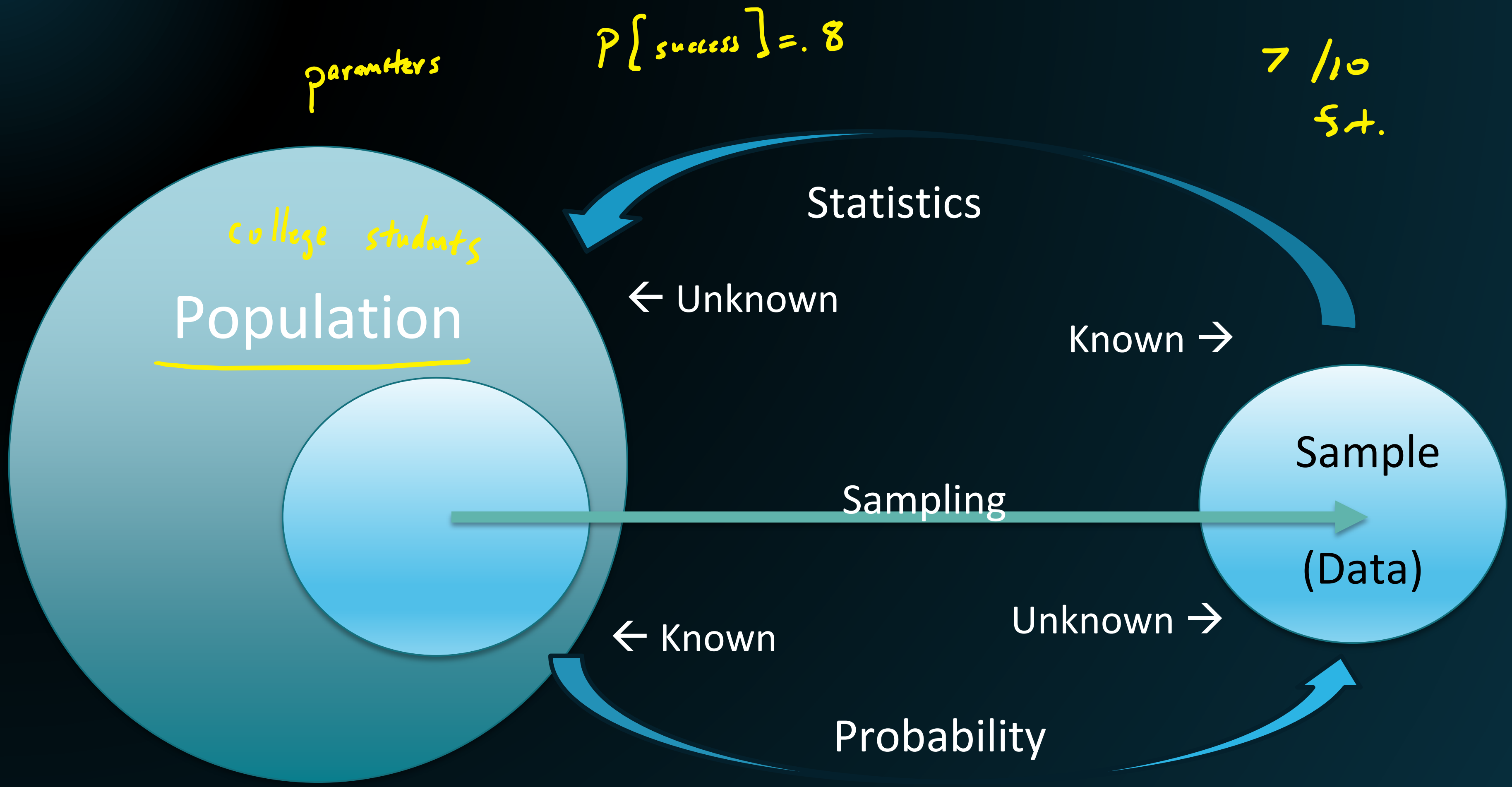
1. Probability

1.1 Properties of Probability
+ Infinite Series

Data → Statistic → Number

What is Statistics?

- What is a statistic? A function of **data**
- **Statistics**: study of the collection, analysis, interpretation, presentation, and organization of **data.**


parameters
P[success] = .8
7/10
St.
college students
Population
Statistics
← Unknown
Known →
Sample
(Data)
Sampling
← Known
Unknown →
Probability

1. Probability

1.1 Properties of Probability

Set : collection of distinct & unique elements

$S = \{A, B, C, \ldots Z\}$

Random Experiments

S = integers > 0

In Statistics, we consider experiments where the outcome can not be predicted with certainty.

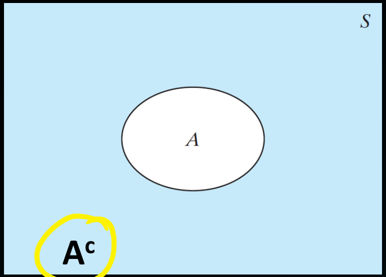
- **Outcome space** or **Sample space**, *S* – collection of all possible outcomes
- An **Event** is a collection of outcomes in *S*. subset of S
- If a random experiment is performed and the outcome of the experiment is in A, we say **event *A* has occurred.**

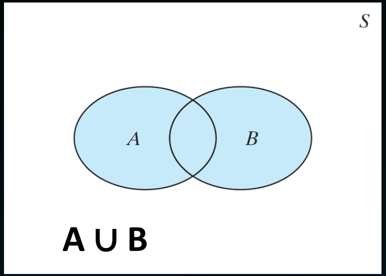
$A = \{1,2,3\}$ $A \cup B = \{1,2,3,4\}$ $A =$ even numbers

$B = \{3,4\}$ $A \cap B = \{3\}$ $\leq$

Set notation and operations

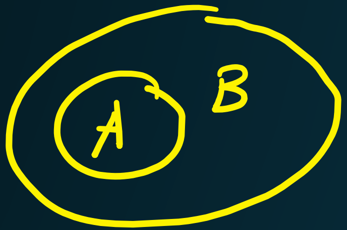
$2 \in A$





	Notation	Meaning
	$\emptyset$, {}	**Null** or empty set
	$x \in A$	x is an element of A
OR	$A \cup B$	the union of A and B
AND	$A \cap B$	the intersection of A and B
	$A \subseteq B$	A is a subset of B
	$A \subset B$	A is a proper subset of B
	A', A^c	the complement of A

$B \subseteq B$



Probability is a real-valued set function P that assigns, to each event A in the sample space S, a number $P(A)$, called the probability of the event A, such that the following properties are satisfied:

6-sided

(a) $P(A) \geq 0$; non-negative

$P[1] = 1/6$ $\quad P[2] = 1/6 \ \ldots$

(b) $P(S) = 1$;

$P[1 \cup 2] = P[1] + P[2]$

(c) if $A_1, A_2, A_3, \ldots$ are events and $A_i \cap A_j = \emptyset$, $i \neq j$, then

$= 2/6$

$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = P(A_1) + P(A_2) + \cdots + P(A_k)$$

for each positive integer k, and

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$$

for an infinite, but countable, number of events.

6 sided die

- S = {1, 2, 3, 4, 5, 6}

- P[S] = 1

$P[\text{even}]$

$\{2,4,6\} \to \boxed{P} \to 0.5$

$y = x^2$

$f(x) = x^2$

Real valued set function

$f(x) = x^2$

$x \to \boxed{} \to x^2$

$\mathbb{R}$

$3 \to \boxed{f(x)} \to 9$

$\text{set} \to \boxed{P} \to \mathbb{R}$

$[0,1]$

Theorem 1.1-1 For each event A,

$$P(A) = 1 - P(A').$$

Proof [See Figure 1.1-1(a).] We have

$$S = A \cup A' \qquad \text{and} \qquad A \cap A' = \emptyset.$$

Thus, from properties (b) and (c), it follows that

$$1 = P(A) + P(A').$$

Hence

$$P(A) = 1 - P(A'). \qquad \square$$

A : even

$\underline{A} \cup \underline{A'} = \underline{S}$

Probability Theorems

- **Theorem 1**
  - P[A’] = 1 – P[A]

$P[A] = .3 \qquad P[A'] = .7$

- **Theorem 2**
  - $\underline{P[\emptyset]=0}$

6-sided die $\qquad P[7] = 0$

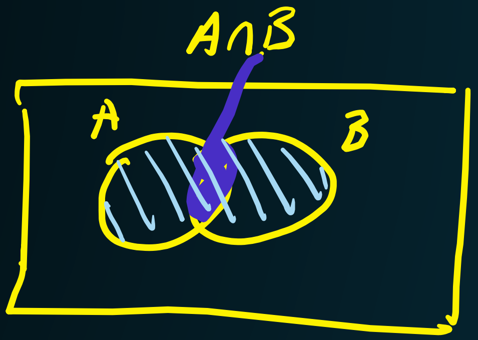
$P[A]$ not $>$ $P[B]$

- **Theorem3**
  - If $\underline{A \subset B}$, then $\underline{P[A]} \leq P[B]$.

$\{1,2,3,4,5\} \subset S$

X	P[x]
1	1/6
2	1/6
3	1/6
4	1/6
5	2/6
6	0/6

Probability Theorems



- **Theorem 4**
  - For any event A, $P[A] \leq 1$
- **Theorem 5**
  - If A and B are any two events, then

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

- **Theorem 6**
  - $P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$

1. Suppose a 6-sided die is rolled:

- **Let event A = {The outcome is even}** $A = \{2, 4, 6\}$
- **Let event B = {The outcome is greater than 3}** $B = \{4, 5, 6\}$
  - a) What are the outcomes in [A ∩ B]? $\{4, 6\}$
    - ii) *What is P[A ∩ B]?* $\underbrace{2/6}$

1. Suppose a 6-sided die is rolled:

- **Let event A = {The outcome is even}**
- **Let event B = {The outcome is greater than 3}**
  - b) What are the outcomes in {A U B}? {2, 4, 5, 6}

    ii) *What is P[A U B]?* 4/6

P( )

P[ ]

1... Suppose the die is loaded so that the probability of an outcome is proportional to the outcome, i.e.

$A = \{2, 4, 6\}$

$B = \{4, 5, 6\}$

$P[1] + P[2] + \ldots P[6] = 1$

$p + 2p + 3p + 4p + 5p + 6p = 1$

$21p = 1 \Rightarrow p = 1/21$

P[1] = *p*, P[2] = *2p*, P[3] = *3p*, P[4] = *4p*, P[5] = *5p*, P[6] = *6p*,

- c) Find the value of *p* that would make this a valid probability model

$1 \le 1$ $\quad$ $\cancel{1 < 1}$

$P[2 \cup 4 \cup 6] = 2p + 4p + 6p = 12p$

- d) Find the following probabilities:
  - i) P[A], ii) P[A'], iii) P[A U B]

$12/21$ $\quad$ $9/21$ $\quad$ 2, 4, 5, 6 $= 17/21$

2. The probability that a randomly selected student at Zoom University owns a bicycle is 0.55, the probability that a student owns a car is 0.30, and the probability that a student owns both is 0.10.

$P[B] = .55$

$= P[B \cap C]$

- A) What is the probability that a student selected at random does not own a bicycle?

  $P[B'] = 1 - P[B] = .45$

- B) What is the probability that a selected student at random owns either a car or a bicycle (or both)?

$$P[C \cup B] = P[C] + P[B] - P[B \cap C]$$

$$= .3 + .55 - .1$$

$$= .75$$

2...

The probability that a randomly selected student at Zoom University owns a bicycle is 0.55, the probability that a student owns a car is 0.30, and the probability that a student owns both is 0.10.

- C) What is the probability that a student selected at random neither has a car nor a bicycle?

$P[B' \cap C'] = 1 - .75 = \underline{.25}$

3. Let a > 2. Suppose S = {0, 1, 2, 3, ... } and

$P[0] = c$, $P[k] = \frac{1}{a^k}$, k = 1, 2, 3...

A) Find the value of c that will make this a valid probability distribution.

B) Find the probability of an odd outcome

$\frac{a}{1-r}$

$P[S] = 1$

$\frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}, \ldots$

$P[0] + \underbrace{P[1] + P[2] + \ldots}$ $= 1$

$c + \underbrace{\sum_{k=1}^{\infty} \frac{1}{a^k}}$ $= 1$

$c + \frac{1/a}{1-\frac{1}{a}}$ $= 1$

$c + \frac{1/a}{1-1/a} = 1$

$c + \frac{1}{a-1} = 1$

$\boxed{c = 1 - \frac{1}{a-1}}$

$P[\text{odd outcome}] = P[1] + P[3] + P[5] + \ldots$

Let a > 2. Suppose S = {0, 1, 2, 3, ... } and

P[0] = c,   P[k] = $\frac{1}{a^k}$, k = 1, 2, 3...

first term

$\frac{a}{1-r}$

$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$

$r = \frac{1}{3}$

$\frac{1}{a} + \frac{1}{a^3} + \frac{1}{a^5} + \ldots$

ratio = $1/a^2$

$\frac{1/a}{1-\frac{1}{a^2}} = \boxed{\frac{a}{a^2-1}}$

$\boxed{\frac{\text{first}}{1-\text{ratio}}}$

4. Suppose S = {0, 1, 2, 3, ... }, P[0] = p, and P[k] = $\frac{1}{2^k k!}$, k = 1, 2, 3...

Find the value of p that will make this a valid probability distribution.

$P[S] = 1$

$P[0] + P[1] + P[2] + \ldots = 1$

$p + \sum_{k=1}^{\infty} \frac{1}{2^k k!} = 1$

$\sum_k \frac{(1/2)^k}{k!}$

$p + \sum_{k=0}^{\infty} \boxed{\frac{1}{2^k k!}} - \frac{1}{2^0 0!} = 1$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$\sum_{k=0}^{\infty} \frac{7^k}{k!} = e^7$

$p + e^{1/2} - 1 = 1 \implies \boxed{2 - e^{1/2} = p}$

R Tutorial

- See .R file on website

- R resource http://www.peterhaschke.com/files/IntroToR.pdf

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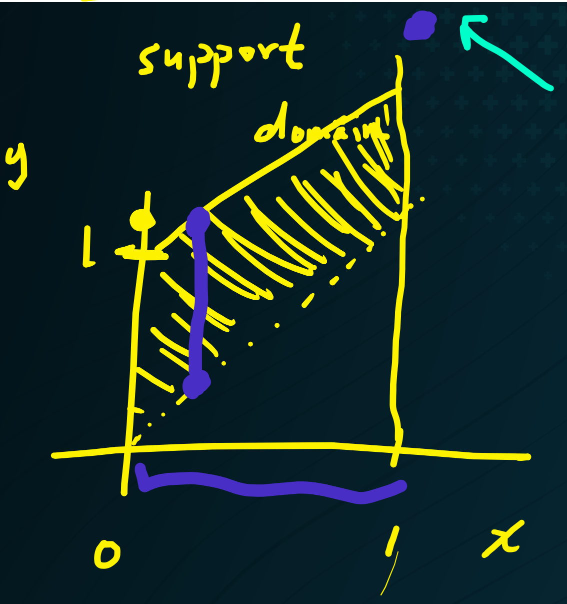
Calc 3 Review

1. Set up a double integral of $f(x, y)$ over the region given by $0 < x < 1, x < y < x + 1$.

$$\int_0^1 \int_x^{x+1} f(x,y)\, dy \quad dx$$



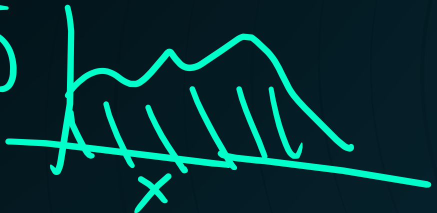
$$\cancel{\iint f(x,y)\, dx\, dy}$$

$x < y$ $(0, 2)$

$(1, 3)$

calc I

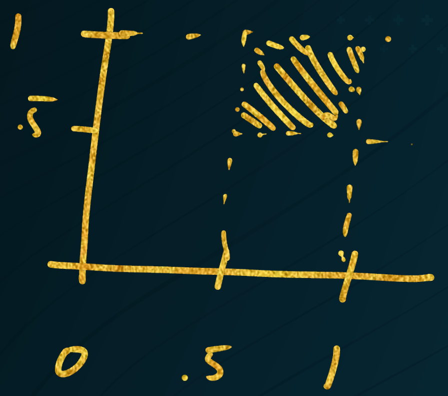
$f(x)$



4. Set up a double integral of $f(x, y)$ over the part of the unit square on which **both** x and y are greater than 0.5.

Solution:

$$\int_{x=1/2}^{1} \int_{y=1/2}^{1} f(x, y)\,dy\,dx$$



7. Set up a double integral of $f(x, y)$ over the set of all points (x, y) in the first quadrant with $|x - y| \leq 1$.

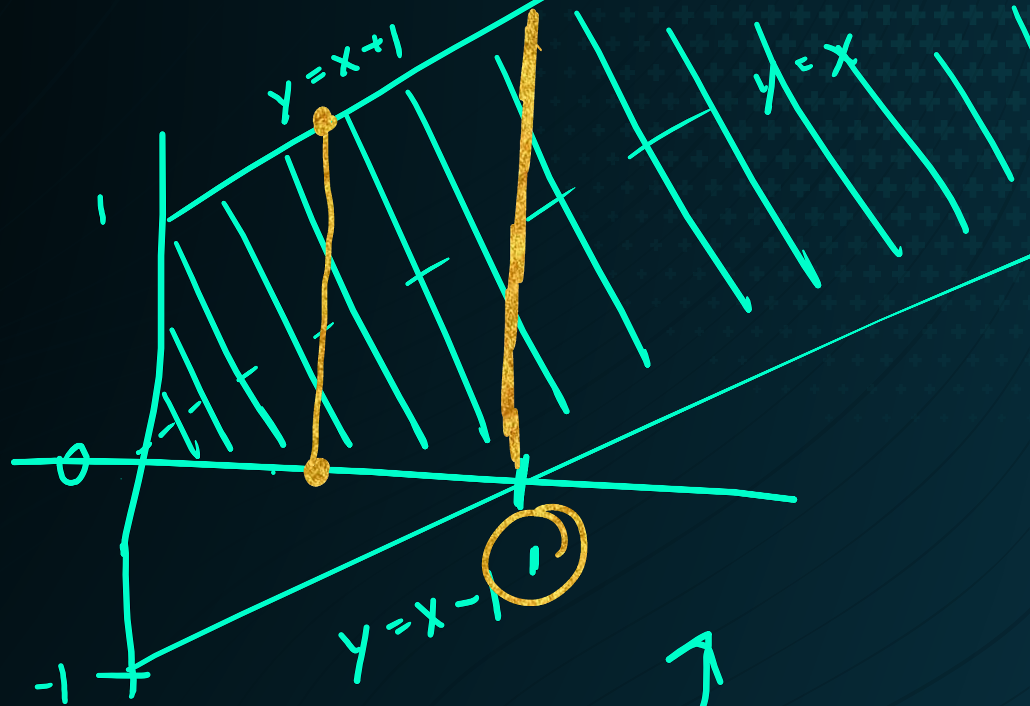
$$\int_0^1 \int_0^{x+1} f(x,y)\,dy\,dx \; + \; \int_1^\infty \int_{x-1}^{x+1} f(x,y)\,dy\,dx$$

$x - 1 \leq y$

$x + 1 \geq y$

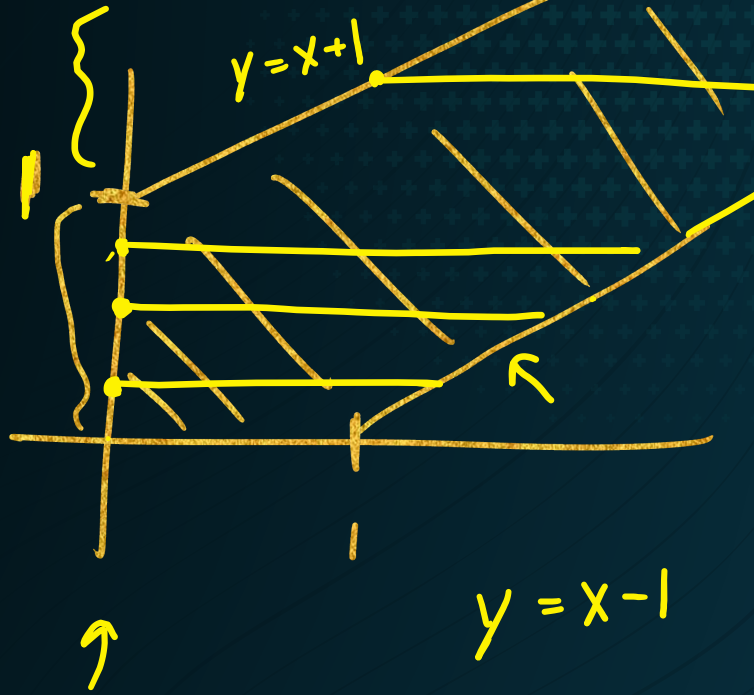
$$\int_0^1 \int_0^{x+1} f(x,y)\,dy\,dx \quad +$$

$$\int_1^{\infty} \int_{x-1}^{x+1} f(x,y)\,dy\,dx$$



$$\int_0^1 \int_0^{y+1} f(x,y)\,dx\,dy \;+$$

$$\int_1^\infty \int_{y-1}^{y+1} f(x,y)\,dx\,dy$$



$y = x-1$

$y+1 = x$

$x = y-1$