Exercise 1

Spring 2021 STAT400 Homework 9 Solutions

oui oui baguette~

Elle needs to eat some chocolate pudding to recharge her superpowers before the government scientists capture her. She finds a stash of "3.25 oz" pudding packs and weighs them on a scale. She takes a sample of 9 puddings and finds that they have the following weights in ounces:

3.1, 3.1, 3.2, 3.3, 3.2, 3.2, 3.0, 3.2, 3.3.

Assume that the weights of puddings are normally distributed

a) (1 point) Construct a 95% Confidence Interval for μ, the true mean of these pudding packs.

b) (0.5 point) Construct a 90% Confidence Interval for μ, the true mean of these pudding packs.

c) (1 point) Construct a 90% Confidence Interval for σ, the population standard deviation of weights of all these pudding packs. (two sided interval)

d) (1 point) Construct a 90% Confidence Upper bound for σ, the population standard deviation of weights of all these pudding packs.

a) $\bar{x} = \frac{28.6}{9} = 143/45 \approx 3.1778$ ounces

$s = \sqrt{\frac{\sum_{i=1}^{9}(x_i-\bar{x})^2}{n-1}} = \sqrt{\frac{17}{1800}} \approx 0.097183 \rightarrow$ standard error $= \sqrt{\frac{17/1800}{9}} \approx 0.03239$

$t_{\alpha/2,df} = t_{0.025,8} \approx 2.3060 \leftarrow$ qt(0.975,8) or -qt(0.025,8), other variants

95% Confidence Interval for μ: $(3.1778 - 2.306 \cdot 0.03239, 3.1778 + 2.306 \cdot 0.03239)$

$(3.1031 \text{ oz}, 3.2525 \text{ oz})$

b) $t_{\alpha/2,df} = t_{0.05,8} \approx 1.8595 \leftarrow$ qt(0.95,8)

$\bar{x}$, s, and standard error remain the same from a)

90% Confidence Interval for μ: $(3.1778 - 1.8595 \cdot 0.03239, 3.1778 + 1.8595 \cdot 0.03239)$

$(3.1175 \text{ oz}, 3.2380 \text{ oz})$

c) $s^2 = \frac{17}{1800} \approx 0.009444$

$\chi^2_{\alpha/2,df} = \chi^2_{0.05,8} \approx 2.7326 \leftarrow$ qchisq(0.05,8)

$\chi^2_{1-\alpha/2,df} = \chi^2_{0.95,8} \approx 15.5073 \leftarrow$ qchisq(0.95, 8)

90% Two-Tailed Confidence Interval for σ^2: $\left(\frac{(9-1)\,17/1800}{15.5073}, \frac{(9-1)\,17/1800}{2.7326}\right) = (0.004872, 0.02765)$

90% Two-Tailed Confidence Interval for σ: $(0.06980 \text{ oz}, 0.16628 \text{ oz})$

d) $\chi^2_{\alpha,df} = \chi^2_{0.10,8} \approx 3.4895 \leftarrow$ qchisq(0.10,8)

90% Upper Bound Confidence Interval for σ^2: $\left(0, \frac{(9-1)\,17/1800}{3.4895}\right) = (0, 0.021652)$

90% Upper Bound Confidence Interval for σ: $(0 \text{ oz}, 0.14715 \text{ oz})$

Exercise 2

Dustin and Lucas decide to investigate Elle's claims about the pudding. They obtain a sample of 100 chocolate pudding packs and find that 70 of them contain less than 3.25oz of pudding.

a) (1 point) Construct a 99% CI for the overall proportion of pudding packs containing less than 3.25oz of pudding.$

b) (1 point) Construct a 92% CI for the overall proportion of pudding packs containing less than 3.25oz of pudding.$

a) $\hat{p} = 70/100 = 0.70$

Standard error $= \sqrt{\dfrac{0.70(1-0.70)}{100}} = \sqrt{\dfrac{21}{10000}} \approx 0.045826$

$Z_{\alpha/2} = Z_{0.005} \approx 2.5758 \leftarrow$ qnorm(0.995)

99% Confidence Interval for p: $(0.70 - 2.5758 \cdot 0.045826, 0.70 + 2.5758 \cdot 0.045826)$

$(0.58196, 0.81804)$

b) $\hat{p}$ and standard error remain the same

$Z_{\alpha/2} = Z_{0.04} \approx 1.7507 \leftarrow$ qnorm(0.96)

92% Confidence Interval for p: $(0.70 - 1.7507 \cdot 0.045826, 0.70 + 1.7507 \cdot 0.045826)$

$(0.61977, 0.78023)$

Exercise 3

Let $0 \leq p \leq 1$ and X be a discrete random variable with probability mass function. Given a random sample (iid) of size n,

x	0	1	2	3
$f(x)$	$\frac{2p}{3}$	$\frac{p}{3}$	$\frac{2(1-p)}{3}$	$\frac{1-p}{3}$

a) (1 point) Find an expression for the Maximum Likelihood Estimator of p, $\hat{p}$.

b) (1 point) Find an expression for the Method of Moments estimator of p, $\tilde{p}$.

a) 1. Identify likelihood function $L(p)$:

$$L(p) = \prod_{i=1}^{n} P(x_i | p) = \left(\frac{2p}{3}\right)^{n_0}\left(\frac{p}{3}\right)^{n_1}\left(\frac{2(1-p)}{3}\right)^{n_2}\left(\frac{1-p}{3}\right)^{n_3}$$

2. Take natural log of $L(p)$:

$$\log(L(p)) = n_0 \log\left(\frac{2p}{3}\right) + n_1 \log\left(\frac{p}{3}\right) + n_2 \log\left(\frac{2(1-p)}{3}\right) + n_3 \log\left(\frac{1-p}{3}\right)$$

$$= n_0 \log\left(\tfrac{2}{3}\right) + n_0 \log(p) + n_1 \log(p) - n_1 \log(3) + n_2 \log\left(\tfrac{2}{3}\right) + n_2 \log(1-p) + n_3 \log(1-p) - n_3 \log(3)$$

3. Take derivative of $\log(L(p))$ w.r.t. p and set to 0:

$$\frac{\partial \log(L(p))}{\partial p} = \frac{n_0}{p} + \frac{n_1}{p} - \frac{n_2}{1-p} - \frac{n_3}{1-p} = \frac{n_0+n_1}{p} - \frac{n_2+n_3}{1-p} = 0$$

4. Solve for $\hat{p}$ in terms of n:

$$(n_0+n_1)(1-p) - (n_2+n_3)p = 0$$

$$n_0 + n_1 - n_0 p - n_1 p - n_2 p - n_3 p = 0$$

$$n_0 + n_1 = p(n_0+n_1+n_2+n_3)$$

$$\hat{p} = \frac{n_0+n_1}{n_0+n_1+n_2+n_3}$$

b) 1. Find $E[X]$

$$E[X] = \sum_{x=0}^{3} x \cdot P(x) = 0 \cdot \frac{2p}{3} + 1 \cdot \frac{p}{3} + 2 \cdot \frac{2(1-p)}{3} + 3 \cdot \frac{1-p}{3} = \frac{p}{3} + \frac{4-4p}{3} + \frac{3-3p}{3} = \frac{7-6p}{3}$$

2. Set $E[X]$ equal to $\bar{x}$ and solve for $\tilde{p}$ in terms of x

$$7 - 6p = 3\bar{x} \rightarrow 6p = 7 - 3\bar{x}$$

$$\tilde{p} = \frac{7-3\bar{x}}{6} = \frac{7}{6} - \frac{\bar{x}}{2}$$

Exercise 4

Peele is interested in the average price of an entree at a French restaurant, but only has a random sample of 9 prices (prices are not listed on the menu). Using this random sample, Key calculates that a 90% confidence interval for the true mean is given by (65, 88). (You can assume the prices follow a normal distribution). Use this information to do the following:

a) (0.5 pt) Calculate $\bar{x}$. (Hint, what does a confidence interval look like?)
b) (1 pt) Calculate s.
c) (1 point) Construct a 95% confidence interval for the true mean price of all the entrees, μ.

a) $\bar{x}$ is the midpoint of a two-tailed Confidence Interval for mean

$$\therefore \bar{x} = \frac{65+88}{2} = \$76.50$$

b) Margin of Error $= t_{\alpha/2,df} \cdot \frac{s}{\sqrt{n}}$

$$= \text{Upper Bound} - \bar{x} = \bar{x} - \text{Lower Bound} = \$11.50$$

$t_{\alpha/2,df} = t_{0.05,8} \approx 1.8595 \leftarrow$ qt(0.95,8)

$$s = \frac{MOE}{t_{0.05,8}} \cdot \sqrt{n} = \frac{11.50}{1.8595} \cdot \sqrt{9} = \$18.5529 \approx \$18.55$$

c) $\bar{x} = \$76.50$, $s = \$18.5529$

$t_{\alpha/2,df} = t_{0.025,8} \approx 2.3060 \leftarrow$ qt(0.975,8)

Standard error $= \frac{s}{\sqrt{n}} = \frac{18.5529}{\sqrt{9}} = \frac{18.5529}{3} = 6.1843$

95% Confidence Interval for μ: $(76.50 - 2.3060 \cdot 6.1843, 76.50 + 2.3060 \cdot 6.1843)$

$$(\$62.23898, \$90.76102)$$

$$(\$62.24, \$90.76)$$