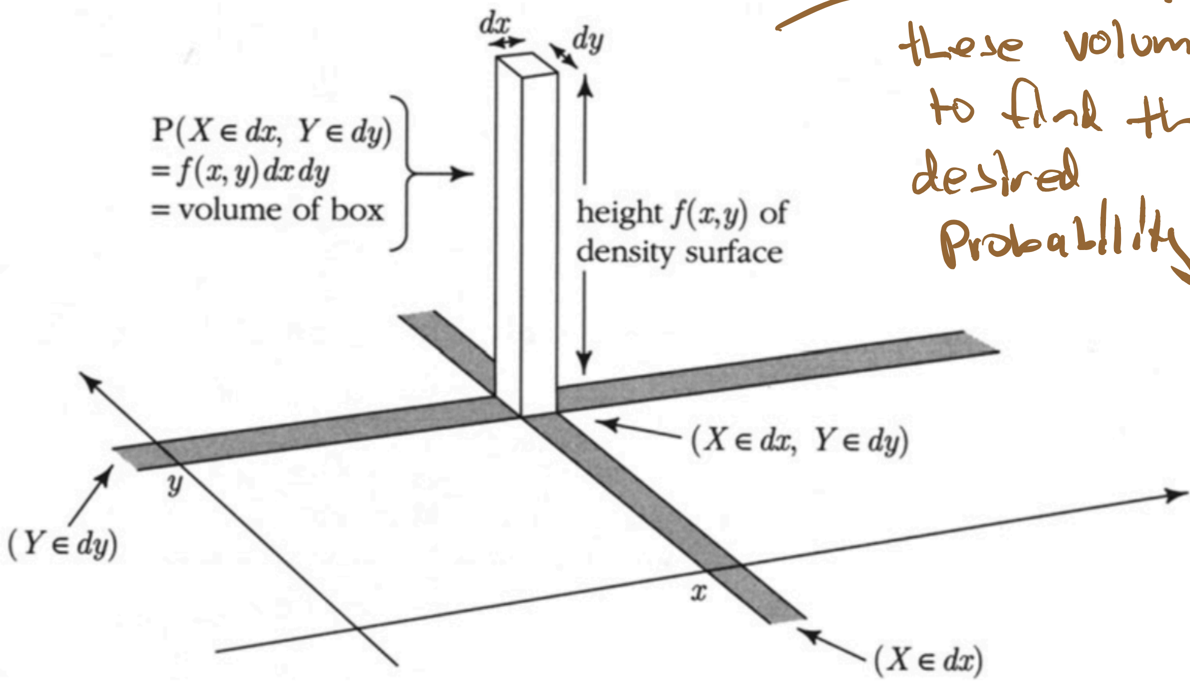
$$\int_y \int_x f(x,y)\,dx\,dy = \int_x \int_y f(x,y)\,dy\,dx = 1$$

Let A be a subset of the plane

$$P((X,Y) \in A) = \iint_A f(x,y)\,dx\,dy$$

$$P(X \in dx, Y \in dy) = f(x,y)\,dx\,dy$$

Picture



— we add up the these volumes to find the desired probability.

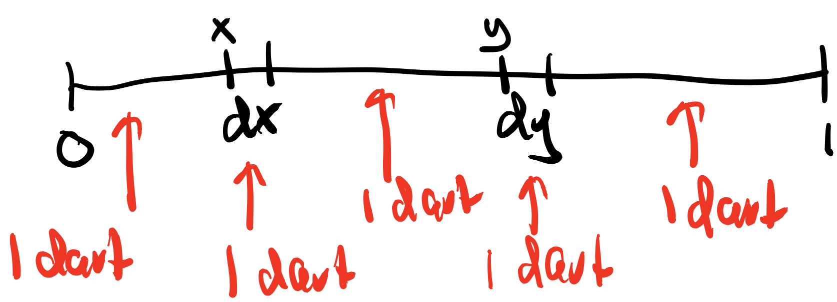
ex Throw down 5 darts on $(0,1)$

Find the joint density of

$X = U_{(2)}$ and $Y = U_{(4)}$

to find density:

$$P(X \in dx, Y \in dy) = f(x,y)\,dx\,dy$$



$$= 5x \cdot 4dx \cdot 3(y-x) \cdot 2dy\,(1-y)$$

$$= \underbrace{5!\,x(y-x)(1-y)}_{f(x,y)}\,dx\,dy$$

this must be a density since it is

$$\frac{P(X \in dx, Y \in dy)}{dx\,dy}$$

and so integrates to 1.

Stat 134

Wednesday March 21 2018

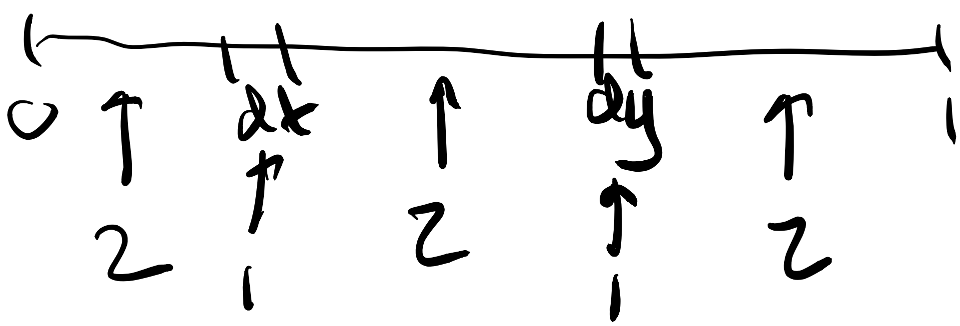
1. I throw down 8 darts on $(0, 1)$. The variable part of the joint density of $X = U_{(3)}$ and $Y = U_{(6)}$ is:

   a $x(y-x)^5(1-y)^2$

   b $x^2(y-x)^2(1-y)^2$

   c $x^4(y-x)^2(1-y)^2$

   d none of the above

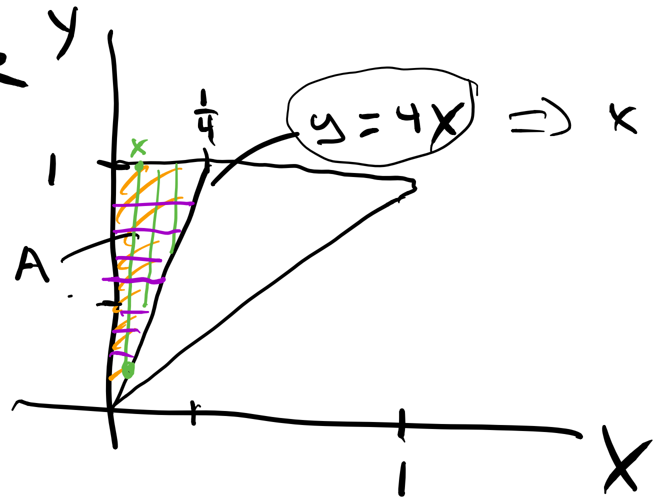


ex $X = U_{(2)}$, $Y = U_{(4)}$

Find $P(Y > 4X)$

$Y = 4X$

Picture



$\Rightarrow x = \frac{y}{4}$

$f(x,y) = 5!\,x\,(y-x)(1-y)$

$$P(Y > 4X) = \iint_A f(x,y)\,dx\,dy$$

usually one order of integration is better than the other

$$P(Y>4X) = \int_{x=0}^{x=1/4}\int_{y=4x}^{y=1} 120x\,(y-x)(1-y)\,dy\,dx$$

or $\circledast$

$$\int_{y=0}^{y=1}\int_{x=0}^{x=\frac{y}{4}} 120x\,(y-x)(1-y)\,dx\,dy$$

easier to integrate since one of the limits of integration is zero.

details:

$$P(Y > 4X) = \int_{y=0}^{y=1} \int_{x=0}^{y/4} 120\, x\, (y-x)(1-y)\, dx\, dy$$

$$= 120 \int_{y=0}^{y=1} (1-y) \int_{x=0}^{x=y/4} (xy - x^2)\, dx\, dy$$

$$= \int_0^1 120\, (1-y) \left(\frac{x^2 y}{2} - \frac{x^3}{3} \right) \Bigg|_0^{y/4}$$

$$= \int_0^1 120\, (1-y) \left(\frac{y^3}{32} - \frac{y^3}{3 \cdot 64} \right) dy$$

$$= \frac{5}{192} \cdot 120 \int_0^1 y^3 - y^4\, dy = \frac{5 \cdot 120}{192} \left(\frac{y^4}{4} - \frac{y^5}{5} \right) \Bigg|_0^1$$

$$= \frac{5 \cdot 120}{192} \left(\frac{1}{4} - \frac{1}{5} \right) = \frac{30}{192} = .156$$