1.4 Independence

Notes

Two events are **independent** if the occurrence of one does not affect the probability of another occurring (and vice versa).

$P[A|B] = P[A]$

$P[B|A] = P[B]$

Independence

Definition 1.4-1
Events A and B are **independent** if and only if $P(A \cap B) = P(A)P(B)$. Otherwise, A and B are called **dependent** events.

Theorem 1.4-1 If A and B are independent events, then the following pairs of events are also independent:

(a) A and B';

(b) A' and B;

(c) A' and B'.

Independence

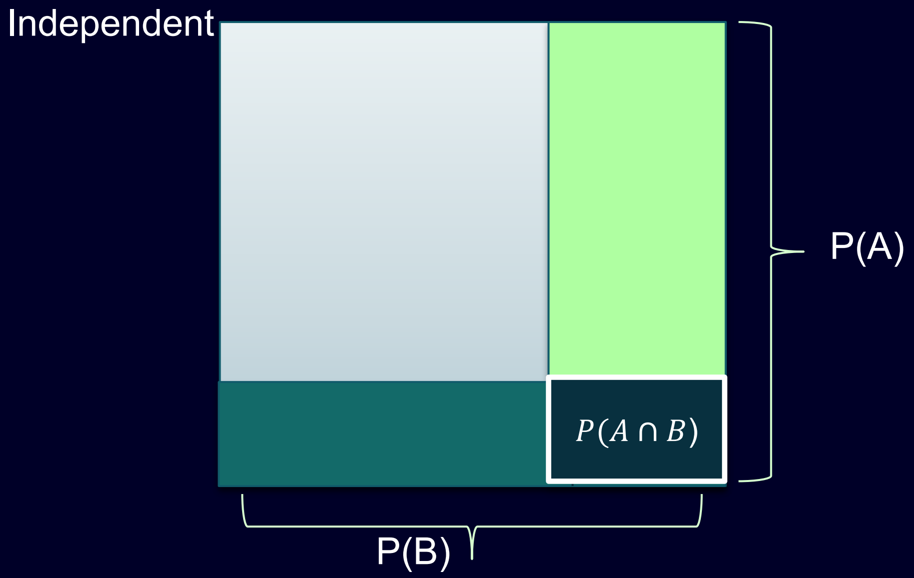
Let A be the event of drawing a queen from a standard deck of cards.
Let B be the event of drawing a spade.

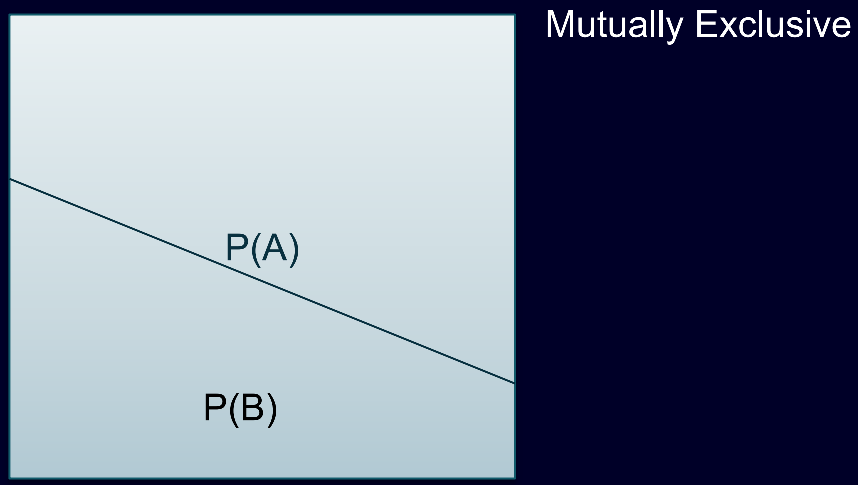
Then (by *definition 1.4-1*), A and B are independent because the probability of their intersection (drawing the queen of spades) is equal to $P(A)P(B)$

$$P(A \cap B) = \frac{1}{52} = \frac{1}{4} \cdot \frac{1}{13} = P(A) \cdot P(B)$$

Note: there are 52 cards total. 4 of each value and 13 of each suit

Mutually Exclusive Events are Not Independent





Suppose two events, A and B, both have nonzero probability and are mutually exclusive. Are they also independent? No

Pairwise vs. Three-way Independence

$A \{(1,6),(3,4),(2,5),(4,3),(5,2),(6,1)\}$

This is a very classic example, reported in any book on Probability:

Example 1. We throw two dice. Let A be the event "the sum of the points is 7", B the event "die #1 came up 3", and C the event "die #2 came up 4". Now, $P[A] = P[B] = P[C] = \frac{1}{6}$. Also,

$$P[A \cap B] = P[A \cap C] = P[B \cap C] = \frac{1}{36}$$

$\left(\frac{1}{6}\right) \cdot \left(\frac{1}{6}\right)$

so that all events are pairwise independent. However,

$(1,1),(1,2)$

$$\boxed{P[A \cap B \cap C]} = P[B \cap C] = \frac{1}{36}$$

$3, 4$

$\frac{1}{6} \cdot \frac{1}{6}$

while

$$P[A]P[B]P[C] = \frac{1}{216}$$

If A and B are independent, are A^C and B^C independent?

prove: $P(A \cap B) = P(A)P(B) \rightarrow P(A^C \cap B^C) = P(A^C)P(B^C)$?

Hint: The union of two complements is the complement of their intersection.

$P[A^c \cup B^c] = P[(A \cap B)^c] = 1 - P[A \cap B]$

$=$

$P[A^c \cup B^c] = P[A^c] + P[B^c] - P[A^c \cap B^c]$

$1 - P[A \cap B] = P[A^c] + P[B^c] - P[A^c \cap B^c]$

$P[A^c \cap B^c] = \cancel{-1} + P[A \cap B] \cancel{+1} - P[A] + 1 - P[B]$

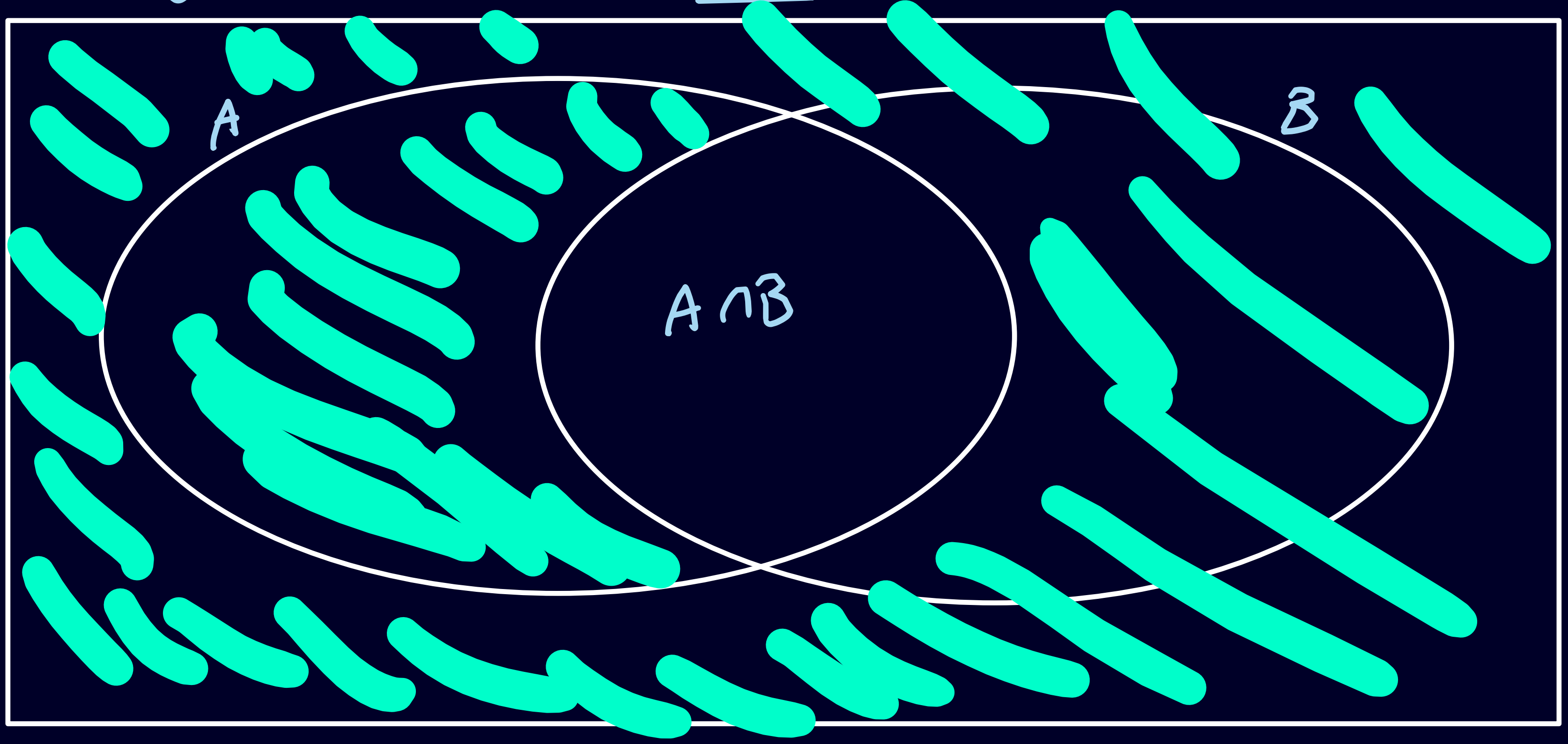
$= P[A \cap B] - P[A] + 1 - P[B]$

$= P[A]P[B] - P[A] + 1 - P(B)$

$= [1 - P[A]][1 - P[B]]$

$= P[A^c] \cdot P[B^c]$ $\square$

$$(A \cap B)^c = \underline{A^c} \cup \underline{B^c}$$



Miscellaneous facts about independence

- Sometimes, the circumstances surrounding two events will make it obvious that the occurrence of one event has absolutely no effect on the occurrence of another. In this case, these events will *necessarily* be independent.
- Another example of independence: Gambler's Fallacy

Can we extend the idea of independence to more than 2 events? What will that look like?

Pairwise vs Mutual Independence

An urn contains 4 numbers .

Let A = {1,2}, B = {1, 3}, C = {1, 4}.

Then P[A] =P[B] = P[C]= ½

$$P(A \cap B) = \frac{1}{4} = P(A)P(B),$$

$$P(A \cap C) = \frac{1}{4} = P(A)P(C),$$

$$P(B \cap C) = \frac{1}{4} = P(B)P(C),$$

A, B, and C, are **pairwise independent**.

Mutual Independence

$.999^{24}$

Definition 1.4-2

Events A, B, and C are **mutually independent** if and only if the following two conditions hold:

(a) A, B, and C are pairwise independent; that is,

$$P(A \cap B) = P(A)P(B), \qquad P(A \cap C) = P(A)P(C),$$

and

$$P(B \cap C) = P(B)P(C).$$

(b) $P(A \cap B \cap C) = P(A)P(B)P(C)$.

Pairwise vs Mutual Independence

An urn contains 4 numbers .

Let A = {1,2}, B = {1, 3}, C = {1, 4}.

Then P[A] = P[B] = P[C] = ½

A, B, and C are pairwise independent, but

$$P(A \cap B \cap C) = \frac{1}{4} \neq \frac{1}{8} = P(A)P(B)P(C).$$

→ A, B, and C, are **not mutually independent**.

1.4 Independence

Examples

1) At Pirdew Unavursety, a student must pass all 3 course exams (arithmetic, alphabet, breathing) to earn a diploma. A practice test given to 4000 seniors resulted in the following number of failures:

Area	Number of failures
Arithmetic	3000
Alphabet	950
Breathing	50

Assume that passing each of these courses are considered mutually independent events. What proportion of these seniors can be expected to **fail** to qualify for a diploma?

2) Home Depot has some amazing sales after Christmas. Suppose a string of Christmas lights has 24 bulbs in a series circuit. If each bulb has a 99.9% chance of being alive, what is the probability that the string (as a whole) will work?