The Gamma and Normal Distributions

3.2, 3.3

The Gamma Distribution

Consider a Poisson process with rate λ:

Let a random variable, X, denote the waiting time until the αth occurrence.

X follows a Gamma Distribution.

The Gamma Function, Γ

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y}\, dy, \qquad 0 < t.$$

← This is the definition of the gamma function

$$\Gamma(t) = \left[-y^{t-1}e^{-y}\right]_0^\infty + \int_0^\infty (t-1)y^{t-2}e^{-y}\, dy$$
$$= (t-1)\int_0^\infty y^{t-2}e^{-y}\, dy \;=\; (t-1)\Gamma(t-1).$$

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\cdots(2)(1)\Gamma(1).$$

$$\Gamma(1) = \int_0^\infty e^{-y}\, dy = 1.$$

When n is an integer, $\Gamma(n) = (n-1)!$

Gamma Distribution $X \sim \text{Gamma}(\alpha, \theta)$

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \quad 0 \leq x < \infty$$

$$E[X] = \alpha\theta$$

$$Var[X] = \alpha\theta^2$$

Gamma Example

Customers arrive in a shop according to a Poisson process with a mean rate of 20 per hour. What is the probability that the shopkeeper will have to wait more than 10 minutes for the arrival of the 4th customer?

$$\int_{10}^{\infty} \frac{1}{\Gamma(4)3^4} x^{4-1} e^{-x/3}\, dx = 0.57$$

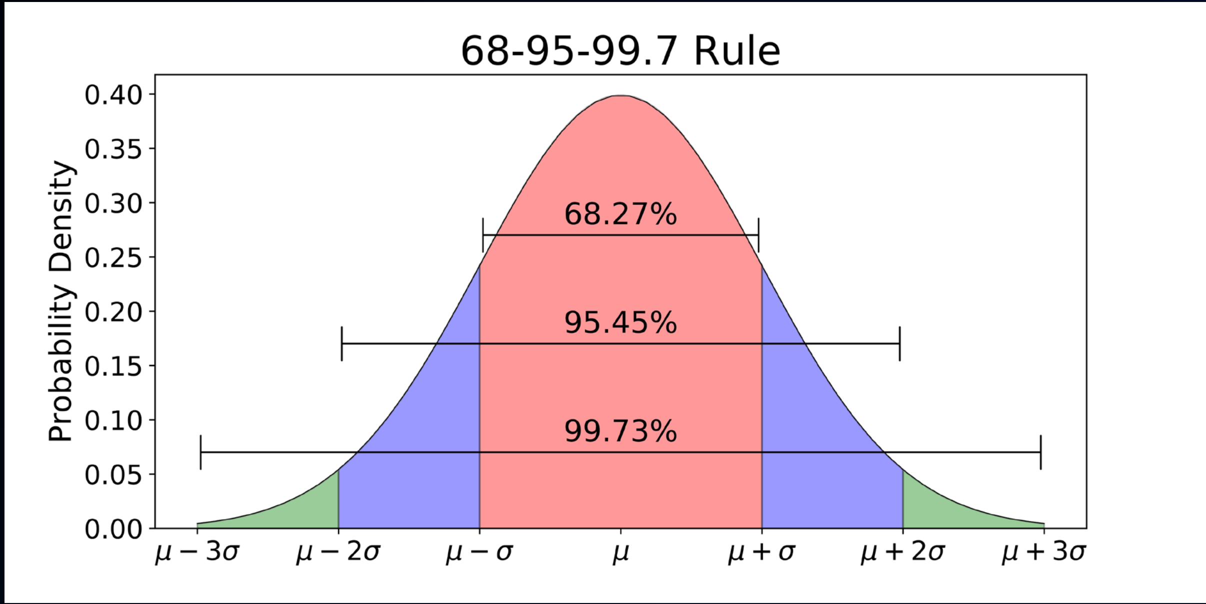
Normal Distribution

Normal Distribution

- Most important distribution in statistics
- Fits many natural phenomena such as IQ, measurement error, height, etc.
- A symmetric distribution with a central peak, and tails that taper off.

Normal Distribution – Empirical Rule

In a normal distribution, approximately 68/95/99.7% of the data falls within 1/2/3 standard deviations of the mean.



Normal Distribution $X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad -\infty < x < \infty$$

$$E[X] = \mu$$

$$Var[X] = \sigma^2$$

Normal Distribution

Let X ~ Normal(μ, σ^2)

- To find the P[a < X < b], one would need to evaluate the integral:

$$\int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

- A closed-form expression for this integral does not exist, so we need to use numerical integration techniques.

Notes about the Normal Distribution

The Normal Distribution is symmetric with a central peak:

- P[X > c] = P[X < -c]
- Mean = Median = Mode
- Half of the area is to the left/right of 0.

Examples: if $X \sim N(0,1)$

- $P[X \leq 0.2] = 0.5 + P[0 \leq X \leq 0.2]$
- $P[X \leq 0.3] = P[X \geq 0.7]$

Examples

Let $Z \sim N(0,1)$

a) Find P[Z >2] (0.0228)

b) Find P[-2 < Z < 2] (0.9544)

c) Find P[0 < Z < 1.73] (0.4582)

Linear Transformation Theorem

Let $X \sim N(\mu, \sigma^2)$. Then $Y = \alpha X + \beta$ follows also a normal distribution.

$$Y \sim N(\alpha\mu+\beta,\ \alpha^2\sigma^2)$$

Can convert any normal distribution to standard normal by subtracting mean and dividing sd:

- $\boxed{Z = \frac{X-\mu}{\sigma}}$

Using this theorem, we can see that $Z \sim N(0,1)$

(Recall) Let X have mean, $E[X]$, and variance, σ^2.
Let $Y = aX + b$. Then, Y has mean $aE[X] + b$, and variance $a^2\sigma^2$.

Example

Suppose the mass of Thor's hammers in kg (he has an infinite number) are distributed $X \sim N(10, 3^2)$.

Find the proportion of Thor's hammers that have mass larger than 13.4 kg. (if we randomly select a hammer, find the probability that its mass > 13.4 kg).

ans. $P[X > 13.4] = P[Z > 1.13] = 0.1292$

What is z?

- The value of z gives the number of standard deviations the particular value of X lies above or below the mean μ.

Examples

Normal Distribution

1 Cream and Fluttr knows that the daily demand for cupcakes is a random variable which follows the normal distribution with mean 43.3 cupcakes and standard deviation 4.6. They would like to make enough so that there is only a 5% chance of demand exceeding the number of cupcakes made. (How many should they make?)

$z=1.645$ $x = 51$

2 Suppose again that Thor's hammers are normally distributed with: E[X] = 10, Var[X] = 9.

Find the 25th percentile of X. (How much mass should a hammer have, in order to have more than 25% of all hammers)

Ans. z = -0.675 $\pi_{0.25}$ = 7.975

3 Stapleton's Auto Park of Urbana believes that total sales for next month will follow the normal distribution, with mean, μ, and a standard deviation, σ= \$300,000. What is the probability that Stapleton's sales will fall within \$150000 of the mean next month?

Ans. 0.6915 – 0.3085 = .383