Sec 6.3 Stat 134 lec 35

Conditioning: density case

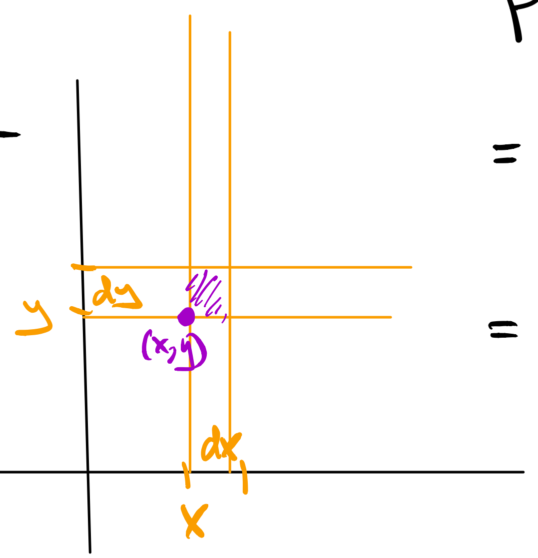
If X and Y are discrete, and x fixed

$$P(Y=y \mid X=x) = \frac{P(X=x, Y=y)}{P(X=x)}$$

Now let X and Y have joint density f

$$P(Y \in dy \mid X=x) = \frac{P(X \in dx, Y \in dy)}{P(X \in dx)}$$

Picture



$$= \frac{f(x,y)\,dx\,dy}{f_X(x)\,dx}$$

$$= \frac{f(x,y)}{f_X(x)}\,dy$$

$\Rightarrow$ define

$$\boxed{f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)}}$$

constant that makes integrate to 1.

ex $f(x,y) = K(y-x)^8 \ , \quad 0<x<y<1$

a) find K

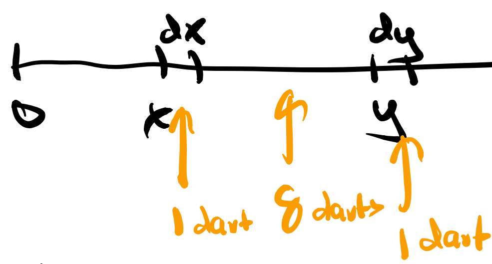
b) find the marginal distribution of X

c) find $P(Y > .7 \mid X = .2)$

Recognize $f(x,y)$ as joint density of two order statistics.

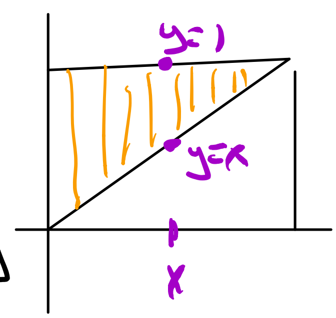
$X = U_{(1)}$

$Y = U_{(10)}$



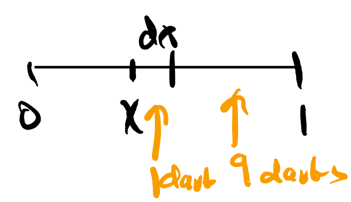
$$f(x,y)\,dx\,dy = 10\,dx \binom{9}{8}(y-x)^8\,dy$$

$$\Rightarrow \boxed{K = 90}$$



b) Density of X;

2 ways: $f_X(x) = \displaystyle\int_{y=x}^{y=1} f(x,y)\,dy$



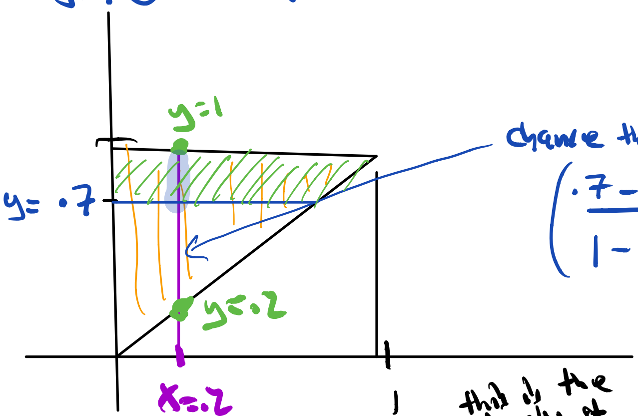
or $P(X \in dx) = 10\,dx \binom{9}{9}(1-x)^9$

$= 10(1-x)^9\,dx$

$$\Rightarrow \boxed{f_X(x) = 10(1-x)^9}$$

Find $P(Y > .7 \mid X = .2)$

two ways: ① From picture



chance the mass is in here is

$$\left(\frac{.7-.2}{1-.2}\right)^9 = \left(\frac{.5}{.8}\right)^9$$

so $P(Y>.7 \mid X=.2)$

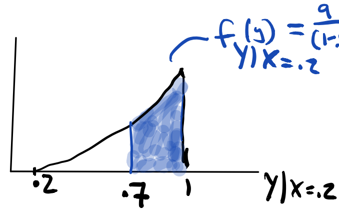
$$= \boxed{1-\left(\frac{.5}{.8}\right)^9}$$

② From conditional density

this is the density of the slice of $f(x,y)$ through $x=x$.

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{90(y-x)^8}{10(1-x)^9}, \quad x<y<1$$

$$f_{Y|X=.2}(y) = \frac{9}{(1-.2)^9}(y-.2)^8$$



$$P(Y>.7 \mid X=.2) = \int_{y=.7}^{y=1} f_{Y|X=.2}(y)\,dy$$

$$= \frac{9}{(1-.2)^9}\int_{0.7}^{1}(y-.2)^8\,dy$$

$$= \boxed{1-\left(\frac{.5}{.8}\right)^9}$$

Multiplication rule

discrete case $P(X=x, Y=y) = P(Y=y \mid X=x) P(X=x)$

Cont case $P(X \in dx, Y \in dy) = P(Y \in dy \mid X=x) P(X \in dx)$

$$\underset{f(x,y)\,dx\,dy}{\overset{\parallel}{}} \qquad \underset{f_{Y|X=x}(y)\,dy}{\overset{\parallel}{}} \qquad \underset{f_X(x)\,dx}{\overset{\parallel}{}}$$

$$\Rightarrow \boxed{f(x,y) = f_{Y|X=x}(y) f_X(x)}$$

Averaging conditional Probabilities

discrete : $P(A) = \sum_{x \in X} P(A \mid X=x) P(X=x)$

Cont : $P(A) = \int_{x \in X} P(A \mid X=x) P(X \in dx)$

$$\underset{f_X(x)\,dx}{\overset{\parallel}{}}$$

$$\Rightarrow \boxed{P(A) = \int_{x \in X} P(A \mid X=x) f_X(x)\,dx}$$ Integral Conditioning Formula.

ex $X \sim Unif(0,1)$

given $X=p$ let $I_1, I_2, \ldots, I_n \overset{iid}{\sim} Bern(p)$ trials.

$P(1^{st}$ toss is heads$)$?

$P(1^{st}$ toss is heads $\mid X=p) = p$

$P(1^{st}$ toss is heads$) = \int_0^1 P(1^{st} H \mid X=p) \underset{1}{\overset{\parallel}{f_X(p)}}\,dp$

$$= \int_0^1 p\,dp = \frac{p^2}{2}\Big|_0^1 = \frac{1}{2}$$

Stat 134

Monday April 16 2018

1. Let $X \sim \text{Unif}(0,1)$. Given X=p, let $I_1, I_2, \ldots, I_n$ be iid Bernouilli(p) trials. What is the probability that the first two tosses are both heads?

   a 1/4

   b 1/3

   c 1/2

   d none of the above

$$P(1^{st} H, 2^{nd} H) = \int_0^1 \underbrace{P(1^{st} H, 2^{nd} H \mid X=p)}_{= P(1^{st} H \mid X=p)\, P(2^{nd} H \mid X=p)} \, dp$$

$$= \int_0^1 p^2 \, dp = \frac{p^3}{3}\Big|_0^1 = 1/3$$

Notice $P(1^{st} H, 2^{nd} H) \neq \frac{1}{2} \cdot \frac{1}{2}$.

Lets confirm this using the multiplication rule:

$$P(1^{st}H, 2^{nd}H) = P(2^{nd}H \mid 1^{st}H)\,P(1^{st}H)$$

$$P(2^{nd}H \mid 1^{st}H) = \int_{p=0}^{p=1} \underbrace{P(2^{nd}H \mid 1^{st}H, X=p)}_{= p}\, f_{X \mid 1^{st}H}(p)$$

$f_{X \mid 1^{st}H}(p)$ posterior density.

$f_X(p)$ prior density.

$P(1^{st}\text{ toss } H \mid X=p)$ likelihood

Have posterior $\propto$ likelihood · prior

proportional to

$$f_{X \mid 1^{st}H}(p) \cdot P(1^{st}H) = f(X=p, 1^{st}H) = P(1^{st}H \mid X=p) \cdot f_X(p)$$

so,

$$\underbrace{f_{X \mid 1^{st}H}(p)}_{\text{posterior}} = \frac{\underbrace{P(1^{st}H \mid X)}_{\text{likelihood} \,=\, p} \cdot \underbrace{f_X(p)}_{\text{prior} \,=\, 1}}{\underbrace{P(1^{st}H)}_{\text{constant.} \,=\, 1/2}}$$

$$= 2p$$

$$P(2^{nd}H \mid 1^{st}H) = \int_0^1 p \cdot 2p\, dp = \int_0^1 2p^2\, dp$$

$$= \frac{2p^3}{3}\Big|_0^1 = \frac{2}{3}$$

$$\Rightarrow P(1^{st}H \mid 2^{nd}H) = P(1^{st}H) \cdot P(2^{nd}H \mid 1^{st}H) = \tfrac{1}{2} \cdot \tfrac{2}{3} = \tfrac{1}{3}$$

as claimed.