FACTORIAL EXPERIMENTS WITH 2 CROSSED FACTORS

Chapter 6

LEARNING OBJECTIVES

- Create and interpret interaction plots
- Derive main effect and interaction contrasts from set of simple effects
- Write main effects and interactions in terms of factorial parameters
- Explain how insignificant effects are pooled into error term
- Fit Model

EMISSIONS EXPERIMENT

- Hunter (1989) describes experiment investigating changes in CO emissions of burning fuel varying
  - E. Amount of Ethanol added (0.1, 0.2, 0.3)
  - F. Air/Fuel Ratio while burning (14, 15, 16)
- Replicate each combination 2 times
- Draw a picture of the design space and describe the randomization

BEGIN ANALYSIS WITH PLOTS!

- Every analysis should begin with a graphical exploration of the data
- Highlights unusual or missing values
- Identify trends in the data that can be incorporated into the model
- For factorial experiments, we are particularly interested in interaction plots

INTERACTION PLOTS

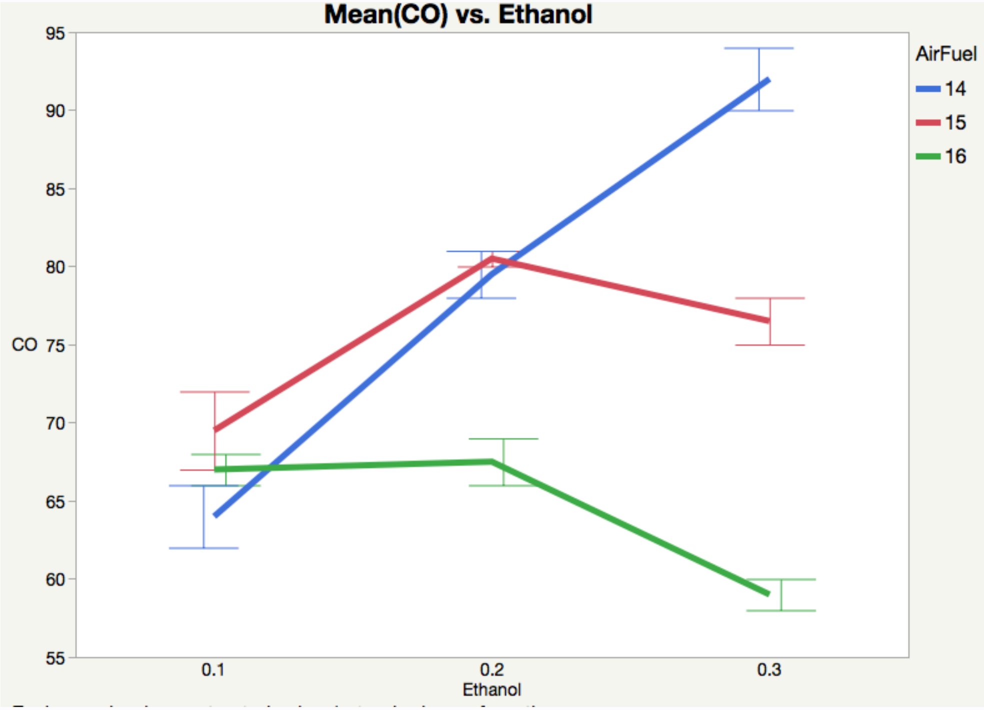
- Interaction plot based on the $\bar{y}_{ij}$.
- Here are $\bar{y}_{ij}$. for the fuel data

		Air/Fuel Ratio (F)		
		14	15	16
Ethanol (E)	0.1	64	69.5	67
	0.2	79.5	80.5	67.5
	0.3	92	76.5	59

- Choose Ethanol as X-axis, plot 3 different lines for each level of Air/Fuel ratio

INTERACTION PLOTS

- Hard to see any consistent behavior



- **Group**: Interpret changes in E for each level of F

ANALYSIS: ONE-FACTOR EFFECTS MODEL

- If we ignore the factorial structure of our treatments we could use the effects model

$$Y_{ijk} = \mu + \tau_{ij} + E_{ijk}$$

$i = 1, .., a; \quad j = 1, \dots, b; \quad k = 1, \dots, r_{ij}$

$E_{ijk} \sim^{iid} N(0, \sigma^2)$

- H_0: All τ_{ij} equal each other, H_A: at least one different
  - Not an interesting hypothesis for factorial experiments

- Like to investigate specific contrasts corresponding to main effects and interaction effects

FACTORIAL EFFECTS MODEL

- Re-express τ_{ij} that connect with the concept of main effects and interactions

$$\tau_{ij} = \alpha_i + \beta_j + \alpha\beta_{ij}$$

α_i = main effect parameter applies every time level *i* is used

β_j = main effect parameter applies every time level *j* is used

$\alpha\beta_{ij}$ = interaction parameter that picks up what's left of $\tau_{ij} - \alpha_i - \beta_j$

- Combining 2 one-way effects models + interaction

ESTIMABLE FUNCTIONS
SIMPLE EFFECTS

- Estimable functions must be some form of a contrast
- Individual parameters are not estimable without additional contraints

- Simple effect contrast (say with *j* level fixed):

$$\tau_{ij} = \alpha_i + \beta_j + \alpha\beta_{ij}$$

$$\sum_i c_i \tau_{ij} = \sum_i c_i \alpha_i + \beta_j \sum_i c_i + \sum_i c_i\, \alpha\beta_{ij}$$

$$= \sum_i c_i (\alpha_i + \alpha\beta_{ij})$$

FACTORIAL EFFECTS MODEL HYPOTHESIS TESTING

- **Ideally want to test following hypotheses**
  - $H_0: \alpha_1 = \cdots = \alpha_a$
  - $H_0: \beta_1 = \cdots = \beta_b$
  - $H_0: \alpha\beta_{11} =..= \alpha_{ab}$

None of these are testable

- **Will automatically test:**
  - $H_0: \alpha_1^* = \cdots = \alpha_a^*$
  - $H_0: \beta_1^* = \cdots = \beta_b^*$
  - $H_0: \alpha\beta_{11}^* =..= \alpha\beta_{ab}^*$

FACTORIAL EFFECTS MODEL HYPOTHESIS TESTING

- Reduced models don't just drop parameters out

- Easier to explain if you rewrite the model as

$$\mu^* + \alpha_i^* + \beta_j^* + \alpha\beta_{ij}^* \quad \text{where } \mu^* = \mu + \overline{\alpha\beta}_{..}$$

<u>Reduced Model</u>

- $H_0: \alpha_1^* = \cdots = \alpha_a^*$ → $(\mu^* + \alpha^*) + \beta_j^* + \alpha\beta_{ij}^*$
- $H_0: \beta_1^* = \cdots = \beta_b^*$ → $(\mu^* + \beta^*) + \alpha_i^* + \alpha\beta_{ij}^*$
- $H_0: \alpha\beta_{11}^* = .. = \alpha_{ab}^*$ → $(\mu^* + \alpha\beta^*) + \alpha_i^* + \beta_j^*$

- Takeaway: interactions never leave the model

FACTORIAL EFFECTS MODEL HYPOTHESIS TESTING

- Each hypothesis is tested using F-statistic approach
  1) A main effects (ssA)
  2) B main effects (ssB)
  3) AB interaction effects (ssAB)

- A df = a-1
- B df = b-1
- AB df = (a-1)(b-1)

These df tell us how many linearly independent contrasts we select for each group

Test on a-1 linearly independent main effects involving factor A

- Formulas for the sum-of-squares easy for balanced data but we won't concern ourselves with that

FACTORIAL EFFECTS MODEL HYPOTHESIS TESTING

- Gives following ANOVA table

Source	DF	SS	MS	F	p-value
A Main	a-1	*ssA*	ssA/(a-1)	msA/msE	p_A
B Main	b-1	*ssB*	ssB/(b-1)	msB/msE	p_B
AB Int	(a-1)(b-1)	*ssAB*	ssAB/ [(a-1)(b-1)]	msAB/msE	p_{AB}
Error	N-ab	*ssE*	ssE/[N-ab]		
Total	N-1	*ssTot*			

- P-values from central *F*-distribution with numerator DF corresponding to their row's DF and denominator DF equal to N-ab

POOLING INTERACTIONS

- If we conclude the interaction contrasts are insignificant then
$$\boldsymbol{E(msAB) = \sigma^2}$$
- We can improve our estimate of σ^2 with

$$\boldsymbol{msE^* = \frac{ssE + ssAB}{N - ab + (a-1)(b-1)} = \frac{ssE + ssAB}{N - a - b + 1}}$$

- Model-based error estimate
- Increases denominator DF which increases power when we test using msE^*
- Done using the main effect model

MAIN EFFECT MODEL

- The main effect model, or the two-way ANOVA model without interactions, is

$$Y_{ijk} = \mu + \alpha_i + \beta_j + E_{ijk}$$

- Testing for main effects same as before but with pooled error estimate

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