Today: Basic notions in topology
Simplicial Complexes / maps
Constructions

Topology: Study of spaces and continuous maps between spaces up to deformation.

topological space: Set X w/ notion of open sets $U \subseteq X$

Map: $f: X \to Y$ is cts. if $U \subseteq Y$ open $\Rightarrow$ $f^{-1}(U) \subseteq X$ is open. $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$

We'll assume all maps are continuous. (not nec. differentiable)

Terminology: Topological spaces & maps form what is called a category in mathematics. Category is composed of:

objects (Top. spaces)

morphisms (Cts. maps)

where you can compose morphism to get a new one. $f: X \to Y$, $g: Y \to Z \Rightarrow g \circ f : X \to Z$ is also a cts map. (Exercise: prove this)

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

also need identity maps/morphisms $i_X : X \to X$
$x \mapsto x$

Exercise prove i_X is cts.

Homotopy: this allows for study of deformation.

Suppose we have 2 maps $f, g: X \to Y$. A homotopy is a continuous map $h: X \times I \to Y$ so that ($I = [0,1]$)

$h(x,0) = f(x)$

$h(x,1) = g(x)$

What is $X \times I$? $= \{(x,t) \mid x \in X, t \in [0,1]\}$ product space.

open sets $U \times V$, U open in X, V open in I.

if such a htpy exists, then we say f, g are homotopic, $f \simeq g$.

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two spaces X, Y are homotopy equivalent if $\exists\, f: X \to Y$, $g: Y \to X$ s.t. $g \circ f \simeq i_X$, $f \circ g \simeq i_Y$.

Example: deformation retraction of X to $A \subset X$ is a map $r: X \to A$. s.t. inclusion map $i: A \to X$ satisfies $r \circ i_{A \subseteq X} = i_A$, and $i_{A \subseteq X} \circ r \simeq i_X$ (deformation)

$r(x) = 0$

$h: (x,t) \mapsto t \cdot x$

$h(x,1) = x$ $\quad$ $h(x, 0.5) = x/2$

$h(x,0) = 0$

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if Y and X are homotopy equivalent, denote $X \simeq Y$

1. Exercise: Show that $\simeq$ is an equiv. relation on spaces

2. Exercise: Show that $\simeq$ is an equiv. relation on maps

$\sim$ is an equiv relation if $a \sim a$ (reflexive)

$a \sim b \Leftrightarrow b \sim a$ (symmetric)

$a \sim b,\ b \sim c \Rightarrow a \sim c$ (transitive)

$f: X \to Y$, $f \simeq f$. $h: X \times I \to Y$

$h(x,t) = f(x)$ $\forall t$. $\square$

$f, g: X \to Y$, $f \simeq g \Leftrightarrow g \simeq f$: $\exists\ h: X \times I \to Y$

$h(x,0) = f(x)$

$h(x,1) = g(x)$

$\Rightarrow h_2(x,t) = h(x, 1-t)$

$h_2(x,0) = g(x)$

$h_2(x,1) = f(x)$

$f, g, k: X \to Y$, $f \underset{h_1}{\simeq} g$, $g \underset{h_2}{\simeq} k$ $h_1(x,0) = f(x)$

$h_1(x,1) = g(x)$

$h_2(x,0) = g(x)$

$h_2(x,1) = h(x)$

$$h_3(x,t) = \begin{cases} h_1(x,2t) & t \leq \frac{1}{2} \\ h_2(x,2t-1) & t \geq \frac{1}{2} \end{cases}$$

$\square$

$f \simeq k$

$[f]_{\simeq} = \{ g: X \to Y \mid g \simeq f \}$

Topological invariants: computable objects that are the same for all elements of the same equivalence class of $\simeq$

Example: # connected components

can't split CC. b/c discontinuous.

What exactly is a topological space:

underlying set of points.

a topology is a collection of "open sets" over this underlying set which is closed under unions and finite intersections.

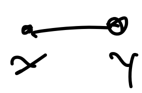
Note: there is a category "Set" which has objects sets and morphisms functions (not just continuous)

notion of open set distinguishes topology from set theory.

Simplicial Complexes:

0-simplex: a point (x) $\quad x$

1-simplex: a line segment (x, y)



2-simplex: triangle (x, y, z)

k-simplex: convex hull of $k+1$ points. $(x_0 \ldots x_k)$

A face of a k-simplex $(x_0 .. x_k)$ is a simplex with some number of vertices removed.

eg. (x_0, x_1) (x_1, x_2), (x_0, x_2) are all faces of
(x_0, x_1, x_2) as well as (x_0), (x_1) and (x_2)

The boundary is the union of faces

$$\partial(x_0 .. x_k) = \bigcup_{i=0}^{k} (x_0 .. \hat{x}_i .. x_k)$$

i removal

A simplicial complex X is a collection of simplices containing all faces, where if σ, τ are simplices, $\sigma \cap \tau \neq \emptyset$ then $\sigma \cap \tau$ must be a face of σ and τ

The k-skeleton $X^{(k)}$ of a simplicial cpx X is the restriction of X to simplices $\leq$ dimension k.

eg. $X^{(0)}$ = set of points
$X^{(1)}$ = points + edges (graph)

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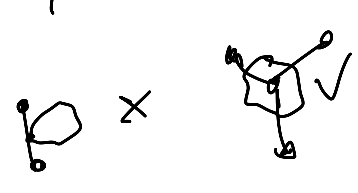
open set: X has "weak topology": $U \subseteq X$ is open iff $U \cap X^{(k)}$ open $\forall k$



other complexes: cubical or cell complexes

Simplicial maps: a map $f: X \to Y$ is simplicial

if $f(x_0 \ldots x_k) = (f(x_0) \ldots f(x_k))$

i.e. extend f linearly from map of 0-skeleton.

a map might map simplices to degenerate simplices

a spx is <u>degenerate</u> if the same vertex appears more than once. e.g. (x, x, y, z)

in this case, it is equivalent to spx on unique set of points. $(x, x, y, z) \sim (x, y, z)$

Example

$x \mapsto a$
$y \mapsto b$
$z \mapsto a$

is simplicial.

$x \mapsto a$
$y \mapsto c$

$(x, y) \mapsto (a, b) \cup (b, c)$

non-simplicial map.

Simplicial maps are cts. under weak topology

Simplicial cpxs & maps are another example of a category (sub-category of Top)

Exercise: Composition of simplicial maps is simplicial.

Examples/ constructions

1) Vietoris-Rips complex:

Start with a finite set of points and metric $d: X \times X \to \mathbb{R}_+$

The Vietoris-Rips at parameter r is the maximal simplicial complex on the 1-skeleton

$R^{(1)}(X; r) = \cup \{(x_i, x_j) : d(x_i, x_j) \leq r\}$

2) A flag complex is a maximal simplicial complex defined on a 1-skeleton e.g. a clique complex.

3) Čech complex: let $X = \{x_0 \ldots x_n\} \subseteq \mathbb{R}^n$ (or other ambient metric space) the simplex $(x_0 \ldots x_n) \in \check{C}(X; r)$ if

$B(x_0; r) \cap B(x_1; r) \cap \cdots \cap B(x_n; r) \neq \emptyset$

Note: $\check{C}(X, r) \subseteq R(X; 2r) \subseteq \check{C}(X; 2r) \subseteq R(X; 4r) \subseteq \cdots$

if $d(x_i, x_j)$ satisfies triangle inequality

pf: exercise.