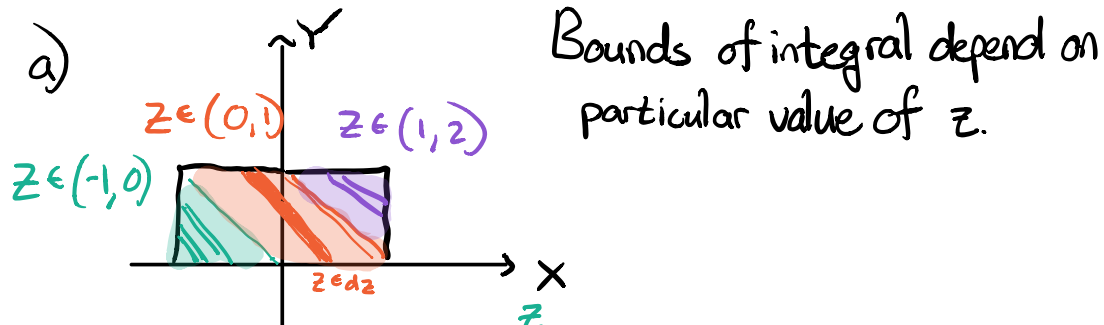*Problem 3*

Suppose $X \sim \text{Unif}(-1,1)$, and $Y \sim \text{Unif}(0,1)$. Find the density of $Z = X + Y$, using:

a. the convolution formula (draw a picture as well);

b. the C.D.F. of Z.

Note $Z \in (-1, 2)$.

a)

$Z \in (0,1)$ $Z \in (1,2)$

$Z \in (-1,0)$

$Z^\epsilon dz$

Bounds of integral depend on particular value of z.

$Z \in (-1, 0):$ $f_Z(z) = \int_{-1}^{z} f_{X,Y}(x, z-x)\, dx$

$$= \int_{-1}^{z} \tfrac{1}{2}\, dx = \frac{z+1}{2}.$$

$Z \in (0,1):$ $f_Z(z) = \int_{z-1}^{z} \tfrac{1}{2}\, dx = \frac{z - z + 1}{2} = \frac{1}{2}.$

$Z \in (1,2):$ $f_Z(z) = \int_{z-1}^{1} \tfrac{1}{2}\, dx = \frac{1 - z + 1}{2} = \frac{2-z}{2}.$

b) Similar cases.

$$F_Z(z) = \begin{cases} \dfrac{\tfrac{1}{2} z(z+1)}{2} & , \ z \in (-1, 0) \\[2mm] \tfrac{1}{4} + \dfrac{z \cdot 1}{4} & , \ z \in (0,1) \\[2mm] \dfrac{\tfrac{3}{4} + \dfrac{(1-z)^2}{2}}{4} & , \ z \in (1,2) \end{cases}$$

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