2.4 The Binomial Distribution

Bernoulli Experiment

A **Bernoulli experiment** is a random experiment where the outcome can be classified as one of two mutually exclusive ways (Heads/Tails, Pass/Fail)

A sequence of **Bernoulli trials** occurs when a Bernoulli experiment is performed several **independent** times, and the success probability, p, remains the same.

Bernoulli experiment

- e.g. Flipping a fair coin. If I count the event “heads” as a success, this is a Bernoulli experiment with $p=0.5$.

- If I toss the coin 10 times, results correspond to 10 Bernoulli trials with $p=0.5$

Bernoulli Distribution

If random variable, X, has a Bernoulli distribution:

- $f(x) = p^x (1-p)^{1-x}$, $\quad x = \{0,1\}$
- $E[X] = \sum_{x=0}^{1} x\, p^x (1-p)^{1-x} = 0(1-p) + 1(p) = p$
- $Var[X] = \sum_{x=0}^{1} (x-p)^2\, p^x (1-p)^{1-x} = p(1-p)$
- $SD[X] = \sqrt{p(1-p)}$

Definition: Random sample

An observed sequence of n Bernoulli trials can be written as a vector of zeroes and ones, with length n. We call this a **random sample** of size n from a Bernoulli distribution.

- X_i denotes the Bernoulli random variable associated with the i^{th} trial.

Example: Sequence of Bernoulli Trials

Suppose 30% of all lottery tickets are winners. If five tickets are purchased, (0, 1, 0, 0, 1) is one possible observed outcome.

Assuming independence, the probability of this exact outcome is $(0.7)(0.3)(0.7)(0.7)(0.3) = (0.7)^3(0.3)^2$

Binomial Distribution

Often, we are interested only in the total number of successes, but not the actual order of occurrence.

If we let X = the # of observed successes in n Bernoulli trials, then the possible values of X are $0,1,2, \ldots n$.

- For x successes, there are $n - x$ failures.
- X has a **binomial distribution.**

Binomial Distribution

X is a binomial random variable if the following are all true

1. A Bernoulli (success/fail) experiment is performed a constant number of times, n.
2. The random variable, X, is the number of successes in n trials.
3. All trials are independent
4. The success probability, p, for every trial is constant. (The failure probability, $1 - p$, is also constant).

Binomial Distribution

$Notation$: $X \sim Binomial(n,p)$

- $f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1,2, \ldots n$

- $E[X] = np$
- $Var[X] = np(1-p)$

https://shiny.rit.albany.edu/stat/binomial/

Lottery Ticket - Binomial Example

Suppose 30% of lottery tickets are winning tickets.

Let X = the number of winning tickets out of n=5 purchased. The probability of purchasing two winning tickets is

$$f(2) = P(X = 2) = \binom{5}{2} 0.3^2 (1 - 0.3)^{5-2}$$

2.5 Negative Binomial & Geometric Distribution

Geometric Distribution

Say we observe a sequence of independent Bernoulli trials until the first success occurs.

If X is the number of trials needed to observe the 1st success, then X follows a **Geometric Distribution** with parameter, p.

Geometric Distribution $X \sim \text{Geom}(p)$

$f(x) = p(1\text{-}p)^{x-1}, \quad x = 1,2,3, \ldots$

- $E[X] = 1/p$
- $Var[X] = \frac{1-p}{p^2}$

Pop Quiz

Can you show that the geometric distribution is a valid pmf?

Negative Binomial Distribution

- More generally, suppose we observe a sequence of independent Bernoulli trials until the r^{th} success occurs. If X is the number of trials needed to observe the r^{th} success, then X follows a **Negative Binomial** distribution with parameters r, p.

Negative Binomial Distribution

- $f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, r+2, \ldots$
- $E[X] = r/p$
- $Var[X] = \frac{r(1-p)}{p^2}$

Examples

2.4 - 2.5

1

A magical beer machine vending machine gives a random beer to the customer. It gives you a stout 30% of the time and an IPA 70% of the time. Let X be the number of stouts you get out of 20 beers.

- What is the distribution of X? What is its pmf?
- What is the probability of getting fewer than 7 stouts?

2

A magical beer machine vending machine gives a random beer to the customer. It gives you a stout 30% of the time and an IPA 70% of the time. Thor gently smashes the machine until it gives him a stout. Let X be a random variable that represents the number of trials required for Thor to get his first stout.

What is the distribution of X? What is its pmf?

What is the probability of getting a stout on the 5^{th} trial?

What is the probability of getting a stout within the first 5 trials?

2

A random variable X has a binomial distribution with $\mu =$ 6, σ^2=3.6.

- Find P(X=4).
- Find F(2).

3

Jacqueline hits her free throws with $p = 0.9$.

- What is the probability that she has her first miss on the 7th free throw?
- What is the probability that she has her first miss on the 12th attempt or later?
- What is the probability that she has her 3rd miss on the 30th free throw?