2.1 Discrete Random Variables

Random Variables

- Review: Sample Space
- Often more convenient to describe the elements of S numerically.

Definition 2.1-1

Given a random experiment with an outcome space S, a function X that assigns one and only one real number $X(s) = x$ to each element s in S is called a **random variable**. The **space** of X is the set of real numbers $\{x : X(s) = x, s \in S\}$, where $s \in S$ means that the element s belongs to the set S.

Random Variables (overview)

- A random variable associates a numerical value to each outcome of a random experiment.
- A random variable is **discrete** if it has a countable number of values. (can be infinite)
- A random variable is **continuous** if it is uncountable. (represents something on a continuous scale and can take any values in an interval)

Random Variables (notes)

- Sometimes, the sample space S has elements that are already real numbers. In this case, we can set the random variable X as the identity function, and the space of X is the same as S.
- Example: Rolling a fair die. S = {1,2,3,4,5,6}. For each element in S, (denoted as s), $X(s) = s$. The space of the random variable is also {1,2,3,4,5,6}.

Random variables – Simple Example

A fair coin is flipped. The set of possible outcomes is Heads and Tails.
$S = \{H, T\}$

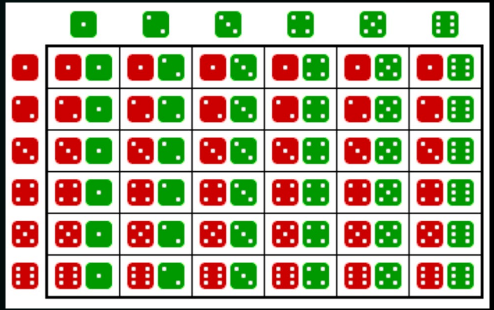
Let X be a function defined on S such that $X(H) = 0$ and $X(T) = 1$.

X is a real valued function that has outcome space S as its domain, and the set of real numbers, $\{x: x = 0,1\}$ as its range.

We call X a random variable, and the space associated with X is the set of numbers $\{0,1\}$.

Example: Defining a Random Variable

Consider the sample space for rolling two die:



If I am interested only in the **sum of the number of spots**, I can call this my random variable, X.

(X assigns a real number to each outcome in the original space. X({1,1})=2, X({1,2}) = 3, X({2,1}) = 3, …X({6,6}) = 12)

Example: Defining a Random Variable (continued)

- The space of X is now: {2,3,4,5,6,7,8,9,10,11,12}
- For convenience, we don't need the original sample space anymore.
  - We can use the space of X instead for this problem.
  - We can even denote it as S.
- S is called the **support** of X.

Discrete random variable

Let S be a one-dimensional sample space.

- If S is a subset of the real numbers, and contains a **countable** number of points, we call S a **discrete sample space**.
- Any random variable X with sample space, S, is called a **discrete random variable**.

Discrete random variable (examples)

- Number of “heads” in 3 flips of a coin S = {0,1,2,3} ✓
- Number of times I will go to Home Depot after quarantine:
  S = {0,1,2,…} ✓
- Hulk starts with 15mL of saliva for Covid testing but accidentally spills a random amount, **exactly** how much saliva does it have left? S = [0, 15) ✗
- Hulk start with 15mL saliva but accidentally spill random amount, how much saliva Hulk have left **rounded to the nearest mL**? S = {0,1,2,…14,15} ✓

Probability Mass Function

- For a random variable X, the probability that the random variable takes a value, x, is $P(X = x)$.
- This is typically denoted by $f(x)$.
- $f(x)$ is called the **probability mass function.**

Properties of a pmf

Definition 2.1-2

The pmf $f(x)$ of a discrete random variable X is a function that satisfies the following properties:

(a) $f(x) > 0, \qquad x \in S;$

(b) $\displaystyle\sum_{x \in S} f(x) = 1;$

(c) $\displaystyle P(X \in A) = \sum_{x \in A} f(x), \qquad \text{where } A \subset S.$

Probability mass function

- If a pmf satisfies all 3 properties in the previous definition, then it is a valid discrete probability distribution.
- If the pmf is simple, it may be written as a table or list.
- Often, it is written as a formula.

x	$f(x)$
x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$
.	.
.	.
.	.
x_n	$f(x_n)$

Quick definition:
A pmf is the probability distribution of a discrete random variable

Cumulative Distribution Function

- The function defined by

$$F(x) = P(X \leq x), -\infty < x < \infty$$

is called the **cumulative distribution function.**

You may also see this referred to as the **distribution function** of a random variable, X.

The most important definitions/terms (re-read these in your book)

- Random variable
- Support
- Probability mass function
- Cumulative distribution function

2.2 Expected Value

Expectation of X

- Let X be a discrete random variable with probability mass function f(x). The **expected value** of X can be denoted $E[X]$, μ, or μ_x, and is given by:

$$\boxed{E[X] = \mu = \mu_x = \sum_{\{all\ x\}} x \cdot f(x)}$$

Expected Value Example

Thanos has an assortment of gummy vitamins. The proportion of gummies is: 1/2 berry, 1/4 orange, 1/4 lemon.

Suppose every gummy Thanos eats will vaporize the following number of people:

Berry - 50 people; Orange - 30 people; Lemon - 40 people.

- Is this a valid probability distribution?
- If Thanos eats a random gummy, find the expected value of the number of people vaporized.

Expected Value Example

- Does the expected value need to be an element of the sample space?

- $E[X] = (50)(\frac{1}{2}) + (30)(\frac{1}{4}) + (40)(\frac{1}{4})$

Expectation of a function of X

Definition 2.2-1

If $f(x)$ is the pmf of the random variable X of the discrete type with space S, and if the summation

$$\sum_{x \in S} u(x)f(x), \qquad \text{which is sometimes written} \qquad \sum_{S} u(x)f(x),$$

exists, then the sum is called the **mathematical expectation** or the **expected value** of $u(X)$, and it is denoted by $E[u(X)]$. That is,

$$E[u(X)] = \sum_{x \in S} u(x)f(x).$$

Some properties of expectation

Theorem 2.2-1 When it exists, the mathematical expectation E satisfies the following properties:

(a) If c is a constant, then $E(c) = c$.

(b) If c is a constant and u is a function, then

$$E[c\,u(X)] = cE[u(X)].$$

(c) If c_1 and c_2 are constants and u_1 and u_2 are functions, then

$$E[c_1u_1(X) + c_2u_2(X)] = c_1E[u_1(X)] + c_2E[u_2(X)].$$

Let $u(X) = X$ for (a) – (c) for simplicity: examples:

a) E[5] = 5

b) E[2X] = 2*E[X]

c) E[2X + 5X] = 2*E[X] + 5*E[X] = 7*E[X]

c*) E[3X + 4X^2] = 3*E[X] + 4*E[X^2]