# Method of Moments (6.4)

# Bias and Variance of an Estimate

# Bias

The **bias** of an estimator, $\hat{\theta}$ is defined as:

$$\boxed{Bias = E[\hat{\theta}] - \theta}$$

If the bias of an estimator equals 0, then it is **unbiased.**

**i.e.** if **E[**$\hat{\theta}$**]** **=** $\theta$ **or** **E [**$\hat{p}$**]** **=** $p$ they are unbiased!

# notes

# Unbiased Estimation

If $\mathrm{E}[u(X_1, X_2, \ldots, X_n)] = \theta$,

Then $u(X_1, X_2, \ldots, X_n)$ is an **unbiased estimator** of $\theta$.

Otherwise, $u(X_1, X_2, \ldots, X_n)$ is biased

# notes

# Example of an Estimator that is Unbiased

MLE of $p$ from a Bernoulli sample of size $n$:

$$\hat{p} = \frac{1}{n}\sum_i X_i$$

If an estimator is unbiased, E$[u(X_1, X_2, \ldots X_n)] = \theta$.

Here, $u(X_1, X_2, \ldots, X_n)$ is $\hat{p}$. ($\theta$ is $p$)

If $X \sim Bern(p)$, E[X] = E[$\bar{X}$]=

$$E[\hat{p}] = E[\frac{1}{n}\sum X_i] = \frac{1}{n}E[\sum X_i]$$

$$= \frac{1}{n}E[X_1 + X_2 + \ldots + X_n]$$

$$= \frac{1}{n}(E[X_1] + E[X_2] + \ldots + E[X_n])$$

$$= \frac{1}{n}(p + p + \ldots + p) = p \quad \checkmark$$

This MLE is unbiased :)

# A Simple Example of a Biased Estimator

Take random Bernoulli samples:

$X_1, X_2, \ldots, X_n \sim \text{Bern}(p)$, where $p$ is unknown.

Instead of using $\bar{X}$ as my **estimator** for $p$, what if I don't care about likelihood and decide to blindly use $\hat{p}$ = ½ as my estimate? (Is that a good idea?)

What is the bias of this estimate?

Let $X_1, X_2, \dots, X_n \sim \text{Bern}(p)$, where $p$ is unknown.

What if we use $\hat{p} = X_1$ instead?

$E[\hat{p}] =$

$\text{Var}[\hat{p}] =$

# Bias

In previous example, if $\hat{p}$ = ½

Bias = E[$\hat{p}$] $- p$

= ½ $- p \neq 0$

it "could" equal 0 if we get lucky, but it's not always 0. biased!

# notes

# Moments Review

If I have a distribution, f(x):

$E[X^k]$ is its $k^{th}$ **raw moment**. e.g. $E[X] = \mu$

Also known as the "**moment about the origin**", or just "**moment**".

theoretical moments

$E[(X-\mu)^k]$ is its $k^{th}$ **central moment.** e.g. $Var[X]$

Also known as "**moment about the mean.**"

# Sample Moment

The k$^{th}$ **sample moment** is defined:

$$\frac{1}{n}\sum_{i=1}^{n} X_i{}^k$$

E.g. $\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$ is the 1$^{st}$ sample moment.

# Method of Moments

- One of the oldest methods to obtaining a parameter estimate (point estimate)
- Usually generates unbiased estimates!

# Method of Moments

1. Set the first sample moment equal to the first theoretical moment: $\frac{1}{n}\sum_{i=1}^{n} X_i \overset{set}{=} E[X]$
2. Set the second sample moment equal to the second theoretical moment: $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \overset{set}{=} E[X^2]$
3. Continue setting the third, fourth, etc. sample moments equal to the theoretical moments until # of equations equals # of parameters.
4. Solve for the parameters

# MOM Steps

- Step 1: Find E[X], the population mean.
  - This will be a function of θ. We will call it g(θ)
- Step 2: Set the population mean equal to the sample mean. g(θ) = $\bar{X}$.
- Step 3: Solve for θ.
- Step 4: Put a tilde over θ to signify that it is an estimator!

that you solved for in step 3

MOM estimator: $\tilde{\theta}$

# Example

**2.** Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x) = \frac{2(\theta - x)}{\theta^2} \qquad 0 < x < \theta \qquad \theta > 0.$$

a) Obtain the method of moments estimator of $\theta$, $\tilde{\theta}$.

b) Is $\tilde{\theta}$ an unbiased estimator for $\theta$?

c) Find $\mathrm{Var}(\tilde{\theta})$.

$X_1, X_2, \ldots X_n \overset{iid}{\sim} f(x) = \frac{2(\theta - x)}{\theta^2} \quad 0 < x < \theta, \quad \theta > 0.$

a)

# notes

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