Exercise 1

Let **X** denote the number of times Ninki Minjaj goes to the beach in a given week. Suppose X has the following probability distribution:

x	f(x)
0	0.2
1	0.2
2	0.2
3	0.3
4	0.1

a) Find the probability that Ninki will go to the beach at least 2 times in any given week. (0.5 points)

b) Find the expected number of times that Ninki will go to the beach in a given week. (0.5 points)

c) Find the standard deviation of beach trips in a week, σ_X. (1 point)

d) The doctor tells Ninki that she is at high risk of skin cancer and should avoid going to the beach more than once per 2 weeks, but Ninki doesn't care. What is the probability that Ninki will go to the beach fewer than 2 times total in any 2 week period? (Assume the frequency of beach visits each week is independent) (1 point)

e) Cole has been to the beach twice this week and has seen Ninki there both times. Based on this information, what is the probability that Ninki went to the beach at least 3 times that week? (1 point)

a) Solve for $P[X \geq 2] = f(2) + f(3) + f(4) = 0.60$

likewise, $= 1 - f(1) - f(0) = 0.60$

b) $E[X] = \sum x \cdot f(x)$ for all possible x.

$= 0[0.2] + 1[0.2] + 2[0.2] + 3[0.3] + 4[0.1] = 1.9$ times

c) $\sigma_X^2 = Var[X] = E\left[(X - E[X])^2\right] = \sum (x - E[X])^2 \cdot f(x)$ for all possible x.

$= (-1.9)^2 \cdot (0.2) + (-0.9)^2 (0.2) + (0.1)^2 (0.2) + (1.1)^2 (0.3) + (2.1)^2 (0.1)$

$Var[X] = 1.69$

$\sigma_X = \sqrt{Var[X]} = 1.3$ beach trips

d) Assuming X_1 = beach trips in week one & X_2 = beach trips in week 2:

$P[X_1 + X_2 < 2] = P[X_1 = 0 \cap X_2 = 0] + P[X_1 = 0 \cap X_2 = 1] + P[X_1 = 1 \cap X_2 = 0]$

$= [0.2 \cdot 0.2] + [0.2 \cdot 0.2] + [0.2 \cdot 0.2] = 0.12$

e) Solve for $P[X \geq 3 \mid X \geq 2] = \dfrac{P[X \geq 3 \cap X \geq 2]}{P[X \geq 2]} = \dfrac{P[X \geq 3]}{P[X \geq 2]} = \dfrac{0.40}{0.60} = \dfrac{2}{3} \approx 0.67$

Exercise 2

Suppose that Miss Fortune is running a booth at the county fair. Guests flip a coin until the first **tails** appears. If the number of tosses equals n, they are paid n dollars. What is the expected value of money that a guest will make? Show your work for full credit. You may solve this question algebraically or using R. (2 points)

Hint: Define a random variable X to represent the number of coin flips.

Assume X is the number of flips until getting one tails, then $n = X$ and:

$$P[X=x] = (1/2)^x \text{ for } x = 1, 2, \ldots \text{ and } E[X] = \sum_{x=1}^{\infty} x(1/2)^x$$

$$E[X] = 1\left(\tfrac{1}{2}\right) + 2\left(\tfrac{1}{2}\right)^2 + 3\left(\tfrac{1}{2}\right)^3 + 4\left(\tfrac{1}{2}\right)^4 + \ldots$$

$$\tfrac{1}{2}E[X] = 1\left(\tfrac{1}{2}\right)^2 + 2\left(\tfrac{1}{2}\right)^3 + 3\left(\tfrac{1}{2}\right)^4 + \ldots$$

$$E[X] - \tfrac{1}{2}E[X] = \tfrac{1}{2}E[X] = 1\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{2}\right)^2 + 1\left(\tfrac{1}{2}\right)^3 + 1\left(\tfrac{1}{2}\right)^4 + \ldots$$

$$\tfrac{1}{2}E[X] = \sum_{n=1}^{\infty}\left(\tfrac{1}{2}\right)^n = \frac{1/2}{1/2} = 1$$

$$\therefore\ E[X] = E[N] = \$2,$$

where N is the value of money a guest will make

Alternatively:

Consider the geometric series: $g(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $-1 < x < 1$

which has a derivative of: $g'(x) = \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$ for $-1 < x < 1$

multiply both sides by x: $g'(x) = \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$ for $-1 < x < 1$

Since $n = X$ and $E[X] = \sum_{x=1}^{\infty} x(1/2)^x$, $E[N] = g'(1/2) = \frac{1/2}{(1/2)^2} = \$2$,

where N is the value of money a guest will make

Exercise 3

Consider a random variable Y with the probability mass function:

$$f(y) = c \cdot \frac{3^y}{y!}, \quad y = 2, 3, 4...$$

a) Find E[Y]. (1.5 points)

b) Find P[Y > 3]. (0.5 points)

c) Find Var[Y]. (2 points)

Hint: Find E[Y(Y-1)].

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

a) $\sum_{y=2}^{\infty} c \cdot \frac{3^y}{y!} = 1 \rightarrow c\left(\sum_{y=0}^{\infty} \frac{3^y}{y!} - 3 - 1\right) = 1 \rightarrow c(e^3 - 4) = 1$

$\therefore \boxed{c = \frac{1}{e^3 - 4}}$

$E[Y] = c\sum_{y=2}^{\infty} y \cdot \frac{3^y}{y!} = c\sum_{y=2}^{\infty} \frac{3^y}{(y-1)!} = 3c\sum_{y=1}^{\infty} \frac{3^y}{y!} = 3c\left(\sum_{y=0}^{\infty} \frac{3^y}{y!} - 1\right)$

$= 3c(e^3 - 1)$

$= 3\left[\frac{e^3 - 1}{e^3 - 4}\right] \approx 3.5595$

b) $P[Y > 3] = 1 - P[Y = 2] - P[Y = 3] = 1 - c\left(\frac{3^2}{2!} + \frac{3^3}{3!}\right)$

$= 1 - 9c = 1 - \frac{9}{e^3 - 4} \approx 0.4405$

c) $E[Y(Y-1)] = \sum_{y=2}^{\infty} y(y-1) \cdot c\frac{3^y}{y!} = c\sum_{y=2}^{\infty} \frac{3^y}{(y-2)!} = 3^2 c\sum_{y=0}^{\infty} \frac{3^y}{y!} = 9e^3 c$

$\hookrightarrow = E[Y^2] - E[Y] \quad \therefore E[Y^2] = 9e^3 c + 3c(e^3 - 1) = 3c(4e^3 - 1)$

so: $Var[Y] = E[Y^2] - (E[Y])^2 = 3c(4e^3 - 1) - 9c^2(e^3 - 1)^2$

≈ 2.1274