Stat 134: Review Session 6.1-6.3

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Conceptual Review

a. Suppose X is a continuous random variable with possible values on (a, b). Let A be an event. Write down a formula for $P(A)$ by conditioning on X.

b. Let Y be a random variable, and let $B_1, B_2, B_3, \ldots$ be events. Write down an expression for $E(Y)$ by conditioning on the events B_i.

c. Let X_1, X_2 be random variables, and suppose h is a two-input function, and consider $\theta = E[h(X_1, X_2)]$. Now let g be a function defined by $g(x) = E[h(x, X_2)] - \theta$. Explain why $g(X_1)$ is not 0 but is a random variable with $E[g(X_1)] = 0$.

d. Let $N \sim \mathit{Geometric}(p)$ on $\{1, 2, \ldots\}$. What is $E[cN + d|N > m]$?

Problem 1

You and your friend are going to play a rock-paper-scissors game all day. You know that you choose rock 1/3 of the time, paper 1/6 of the time, and scissors 1/2 of the time. However, your friend does not like paper. Instead, your friend plays rock 1/2 of the time and scissors half of the time. Each time you duel, each player randomly picks one of rock, paper, or scissors independently. A duel corresponds to a pair, for example (Rock, Rock).

a. Find the expected number of duels it takes you to get:
   i. (Rock, Rock)
   ii. (Scissor, Scissor)

b. Find the expected number of duels it takes to get either two (Rock, Rock) or two (Scissor, Scissor) in a row, whichever comes first. After a (R,R) stalemate occurs, neither player will pick Scissor. Instead, Paper absorbs this probability. The same works for a (S, S) stalemate.

Problem 2: Conjugate Priors "Gamma-Poisson"

Suppose the number of car accidents on the Bay Bridge every year, X, follows a $Poisson(\mu)$ distribution given μ. The rate μ is a realization from another distribution (called a prior) $M \sim Gamma(r, \lambda)$.

a. Given that X = k, what is the posterior distribution of the rate μ? (i.e. find the distribution of $M|X = k$)

b. Compare the expectation of M and $M|X = k$.
Plug in $r = 1,\ \lambda = 1,\ k = 5$

Problem 3: Branching Process

Suppose there is a certain species that reproduces as follows: At time $n = 0$, there is one organism, and for each time step $n \geq 0$, each organism independently gives birth to 0 organisms with probability 1/4, or 2 organisms with probability 3/4, then dies. Let X_n denote the number of organisms alive at time n. We wish to find the probability that the species goes extinct (and thus the probability that it survives forever).

a. Show that $E(X_{n+1}) = \frac{3}{2}E(X_n)$. Note that this implies that $E(X_n) = (3/2)^n$.

b. Let $u_n = P(X_n = 0)$. Show that $u_n = \frac{1}{4} + \frac{3}{4}u_{n-1}^2$.

c. It takes some additional analysis to show that the sequence $\{u_n\}$ converges (indeed, it is an increasing sequence bounded above by 1). Taking this for granted, what is the probability that the species will eventually go extinct? *Hint:* You should arrive at 2 possible answers. Argue intuitively why one of them should be the correct answer.

Prepared by Peter, Joseph and Yann