Stat 134 lec 32

Last time

sec 5.4 Convolution formula for density of sum

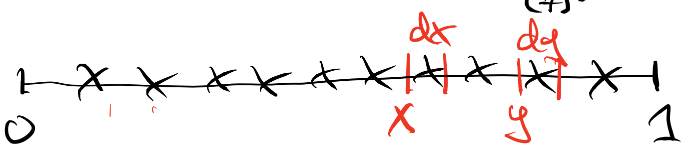
Let X, Y be RVs and $S = X + Y$.

if $X > 0, Y > 0$ — if X,Y independent

$$f_S(s) = \int_{x=0}^{x=s} f_{(X,Y)}(x, s-x)\,dx = \int_{x=0}^{x=s} f_X(x) f_Y(s-x)\,dx$$

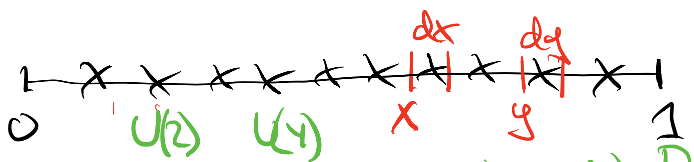
ex (Uniform spacing) (see #13 p 355)

Throw 10 darts. $X = U_{(7)}$, $Y = U_{(9)}$



we showed $Y - X \sim \text{Beta}(2, 9)$ $(k, n-k+1)$

Notice $U_{(9)} - U_{(7)} \overset{D}{=} U_{(2)} \sim \text{Beta}(2, 10-2+1)$



$U(4) - U(2) \overset{D}{=} U_{(2)}$

Today

① Go over student comments from concept test last time.

sec 5.4

② triangular density X, Y iid $U(0,1)$, $S = X + Y$.

③ Convolution formula of ratio Y/X

1

1. Let $X \sim N(0,1)$ and $Y \sim N(0,1)$ be independent. $P(X > 2Y)$ equals

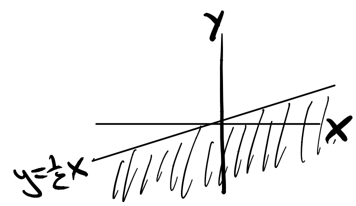
   a $1 - \Phi(0)$

   b $1 - \Phi(\frac{1}{\sqrt{3}})$

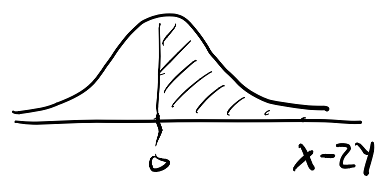
   c $1 - \Phi(\sqrt{3})$

   d none of the above

Discuss with neighbor how you did this for 1 minute.



a	Wait, the image is symmetry on the plane!



a	the mean of X-2Y will still be 0

a	we know the dist of x-2y so it's easier to solve that way

a	Adding together the normals, P(X-2Y > 0) X-2Y ~ N(0, 5) 1-Phi(0/5) = 1-Phi(0)

$$P(X-2Y>0) = P\left(\frac{X-2Y}{\sqrt{5}} > \frac{0}{\sqrt{5}}\right) = P(Z>0)$$
$$= \boxed{1-\Phi(0)}$$

② Sec 5.4 Triangular density ← convolution of sum of 2 iid $U(0,1)$

Let $X, Y \overset{iid}{\sim} U(0,1)$

$S = X + Y$ — takes values between 0 and 2

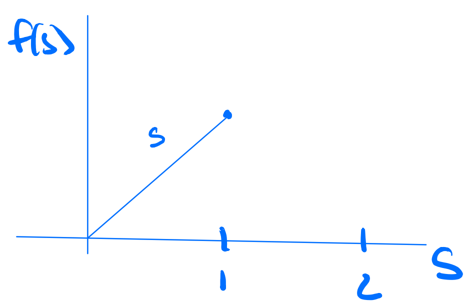
recall, if $X>0, Y>0$

if X, Y independent

$$f_S(s) = \int_{x=0}^{x=s} f_{(X,Y)}(x, s-x)\,dx = \int_{x=0}^{x=s} f_X(x) f_Y(s-x)\,dx$$

a) for $0<s<1$ find $f_S(s)$.

$$f_S(s) = \int_{x=0}^{x=s} 1 \cdot 1\,dx = \boxed{s}$$



if $1<S<2$,

let $T = 1-X + 1-Y = 2-S$

$1-X \overset{D}{=} X$, $1-Y \overset{D}{=} Y$

$\Rightarrow$ T is sum of 2 iid $U(0,1)$ and $0<T<1$
since $1<S<2$.

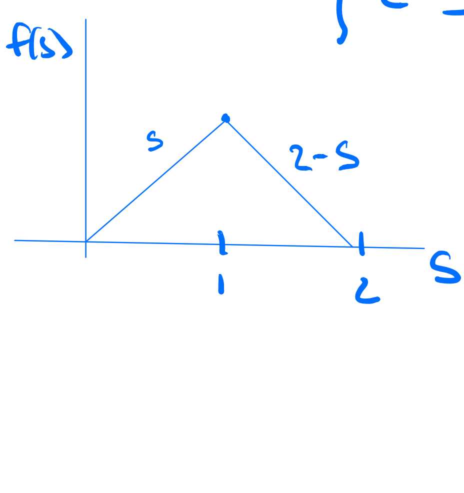
$\Rightarrow f_T(t) = t$ by part 1.

b) For $1<s<2$ find $f_S(s)$

$S = 2-T$

$$f_S(s) = \frac{1}{|-1|}\cdot f_T(t)\Big|_{t=2-s} = t\Big|_{t=2-s} = \boxed{2-s}$$

$$\Rightarrow f(s) = \begin{cases} s & 0<s<1 \\ 2-s & 1<s<1 \end{cases}$$



③

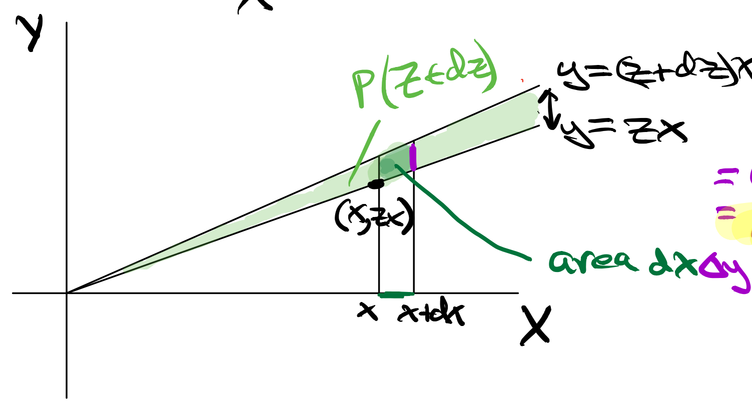
Convolution formula for density of ratio Y/X

$X > 0,\ Y > 0$

let $Z = \dfrac{Y}{X}$.

Find $f_Z(z)$.

Picture $\quad Z = \dfrac{Y}{X} \Rightarrow Y = ZX$ ← slope.



$$\Delta y = (z+dz)x - zx = x\,dz$$

$$P(Z\in dz) = \int_{x=0}^{x=\infty} P(Z\in dz, X\in dx)$$

$$\| $$

$$f(z)\,dz = \int_{x=0}^{x=\infty} f_{(X,Y)}(x, zx)\, x\,dz\, dx$$

$$\Rightarrow f(z) = \int_{x=0}^{x=\infty} f(x,zx)\,x\,dx = \int_{x=0}^{x=\infty} f_X(x) f_Y(zx)\,x\,dx$$

– if X, Y indep.

Convolution formula.

__ex__

Recall gamma density

$X \sim \text{Gamma}(r, \lambda)$

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} \underbrace{x^{r-1} e^{-\lambda x}}_{\text{variable part}}$$

so
$$\boxed{\int_0^\infty x^{r-1} e^{-\lambda x} = \frac{\Gamma(r)}{\lambda^r}}$$

Let $X, Y \overset{iid}{\sim} \text{Exp}(1)$. $Z = \frac{Y}{X}$

Find $f_Z(z)$.

__Soln__

$$f_Z(z) = \int_{x=0}^\infty \overbrace{f_X(x)}^{e^{-x}} \overbrace{f_Y(zx)}^{e^{-zx}} \, x \, dx$$

$$= \int_{x=0}^\infty e^{-x} e^{-zx} x \, dx$$

$$= \int_{x=0}^\infty \underbrace{x \, e^{-(1+z)x}}_{\text{gamma}(r=2, \lambda = 1+z)} dx$$

$$= \frac{\Gamma(2)}{(1+z)^2} = \boxed{\frac{1}{(1+z)^2} \text{ for } 0 < z < \infty.}$$

tinyurl http://tinyurl.com/april15-pt1

http://tinyurl.com/april15-pt2

Stat 134

Monday April 15 2019

1. Let $X \sim U(0,1)$ and $Y \sim U(0,1)$ be independent. The density of $Z = Y/X$ for $0 < z < 1$ is:

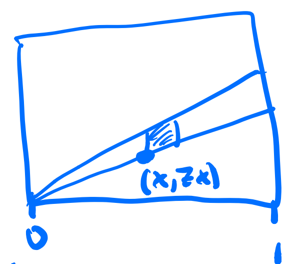
   a $1/(2z)$

   b $1/2$

   c $1/(2z^2)$

   d none of the above

$0<z<1$



$z = \frac{Y}{X}$

$f_Z(z) = \int_{x=0}^{x=1} f_X(x) f_Y(zx)\, x\, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

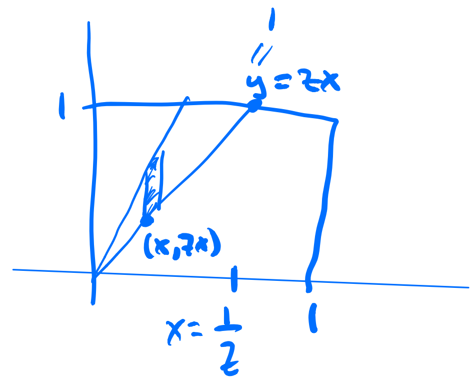
2. Let $X \sim U(0,1)$ and $Y \sim U(0,1)$ be independent. The density of $Z = Y/X$ for $z > 1$ is:

a $1/(2z)$

b $1/2$

c $1/(2z^2)$

d none of the above

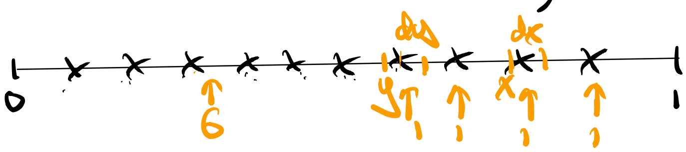


$$f_Z(z) = \int_{x=0}^{x=\frac{1}{z}} f_X(x) f_Y(zx)\, x\,dx$$

$$= \int_0^{\frac{1}{z}} 1 \cdot 1\, x\,dx = \frac{x^2}{2}\Big|_0^{\frac{1}{z}} = \boxed{\frac{1}{2z^2}}$$

ex Let $Y \sim U_{(7)}$, $X \sim U_{(9)}$ for 10 iid $U(0,1)$.

The joint density $f_{X,Y}(x,y) = \binom{10}{6,1,1,1,1} y^6 (x-y)(1-x)$ for $0<y<x<1$.



Let $Z = \frac{Y}{X}$ — values between 0 and 1.

Prove via the convolution formula that the density of $Z = \frac{Y}{X}$ is a Beta distribution and find the parameters.

$$f_Z(z) = \int_{x=0}^{x=1} f_{X,Y}(x, zx)\, x\, dx$$

$$c = \binom{10}{6\,1\,1\,1\,1}$$

$$f(z) = c\int_{x=0}^{x=1} (zx)^6 (x-zx)(1-x)\,dx$$

$$= c(1-z)z^6 \int_0^1 x^7(1-x)\,dx$$

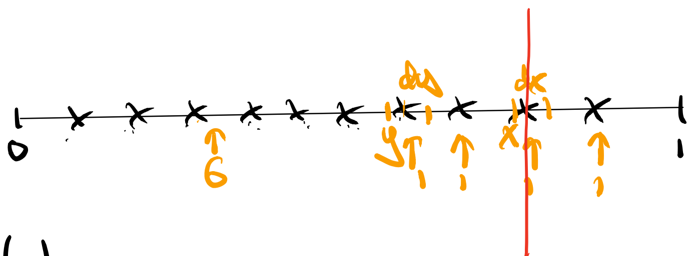
$$= \underbrace{c\int_0^1 x^7(1-x)\,dx}_{\text{constant}} \cdot \boxed{z^6(1-z)}$$

$$\Rightarrow \boxed{Z \sim Beta(7,2)}$$

$Z \overset{D}{=} U_{(7)}$ when throw 8 darts.

Chk $U_{(7)} \sim Beta(7, 8-7+1) = Beta(7,2)$

interpretation:



Y u

$$X = \frac{U_{(7)} \text{ of } 10}{U_{(9)} \text{ of } 10}$$

Make the location of the 9th dart 1 (see red line). There are 8 darts before it.

The location of the 7th of the 8 darts is $U_{(7)}$ of 8.

we have $\dfrac{U_{(7)} \text{ of } 10}{U_{(9)} \text{ of } 10} \overset{D}{=} U_{(7)} \text{ of } 8$

You can check that they are both Beta(7, 2) since we know

$$U_{(k)} \text{ of } n \sim \text{Beta}(k, n-k+1) \quad \text{so}$$

$$U_{(7)} \text{ of } 8 \sim \text{Beta}(7, \underbrace{8-7+1}_{2}) \checkmark$$

ex You throw 10 darts
what distribution is

$$\frac{U_{(4)}^{\text{ of } 10}}{U_{(8)}^{\text{ of } 10}} \text{ ?}$$

$$\boxed{\frac{U_{(4)}^{\text{ of } 10}}{U_{(8)}^{\text{ of } 10}} \overset{D}{=} U_{(4)} \text{ of } 7}$$

More generally

If $U_1, \ldots, U_n \overset{iid}{\sim} U(0,1)$

Then for $k < m$

$$\boxed{\frac{U_{(k)}^{\text{ of } n}}{U_{(m)}^{\text{ of } n}} \overset{D}{=} U_{(k)} \text{ of } m-1}$$