Sample Size (7.4),
Confidence Intervals for Variance and Standard Deviation

Stat 400 - April 1, 2021

Today's topics

Review:

- Chi Squared Distribution, t distribution, CI for mean

New:

- Confidence Interval for variance (or sd)
- Required Sample Size

Chi-squared distribution (review) χ^2

The Chi-Squared distribution is a special case of the Gamma distribution where θ =2.

Also, sum of squares of normal distributions

https://homepage.divms.uiowa.edu/~mbognar/applets/chisq.html

notes

notes

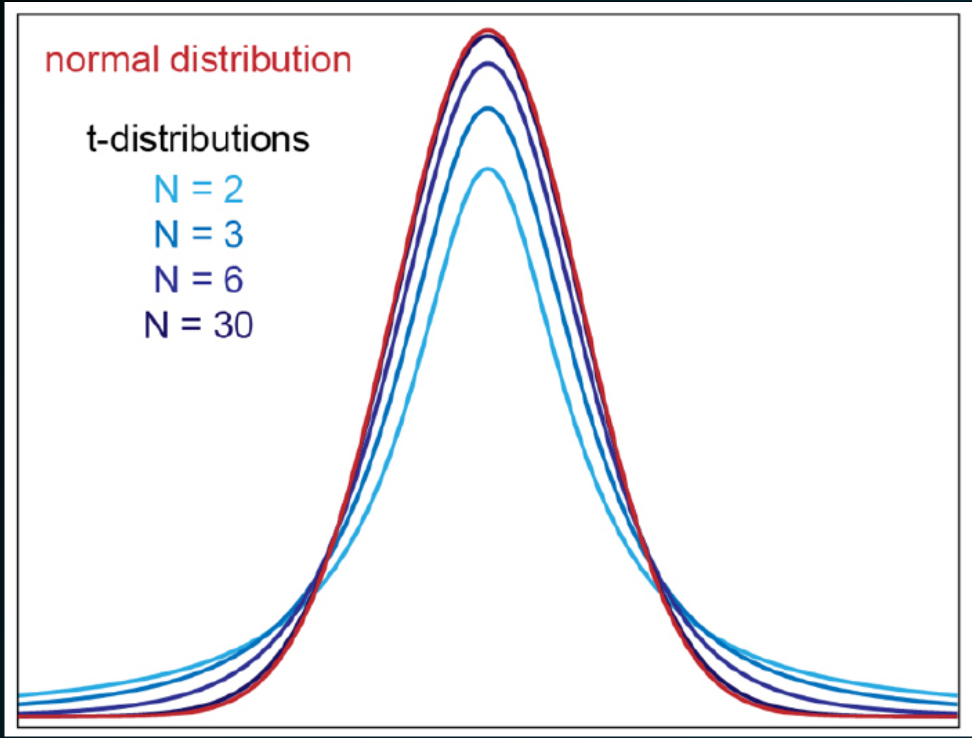
t distribution:

If σ is unknown:

Use $\boldsymbol{s}$ instead of σ

$$T = \boxed{\frac{\bar{X} - \mu}{s/\sqrt{n}} \sim t_{n-1}}$$

if I "standardize" the sample mean using 's' instead of 'σ'.



t distribution

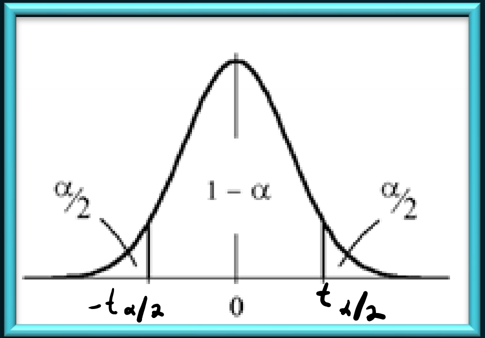
Theorem 5.5-3 **(Student's *t* distribution)** Let

$$T = \frac{Z}{\sqrt{U/r}},$$

where Z is a random variable that is $N(0,1)$, U is a random variable that is $\chi^2(r)$, and Z and U are independent. Then T has a t distribution with pdf

$$f(t) = \frac{\Gamma((r+1)/2)}{\sqrt{\pi r}\,\Gamma(r/2)} \frac{1}{(1+t^2/r)^{(r+1)/2}}, \qquad -\infty < t < \infty.$$

If interested, please refer to textbook for proof. (You are not expected to know how to do it).



Review: CI for mean

Given the following sample: {16, 12, 18, 13, 21, 15, 8, 17} $n=8$

Construct a 92% confidence interval for the true mean.

$$\bar{x} \pm t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}.$$

$\bar{x} = 15,$ $s^2 = \frac{\sum(x_i-\bar{x})^2}{n-1} = \frac{112}{7} = 16,$ $s=4$

$\alpha = 0.08, \alpha/2 = 0.04$

df = n -1 = 7, $t_{7,\,0.04} = 2.046$

x	$x-\bar{x}$	$(x-\bar{x})^2$
16	1	1
12	−3	9
18	3	9
13	−2	4
21	6	36
15	0	0
8	−7	49
17	2	4
	0	112

CI: $\mathbf{15 \pm 2.046 \cdot \frac{4}{\sqrt{8}}}$ = (12.107 , 17.893)

notes

General Form of Confidence Interval

Estimate $\pm$ (Critical Value * SE of estimate)

e.g. if σ is known:

$$\bar{x} \pm z_{\alpha/2} * \frac{\sigma}{\sqrt{n}}$$

if σ is unknown:

$$\bar{x} \pm t_{n-1,\alpha/2} * \frac{s}{\sqrt{n}}$$

Required Sample Size

- $\bar{x} \pm z_{\alpha/2} * \frac{\sigma}{\sqrt{n}}$

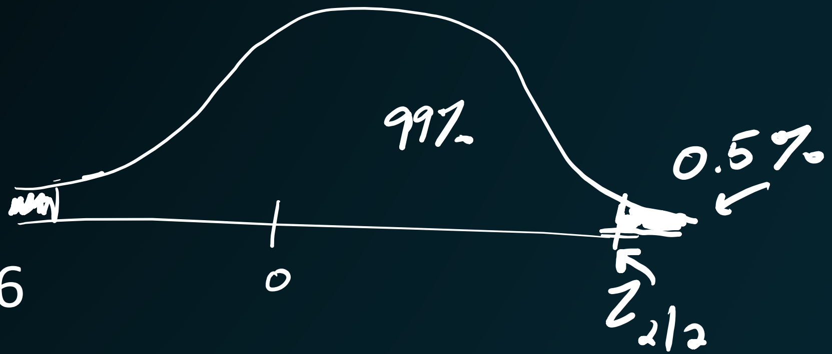
- $\varepsilon = z_{\alpha/2} * \frac{\sigma}{\sqrt{n}}$

$$\boxed{n = \left[\frac{z_{\alpha/2} * \sigma}{\varepsilon}\right]^2}$$

Required Sample Size

How many test runs of an automobile are required for determining its average miles-per-gallon rating on the highway to within 0.5 miles per gallon with 99% confidence? Suppose that the variance of the population in mpg^2 is 6.25.

- $$\boxed{n = \left[\frac{z_{\alpha/2} * \sigma}{\varepsilon}\right]^2}$$



- $\alpha = 0.01 \quad z_{\alpha/2} = 2.576$

$n =$ $= 165.89$ $\boxed{n = 166}$

Round up!

last example, found $s^2 = 16$ ← point estimate

Confidence Interval for σ^2

Now we want to make a confidence interval for σ^2 based on s^2.

- The distribution of s^2 is not Normal.

- It can be shown that $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$

will not need to prove this

χ

Proof of $\frac{(n-1)S^2}{\sigma^2} \backsim \chi^2_{n-1}$

$S^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$

Assume $\left\{ \right.$

1. $\bar{X}$ (the sample mean) and S^2 are independent.
2. If $Z \sim N(0,1)$ then $Z^2 \sim \chi^2(1)$.
3. If $X_i \sim \chi^2(1)$ and the X_i are independent then $\sum_{i=1}^n X_i \sim \chi^2(n)$.
4. A $\chi^2(n)$ random variable has the moment generating function $(1-2t)^{-n/2}$.

With some algebra, you can show, by adding $-\bar{X} + \bar{X}$ inside the parentheses and grouping appropriately, that $\sum_{i=1}^n (X_i - \mu)^2 = \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$. Then, dividing through by σ^2 yields

$$\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 = \sum_{i=1}^n \left(\frac{X_i - \bar{X}}{\sigma}\right)^2 + \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2.$$

Denote these expressions by U, V, and W, respectively, so that the formula reads $U = V + W$. By facts (2) and (3) above, $U \sim \chi^2(n)$ and $W \sim \chi^2(1)$. Also, $V = \frac{(n-1)S^2}{\sigma^2}$.

Since $\bar{X}$ and S^2 are independent, so are V and W. Thus $M_U(t) = M_V(t)M_W(t)$, where $M_X(t)$ denotes the moment generating function of the random variable X. By fact (4) above, this says that

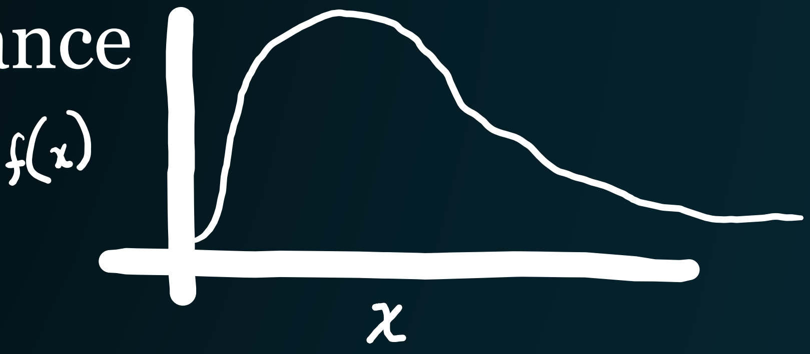
$$\frac{1}{(1-2t)^{n/2}} = M_V(t)\frac{1}{(1-2t)^{1/2}}.$$

$M_V(t) = \frac{1}{(1-2t)^{(n-1)/2}}$, so $V \sim \chi^2_{(n-1)}$

notes

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$ e.g., $\chi^2_{(10)}$

Derivation of CI for Variance



$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

$$P\left[\chi^2_{n-1,\,1-\alpha/2} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{n-1,\,\alpha/2}\right] = 1-\alpha$$

on the next slide

$$P\left[\frac{\chi^2_{n-1,\,1-\alpha/2}}{(n-1)S^2} < \frac{1}{\sigma^2} < \frac{\chi^2_{n-1,\,\alpha/2}}{(n-1)S^2}\right] = 1-\alpha \Rightarrow \boxed{P\left[\frac{(n-1)S^2}{\chi^2_{n-1,\,\alpha/2}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{n-1,\,1-\alpha/2}}\right] = 1-\alpha}$$

Confidence Interval for σ^2

Confidence Interval for σ^2 :

$$\left(\frac{(n-1)\cdot s^2}{\chi^2_{\alpha/2}}, \frac{(n-1)\cdot s^2}{\chi^2_{1-\alpha/2}} \right)$$

CI for Variance

Confidence Interval for σ^2 : $\left(\dfrac{(n-1)\cdot s^2}{\chi^2_{\alpha/2}}, \dfrac{(n-1)\cdot s^2}{\chi^2_{1-\alpha/2}}\right)$

Given the following sample: {16, 12, 18, 13, 21, 15, 8, 17}

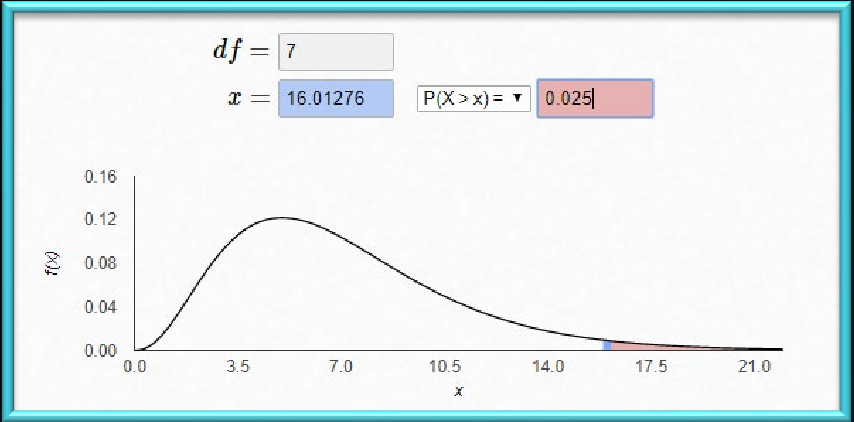
Construct a 95% CI for the true variance.

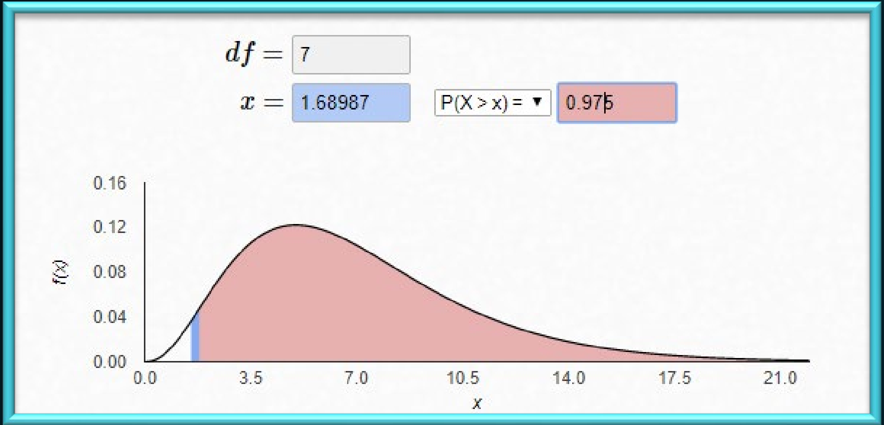
$s^2=16$ $\alpha = 0.05$ $\alpha/2 = 0.025$ $1-\alpha/2 = 0.975$ df = 7

$\chi^2_{(7,\,0.025)} = 16.013$ $\chi^2_{(7,\,0.975)} = 1.690$

$$\left(\frac{(8-1)16}{16.013}, \frac{(8-1)16}{16.013}\right)$$

$= \boxed{(6.996, 66.272)}$





1-sided Confidence Intervals for σ^2

95% confidence lower bound:

$$\boxed{\left(\frac{(n-1)s^2}{\chi^2_{(df,\alpha)}}, \infty\right)}$$

95% confidence upper bound:

$$\boxed{\left(-\infty, \frac{(n-1)s^2}{\chi^2_{(df,1-\alpha)}}\right)}$$

Blank page

notes