Stat 134 lec 31

Last time

sec 5.3 independent normal variables

a) $T \sim Exp(\frac{1}{2}) \Rightarrow R = \sqrt{T}$ has density $f(r) = re^{-\frac{1}{2}r^2}$, $r > 0$

called Rayleigh RV

b) Using Rayleigh distribution we showed if $Z_1, Z_2 \overset{iid}{\sim} N(0,1)$

① $Z_1^2 + Z_2^2 = T \sim Exp(\frac{1}{2})$.

② $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$, $E(Z) = 0$, $SD(Z) = 1$

c) If $Z \sim N(0,1)$, $X = \mu + \sigma Z \sim N(\mu, \sigma^2)$

with density $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

d) $Z_1 + Z_2 \sim N(0, 2)$

Today

sec 5.3

⓪ Review MGF

① A linear combination of normal is normal

② Chi square distribution

sec 5.4

③ Convolution formula for the density of $X+Y$

MGF Review (lecture 25)

Recall the MGF of a RV X is $M(t) = E(e^{tX})$ for some interval of t containing 0.

Common MGF

$X \sim N(0,1)$, $M(t) = e^{\frac{t^2}{2}}$ for all t

$T \sim \text{Gamma}(r, \lambda)$, $M(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$ for $t < \lambda$

Properties of MGF

if X, Y are independent $M_{X+Y}(t) = M_X(t) M_Y(t)$

if $M_X(t) = M_Y(t)$ then $X = Y$

$M_{b+aX}(t) = e^{bt} M(at)$

ex Use these properties to find the MGF of $X \sim N(\mu, \sigma^2)$

Soln

For $Z \sim N(0,1)$

$X = \mu + \sigma Z$ and $M_Z(t) = e^{\frac{t^2}{2}}$

so $M_X(t) = e^{\mu t} \underbrace{e^{\frac{\sigma^2 t^2}{2}}}_{M_Z(\sigma t) = e^{\frac{(\sigma t)^2}{2}}}$

ex Use MGF to prove that the sum of two independent std normals is $N(0,2)$.

soln $Z_1, Z_2 \overset{iid}{\sim} N(0,1)$

$$M_{Z_1+Z_2}(t) = M_{Z_1}(t) M_{Z_2}(t)$$

$$= e^{\frac{t^2}{2}} \cdot e^{\frac{t^2}{2}} = e^{\frac{2t^2}{2}}$$

σ^2

by uniqueness of MGF

$Z_1 + Z_2 \sim N(0, 2)$

Thm Let $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ $\Big\}$ inden.

then $aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

pf at end of lecture.

Stat 134

Friday November 2 2018

1. Let $X \sim N(0,1)$ and $Y \sim N(0,1)$ be independent. $P(X > 2Y)$ equals

   a $1 - \Phi(0)$ $= 1/2$

   b $1 - \Phi(\frac{1}{\sqrt{3}})$

   c $1 - \Phi(\sqrt{3})$

   d none of the above

soln

$Y \sim N(0,1) \Rightarrow 2Y \sim N(0,2)$

and $X - 2Y \sim N(0,3)$

$P(X > 2Y) = P(X - 2Y > 0)$

normalize $= P\left(\frac{(X-2Y)-0}{\sqrt{3}} > \frac{0-0}{\sqrt{3}}\right)$

$= P(Z > 0) = 1 - \Phi(0) = 1/2$

2. Let $X \sim N(68, 3^2)$ and $Y \sim N(66, 2^2)$ be independent. $P(X > Y)$ equals

a $1 - \Phi(\frac{0-2}{\sqrt{3^2+2^2}})$

b $1 - \Phi(\frac{0-2}{3^2+2^2})$

c $1 - \Phi(\frac{68-66}{\sqrt{3^2+2^2}})$

d none of the above

soln

$$X-Y \sim N(68-66,\ 3^2+2^2) = N(2, 13).$$

$$P(X>Y) = P(X-Y>0)$$

$$= P\left(\underbrace{\frac{(X-Y)-2}{\sqrt{13}}}_{=Z} > \frac{0-2}{\sqrt{13}}\right)$$

$$= 1 - P\left(\underbrace{\frac{(X-Y)-2}{\sqrt{13}}}_{=Z} < \frac{-2}{\sqrt{13}}\right)$$

$$= 1 - \Phi\left(\frac{-2}{\sqrt{13}}\right)$$

Sec 5.3 Chi-square distribution

see end of lecture notes for proof.

Fact if $Z \sim N(0,1)$ then

$$Z^2 \sim \text{Gamma}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Recall that if $T \sim \text{Gamma}(r, \lambda)$, $M_T(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$, $t < \lambda$

Hence $M_{Z^2}(t) = \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{1/2}$, $t < 1/2$

Let $Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$

$$M_{Z_1^2 + \cdots + Z_n^2}(t) = \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{1/2 \cdot n}, \quad t < \tfrac{1}{2}$$

By uniqueness of MGF

$$Z_1^2 + \cdots + Z_n^2 \sim \text{Gamma}\left(\tfrac{n}{2}, \tfrac{1}{2}\right)$$

called chi square distribution with n degrees of freedom.

Sec 5.4 The Density Convolution Formula

ex Let $X \sim \text{Unif}\{0,1,2,3,4,5,6\}$

$Y \sim \text{Unif}\{0,1,2,3,4,5,6\}$

Let $S = X+Y$.

Find the probability mass function of S.

$$P(S=3) = P(X=0, Y=3-0) + P(X=1, Y=3-1) + P(X=2, Y=3-2) + P(X=3, Y=3-3)$$

$$P(S=s) = \sum_{x=0}^{s} P(X=x, Y=s-x)$$

convolution formula.

Let $X>0$, $Y>0$ be continuous RVs.

Let $S = X+Y$

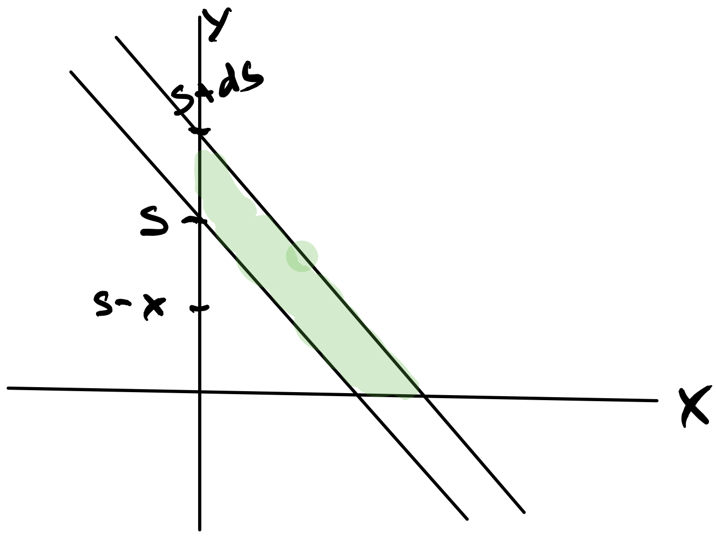
Find the density of S

$s = x+y$

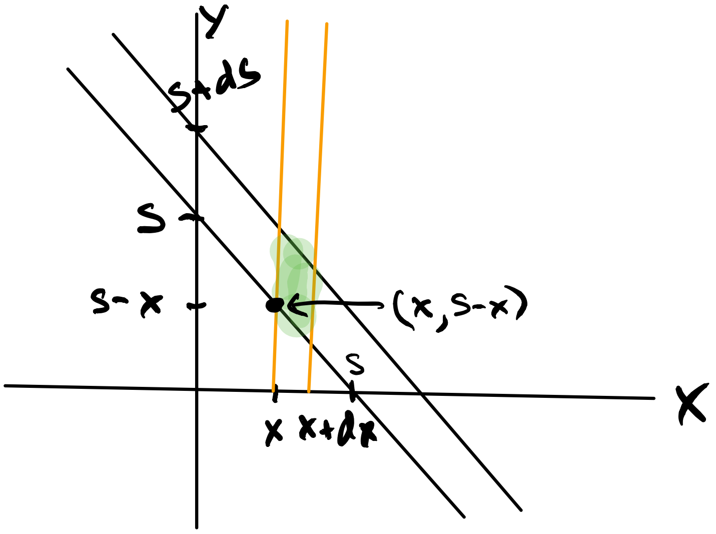
$y = s-x$

y intercept.

$P(S \in ds)$ is the volume under $f(x,y)$ over the green region.



$P(S \in ds, X \in dx)$ is the volume under $f(x,y)$ over the green region.



$f(x,s-x)\,dx\,ds$

$$P(S \in ds) = \int_{x=0}^{x=s} P(S \in ds, X \in dx)$$

$$\| \quad f(s)\,ds$$

$$= \int_{x=0}^{x=s} f(x, s-x)\,dx\,ds$$

$x=s$

$$\Rightarrow f(s) = \int_{x=0}^{s} f(x, s-x)\,dx$$

convolution formula for densities.

$\underline{\text{ex}}$ $X, Y \overset{\text{iid}}{\sim} \text{expon}(\lambda)$ $S = X+Y$

$$f_S(s) = \int_0^s f(x, s-x)\,dx$$

$$= \int_0^s f_X(x)\, f_Y(s-x)\,dx$$

independence

$$= \int_0^s \lambda e^{-\lambda x} \lambda e^{-\lambda(s-x)}\,dx$$

$$= \int_0^s \lambda^2 e^{-\lambda s}\,dx = \lambda^2 e^{-\lambda s} x \Big|_0^s$$

$$= \boxed{\lambda^2 e^{-\lambda s} \cdot s}$$

variable part of gamma(2, λ)

$\Rightarrow S \sim \text{gamma}(2, \lambda)$.

Appendix

Thm Let $\left.\begin{array}{l} X_1 \sim N(\mu_1, \sigma_1^2) \\ X_2 \sim N(\mu_2, \sigma_2^2) \end{array}\right\}$ indep.

then $aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

Pf/ $M_{X_1}(t) = e^{\mu_1 t} e^{\sigma_1^2 \frac{t^2}{2}}$

$M_{aX_1}(t) = M_{X_1}(at) = e^{\mu_1 at} e^{\sigma_1^2 a^2 \frac{t^2}{2}}$

Then $M_{aX_1+bX_2}(t) = M_{aX_1}(t) M_{bX_2}(t)$

$= e^{\mu_1 at} e^{\sigma_1^2 a^2 \frac{t^2}{2}} \cdot e^{\mu_2 bt} e^{\sigma_2^2 b^2 \frac{t^2}{2}}$

$= e^{(\mu_1 a + \mu_2 b)t} e^{(\sigma_1^2 a^2 + \sigma^2 b^2)\frac{t^2}{2}}$

by uniqueness of MGF

$aX_1 + bX_2 \sim N(\mu_1 a + \mu_2 b, \sigma_1^2 a^2 + \sigma^2 b^2)$

$\square$

If $Z \sim N(0,1)$, then $Z^2 \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

Proof/

$\lambda > 0$, pos integer r,

gamma (r, λ) density

$$f(t) = \frac{\lambda^r}{\Gamma(r)} t^{r-1} e^{-\lambda t}, \quad t > 0$$

let Z = std normal

$X = Z^2$

change of variable rule.

Find $f_X(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-\frac{1}{2}x}, \quad x > 0$

$$= \frac{\left(\frac{1}{2}\right)^{1/2}}{\sqrt{\pi}} x^{\frac{1}{2}-1} e^{-\frac{1}{2}x}$$

$\Gamma(\frac{1}{2})$

$\Rightarrow X \sim \text{gamma}(\frac{1}{2}, \frac{1}{2})$

$\square$