Various Topics: Tues Mar 16, 2021

- “independent and identically distributed”
- Variance and Covariance Properties
- Chebyshev and Markov’s Inequalities
- Weak Law of Large Numbers
- Review problems – CLT, Normal Distributions

Чебышёв

i.i.d.

n R.V.s CLT

If we have multiple random variables that come from exactly the same distribution and are (mutually) independent, we can say that these random variables are **i.i.d**.

"iid" means "**independent** and **identically distributed**" e.g.

- $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{f(x)}$
- $X_1, X_2, \ldots, X_n \overset{iid}{\sim} Poisson(4)$
- $X_1, X_2, \ldots, X_n \overset{iid}{\sim} Bernoulli(.2)$

$X_1, X_2, \ldots X_n \overset{iid}{\sim} N(1200, 150^2)$

~~Markov's Inequality~~ $x_1, x_2, \dots x_n$ $\mathcal{X}$ $\mathcal{S}$

Markov's inequality: for any nonnegative random variable X, and for any $t > 0$,

P Pr

$$\to \underline{\mathbf{Pr}[X \geq t]} \leq \frac{\mathbf{E}[X]}{t}.$$

Discrete

Proof: say x can take values $x_1 < x_2 < \dots \boxed{x_j = t} < \dots x_n$

$$\to \underbrace{E[x]} = \sum_{i=1}^{n} x_i \cdot f(x_i) \geq \sum_{i=j}^{n} \boxed{x_i} \cdot f(x_i) \geq \sum_{i=j}^{n} \boxed{t} \cdot f(x_i)$$

$$= \underbrace{t \sum_{i=j}^{n} f(x_i)}_{t \cdot P[X \geq t]}$$

Continuous proof is similar to this, but with integrals

$$E[x] \geq t \cdot P[X \geq t]$$

Markov's inequality: for any nonnegative random variable X, and for any $t > 0$,

$$\boxed{\mathbf{Pr}[X \geq t] \leq \frac{\mathbf{E}[X]}{t}.}$$

let $t = s \cdot E[X]$:

Markov's

$$\mathbf{Pr}[X \geq \underline{s \cdot \mathbf{E}[X]}\,] \leq \frac{1}{s}. \quad ①$$

Chebyshev's Inequality

① $P[Y \geq \underline{s} \cdot E[Y]] \leq \frac{1}{\underline{s}}$ let $s = k^2$:

② $P[\underline{Y} \geq \underline{k^2} \cdot E[Y]] \leq \underline{1/k^2}$ let $\underline{Y} = \underline{(X-\mu)^2}$

③ $P[\underline{(X-\mu)^2} \geq k^2 \cdot \underbrace{E[X-\mu]^2}] \leq \frac{1}{k^2}$

σ_x^2

$\underline{k^2 \sigma^2}$

For any random variable **X**, and scalars {*t, a*} ∈ R with t > 0:

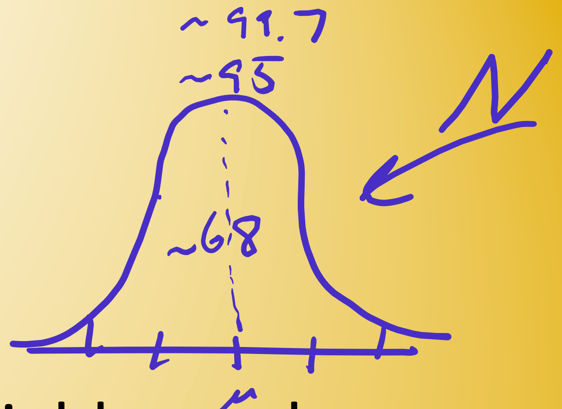
$$\mathbf{Pr}[\,\underline{|X-a|} \geq \underline{t}\,] = \mathbf{Pr}[\,\underline{(X-a)}^2 \geq t^2]$$

$$\boxed{P[\underline{|X-\mu|} > \underline{k\sigma}] \leq \frac{1}{k^2}}$$

Chebyshev's Inequality

Chebyshev Example

$$\boxed{P[\,|X-\mu| > k\sigma\,] \le \frac{1}{k^2}}$$



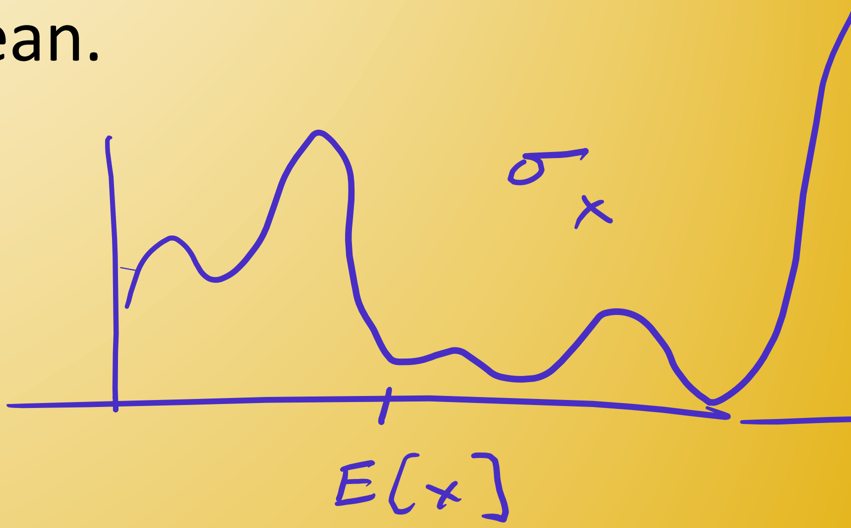
Ex. Find an **upper bound** for the probability that a random variable can be more than 2 standard deviations away from its mean.

$k=2$

$$P[\,|X-\mu| > 2\sigma\,] \le \frac{1}{2^2} = \frac{1}{4}$$

$$P[\,|X-\mu| > 3\sigma\,] \le \frac{1}{3^2} = \frac{1}{9}$$

$$P[\,|X-\mu| < 3\sigma\,] \ge 1 - \frac{1}{9} = \quad \ge \frac{8}{9}$$



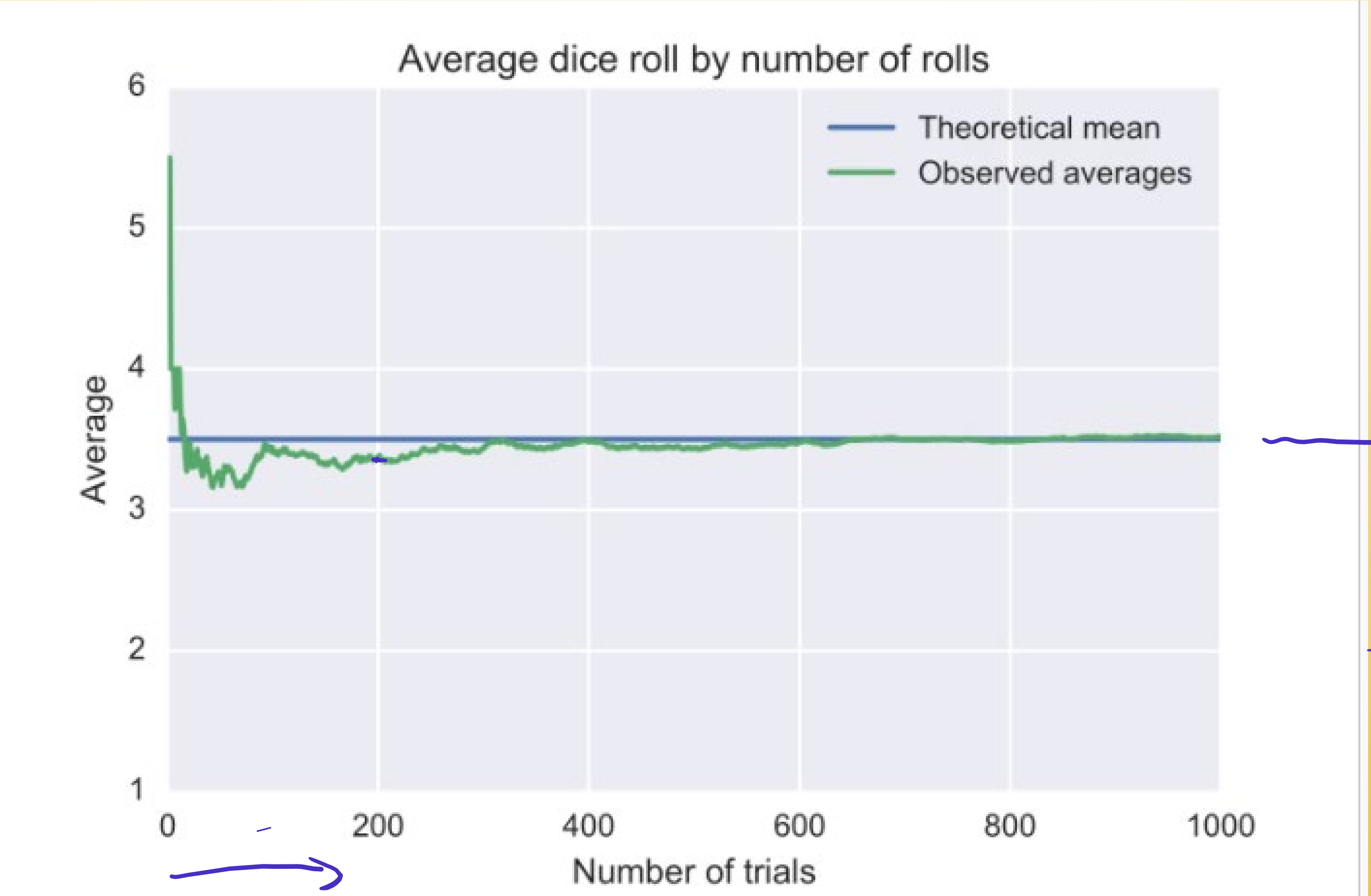
notes

in probabity

Weak Law of Large Numbers (WLLN)

$$\overline{X}_n \xrightarrow{P} \mu \qquad \text{when } n \to \infty.$$

$\overline{X} \xrightarrow{P} \mu$



X

$E[X] = 3.5$

1
2
3
4
5
6

$1/6$

https://en.wikipedia.org/wiki/Law_of_large_numbers if $Var[X] = \sigma^2$,

Weak Law of Large Numbers (WLLN)

$\boxed{\overline{X}_n} \xrightarrow{P} \boxed{\mu}$ when $n \to \infty$.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with a finite expected value $EX_i = \mu < \infty$. Then, for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(|\overline{X} - \mu| \geq \epsilon) = 0.$$

eg. 0.000001

converge in prob

σ^2/n

$$P[|X-\mu| > k\sigma] \leq \frac{1}{k^2}$$

Let $\varepsilon = k\sigma$. Then, $k = \varepsilon/\sigma$

$$P[|X-\mu| > \varepsilon] \leq \frac{\sigma^2}{\varepsilon^2}$$

$$\frac{1}{(\varepsilon/\sigma)^2}$$

$$\boxed{Var[\bar{X}] = \frac{\sigma^2}{n}}$$

$$P(|\overline{X}_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}.$$

$n \to 0$

More Examples (old material)

Variance & Covariance

Functions of Normal Distribution

Central Limit Theorem

Variance Examples (assuming independence)

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathrm{N}(\mu_X, \sigma_X^2)$.

Find in terms of σ_X^2:

1. $\mathrm{Var}[X_1 + X_2]$

$= \mathrm{Var}[(1)X_1 + (1)X_2]$

$= (1)^2\mathrm{Var}[X_1] + (1)^2\mathrm{Var}[X_2]$

$= \mathrm{Var}[X_1] + \mathrm{Var}[X_2]$

$= \sigma_X^2 + \sigma_X^2$

$= 2\sigma_X^2$

iid

indep $\rightarrow$ cov $= 0$

2. $\mathrm{Var}[X_1 + X_2 + X_3]$

$= \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \mathrm{Var}[X_3]$

$= \sigma_X^2 + \sigma_X^2 + \sigma_X^2$

$= 3\sigma_X^2$

$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$

if $X \perp Y$

Variance Examples

iid

Let $X_1, X_2, \ldots X_n \overset{iid}{\sim} \mathrm{N}(\mu_X, \sigma_X^2)$.

Find in terms of σ_X^2:

3. $\mathrm{Var}[X_1 - X_2]$

$= \mathrm{Var}[(1)X_1 + (-1)X_2]$

$= (1)^2\mathrm{Var}[X_1] + (-1)^2\mathrm{Var}[X_2]$

$= \mathrm{Var}[X_1] + \mathrm{Var}[X_2]$

$= \sigma_X^2 + \sigma_X^2 = 2\sigma_X^2$

4. $\mathrm{Var}[X_1 + X_2 + \ldots + X_n] = n\sigma_X^2$

5. $\mathrm{Var}[3X_1] = (3)^2\mathrm{Var}[X_1] = 9\sigma_X^2$

6. $\mathrm{Var}[3X_1 - 2X_2 - X_3]$

$= \mathrm{Var}[(3)X_1 + (-2)X_2 + (-1)X_3]$

$= (3)^2\mathrm{Var}[X_1] + (-2)^2\mathrm{Var}[X_2] + (-1)^2\mathrm{Var}[X_3]$

$= 9\sigma_X^2 + 4\sigma_X^2 + 1\sigma_X^2$

$= 14\sigma_X^2$

Variance Examples

Let $X_1, X_2, \ldots X_n \overset{iid}{\sim} \mathrm{N}(\mu_X, \sigma_X^2)$.

Find in terms of σ_X^2:

7. $\mathrm{Var}[aX_1 + \mathrm{b}X_5 - \mathrm{c}X_2]$

$= (\mathrm{a})^2\mathrm{Var}[X_1] + (\mathrm{b})^2\mathrm{Var}[X_5] + (\text{-c})^2\mathrm{Var}[X_2]$

$= \mathrm{a}^2\sigma_X^2 \quad + \quad \mathrm{b}^2\sigma_X^2 \quad + \quad \mathrm{c}^2\sigma_X^2$

$= (\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2)\,\sigma_X^2$

$\boxed{8.\ \mathrm{Var}[\bar{X}]}$

$= \mathrm{Var}\left[\frac{1}{n}(X_1 + X_2 + \cdots + X_n)\right]$

$= \mathrm{Var}\left[\left(\frac{1}{n}X_1 + \frac{1}{n}X_2 + \cdots + \frac{1}{n}X_n\right)\right]$

$= \frac{1}{n^2}\mathrm{Var}[X_1] + \frac{1}{n^2}\mathrm{Var}[X_2] + \ldots + \frac{1}{n^2}\mathrm{Var}[X_2]$

$= \frac{1}{n^2}\underbrace{(\mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \ldots + \mathrm{Var}[X_n])}_{n \text{ terms}}$

$= \frac{1}{n^2}(\mathrm{n}\,\sigma_X^2)$

$= \frac{\sigma_X^2}{n}$ ← important

Covariance Practice

Find in terms of $\sigma_X^2,\ \sigma_Y^2,\ and\ \ \sigma_{XY}$:

1. Var[X+Y] = Cov[X+Y, X+Y]

= Cov[X,X]+Cov[X,Y]+Cov[Y,X]+Cov[Y,Y]
= σ_X^2 + $2\sigma_{XY}$ + σ_Y^2

2. Var[2X -3Y] = Cov[2X + (-3)Y, 2X + (-3)Y]

= Cov[2X, 2X] + Cov[2X, -3Y] + Cov[-3Y, 2X] + Cov[-3Y, -3Y]

= (2)(2)Cov[X,X] + 2(2)(-3)Cov[X,Y] + (-3)(-3)Cov[Y, Y]

= $4\sigma_X^2$ $-12\sigma_{XY}$ $+9\sigma_Y^2$

Covariance Practice

Find in terms of $\sigma_X^2, \sigma_Y^2, \sigma_{XY}$:

3. Cov[3X-Y, 2X + 5Y]

= Cov[3X,2X] + Cov[3X, 5Y] + Cov[-Y, 2X] + Cov[-Y, 5Y]

= $6\sigma_X^2$ + $15\sigma_{XY}$ + $-2\sigma_{XY}$ + $-5\sigma_Y^2$

= $6\sigma_X^2 + 13\sigma_{XY} - 5\sigma_Y^2$

$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ $\frac{1}{9}$ $\sigma_{\bar{x}} = \frac{1}{3}$

Normal Example

$N(\mu, 1)$

A bartender has Guinness on tap, and is instructed to fill glasses up to an average of μ ounces per glass. The amount that she fills is normally distributed with $\mathbf{mean} = \boldsymbol{\mu}$ ounces and $\boldsymbol{\sigma} = \mathbf{1.0}$ ounces.

Mr. Kam, the owner, randomly selects a sample of 9 glasses and measures the amount of Guinness in each glass. Find the probability that the sample mean will be within 0.3 ounces of $\boldsymbol{\mu}$.

$n = 9$

$P[\mu - 0.3 < \bar{X} < \mu + 0.3]$

P[-0.9 < Z < 0.9] = 1 – 2(0.1841) = **0.6318**

- What if n = 16? n = 49? n = 100?

$= P\left[\frac{\mu - 0.3 - \mu}{1/3} < Z < \frac{\mu + 0.3 - \mu}{1/3}\right] = P[-0.9 < Z < 0.9]$

notes

Normal Example

$W = Y - X$

$W \sim N(4-5,\ 5)$

$\boxed{\sim N(-1,\ 5)}$

$P[W > 0]$

$= P\left[Z > \frac{0 - -1}{\sqrt{5}}\right]$

Let $X \sim N(5,4)$

$Y \sim N(4,1)$

Find P[Y > X]? $= P[\underbrace{Y - X}_{} > 0]$

$P[Z > \frac{1}{\sqrt{5}}] = P[Z > 0.447] =$ **0.327**

Find P[Y > 2X]? $= P[Y - 2X > 0]$

$Q = Y - 2X$

$P[Z > \frac{6}{\sqrt{17}}] = P[Z > 1.455] =$ **0.0728**

$Q \sim N\left(4 - (2)(5),\ 1 + (-2)^2(4)\right) \sim N(-6,\ 17)$

CLT Example

$\bar{X} = 58$

$\mu_x = 60$

$\sigma_x^2 = 64$

Continuous

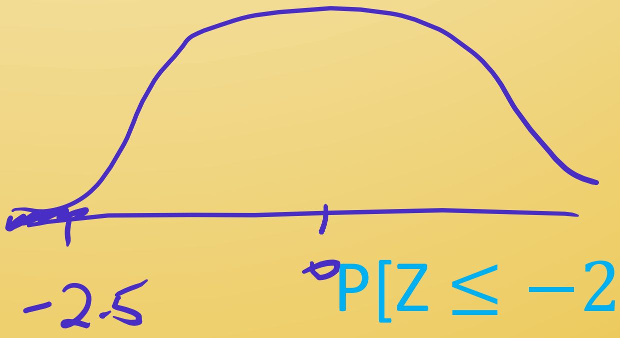
Suppose that achievement test scores of all college freshmen Illinois have a mean of 60, and variance 64. A random sample of 100 students from Pirdew had a mean score of 58. What is the probability that the mean score would be 58 or less if Pirdew is on par with other schools (i.e. if Pirdew has mean = 60).

$$\bar{X} \sim N\left(60, \frac{64}{100}\right) \qquad \sigma_x = \frac{8}{10}$$

$$P[\bar{X} \le 58] = ?$$

$$P\left[\underbrace{\frac{\bar{X} - 60}{8/10}} \le \frac{58 - 60}{8/10}\right]$$

$$P[Z \le -2.5]$$



$P[Z \le -2.5] = 0.0062$ ~~Yes~~

notes

n large, iid

by CLT,

$\bar{X} \sim N(\_\_\_, \_\_\_\_\_)$