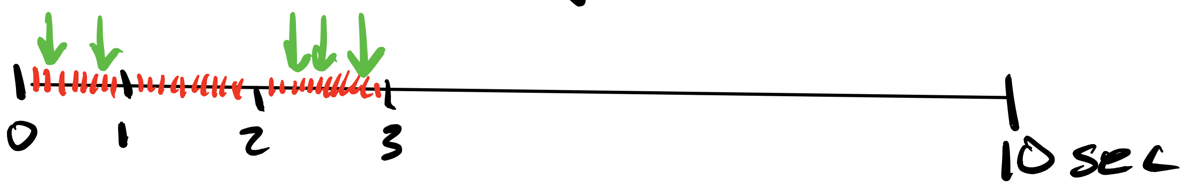
Stat 134 lec 17

Last time Sec 3.5 Poisson random scatter (PRS)

ex radioactive decay of Americium 241 in 10 seconds



Assumptions

① no two particles arrive at the same time.
(this allows us to divide 10 sec into n small time intervals each with at most one arrival.

② X is a sum of n ← large iid Bernoulli(p) trials.
↖ small

$\mu = np$ is avg # of arrivals in 10 sec.

$\lambda = \mu/10$ is the arrival rate per second.

X = # arrivals in 10 seconds.

Suppose $\lambda = 4$ arrivals/sec

then $\mu = \lambda \cdot 10 = 40 \Rightarrow X \sim Pois(40)$

Americium has a long half life.

Y = # arrivals in 12070 sec.

$Y \sim Pois(\lambda \cdot 12070)$
↖ 4

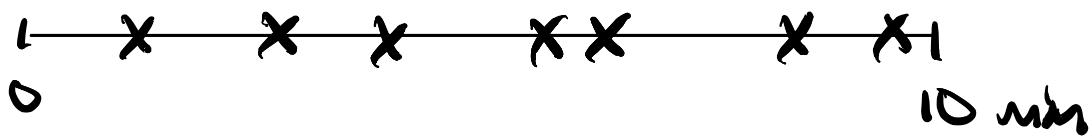
Today Sec 3.5 Poisson thinning

review

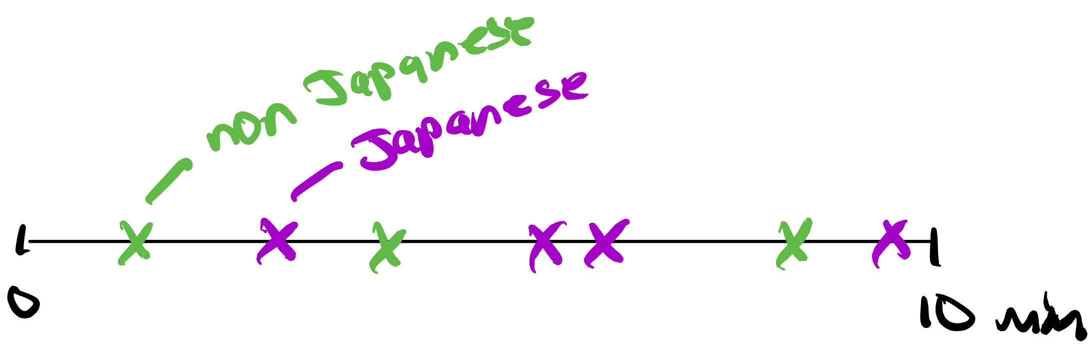
Sec 3.5 Poisson thinning

Cars arrive at a toll booth according to a Poisson process at a rate $\lambda = 3$ arrivals/min

X = # cars arriving at a toll booth in 10 min. $X \sim \mathrm{Pois}(\underbrace{\lambda \cdot 10}_{30})$



Of the cars arriving, it is known over the long term that 60% are Japanese imports.



Call Japanese cars a success and non Japanese a failure.

Each hit is a success w/ prob .6 independent of all other hits.

Then the process of "success" hits in your PRS is a PRS with intensity λp, and the process of "failure" hits

in your PRS is an independent PRS
w/ intensity λq.

Ex What is the prob that in a given 10 min interval 15 cars arrive at the booth and 10 are Japanese imports?

X = # cars in 10 min $\sim \text{Pois}(30)$

J = # Japanese cars in 10 min

$\sim \text{Pois}(\underbrace{\lambda p \cdot 10}_{18})$

nJ = # nonJapanese $\sim \text{Pois}(12)$

we assume J and nJ are indep.

$$P(X=15, J=10) = P(nJ=5, J=10)$$

$$= P(nJ=5)P(J=10)$$

$$= \frac{e^{-12}\, 12^5}{5!} \cdot \frac{e^{-18} \cdot 18^{10}}{10!}$$

ex Phone calls arrive into a telephone exchange according to a PRS at rate λ. This exchange serves 3 regions, and an incoming call gets routed to region i with probability p_i for $i=1,2,3$. ($p_1+p_2+p_3=1$).

Let N_t^i be the number of calls routed to region i in time t starting from time 0.

Find $P(N_t^1=j, N_t^2=k)$

Soln

$$\left.\begin{aligned} N_t^1 &\sim \text{Pois}(\lambda p_1 t) \\ N_t^2 &\sim \text{Pois}(\lambda p_2 t) \end{aligned}\right\} \text{indep.}$$

$$P(N_t^1=j, N_t^2=k) = P(N_t^1=j)P(N_t^2=k)$$

$$= \boxed{\frac{e^{-\lambda p_1 t}(\lambda p_1 t)^j}{j!} \frac{e^{-\lambda p_2 t}(\lambda p_2 t)^k}{!k}}$$

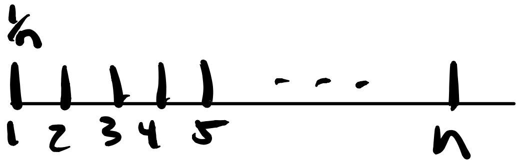
Discrete

name and range	$P(k) = P(X = k)$ for $k \in$ range	mean	variance
uniform on $\{a, a+1, \ldots, b\}$ $\{1, 2, \ldots, n\}$	$\frac{1}{b-a+1}$	$\frac{a+b}{2}$ $\frac{n+1}{2}$	$\frac{(b-a+1)^2-1}{12}$
Bernoulli (p) on $\{0, 1\}$	$P(1) = p$; $P(0) = 1-p$	p	$p(1-p)$
binomial (n, p) on $\{0, 1, \ldots, n\}$	$\binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
Poisson (μ) on $\{0, 1, 2, \ldots\}$	$\frac{e^{-\mu}\mu^k}{k!}$	μ	μ
hypergeometric (n, N, G) on $\{0, \ldots, n\}$	$\frac{\binom{G}{k}\binom{N-G}{n-k}}{\binom{N}{n}}$	$\frac{nG}{N}$	$n\left(\frac{G}{N}\right)\left(\frac{N-G}{N}\right)\left(\frac{N-n}{N-1}\right)$
geometric (p) on $\{1, 2, 3 \ldots\}$	$(1-p)^{k-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
geometric (p) on $\{0, 1, 2 \ldots\}$	$(1-p)^k p$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$
negative binomial (r, p) on $\{0, 1, 2, \ldots\}$	$\binom{k+r-1}{r-1} p^r (1-p)^k$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$
normal	$\Phi(x)$	μ	σ

Uniform

X = face of a n sided fair die

$X \sim \text{Unif}(1, 2, \dots, n)$



Note $1+2+\dots+n = \frac{n(n+1)}{2}$

$$E(X) = \sum_{i=1}^{n} i \cdot P(X=i) = \frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n}\left(\frac{n(n+1)}{2}\right) = \frac{n+1}{2}$$

Bernoulli

X = # heads when flip a p-coin once.

$$E(X) = 1 \cdot p + 0 \cdot (1-p) = \boxed{p}$$

Binomial

X = # heads when flip a p coin n times.

$$X = \underbrace{I_1 + \dots + I_n}_{\text{indep}}$$

$$E(X) = \boxed{n \cdot p}$$

Poisson

X = # calls in 10 min

$$X = \underbrace{I_1 + \dots + I_n}_{\text{indep}}$$

n large
p small.
$np \to \mu$.

$$E(X) = np = \boxed{\mu.}$$

hypergeometric

X = # aces drawn w/o replacement in a sample of size n.

$G=4$
$N=52$
$n=5$

$X = \underbrace{I_1 + \dots + I_n}_{\text{dep}}$

$E(X) = \boxed{n\left(\frac{G}{N}\right)}$

geometric on $\{1,2,\dots\}$

X = # trials until 1^{st} success

$P(X>k) = q^k$

$E(X) = \sum_{k=1}^{\infty} P(X \geq k) = \sum_{k=0}^{\infty} P(X>k) = \sum_{k=0}^{\infty} q^k$

$= \frac{1}{1-q} = \boxed{\frac{1}{p}}$

geometric on $\{0,1,2,\dots\}$

Y = # failures until 1^{st} success.

$Y = X-1$

$E(Y) = E(X) - 1 = \frac{1}{p} - 1 = \boxed{\frac{1-p}{p}}$

neg. binomial on $\{0,1,2,\dots\}$

Y = # trials until the r^{th} success.

$Y = Y_1 + \dots + Y_r$ — sum of r indep geom on $\{0,1,\dots\}$

$\Rightarrow E(Y) = rE(Y_1) = \boxed{r\left(\frac{1-p}{p}\right)}$

Normal

X = height of boys in the class.

$\text{Bin}(n,p) \xrightarrow{\text{CLT}} \text{Normal}(np, npq)$

$\text{Hypergeom}(N,G,n) \xrightarrow{\text{CLT}}$

$\text{Normal}\left(\frac{nG}{N}, \frac{nG}{N}\left(1-\frac{G}{N}\right)\left(\frac{N-n}{N-1}\right)\right)$

ex (min of independent geom)

Adam, Beth and John independently flip a p_1, p_2, p_3 coin respectively. let X = # trials until Adam, Beth or John get a heads.

ex
A TTT
B TTT
J TTH

$\underbrace{\quad\quad}_{X=3}$

what distribution is X?

what is probability Adam, Beth or John get a head?

$X_1 \sim \text{Geom}(p_1)$
$X_2 \sim \text{Geom}(p_2)$
$X_3 \sim \text{Geom}(p_3)$

$X \sim \text{Geom}(p)$

$p = \text{Prob}(A \text{ or } B \text{ or } J \text{ get } H)$

$= 1 - \text{Prob}(A, B, J \text{ don't get } H)$

$X = \min(X_1, X_2, X_3) \sim \boxed{\text{Geom}(1 - q_1 q_2 q_3)}$ $\quad q_1 q_2 q_3$

ex Let X_1 and X_2 be independent RVs such that for $i = 1, 2$ the distribution of X_i is Poisson (μ_i). Let m be a fixed positive integer. Find the distribution of X_1 given that $X_1 + X_2 = m$. Recognize this distributions as one of the famous ones, and provide its name and parameters.

$$\left.\begin{array}{l} X_1 \sim \text{Pois}(\mu_1) \\ X_2 \sim \text{Pois}(\mu_2) \end{array}\right\} \text{ indep.}$$

$X_1 \mid X_1 + X_2 = m$ takes values $0, 1, \ldots, m$.

Suggest binomial (m, p)

$$P(X_1 = k \mid X_1 + X_2 = m) = \frac{P(X_1 = k, X_2 = m-k)}{P(X_1 + X_2 = m)}$$

$$= \frac{P(X_1 = k) P(X_2 = m-k)}{P(X_1 + X_2 = m)}$$

$$= \frac{\dfrac{\cancel{e^{-\mu_1}} \mu_1^k}{k!} \dfrac{\cancel{e^{-\mu_2}} \mu_2^{m-k}}{(m-k)!}}{}$$

$$\frac{\cancel{e^{-(\mu_1+\mu_2)}}\,(\mu_1+\mu_2)^n}{n!}$$

$$= \binom{n}{k}\left(\frac{\mu_1}{\mu_1+\mu_2}\right)^k\left(\frac{\mu_2}{\mu_1+\mu_2}\right)^{n-k}$$

$$\Rightarrow \boxed{X_1 \mid X_1+X_2=n \sim \mathrm{Bin}\left(n, \frac{\mu_1}{\mu_1+\mu_2}\right)}$$