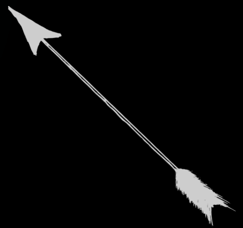
Stat 400 / Math 463
Spring 2021

1.5 Bayes Rule

Introductory Example (Bayes Rule)

Hawkeye purchases arrows from 3 vendors: A, B and C. Out of his entire collection 25% of his arrows are from **vendor A**, 35% from **vendor B**, and the rest are from **vendor C**. Assume we know:

- 5% of the arrows from Vendor A are defective
- 4% from vendor B are defective
- 2% from vendor C are defective.



After a battle, Ted Mosby picks up an arrow at random and finds that it is defective. What is the probability that it came from vendor A?

Law of Total Probability

Let $B_1, B_2, \ldots, B_m$ be a **partition** of the sample space, *S.*

$$S = B_1 \cup B_2 \cup \cdots \cup B_m \text{ and } B_i \cap B_j = \emptyset, i \neq j.$$

Now, let A be an event.

We can write A as the union of *m* mutually exclusive events:

$$A = (B_1 \cap A) \cup (B_2 \cap A) \cup \cdots \cup (B_m \cap A).$$

P[A] = $\sum_{i=1}^{m} P(B_i \cap A)$ using Multiplication Rule,

$\boxed{\textbf{P[A]} = \sum_{i=1}^{m} P(B_i)P(A|B_i)}$ ← **Law of Total Probability**

Bayes Theorem

- Recall: if P[A] > 0,

$$P(B_k \mid A) = \frac{P(B_k \cap A)}{P(A)}, \qquad k = 1, 2, \ldots, m.$$

Using the Law of Total Probability to re-write P[A], we can re-write **Bayes Theorem:**

$$P(B_k \mid A) = \frac{P(B_k)P(A \mid B_k)}{\sum_{i=1}^{m} P(B_i)P(A \mid B_i)}, \qquad k = 1, 2, \ldots, m.$$

Bayes Theorem Shortcut for 2 cases

If we are only considering B and B^c,
we can write Bayes rule:

$$\boxed{P[B|A] = \frac{P(B)P(A|B)}{P(B)P(A|B)+P(B^C)P(A|B^C)}}$$

Introductory Example (Bayes Rule)

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After a battle, Ted Mosby picks up an arrow at random and finds that it is defective. What is the probability that it came from vendor A?

$$
\begin{aligned}
\mathsf{P}(A|D) &= \frac{\mathsf{P}(D|A)\mathsf{P}(A)}{\mathsf{P}(D|A)\mathsf{P}(A) + \mathsf{P}(D|B)\mathsf{P}(B) + \mathsf{P}(D|C)\mathsf{P}(C)} \\
&= \frac{0.05 \times 0.25}{0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.4} \\
&= 0.362
\end{aligned}
$$

1.5 Bayes' Rule

Examples

After spending a few gap years as a pirate, Jack Sparrow is now a senior at UIUC. He believes he has a 25% chance of ending up with a GPA between 3.5 - 4.0, a 35% chance of ending with a 3.0 – 3.5, and 40% chance of ending up with a GPA less than 3.0. His advisor, Captain Hook, tells him that Jack has the following chances of getting into grad school:

- 3.5 – 4.0 GPA: *p = 0.8*
- 3.0-3.5 GPA: *p= 0.5*
- Below 3.0 GPA: *p= 0.1*

1) Based on this information, what is the probability that Jack gets into grad school?

After spending a few gap years as a pirate, Jack Sparrow is now a senior at UIUC. He believes he has a 25% chance of ending up with a GPA between 3.5 - 4.0, a 35% chance of ending with a 3.0 – 3.5, and 40% chance of ending up with a GPA less than 3.0. His advisor, Captain Hook, tells him that Jack has the following chances of getting into grad school:

- 3.5 – 4.0 GPA: *p = 0.8*
- 3.0-3.5 GPA: *p= 0.5*
- Below 3.0 GPA: *p= 0.1*

2) Suppose Jack has been accepted into grad school. What is the probability that Jack ended up with a GPA between 3.0 and 3.5?

Bayes Hogwarts Example

Harry, Ron, and Hermione failed their class. Snape has (randomly) chosen one of them to pass anyway. Their TA, knows which one is going to pass but is not allowed to tell Ron what happens to him.

Ron asks the TA to give him the name of someone that will **fail:**

“-If Harry is going to pass, tell me Hermione is going to fail.

-If Hermione is going to pass, tell me Harry is going to fail.

-If I am going to pass, flip a coin to decide which name (Harry/Hermione) you will give me.”

The TA tells Ron that Harry will fail. Ron is excited and thinks that his probability of passing is now 1/2. Is he correct?

Bayes Rule

- Let A = Ron passes
- B = Harry passes — b = TA says Harry fails
- C = Hermione passes — c = TA says Herm. Fails

Ron:

$$P(A|b) = \frac{P(b|A)P(A)}{P(b|A)P(A) + P(b|B)P(B) + P(b|C)P(C)}$$

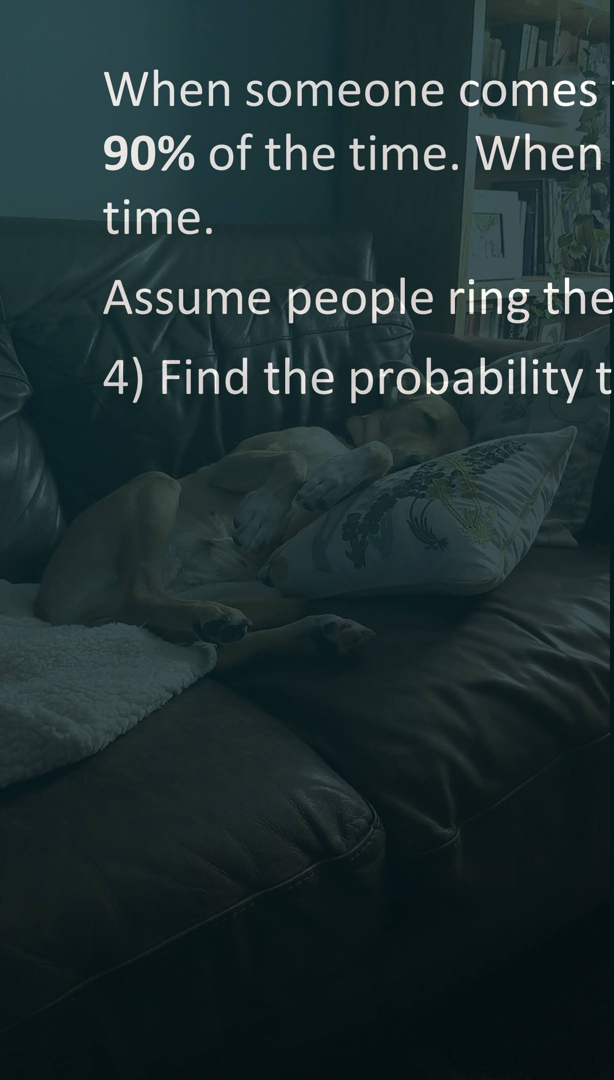
$$= \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3}} = \frac{1}{3}.$$

Bayes Rule

- Let A = Ron passes
- B = Harry passes — b = TA says Harry fails
- C = Hermione passes — c = TA says Herm. Fails

Hermione:

$$P(C|b) = \frac{P(b|C)P(C)}{P(b|A)P(A) + P(b|B)P(B) + P(b|C)P(C)}$$



When someone comes to the house, if they ring the doorbell, Chloe barks **90%** of the time. When they do **not** ring the doorbell, she barks 30% of the time.

Assume people ring the doorbell **80%** of the time.

4) Find the probability that Chloe barks when someone comes to the door.

5) Given that Chloe barked when someone came to the door, what is the probability that they rang the bell?