STAT 400 Discussion 7

1. An aerospace company is mass producing titanium panels for a prototype space elevator. Since the size of the panels is critical to the design, the company wants to determine if they are mass producing the panels of the correct size, on average. The company randomly selects 30 panels of the assembly line and records the area of the outer face in square inches. The average and standard deviation of the sample were recorded as 64.01 and .37. Construct an appropriate 98% confidence interval for the true average size of the panels.

   Solution: Given that we do not know the population standard deviation we should build a T interval which takes the form

   $$[\bar{x} - t_{\alpha/2,n-1} * \frac{s}{\sqrt{n}}, \bar{x} + t_{\alpha/2,n-1} * \frac{s}{\sqrt{n}}]$$

   First, we need to find the appropriate value for t

   $$t_{\alpha/2,n-1} = t_{.01,29} = 2.46$$

   Plugging into our interval we have

   $$64.01 \pm 2.46 * \frac{.37}{\sqrt{30}} \rightarrow [63.84, 64.18]$$

2. After showing your interval to to the team, they suggest that a more precise interval might be more helpful. They suggest you collect another sample of an appropriate size such that you can estimate the mean within .1. Assume that the standard deviation of the population is .37 and we again use 98% confidence.

   Solution: To create an interval that estimates the mean within .1 requires that our margin of error for our interval is .1. We know that margin of error is

$$MOE = Z^* \frac{\sigma}{\sqrt{n}}$$

Solving for n gives us

$$n \geq (\frac{Z^*\sigma}{MOE})^2$$

plugging in our knowns gives us

$$n \geq (\frac{2.33 * .37}{.1})^2 = 74.32$$

Since we need n to be a whole number this gives us that the minimum sample size must be 75.

3. Suppose $X_1, X_2, ..., X_n$ are iid with mean θ and variance θ.

   Now Lets suppose that we have the following estimators:

$$\hat{\theta}_1 = X_1 + X_2^2$$
$$\hat{\theta}_2 = X_4 + \frac{3X_5}{5}$$
$$\hat{\theta}_3 = \frac{\bar{X}}{2}$$

   (a) Find the Bias of each estimator

   Solution:

$$E[\hat{\theta}_1] - \theta = \theta + \theta^2 + \theta - \theta = \theta^2 + \theta$$
$$E[\hat{\theta}_2] - \theta = \theta + \frac{3\theta}{5} - \theta = \frac{3\theta}{5}$$
$$E[\hat{\theta}_3] - \theta = \frac{\theta}{2} - \theta = -\frac{\theta}{2}$$

   (b) Find the Variance of $\hat{\theta}_2$ and $\hat{\theta}_3$(Note, not enough info to solve for $\hat{\theta}_1$)

Solution:

$$Var[\hat{\theta}_2] = \theta + \frac{9\theta}{25} = \frac{34\theta}{25}$$

$$Var[\hat{\theta}_3] = \frac{\theta}{4n}$$

4. We would like to calculate MLE and MOM estimators for a discrete distribution. Assume that f(x) follows the distribution in the table below

X	1	2	3
P(X)	$\theta/2$	$\theta/2$	$1-\theta$

(a) Find the MOM for θ

Solution:

$$E(X) = \sum x * P(x) = \frac{\theta}{2} + \frac{2\theta}{2} + (3 - 3\theta) = 3 - \frac{3\theta}{2}$$

$$\bar{X} = 3 - \frac{3\theta}{2}$$

$$\tilde{\theta} = 2 - \frac{2\bar{X}}{3}$$

(b) Find the MLE for θ

Solution: For ease, we let n_1, n_2, n_3 be the number of observations in a random sample of size n that are of the corresponding X value. That is, n_1 is the number of x=1 in the sample. Now we can define the likelihood

$$L(\theta) = \prod_1^n \frac{\theta}{2}\frac{\theta}{2}(1-\theta) = (\frac{\theta}{2})^{n_1}(\frac{\theta}{2})^{n_2}(1-\theta)^{n_3}$$

$$l(\theta) = n_1 log(\frac{\theta}{2}) + n_2 log(\frac{\theta}{2}) + n_3 log(1-\theta)$$

Taking the derivative and then setting equal to zero gives

$$\frac{d}{d\theta} = \frac{n_1}{\theta} + \frac{n_2}{\theta} - \frac{n_3}{1-\theta}$$

$$\boxed{\hat{\theta} = \left(\frac{n_3}{n_1 + n_2} + 1\right)^{-1}}$$