Spring 2021 STAT400 Homework 6 Solutions

Exercise 1

Let X $\sim Uniform(1,2)$, and

$$f_Y(y) = \frac{3y^2}{8}, \qquad 0 \le y \le 2.$$

Suppose X and Y are independent.

a) (0.5 pts) Calculate $P[X > Y]$.

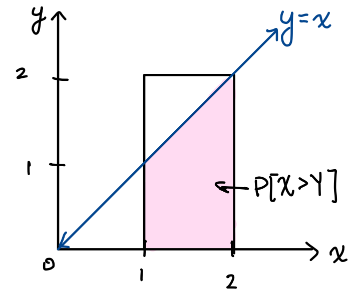
b) (0.5 pts) Calculate $P[X + Y > 3]$.

c) (0.5 pts) Calculate $P[X \times Y > 3]$.

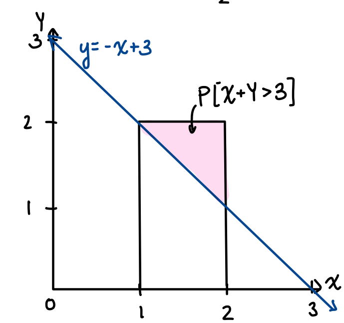
Because X and Y are independent, we know that $f_X(x) \cdot f_Y(y) = f_{X,Y}(x,y)$.

Therefore: $f_{X,Y}(x,y) = \frac{3y^2}{8}$, $1 \le x \le 2$, $0 \le y \le 2$

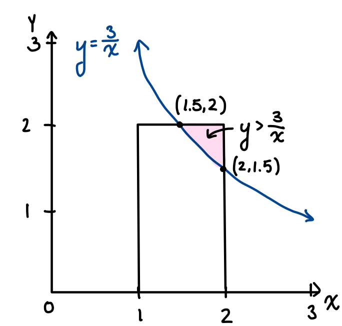
a) $$P[X>Y] = \int_1^2 \int_0^x \frac{3y^2}{8}\,dy\,dx$$
$$= \int_1^2 \left[\frac{y^3}{8}\right]_0^x dx = \int_1^2 \frac{x^3}{8}\,dx$$
$$= \left[\frac{x^4}{32}\right]_1^2 = \frac{16}{32} - \frac{1}{32} = \frac{15}{32} = 0.46875$$



b) $$P[X+Y>3] = \int_1^2 \int_{-x+3}^2 \frac{3y^2}{8}\,dy\,dx$$
$$= \int_1^2 \left[\frac{y^3}{8}\right]_{-x+3}^2 dx = \int_1^2 \frac{x^3 - 9x^2 + 27x - 19}{8}\,dx$$
$$= \frac{1}{8}\left[\frac{x^4}{4} - 3x^3 + \frac{27x^2}{2} - 19x\right]_1^2$$
$$= \frac{1}{8}\left[-4 - (-8.25)\right] = \frac{17}{32} = 0.53125$$



c) $$P[XY>3] = \int_{1.5}^2 \int_{\frac{3}{x}}^2 \frac{3y^2}{8}\,dy\,dx$$
$$= \int_{1.5}^2 \left[\frac{y^3}{8}\right]_{\frac{3}{x}}^2 dx = \int_{1.5}^2 1 - \frac{27}{8x^3}\,dx$$
$$= x + \left.\frac{27}{16x^2}\right]_{1.5}^2 = \left(2 + \frac{27}{64}\right) - (1.5 + 0.75)$$
$$= \frac{11}{64} = 0.171875$$



Exercise 2

Let the joint pdf for (X,Y) be

$$f(x,y) = \frac{x+y}{2}, \qquad x > 0, \quad y > 0, \quad 3 - 3y > x.$$

a) (1 pt) Find the probability $P(X < 2Y)$.

b) (0.75 pts) Find the marginal pdf of X, $f_X(x)$.

c) (0.75 pts) Find the marginal pdf of Y, $f_Y(y)$.

d) (1 pt) Find $Cov[X,Y]$. Are X and Y independent?

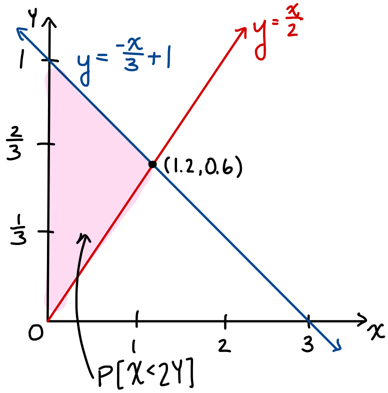
a) $$P[X<2Y] = \int_0^{1.2}\int_{\frac{x}{2}}^{-\frac{x}{3}+1} \frac{x+y}{2}\,dy\,dx$$

$$= \int_0^{1.2}\left[\frac{xy}{2} + \frac{y^2}{4}\right]_{\frac{x}{2}}^{-\frac{x}{3}+1} dx$$

$$= \frac{1}{144}\int_0^{1.2} -65x^2 + 48x + 36\,dx$$

$$= \frac{1}{144}\left[-\frac{65}{3}x^3 + 24x^2 + 36x\right]_0^{1.2}$$

$$= 7/25 = 0.28$$



b) $$f_X(x) = \int_0^{-\frac{x}{3}+1} \frac{x+y}{2}\,dy = \frac{1}{2}\int_0^{-\frac{x}{3}+1} x+y\,dy = \frac{1}{2}\left[xy + \frac{y^2}{2}\right]_0^{-\frac{x}{3}+1}$$

$$= \frac{1}{2}\left[\frac{-5x}{18} + \frac{2x}{3} + \frac{1}{2}\right] = \frac{-5x}{36} + \frac{x}{3} + \frac{1}{4}$$

$$f_X(x) = \frac{-5x^2}{36} + \frac{x}{3} + \frac{1}{4}, \quad 0 < x < 3$$

c) $$f_Y(y) = \int_0^{3-3y} \frac{x+y}{2}\,dx = \frac{1}{2}\int_0^{3-3y} x+y\,dx = \frac{1}{2}\left[\frac{x^2}{2} + xy\right]_0^{3-3y}$$

$$= \frac{1}{2}\left[\frac{9-18y+9y^2}{2} + \frac{6y-6y^2}{2}\right] = \frac{3y^2-12y+9}{4}$$

$$f_Y(y) = \frac{3y^2}{4} - 3y + \frac{9}{4}, \quad 0 < y < 1$$

d) $\text{Cov}[X,Y] = E[XY] - E[X]E[Y]$

$$E[X] = \int_0^3 \frac{-5x^3+12x^2+9x}{36}\,dx = 21/16 = 1.3125$$

$$E[Y] = \int_0^1 \frac{3y^3-12y^2+9y}{4}\,dy = 5/16 = 0.3125$$

$$E[X \cdot Y] = \iint_A xy \cdot f(x,y)\,dA = 0.5\int_0^3\int_0^{-\frac{1}{3}x+1} x^2y + xy^2\,dy\,dx$$

$$\left(-\frac{x}{3}+1\right)^2 = \frac{x^2}{9} - \frac{2x}{3} + 1$$

$$\left(\frac{-x}{3}+1\right)^3 = \frac{-x^3}{27} + \frac{x^2}{3} - x + 1$$

$$= 0.5\int_0^3 \left[\frac{1}{2}x^2y^2 + \frac{1}{3}xy^3\right]_0^{-\frac{1}{3}x+1} dx$$

$$= 0.5\int_0^3 \frac{1}{2}\left(\frac{x^4}{9} - \frac{2x^3}{3} + x^2\right) + \frac{1}{3}x\left(\frac{-x^3}{27} + \frac{x^2}{3} - x + 1\right) dx$$

$$= 0.5\int_0^3 \frac{1}{18}(x^4 - 6x^3 + 9x^2) + \frac{1}{81}(-x^4 + 9x^3 - 27x^2 + 27x)\,dx$$

$$= 0.5\int_0^3 \frac{9}{162}(x^4 - 6x^3 + 9x^2) + \frac{2}{162}(-x^4 + 9x^3 - 27x^2 + 27x)\,dx$$

$$= 0.5\int_0^3 \frac{1}{162}(9x^4 - 54x^3 + 81x^2) + \frac{1}{162}(-2x^4 + 18x^3 - 54x^2 + 54x)\,dx$$

$$= 0.5\int_0^3 \frac{1}{162}(7x^4 - 36x^3 + 27x^2 + 54x)\,dx = \frac{3}{10} = 0.30$$

$$\text{Cov}[X,Y] = E[X \cdot Y] - E[X] \cdot E[Y] = \frac{3}{10} - \left(\frac{21}{16} \cdot \frac{5}{16}\right) = -141/1280 \approx -0.11015625$$

Because $\text{Cov}[X,Y] \neq 0$, X and Y are not independent; a correlation exists between the two.

Exercise 3

Let H denote the number of times Harry irks Snape in one day. Let R denote the number of times Ron irks Snape in one day. Let the joint probability mass function for (H,R) be given as

$$f_{H,R}(h,r) = \frac{r+h}{24}, \qquad h \in \{1,2\}, \quad r \in \{0,1,2,3\}.$$

a) (0.5 pts) Find the marginal distribution of the number of times Harry irks Snape in one day $f_H(h)$.

b) (0.5 pts) Find the expected value of the number of times Harry irks Snape in one day, $E[H]$.

c) (0.5 pts) Find the variance of the number of times Harry irks Snape in one day, $Var[H]$.

d) (1 pt) Find the covariance of the number of times that Harry and Ron irk Snape in one day, $Cov[H,R]$.

e) (0.5 pts) Every time Ron irks Snape, he takes thirty-three (33) points away. Every time Harry irks Snape, he takes thirteen (13) points away. Assume that Snape starts every day by taking away thirty (30) points when he wakes up. Let X be the number of points that Snape will take away each day. Find $E[X]$.

a) $f_H(h) = \sum_{r=0}^{3} \frac{r+h}{24} = \frac{h}{24} + \frac{1+h}{24} + \frac{2+h}{24} + \frac{3+h}{24} = \frac{6+4h}{24},\ h \in \{1,2\}$

b) $E[H] = \sum_{h=1}^{2} \frac{6h+4h^2}{24} = \frac{10}{24} + \frac{28}{24} = 19/12 \text{ times} \approx 1.5833 \text{ times}$

c) $Var[H] = E[H^2] - (E[H])^2$

$E[H^2] = \sum_{h=1}^{2} \frac{6h^2+4h^3}{24} = \frac{10}{24} + \frac{56}{24} = 66/24 \text{ times}^2 = 2.75 \text{ times}^2$

$Var[H] = \frac{66}{24} - \left(\frac{19}{12}\right)^2 = 35/144 \text{ times}^2 \approx 0.2431 \text{ times}^2$

d) $Cov[H,R] = E[HR] - E[H]E[R]$

$f_R(r) = \sum_{h=1}^{2} \frac{r+h}{24} = \frac{r+1}{24} + \frac{r+2}{24} = \frac{2r+3}{24},\ r \in \{0,1,2,3\}$

$E[R] = \sum_{r=0}^{3} \frac{2r^2+3r}{24} = \frac{23}{12}$

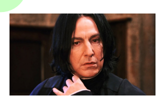
$E[HR] = \sum_{h=1}^{2} \sum_{r=0}^{3} hr\left(\frac{h+r}{24}\right) = 3$

$Cov[H,R] = 3 - \left(\frac{19}{12}\right)\left(\frac{23}{12}\right) = -5/144 \approx -0.03472$

e) $E[\text{Total Points}] = E[X] = E[30 + 13H + 33R] = 30 + 13E[H] + 33E[R]$

$= 30 + 13\left(\frac{19}{12}\right) + 33\left(\frac{23}{12}\right)$

$= 683/6 \text{ points} \approx 113.83 \text{ points}$



Exercise 4

Terry goes to a yogurt bar daily and loads up on his favorite flavors. Suppose the daily amount Terry scoops follows a normal distribution with mean of 1200 grams and a standard deviation of 150 grams.

a) (0.75 pts) For a random sample of 5 days, find the probability that the total amount of yogurt Terry scoops will exceed 5000g.

b) (0.5 pts) Find the probability that on any given day, Terry will scoop more than 1100g.

c) (0.75 pts) Let X represent the number of days in a year (365 days) that Terry scoops more than 1100g of yogurt. What is $P[80 \leq X \leq 90]$? (Use R for this question)

a) Let Y be total amount of yogurt scooped over 5 random days.

$Y \sim \text{Normal}(\mu = 6000g, \sigma^2 = 112500g^2)$

$P[Y > 5000g] = P\left[z > \frac{5000-6000}{\sqrt{112{,}500}}\right] = P[z > -2.9814]$

$P[z > -2.9814] \approx 0.9986$

b) Let X be amount of yogurt scooped in one day.

$X \sim \text{Normal}(\mu = 1200g, \sigma^2 = 22500 g^2)$

$P[X > 1100] = P\left[z > \frac{1100-1200}{150}\right] = P\left[z > -\frac{2}{3}\right] \approx 0.7475$

Exercise 4c)

(0.75 pts) Let X represent the number of days in a year (365) days that Terry scoops more than 1100g of yogurt. What is $P[80 \leq X \leq 90]$?

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prob_over_1100 = 1 - pnorm(1100, 1200, 150) #answer from b) ~0.7475

answer = pbinom(90, 365, prob_over_1100) - pbinom(79, 365, prob_over_1100)
answer

## [1] 3.294165e-89
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$P[80 \leq X \leq 90] = 3.2941648 \times 10^{-89}$