Exercise 1

Suppose the moment-generating function of X is

$$M_X(t) = 0.2 + 0.1 \cdot e^t + 0.3 \cdot e^{2t} + 0.4 \cdot e^{3t}$$

a) Calculate $E[X]$. (0.5 pt)

b) Calculate $Var[X]$. (0.5 pt)

a) $E[X] = M_X'(0)$

$$M_X'(t) = 0.1 \cdot e^t + 0.6 \cdot e^{2t} + 1.2 \cdot e^{3t}$$

$$E[X] = M_X'(0) = 0.1 + 0.6 + 1.2 = 1.9$$

b) $Var[X] = E[X^2] - (E[X])^2$

$\hookrightarrow E[X^2] = M_X''(0)$

$$M_X''(t) = 0.1 \cdot e^t + 1.2 \cdot e^{2t} + 3.6 \cdot e^{3t}$$

$$E[X^2] = 0.1 + 1.2 + 3.6 = 4.9$$

$$Var[X] = 4.9 - 1.9^2 = 1.29$$

Exercise 2

Let X ~ Bernoulli(p).

a) Derive an expression for the moment generating function $M_x(t)$. Note: you can check your answer with the given mgf in the table (textbook). (1 pt)

b) Use the mgf and calculus to derive expressions for E[X] and Var[X]. Show all steps for credit. (1 pt)

a) $M_X(t) = E[e^{tX}] = e^{t \cdot 0}(1-p) + e^{t \cdot 1}p = (1-p) + e^t p = 1 - p + p \cdot e^t$

b) $M_X'(t) = p \cdot e^t$

$M_X''(t) = p \cdot e^t$

$$E[X] = M_X'(0) = p$$

$$E[X^2] = M_X''(0) = p$$

$$Var[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

$$E[X] = p, \quad Var[X] = p(1-p)$$

Exercise 3

Let random variable X have probability mass function

$$f(x) = \frac{54}{3^x}, \quad x = 4, 5, 6, \ldots$$

a) Derive and simplify an expression for the moment-generating function of X, $M_X(t)$. (1 pt)

b) Calculate E[X] using the definition of E[X]. (The expected value formula) (0.5 pt)

c) Calculate $E[X]$ using the MGF method. (0.5 pt)

a) $$M_x(t) = E[e^{tX}] = \sum_{x=4}^{\infty} e^{tx} \cdot \frac{54}{3^x} = 54 \sum_{x=4}^{\infty} \frac{e^{tx}}{3^x} = 54 \sum_{x=4}^{\infty} \left(\frac{e^t}{3}\right)^x$$

$$= 54\left[\frac{(e^t/3)^4}{1 - e^t/3}\right] = 54\left[\frac{e^{4t}}{81} \cdot \frac{3}{3-e^t}\right]$$

$$M_x(t) = \frac{2e^{4t}}{3-e^t}$$

(Note that the sum converges when $e^t/3 < 1 \iff t < \log 3$, therefore this moment generating function exists for $t < \log 3$)

b) $$E[X] = \sum_{x=4}^{\infty} x \cdot \frac{54}{3^x} = 54 \sum_{x=4}^{\infty} \frac{x}{3^x}$$

$$E[X] = 54\left[4(1/3)^4 + 5(1/3)^5 + 6(1/3)^6 + \ldots\right]$$

$$\tfrac{1}{3}E[X] = 54\left[\qquad 4(1/3)^5 + 5(1/3)^6 + \ldots\right]$$

$$E[X] - \tfrac{1}{3}E[X] = 54\left[4(1/3)^4 + 1(1/3)^5 + 1(1/3)^6 + \ldots\right]$$

$$\tfrac{2}{3}E[X] = 54\left[4(1/3)^4 + \frac{(1/3)^5}{1-1/3}\right] = 3$$

$$\therefore E[X] = 9/2 = 4.5$$

c) $$M_x(t) = \frac{2e^{4t}}{3-e^t}$$

$$M_x'(t) = \frac{(3-e^t)(8e^{4t}) - (2e^{4t})(-e^t)}{(3-e^t)^2}$$

$$E[X] = M_x'(0) = \frac{(3-1)(8) - (2)(-1)}{(3-1)^2} = 18/4 = 9/2 = 4.5$$

Exercise 4

Students often worry about the time it takes to complete the final exam. Suppose students begin taking the exam at 8:00 am. Suppose that the completion time in hours, T, for a STAT 400 final exam follows a distribution with density

$$f(t) = \frac{2}{9} \cdot (3t - t^2),\ 0 \le t \le 3.$$

a) What is the probability that a randomly chosen student finishes the exam **beween** 9:00 am and 11:00am? (0.5 pt)

b) What is the probability that a randomly chosen student finishes the exam **during** the first hour of the exam? (0.5 pt)

c) Calculate the expected value of the time it take to finish the final, $E[T]$. (0.5 pt)

d) Calculate the 20th percentile of the time it takes to finish this exam. (0.5 pt)

e) **Use R:** Suppose 600 students take the final exam. Find the probability that **at least** 450 finish the exam between 9:00 and 11:00 am. **Show your code and output**. For partial credit, feel free to write comments explaining what you did. (1 pt)

a) P[finish between 9:00am and 11:00am] $= P[1 \le T \le 3]$

$$P[1 \le T \le 3] = \int_1^3 \frac{2}{9}(3t - t^2)\,dt = \frac{2}{9}\left[\frac{3}{2}t^2 - \frac{1}{3}t^3\right]_1^3$$

$$= \frac{2}{9}\left[\left(\frac{27}{2} - 9\right) - \left(\frac{3}{2} - \frac{1}{3}\right)\right] = 20/27 \approx 0.7407$$

b) P[finish during first hour] $= P[0 \le T \le 1]$

$$P[0 \le T \le 1] = \int_0^1 \frac{2}{9}(3t - t^2)\,dt = \frac{2}{9}\left[\frac{3}{2}t^2 - \frac{1}{3}t^3\right]_0^1$$

$$= \frac{2}{9}\left[\frac{3}{2} - \frac{1}{3} - 0\right] = 7/27 \approx 0.2593$$

Alternatively, because $0 \le t \le 3$,

P[finish during first hour] = 1 − P[finish between 9:00am and 11:00am]

$$= 1 - 20/27 = 7/27 \approx 0.2593$$

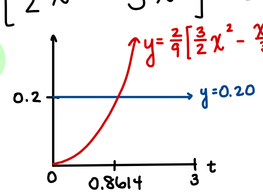
c)
$$E[T] = \int_{-\infty}^{\infty} t \cdot f(t)\,dt = \int_0^3 \frac{2}{9}(3t^2 - t^3)\,dt = \frac{2}{9}\left[t^3 - \frac{1}{4}t^4\right]_0^3$$

$$= \frac{2}{9}\left[27 - \frac{81}{4}\right] = 3/2 \text{ hours} = 1.5 \text{ hours}$$

d) We want to find a time $x \in [0,3]$ such that $P[T \le x] = 0.2$

$$P[T \le x] = \int_0^x \frac{2}{9}(3t - t^2)\,dt = \frac{2}{9}\left(\frac{3}{2}t^2 - \frac{1}{3}t^3\right]_0^x = \frac{2}{9}\left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right] = 0.20$$

Solve for x through graphing calculator: $x \approx 0.8614$



e) For $i \in \{1, 2, \ldots 599, 600\}$, let T_i be the number of hours it takes the ith student to finish.

Define: $X_i = \begin{cases} 1, & T_i \in [1,3] \\ 0, & \text{otherwise} \end{cases}$, so $X_i \sim Bern(p = \frac{20}{27})$

Because T_i are independent for all i, so are X_i.

Therefore, $\sum_{i=0}^{600} X_i \sim Binom(n = 600, p = \frac{20}{27})$.

$P\left[\sum_{i=0}^{600} X_i \geq 450\right] = 1 - P\left[\sum_{i=0}^{600} X_i \leq 449\right]$

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> 1 - pbinom(q = 449, size = 600, prob = 20/27)
[1] 0.3209366
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Alternatively:

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> pbinom(q = 449, size = 600, prob = 20/27, lower.tail = FALSE)
[1] 0.3209366
```

Exercise 5

Suppose the scores on a certain midterm were not very high and their probability density function was

$$f(s) = \frac{1}{14480} \cdot (3s+1),\ 20 \le s \le 100.$$

a) What is the expected value of the scores, $E[S]$? (0.5 pt)

b) What is the standard deviation of the scores, $SD[S]$? (1 pt)

c) What score is in the 90th percentile? (0.5 pt)

a) $E[S] = \int_{-\infty}^{\infty} s \cdot f(s)\, ds = \int_{20}^{100} \frac{1}{14480} \cdot (3s^2+s)\, ds = \frac{1}{14480}\left[s^3 + \frac{1}{2}s^2\right]_{20}^{100}$

$= \frac{1}{14480}\left[(1{,}000{,}000 + 5{,}000) - (8{,}000 + 200)\right]$

$= \frac{996800}{14480} = \frac{12460}{181}$ points $\approx$ 68.84 points

b) $SD[S] = \sqrt{Var[S]}$, $Var[S] = E[S^2] - (E[S])^2$

$E[S^2] = \int_{-\infty}^{\infty} s^2 \cdot f(s)\, ds = \int_{20}^{100} \frac{1}{14480}(3s^3+s^2)\, ds = \frac{1}{14480}\left[\frac{3}{4}s^4 + \frac{1}{3}s^3\right]_{20}^{100}$

$= \frac{1}{14480}\left[\left(75{,}000{,}000 + \frac{1{,}000{,}000}{3}\right) - \left(120{,}000 + \frac{8{,}000}{3}\right)\right]$

$= \frac{2{,}820{,}400}{543}$ points^2 ≈ 5194.11 points^2

$Var[S] = \frac{2{,}820{,}400}{543} - \left(\frac{12460}{181}\right)^2 = \frac{44{,}737{,}600}{98283}$ points^2

$SD[S] = \sqrt{\frac{44{,}737{,}600}{98283}} \approx$ 21.3352 points

c) We want to find $x \in [20,100]$ such that $P[S \le x] = 0.90$

$P[S \le x] = \int_{20}^{x} \frac{1}{14480}(3s+1) = 0.90$

$\frac{1}{14480}\left[\frac{3}{2}s^2 + s\right]_{20}^{x} = \frac{1}{14480}\left[\left(\frac{3}{2}x^2 + x\right) - (600+20)\right] = 0.90$

$\frac{3}{2}x^2 + x - 620 = 13032$

$\frac{3}{2}x^2 + x - 13652 = 0$

$x = \frac{-1 \pm \sqrt{1 - (4)(\frac{3}{2})(-13652)}}{3}$

$x = \frac{-1 + \sqrt{81913}}{3} \approx$ 95.0682 points

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