STAT400 – Homework 1 Solutions

Exercise 1

The probability that Chloe acts hungry at 7pm given that she has already eaten dinner is 0.5. The probability that Chloe acts hungry at 7pm given that she has not had dinner is 0.99. Assume that there is probability 0.9 that Chloe has eaten dinner by 7pm on a given day.

a) (1 pt) If Chloe is acting hungry at 7pm, find the probability that she has not had dinner yet.

b) (1 pt) If Chloe is not acting hungry at 7pm, find the probability that she has already had dinner.

Givens: $P(\text{Hungry} \mid \text{Dinner}) = 0.50$

$P(\text{Hungry} \mid \text{Dinner}') = 0.99$

$P(\text{Dinner}) = 0.90$

a) Solve for $P(\text{Dinner}' \mid \text{Hungry})$

$$P(\text{Dinner}' \mid \text{Hungry}) = \frac{P(\text{Dinner}' \cap \text{Hungry})}{P(\text{Hungry})}$$

$$P(\text{Dinner}' \cap \text{Hungry}) = P(\text{Hungry} \mid \text{Dinner}') \cdot P(\text{Dinner}')$$

$$P(\text{Dinner}') = 1 - P(\text{Dinner}) = 1 - 0.90 = 0.10$$

$$P(\text{Dinner}' \cap \text{Hungry}) = 0.99 \cdot 0.10 = \boxed{0.099}$$

$$P(\text{Hungry}) = P(\text{Hungry} \cap \text{Dinner}) + P(\text{Hungry} \cap \text{Dinner}')$$
$$= P(\text{Dinner}) \cdot P(\text{Hungry} \mid \text{Dinner}) + P(\text{Dinner}') \cdot P(\text{Hungry} \mid \text{Dinner}')$$
$$= 0.90 \cdot 0.50 + 0.10 \cdot 0.99$$

$$P(\text{Hungry}) = \boxed{0.549}$$

$$P(\text{Dinner}' \mid \text{Hungry}) = \frac{0.099}{0.549} = 11/61 \approx 0.1803$$

b) Solve for $P(\text{Dinner} \mid \text{Hungry}')$

$$P(\text{Dinner} \mid \text{Hungry}') = \frac{P(\text{Dinner} \cap \text{Hungry}')}{P(\text{Hungry}')}$$

$$P(\text{Dinner} \cap \text{Hungry}') = P(\text{Hungry}' \mid \text{Dinner}) \cdot P(\text{Dinner})$$

$$P(\text{Hungry}' \mid \text{Dinner}) = 1 - P(\text{Hungry} \mid \text{Dinner}) = 1 - 0.50 = 0.50$$

$$P(\text{Dinner} \cap \text{Hungry}') = 0.50 \cdot 0.90 = \boxed{0.45}$$

$$P(\text{Hungry}') = 1 - P(\text{Hungry}) = 1 - 0.549 = \boxed{0.451}$$

$$P(\text{Dinner} \mid \text{Hungry}') = \frac{0.45}{0.451} = 450/451 \approx 0.9978$$

Exercise 2

Solve for **c** in the following double integral:

$$\iint_A cx^3y^2\,dx\,dy = 1,$$

where $A = \{(x,y) : 0 < x < \sqrt{y}, 0 < y < 4\}$. Show all your work. Do not use a calculator or computer to solve the integrals except to check your work. (2 pts)

$$\iint_A cx^3y^2\,dx\,dy = c\int_0^4\int_0^{\sqrt{y}} x^3y^2\,dx\,dy = 1$$

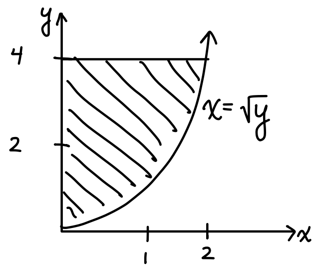
$$c\int_0^4 \left[\frac{1}{4}x^4y^2\right]_0^{\sqrt{y}} dy = 1$$

$$c\int_0^4 \frac{1}{4}y^4\,dy = 1$$

$$c\left[\frac{1}{20}y^5\right]_0^4 = 1$$

$$\frac{1024}{20}c = 1$$

$$c = \frac{20}{1024} = \frac{5}{256} \approx 0.01953$$



Exercise 3

Suppose $S = \{1, 2, 3, 4, 5, ...\}$ and for any element of S,

$$P[k] = c\frac{5^k}{k!}$$

Find the value of c that makes this a valid probability distribution. (2 pts)

$$\sum_{k=1}^{\infty} P[k] = 1 \longrightarrow \sum_{k=1}^{\infty} c\frac{5^k}{k!} = 1$$

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\sum_{k=1}^{\infty} c\frac{5^k}{k!} = c\sum_{k=0}^{\infty}\frac{5^k}{k!} - \frac{5^0}{0!} = c\sum_{k=0}^{\infty}\frac{5^k}{k!} - 1 = 1$$

$$c(e^5 - 1) = 1 \longrightarrow c = \frac{1}{e^5 - 1} \approx 0.006784$$

Exercise 4

Suppose $S = \{2, 3, 4...\}$ and

$$P[k] = \frac{c}{(5/2)^k}$$

a) (1.5 pts) Find a value of c that makes this a valid probability distribution.

b) (1 pt) Find P[Outcome is *greater* than 3].

a) $\sum_{k=2}^{\infty} P[k] = c\sum_{k=2}^{\infty} \frac{1}{(5/2)^k} = c\sum_{k=2}^{\infty}\left(\frac{2}{5}\right)^k = 1$

The sum of a geometric series with ratio, r, and first term, a, is $\frac{a}{1-r}$.

first term of series $= \left(\frac{2}{5}\right)^2 = \frac{4}{25} \rightarrow \sum_{k=2}^{\infty}\left(\frac{2}{5}\right)^k = \frac{4/25}{1-2/5} = \frac{4/25}{3/5} = \frac{4}{15}$

$\frac{4}{15}c = 1$

$c = \frac{15}{4} = 3.75$

b) $P[k>3] = 1 - P[k \leq 3] = 1 - P[3] - P[2]$

$= 1 - \frac{15/4}{(5/2)^3} - \frac{15/4}{(5/2)^2} = \frac{4}{25} = 0.16$

Exercise 5

Suppose P[A] = 0.5, P[B'] = 0.3, and P[A ∩ B] = 0.2

a) (0.5 pts) Find P[B | A]

b) (0.5 pts) Find P[B'| A']

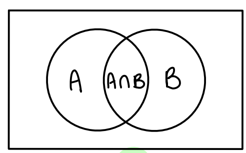
c) (0.5 pts) Find P[A' | B]

So: $P[A'] = 0.50$

$P[B] = 0.70$

$P[A' \cap B'] = 1 - P[A \cup B] = 0$

a) $P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{0.20}{0.50} = \frac{2}{5} = 0.40$



b) $P[B'|A'] = \frac{P[A' \cap B']}{P[A']} = \frac{0}{0.50} = 0$

c) $P[A'|B] = \frac{P[A' \cap B]}{P[B]} = \frac{P[B] - P[A \cap B]}{P[B]} = \frac{0.70 - 0.20}{0.70} = \frac{5}{7} \approx 0.7143$