# Sample Size (7.4),
# Confidence Intervals for Variance and Standard Deviation

Stat 400 - April 1, 2021

# Today’s topics

Review:

- Chi Squared Distribution, t distribution, CI for mean

New:

- Confidence Interval for variance (or sd)
- Required Sample Size

# Chi-squared distribution (review) $\chi^2$

The Chi-Squared distribution is a special case of the Gamma distribution where $\theta = 2$.

Also, sum of squares of normal distributions

https://homepage.divms.uiowa.edu/~mbognar/applets/chisq.html

# notes

# notes

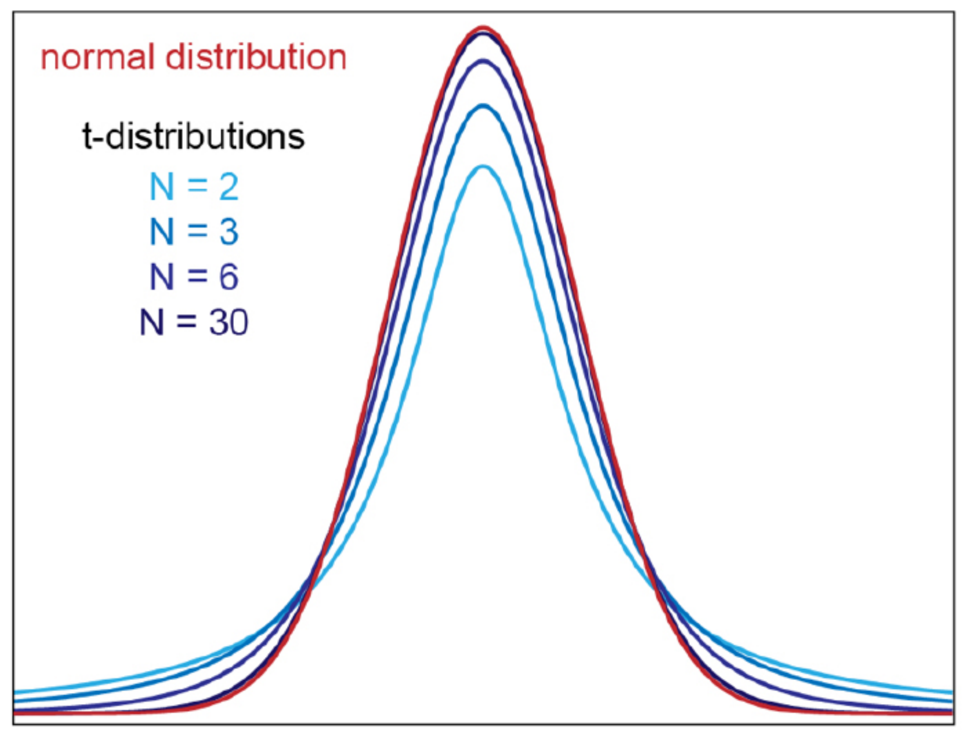
# t distribution:

If $\sigma$ is unknown:

Use $\boldsymbol{s}$ instead of $\sigma$

$$T = \boxed{\frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t_{n-1}}$$

if I "standardize" the sample mean using 's' instead of 'σ'.



# t distribution

**Theorem 5.5-3** **(Student's $t$ distribution)** Let

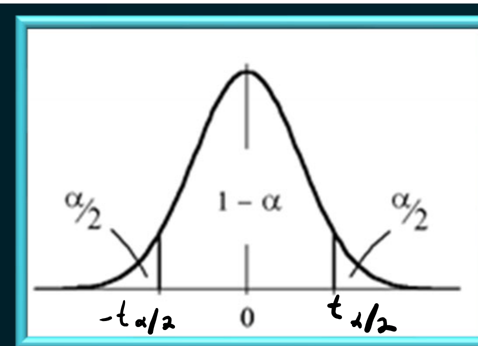
$$T = \frac{Z}{\sqrt{U/r}},$$

where $Z$ is a random variable that is $N(0,1)$, $U$ is a random variable that is $\chi^2(r)$, and $Z$ and $U$ are independent. Then $T$ has a $t$ distribution with pdf

$$f(t) = \frac{\Gamma((r+1)/2)}{\sqrt{\pi r}\,\Gamma(r/2)} \frac{1}{(1+t^2/r)^{(r+1)/2}}, \qquad -\infty < t < \infty.$$

If interested, please refer to textbook for proof. (You are not expected to know how to do it).

# Review: CI for mean




Given the following sample: {16, 12, 18, 13, 21, 15, 8, 17}   $n=8$

Construct a 92% confidence interval for the true mean.

$\bar{x} = 15$,   $s^2 = \frac{\sum(x_i - \bar{x})^2}{n-1} = \frac{112}{7} = 16$,   s=4

$$\bar{x} \pm t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}.$$

$\alpha = 0.08, \alpha/2 = 0.04$

df = n -1 = 7,   $t_{7,\,0.04} = 2.046$

| $x$ | $x-\bar{x}$ | $(x-\bar{x})^2$ |
|---|---|---|
| 16 | 1 | 1 |
| 12 | −3 | 9 |
| 18 | 3 | 9 |
| 13 | −2 | 4 |
| 21 | 6 | 36 |
| 15 | 0 | 0 |
| 8 | −7 | 49 |
| 17 | 2 | 4 |
| | 0 | 112 |

CI: $\mathbf{15 \pm 2.046 \cdot \frac{4}{\sqrt{8}}}$ = (12.107 , 17.893)

# notes

# General Form of Confidence Interval

Estimate $\pm$ (Critical Value * SE of estimate)

e.g. if $\sigma$ is known:

$$\bar{x} \;\pm\; z_{\alpha/2} * \frac{\sigma}{\sqrt{n}}$$

if $\sigma$ is unknown:

$$\bar{x} \;\pm\; t_{n-1,\alpha/2} * \frac{s}{\sqrt{n}}$$

# Required Sample Size

- $\bar{x} \pm z_{\alpha/2} * \frac{\sigma}{\sqrt{n}}$

- $\varepsilon = z_{\alpha/2} * \frac{\sigma}{\sqrt{n}}$

$$\boxed{n = \left[\frac{z_{\alpha/2} * \sigma}{\varepsilon}\right]^2}$$

# Required Sample Size

How many test runs of an automobile are required for determining its average miles-per-gallon rating on the highway to within 0.5 miles per gallon with 99% confidence? Suppose that the variance of the population in mpg$^2$ is 6.25.

- $$\boxed{n = \left[\frac{z_{\alpha/2} * \sigma}{\varepsilon}\right]^2}$$


- $\alpha = 0.01 \quad z_{\alpha/2} = 2.576$

$n =$ $= 165.89$ $\boxed{n = 166}$

Round up!

last example, found $s^2 = 16$ ← point estimate

# Confidence Interval for $\sigma^2$

Now we want to make a confidence interval for $\sigma^2$ based on $s^2$.

- The distribution of $s^2$ is not Normal.
- It can be shown that $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$

will not need to prove this

$\chi$

# Proof of $\frac{(n-1)S^2}{\sigma^2} \backsim \chi^2_{n-1}$

$S^2 = \frac{1}{n-1}\sum (X_i - \bar{X})^2$

Assume
1. $\bar{X}$ (the sample mean) and $S^2$ are independent.
2. If $Z \sim N(0,1)$ then $Z^2 \sim \chi^2(1)$.
3. If $X_i \sim \chi^2(1)$ and the $X_i$ are independent then $\sum_{i=1}^n X_i \sim \chi^2(n)$.
4. A $\chi^2(n)$ random variable has the moment generating function $(1-2t)^{-n/2}$.

With some algebra, you can show, by adding $-\bar{X} + \bar{X}$ inside the parentheses and grouping appropriately, that $\sum_{i=1}^n (X_i - \mu)^2 = \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$. Then, dividing through by $\sigma^2$ yields

$$\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 = \sum_{i=1}^n \left(\frac{X_i - \bar{X}}{\sigma}\right)^2 + \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2.$$

Denote these expressions by $U$, $V$, and $W$, respectively, so that the formula reads $U = V + W$. By facts (2) and (3) above, $U \sim \chi^2(n)$ and $W \sim \chi^2(1)$. Also, $V = \frac{(n-1)S^2}{\sigma^2}$.

Since $\bar{X}$ and $S^2$ are independent, so are $V$ and $W$. Thus $M_U(t) = M_V(t)M_W(t)$, where $M_X(t)$ denotes the moment generating function of the random variable $X$. By fact (4) above, this says that

$$\frac{1}{(1-2t)^{n/2}} = M_V(t)\frac{1}{(1-2t)^{1/2}}.$$

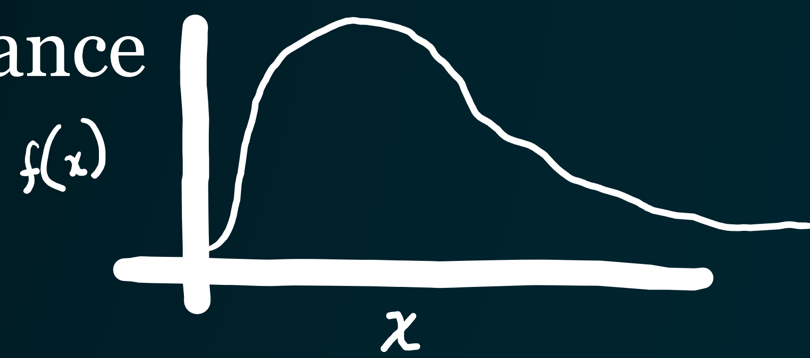
$M_V(t) = \frac{1}{(1-2t)^{(n-1)/2}}$, so $V \sim \chi^2_{(n-1)}$

# notes

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)} \quad \text{e.g., } \chi^2_{(10)}$$

# Derivation of CI for Variance

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$




$$P\left[\chi^2_{n-1,\,1-\alpha/2} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{n-1,\,\alpha/2}\right] = 1-\alpha$$

on the next slide ↓

$$P\left[\frac{\chi^2_{n-1,\,1-\alpha/2}}{(n-1)S^2} < \frac{1}{\sigma^2} < \frac{\chi^2_{n-1,\,\alpha/2}}{(n-1)S^2}\right] = 1-\alpha \Rightarrow \boxed{P\left[\frac{(n-1)S^2}{\chi^2_{n-1,\,\alpha/2}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{n-1,\,1-\alpha/2}}\right] = 1-\alpha}$$

# Confidence Interval for $\sigma^2$

Confidence Interval for $\sigma^2$ : $\left( \frac{(n-1)\cdot s^2}{\chi^2_{\alpha/2}}, \frac{(n-1)\cdot s^2}{\chi^2_{1-\alpha/2}} \right)$

# CI for Variance

Confidence Interval for $\sigma^2$ : $\left( \frac{(n-1)\cdot s^2}{\chi^2_{\alpha/2}}, \frac{(n-1)\cdot s^2}{\chi^2_{1-\alpha/2}} \right)$

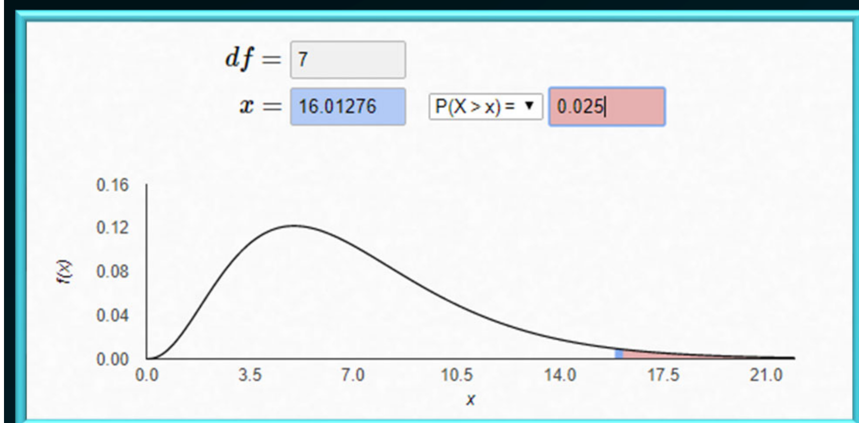
Given the following sample: {16, 12, 18, 13, 21, 15, 8, 17}

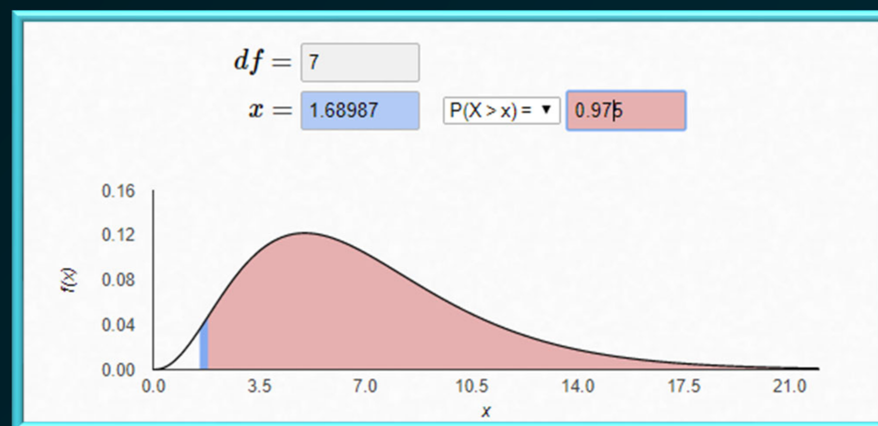
Construct a 95% CI for the true variance.

$s^2=16$ $\alpha = 0.05$ $\alpha/2 = 0.025$ $1-\alpha/2 = 0.975$ df = 7

$\chi^2_{(7,\, 0.025)} = 16.013$ $\chi^2_{(7,\, 0.975)} = 1.690$

$$\left( \frac{(8-1)16}{16.013}, \frac{(8-1)16}{1.690} \right) = \boxed{(6.996, 66.272)}$$



# 1-sided Confidence Intervals for $\sigma^2$

95% confidence lower bound:

$$\boxed{\left(\frac{(n-1)s^2}{\chi^2_{(df,\alpha)}}, \infty\right)}$$

95% confidence upper bound:

$$\boxed{\left(-\infty, \frac{(n-1)s^2}{\chi^2_{(df,1-\alpha)}}\right)}$$

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# notes