Chapter 2: Probability for Statistical Computing

just like we did for R

We will **briefly** review some definitions and concepts in probability and statistics that will be helpful for the remainder of the class.

Just like we reviewed computational tools (`R` and packages), we will now do the same for probability and statistics.

Note: This is not meant to be comprehensive. I am assuming you already know this and maybe have forgotten a few things.

i.e. you may need to do some refreshing outside of class as well

https://xkcd.com/892/

Alternative text: "Hell, my eighth grade science class managed to conclusively reject it just based on a classroom experiment. It's pretty sad to hear about million-dollar research teams who can't even manage that."

1 Random Variables and Probability

Definition 1.1 A *random variable* is a function that maps sets of all possible outcomes of an experiment (sample space Ω) to $\mathbb{R}$.

Example 1.1

Toss 2 dice

X = sum of the dice

↑ r.v.

Example 1.2

Randomly select 25 deer & test for CWD (chronic wasting disease)

Sample space: {+, - CWD test}

X_i {0 or 1} observe $X_1, \dots, X_{25}$

Note $p = \sum_{i=1}^{25} X_i / 25$ is also a r.v.!

Example 1.3

Today's high temperature = X_i

Types of random variables –

Discrete take values in a countable set.

Ex. 1.1 and X_i from Ex 1.2

Continuous take values in an uncountable set (like $\mathbb{R}$)

↓ real numbers $(-\infty, \infty)$

Ex 1.3 $X_i \in \mathbb{R}$

p from Ex 1.2, $p \in [0,1]$

1.1 Distribution and Density Functions

Definition 1.2 The *probability mass function (pmf)* of a random variable X is f_X defined by

$$f_X(x) = P(X = x)$$

sometimes when the r.v. is obvious, we will omit the subscript.

where $P(\cdot)$ denotes the probability of its argument.

There are a few requirements of a **valid** pmf

1. $f(x) \geq 0$ for all x
2. $\sum_x f(x) = 1$
3. We call $\mathcal{X} = \{x : f(x) > 0\}$ the "support" of X.

not a requirement

Example 1.4 Let Ω = all possible values of a roll of a single die = $\{1, \ldots, 6\}$ and X be the outcome of a single roll of one die $\in \{1, \ldots, 6\}$.

$f(1) = P(X=1) = \frac{1}{6}$

$\vdots$

$f(6) = \frac{1}{6}$

$\Rightarrow \sum_{x \in \mathcal{X}} f(x) = \sum_{i=1}^{6} \frac{1}{6} = 1$ ✓ valid pmf

A pmf is defined for **discrete variables**, but what about **continuous**? Continuous variables do not have positive probability pass at any single point.

Definition 1.3 The *probability density function (pdf)* of a random variable X is f_X defined by

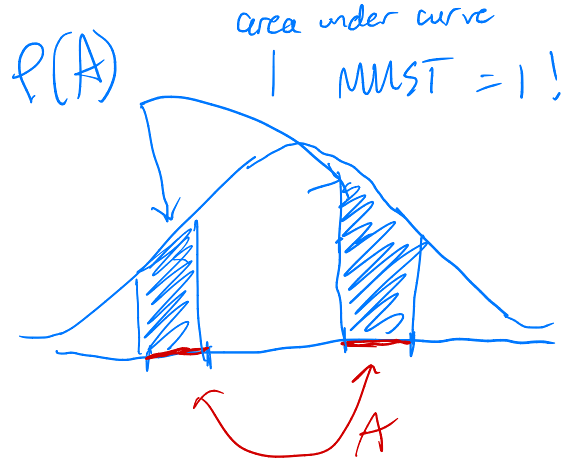
$$P(X \in A) = \int_{x \in A} f_X(x)dx.$$

where $A \subset \mathbb{R}$.

X is a continuous random variable if there exists this function $f_X \geq 0$ such that for all $x \in \mathbb{R}$, this probability exists.

For f_X to be a valid pdf,

1. $f(x) \geq 0$ for all x
2. $\int_{\mathbb{R}} f(x)\,dx = 1$



Again $\mathcal{X} = \{x : f(x) > 0\}$ is the "support" of X

Link to statcomp.

There are many named pdfs and cdfs that you have seen in other class, e.g.

Normal, Beta, Binomial, exponential, Poisson, hypergeometric, etc. ask them for examples

Example 1.5 Let

$$f(x) = \begin{cases} c(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

$= \mathcal{X}$ the support

Find c and then find $P(X > 1)$

$$1 = \int_0^2 c(4x - 2x^2)\,dx = c\left[2x^2 - \frac{2x^3}{3}\right]_0^2 = c\left[\frac{8}{3}\right] \Rightarrow c = \frac{3}{8}$$

"normalizing constant"

$$P(X > 1) = \int_1^\infty f(x)\,dx = \int_1^2 \frac{3}{8}(4x - 2x^2)\,dx = \frac{3}{8}\left[2x^2 - \frac{2x^3}{3}\right]_1^2 = \frac{1}{2}$$

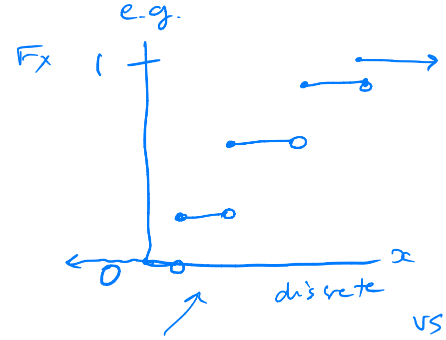
Definition 1.4 The *cumulative distribution function (cdf)* for a random variable X is F_X defined by

for both cts and discrete.

$$F_X(x) = P(X \le x), \quad x \in \mathbb{R}.$$

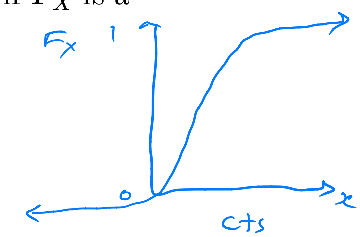
The cdf has the following properties

e.g.



1. F_X is non-decreasing
2. F_X is right-continuous
3. $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$

A random variable X is *continuous* if F_X is a continuous function and *discrete* if F_X is a step function.



Example 1.6 Find the cdf for the previous example.

$F_X(x) = P(X \le x)$

for $x \in [0,2)$, $P(X \le x) = \int_0^x \frac{3}{8}(4y - 2y^2)\,dy = \frac{3}{8}\left[2y^2 - \frac{2y^3}{3}\right]_0^x = \frac{3}{4}x^2\left(1 - \frac{x}{3}\right)$

So $F_X(x) = \begin{cases} 0 & x \le 0 \\ 3/4\, x^2\left(1 - \frac{x}{3}\right) & x \in [0,2) \\ 1 & x \ge 2 \end{cases}$

Note $f(x) = F'(x) = \frac{dF(x)}{dx}$ in the continuous case.

Move section 7 example Here

1.2 Two Continuous Random Variables

Definition 1.5 The *joint pdf* of the continuous vector (X, Y) is defined as

$$P((X,Y) \in A) = \iint_A f_{X,Y}(x,y)dxdy$$

for any set $X \subset \mathbb{R}^2$.

Note, we can also have joint discrete r.v.'s where

$$\sum_{\mathcal{X}}\sum f(x,y) = 1.$$

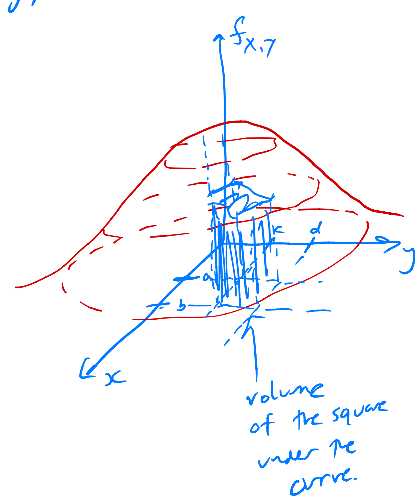
Joint pdfs have the following properties

1. $f_{X,Y}(x,y) \geq 0$ for all x, y
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$

and a support defined to be $\{(x,y) : f_{X,Y}(x,y) > 0\}$. $= \mathcal{X}$

Example 1.7

$$P(a \leq X \leq b,\ c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x,y)\, dy\, dx$$

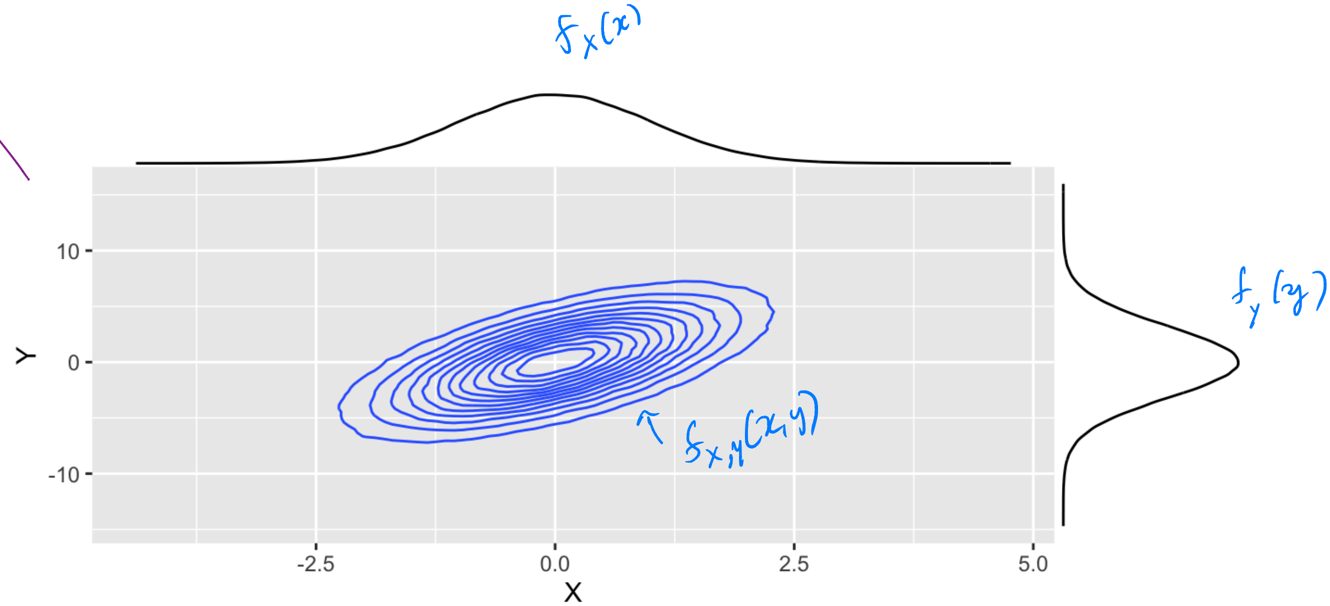


The *marginal densities* of X and Y are given by

$$f_X(x) = \int_{\infty}^{\infty} f_{X,Y}(x,y)dy \quad \text{and} \quad f_Y(y) = \int_{\infty}^{\infty} f_{X,Y}(x,y)dx;$$

$x \in \mathbb{R}$ $\qquad$ $y \in \mathbb{R}$

move image here!



Example 1.8 (From Devore (2008) Example 5.3, pg. 187) A bank operates both a drive-up facility and a walk-up window. On a randomly selected day, let X be the proportion of time that the drive-up facility is in use and Y is the proportion of time that the walk-up window is in use.

The the set of possible values for (X, Y) is the square $D = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$. Suppose the joint pdf is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{5}(x^{\cancel{2}} + y^2) & x \in [0,1], y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Note $\int_0^1 \int_0^1 \frac{6}{5}(x+y^2)\,dx\,dy = 1.$

Evaluate the probability that both the drive-up and the walk-up windows are used a quarter of the time or less.

$$P\left(0 \le X \le \tfrac{1}{4}, 0 \le Y \le \tfrac{1}{4}\right) = \int_0^{1/4}\int_0^{1/4} \frac{6}{5}(x+y^2)\,dx\,dy$$

$$= \int_0^{1/4} \frac{6}{5}\left[\frac{x^2}{2} + xy^2\right]_{x=0}^{x=1/4} dy$$

$$= \int_0^{1/4} \frac{6}{5}\left[\frac{1}{32} + \frac{y^2}{4}\right] dy$$

$$= \frac{6}{5}\left[\frac{y}{32} + \frac{y^3}{12}\right]_0^{1/4} = \frac{6}{5}\left[\frac{1}{32}\cdot\frac{1}{4} + \frac{1}{12}\left(\frac{1}{4}\right)^3\right] = \frac{7}{640} = 0.0109$$

Find the marginal densities for X and Y.

$$f_X(x) = \int_{y=0}^1 \frac{6}{5}(x+y^2)\,dy = \frac{6}{5}\left[xy + \frac{y^3}{3}\right]_{y=0}^1 = \begin{cases} \frac{6}{5}\left(x+\frac{1}{3}\right) & \text{for } x \in [0,1] \\ 0 & \text{o.w.} \end{cases}$$

$$f_Y(y) = \int_{x=0}^1 \frac{6}{5}(x+y^2)\,dx = \frac{6}{5}\left[\frac{x^2}{2} + xy^2\right]_{x=0}^1 = \begin{cases} \frac{6}{5}\left(\frac{1}{2}+y^2\right) & \text{for } y \in [0,1] \\ 0 & \text{o.w.} \end{cases}$$

X

Compute the probability that the drive-up facility is used a quarter of the time or less.

$$P\left(X \leq \frac{1}{4}\right) = \int_0^{1/4} f_X(x)\,dx = \int_0^{1/4} \frac{6}{5}\left(x + \frac{1}{3}\right)dx = \frac{6}{5}\left[\frac{x^2}{2} + \frac{1}{3}x\right]_0^{1/4} = \frac{6}{5}\left[\frac{1}{4^2}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{4}\right]$$

$$= \frac{11}{80} = 0.1375$$

2 Expected Value and Variance

Definition 2.1 The *expected value* (average or mean) of a random variable X with pdf or pmf f_X is defined as

$$E[X] = \begin{cases} \sum_{x \in \mathcal{X}} x f_X(x_i) & X \text{ is discrete} \\ \int_{x \in \mathcal{X}} x f_X(x) dx & X \text{ is continuous.} \end{cases}$$

Where $\mathcal{X} = \{x : f_X(x) > 0\}$ is the support of X.

This is a weighted average of all possible values $\mathcal{X}$ by the probability distribution.

what is the r.v.? what is the pdf? what is the parameter? what is the support?

Example 2.1 Let $X \sim \text{Bernoulli}(p)$. Find $E[X]$.

$\Rightarrow X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{o.w.} \end{cases} \Rightarrow f(x) = \begin{cases} p & \text{when } x=1 \\ 1-p & \text{when } x=0 \end{cases}$ or $f(x) = p^x (1-p)^{1-x}$ for $x \in \{0,1\}$

$$E[X] = \sum_{x \in \{0,1\}} x f(x) = 0(1-p) + 1 \cdot p = p$$

Example 2.2 Let $X \sim \text{Exp}(\lambda)$. Find $E[X]$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

integration by parts (HW 1)

$\int u\,dv = uv - \int v\,du$

pick u and dv

$$E[X] = \int_0^\infty \underbrace{x}_{u} \underbrace{\boxed{\lambda e^{-\lambda x} dx}}_{dv} = \frac{1}{\lambda}$$

Definition 2.2 Let $g(X)$ be a function of a continuous random variable X with pdf f_X. Then,

$$E[g(X)] = \int_{x \in \mathcal{X}} g(x) f_X(x) dx.$$

sometimes this is hard to compute by hand!
$\Rightarrow$ will need stat computing to estimate (Ch. 5)

Definition 2.3 The *variance* (a measure of spread) is defined as

$$\begin{aligned} Var[X] &= E\left[(X - E[X])^2\right] \\ &= E[X^2] - (E[X])^2 \end{aligned}$$

Example 2.3 Let X be the number of cylinders in a car engine. The following is the pmf function for the size of car engines.

x	4.0	6.0	8.0
f	0.5	0.3	0.2

Who might care about this?

Car parts distributor? car parts orderer?

Find

$E[X]$ $= \sum_x x f(x) = 4(0.5) + 6(0.3) + 8(0.2) = 5.4$

$Var[X]$ $= E(X^2) + (EX)^2$

$E(X^2) = \sum_x x^2 f(x) = 4^2(0.5) + 6^2(0.3) + 8^2(0.2) = 31.6$

$g(x) = x^2$

$\Rightarrow Var[X] = 31.6 - (5.4)^2 = 2.44$ easier to interpret : $sd = \sqrt{Var X} = 1.56$

Covariance measures how two random variables vary together (their linear relationship).

$x \leftarrow -5:5$ plot x,y

$y \leftarrow x^2$ $Cov(x,y) = 0.$

$Cov[X,Y] > 0$

$Cov[X,Y] \approx 0$

"random noise"

Definition 2.4 The *covariance* of X and Y is defined by

$$
\begin{aligned}
Cov[X,Y] &= E\left[(X - E[X])(Y - E[Y])\right] \\
&= E[XY] - E[X]E[Y]
\end{aligned}
$$

Note : for 2 r.v.'s,

$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy$

and the *correlation* of X and Y is defined as

move formula up

$$\rho(X,Y) = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}}.$$

Two variables X and Y are *uncorrelated* if $\rho(X,Y) = 0$.

no linear relationship.

3 Independence and Conditional Probability

In classical probability, the *conditional probability* of an event A given that event B has occured is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

$P(A|B)P(B) = P(A \cap B)$

Definition 3.1 Two events A and B are *independent* if $P(A|B) = P(A)$. The converse is also true, so

$$A \text{ and } B \text{ are independent} \Leftrightarrow P(A|B) = P(A) \Leftrightarrow P(A \cap B) = P(A)P(B)$$

Theorem 3.1 (Bayes' Theorem) Let A and B be events. Then,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

3.1 Random variables

The same ideas hold for random variables. If X and Y have joint pdf $f_{X,Y}(x,y)$, then the conditional density of X given $Y = y$ is

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

Thus, two random variables X and Y are independent if and only if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

Also, if X and Y are independent, then

$$f_{X|Y=y}(x) \underset{\text{def'n}}{=} \frac{f_{X,Y}(x,y)}{f_Y(y)} \overset{\text{indep.}}{=} \frac{f_X(x)\cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x).$$

4 Properties of Expected Value and Variance

Suppose that X and Y are random variables, and a and b are constants. Then the following hold:

1. $E[aX + b] =$ $aEX + b$

2. $E[X + Y] =$ $EX + EY$

3. If X and Y are independent, then $E[XY] =$ $EXEY$

4. $Var[b] =$ 0

5. $Var[aX + b] =$ $a^2 \, Var\,X$

6. If X and Y are independent, $Var[X + Y] =$ $Var\,X + Var\,Y$

5 Random Samples

Definition 5.1 Random variables $\{X_1, \ldots, X_n\}$ are defined as a *random sample* from f_X if $X_1, \ldots, X_n \overset{iid}{\sim} f_X$.

" independent and identically distributed "

Example 5.1

$X_1, \ldots, X_n \overset{iid}{\sim} N(0, \sigma^2)$ vs. $X_1 \sim N(\mu_1, \sigma^2)$, $X_2 \sim N(\mu_2, \sigma^2)$ } may be independent, but NOT identically distributed

Theorem 5.1 If $X_1, \ldots, X_n \overset{iid}{\sim} f_X$, then

$$\underbrace{f(x_1, \ldots, x_n)}_{\text{joint pdf}} = \underbrace{\prod_{i=1}^{n} f_X(x_i)}_{\text{product of marginals, easier to deal with}}.$$

Example 5.2 Let $X_1, \ldots, X_n$ be iid. Derive the expected value and variance of the sample mean $n = {i = 1}^n X_i.

X_i's iid $\Rightarrow EX_1 = \ldots = EX_n$

$$E\left(\frac{1}{n}\sum_{i=1}^n X_i\right) \underset{\text{property 1}}{=} \frac{1}{n} E\left[\sum_{i=1}^n X_i\right] = \frac{1}{n}\sum_{i=1}^n EX_i = \frac{1}{n}\cdot n\,EX_1 = EX_1$$

$$Var\left(\frac{1}{n}\sum_{i=1}^n X_i\right) \underset{\text{property 5}}{=} \frac{1}{n^2} Var\left(\sum_{i=1}^n X_i\right) \underset{\substack{X_i \text{ independent} \\ \text{and property 6}}}{=} \frac{1}{n^2}\sum_{i=1}^n Var X_i \underset{\substack{X_i \text{ iid} \Rightarrow \\ Var X_1 = \ldots = Var X_n}}{=} \frac{1}{n^2}\cdot n\, Var X_1 = \frac{1}{n} Var X_1$$

6 R Tips

From here on in the course we will be dealing with a lot of **randomness**. In other words, running our code will return a **random** result.

> But what about reproducibility??

When we generate "random" numbers in R, we are actually generating numbers that *look* random, but are *pseudo-random* (not really random). The vast majority of computer languages operate this way.

This means all is not lost for reproducibility!

```
set.seed(400)
```

Before running our code, we can fix the starting point (`seed`) of the pseudorandom number generator so that we can reproduce results.

Speaking of generating numbers, we can generate numbers (also evaluate densities, distribution functions, and quantile functions) from named distributions in R.

```
rnorm(100)
dnorm(x)
pnorm(x)
qnorm(y)
```

may be useful for future homework

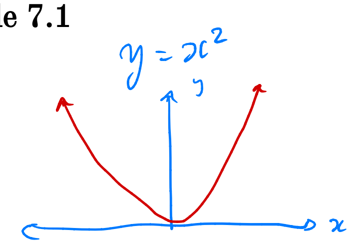
7 Food for thought

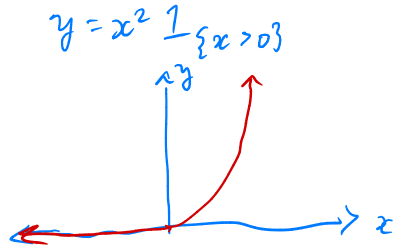
More to pg. 4. example.

Recall an indicator function is defined as

$$1_{\{A\}} = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{otherwise} \end{cases}.$$

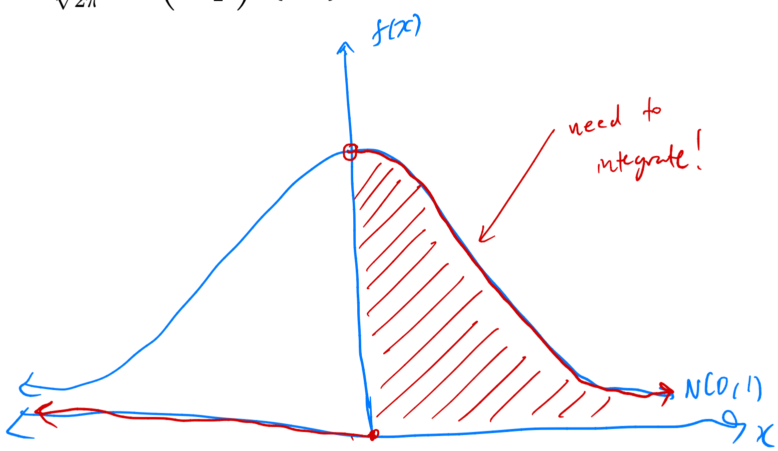
Example 7.1





Example 7.2 If $X \sim N(0,1)$, the pdf is $f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ for $-\infty < x < \infty$.

If $f(x) = \frac{c}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) 1_{\{x>0\}}$, what is c?



We know symmetry

$$\Rightarrow \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) = \frac{1}{2}$$

$$\Rightarrow c \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) = \frac{c}{2}$$

Need $1 = \int_0^\infty \frac{c}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$

$\Rightarrow$ need $c = 2$.