2.3 Moment Generating Function

What is a Moment?

The $\boldsymbol{r^{th}}$ **moment** of a random variable, X, is $\boxed{E[X^r].}$

- Also called: **moment about the origin, raw moment**

The $\boldsymbol{r^{th}}$ **central moment** of a random variable, X, is the expected value of the rth power of the deviation of a random variable from its mean: $\boxed{E[(X-\mu_X)^r]}$

Moments

Moment	Real World	Statistics
0th	Total mass	Total Probability
1st	Center of Mass	Expected Value
2nd	Rotational Inertia (torque required for desired angular acceleration)	Variance (2nd central moment)
3rd	---	Skewness (3rd standardized moment)
4th	---	Kurtosis (4th standardized moment)

Moment Generating Function

Definition 2.3-1

Let X be a random variable of the discrete type with pmf $f(x)$ and space S. If there is a positive number h such that

$$E(e^{tX}) = \sum_{x \in S} e^{tx} f(x)$$

exists and is finite for $-h < t < h$, then the function defined by

$$\boxed{M(t) = E(e^{tX})}$$

is called the **moment-generating function of** X (or of the distribution of X). This function is often abbreviated as mgf.

Moment Generating Function

Suppose the sample space of X is S = $\{x_1, x_2, x_3, \ldots\}$

What is an expression for mgf?

$M(t) = e^{tx_1} f(x_1) + e^{tx_2} f(x_2) + e^{tx_3} f(x_3) + \ldots$

The coefficient of each e^{tx_i} is the probability, $f(x_i) = P(X = x_i)$

Simple mgf example

Example 2.3-5 If X has the mgf

$$M(t) = e^t\left(\frac{3}{6}\right) + e^{2t}\left(\frac{2}{6}\right) + e^{3t}\left(\frac{1}{6}\right), \qquad -\infty < t < \infty,$$

then the support of X is $S = \{1, 2, 3\}$ and the associated probabilities are

$$P(X = 1) = \frac{3}{6}, \qquad P(X = 2) = \frac{2}{6}, \qquad P(X = 3) = \frac{1}{6}.$$

Or, we could write f(x) = $\frac{4-x}{6}$, x = 1,2,3.

More Properties of MGFs

1. If two random variables have the same MGF, then they have the same distribution. i.e. if X and Y are random variables that have the same MGF: $M_X(t) = M_Y(t)$, then X and Y have the exact same distribution (pmf, cdf, etc)
2. For two independent random variables, X and Y, the MGF of their sum is the product of their MGFs:

$M_{X+Y}(t) = M_X(t)M_Y(t)$ (works for more than 2 as well)

More Properties of MGFs

3. The n^{th} derivative of $M_X(t)$ evaluated at t=0 is equal to the n^{th} moment, $E[X^n]$.

Examples

Moment Generating Function

1 Let X ~ Binom(n, p). The mgf is:

$M_x(t)$ = E[e^{tX}] = $\sum_{x=0}^{n} e^{tx}\binom{n}{x} p^x (1-p)^{n-x}$

= $\sum_{x=0}^{n}\binom{n}{x}(pe^t)^x (1-p)^{n-x}$ = $\boxed{[pe^t + (1-p)]^n}$

Computing the first two moments using the mgf:

$M'(t) = n[pe^t + (1-p)]^{n-1}pe^t.$

$M''(t) = n(n-1)[pe^t + (1-p)]^{n-2}p^2e^{2t} + n[pe^t + (1-p)]^{n-1}pe^t.$

$E[X] = M'(0) = np$

$E[X^2] = M''(0) = n(n-1)p^2 + np$

$Var[X] = E[X^2] - (E[X])^2 = n(n-1)p^2 + np - (np)^2$

$= np - np^2 = np(1-p)$

2 Example: known distributions

Suppose a random variable X has moment generating function: $M(t) = (\frac{2}{3} + \frac{1}{3}e^{t})^{10}$

What is the pmf of X?

$$f(x) = \binom{10}{x}\left(\frac{1}{3}\right)^{x}\left(\frac{2}{3}\right)^{10-x}$$

3

Say we have 3 random variables, W, X, Y $\sim$ Bernoulli(p)

Let Z be the sum of all three: Z = W + X + Y.

Show that Z $\sim$ Binom(n,p)

Note: The mgf of a Bernoulli random variable is

$M(t) = (1 - p + pe^t)$

Q: Can you do the same for $\sum Geometric$ = NB at home?