Hypothesis Testing – Means and Proportions

8.1, 8.3

Today's topics

Hypothesis Testing

- Definitions
- Testing for one mean
- p-value
- Testing for one proportion

Statistics overview

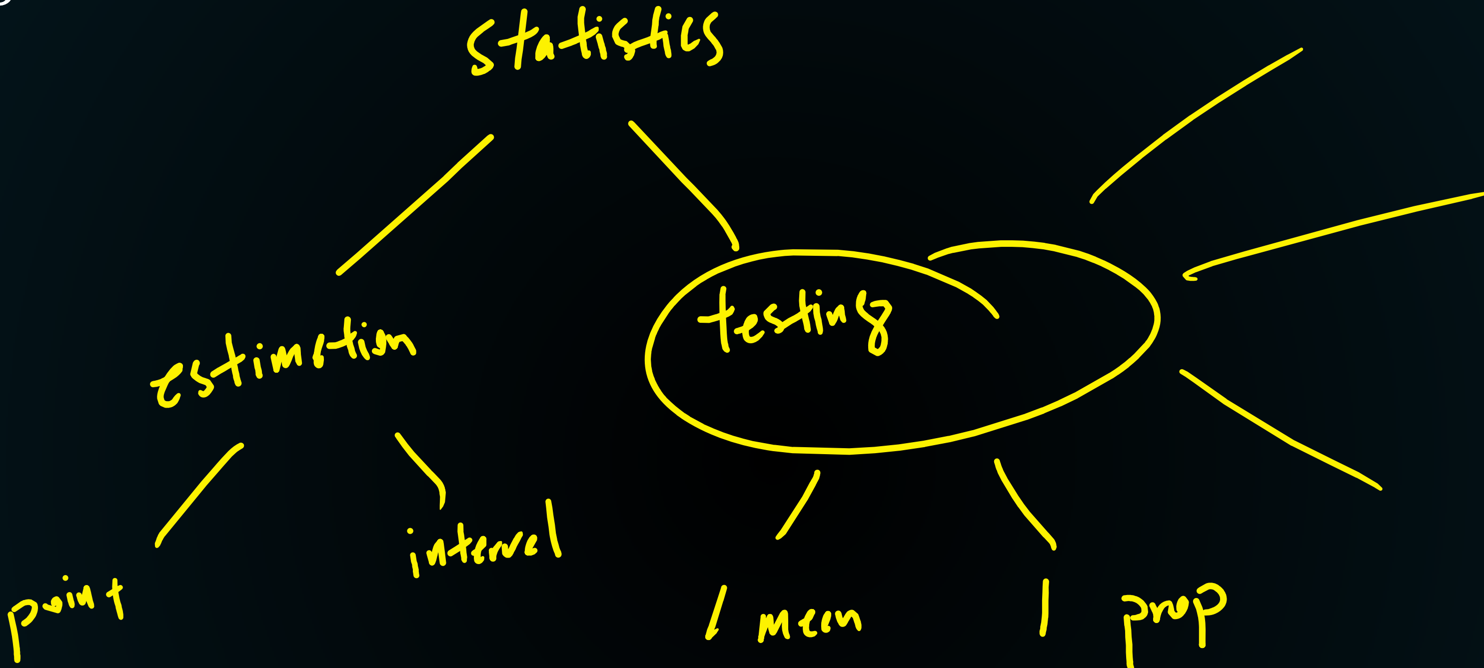
One goal in Statistics is to make *inferences* about populations based on samples taken from the population.

Previously, we estimated population parameters:

- Point estimates (MLE, MOM)
- Interval estimates (Confidence Intervals)

$\underbrace{100(1-\alpha)\%}$

notes


statistics
estimation
testing
point
interval
1 mean
1 prop

Testing

Another way to do inference, is to **make a decision** about a parameter.

Examples:

test claim

my Mazda 3 manual claims that it gets 35 highway mpg

Dustin’s pudding packs actually contain 3.25 oz

Terms

- Null hypothesis, H_0
- Alternative hypothesis, H_A or H_1
- Type I error:
- Type II error
- Simple hypothesis
- Compound hypothesis

$H_0: \mu = 35$ $\qquad$ $H_A: \mu < 35$

$\mu_0 = 35$

Null and Alternative Hypotheses

Say an experimenter wants to test the plausibility of the statement $\mu = \mu_0$. a fixed value e.g. 50

We can formally describe this as a **null hypothesis.**

- **H_0 :** $\mu = \mu_0$ mean from null
- The word "hypothesis" indicates that we will be testing this statement (with data).

We will associate the null hypothesis with a different one that we are testing 'for', called the "alternative hypothesis".

- **H_A :** $\mu \neq \mu_0$ or **H_A :** $\mu > \mu_0$ or **H_A :** $\mu < \mu_0$

notes

Type 1 and Type 2 Error

	Do not reject H_0	Reject H_0
H_0 true	Correct	Type I Error
H_0 not true	Type II Error	Correct

incorrect

Hypothesis Test Example – “Compound” H_A

Perdont university claims that students at their school are above average intelligence. A random sample of thirty students IQ scores have a mean score of **102.5**.

$\bar{x} = 102.5$

Suppose the mean population IQ score is 100 with a standard deviation of 15. Is there sufficient evidence to support this claim?

H_0 : $\mu = 100$

H_A : $\mu > 100$

notes

Hypothesis Test Example – "Simple" H_A

Example 8.1-1 Let X equal the breaking strength of a steel bar. If the bar is manufactured by process I, X is $N(50, 36)$, i.e., X is normally distributed with $\mu = 50$ and $\sigma^2 = 36$. It is hoped that if process II (a new process) is used, X will be $N(55, 36)$. Given a large number of steel bars manufactured by process II, how could we test whether the five-unit increase in the mean breaking strength was realized?

H_0: $\mu = 50$ $H_0 : \mu \leq 50$ H_1: $\mu = 55$ $H_A : \mu = 55$

We want to set up a "rule" to determine whether to stick with H_0 or not.
This rule will lead to a decision about what to do with H_0.

Partition sample space into 2 parts: C and C'.

If $(x_1, x_2, \ldots x_n) \in C$, reject H_0

If $(x_1, x_2, \ldots x_n) \in C'$, do not reject H_0 don't reject H_0

e.g. $\mathbf{x} = \{64.4, 54.7, 57.2, 61.6, 51.3\}$ or $\mathbf{x} = \{51.2, 54.7, 47.2, 51.6, 46.3\}$

We often partition the sample space in terms of values of a statistic called a **test statistic.**

$H_0 : \mu = 50$

$\boxed{H_1 : \mu = 55}$ AH

Test statistic

$\bar{X}$

Often, we partition the sample space based on the value of a statistic called the **test statistic**.

One common example is $\bar{X}$ (for testing the mean).

We might want to reject the null hypothesis if the sample average is larger or smaller than a certain number. E.g. $\boxed{\bar{X} > 53.}$

i.e. $C = \{(x_1, x_2, \ldots x_n) : \bar{x} > 53\}$

$C : \bar{x} > 54$

$C : \bar{x} > 53$

C is referred to as the **rejection region**, or the **critical region.**

p value

$P[\text{data} \mid H_0 \text{ true}]$

The plausibility of a null hypothesis can be measured with a **p-value,** which is a number between 0 and 1.

- A p-value is sometimes referred to as the *observed level of significance*
- The smaller the p-value, the less plausible H_0 is.

Definition of a p-value:

"Probability of observing data at least as extreme as the observed sample given that H_0 is true."

p-value illustration

e.g. $\bar{x} = 80$

60

0.00000000

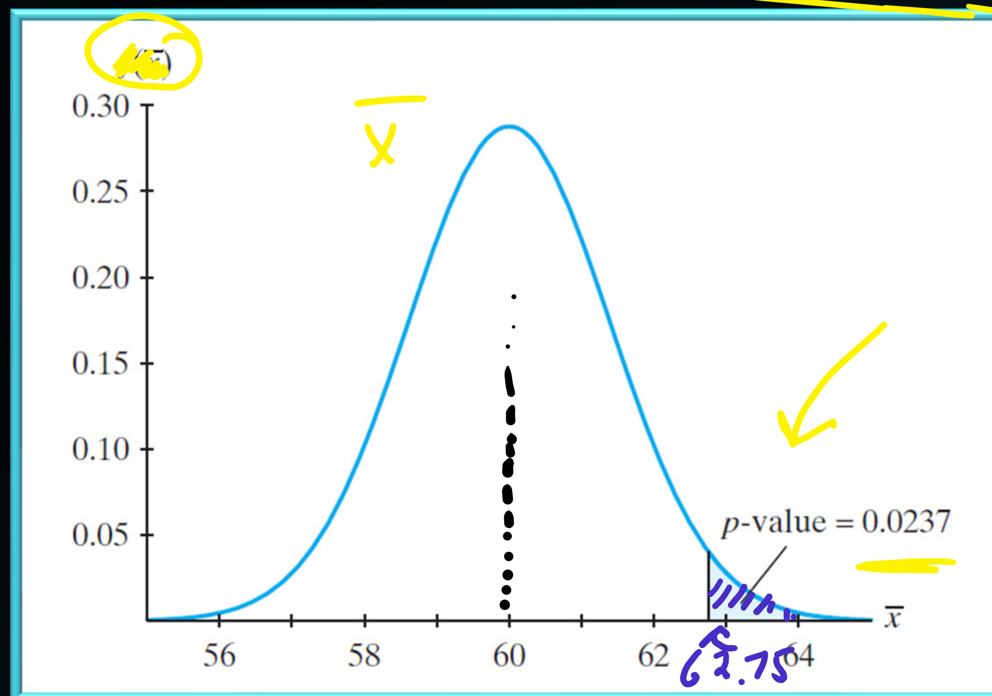
80

H_0: $\mu = 60$ vs

H_A: $\mu > 60$

Suppose $\bar{x} = 62.75$

$n = ?$



Hypothesis Testing (Steps)

$\bar{x}$ σ

1. Formulate H_0 and H_A (based on the scenario)
2. Identify a test statistic to use and its distribution under H_0
3. Evaluate the test statistic
4. Calculate a p-value, compare to α.
5. Make a decision
   - if $p < \alpha$, reject H_0. Otherwise, (if $p > \alpha$), do not reject H_0.
6. State conclusion **in the context of the original question.**
   - "There is/isn't enough evidence to show that…"

Hypothesis Test Example – "Compound" H_A

normally distributed

Perdont university claims that students at their school are above average intelligence. A random sample of thirty students IQ scores have a mean score of **102.5**. $\bar{X} = 102.5$

Suppose the mean population IQ score is 100 with a standard deviation of 15. Is there sufficient evidence to support this claim? test at $\boxed{\alpha = 0.05}$

$\alpha = 0.05$ $\sigma = 15$ $n = 30$

default H

H_0 : $\mu = 100$

H_A : $\mu > 100$

competing claim

$\bar{X}$

test statistic

$$\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim Z$$

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$$

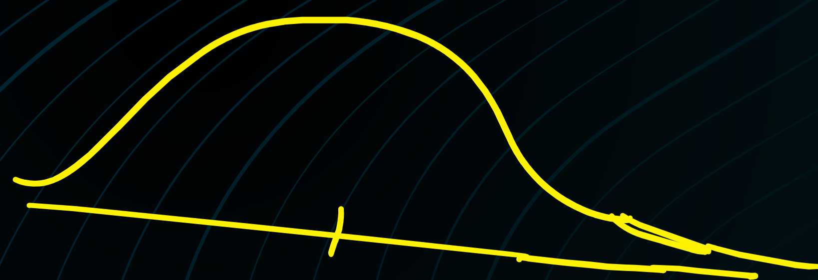
$$Z = \frac{102.5 - 100}{15 / \sqrt{30}} = 0.9128 \sim N(0,1)$$

$p = .18 > \alpha$ Do not reject H_0

$H_0: \mu = 100$ $H_A: \mu > 100$ $\bar{X} > 500$

- Alpha is the level of significance 0.05 0.01 0.1
- "probability of a false positive" 0.0001

$\bar{X} > 100.5$



Two ways to perform a hypothesis test

given

→ 1. Calculate a p-value and compare to significance level

"observed significance level"

.0499

.0501

α

if $p < \alpha$, reject H_0

if $p > \alpha$, DNR H_0

2. Define a rejection region (RR) and see if sample falls in RR. (also known a critical region)

e.g. $\alpha = 0.05$ 1-sided

$H_A: \mu > 100$

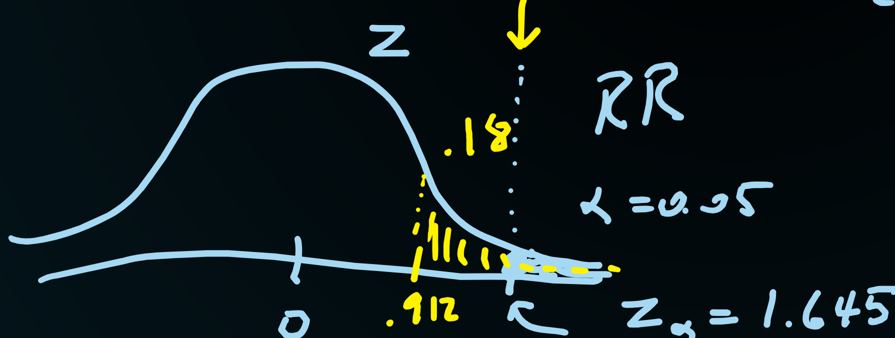


Table 8.1-1 Tests of hypotheses about one mean, variance known

H_0	H_1	Critical Region
$\mu = \mu_0$ 100	$\mu > \mu_0$ 100	test stat 0.9128 $\boxed{z \geq z_\alpha}$ 1.645 crit. value or $\bar{x} \geq \mu_0 + z_\alpha \sigma/\sqrt{n}$
$\mu = \mu_0$	$\mu < \mu_0$	$z \leq -z_\alpha$ or $\bar{x} \leq \mu_0 - z_\alpha \sigma/\sqrt{n}$
$\mu = \mu_0$	$\mu \neq \mu_0$	$\lvert z \rvert \geq z_{\alpha/2}$ or $\lvert \bar{x} - \mu_0 \rvert \geq z_{\alpha/2} \sigma/\sqrt{n}$

$$Z = \frac{\overline{X} - \mu_0}{\sqrt{\sigma^2/n}} = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$$

Table 8.1-2 Tests of hypotheses for one mean, variance unknown

H_0	H_1	Critical Region
$\mu = \mu_0$	$\mu > \mu_0$	$t \ge t_\alpha(n-1)$ or $\bar{x} \ge \mu_0 + t_\alpha(n-1)s/\sqrt{n}$
$\mu = \mu_0$	$\mu < \mu_0$	$t \le -t_\alpha(n-1)$ or $\bar{x} \le \mu_0 - t_\alpha(n-1)s/\sqrt{n}$
$\mu = \mu_0$	$\mu \ne \mu_0$	$\lvert t\rvert \ge t_{\alpha/2}(n-1)$ or $\lvert\bar{x} - \mu_0\rvert \ge t_{\alpha/2}(n-1)s/\sqrt{n}$

$$T = \frac{\overline{X} - \mu}{\sqrt{S^2/n}} = \frac{\overline{X} - \mu}{S/\sqrt{n}}.$$

Example

$\bar{x} = 2.879$ H_0

A machine that pours beer is supposed to fill buckets with a mean of 3.0 gallons. A sample of 13 buckets gives a sample mean of 2.879 gallons and s = 0.325. Perform a hypothesis test at α= 0.05 to see if this machine is **accurately doing its job**.

$\frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \sim Z$ $\boxed{\frac{\bar{x} - \mu}{s / \sqrt{n}}} \sim t_{n-1}$

H_0: $\mu = 3.0$

H_A: $\mu \neq 3.0$ 2-sided test

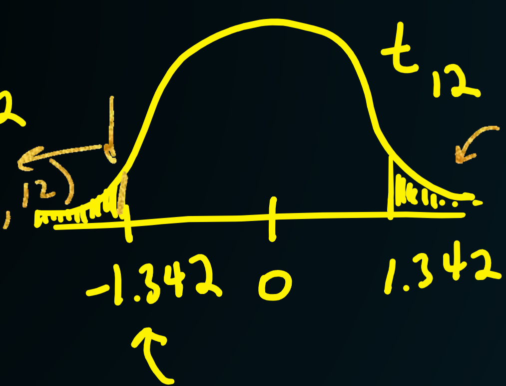
test stat $\rightarrow$ $t = \frac{2.879 - 3}{.325 / \sqrt{13}} \sim t_{12}$

$t = -1.342 \sim t_{12}$

p-value = .204 $2 \cdot pt(-1.342, 12)$

pnorm ~~tnorm~~

Decision: $p > \alpha$ DNR H_0



Conclusion: There is not enough evidence to suggest that the machine is not accurately doing its job

Example

is $\bar{x} = 3.1$

A HAL-8000 machine that pours beer is supposed to fill buckets with a mean of 3.0 gallons. Dave takes a sample of 13 buckets and finds a sample mean of $\boxed{2.879}$. Suppose the true standard deviation of these machines is 0.2 gallons. Perform a hypothesis test at α= 0.05 to see if this machine is **underfilling**.

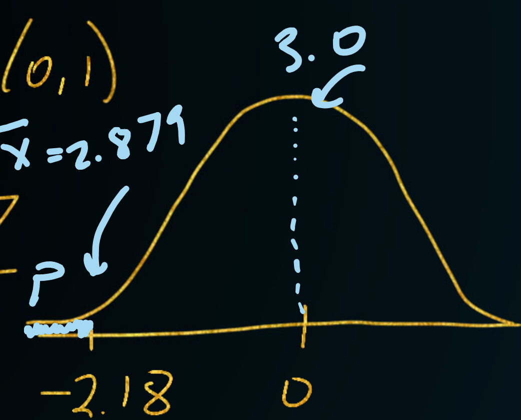
$\bar{x} =$

$\sigma = 0.2$

H_0: $\mu = 3.0$

H_A: $\mu < 3.0$ ← 1-sided test

z = $\dfrac{2.879 - 3.0}{0.2/\sqrt{13}} = -2.18 \sim Z$



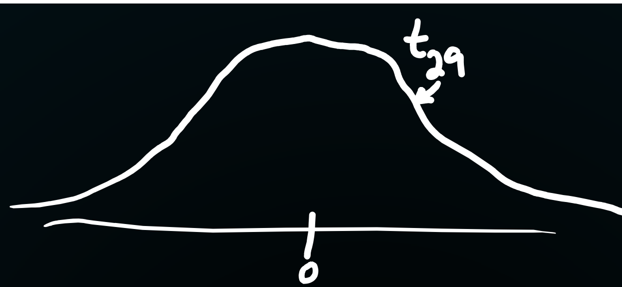
p-value = $0.015 < 0.05$

Decision: $p < \alpha$ Reject H_0

Conclusion: There is enough evidence to suggest that this machine is underfilling. ← (H_0 not true)

$$T = \frac{\overline{X} - \mu}{\sqrt{S^2/n}} = \frac{\overline{X} - \mu}{S/\sqrt{n}}.$$

Example



normally distributed

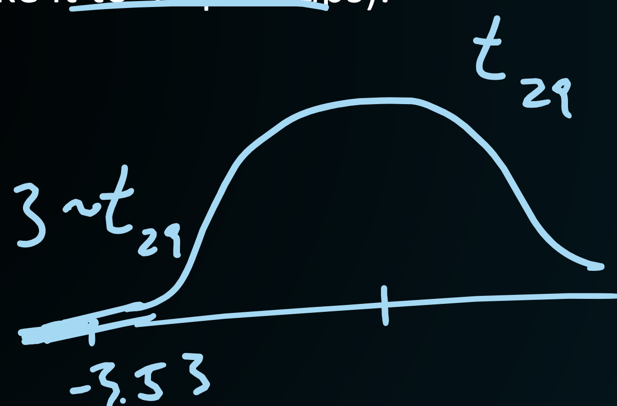
Nick Fury claims that the (true) mean number of push-ups his superheroes can do is at least 40.0. A random sample of 30 superheroes gives $\bar{x} =$ 38.518 pushups, s = 2.299. Perform a hypothesis test at α = 0.01 to determine if this is true (or if they can't really make it to 40 pushups).

H_0: $\mu \geq 40$ or $\mu = 40$

H_A: $\mu < 40$

t = $\dfrac{38.518 - 40}{2.299/\sqrt{30}}$ ~ t_{29}

$t = -3.53 \sim t_{29}$



p-value: $0.0007 < 0.01$ (α)

Decision: Reject H_0

Conclusion: There is significant evidence to suggest that they can NOT do 40 pushups.

Table 8.3-1 Tests of hypotheses for one proportion

H_0	H_1	Critical Region
$p = p_0$	$p > p_0$	$z = \dfrac{y/n - p_0}{\sqrt{p_0(1-p_0)/n}} \geq z_\alpha$
$p = p_0$	$p < p_0$	$z = \dfrac{y/n - p_0}{\sqrt{p_0(1-p_0)/n}} \leq -z_\alpha$
$p = p_0$	$p \neq p_0$	$\lvert z \rvert = \dfrac{\lvert y/n - p_0 \rvert}{\sqrt{p_0(1-p_0)/n}} \geq z_{\alpha/2}$

$$\boxed{Z = \frac{Y/n - p_0}{\sqrt{p_0(1-p_0)/n}}}$$

Hypothesis test for proportions

Aladdin is a frequent flyer. He thinks he gets security-screened more than normal at the magic-carpet-port. Assume security randomly screens 10% of all people (so he should be screened 10% of the time). In the past few years, he has been (randomly?) selected 16 out of 100 times.

Perform a hypothesis test at 0.05 significance to see if the screening process is random, or biased towards screening him more.

$$Z = \frac{Y/n - p_0}{\sqrt{p_0(1-p_0)/n}}$$

Hypothesis test for proportions

$$Z = \frac{Y/n - p_0}{\sqrt{p_0(1-p_0)/n}}$$

$\hat{p} = \frac{Y}{n}$

Assume security randomly screens 10% of all people (so he should be screened 10% of the time). Aladdin has been selected 16 out of 100 times. Is process biased towards screening him more? Test at α=0.05.

$\hat{p} = .16$

H_0: $p = 0.1$

H_A: $p > 0.1$

Test statistic: $Z = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{.16 - .1}{\sqrt{.1(.9)/100}} = 2 \sim N(0,1)$

p-value: 0.0227

1-pnorm(2)

Decision: $p < \alpha$ Reject H_0

Conclusion:

There is significant evidence to suggest that the

Screening process is biased towards screening Aladdin more