Spring 2021 STAT400 Homework 5 Solutions (TIGHT TIGHT TIGHT!)

Exercise 1

Let X be a Normal random variable that has moment generating function

$$M_X(t) = e^{t+t^2}$$

(1.5 points) Find $P[-2 < X < 1]$.

MGF for a Normal Random Variable:

$$M_X(t) = e^{\mu t + \sigma^2 t^2/2}$$

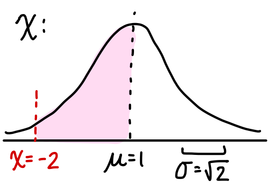
$$\therefore \mu = 1, \ \sigma^2 = 2$$

Thus, $P[-2 < X < 1] = P\left[\frac{-2-1}{\sqrt{2}} < Z < \frac{1-1}{\sqrt{2}}\right] = P\left[\frac{-3\sqrt{2}}{2} < Z < 0\right]$

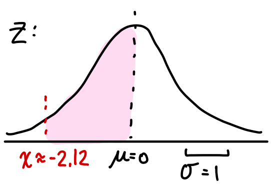
$$= P[Z<0] - P\left[Z < \frac{-3\sqrt{2}}{2}\right]$$

$$= 0.50 - 0.01695 \approx \boxed{0.4831}$$

X:



Z:



Alternatively:

TI-84 : 2nd → Vars → normalcdf(lower = $\frac{-3\sqrt{2}}{2}$, upper = 0, $\mu=0, \sigma=1$)

R: pnorm(x=0) − pnorm(x=-3/sqrt(2))

Exercise 2

Let T denote the time it takes for a computer to shut down. Suppose T follows an Exponential distribution with mean 15 seconds. A computer lab has 10 independent computers that must all be shut down at the end of the day.

a) (0.5 points) What is the probability that it takes any given computer **at least** 10 seconds to shut down?

b) (0.5 points) What is the probability that it takes any given computer at least 1 minute to shut down?

c) (0.5 points) What is the probability that all 10 computers successfully shut down in under a minute?

Given: $T \sim$ Exponential ($\theta = 15$ seconds) $\rightarrow f(x) = \frac{1}{15} e^{-x/15}$

a) $P[T \geq 10 \text{ seconds}] = 1 - \int_0^{10} \frac{1}{15} e^{-x/15}\,dx = e^{-10/15} = \boxed{1/e^{2/3}} \approx \boxed{0.5134}$

b) $P[T \geq 1 \text{ minute}] = P[T \geq 60 \text{ seconds}] = e^{-60/15} = \boxed{1/e^4} \approx \boxed{0.01832}$

c) $P[1 \text{ computer} < 60 \text{ seconds}] = 1 - P[T \geq 1 \text{ minute}] = 1 - 1/e^4$

Let X be # of computers that shut down in under 60 seconds

$P[X=10] = (1 - 1/e^4)^{10} = (1 - 1/e^4)^{10} \approx \boxed{0.8312}$

Exercise 3

Let $\theta > 0$. Suppose X has a uniform distribution on the interval $(\theta, 2\theta)$.

a) (0.5 points) What is $E[X]$? (in terms of θ)

b) (0.5 points) Find an expression for $Var[X]$.

c) (1 point) Assume that there is a 20% chance that $\theta = 2$, and a 80% chance that $\theta = 2.5$. Based on this information and the above Uniform distribution, What is $P[0 < X < 4]$?

Given: $X \sim \text{Uniform}(a = \theta,\ b = 2\theta)$

a) $E[X] = \mu = \dfrac{a+b}{2} = \dfrac{\theta + 2\theta}{2} \rightarrow \boxed{E[X] = \frac{3}{2}\theta}$

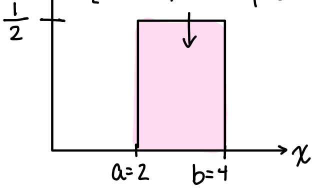
b) $Var[X] = \sigma^2 = \dfrac{(b-a)^2}{12} \rightarrow \boxed{Var[X] = \dfrac{\theta^2}{12}}$

c) $P[0<X<4] = P[0<X<4 \cap \theta = 2] + P[0<X<4 \cap \theta = 2.5]$

@ $\theta = 2$, $X \sim \text{Uniform}(a=2, b=4)$

$P[\theta = 2] = 0.20$

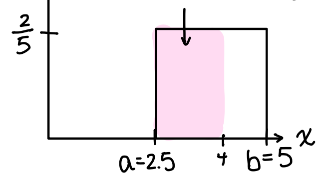
$P[0<X<4 \mid \theta = 2] = 1$



@ $\theta = 2.5$, $X \sim \text{Uniform}(a=2.5, b=5)$

$P[\theta = 2.5] = 0.80$

$P[0<X<4 \mid \theta = 2.5] = \dfrac{4-2.5}{5-2.5} = \dfrac{3}{5} = 0.60$



$P[0<X<4] = P[0<X<4 \mid \theta = 2] \cdot P[\theta = 2] + P[0<X<4 \mid \theta = 2.5] \cdot P[\theta = 2.5]$

$P[0<X<4] = 1 \cdot 0.20 + 0.60 \cdot 0.80 = \boxed{\dfrac{17}{25}} = \boxed{0.68}$

Exercise 4

Suppose Walter White runs into Tuco Salamanca according to a Poisson process with an average of 1.5 run-ins per day. Assume that the week starts on Sunday at midnight (00:00).

Hint: Sunday is equivalent to time, T in (0<T<1)

a) (0.5 points) Walter is trying to avoid Tuco. What is the probability that he does not run into Tuco next week?

b) (0.5 points) What is the probability that he runs into Tuco **before** (not including) Wednesday for the first time? (i.e. Sunday/Monday/Tuesday)

c) (0.5 points) What is the probability that Walter has his third run-in with into Tuco on either Thursday or Friday? $(i.e., Thursday \cup Friday)$

d) (0.5 points) What is the probability that the 6th run-in occurs within the second week?

Given: $\lambda = 1.5$ run-ins/day

a) Regardless of 0-7 or 7-14, it is still just 7 days.

$P[0 \text{ run-ins in } w \text{ days}] = e^{-\lambda w}$

$P[0 \text{ run-ins in } 7 \text{ days}] = e^{-1.5 \cdot 7} = 1/e^{10.5} \approx 2.7536 \times 10^{-5}$

b) $P[T<3] = \int_0^3 1.5e^{-1.5x}\,dx = -e^{-1.5x}\Big]_0^3 = 1 - 1/e^{4.5} \approx 0.9889$

c) $\lambda = 1.5$ run-ins/day $\rightarrow \theta = \frac{2}{3}$, $\alpha = 3$

Thursday = (4,5), Friday (5,6)

$P[3^{rd} \text{ run-in on Thursday or Friday}] = \int_4^6 \frac{1}{\Gamma(3)(\frac{2}{3})^3} x^{3-1} e^{-1.5x}\,dx$

$= \frac{27}{16}\int_4^6 x^2 e^{-1.5x}\,dx$

Integration by Parts: $u = x^2$, $dv = e^{-1.5x}$

$du = 2x \quad v = -\frac{2}{3}e^{-1.5x}$

$P[3^{rd} \text{ run-in on Thursday or Friday}] = \frac{27}{16}\left[-\frac{2}{3}x^2e^{-1.5x} + \frac{4}{3}\underbrace{\int_4^6 xe^{-1.5x}\,dx}\right]$

$u = x \quad v = -\frac{2}{3}e^{-1.5x}$

$du = 1 \quad dv = e^{-1.5x}$

$= \frac{27}{16}\left[-\frac{2}{3}x^2e^{-1.5x} + \frac{4}{3}\left[-\frac{2}{3}xe^{-1.5x} + \frac{2}{3}\int_4^6 e^{-1.5x}\,dx\right]\right]$

$= \frac{27}{16}\left[-\frac{2}{3}x^2e^{-1.5x} + \frac{16}{27}\left[-1.5xe^{-1.5x} - e^{-1.5x}\right]\right]_4^6$

$P[3^{rd} \text{ run-in on Thursday or Friday}] = \frac{27}{16}[-0.003693 + 0.03672] \approx 0.05574$

d) $P[6^{th} \text{ run-in during } 7\text{-}14] = \int_7^{14} \frac{1}{\Gamma(6)(\frac{2}{3})^6} x^{6-1} e^{-1.5x}\,dx \approx 0.05035$

Exercise 5 (Refers to Exercise 4c)

a) **Write a function in R** that will calculate the probability that the kth run-in will occur on either Thursday or Friday. This function should take 2 arguments, (k, the kth run-in, and λ), the Poisson rate), and return the probability that this run-in occurs on Thursday or Friday.

```
gamma_thurs_fri_func = function(k, lambda = 1.5) {
  pgamma(6, shape = k, rate = lambda) - pgamma(4, shape = k, rate = lambda)
}
```

b) Use the function you wrote to calculate the probability that the 3rd run-in will occur on either Thursday or Friday.

```
gamma_thurs_fri_func(k = 3)

## [1] 0.05573661
```