Bivariate Distributions (Discrete)

4.1

Bivariate Distributions

Univariate: One measurement for observed items. (outcome associated with one variable). E.g.

- Waiting time
- Number of successes in n trials
- Number of occurrences in a unit time, etc.

Bivariate: Use 2 variables to predict an outcome.

E.g. Predict college GPA, z, using HS class rank, x, and ACT score, y,

$$z = f(x, y)$$

Discrete Bivariate Distributions

Definition 4.1-1

Let X and Y be two random variables defined on a discrete space. Let S denote the corresponding two-dimensional space of X and Y, the two random variables of the discrete type. The probability that $X = x$ and $Y = y$ is denoted by $f(x, y) = P(X = x, Y = y)$. The function $f(x, y)$ is called the **joint probability mass function** (joint pmf) of X and Y and has the following properties:

(a) $0 \le f(x, y) \le 1$.

(b) $\displaystyle\sum\sum_{(x,y)\in S} f(x, y) = 1$.

(c) $P[(X, Y) \in A] = \displaystyle\sum\sum_{(x,y)\in A} f(x, y)$, where A is a subset of the space S.

Discrete Bivariate Example

$$f(x,y) = \frac{xy^2}{30}, \quad x = 1,2,3 \;\; y = 1,2.$$

(a) $0 \le f(x,y) \le 1$.

(b) $\sum\sum_{(x,y)\in S} f(x,y) = 1$.

(c) $P[(X,Y) \in A] = \sum\sum_{(x,y)\in A} f(x,y)$, where A is a subset of the space S.

Discrete Bivariate Example

Let X and Y be two discrete random variables such that their joint distribution is given below:

e.g. f(3,0) = 0.31

		X			
		3	4	5	
Y	0	0.31	0.21	0.21	0.73
	1	0.03	0.04	0.05	0.12
	2	0.02	0.03	0.04	0.09
	3	0.01	0.02	0.03	0.06
		0.37	0.30	0.33	1.0

Marginal pmf

Definition 4.1-2

Let X and Y have the joint probability mass function $f(x, y)$ with space S. The probability mass function of X alone, which is called the **marginal probability mass function of** X, is defined by

$$f_X(x) = \sum_y f(x, y) = P(X = x), \qquad x \in S_X,$$

where the summation is taken over all possible y values for each given x in the x space S_X. That is, the summation is over all (x, y) in S with a given x value. Similarly, the **marginal probability mass function of** Y is defined by

$$f_Y(y) = \sum_x f(x, y) = P(Y = y), \qquad y \in S_Y,$$

Marginal probability

- $f(y) = \begin{cases} 0.73, & y = 0 \\ 0.12, & y = 1 \\ 0.09, & y = 2 \\ 0.06, & y = 3 \end{cases}$

		X			
		3	4	5	
Y	0	0.31	0.21	0.21	0.73
	1	0.03	0.04	0.05	0.12
	2	0.02	0.03	0.04	0.09
	3	0.01	0.02	0.03	0.06
		0.37	0.30	0.33	1.0

Independence of X and Y

X and Y are independent **iff:**

- for every x ∈ S_x and y ∈ S_y,

$$\mathrm{P}[\mathrm{X} = \mathrm{x}, \mathrm{Y} = \mathrm{y}] = \mathrm{P}[\mathrm{X} = \mathrm{x}]\mathrm{P}[\mathrm{Y} = \mathrm{y}]$$

i.e.,

$$\boxed{f_{XY}(x, y) = f_X(x) f_Y(y)}$$

Examples

Bivariate Discrete

1

Let $f(x,y) = \frac{xy^2}{30}$, $x = 1,2,3 \;\; y = 1,2.$

A) Find the marginal pmf of X: $f_X(x) = \frac{x}{6}, \quad x = 1,2,3.$

B) Find the marginal pmf of Y: $f_Y(y) = \frac{y^2}{5}, \quad y = 1,2.$

C) Find P[X=Y] : 9/30

D) Are X and Y independent? (Yes)

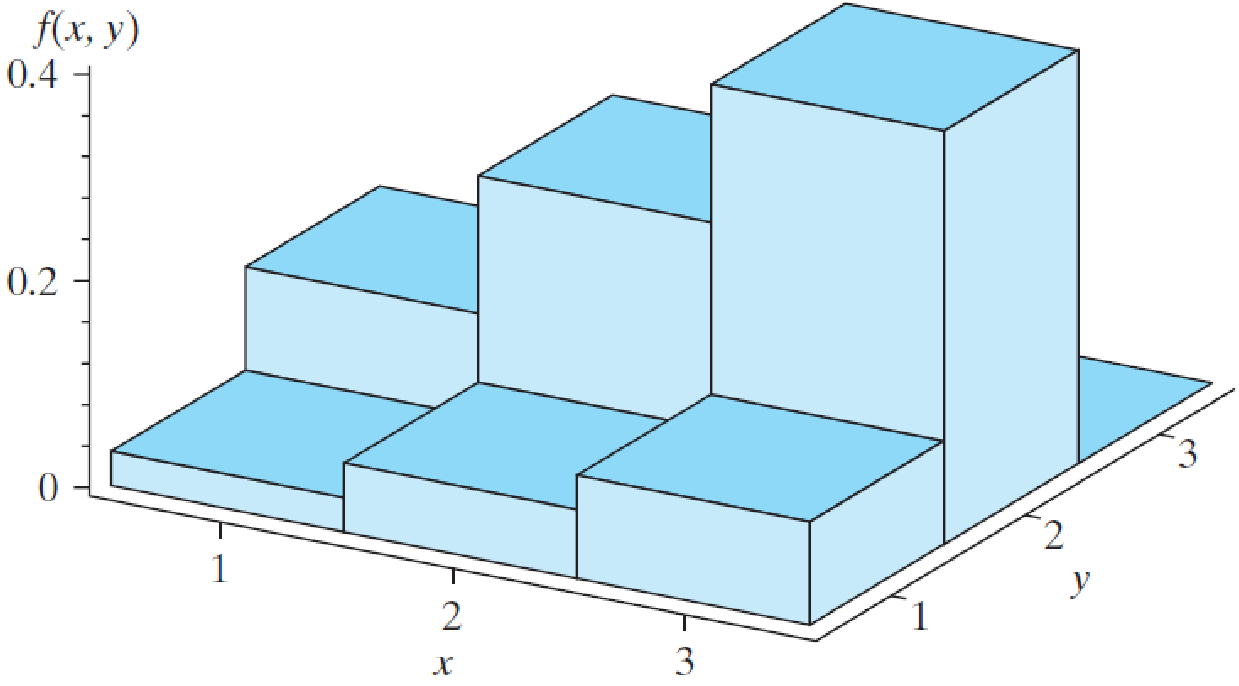

Figure 4.1-3 Joint pmf $f(x, y) = \dfrac{xy^2}{30}$, $x = 1, 2, 3$ and $y = 1, 2$

2 Let $f(x, y) = c(x + 2y),\ x = 1{,}2 \quad y = 1{,}2{,}3$

What value must the constant c take, so that $f(x, y)$ is a valid joint pmf?

(1/33)

3 Let $f(x, y) = 6\left(\frac{1}{4}\right)^x \left(\frac{1}{3}\right)^y$, x = 1,2,3,… y = 1,2,3,…

A) Find an expression for the marginal pmf of x. $f_X(x) = 3\left(\frac{1}{4}\right)^x$

B) Show that the marginal pmf of x is a valid probability distribution.