Stat 134: Section 22 Solution

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April 15, 2020

Conceptual Review

Please discuss these short questions with those around you in section. These problems are intended to highlight concepts that will be relevant for today's problems.

a. What is conditional expectation? Can you give a concrete example?

b. Why is the expectation equal to the expectation of conditional expectation? Try to give an intuitive explanation and then a rigorous proof.

Problem 1

Let X_1, X_2 be independent uniform variables on $\{1, \cdots, n\}$. Let U, V be the minimum and maximum of X_1, X_2, respectively. Compute $\mathbb{E}[V|U = u]$ and $\mathbb{E}[U|V = v]$.What if X_1, X_2 are draws without replacement but still independent?

Ex 6.2.2 and 6.2.3 in Pitman's Probability

Soln: Conditional on $U = u$, (X_1, X_2) must be $(u, u), \cdots, (u, n), (n, u), \cdots, (u + 1, u)$, where we note that the pair (u, u) appears once while the others appear twice. Hence, $P(V = v) = \frac{1}{2(n-u+1)-1}$ if $v = u$ and $\frac{2}{2(n-u+1)-1}$ if $v > u$. Hence its conditional expectation is just $\mathbb{E}[V|U = u] = (u + n)/2$. Hence after simplifing the terms we can find the expected value to be:

$$\frac{n^2 + n - u^2}{2n - 2u + 1}.$$

Similarly, we can compute $\mathbb{E}[U|V = v] = (v^2)/(2v - 1)$.

The case where these draws are without replacement is similar. If the minimum is u then the maximum is uniform on $[u + 1, \cdots, n]$, so the conditional expectation is $(u + n + 1)/2$. The other conditional expectation is $v/2$.

Problem 2

A deck of cards is cut into two halves of 26 cards each. As it turns out, the top half contains 3 aces and the bottom half just one ace. The top half is shuffled, then cut into two halves of 13 cards each. One of these packs of 13 cards is shuffled into the bottom half of 26 cards, and from this pack of 39 cards, 5 cards are dealt. What is the expected number of aces among these 5 cards?
Ex 6.2.11 in Pitman's Probability

Soln: Let A_k denote the event that the 13 cards added to the pack of 39 cards contains the k-th ace, so $P(A_k) = P(A_k^c) = 0.5$. The number of aces in the 5 cards dealt, denoted N, is equal to $I_1 + \cdots + I_4$, where I_k represents the indicator that the k-th ace is in the 5 cards. Without loss of generality, we may assume that the first ace is already in the original bottom half. Hence the remaining three aces have the same probability of showing up in the final 5 cards, but the first ace is different.

We may compute:

$$
\begin{aligned}
\mathbb{E}[N] &= \mathbb{E}[I_1] + 3\mathbb{E}[I_2] \\
&= \frac{5}{39} + 3 \cdot \left(\mathbb{E}[I_2|A_2]P(A_2) + \mathbb{E}[I_2|A_2^c]P(A_2^c) \right) \\
&= \frac{5}{39} + 3 \cdot \left(\frac{5}{39} \cdot 0.5 + 0 \cdot 0.5 \right) \\
&= \frac{25}{78}.
\end{aligned}
$$

Problem 3

Define the conditional variance of Y on $X = x$ as

$$\mathrm{Var}[Y|X = x] = \mathbb{E}[Y^2|X = x] - (\mathbb{E}[Y|X = x])^2.$$

Now, show that:

$$\mathrm{Var}(Y) = \mathbb{E}[\mathrm{Var}(Y|X)] + \mathrm{Var}[\mathbb{E}(Y|X)].$$

Ex 6.2.18 in Pitman's Probability

Soln: Using the definition of conditional variance, we can show that:

$$\mathbb{E}[\mathrm{Var}(Y|X)] = \mathbb{E}[Y^2] - \left(\mathrm{Var}[\mathbb{E}[Y|X]] + \mathbb{E}[Y]^2 \right).$$

The result then follows naturally.

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