Stat 134 lec 18

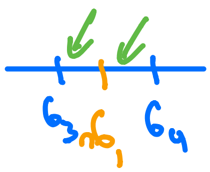
Warmup 1:00–1:10

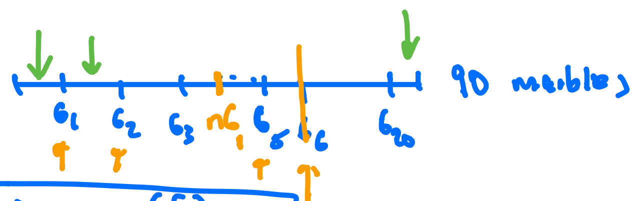
An urn contains 90 marbles, of which there are 20 greens, 20 blacks and 50 red marbles. Tom draw marble without replacement until the 6^{th} green marble. Let $X = \#$ of marbles drawn. Example: **GGG**BRB**GG**BR**G** with $x = 11$. Find $\mathbb{E}[X]$.

$p = \frac{6}{21}$

$X = I_1 + \cdots + I_{70} + 6$

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ nongreen before } 6^{th} \text{ green} \\ 0 & \text{else} \end{cases}$





$$\boxed{E(X) = 70\left(\frac{6}{21}\right) + 6}$$

$p_{12} = \frac{6}{21} \cdot \frac{7}{22}$

Var(X)

$I_{12} = \begin{cases} 1 & \text{if } 1^{st} \text{ and } 2^{nd} \text{ nongreen before } 6^{th} \text{ green} \\ 0 & \text{else} \end{cases}$

let $Y = I_1 + \cdots + I_{70}$

$Var(Y) = Var(X)$

$E(Y^2) = 70p_1 + 70 \cdot 69 p_{12}$

$E(Y) = E(X) - 6 = 70\left(\frac{6}{21}\right)$

$Var(X) = Var(Y) = E(Y^2) - (E(Y))^2.$

ex Chebyshev

Ashley Jung
5:43pm

Can we go over the Chebyshev problem from the problems we couldn't go over from Tuesdays's lecture?

Reply (1 like)

You are using a telescope to measure the speed at which the planet Saturn crosses the night sky. To do this you draw two lines on your lens, and measure the time it takes for Saturn to cross between the two lines. However, your time measurement is noisy, so you will conduct this observation several times and average their results.

Let X_i represent the time measurement from the ith observation. Your measurements are well calibrated, so for each i, $E(X_i) = \mu_X$, where μ_X is the true time it takes Saturn to cross between the lines. Each measurement also has standard deviation $SD(X_i) = 0.03$ seconds.

a You will take n measurements, $X_1, \ldots, X_n$, using the same procedure, and use the sample average $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ to estimate μ_X. In terms of n, what is $SD(\bar{X})$?

b What is the smallest number of measurements you will need to take so that your estimate $\bar{X}$ has at most a $\frac{1}{25}$ probability of falling outside the interval $\mu_X \pm 0.003$ seconds? (Hint: Chebyshev)

a) $\bar{X} = \frac{1}{n}(X_1 + \cdots + X_n)$

$Var(X_1) + Var(X_2) + \cdots + Var(X_n) = n\,Var(X_1)$

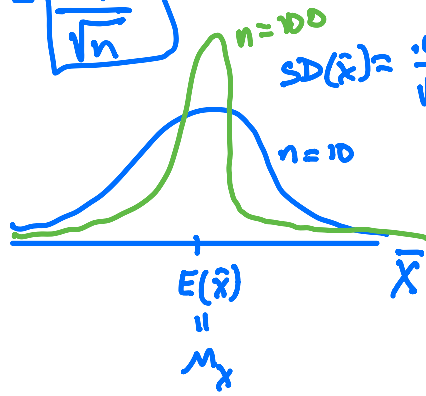
$Var(\bar{X}) = \frac{1}{n^2} \cdot n\,Var(X_1) = \frac{1}{n} Var(X_1)$

$SD(\bar{X}) = \sqrt{\frac{1}{n} Var(X_1)} = \frac{1}{\sqrt{n}} \cdot SD(X_1) = \boxed{\frac{.03}{\sqrt{n}}}$

$E(\bar{X}) = E\left(\frac{1}{n}(X_1 + \cdots + X_n)\right)$

$= \frac{1}{n} \cdot n E(X_1) = E(X_1) = \mu_X$

$SD(\bar{X}) = \frac{.03}{\sqrt{n}}$



b) $P(|\bar{X} - \mu_X| \geq .003) \leq \frac{1}{5^2}$

$.003 = 5 \cdot SD(\bar{X}) = 5 \cdot \frac{.03}{\sqrt{n}}$

$\sqrt{n} = \frac{5 \cdot .03}{.003} = 50 \Rightarrow \boxed{n = 2500}$

Yuen Chen

9:24am

Can we go over $\sum_0$ $\sum_1$ $\sum_2$ on page 212 and 213 ?

Reply (1 like)

From lecture 15 :

Appendix

Fact $Var(X) = \frac{q}{p^2}$

To find $Var(X)$ we need an identity :

Σ_0 $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$ geometric sum $\sum_{k=0}^{\infty} q^k = 1 + q + q^2 + q^3 + \dots = \frac{1}{1-q}$

$\frac{d}{dq}$

Σ_1 $\sum_{k=0}^{\infty} k q^{k-1} = \frac{1}{(1-q)^2}$ $\sum_{k=0}^{\infty} 1 + 2q + 3q^2 + 4q^3 + \dots$

$\frac{d}{dq}$

$$\boxed{\sum_{k=0}^{\infty} k(k-1) q^{k-2} = \frac{2}{(1-q)^3} = \frac{2}{p^3}}$$

$\sum_{k=0}^{\infty} k(k-1) q^{k-2} = 2 + 6q + 12q^2$

$$\mathrm{Var}(X) = E(X^2) - E(X)^2$$

$$= E(X^2) - E(X) + E(X) - E(X)^2$$

$$= E(\underbrace{X^2 - X}_{X(X-1)}) + \underbrace{E(X)}_{= \frac{1}{p}} - \underbrace{E(X)^2}_{= \frac{1}{p^2}}$$

$$E(X(X-1)) = \sum_{k=1}^{\infty} k(k-1)\underbrace{P(X=k)}_{= q^{k-1}p}$$

$$E(g(X)) = \sum_{x \in X} g(x) P(X=x)$$

$$= qp \sum_{k=1}^{\infty} k(k-1) q^{k-2} = qp \underbrace{\sum_{k=0}^{\infty} k(k-1) q^{k-2}}_{= \frac{2}{p^3}}$$

(see above)

$$= \frac{2q}{p^2}$$

so $\mathrm{Var}(X) = \underbrace{\frac{2q}{p^2}}_{E(X(X-1))} + \underbrace{\frac{1}{p}}_{E(X)} - \underbrace{\frac{1}{p^2}}_{E(X)^2} = \boxed{\frac{q}{p^2}}$

Meifeng Huang

Yesterday

I would like to go over chapter 3.3, question 8.

Reply (1 like)

8. Let A_1, A_2, and A_3 be events with probabilities $\frac{1}{5}$, $\frac{1}{4}$, and $\frac{1}{3}$, respectively. Let N be the number of these events that occur.

a) Write down a formula for N in terms of indicators.

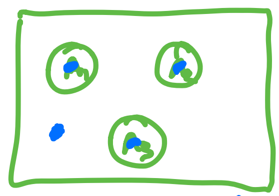
b) Find $E(N)$.

In each of the following cases, calculate $Var(N)$:

c) A_1, A_2, A_3 are disjoint;

d) they are independent;

e) $A_1 \subset A_2 \subset A_3$.



$1/4$

a) $N = I_1 + I_2 + I_3$ $\quad I_2 = \begin{cases} 1 & \text{if } A_2 \\ 0 & \text{else} \end{cases}$

b) $E(N) = \frac{1}{5} + \frac{1}{4} + \frac{1}{3}$

$P(N=1) = \frac{1}{5} + \frac{1}{4} + \frac{1}{3} = \frac{47}{60}$

$P(N=0) = 1 - \frac{47}{60} = \frac{13}{60}$

$N = \begin{cases} 1 & \text{Prob } \frac{47}{60} \\ 0 & \text{Prob } 13/60 \end{cases}$

$Var(N) = p \cdot q = \boxed{\frac{47}{60} \cdot \frac{13}{60}}$

d) $N = I_1 + I_2 + I_3$

$Var(N) = \underset{\frac{1}{5}\frac{4}{5}}{\underset{\parallel}{Var(I_1)}} + \underset{\frac{1}{4}\frac{3}{4}}{\underset{\parallel}{Var(I_2)}} + \underset{\frac{1}{3}\cdot\frac{2}{3}}{\underset{\parallel}{Var(I_3)}}$

e) $E(N^2) - E(N)^2$

$(I_1 + I_2 + I_3)^2 = I_1^2 + I_2^2 + I_3^2 + \overset{2I_1^2}{\overset{\parallel}{2I_{12}}} + \overset{2I_1^2}{\overset{\parallel}{2I_{13}}}$

$+ \underset{I_2^2}{\underset{\parallel}{2I_{23}}}$

$$= 5I_1 + 3I_2 + I_3$$

$$E\left((I_1+I_2+I_3)^2\right) = 5E(I_1) + 3E(I_2) + E(I_3)$$

$$= \frac{25}{12}$$

$$\Rightarrow \mathrm{Var}(N) = \frac{25}{12} - \left(\frac{47}{60}\right)^2$$

$E(N)^2$

Brandon Byrne

6:12pm

— lets discuss in OH

Could we please do 3.rev.23 a and 3.rev.27? Thanks

Reply (1 like)

Extra practice

(5 pts) Suppose that on average, 2 moths per 12-hour night are killed by a particular hanging bug zapper. Assume that conditions are the same across different nights and different times of the night, and that moths arrive independently of one another. Find the chance that more than 7 moths are killed in a period of three nights.

Under these assumptions, the number of moths killed in a certain amount of time follows a Poisson Scatter. Let X represent the number of moths killed in one night. We are told that in one night, 2 moths on average appear, so over three nights, 6 moths on average should appear; thus $X \sim \text{Pois}\ (6)$. And so,

$$\begin{aligned} P(X > 7) &= 1 - P(X \leq 7) \\ &= 1 - \sum_{k=0}^{7} P(X = k) \\ &= 1 - \sum_{k=0}^{7} \frac{e^{-6} 6^k}{k!} \end{aligned}$$

Extra Practice

A die has 2 red faces and 4 green faces. The die is rolled 13 times. Given that green faces appeared exactly 7 times in 13 rolls, what is the chance that the green faces appeared exactly 3 times in the first 5 rolls?

Please look to see if you can simplify your answer algebraically.

Let X= the number of green faces in the first 5 rolls.
Let Y= the number of green faces in all 13 rolls.
You can think of your population as 13 rolled die, 7 of which have a green face (Good) lined up randomly (SRS) into a group of 5 followed by a group of 8. We are asked to find the chance that 3 out of the group of 5 are green. Since we have population of size 13, 7 of which are Good, and a sample size of 5 we may apply the HG formula. More precisely, $X|Y=7 \sim HG(13,7,5)$ so by HG formula

$$P(X=3|Y=7) = \frac{\binom{7}{3}\binom{6}{2}}{\binom{13}{5}}.$$

Alternatively, the answer can be worked out using Baye's rule and the binomial formula.

$$P(X=3|Y=7) = \frac{P(Y=7|X=3)P(X=3)}{P(Y=7)}.$$

Notice that $P(Y=7|X=3)$ is the probability of 4 green in the last 8 rolls. Each of the terms can be found with the binomial formula.

$$P(X=3|Y=7) = \frac{\binom{8}{4}(2/3)^4(1/3)^4\binom{5}{3}(2/3)^3(1/3)^2}{\binom{13}{7}(2/3)^7(1/3)^6} = \frac{\binom{8}{4}\binom{5}{3}}{\binom{13}{7}} = \frac{\binom{7}{3}\binom{6}{2}}{\binom{13}{5}}.$$

The last equality can be seen by writing out the definition of the choose terms and doing some algebra, but either answer will receive full credit.