# Bivariate Distributions (Discrete)

4.1

$\mu_X$

# Bivariate Distributions

$f(x)$

$X$

**Univariate**: One measurement for observed items. (outcome associated with one variable). E.g.

- Waiting time Exp
- Number of successes in n trials Bin
- Number of occurrences in a unit time, etc. Pois

**Bivariate**: Use 2 variables to predict an outcome.

E.g. Predict college GPA, $z$, using HS class rank, $x$, and ACT score, $y$,

$$z = f(x, y)$$

Roll 1 die: $S = \{1, 2, 3, 4, 5, 6\}$

$X$

2 die

$S = \{(1,1), (1,2), \ldots$

# Discrete Bivariate Distributions

**Definition 4.1-1**

Let $X$ and $Y$ be two random variables defined on a discrete space. Let $S$ denote the corresponding two-dimensional space of $X$ and $Y$, the two random variables of the discrete type. The probability that $X = x$ and $Y = y$ is denoted by $f(x, y) = P(X = x, Y = y)$. The function $f(x, y)$ is called the **joint probability mass function** (joint pmf) of $X$ and $Y$ and has the following properties:

(a) $0 \le \boxed{f(x, y)} \le 1$.

(b) $\displaystyle\sum\sum_{(x,y)\in S} f(x, y) = 1$.

(c) $P[(X, Y) \in A] = \displaystyle\sum\sum_{(x,y)\in A} f(x, y)$, where $A$ is a subset of the space $S$.

# Discrete Bivariate Example

$$\boxed{f(x,y)} = \boxed{\frac{xy^2}{30},}\quad x = 1,2,3 \quad y = 1,2.$$

(a) $0 \le f(x,y) \le 1$.

(b) $\sum\sum_{(x,y)\in S} f(x,y) = 1$.

(c) $P[(X,Y) \in A] = \sum\sum_{(x,y)\in A} f(x,y)$, where $A$ is a subset of the space $S$.

$$P[X=2, Y=1] = \frac{2(1)^2}{30} = \frac{1}{15}$$

# Discrete Bivariate Example

$P[A]$ $P[A|B]$

Let X and Y be two discrete random variables such that their joint distribution is given below:

e.g. f(3,0) = 0.31

$P[X=3, Y=0] = 0.31$

$\underbrace{P[Y=0] = 0.73}_{\text{marginal}}$

| | | X | | | |
|---|---|---|---|---|---|
| | | 3 | 4 | 5 | |
| | 0 | 0.31 | 0.21 | 0.21 | 0.73 |
| Y | 1 | 0.03 | 0.04 | 0.05 | 0.12 |
| | 2 | 0.02 | 0.03 | 0.04 | 0.09 |
| | 3 | 0.01 | 0.02 | 0.03 | 0.06 |
| | | 0.37 | 0.30 | 0.33 | 1.0 |

# Marginal pmf

**Definition 4.1-2**

Let $X$ and $Y$ have the joint probability mass function $f(x, y)$ with space $S$. The probability mass function of $X$ alone, which is called the **marginal probability mass function of** $X$, is defined by

$$f_X(x) = \sum_y f(x, y) = P(X = x), \qquad x \in S_X,$$

where the summation is taken over all possible $y$ values for each given $x$ in the $x$ space $S_X$. That is, the summation is over all $(x, y)$ in $S$ with a given $x$ value. Similarly, the **marginal probability mass function of** $Y$ is defined by

$$f_Y(y) = \sum_x f(x, y) = P(Y = y), \qquad y \in S_Y,$$

# Marginal probability

- $f(y) = \begin{cases} 0.73, & y = 0 \\ 0.12, & y = 1 \\ 0.09, & y = 2 \\ 0.06, & y = 3 \end{cases}$

| | | X | | | |
|---|---|---|---|---|---|
| | | 3 | 4 | 5 | |
| Y | 0 | 0.31 | 0.21 | 0.21 | 0.73 |
| | 1 | 0.03 | 0.04 | 0.05 | 0.12 |
| | 2 | 0.02 | 0.03 | 0.04 | 0.09 |
| | 3 | 0.01 | 0.02 | 0.03 | 0.06 |
| | | 0.37 | 0.30 | 0.33 | 1.0 |

$f(x) =$

| x | f(x) |
|---|---|
| 3 | |
| 4 | |
| 5 | |

# Independence of X and Y

X and Y are independent **iff:**

- for every x $\in S_x$ and y $\in S_y$,

$$\underbrace{P[X = x, Y = y]}_{} = \underbrace{P[X = x]P[Y = y]}_{}$$

i.e.,

iff joint pmf = product of marginals

$$\boxed{f_{XY}(x, y) = f_X(x) f_Y(y)}$$

# Examples

Bivariate Discrete

1

$$f_X(x) = \sum_y \frac{xy^2}{30} = \sum_{y=1}^{2} \frac{xy^2}{30} = \frac{x(1)^2}{30} + \frac{x(2)^2}{30} = \frac{1(x)}{30} + \frac{4(x)}{30} = \frac{x}{6}$$

Let $f(x,y) = \boxed{\frac{xy^2}{30}}$, $x = \boxed{1,2,3}$ $y = \boxed{1,2.}$

$$\sum_{x=1}^{3} \frac{xy^2}{30} =$$

A) Find the marginal pmf of X: $\boxed{f_X(x) = \boxed{\frac{x}{6}}, \quad x = 1,2,3.}$

B) Find the marginal pmf of Y: $\boxed{f_Y(y) = \boxed{\frac{y^2}{5}}, \quad y = 1,2.}$

C) Find P[X=Y] : $P[(1,1),(2,2)]$ 9/30

D) Are X and Y independent? (Yes)

$$= \frac{1(1)^2}{30} + \frac{2(2)^2}{30} = \frac{1}{30} + \frac{8}{30} = \frac{9}{30}$$

$$\frac{x}{6} \cdot \frac{y^2}{5} = \frac{xy^2}{30} \checkmark$$

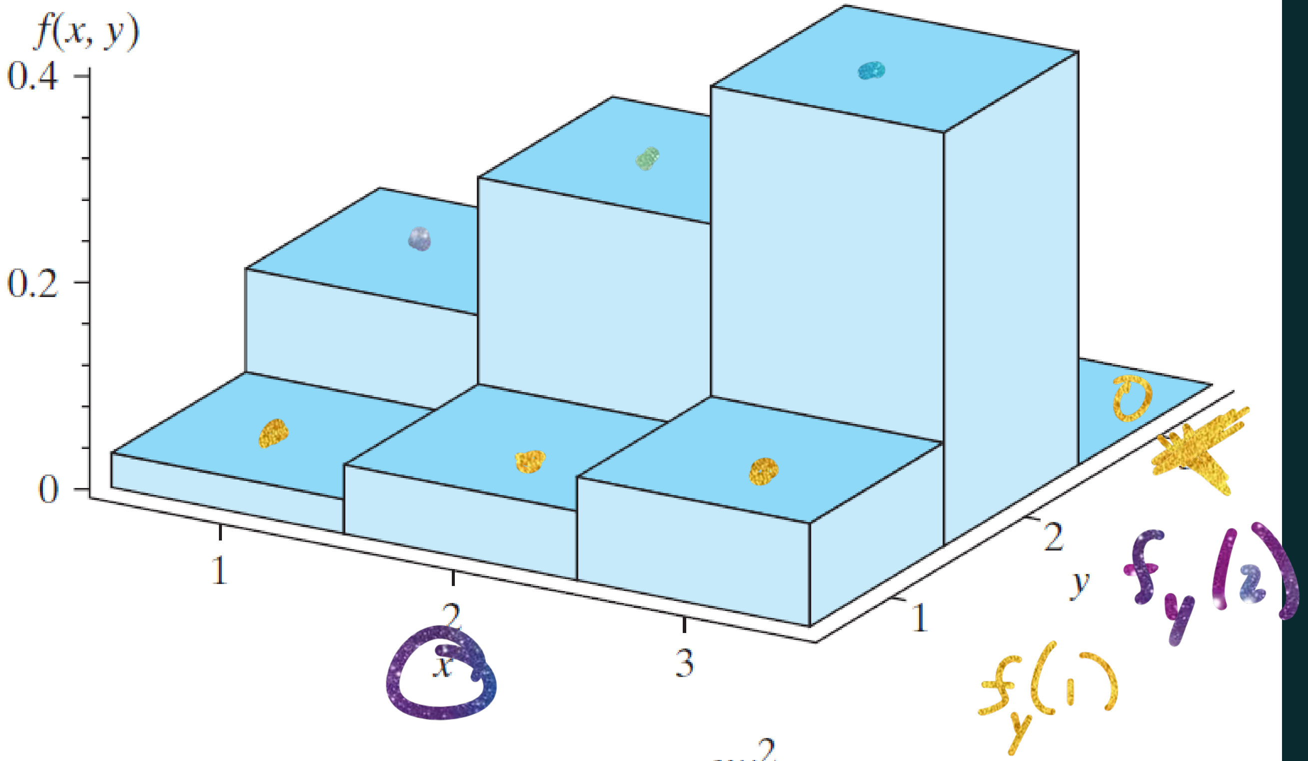


**Figure 4.1-3** Joint pmf $f(x, y) = \dfrac{xy^2}{30}$, $x = 1, 2, 3$ and $y = 1, 2$

**2** Let $f(x, y) = c(x + 2y)$, $x = 1,2$ $y = 1,2,3$

What value must the constant $c$ take, so that $f(x, y)$ is a valid joint pmf?

$f(1,1) + f(1,2) + \ldots$

$f(1,1) = c(3)$ $\quad f(2,1) = c(2+2(1)) = 4c$

$f(1,2) = c(5)$ $\quad 6c$

$f(1,3) = c(7)$ $\quad 8c$

(1/33)

$33c = 1 \longrightarrow c = 1/33$

3 Let $f(x,y) = 6\left(\frac{1}{4}\right)^x \left(\frac{1}{3}\right)^y$, $x = 1,2,3,\ldots$ $y = 1,2,3,\ldots$

$\frac{a}{1-r}$

A) Find an expression for the marginal pmf of x. $f_X(x) = 3\left(\frac{1}{4}\right)^x$

$$f_X(x) = \sum_{y=1}^{\infty} \underbrace{6\left(\frac{1}{4}\right)^x}\left(\frac{1}{3}\right)^y = 6\left(\frac{1}{4}\right)^x \sum_{y=1}^{\infty}\left(\frac{1}{3}\right)^y$$

$$= 6\left(\frac{1}{4}\right)^x \cdot \frac{1/3}{1-1/3} = 3\left(\frac{1}{4}\right)^x, \quad x = 1,2,3,\ldots$$

B) Show that the marginal pmf of x is a valid probability distribution.

show $\sum_{x=1}^{\infty} 3\left(\frac{1}{4}\right)^x = 1$

$$= 3\sum_{x=1}^{\infty}\left(\frac{1}{4}\right)^x = 3\,\frac{(1/4)}{1-1/4} = 3\left(\frac{1}{3}\right) = 1 \checkmark$$