Uniform and Exponential Distributions

3.2

Uniform Distribution $X \sim Unif(a, b)$

A random variable, X, has a **uniform distribution** if its pdf is equal to a constant on its support.

$$f(x) = \frac{1}{b-a}, \qquad a \le x \le b.$$

$$E[X] = \frac{a+b}{2}$$

$$Var[X] = \frac{(b-a)^2}{12}$$

$$f(x) = \frac{1}{b-a}, \qquad a \le x \le b$$

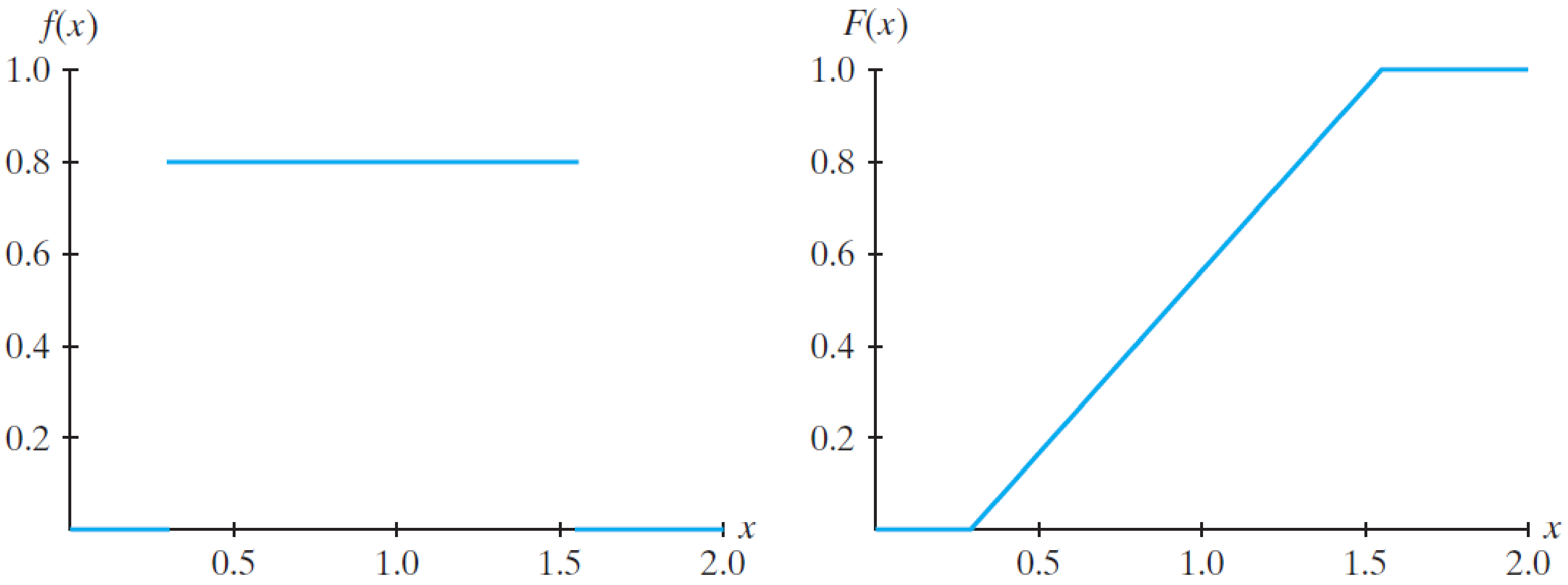

Figure 3.1-1 Uniform pdf and cdf

Exponential Distribution

Consider a Poisson process with an expected number of occurrences, λ, in a given interval.

Let W be the waiting time until the first occurrence. Then W follows an **exponential distribution**.

Instead, if we count the number of these occurrences, X, $X \sim \text{Pois}(\lambda)$.

Exponential Distribution

Given a Poisson Process with rate λ,

# of occurrences in a time of length *w* follows $X \sim Poisson(\lambda w)$.

$P[X = 0] = e^{-\lambda w}$ (no occurrences in an interval of length w)

Let W be the *waiting time* until the first occurrence.

$$\boxed{\begin{aligned} F(w) = P(W \le w) &= 1 - P(W > w) \\ &= 1 - P(\text{no occurrences in } [0, w]) \\ &= 1 - e^{-\lambda w}, \end{aligned}}$$

$$F'(w) = f(w) = \lambda e^{-\lambda w}$$

Exponential Distribution $X \sim Exp(\theta)$

We often parameterize the exponential distribution with $\theta = 1 / \lambda$. If X has an exponential distribution,

$$\boxed{f(x) = \frac{1}{\theta} e^{-x/\theta} \ , \quad 0 \leq x < \infty}$$

Alternatively,

$$f(x) = \lambda e^{-x\lambda} \ , \quad 0 \leq x < \infty$$

Finding E[X] and σ^2 for an exponential

$$M(t) = \int_0^\infty e^{tx}\left(\frac{1}{\theta}\right)e^{-x/\theta}\,dx = \lim_{b\to\infty}\int_0^b \left(\frac{1}{\theta}\right)e^{-(1-\theta t)x/\theta}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{e^{-(1-\theta t)x/\theta}}{1-\theta t}\right]_0^b = \frac{1}{1-\theta t}, \qquad t < \frac{1}{\theta}.$$

$$M'(t) = \frac{\theta}{(1-\theta t)^2} \qquad M''(t) = \frac{2\theta^2}{(1-\theta t)^3}.$$

$$E[X] = M'(0) = \theta$$
$$E[X^2] = M''(0) = 2\theta^2 \qquad Var[X] = \theta^2$$

Memoryless Property

The **exponential** and **geometric** distributions are **memoryless**.

Exponential Distribution

- Radioactive decay
- How long a fly will stay on a table until it takes off?

Geometric Distribution

- How many more times do I need to roll a die until my first success

Memoryless Property

Chloe walks down an infinite hallway of safes.

- Each safe has 1000 possible codes $p = 1/1000$
- Each safe is different, and has a different code
- Chloe only tries one code per safe
- Let X be the number of safes Chloe **still** needs to try before she successfully opens one. $E[X] = 1000$

Memoryless Property

Discrete Memorylessness:

if X is the **total** number of trials required for the first success,

$$P[X > m + n \mid X \geq m] = P[X > n]$$

Continuous Memorylessness:

if X is the **total** time required for the first success,

$$P[X > t + s \mid X \geq t] = P[X > s]$$

Memoryless Property

*Most phenomena are **not** memoryless. We generally obtain and update information over time.

Examples **(<u>not memoryless</u>)**:

Let X be a random variable that describes...

- A car engine's remaining life (how many miles it has left).
- How many more miles can Albert run if he has run m miles.
- The amount of time left until class ends.

Examples

Uniform & Exponential

Example 1 (textbook)

Example 3.2-2 Customers arrive in a certain shop according to an approximate Poisson process at a mean rate of 20 per hour. What is the probability that the shopkeeper will have to wait more than 5 minutes for the arrival of the first customer? Let X denote the waiting time in minutes until the first customer arrives, and note that $\lambda = 1/3$ is the expected number of arrivals per minute. Thus,

$$\theta = \frac{1}{\lambda} = 3$$

$$f(x) = \frac{1}{3}e^{-(1/3)x}, \qquad 0 \le x < \infty.$$

$$P(X > 5) = \int_5^\infty \frac{1}{3}e^{-(1/3)x}\,dx = e^{-5/3} = 0.1889.$$

Example 2

Once you arrive at a stoplight, it takes between 0 and 30 seconds for you to hear: "*Walk sign is on to cross Green St.*"

Assume that the time it takes, X, follows a uniform distribution.

What is the probability that it takes between 10 and 20 seconds for you to hear this message? (1/3)

What are E[X] and Var[X]? (15), $\left(\frac{30^2}{12}\right)$

Example 3

Suppose the length of time, X, that it takes someone to find their phone follows an exponential distribution with mean = 5 sec.

- A) Give the pdf and support (sample space) of X.
- B) What is the probability that it will take someone more than 10 seconds to find their phone? (0.135)
- C) Suppose this person will give up after 10 seconds. Assume that they try twice in afternoon. Let the random variable Y be the number of times they find their phone. Calculate $f_Y(1)$. (0.233)

Example 4

Suppose an electronic component has a lifespan which can be modeled as an exponential distribution, with mean = 500 hours.

A) Find the pdf and cdf of this distribution

B) Find P[X > x] ($e^{-x/500}$)

C) If this component has already lasted 200 hours, find the probability that it will last at least 600 hours total. ($e^{-4/5}$)