Stat 134 lec 5

last time: sec 2.1 mode of the binomial distribution

Binomial formula: $P(k) = \binom{n}{k} p^k (1-p)^{n-k}$

in book int(np+p)

$m = \lfloor np+p \rfloor$ is the mode unless $np+p \in \mathbb{Z}$

in which case $m-1, m$ are mode.

ex For $n=10$, $p=\frac{1}{3}$ the mode is

$m = \lfloor np+p \rfloor = \lfloor 3\tfrac{2}{3} \rfloor = 3$

Today

Finish 2.1

Start 2.2 Normal approximation to binomial

sec 2.1

ex for each n, what is the smallest value of p such that n is the most likely number of successes (i.e part of the mode)?

Picture

n is mode if

$n \le np+p < n+1$

when $np+p=n$ we have both n and $n-1$ the mode.
For p any smaller, $np+p<n$ and n wont be the mode.

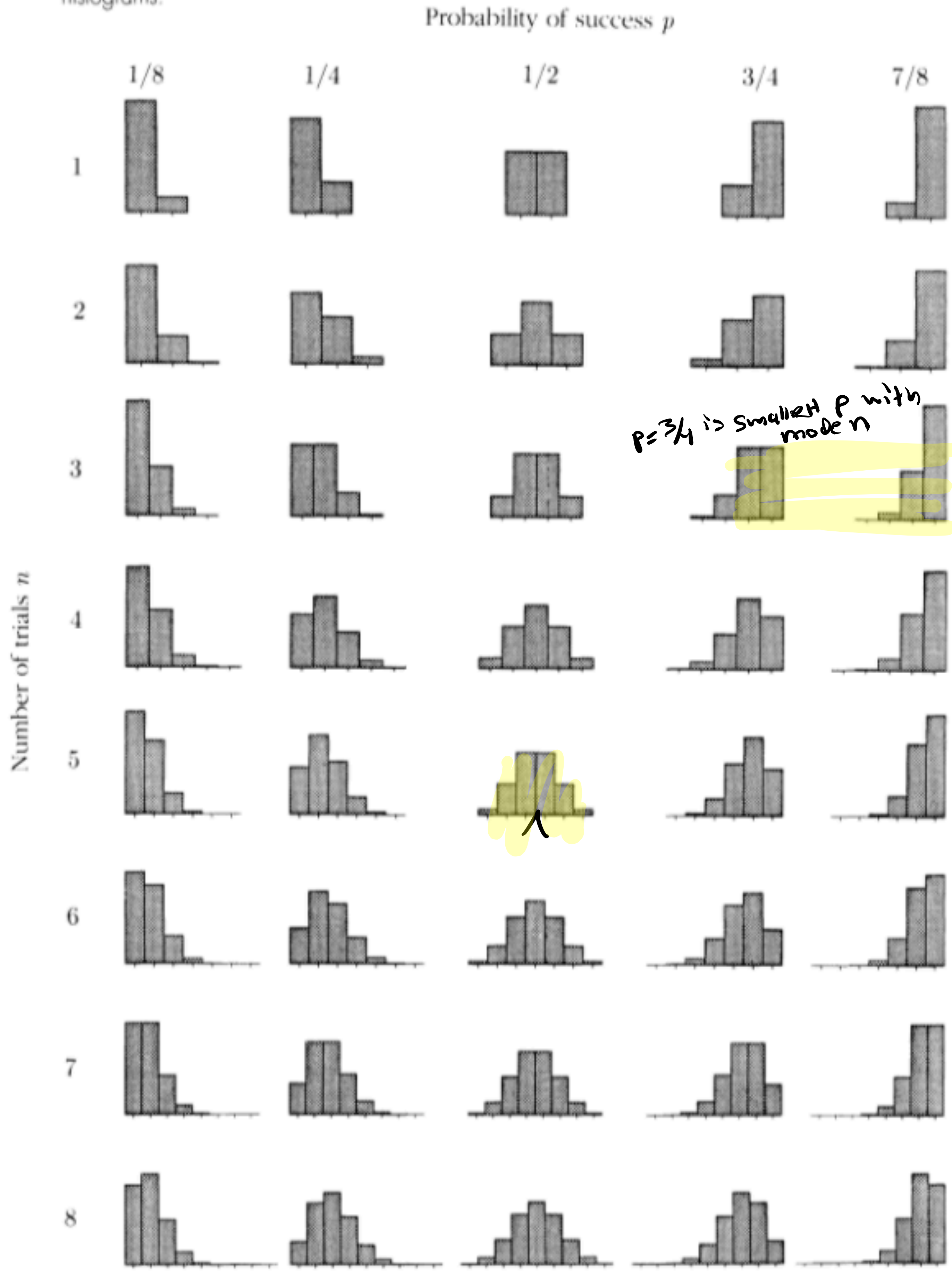
Hence solve $n = np+p$ for n.

$n = np+p = p(n+1)$

$$\boxed{p = \frac{n}{n+1}}$$

P87

FIGURE 3. **Histograms of some binomial distributions.** The histogram in row n, column p shows the binomial (n, p) distribution for the number of successes in n independent trials, each with success probability p. In row n, the range of values shown is 0 to n. The horizontal scale changes from one row to the next, but equal probabilities are represented by equal areas, even in different histograms.



The mode is a measure of the center of your data.
the true center of your data is the expectation or mean.

Fact the expected (mean) number of successes is $\boxed{\mu = np}$ ← shown in chap 3

This isn't usually an integer.

Ex $n=5$, $p=1/2$

$\mu = 5/2$

If the mean is an integer, is it the mode? — yes.

if np is an integer

then $m = \lfloor np+p \rfloor = np$ since

$np+p$ is np plus a little bit.

Fact The average spread around the mean (standard deviation) is $\sigma = \sqrt{npq}$, $q = 1-p$ ← shown in chap 3

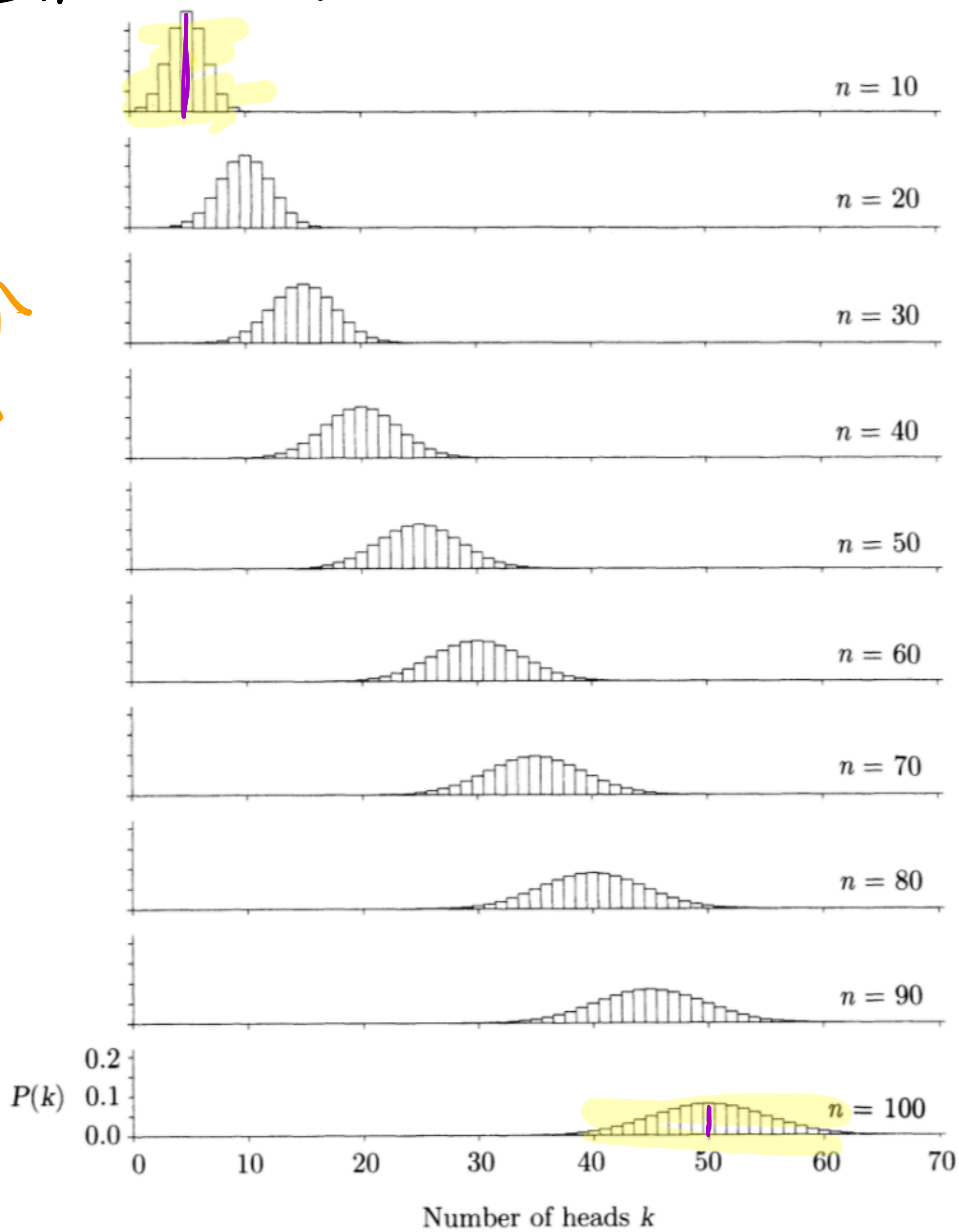
Notice that the spread around the mean gets larger as n gets bigger.

P88

Binomial $(n, \frac{1}{2})$

$$\sigma = \sqrt{npq} = \sqrt{n} \cdot \frac{1}{2}$$

Notice spread ↑ as n ↑



Stat 134

Chapter 2 Friday August 31 2018

1. A fair coin is tossed, and you win a dollar if there are exactly 50% heads. Which is better?

   a 10 tosses

   b 100 tosses

As n increases the spread increases and the probabilities go down.

In this example $P(5) = \binom{10}{5}\left(\frac{1}{2}\right)^{10} = .246$ and

$P(50) = \binom{100}{50}\left(\frac{1}{2}\right)^{100} = .08$

Some of you said b because of the law of large numbers. WRONG.

The law of large numbers says the spread of of the proportion of successes goes down as $n\uparrow$.

Here is why: $\sigma\left(\frac{\#\text{heads}}{n}\right) = \frac{\sigma(\#\text{heads})}{n} = \frac{\sqrt{npq}}{n}$

$= \frac{\sqrt{pq}}{\sqrt{n}}$ which

goes down as $n\uparrow$

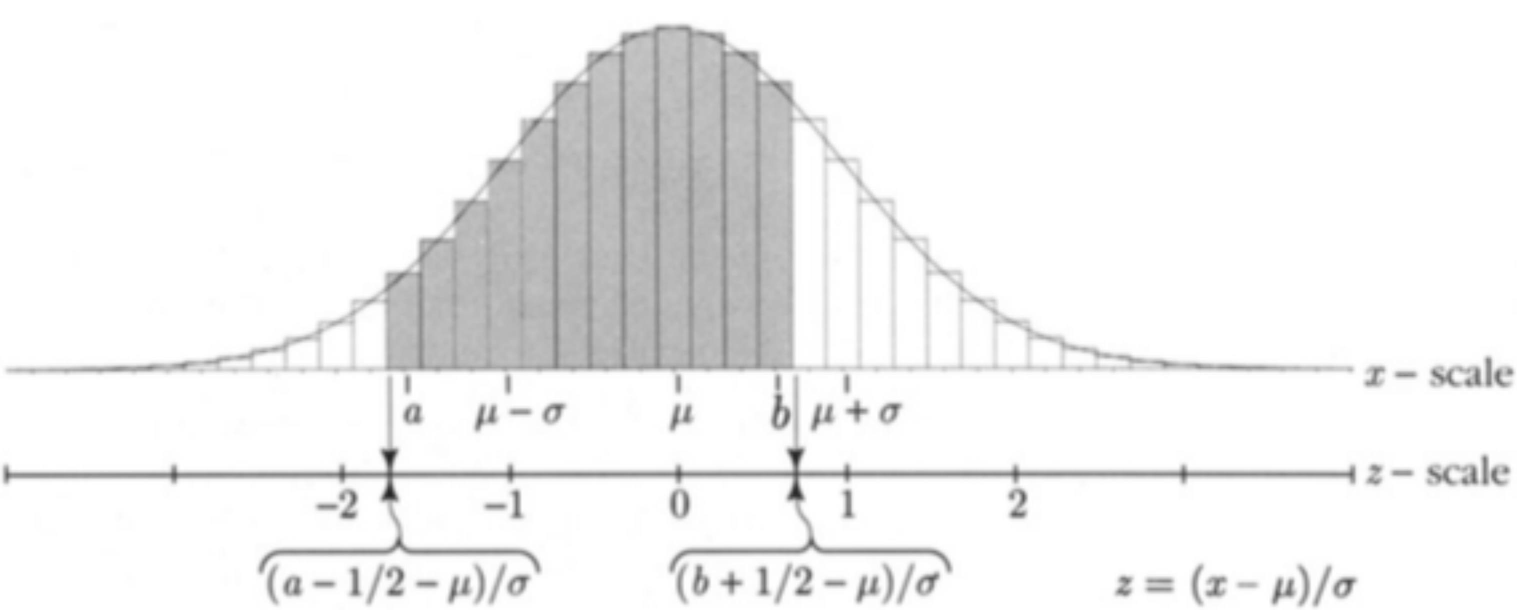
In this problem we are concerned with the spread of the # successes not the proportion of successes.

Sec 2.2 Normal approx.

Motivation. It is difficult to calculate exact probabilities with binomial formula. It is easier to calculate the area under the normal curve.

p98

FIGURE 4. A binomial histogram, with the normal curve superimposed. Both the x scale (number of successes) and the z scale (standard units) are shown.



The normal curve is

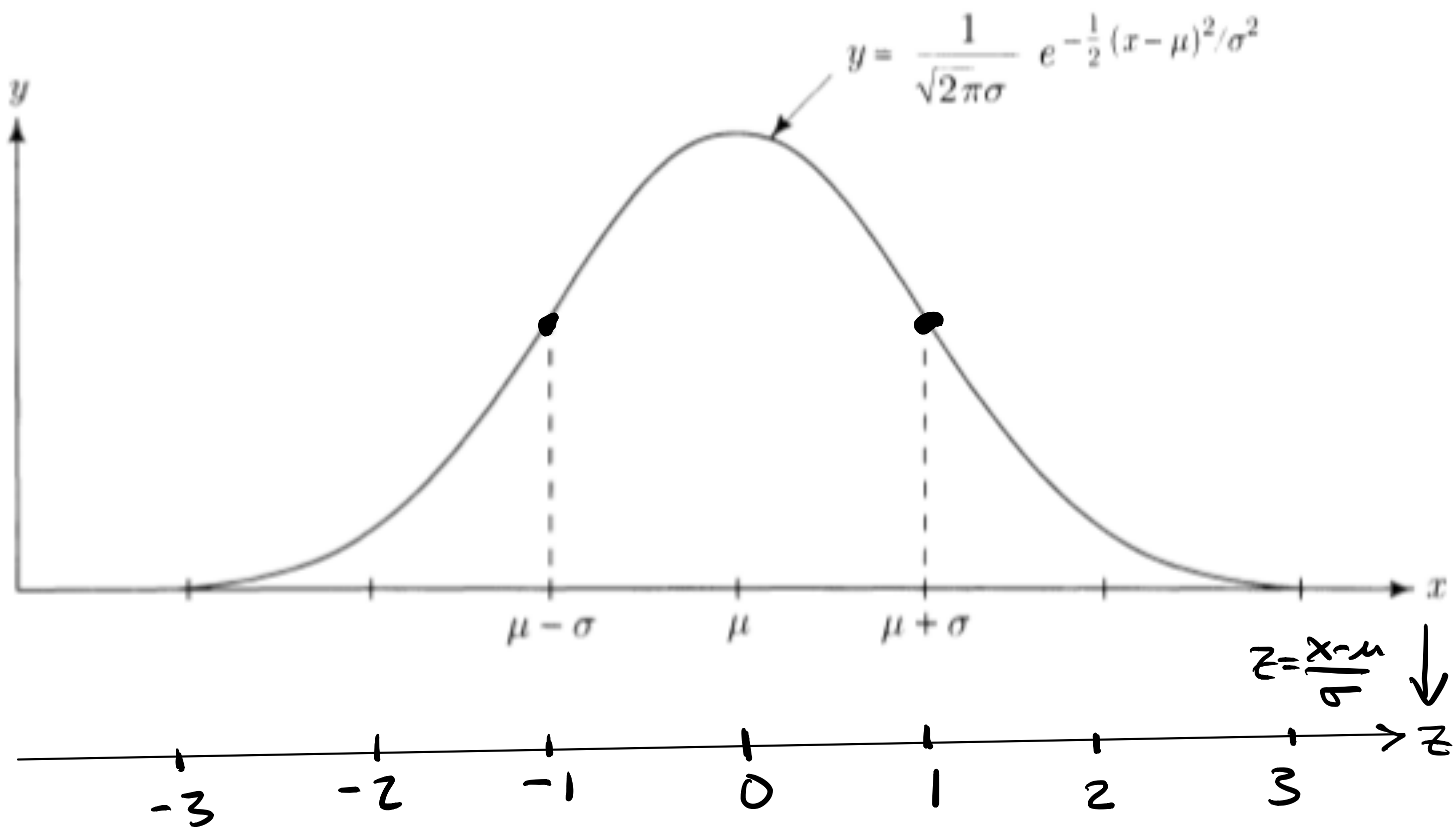
$$y = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Notice :

1. two param μ = mean
   σ = std dev
2. inflection pts $\mu \pm \sigma$
3. almost all data between $\mu \pm 3\sigma$

P93

FIGURE 1. The normal curve.



To find area under curve convenient to make a change of coords.

$$z = \frac{x-\mu}{\sigma}$$

This makes $\mu = 0$ and $\sigma = 1$

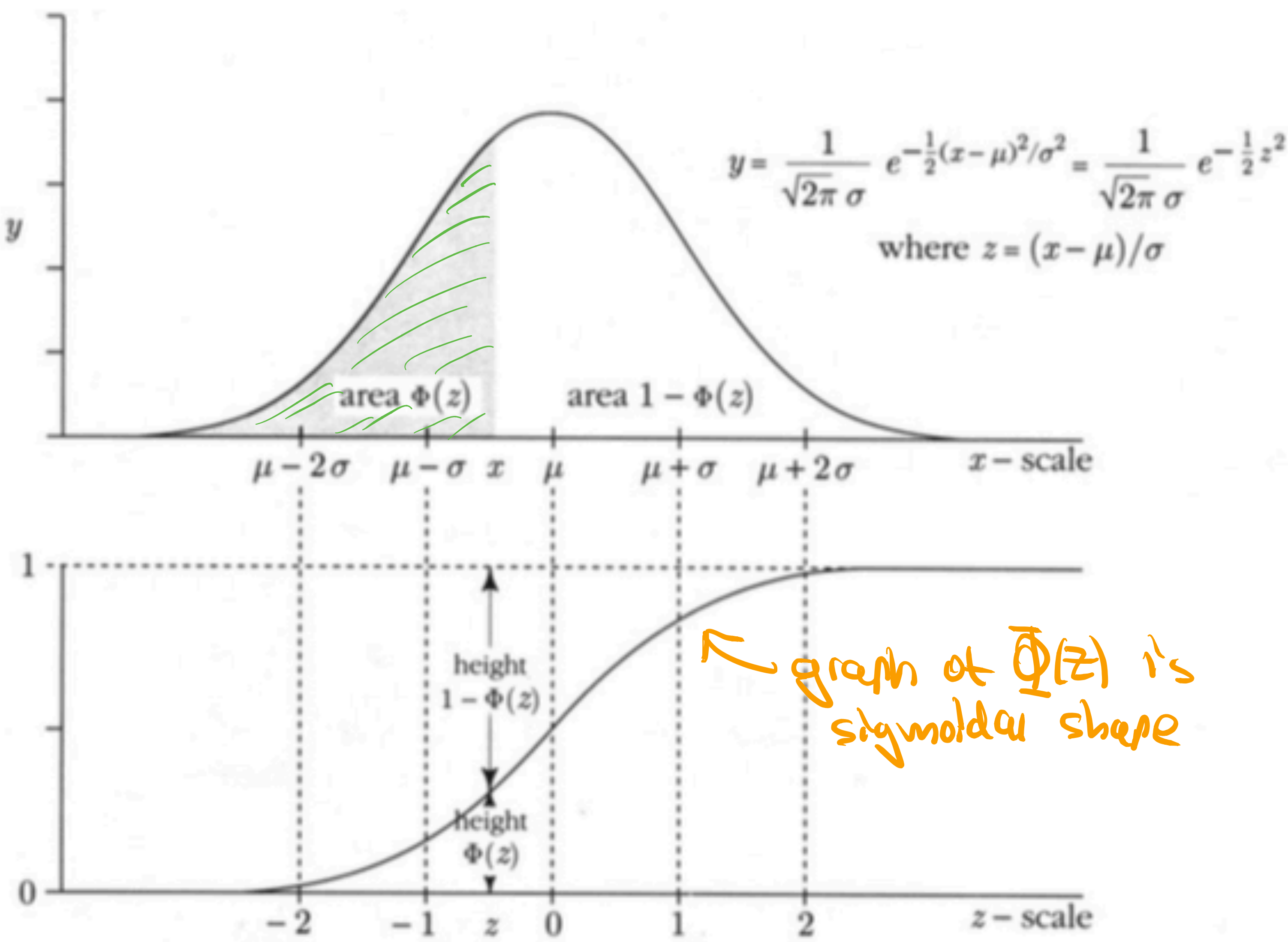
density function

$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ std normal curve

Define cummulative distribution function (cdf)

as $\Phi(z) = \int_{-\infty}^{z} \phi(z)\,dz$ ← area between $-\infty$ and z

P95

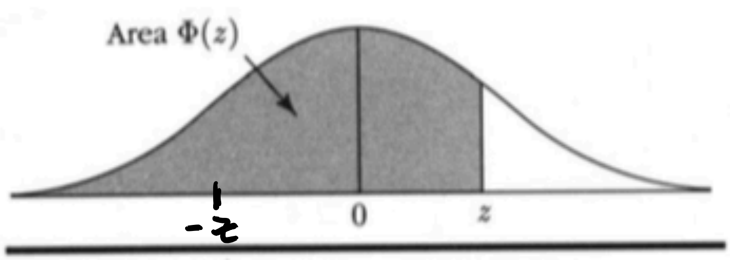


← graph of $\Phi(z)$ is sigmoidal shape

we can't solve integral $\Phi(z)$ but instead use look up table.

Notice table only given values for $z \geq 0$

$\Phi(-z) = 1 - \Phi(z)$



Appendix 5

Normal Table

Table shows values of $\Phi(z)$ for z from 0 to 3.59 by steps of .01. Example: to find $\Phi(1.23)$, look in row 1.2 and column .03 to find $\Phi(1.2 + .03) = \Phi(1.23) = .8907$. Use $\Phi(z) = 1 - \Phi(-z)$ for negative z.

	.0	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998

lets find area between

1 and -1

$$\Phi(1) - \Phi(-1) = \Phi(1) - (1 - \Phi(1))$$

$$= 2\Phi(1) - 1$$

$$= 2(.8413) - 1 = \boxed{.68}$$

2 and -2

$$\Phi(2) - \Phi(-2) = \boxed{.95}$$

3 and -3

$$\Phi(3) - \Phi(-3) = \boxed{.997}$$

empirical rule

Normal approx to binomial dist.

2 questions,

1. How do we write μ and σ in terms of n, p to match normal curve w/ binomial dist?
2. For what n, p is it ok to approx the binomial(n,p) by a normal distribution.

Answ.

$\mu = np$

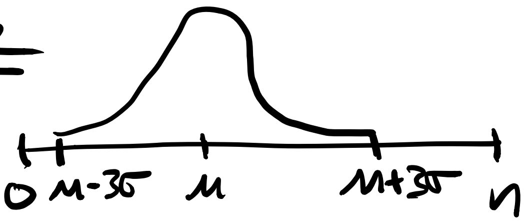
$\sigma = \sqrt{npq}$

lets require $n \geq 20$ since for fixed p binomial more normal as $n \uparrow$ by central limit theorem (chap 4).

Outcomes between 0 and n

All data is between $\mu \pm 3\sigma$ so we require $0 < \mu - 3\sigma$ and $\mu + 3\sigma < n$

<u>Picture</u>



ex $n = 20$
$p = 1/10$

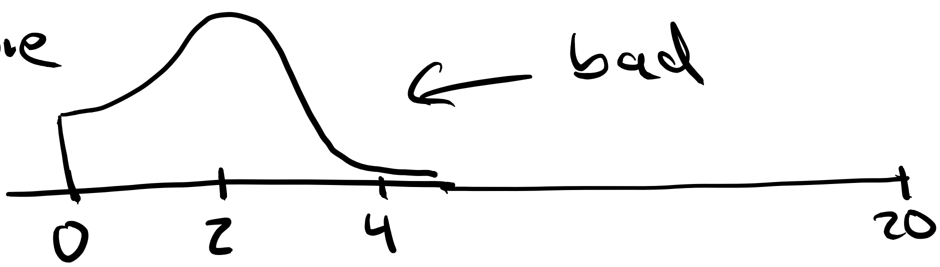
$\mu = 2$

$\sigma = \sqrt{20 \cdot \frac{1}{10} \cdot \frac{9}{10}} = 1.34 \Rightarrow 3\sigma = 4$

$2 - 3\sigma = 2 - 4 = -2 < 0$ ✗

$2 + 3\sigma = 2 + 4 = 6 < 30$ ✓

Picture



← bad

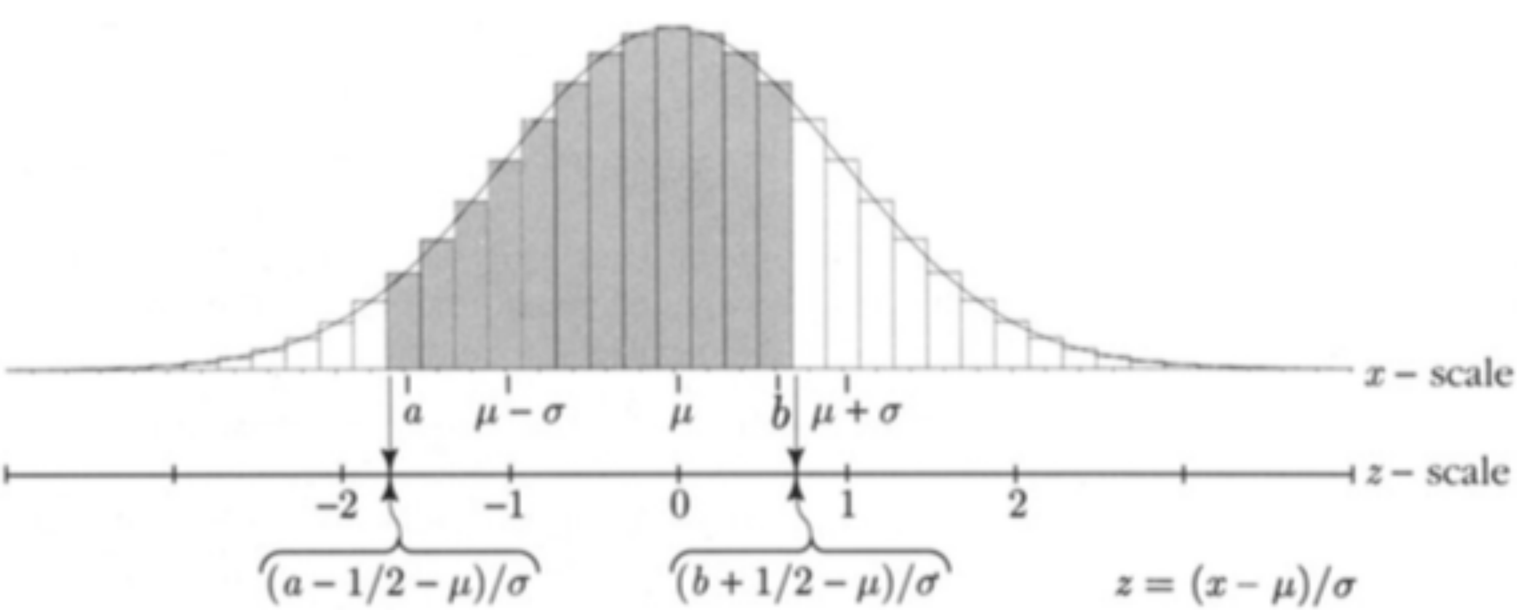
Normal approx to binomial distr

$$\boxed{P(a \text{ to } b \text{ successes}) \approx \Phi\left(\frac{b+\frac{1}{2}-\mu}{\sigma}\right) - \Phi\left(\frac{a-\frac{1}{2}-\mu}{\sigma}\right)}$$

Picture:

p98

FIGURE 4. A binomial histogram, with the normal curve superimposed. Both the x scale (number of successes) and the z scale (standard units) are shown.



why 1/2?

each block has width 1 and a is in the middle of the block so we subtract by 1/2 to get to the edge of the block.

This is called a <u>continuity correction</u>

as we are approximating the discrete distribution binomial(n,p) by a continuous normal distribution.

ex Find the approximate chance of getting 75 sixes in 600 rolls of a die.

soln let $a = b = 75$ then

$$\Phi\left(\frac{75.5-100}{9.1}\right) - \Phi\left(\frac{74.5-100}{9.1}\right) = .00101$$

The exact answer is $\binom{600}{75}\left(\frac{1}{6}\right)^{75}\left(\frac{5}{6}\right)^{525} = .00087$