Stat 134 lec 13

warmup 1:00–1:10

$X \sim \text{Pois}(\tfrac{1}{3})$

Find $E(X!)$

$X \sim \text{Pois}(\mu)$

$P(X=k) = \dfrac{e^{-\mu}\mu^k}{k!}$

$k = 0, 1, 2, \ldots$

$$E(g(x)) = \sum_{\text{all } x} g(x) P(X=x)$$

$$E(X!) = \sum_{k=0}^{\infty} \cancel{k!}\, P(X=k) = \frac{e^{-1/3}\,\frac{1}{3}^{k}}{\cancel{k!}} = e^{-1/3} \sum_{k=0}^{\infty} \left(\tfrac{1}{3}\right)^k$$

$$= e^{-1/3}\left[1 + \tfrac{1}{3} + \left(\tfrac{1}{3}\right)^2 + \cdots\right] = \boxed{\tfrac{3}{2} e^{-1/3}}$$

$$\left[1 + \tfrac{1}{3} + \left(\tfrac{1}{3}\right)^2 + \cdots\right] = \frac{1}{1 - 1/3} = \tfrac{3}{2}$$

last time

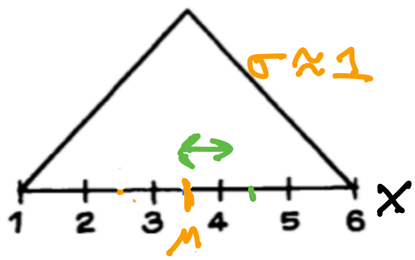
Sec 3.3

SD(X) is the average deviation from the mean

i.e. $SD = \sigma = \sqrt{E((X-\mu)^2)}$

$Var = \sigma^2 = E((X-\mu)^2)$

often write $E(X-\mu)^2$



on average X values are distance 1 from $\mu = 3.5$

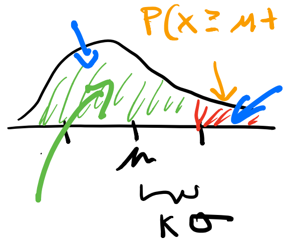
Tail bounds

Markov's inequality

$P(X \geq a) \leq \frac{\mu}{a}$

=

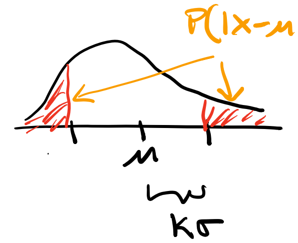
$P(X \geq \mu + k\sigma) \leq \frac{\mu}{\mu + k\sigma}$



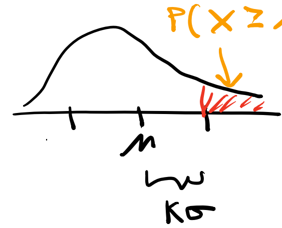
$P(X < \mu + k\sigma) \geq 1 - \frac{\mu}{\mu + k\sigma}$

Chebyshev's inequality

$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$



$P(X \geq \mu + k\sigma) \leq \frac{1}{k^2}$



Today finish sec 3.3

1. review student comments concept test
2. Proof of Chebyshev inequality
3. another formula for variance
4. Properties of variance
5. Central limit theorem (CLT)

① Concept test from last time

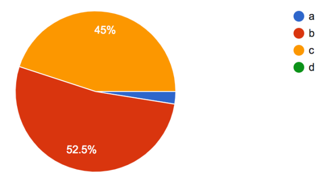
1. A list of non negative numbers has an average of 1 and an SD of 2. Let p be the proportion of numbers ~~over~~ $\geq$ 5. To get an upper bound for p, you should:

   a Assume a normal distribution

   b Use Markov's inequality

   c Use Chebyshev's inequality

   d none of the above



b	The upper bound from Markovs inequality is 1/5, which is lower than the upper bound from chebyshevs inequality, 1/4.

$$P(X \geq 5)$$

$$\underset{=1}{\mu} + k\underset{=2}{\sigma} \Rightarrow k = 2$$

$$M: \frac{\mu}{\mu + k\sigma} = \frac{1}{5} \leftarrow \text{smaller}$$

$$C: \frac{1}{k^2} = \frac{1}{4}$$

(2)

Proof of Chebyshev

For any random variable X, and any $K>0$

$$P\left(|X-E(X)| \geq K\,SD(X)\right) \leq \frac{1}{K^2}$$

Pf/

By Markov

$$P(Y \geq a) \leq \frac{E(Y)}{a} \quad \text{for } Y \geq 0,\ a>0$$

$$(\cancel{SD(X)})^2 = \text{var}(X)$$

let $Y = (X-E(X))^2$ ← nonneg

$a = (K\,SD(X))^2$ ← pos

$$P\left((X-E(X))^2 \geq (K\,SD(X))^2\right) \leq \frac{E\left((X-E(X))^2\right)}{K^2(\cancel{SD(X)})^2} = \frac{1}{K^2}$$

$$\| $$

$$P\left(\underbrace{\sqrt{(X-E(X))^2}}_{=\,|X-E(X)|} \geq \underbrace{\sqrt{(K\,SD(X))^2}}_{=\,K\,SD(X)}\right)$$

$\square$

③

Sec 3.3 Another formula for $Var(X)$.

Recall $E(cX) = cE(X)$ ←

so $E(E(X)X) = E(X)E(X)$

constant

$E(2E(X)X)$

$2E(X)E(X)$

$$Var(X) = E((X-E(X))^2)$$

FOIL

$$= E(X^2 - 2E(X)X + (E(X))^2)$$

$E(cX) = cE(X)$

$$= E(X^2) - \underbrace{2E(X)E(X) + (E(X))^2}_{= -E(X)^2}$$

$$\Rightarrow \boxed{Var(X) = E(X^2) - (E(X))^2}$$

$$\text{or)} \quad \boxed{E(X^2) = Var(X) + E(X)^2}$$

$(E(X))^2$

ex let $X = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{with prob } q \end{cases}$

$$E(X^2) = \sum_{\text{all } x} x^2 P(X=x) = 1^2 \cdot p + 0 \cdot q = \boxed{p}$$

$Var(X)$? — $E(X^2) - (E(X))^2 = p - p^2 = p(1-p)$

$= \boxed{pq}$

variance of indicator

ex. Let X be a nonnegative RV such that

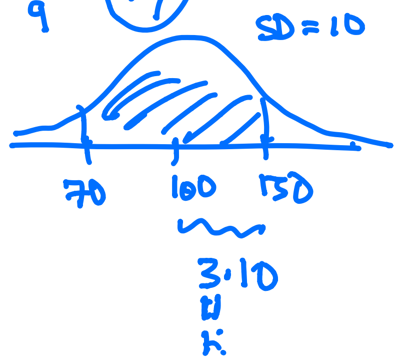
$E(X) = 100 = Var(X)$

a) Can you find $E(X^2)$ exactly? If not what can you say.

$$E(X^2) = Var(X) + E(X)^2$$
$$= 100 + 10{,}000 = \boxed{10{,}100}$$

b) Can you find $P(70^2 < X^2 < 130^2)$ exactly? If not what can you say?

$$P(70 < X < 130) \geq 1 - \frac{1}{k^2} = 1 - \frac{1}{9} = 8/9$$



tinyurl.com/sept25-pt1a

Stat 134

1. X is nonnegative random variable with $E(X) = 3$ and $SD(X) = 2$. True, False or Maybe:

$$P(X^2 \geq 40) \leq \frac{1}{3}$$

a True

b False

c Maybe

C: $P(X \geq \sqrt{40}) \leq \frac{1}{\left(\frac{\sqrt{40}-3}{2}\right)^2} = \underline{0.36}$

$\sqrt{40} = 3 + K \cdot 2$

$K = \frac{\sqrt{40}-3}{2}$

M: $P(X \geq \sqrt{40}) \leq \frac{E(X)}{\sqrt{40}} = \frac{3}{\sqrt{40}} = \underline{.47}$

$P(X^2 \geq 40) \leq \frac{E(X^2)}{40} = \frac{13}{40} < \frac{13}{39} = 1/3$

maybe

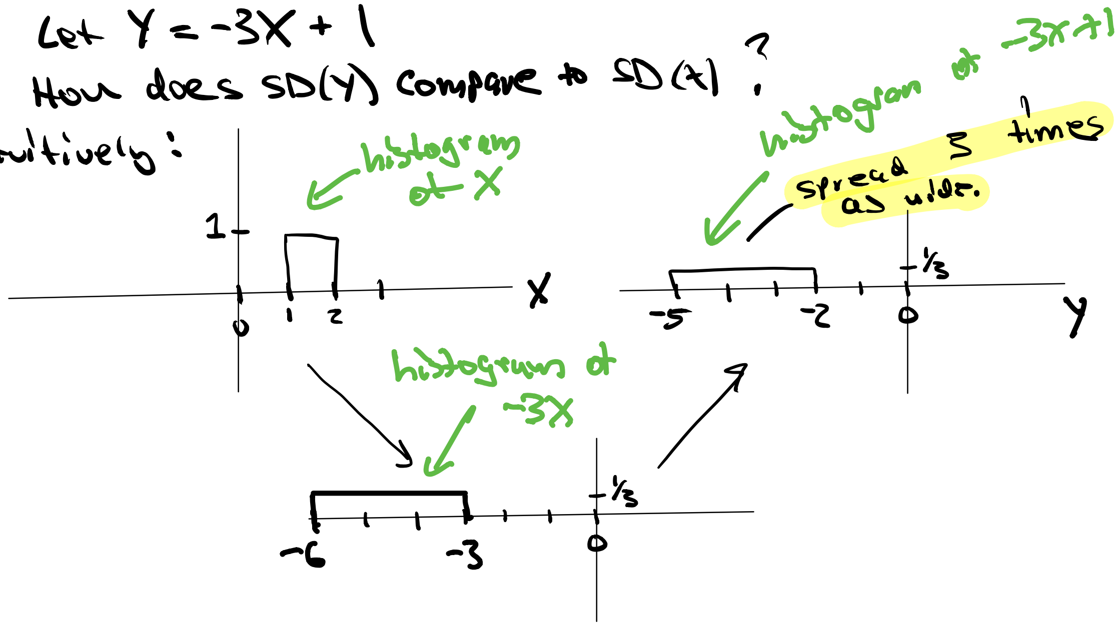
$E(X^2) = Var(X) + (E(X))^2 = 4 + 9 = 13$

④ Properties of Variance

Let $Y = -3X + 1$

How does $SD(Y)$ compare to $SD(X)$?

intuitively:



$$\boxed{\begin{aligned} SD(aX+b) &= |a|\,SD(X) \\ Var(aX+b) &= a^2\,Var(X) \end{aligned}}$$

see p 193 Pitman for the proof.

Thm $Var(X+Y) = Var(X) + Var(Y)$ if X, Y are independent.

ex X = # hours a student is awake a day
Y = # hours a student is asleep a day.
$X + Y = 24 \Rightarrow Var(X+Y) = Var(24) = 0 \neq Var(X) + Var(Y)$

so variance formula needs X, Y to be independent.