2.3 Moment Generating Function

What is a Moment?

$E[X]$ 1st moment

$E[X^2]$ 2nd moment

The r^{th} **moment** of a random variable, X, is $\boxed{E[X^r]}$.

- Also called: **moment about the origin, raw moment**

Variance: 2nd central moment $E[(X-\mu_X)^2]$

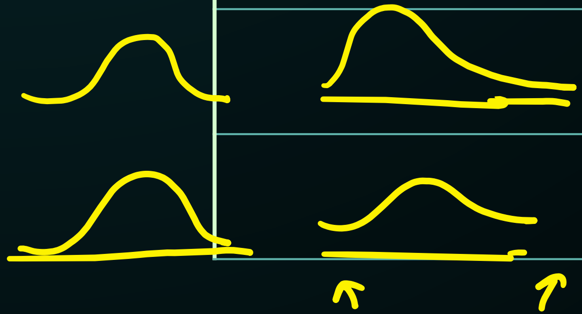
The r^{th} **central moment** of a random variable, X, is the expected value of the rth power of the deviation of a random variable from its mean: $\boxed{E[(X-\mu_X)^r]}$

Moments

$E[x^0] = E[1] = 1$

Moment	Real World	Statistics
0th	Total mass	Total Probability
1st	Center of Mass	Expected Value
2nd	Rotational Inertia (torque required for desired angular acceleration)	Variance (2nd central moment)
3rd	---	Skewness (3rd standardized moment)
4th	---	Kurtosis (4th standardized moment)

$E[x^3$



FYI

Moment Generating Function

Definition 2.3-1

Let X be a random variable of the discrete type with pmf $f(x)$ and space S. If there is a positive number h such that

$$E(e^{tX}) = \sum_{x \in S} e^{tx} f(x)$$

exists and is finite for $-h < t < h$, then the function defined by

$$\boxed{M(t) = E(e^{tX})}$$

is called the **moment-generating function of** X (or of the distribution of X). This function is often abbreviated as mgf.

$$M(t) = E\left[e^{tx}\right]$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{tx} = 1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots$$

$$M(t) = E\left[e^{tx}\right] = E\left[1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots\right]$$

$$E[X^1]$$

$$M'(t) = E\left[0 + x + \frac{2(tx)^1 x}{2!} + \frac{3(tx)^2(x)}{3!} + \dots\right]$$

$$M'(t)\Big|_{t=0} = E\left[X + 0 + 0 + 0\right]$$

Moment Generating Function

Suppose the sample space of X is $S = \{x_1, x_2, x_3, \ldots\}$

What is an expression for mgf?

x

$M(t) = E[e^{tx}] = \sum_{x} e^{tx} \cdot f(x)$

$M(t) = e^{tx_1} f(x_1) + e^{tx_2} f(x_2) + e^{tx_3} f(x_3) + \ldots$

The coefficient of each e^{tx_i} is the probability, $f(x_i) = P(X = xi)$

Simple mgf example

Example 2.3-5 If X has the mgf

e^{1t} e^{2t} e^{3t}

$$M(t) = e^t\left(\frac{3}{6}\right) + e^{2t}\left(\frac{2}{6}\right) + e^{3t}\left(\frac{1}{6}\right), \qquad -\infty < t < \infty,$$

then the support of X is $S = \{1, 2, 3\}$ and the associated probabilities are

$$P(X = 1) = \frac{3}{6}, \qquad P(X = 2) = \frac{2}{6}, \qquad P(X = 3) = \frac{1}{6}.$$

Or we could write f(x) = $\frac{4-x}{6}$, x = 1,2,3.

$M_X(t) = E[e^{tX}]$ $M_{X+Y}(t) = E[e^{t(X+Y)}]$

More Properties of MGFs

1. If two random variables have the same MGF, then they have the same distribution. i.e. if X and Y are random variables that have the same MGF: $M_X(t) = MY(t)$, then X and Y have the exact same distribution (pmf, cdf, etc)
2. For two independent random variables, X and Y, the MGF of their sum is the product of their MGFs:

$$M_{X+Y}(t) = M_X(t)MY(t)$$ (works for more than 2 as well)

$M(t)$

More Properties of MGFs

w.r.t. t

3. The n^{th} derivative of $M_X(t)$ evaluated at t=0 is equal to the n^{th} moment, $E[X^n]$.

Examples

Moment Generating Function

1 Let X ~ Binom(n, p). The mgf is:

what is the mgf of X?

$M_x(t)$ = E[e^{tX}] = $\sum_{x=0}^{n} e^{tx}\binom{n}{x} p^x(1-p)^{n-x}$

= $\sum_{x=0}^{n}\binom{n}{x}(pe^t)^x\,(1-p)^{n-x}$ = $\boxed{[pe^t+(1-p)]^n}$

mgf

binomial expansion

Computing the first two moments using the mgf:

$M'(t) \;= n[pe^t + (1-p)]^{n-1}pe^t.$

ignore

$M''(t) = n(n-1)[pe^t + (1-p)]^{n-2}p^2e^{2t} + n[pe^t + (1-p)]^{n-1}pe^t.$

$E[X] \;=\; M'(0) \;=\; np$

$E[X^2] \;=\; \boxed{M''(0)} = n(n-1)p^2 + np$

$Var[X] = E[X^2]- (E[X])^2 = n(n-1)p^2 + np \;-(np)2$

$=\; np - np^2 =\; np(1-p)$

2 Example: known distributions

Suppose a random variable X has moment generating function: $M(t) = (\frac{2}{3} + \frac{1}{3}e^t)^{10}$

Binomial	$f(x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$,	
$b(n,p)$		
$0 < p < 1$	$M(t) = (1-p+pe^t)^n$,	$-\infty$

What is the pmf of X?

$X \sim \text{Binom}(n=10,\ p=1/3)$

$$f(x) = \binom{10}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{10-x}$$

3

Say we have 3 random variables, W, X, Y ~ Bernoulli(p)

Let Z be the sum of all three: Z = W + X + Y. $\sim \text{Binom}(n=3, p)$

Show that Z ~ Binom(n,p)

Note: The mgf of a Bernoulli random variable is

$M(t) = (1 - p + pe^t)$

$$M_{w+x+y}(t) = M_w(t)\,M_x(t)\,M_y(t)$$

Q: Can you do the same for $\sum Geometric$ = NB at home?

$\rightarrow \quad (1-p+pe^t)^3$ binom mgf