Spring 2021 STAT400 Homework 3 Solutions (SO Fetch!)

Exercise 1

While wandering around the halls of North Shore High School after hours, Regina George encounters lots of enemies. Suppose enemies appear one-at-a-time. You may assume that these encounters are independent and that these are there are only 4 types of enemies she may run into:

Enemy	Probability
Freshman	0.5
Sophomore	0.2
Junior	0.2
Senior	0.1

a) Suppose she encounters 10 enemies. What is the probability that at least 1 is a Junior? (0.4 pt)

b) What is the probability that the first 8 enemies she encounters are all Freshmen? (0.4 pt)

c) Suppose she encounters 10 enemies. What is the probability that more than 8 are Freshmen? (0.4 pt)

d) What is the probability that she sees her first Senior **after** her tenth encounter? (0.4 pt)

e) What is the probability that she sees her third Senior **on** (exactly) her thirtieth encounter? (0.4 pt)

f) What is the variance of the number of Sophomore she sees in her first 20 encounters? (0.5 pt)

g) What is the variance of the number of trials required to find her first Junior? (0.5 pt)

a) $P[\#\text{Juniors} \geq 1] = 1 - P[\#\text{Juniors in 10 Encounters} = 0]$

$= 1 - P[\text{Junior}']^{10} = 1 - (1-0.20)^{10} = 1 - 0.80^{10} \approx 0.8926$

b) $P[\text{first 8 enemies all freshmen}] = \prod_{i=1}^{8} P[i\text{-th enemy is a freshman}]$

$= (0.50)^8 = 1/256 \approx 0.003906$

c) Let # of freshmen be X, $X \sim \text{binomial}(n=10, p=0.50)$

$P[X>8] = P[X=9] + P[X=10]$

$= \binom{10}{9}(0.50)^9(1-0.50)^1 + \binom{10}{10}(0.50)^{10}(1-0.50)^0$

$= 5/512 + 1/1024$

$= 11/1024 \approx 0.01074$

d) $P[1^{st} \text{ senior after } 10^{th} \text{ encounter}] = P[\text{no seniors in first 10 encounters}]$

$= (1-0.10)^{10} = 0.90^{10} \approx 0.3487$

e) $P[3^{rd} \text{ senior on } 30^{th} \text{ encounter}] = P[2 \text{ seniors before } 30^{th}] \cdot P[1 \text{ senior on } 30^{th}]$

(creates negative binomial f(x)) $= \binom{29}{2}(0.10)^2(1-0.10)^{27} \cdot (0.10)$

$= \binom{29}{2}(0.10)^3(1-0.10)^{27} \approx 0.02361$

f) Let # of Sophomores in first 20 encounters be X, $X \sim \text{binomial}(n=20, p=0.20)$

$\sigma^2 = np(1-p) \rightarrow \text{Var}[X] = 20(0.20)(1-0.20) = 3.2 \text{ sophomores}^2$

g) Let # of trials required to find first Junior be X, $X \sim \text{geometric}(p=0.20)$

$\sigma^2 = \frac{1-p}{p^2} \rightarrow \text{Var}[X] = \frac{1-0.20}{0.20^2} = 20 \text{ trials}^2$

Exercise 2 (Use R to find the following. Show your commands)

(Use the same probabilities shown above for #1) **Must use R** to get credit on these!

a) Find the probability that it takes fewer than 8 trials to find the first Sophomore. (0.5 pt)

The number of trails follows a negative binomial distribution. Therefore, we use the `pnbinom()` function in R. Be aware that the `q` is the number of failures and `n` is the number of sccusses in `pnbinom()`. You may find these details by `?pnbinom`.

```
pnbinom(q = 6, size = 1, prob = 0.2)
```

```
## [1] 0.7902848
```

We can also check the solution by the explicit formula of probability mass function.

```
trial <- 1:7
sum(0.2 * 0.8 ^ (trial - 1))
```

```
## [1] 0.7902848
```

b) Suppose she encounters 100 enemies. What is the probability that fewer than 18 are a Juniors? (0.5 pt)

Use the `pbinom` function to find the cummulative probability.

```
pbinom(q = 17, size = 100, prob = 0.2)
```

```
## [1] 0.271189
```

We can also check the solution by the explicit formula of probability mass function.

```
n.juniors <- 0:17
n <- 100
sum( choose(n, n.juniors) * 0.2 ^ n.juniors * 0.8 ^ (n - n.juniors) )
```

```
## [1] 0.271189
```

c) Suppose she encounters 100 enemies. What is the probability that at least 65 are Freshmen? (0.5 pt)

```
1 - pbinom(q = 64, size = 100, prob = 0.5)
```

```
## [1] 0.001758821
```

We can also check the solution by the explicit formula of probability mass function.

```
n.freshmen <- 65:100
n <- 100
sum( choose(n, n.freshmen) * 0.5 ^ n.freshmen * 0.5 ^ (n - n.freshmen) )
```

```
## [1] 0.001758821
```

d) What is the probability that she sees her 4th Junior by (including) her twentieth encounter? (0.5 pt)

She may see her 4th Junior at 20th, 19th, $\ldots$, 4th encounter. Until the 20th encounter, there might be greater or equal to 4 many Juniors. If we choose to follow a negative binomial distribution, it would look like the following.

```
pnbinom(q = 16, size = 4, prob = 0.2)
```

```
## [1] 0.5885511
```

If we consider the opposite event: there are no more than 3 Juniors by the twentieth encounters, the number of Juniors follows the binomial distribution. Taking the complement results in the same probability value as shown above.

```
1 - pbinom(q = 3, size = 20, prob = 0.2)
```

```
## [1] 0.5885511
```

Exercise 3

In a football league, suppose a particular team's offensive line allows their quarterback to be sacked on 40% of their plays if they are not playing against the Bucs. On any given night, there is a 10% chance that they are playing against the Bucs, in which case the team will allow their quarterback to be sacked on 90% of the plays. *Note: Assume that a play either does or does not result in sack. You can only be sacked once in a play.*

a) Suppose your roommate saw a randomly selected highlight reel (play) where the quarterback got sacked. Based on this information, what is the probability that this team was playing against the Bucs? (1 pt)

b) Assume that the result of every play is independent. Out of the first 10 plays, your roommate says that the quarterback got sacked 7 times. What is the probability that this team was playing against the Bucs? (1.5 pt)

Givens:

$P[\text{Sacked} \mid \text{Bucs}'] = 0.40$

$P[\text{Bucs}] = 0.10$

$P[\text{Sacked} \mid \text{Bucs}] = 0.90$

a) Solve for $P[\text{Bucs} \mid \text{Sacked}]$

Through Bayes' Theorem:

$$P[\text{Bucs} \mid \text{Sacked}] = \frac{P[\text{Sacked} \mid \text{Bucs}] \cdot P[\text{Bucs}]}{P[\text{Sacked} \mid \text{Bucs}] \cdot P[\text{Bucs}] + P[\text{Sacked} \mid \text{Bucs}'] \cdot P[\text{Bucs}']}$$

$$= \frac{0.90 \cdot 0.10}{0.90 \cdot 0.10 + 0.40 \cdot (1 - 0.10)} = 1/5 = 0.20$$

b) Solve for $P[\text{Bucs} \mid \text{Sacked } 7/10 \text{ Plays}]$

Assuming independence, the number of times the quarterback is sacked, X, follows a binomial distribution.

- If it's the Bucs, $X \sim \text{binomial}(n=10, p=0.90)$

  $\therefore P[\text{Sacked } 7/10 \text{ Plays} \mid \text{Bucs}] = \binom{10}{7}(0.90)^7(1-0.90)^3 \approx \boxed{0.05740}$

- If it's not the Bucs, $X \sim \text{binomial}(n=10, p=0.40)$

  $\therefore P[\text{Sacked } 7/10 \text{ Plays} \mid \text{Bucs}'] = \binom{10}{7}(0.40)^7(1-0.40)^3 \approx \boxed{0.04247}$

Through Bayes' Theorem:

$$P[\text{Bucs} \mid \text{Sacked } 7/10 \text{ Plays}] = \frac{P[\text{Sacked } 7/10 \text{ Plays} \mid \text{Bucs}] \cdot P[\text{Bucs}]}{P[\text{Sacked } 7/10 \text{ Plays} \mid \text{Bucs}] \cdot P[\text{Bucs}] + P[\text{Sacked } 7/10 \text{ Plays} \mid \text{Bucs}'] \cdot P[\text{Bucs}']}$$

$$= \frac{0.05740 \cdot 0.10}{0.05740 \cdot 0.10 + 0.04247 \cdot (1 - 0.10)}$$

$$\approx 0.1306$$

Exercise 4

While taking Chloe for a bike ride, we encounter squirrels according to a Poisson process with a rate of 4 squirrels per 10 blocks.

a) What is the probability of not encountering any squirrels in 1 block? (0.5 pt)

b) What is the probability of not encountering any squirrels in 20 blocks? (0.5 pt)

c) What is the probability of encountering 4 squirrels in 10 blocks? (0.5 pt)

d) What is the probability of encountering at least 2 squirrels in 10 blocks? (0.5 pt)

e) What is the probability that exactly 4 of the 10 blocks contain no squirrels? (0.5 pt)

Let X_n be the number of squirrel encounters in n blocks.
The probability mass function for a Poisson distribution is: $f(x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}$

Given: λ follows a rate of $\frac{4 \text{ squirrels}}{10 \text{ blocks}}$

a) $\lambda = 0.40$ (squirrels / 1 block)

$$P[X_1 = 0] = \frac{(0.40)^0 \cdot e^{-0.40}}{0!} = e^{-0.40} = 1/e^{0.40} \approx 0.6703$$

b) $\lambda = 8$ (squirrels / 20 blocks)

$$P[X_{20} = 0] = \frac{8^0 \cdot e^{-8}}{0!} = e^{-8} = 1/e^8 \approx 0.0003355$$

c) $\lambda = 4$ (squirrels / 10 blocks)

$$P[X_{10} = 4] = \frac{4^4 \cdot e^{-4}}{4!} = 32/3e^4 \approx 0.1954$$

d) $\lambda = 4$ (squirrels / 10 blocks)

$$P[X_{10} \geq 2] = 1 - P[X_{10} = 1] - P[X_{10} = 0]$$

$$= 1 - \frac{4^1 \cdot e^{-4}}{1!} - \frac{4^0 \cdot e^{-4}}{0!} = 1 - 4/e^4 - 1/e^4$$

$$= \frac{e^4 - 5}{e^4} \approx 0.9084$$

e) By independence of the Poisson process, # of blocks with no squirrels follows a binomial distribution $\sim \text{binomial}(n = 10, p = 1/e^{0.40})$

$\therefore$ P[exactly 4/10 blocks have no squirrels] $= \binom{10}{4}\left(1/e^{0.40}\right)^4\left(1 - 1/e^{0.40}\right)^6$

≈ 0.05444