2.4 The Binomial Distribution

Bernoulli Experiment

A **Bernoulli experiment** is a random experiment where the outcome can be classified as one of two mutually exclusive ways (Heads/Tails, Pass/Fail)

A sequence of **Bernoulli trials** occurs when a Bernoulli experiment is performed several **independent** times, and the success probability, *p*, remains the same.

Bernoulli experiment

- e.g. Flipping a fair coin. If I count the event “heads” as a success, this is a Bernoulli experiment with p=0.5.

- If I toss the coin 10 times, results correspond to 10 Bernoulli trials with p=0.5

Bernoulli Distribution $X \sim Bernoulli(p)$

If random variable, X, has a Bernoulli distribution:

$f(x) = p^x (1-p)^{1-x}, \qquad x = \{0,1\}$

- $E[X] = \sum_{x=0}^{1} x\, p^x (1-p)^{1-x} = 0(1-p) + 1(p) = p$

- $Var[X] = \sum_{x=0}^{1} (x-p)^2\, p^x (1-p)^{1-x} = p(1-p)$
- $SD[X] = \sqrt{p(1-p)}$

Definition: Random sample

An observed sequence of n Bernoulli trials can be written as a vector of zeroes and ones, with length n. We call this a **random sample** of size n from a Bernoulli distribution.

- X_i denotes the Bernoulli random variable associated with the i^{th} trial.

Example: Sequence of Bernoulli Trials

Suppose 30% of all lottery tickets are winners. If five tickets are purchased, (0, 1, 0, 0, 1) is one possible observed outcome.

Assuming independence, the probability of this exact outcome is $(0.7)(0.3)(0.7)(0.7)(0.3) = (0.7)^3(0.3)^2$

Binomial Distribution

Often, we are interested only in the total number of successes, but not the actual order of occurrence.

If we let X = the # of observed successes in n Bernoulli trials, then the possible values of X are $0,1,2, \ldots n$.

- For x successes, there are $n - x$ failures.
- X has a **binomial distribution.**

Binomial Distribution

X is a binomial random variable if the following are all true

1. A Bernoulli (success/fail) experiment is performed a constant number of times, n.
2. The random variable, X, is the number of successes in n trials.
3. All trials are independent
4. The success probability, p, for every trial is constant. (The failure probability, $1 - p$, is also constant).

Binomial Distribution

$Notation$: $X \sim Binomial(n,p)$

- $f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1,2,\ldots n$

- $E[X] = np$
- $Var[X] = np(1-p)$

https://shiny.rit.albany.edu/stat/binomial/

Lottery Ticket - Binomial Example

Suppose 30% of lottery tickets are winning tickets.

Let X = the number of winning tickets out of n=5 purchased. The probability of purchasing two winning tickets is

$$f(2) = P(X = 2) = \binom{5}{2} 0.3^2 (1 - 0.3)^{5-2}$$

2.5 Negative Binomial & Geometric Distribution

Geometric Distribution

Say we observe a sequence of independent Bernoulli trials until the first success occurs.

If X is the number of trials needed to observe the 1st success, then X follows a **Geometric Distribution** with parameter, p.

Geometric Distribution $X \sim \text{Geom}(p)$

$f(x) = p(1\text{-}p)^{x-1}$, $x = 1,2,3, \ldots$

- $E[X] = 1/p$
- $Var[X] = \frac{1-p}{p^2}$

Pop Quiz

Can you show that the geometric distribution is a valid pmf?

Negative Binomial Distribution

More generally, suppose we observe a sequence of independent Bernoulli trials until the r^{th} success occurs.

If X is the number of trials needed to observe the r^{th} success, then X follows a **Negative Binomial** distribution with parameters r, p.

Negative Binomial Distribution

$$X \sim NB(r, p)$$

$$f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, r+2, \ldots$$

- $E[X] = r/p$
- $Var[X] = \frac{r(1-p)}{p^2}$

Examples

2.4 - 2.5

1 A magical beer machine vending machine gives a random beer to the customer. It gives you a stout 30% of the time and an IPA 70% of the time. Let X be the number of stouts you get out of 20 beers.

- What is the distribution of X? What is its pmf?

- What is the probability of getting fewer than 7 stouts?

2 A magical beer machine vending machine randomly gives you a stout 30% of the time and an IPA 70% of the time. Thor gently smashes the machine until it gives him a stout. Let X represent the number of trials required for Thor to get his first stout.

What is the distribution of X? What is its pmf?

What is the probability of getting a stout on the 5^{th} trial?

What is the probability of getting a stout within the first 5 trials?

3 A random variable $X \sim$ has a binomial distribution with $\mu = 6, \ \sigma^2 = 3.6$.

What is the distribution of X?

Find $P(X = 4)$.

Find $F(2)$.

4 Jacqueline hits her free throws with $p = 0.9$.

What is the probability that she has her first miss on the 7^{th} free throw?

What is the probability that she has her first miss on the 12^{th} attempt or later?

What is the probability that she has her 3^{rd} miss on the 30^{th} free throw?