Stat 400 Discussion 6

Spring 2021 (Yu)

Exercise 1

Let $X_1, ..., X_n$ be an independent sample where the pdf of each X_i is $f(X_i|\theta) = \frac{1}{\theta}x^{\frac{1-\theta}{\theta}}; 0 < X_i < 1, 0 < \theta < \infty$.

a) What is the Method of Moments estimator, $\tilde{\theta}$ of θ?

Solution:

$$\mathbb{E}[X] = \int_0^1 x * f(x)dx$$
$$\mathbb{E}[X] = \int_0^1 x * \frac{1}{\theta}x^{\frac{1-\theta}{\theta}}dx$$
$$\mathbb{E}[X] = \frac{1}{\theta}\int_0^1 x^{\frac{1-\theta}{\theta}+1}dx$$
$$\mathbb{E}[X] = \frac{1}{\theta} * (\frac{x^{\frac{1-\theta}{\theta}+2}}{\frac{1-\theta}{\theta}+2}\Big|_0^1)$$
$$\mathbb{E}[X] = \frac{1}{\theta}(\frac{1}{\frac{1-\theta}{\theta}+2} - 0)$$
$$\mathbb{E}[X] = \frac{1}{\theta}(\frac{1}{\frac{1+\theta}{\theta}})$$
$$\mathbb{E}[X] = \frac{1}{1+\theta}$$
$$\text{plug in } \bar{X} \text{ for } \mathbb{E}[X]$$
$$\bar{X} = \frac{1}{1+\tilde{\theta}}$$
$$\text{solve for } \tilde{\theta}$$
$$\tilde{\theta} = \boxed{\frac{1}{\bar{X}} - 1}$$

b) What is the Maximum Likelihood estimator, $\hat{\theta}$ of θ?

Solution:

Let $L(\theta)$ be the likelihood function for the parameter θ. Then:

$$L(\theta) = f(\mathbf{X}|\theta) = \prod_{i=1}^{n} f(X_i|\theta)$$

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} x_i^{\frac{1-\theta}{\theta}}$$

$$L(\theta) = \theta^{-n} (\prod_{i=1}^{n} x_i)^{\frac{1}{\theta}-1}$$

$$l(\theta) = log(L(\theta)) = -nlog(\theta) + (\frac{1}{\theta} - 1) \sum_{i=1}^{n} log(x_i)$$

$$l(\theta) = -nlog(\theta) + \frac{1}{\theta} \sum_{i=1}^{n} log(x_i) - \sum_{i=1}^{n} log(x_i)$$

$$\frac{dl(\theta)}{d\theta} = \frac{-n}{\theta} - \frac{1}{\theta^2} \sum_{i=1}^{n} log(x_i)$$

Set $\frac{dl(\theta)}{d\theta}$equal to 0 and solve for $\hat{\theta}$

$$\frac{-n}{\hat{\theta}} - \frac{1}{\hat{\theta}^2} \sum_{i=1}^{n} log(x_i) = 0$$

$$\frac{-1}{\hat{\theta}} (n + \frac{1}{\hat{\theta}} \sum_{i=1}^{n} log(x_i)) = 0$$

$$\frac{1}{\hat{\theta}} \sum_{i=1}^{n} log(x_i) = -n$$

$$\hat{\theta} = \frac{-\sum_{i=1}^{n} log(x_i)}{n}$$

Let's make sure the second derivative is less than 0 at our proposed solution $\hat{\theta} = \frac{-\sum_{i=1}^{n} log(x_i)}{n}$.

$$\frac{d^2 l(\theta)}{d\theta^2} = \frac{d}{d\theta}\frac{dl(\theta)}{d\theta}$$

$$\frac{d^2 l(\theta)}{d\theta^2} = \frac{d}{d\theta}(\frac{-n}{\theta} - \frac{1}{\theta^2}\sum_{i=1}^{n} log(x_i))$$

$$\frac{d^2 l(\theta)}{d\theta^2} = \frac{n}{\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^{n} log(x_i)$$

Plug in $\hat{\theta}$ for θ :

$$\frac{d^2 l(\hat{\theta})}{d\hat{\theta}^2} = \frac{n}{\hat{\theta}^2} + \frac{2}{\hat{\theta}^3}\sum_{i=1}^{n} log(x_i)$$

Remember that $\frac{-\sum_{i=1}^{n} log(x_i)}{n} = -n\hat{\theta}$, so we have:

$$\frac{d^2 l(\hat{\theta})}{d\hat{\theta}^2} = \frac{n}{\hat{\theta}^2} - \frac{2}{\hat{\theta}^3}n\hat{\theta}$$

$$\frac{d^2 l(\hat{\theta})}{d\hat{\theta}^2} = \frac{n}{\hat{\theta}^2} - \frac{2n}{\hat{\theta}^2}$$

$$\frac{d^2 l(\hat{\theta})}{d\hat{\theta}^2} = \frac{-n}{\hat{\theta}^2}$$

Now since $0 < \theta < \infty$, $\frac{d^2 l(\hat{\theta})}{d\hat{\theta}^2} = \frac{-n}{\hat{\theta}^2} < 0$, so $\hat{\theta} = \boxed{\frac{-\sum_{i=1}^{n} log(x_i)}{n}}$ is indeed the MLE.

Exercise 2

The NCAA tournament, beginning with 64 teams, has 63 games. Razia has a lot of time on her hands, and she watches every one. Suppose, just before each game, Razia randomly picks a winner, believing each game is an i.i.d. Bernoulli(1/2). For the purposes of this question, let's run with that assumption.

Razia figures she will predict at least 40 games correctly. Let X denote the number of games she calls correctly. Use the Central Limit Theorem to approximate $P(X \geq 40)$.

Solution:

We know that X ~ Bernoulli(p) has E[X] = p and Var[X] = p(1-p). We also know the distribution of the *sum* of Bernoullis, right?

Therefore, since n = 63, we can use the Normal approximation to the binomial, where our mean is $63 * p = 63/2$ and our variance is $63 * p(1-p) = 63/4$.

Then

$$P(X \geq 40) = P(X - 63/2 \geq 40 - 63/2) = P(\frac{X - 63/2}{\sqrt{63/4}} \geq \frac{40 - 63/2}{\sqrt{63/4}}) = P(Z \geq \frac{40 - 63/2}{\sqrt{63/4}}) = .016$$

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