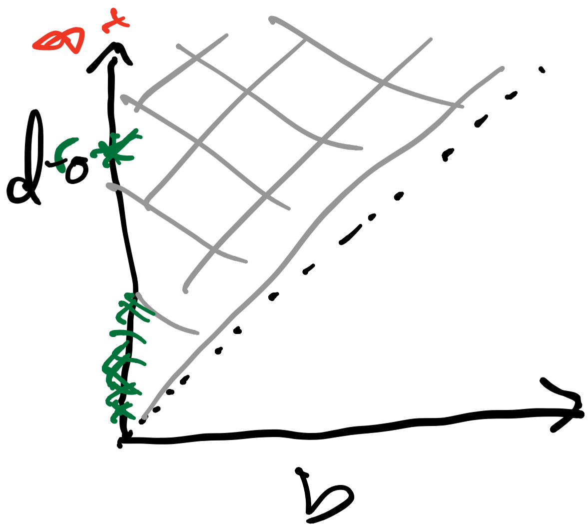
Lecture 7

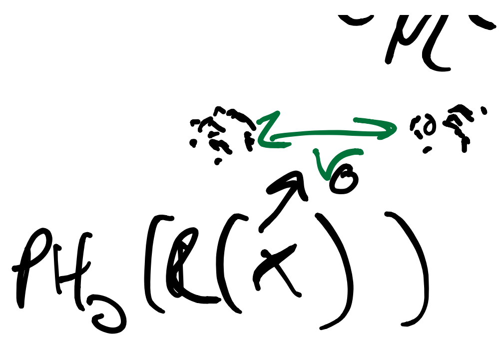
Announcements
OH moved to Thurs 10 am
HW due Friday
project proposals next week

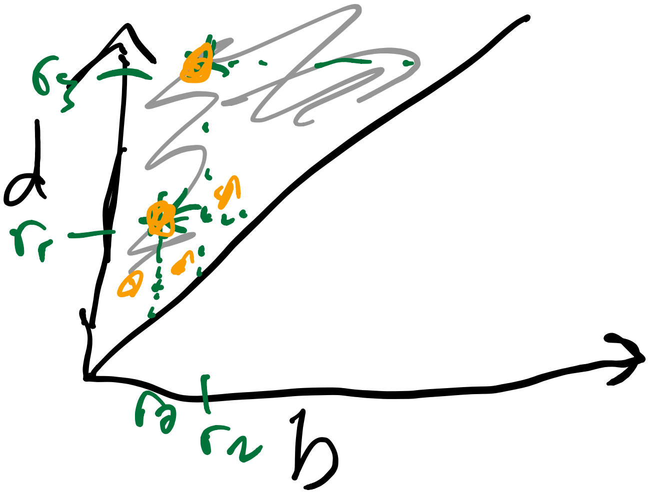
Today:
Pours Barcodes, Diagrams
Bottleneck Distance
Algebraic Features, Landscapes

Persistent Homology:
$X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ Filtrations
$R(X; r_0) \subseteq R(X, r_1) \subseteq \dots$
Persistent homology:
$PH_k(X) : \{(b_i, d_i)\}$
$d_i = \infty$ if homology class at end
"persistence Pair"

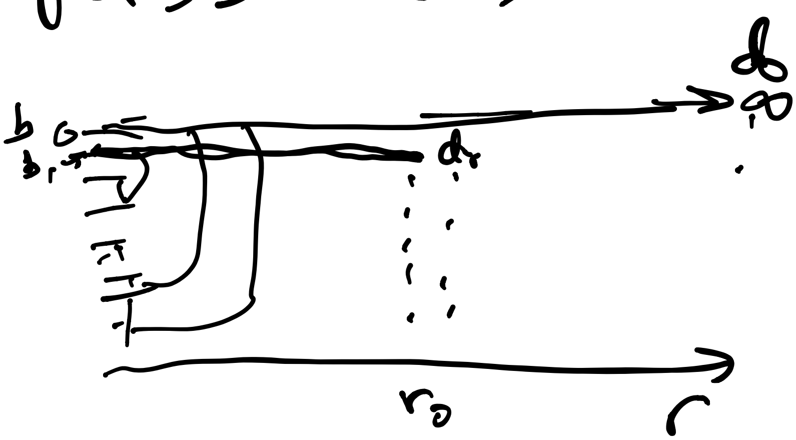
Persistence Diagram: $\{(b_i, d_i)\}$
$\in \mathbb{R}^2$

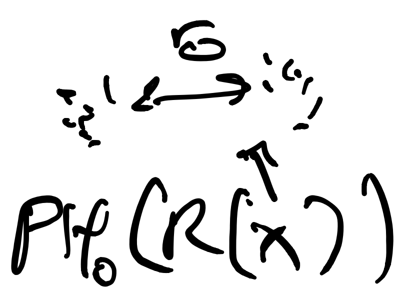


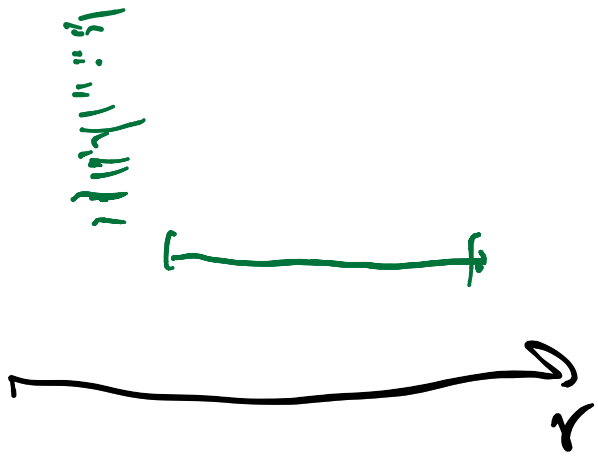


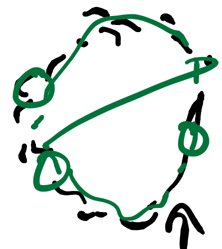


Persistence Barcode: $\{(b_i, d_i)\}$



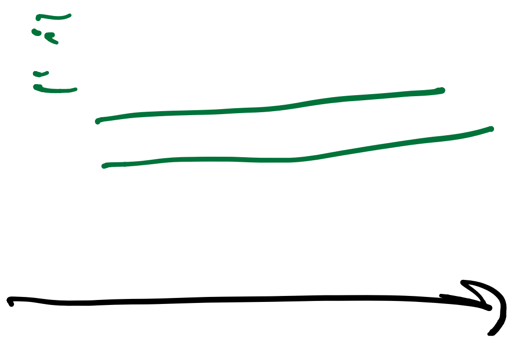


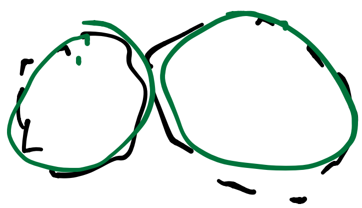




$PH_1(R(X))$

$\{(b_i, d_i)\}$





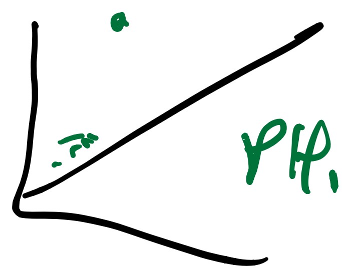
Pairs $\longleftrightarrow$ Diagrams $\longleftrightarrow$ Barcodes

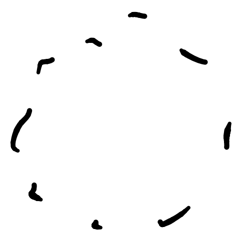
Applied Math:

Define something computable

Perturbation Theory

How robust is our procedure?





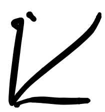
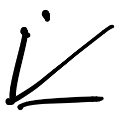
X_0

X_1

$X_0, X_1 \simeq S^1$

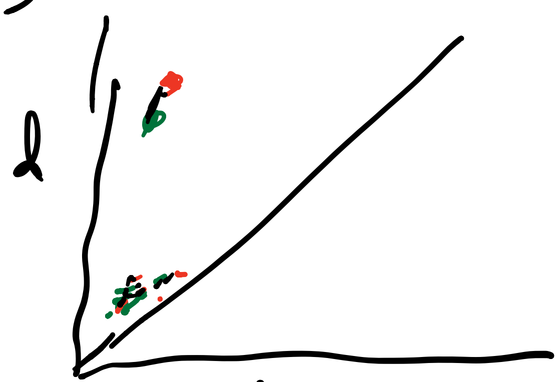
Bottleneck distance on Persistence D.

$d_B(D_1, D_2)$

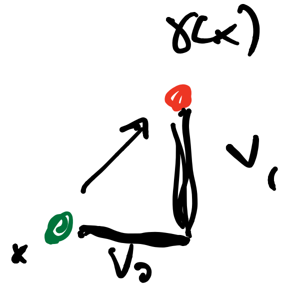
$d_B($  $,$  $)$

$d_B = \inf_{\gamma} \underbrace{\sup_x \|x - \gamma(x)\|_\infty}_{\text{distortion}}$

γ matching between pairs



$x\ (b_0, d_0)$	(b_0, d_0)
(b_1, d_1)	(b_1, d_1)
(b_2, d_2)	(b_2, d_2)
$\vdots$	$\vdots$



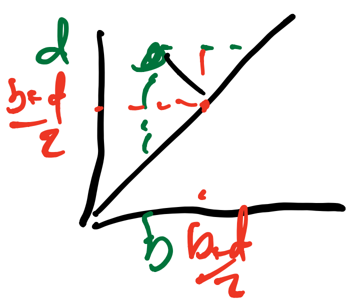
$$\|v\|_\infty = \max_i |v_i|$$

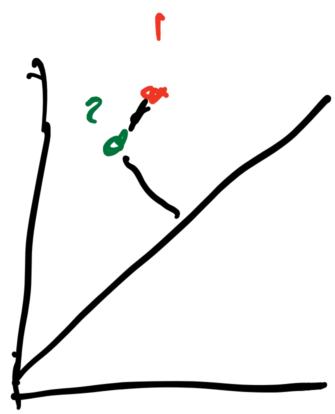
$$\max\{|b - \gamma b|, |d - \gamma d|\}$$

What if the pairs is not the same?

- add all points on the diagonal

$$d_B(K', K) = \sup_{(b_i, d_i)} \frac{d_i - b_i}{2}$$





$$\{(\tilde{b}, d), (b, d)\}$$

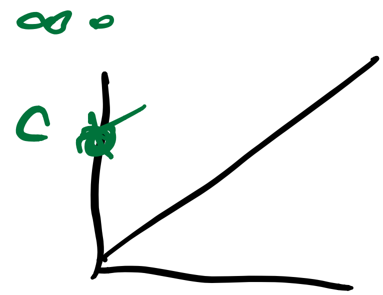
$$d_B = \inf_\gamma \sup_{x \in \{(b_i, d_i)\} \cup \Delta} \|x - \gamma(x)\|$$

notion of distance

perturbation theory?

sub-levelset filtrations

$f : X \to \mathbb{R}$



$f^{-1}((-\infty, a]) : X_a$

$X_a \subseteq X_b$ if $a \leq b$

Def: Let X be a top. space, $f : X \to \mathbb{R}$

A homological critical value of f is a real number a for which $\exists$ an integer k, for which $\forall \varepsilon > 0$ the map induced by inclusion

$$H_k(f^{-1}(-\infty, a-\varepsilon]) \longrightarrow H_k(f^{-1}(-\infty, a+\varepsilon])$$

is not an isomorphism

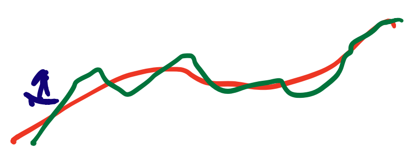
Def: A function $f : X \to \mathbb{R}$ is tame if it has a finite number of hom. crit. values, and and the homology groups $H_k(f^{-1}((-\infty, a]))$ are finite dimensional $\forall k \in \mathbb{Z}$ $\forall a \in \mathbb{R}$

Cohen-Steiner, Edelsbrunner, Harer 2007

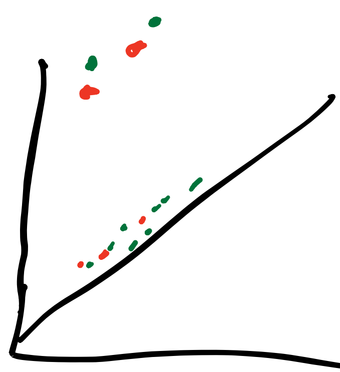
Let X be a triangulable space w/ cts. tame functions $f, g : X \to \mathbb{R}$

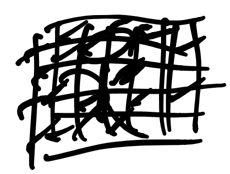
Then

$$d_B(D_k(f), D_k(g)) \le \|f-g\|_\infty$$



$\|f-g\|_\infty : \sup_{x \in X} |f(x)-g(x)|$





Gromov-Hausdorff Stable Signatures for Shapes using Persistence

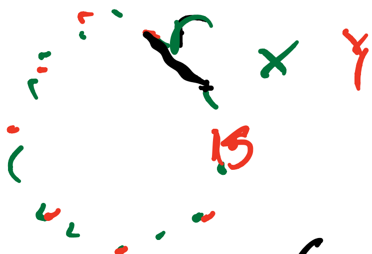
Chazal et al 2009

$X, Y \subseteq \mathbb{R}^2$
$\quad\quad Z$

def: Hausdorff distance

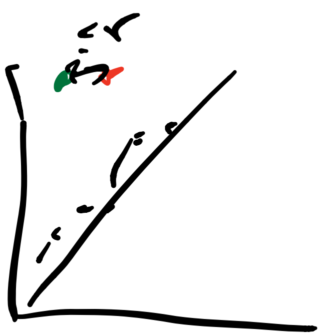
Let $X, Y \subseteq Z$ (some ambient metric space)
so we can compute $d_Z(x,y)$ $\forall x \in X$, $y \in Y$

$$d_H = \max\left\{ \sup_{x} \inf_{y} \|x-y\|_\infty , \sup_{y} \inf_{x} \|x-y\|_\infty \right\}$$



$$\max\{\underline{r}, s\}$$
$$\shortparallel$$

thm: $d_B(D_k(R(X)), D_k(R(Y))) \leq d_H(X,Y)$



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def: The Gromov-Hausdorff distance btw compact metric spaces (X, d_X), (Y, d_Y)

$d_{GH}((X,d_X),(Y,d_Y))$:

$d_{GH}(X, Y) =$

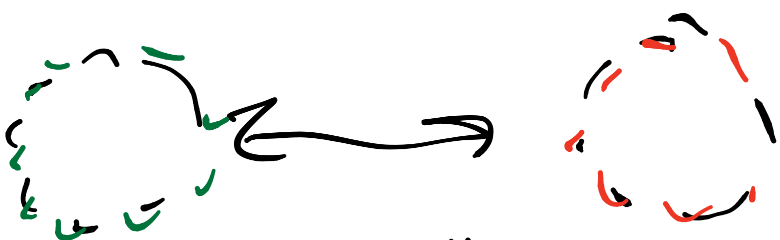
$$\inf_{Z, \gamma_X, \gamma_Y} d_H^Z(\gamma_X(X), \gamma_Y(Y))$$

Z some metric space

γ_X: isometric embedding $X \to Z$

γ_Y " " $Y \to Z$

$$d_{GH}((X, d_{L_1}), (X, d_{L_2}))$$



d_{GH} small

Thm: $d_B(D_k(R(X, d_X)), D_k(R(Y, d_Y)))$
$$\leq d_{GH}((X, d_X), (Y, d_Y))$$

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comment: d_B is a version of d_H for persistence diagrams.

Exercise: prove d_B is a metric

Challenge of using PDs:

are multisets of points in $\mathbb{R}^2$

are multisets of points

stats/ML assumes data in $\mathbb{R}^n$

Idea 1: Adcock, Carlsson, Carlsson 2014

Algebraic fns of barcodes

$$\{(b_i, d_i)\} \to \sum_i (d_i - b_i)^p (d_i + b_i)^q$$

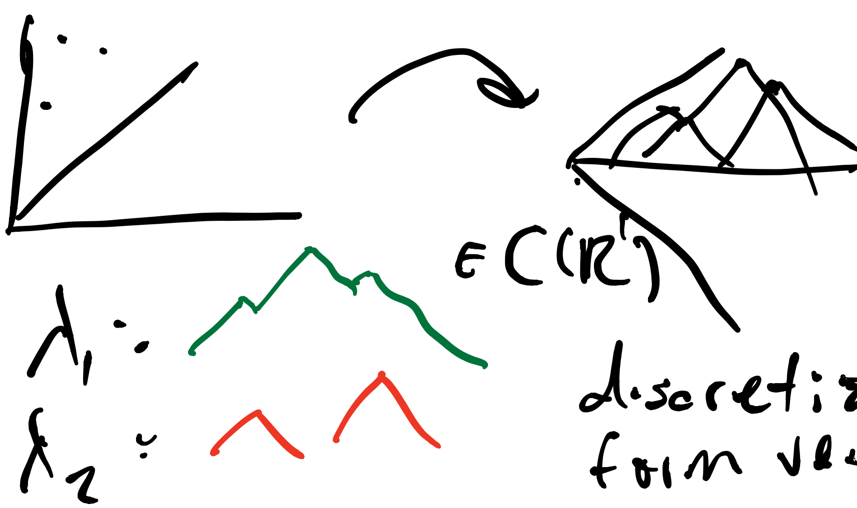
p, q can be anything

choose a couple values of p, q

$\Rightarrow$ PDs $\to \mathbb{R}^{n^2} \to$ ML pipeline

How to take avgs of PDs?

Persistence Landscapes: Bubenik 2016



discretize to form vector in $\mathbb{R}^n$