2.4 The Hypergeometric & Multinomial Distributions

- Bernoulli Binomial Geometric Negative Binomial

trials until S

1st S

r th

$N = 30$ 10 red 20 blue $\frac{10}{30}$

interested in red $N_1 = 10$

Hypergeometric Distribution

e.g. pick 5 marble

Out of a population of size N, suppose we have N_1 successes and N_2 failures.

$n = 5$

(note, $N_1 + N_2 = N$, the probability of a success, $p = N_1 / N$)

Define a random variable X:

the number of successes in a random sample of size n.

dependent

independent

If sampling is done without replacement, X follows a **hypergeometric distribution.**

if with replacement

binomial

$N=30$ $\quad n=5$ $\quad \frac{\binom{10}{x}\binom{20}{5-x}}{\binom{30}{5}}$

$N_1=10$

total succ sample size

Hypergeometric Distribution

prob exact 2 red?

$$X \sim Hypergeom(N, N_1, n)$$

(remember: $N = N_1 + N_2$)

$$f(x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}, \qquad x \le n,\ x \le N_1,\ n - x \le N_2$$

- $E[X] = n\frac{N_1}{N}$ $\qquad n \cdot p$
- $Var[X] = n\frac{N_1}{N}\frac{N_2}{N}\frac{N-n}{N-1}$

$x=2$ $\qquad f(2) = \frac{\binom{10}{2}\binom{20}{5-2}}{\binom{30}{5}} =$

dep indep

w/o rep w rep

$\binom{20}{x} p^{x} (1-p)^{20-x}$

Hypergeometric vs Binomial

If instead, sampling is done one at a time with replacement, X ~ Binomial(n,p) e.g.

$p = .3$

- **Binomial:** A magical beer machine gives the user a stout 30% of the time and an IPA 70% of the time. Let X be the number of stouts you get out of 20 beers.

  $X \sim Binom(n=20, p=.3)$

- **Hypergeometric:** A nice minibar has 9 stouts and 21 IPAs. Let X be the number of stouts you get if you randomly select 20 beers.

  don't replace

  - What is the pmf of X? $f(x) = \frac{\binom{9}{x}\binom{21}{20-x}}{\binom{30}{20}}, \quad x \le 9$

  typo

- Roll a die: independent

- Drawing from a bag: replace -- > independent
- Not replacing → dependent

bi

Multinomial Distribution

Similar to binomial distribution, but for more than 2 groups. E.g.

- Color – Red/Green/Blue
- Your Major – Stats/Math/Engineering/Other

Stat / not Stat $\rightarrow$ binomial

$$\frac{10!}{2!\,2!\,6!}$$

$(.1)(.1)(.2)(.2)(.7)(.7)(.7)(.7)(.7)(.7)$

Multinomial Distribution

$$X = (X_1, X_2, \ldots X_k) \sim Multinomial(n, p_1, p_2, \ldots, pk)$$

$\binom{n}{x} = \frac{n!}{x!(n-x)!}$

$$f(x_1, x_2, \ldots x_k) = \frac{n!}{x_1!x_2!\ldots x_k!} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k}$$

indep

- $E[X_i] = np_i$
- $Var[X_i] = np_i(1\text{-}p_i)$

Examples

2.4

1 A deck of cards contains 20 cards: 6 red cards and 14 black cards. 5 cards are drawn randomly *without replacement*.

$X \sim HG(20, 6, 5)$

$N=20,\ N_1=6,\ n=5$

What is the probability that exactly 4 red cards are drawn?

$$f(4) = \frac{\binom{6}{4}\binom{14}{1}}{\binom{20}{5}}$$

$X \sim HG(N=20,\ N_1=14,\ n=5)$

What is the probability that at least 2 black cards are drawn?

$$P[X \geq 2] = 1 - \underbrace{P[X \leq 1]}_{F(1)} = 1 - f(0) - f(1)$$

$$= 1 - \frac{\binom{14}{0}\binom{6}{5}}{\binom{20}{5}} - \frac{\binom{14}{1}\binom{6}{4}}{\binom{20}{5}}$$

2 Suppose the majors of students taking Stat 400 can be broken down as follows:

X_1	X_2	X_3
Math	**Statistics**	**Other**
10% p_1	20% p_2	70% p_3

$n=10$

Out of 10 randomly sampled students, calculate the probability that this group contains:

- A) 2 Math, 2 Stats, and 6 Other $\Rightarrow P[X_1=2,\ X_2=2,\ X_3=6]$ (x_1, x_2, x_3)
- B) At least one Stats student ($p=.2$)

$$\boxed{\frac{10!}{2!\,2!\,6!}\,.1^2\,.2^2\,.7^6}$$

$\rightarrow X \sim \text{Binom}(10, .2)$

$P[X \geq 1] = 1 - P[X=0]$ = 1 – .8^10 = .8926

- $E[X_i] = np_i$

3 When Iron Man and Captain America play 

against each other, Iron Man wins 40% of the time, loses 35% of the time and draws 25% of the time. Assume results of games are independent.

$p_3 = .25$

$n=12$

If they play 12 games, what is the probability that Iron Man wins 7, loses 2, and draws 3 games?

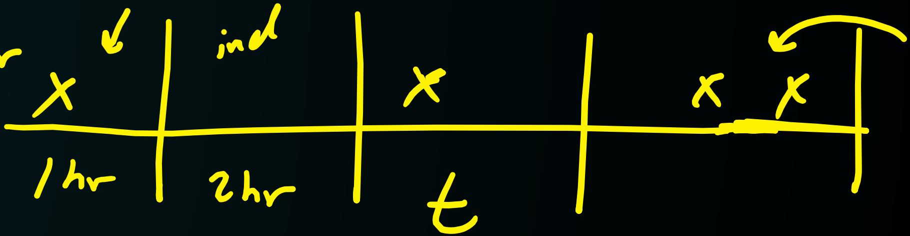
$$f(x_1, x_2, \dots x_k) = \frac{n!}{x_1!x_2!\dots x_k!} p_1^{x_1}\, p_2^{x_2} \dots p_k^{x_k} = \frac{12!}{7!2!3!} \times .40^7 \times .35^2 \times .25^3 = 0.0248$$

If they play 12 games, what is the expected value of the number of games that they will tie?

$np_3 = 12(.25) = 3$

2.6 The Poisson Distribution

$\boxed{\lambda = 10}$ / min $\boxed{\lambda = 600}$ /hr



Poisson Process

Definition 2.6-1

Let the number of occurrences of some event in a given continuous interval be counted. Then we have an **approximate Poisson process** with parameter $\lambda > 0$ if the following conditions are satisfied:

(a) The numbers of occurrences in nonoverlapping subintervals are independent.

(b) The probability of exactly one occurrence in a sufficiently short subinterval of length h is approximately λh.

(c) The probability of two or more occurrences in a sufficiently short subinterval is essentially zero.

Poisson Process Examples

- # of cell phone calls passing through a relay tower between 9 and 11 a.m.
- Number of customers that show up to Oberweis between 5-6pm.
- Number of defects in a 100-foot roll of aluminum screen that is 2 feet wide.

X X X X X X

t

Poisson Distribution

$$\sum_{x=0}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{x!} = E[X]$$

$X \sim Poisson(\lambda = 5)$

$$\boxed{X \sim Poisson(\lambda)}$$

rate

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0,1,2,\ldots$$

$$f(3) = \frac{5^3 e^{-5}}{3!} = 0.14$$

- $E[X] = \boxed{\lambda}$
- $Var[X] = \boxed{\lambda}$

$P[X \le 8] = ?$

ppois(8,5) = 0.931

dpois(3,5) = 0.14

Note: λ is the Poisson rate.

Poisson Parameter Scaling

If events occur according to a Poisson process with rate λ, then the rate for a Poisson process in an interval of length t is λt. e.g. :

$\lambda = 3$

minute

Every minute, cars pull up to a drive-through according to a Poisson process with rate $\lambda = 3$.

- In an interval of length 1 hour, the rate is $\lambda = 180$.

3 * 60 (minutes in an hour) = 180

Examples

2.6

in R: d → density
pmf
pdf

p → prob. (cumulative)
cdf

2

Suppose Albert receives spam mail every day according to a Poisson Process with an expected value of 10.

What is the distribution of X? $X \sim Poisson(\lambda = 10)$

$P[X = 8]$ $E[X]$

What is the probability that Albert receives 8 items of spam in a given day? $f(8) = \frac{e^{-10} 10^8}{8!} = 0.112599$

What is the probability that Albert receives 10 items of spam in a given day? $f(10) = \frac{e^{-10} 10^{10}}{10!} = 0.12511$

year: $\lambda = 3650$

2 Suppose Albert receives spam mail every day according to a Poisson Process with an expected value of 10.

$X \sim Pois(10)$

$\boxed{\lambda = 10}$

$\lambda \to \lambda t$

~~Find P[Albert receives 10 items of spam in a given day].~~

hour

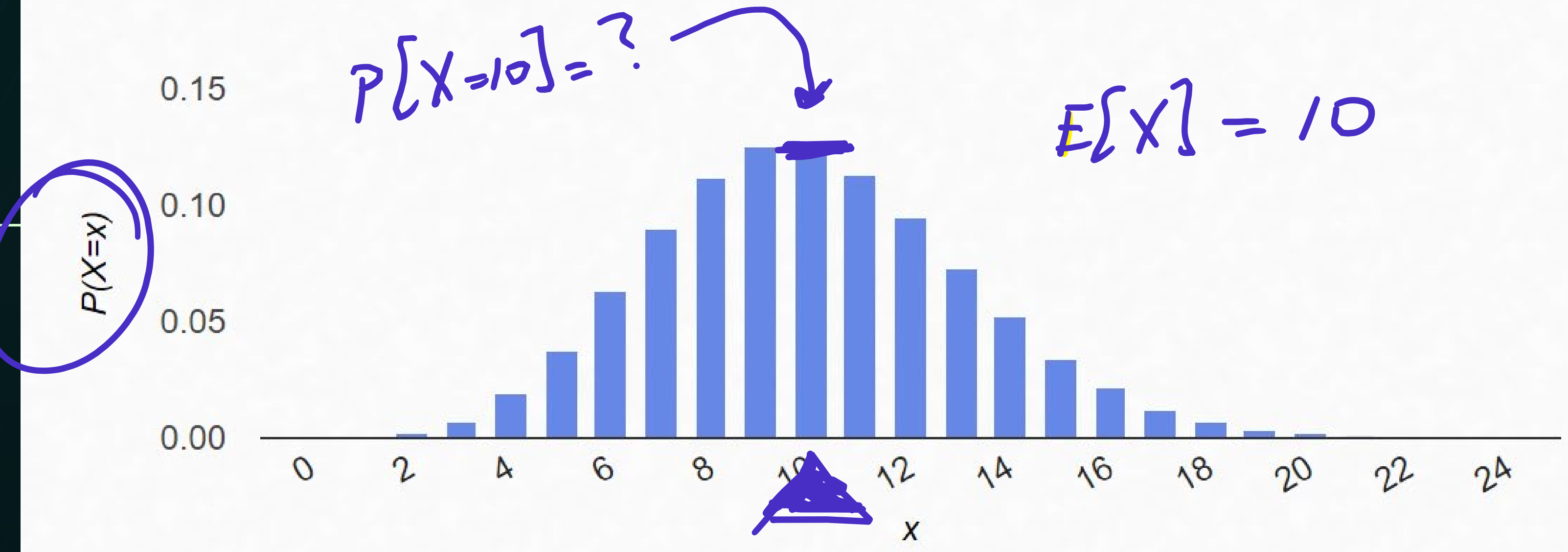
$t = \frac{1}{24}$

Find P[0 items of spam in a given day]? $\underline{f(0)} = \frac{e^{-10}\, \cancel{10}^{0} {}^{1}}{\cancel{0}!} = \underline{e^{-10}}$

Find P[Albert receives 1 item of spam in a given $\boxed{\text{hour}}$]?

$X \sim Pois\left(\lambda = \frac{10}{24}\right) \leftarrow X \sim Pois(\underline{.416\bar{6}})$

$f(1) = \frac{e^{-.4166}\, .4166^{1}}{1!} = .2747$



$\mu = E(X) = 10 \quad \sigma = SD(X) = 3.162 \quad \sigma^2 = Var(X) = 10$