Exercise 1

Let $X_1, X_2, ...X_5 \sim Poisson(\lambda)$

a) (1.5 point) Find an expression for the MLE of λ, $\hat{\lambda}$.

b) (1.5 point) Find an expression for the MOM of λ, $\tilde{\lambda}$.

c) (0.5 point) Find an estimate of λ when $x_1 = 2, x_2 = 7, x_3 = 3, x_4 = 1, x_5 = 2$

$$f(x;\lambda) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

a) 1. Find expression for likelihood:

$$L(\lambda) = \prod_{i=1}^{5} f(x_i;\lambda) = \frac{e^{-\lambda} \cdot \lambda^{x_1}}{x_1!} \cdot \frac{e^{-\lambda} \cdot \lambda^{x_2}}{x_2!} \cdot \ldots \cdot \frac{e^{-\lambda} \cdot \lambda^{x_5}}{x_5!} = \frac{e^{-5\lambda} \cdot \lambda^{\sum_{i=1}^{5} x_i}}{\prod_{i=1}^{5} x_i!}$$

2. Take natural log of likelihood expression:

$$\log(L(\lambda)) = -5\lambda + \sum_{i=1}^{5} x_i \cdot \log(\lambda) - \sum_{i=1}^{5} \log(x_i!)$$

3. Take derivative of $\log(L(\lambda))$ and set to 0:

$$\frac{d\log(L(\lambda))}{d\lambda} = -5 + \sum_{i=1}^{5} x_i \cdot \frac{1}{\lambda} - 0 = 0$$

4. Solve for $\hat{\lambda}$ in terms of x:

$$\frac{1}{\lambda} \sum_{i=1}^{5} x_i = 5$$

$$\hat{\lambda} = \frac{\sum_{i=1}^{5} x_i}{5} = \bar{x}$$

$$\hat{\lambda} = \bar{x}$$

b) 1. Find $E[X]$, the population mean μ.

$$\mu = E[X] = \lambda$$

2. Set the population mean equal to the sample mean, $\bar{x}$.

$$\lambda = \bar{x}$$

3. Solve for $\tilde{\lambda}$ in terms of x

$$\tilde{\lambda} = \bar{x}$$

c) $\lambda = \bar{x} = \frac{2+7+3+1+2}{5} = 3 \rightarrow \lambda = 3$

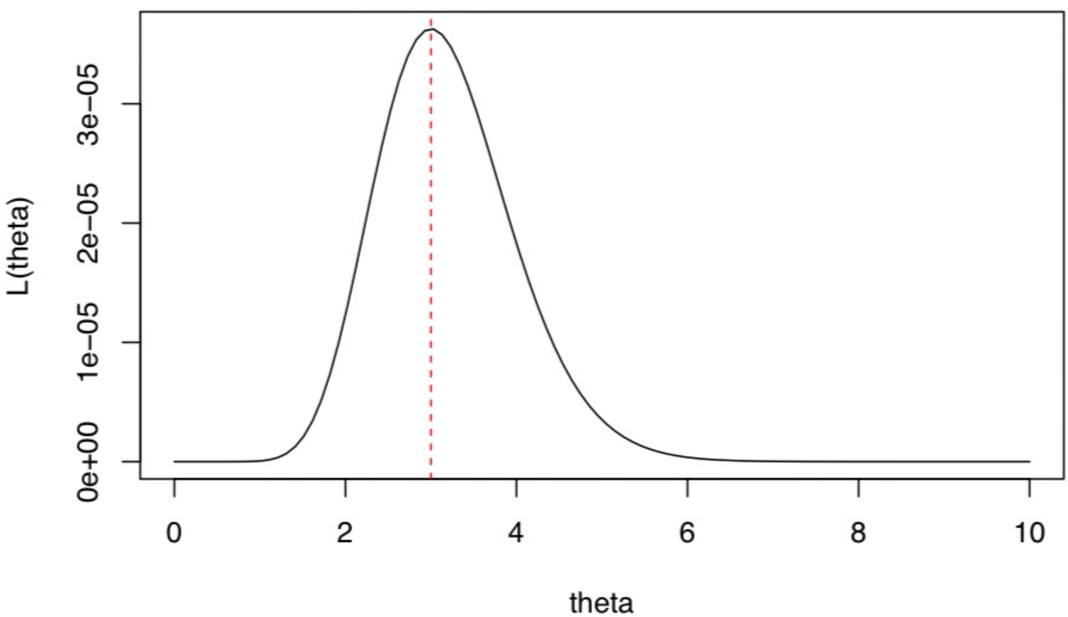
Exercise 1

d) (1.5 points) Using the 5 data points in 1(c) and the likelihood from 1(a), use R to plot the Likelihood as a function of θ. Must show code and plot output from R!

Plot θ on the x-axis: $0 < \theta < 10$. The y-axis is Likelihood, $L(\theta)$.

```r
L = function(lambda) {
  x = c(2, 7, 3, 1, 2)
  x_factorials = factorial(x)
  return(exp(-5 * lambda) * lambda ^ (sum(x)) / prod(x_factorials))
  }
```

```r
theta = seq(0, 10, length = 100)
plot(x = theta, y = L(theta), ylab = "L(theta)", xlab = "theta", type = "l")
abline(v = 3, col = "red", lty = 2)
```



Exercise 2

Let

$$f(x) = \frac{x}{\theta} e^{\frac{-x^2}{2\theta}}, \quad x > 0, \qquad \theta > 0.$$

(2 points) Find an estimator of θ using the MLE method.

1. Find expression for likelihood:

$$L(\theta) = \prod_{i=1}^{n} \frac{x_i}{\theta} e^{\frac{-x_i^2}{2\theta}} = \frac{x_1}{\theta} e^{\frac{-x_1^2}{2\theta}} \cdot \frac{x_2}{\theta} e^{\frac{-x_2^2}{2\theta}} \cdot \ldots \cdot \frac{x_n}{\theta} e^{\frac{-x_n^2}{2\theta}} = \frac{\prod_{i=1}^{n} x_i}{\theta^n} e^{\frac{-\sum_{i=1}^{n} x_i^2}{2\theta}}$$

2. Take natural log of likelihood expression:

$$\log(L(\theta)) = \log\left(\frac{\prod_{i=1}^{n} x_i}{\theta^n}\right) + \log\left(e^{\frac{-\sum_{i=1}^{n} x_i^2}{2\theta}}\right)$$

$$\log(L(\theta)) = \sum_{i=1}^{n} \log(x_i) - n\log(\theta) - \frac{\sum_{i=1}^{n} x_i^2}{2\theta}$$

3. Take derivative of $\log(L(\theta))$ and set to 0:

$$\frac{\partial \log(L(\theta))}{\partial \theta} = \frac{\sum_{i=1}^{n} x_i^2}{2\theta^2} - \frac{n}{\theta} = 0$$

4. Solve for $\hat{\theta}$ in terms of x:

$$\frac{\sum_{i=1}^{n} x_i^2}{2\theta^2} = \frac{n}{\theta}$$

$$\sum_{i=1}^{n} x_i^2 = 2n\theta$$

$$\boxed{\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2}{2n}}$$

Exercise 3

Suppose $X_1, X_2, ..., X_n$ are iid with mean θ and variance θ^2.

Suppose super stats slueths Peralta, Boyle, Santiago, and Diaz propose the following estimators:

$$\hat{\theta}_{Peralta} = X_1, \quad \hat{\theta}_{Boyle} = \frac{X_1 + 3X_2}{6}, \quad \hat{\theta}_{Santiago} = \bar{X}, \quad \hat{\theta}_{Diaz} = \frac{X_1 + X_2 + X_3}{3}.$$

a) (1 point) Find the Bias of each estimator (please provide 4 answers, 1 for each detective).
b) (2 points) Find the Variance of each estimator (4 answers).

a) $\text{Bias} = E[\hat{\theta}] - \theta$

Peralta: $E[\hat{\theta}] = \theta \rightarrow \text{Bias} = \theta - \theta = 0$

Boyle: $E[\hat{\theta}] = \frac{E[X_1] + 3E[X_2]}{6} = \frac{4\theta}{6} = \frac{2}{3}\theta \rightarrow \text{Bias} = \frac{2}{3}\theta - \theta = -\frac{\theta}{3}$

Santiago: $E[\hat{\theta}] = E[\bar{x}] = \theta \rightarrow \text{Bias} = \theta - \theta = 0$

Diaz: $E[\hat{\theta}] = E\left[\frac{X_1 + X_2 + X_3}{3}\right] = \frac{E[X_1] + E[X_2] + E[X_3]}{3} \rightarrow \text{Bias} = \theta - \theta = 0$

Boyle's estimator is the only biased one. The rest are unbiased.

b) Peralta: $\text{Var}[\hat{\theta}] = \text{Var}[X_1] = \theta^2$

Boyle: $\text{Var}[\hat{\theta}] = \text{Var}\left[\frac{1}{6}(X_1 + 3X_2)\right] = \left(\frac{1}{6}\right)^2 \text{Var}[X_1 + 3X_2]$

$= \frac{1}{36}\left(\text{Var}[X_1] + 3^2 \text{Var}[X_2]\right)$

$= \frac{1}{36}(\theta^2 + 9\theta^2) = \frac{5}{18}\theta^2$

Santiago: Because it is i.i.d, $\bar{x} = \frac{X_1 + X_2 + ... + X_n}{n} \rightarrow \text{Var}[\bar{x}] = \frac{1}{n^2}\text{Var}\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2} \cdot n\theta^2$

$\text{Var}[\hat{\theta}] = \text{Var}[\bar{x}] = \frac{\theta^2}{n}$

Diaz: $\text{Var}[\hat{\theta}] = \text{Var}\left[\frac{1}{3}(X_1 + X_2 + X_3)\right] = \frac{1}{9}\left[\theta^2 + \theta^2 + \theta^2\right]$

$\text{Var}[\hat{\theta}] = \frac{\theta^2}{3}$