2.3 Discrete Random Variables

(Variance)

Review

X r.v.

x values

S or $\mathcal{X}$ all possible values "space"

$E[X]$ μ μ_X center of mass of r.v.

Variance

- One way to characterize a random variable is by its location (**mean, median**).
- Another way is to describe how spread out it is (**variance**).

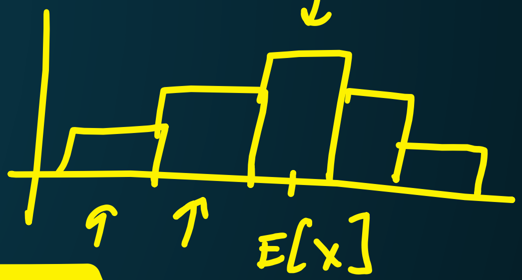
5, 8, 7, 6, 7

0, 14, 2, ~~[illegible]~~, 15

For a random variable, X, we can say $Var[X]$, σ^2, or $\sigma^2{}_X$

sigma

$E[x] \equiv \mu$



$E[u(x)] = \sum_{x} u(x) f(x)$

$E[x^2] = \sum x^2 f(x$

Variance

$\boxed{\sigma^2 = E[(X-\mu)^2] = \sum_{all\ x}(x-\mu)^2 f(x)}$

x

Also,

c

$$\sigma^2 = E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2]$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) - \mu^2.$$

$-2\mu \underbrace{E[X]}_{=\mu}$

$= -2\mu^2$

$-2\mu^2$ $+\mu^2$

, or $\boxed{\sigma^2 = E[X^2] - (E[X])^2}$

shortcut formula

thiccc

transform X

$\underline{aX + b}$

Linear Transformation of a Random Variable – Basic Properties

- $E[aX + b] = a \cdot E[X] + b$
- $Var[aX + b] = a^2 \cdot Var[X]$
- $SD[aX + b] = |a| \cdot SD[X]$

$SD[X] = \sqrt{10}$

$Var[X] = 10$

$Var[5X] = 5^2 \cdot 10 = 250$

$SD[5X] = \cancel{250} = 5\sqrt{10}$

X 0, 5, 2, 5

$X+1$ 1, 6, 3, 6

standard deviation $= \sqrt{\text{Variance}}$

$SD^2 = Var$

2.1 – 2.3

Examples

$${}_nC_r = \frac{n!}{(n-r)!\,r!}$$

1) A pocket contains 5 billiard balls numbered 1 to 5. Jake reaches in and pulls out two of them randomly.

- a) How many different subsets of 2 billiards are there in this pocket? ${}_5C_2 = \frac{5!}{3!\,2!} = \frac{120}{6 \cdot 2} = 10$
- b) Let X be the larger of the two numbers drawn. What is the pmf of X?
- c) What is E[X]?
- d) What is Var[X]?

Let X be larger of 2 numbers

Sample space

S

Outcome	X
1,2	2
1,3	3
1,4	4
1,5	5
2,3	3
2,4	4
2,5	5
3,4	4
3,5	5
4,5	5

x^2	x	$f(x)$	$x^2 \cdot f(x)$
4	2	1/10	$2^2(1/10) = 4/10$
9	3	2/10	$3^2(2/10) = \frac{9\cdot 2}{10} = \frac{18}{10}$
16	4	3/10	$16 \cdot 3/10 = 48/10$
25	5	4/10	$25 \cdot 4/10 = 100/10$

- b) Let X be [illegible] of the two numbers drawn. What is the pmf of X?
- c) What is E[X]?
- d) What is Var[X]?

$2(1/10) + 3(2/10) + 4(3/10) + 5(4/10) = 4 \leftarrow E[X]$

$E[X^2] = \frac{170}{10} = 17$

$E[X^2] = \frac{170}{10} = 17$ $\qquad E[X] = 4$

$* \; Var[X] = E[X^2] - (E[X])^2$

$= 17 - 4^2 = 1$

OR

$Var[X] = \sum_x (x-\mu)^2 f(x) = (2-4)^2 \cdot (\frac{1}{10}) + \ldots$

6 how many elements of S? $6^3 = 216$

2) Suppose a fair die is tossed 3 times. Let X be the largest number that shows up.

$\mathcal{X} = \{1,2,3,4,5,6\}$

$f(1) =$

a) Find an **expression** for $\boxed{F(x)}$. cdf $P[X \le x]$

$P[X \le 1] = 1^3/6^3$

$P[X \le 2] = 2^3/6^3$

cdf

$$F(x) = P[X \le x] = \frac{x^3}{6^3}$$

x	$F(x)$
1	1/216
2	8/216
3	27/216
4	64/216
5	125/216
6	216/216

b) Find an expression for f(x).

$P[X = x]$

$f(x) = F(x) - F(x-1)$

x	$f(x)$
1	1/216
2	7/216
3	19/216
4	37/216

$$f(x) = \frac{x^3}{216} - \frac{(x-1)^3}{216}$$

Note: the following applies to (discrete) p.m.f.'s

$$\boxed{f(x) = P[X = x] = P[X \le x] - P[X \le (x-1)]}$$

$f(3) = P[X = 3] = P[X \le 3] - P[X \le 2]$

- Lowercase f(x) → pmf
- Uppercase F(x) → cdf

- Argument (x) is lowercase because it's a value

HTH $\to$ 1 TTT $\to$ -3

3) A fair coin is tossed three times. Let X be $\# \; of \; heads \; - \; \# \; of \; tails$ in the three tosses.

a) What is the space of X? $\{-3, -1, 1, 3\}$

x	$f(x)$	$x^2 \cdot f(x)$
-3	1/8	9/8
-1	3/8	3/8
1	3/8	3/8
3	1/8	9/8
		24/8

b) What is the pmf of X?

c) What is $E[X]$? $= 0$

d) What is $Var[X]$? $= E[X^2] - \mu^2 = 3 - 0 = 3$

can neg? NO

3) A fair coin is tossed three times. Let X be $\#\ of\ heads\ -\ \#\ of\ tails$ in the three tosses.

c) What is $E[X]$?

d) What is $Var[X]$?

σ^2 Var if had $f(x)$

$\sum_x (3x+1) f(x) = E[Y]$

$\mu = 20$ σ_x

$\sigma_x^2 = 4$

Suppose $E(X) = 20$, $SD(X) = 2$

Let $Y = 3X + 1$.

- Find $E[Y]$ and $Var[Y]$

$E[Y] = E[3X+1]$
$= E[3X] + E[1]$
$= 3E[X] + 1 = 3(20)+1$
$= 61$

Let $Z = 3 - X$

- Find $E[Z]$ and $SD[Z]$

$E[3-X] = 3 - E[X] = -17$

$\sigma_z = 2$

$Var[Y] = Var[3X+1]$
$= 3^2 Var[X] = 9 \times 4$
$= 36$

$E[X^3]$

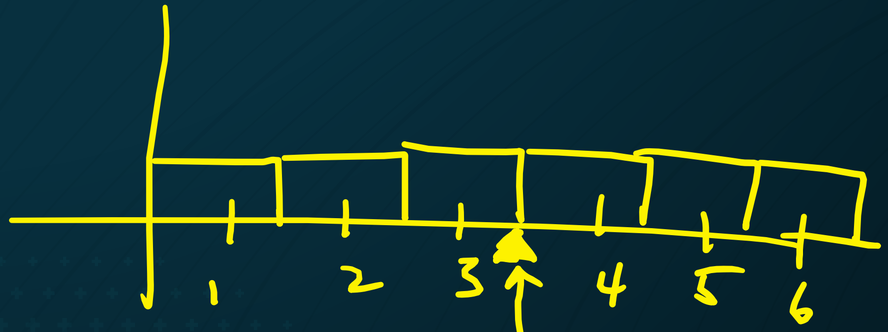
$E[X] = \sum_x x \cdot f(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \dots$

$= 3.5$

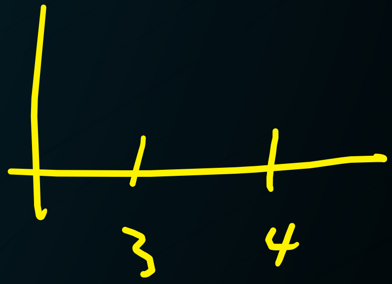
Additional Examples

$Var[X] = E[(X-\mu)^2] = \sum (x-\mu)^2 f(x)$

x	$f(x)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6



$E[X] = 3.5$



$= (1-3.5)^2\left(\frac{1}{6}\right) + (2-3.5)^2\left(\frac{1}{6}\right)$
$+ (3-3.5)^2\left(\frac{1}{6}\right) + \dots$

$3x \quad +1$

Exercise 2

Suppose that Miss Fortune is running a booth at the county fair. Guests flip a coin until the first **tails** appears. If the number of tosses equals n, they are paid n dollars. What is the expected value of money that a guest will make? Show your work for full credit. You may solve this question algebraically or using R. (2 points)

Hint: Define a random variable X to represent the number of coin flips.

pmf

x	$f(x)$
1	1/2
2	
3	
$\vdots$	

→ eqn

$E[X]$

ex. $f(x) = \frac{1/3}{(3/2)^x}$, $\mathcal{X} = \{0, 1, 2, \ldots\}$

$3 = x$

$9 = 3x$ ✓

$$E[x] = \sum_{x=0}^{\infty} x \cdot \frac{1/3}{(3/2)^x}$$

$4 = 4$

$8 = 8$ ✓

$$E[x] = 0 \cdot \frac{1/3}{(3/2)^0} + 1 \cdot \frac{1/3}{(3/2)^1} + 2 \cdot \frac{1/3}{(3/2)^2} + \ldots$$ ✓

$$\frac{1}{3/2} E[x] = 0 \cdot \frac{1/3}{(3/2)^1} + 1 \cdot \frac{1/3}{(3/2)^2} + \ldots$$

$\frac{a}{1-r}$

$$\frac{1}{3} E[x] = +1 \frac{1/3}{(3/2)^1} + 1 \cdot \frac{1/3}{(3/2)^2} + \ldots$$

Exercise 3

Consider a random variable Y with the probability mass function:

$y \cdot y-1$

$$f(y) = c \cdot \frac{3^y}{y!}, \quad y = 2, 3, 4...$$

$y-1$

a) Find E[Y]. (1.5 points)

b) Find P[Y > 3]. (0.5 points)

c) Find Var[Y]. (2 points) $= E[Y^2] - (E[Y])^2$

Hint: Find E[Y(Y-1)].

1

$E[Y^2 - Y]$

$E[Y^2] = \qquad + E[Y]$