Homework 3

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Chapter 1

Exercise 3.1

1.1 Get Fisher Information

$$
\begin{aligned}
f(x;\theta) &= \frac{1}{\pi(1+(x-\theta)^2)} \\
\Rightarrow l(\theta) &= \sum_{i=1}^{n} \ln(f(X_i;\theta)) = -n\ln(\pi) - \sum_{i=1}^{n} \ln(1+(X_i-\theta)^2) \\
\Rightarrow l'(\theta) &= -2\sum_{i=1}^{n} \frac{\theta - X_i}{1+(\theta-X_i)^2} \\
\Rightarrow l''(\theta) &= -2\sum_{i=1}^{n}[\frac{1}{1+(\theta-X_i)^2} - \frac{2(\theta-X_i)^2}{[1+(\theta-X_i)^2]^2}] = -2\sum_{i=1}^{n} \frac{1-(\theta-X_1)^2}{[1+(\theta-X_i)^2]^2} \\
\Rightarrow I_n(\theta) &= -El''(\theta) \\
&= 2nE\frac{1-(\theta-X)^2}{[1+(\theta-X)^2]^2} \\
&= \frac{2n}{\pi}\int_R \frac{1-(x-\theta)^2}{(1+(x-\theta)^2)^3}dx \\
&= \frac{2n}{\pi}\int_R \frac{1-x^2}{(1+x^2)^3}dx \\
&= \frac{2n}{\pi}\int_R \frac{-1}{(1+x^2)^2} + 2\frac{2}{(1+x^2)^3}dx
\end{aligned}
$$

Also:

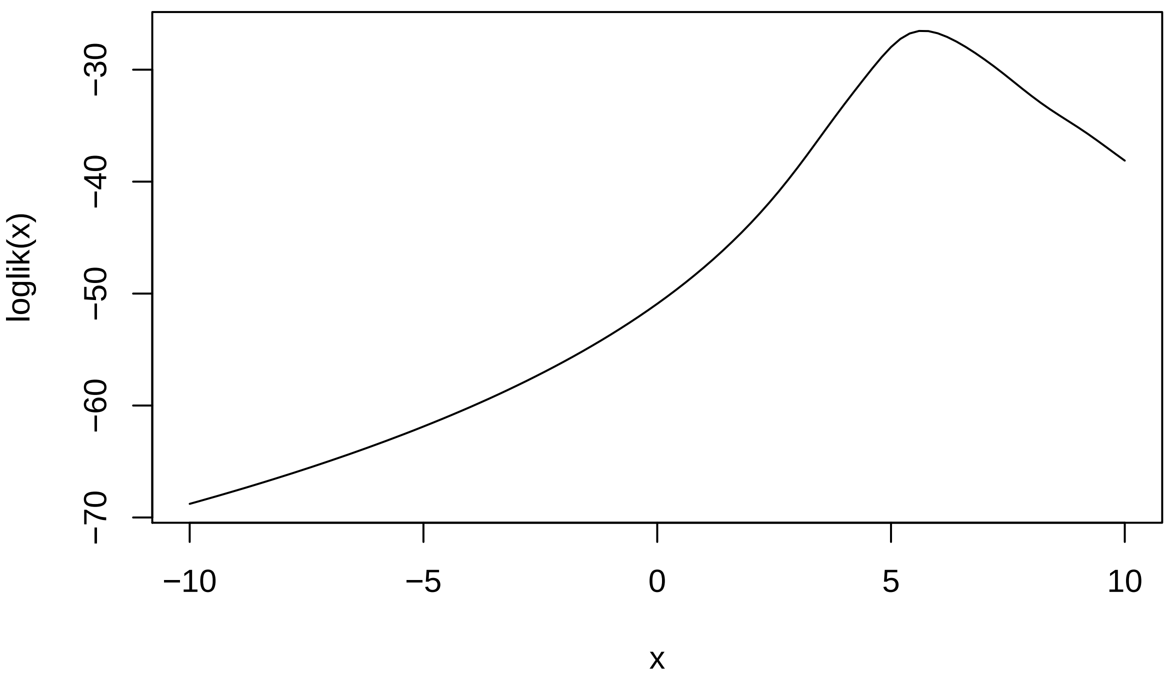
$$
\begin{aligned}
M_k &= \int_R \frac{1}{(1+x^2)^k}dx \\
&= \int_R \frac{1+x^2}{(1+x^2)^{k+1}}dx \\
&= M_{k+1} + \int_R \frac{x^2}{(1+x^2)^{k+1}}dx \\
&= M_{k+1} + \int_R \frac{2kx}{(1+x^2)^{k+1}}\frac{x}{2k}dx = M_{k+1} + \frac{1}{2k}M_k
\end{aligned}
$$

Since $M_1 = \pi$, we have $M_2 = \pi/2, M_3 = 3\pi/8$, then $I_n(\theta) = n/2$.

1.2 Implement loglikelihood with a random sample and plot against θ

Use the loglikelihood function we got from above, set n = 10 and plug in the generated sample value X_i, we can get the loglikelihood function. When generating sample, the location parameter was set to be $\theta = 5$. The loglikelihood function curve against θ are shown in Figure **??**:

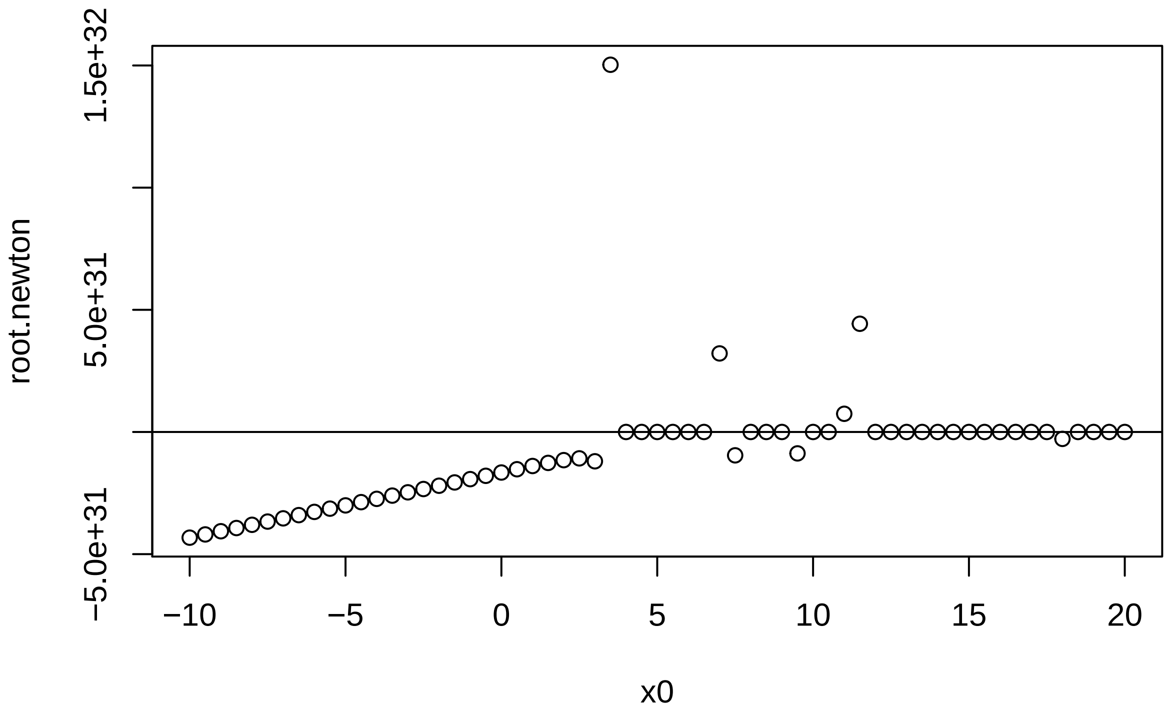
```
set.seed(20180909)
n <- 10
X <- rcauchy(n, location = 5, scale = 1)
loglik.0 <- function(theta) {
  l <- sum(dcauchy(X, location = theta, log = TRUE))
  l
}
loglik <- function(theta) {
  l <- sapply(theta, FUN = loglik.0)
  l
}
curve(loglik, -10, 10)
```



1.3 Newton-Raphson method

```
library(pracma)
## define the derivitive function
dev.loglik <- function(theta) {
  dev.l <- -2 * sum((theta-X)/(1+(theta-X)^2))
  dev.l
```

```
}
## define the hessian function
hessian.loglik <- function(theta) {
  h <- -2 * sum((1-(theta-X)^2) * (1+(theta-X)^2)^(-2))
  h
}
x0 <- seq(-10, 20, by = 0.5)
root.newton <- rep(0, length(x0))
for (i in 1:length(x0)) {
  root.newton[i] <- newtonRaphson(dev.loglik, x0 = x0[i], dfun = hessian.loglik)$root
}
plot(x0, root.newton)
abline(h = 5.442)
```



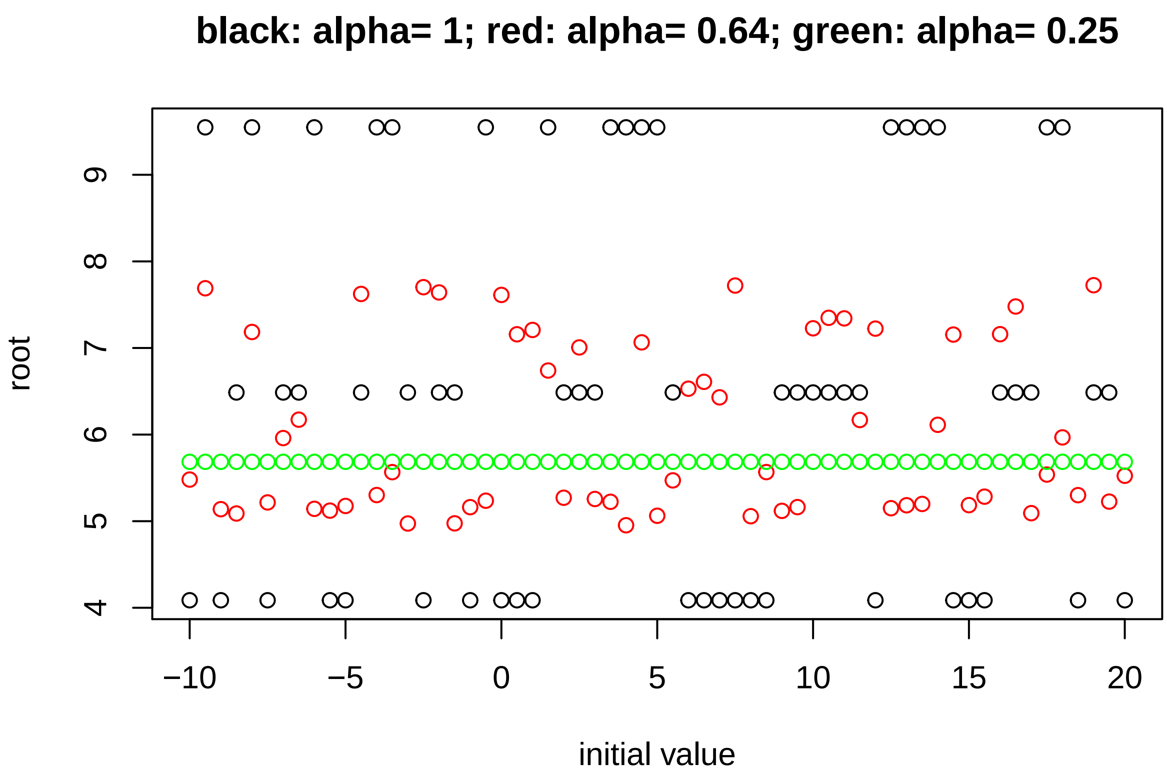

```
root.newton
```

```
##  [1] -4.324741e+31 -4.193577e+31 -4.062249e+31 -3.930748e+31 -3.799064e+31
##  [6] -3.667185e+31 -3.535100e+31 -3.402796e+31 -3.270261e+31 -3.137479e+31
## [11] -3.004436e+31 -2.871118e+31 -2.737510e+31 -2.603599e+31 -2.469374e+31
## [16] -2.334832e+31 -2.199981e+31 -2.064850e+31 -1.929508e+31 -1.794100e+31
## [21] -1.658922e+31 -1.524582e+31 -1.392396e+31 -1.265439e+31 -1.151924e+31
## [26] -1.079358e+31 -1.199750e+31  1.502957e+32  2.056366e+01  2.108229e+01
## [31]  5.685422e+00  5.685422e+00  5.685422e+00  5.685422e+00  3.215974e+31
## [36] -9.558888e+30  1.937744e+01  2.108229e+01  5.685422e+00 -8.759488e+30
## [41]  2.108229e+01  5.685422e+00  7.439560e+30  4.429077e+31  2.056366e+01
## [46]  2.056366e+01  2.056366e+01  2.108229e+01  2.108229e+01  2.108230e+01
## [51]  1.937743e+01  2.056366e+01  2.056366e+01  1.937744e+01  1.937743e+01
## [56]  1.937744e+01 -2.825479e+30  2.056366e+01  2.056366e+01  2.056366e+01
## [61]  2.056366e+01
```

We can see that Newton method doesn't converge when initial value is not close to the real root.

1.4 Fixed point method

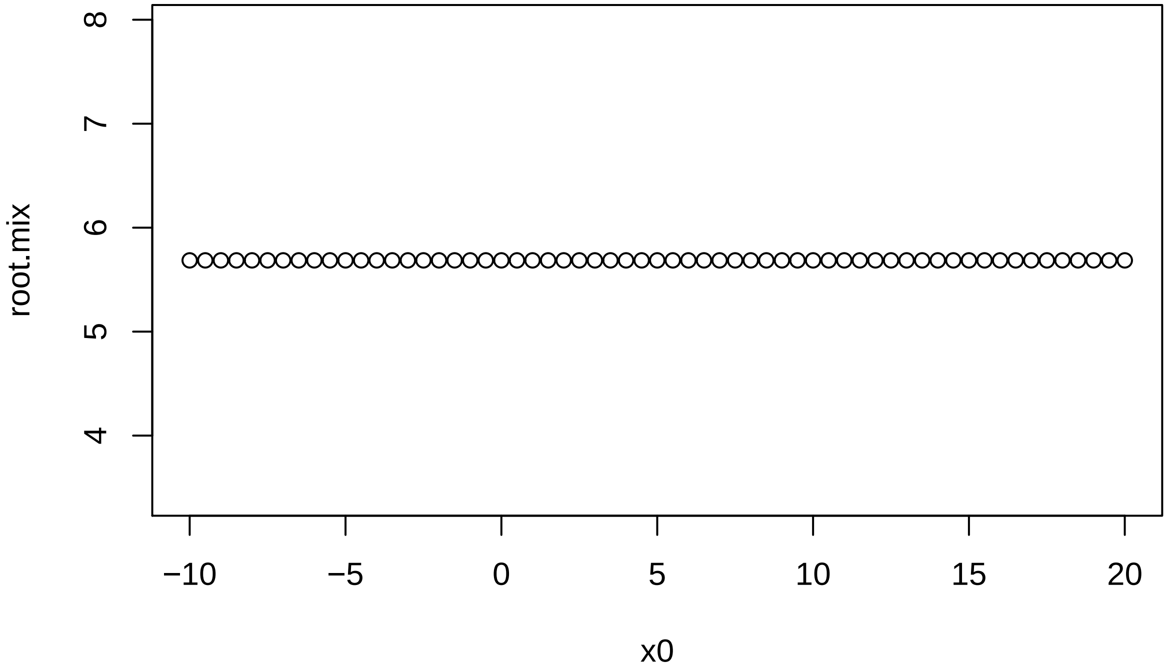
```
## self-defined fixed point methods to find mle
## input gradiant of loglikelihood function, x0 is initial value
FixPoint.mle <- function(dev.loglik, alpha, x0, maxiter = 100,
                         tol = .Machine$double.eps^0.5){
  x <- x0
  for (i in 1:maxiter) {
   x.new <- alpha * dev.loglik(x) + x
   if (abs(x.new - x) < tol) break
   x <- x.new
  }
  if (i == maxiter) warning("maximum iteration has reached")
  return(list(root = x, niter = i))
}
alpha <- c(1, 0.64, 0.25)
root.fixpoint <- matrix(0, ncol = length(alpha), nrow = length(x0))
for (i in 1:length(alpha)) {
  for (j in 1:length(x0)) {
    root.fixpoint[j, i] <- FixPoint.mle(dev.loglik = dev.loglik, alpha = alpha[i],
                                        x0 = x0[j])$root
  }
}
plot(x0, root.fixpoint[, 1], ylim = c(min(root.fixpoint), max(root.fixpoint)),
     ylab = "root", xlab = "initial value",
     main = paste0("black: ", expression(alpha), "= 1; red: ", expression(alpha),
     "= 0.64; green: ", expression(alpha), "= 0.25"))
points(x0, root.fixpoint[, 2], col = "red")
points(x0, root.fixpoint[, 3], col = "green")
```



1.5 Fisher scoring and Newton-Raphson

```
## Self-defined fisher scoring method to find mle.
## input gradiant of loglikelihood and sample fisher information.
FisherScore.mle <- function(dev.loglik, information, x0, maxiter = 100,
                            tol = .Machine$double.eps^0.5) {
  x <- x0
  for (i in 1:maxiter) {
    x.new <- x + dev.loglik(x) / information(x)
    if (abs(x.new - x) < tol) break
    x <- x.new
   }
   if (i == maxiter) warning("maximum iteration has reached")
   return(list(root = x, niter = i))
}
FisherNewton.mle <- function(dev.loglik, information, dfun, x0, maxiter = 100,
                             tol = .Machine$double.eps^0.5) {
  method.fisher <- FisherScore.mle(dev.loglik = dev.loglik, information = information,
                              x0 = x0, maxiter = maxiter, tol = tol)
  x.fisher <- method.fisher$root
  niter.fisher <- method.fisher$niter
  method.newton <- newtonRaphson(fun = dev.loglik, x0 = x.fisher, dfun = dfun, maxiter = maxiter,
                       tol = tol)
  return(list(root = method.newton$root, niter.fisher = niter.fisher,
              niter.newton = method.newton$niter))
}
inf.cauchy <- function(x) n/2
```

```
root.mix <- rep(0, length(x0))
for (i in 1:length(x0)) {
  root.mix[i] <- FisherNewton.mle(dev.loglik, inf.cauchy, dfun = hessian.loglik,
                                  x0 = x0[i])$root
}
plot(x0, root.mix)
```



1.6 comparing the different methods

```
library(microbenchmark)
## comparing the speed of different methods
## use starting point 5, alpha = 0.25 for fixed point method
newton.method <- newtonRaphson(fun = dev.loglik, x0 = 5, dfun = hessian.loglik)
fixpoint.method <- FixPoint.mle(dev.loglik = dev.loglik, alpha = 0.25, x0 = 5)
fishernewton.method <- FisherNewton.mle(dev.loglik = dev.loglik, information = inf.cauchy,
                                dfun = hessian.loglik, x0 = 5)
list(newton.niter = newton.method$niter, fixpoint.niter = fixpoint.method$niter,
     fishernewton.niter = c(fishernewton.method$niter.fisher,
                            fishernewton.method$niter.newton))
```

```
## $newton.niter
## [1] 6
##
## $fixpoint.niter
## [1] 17
##
## $fishernewton.niter
## [1] 8 1
```

Fixed point method is most stable but converges slowly compare to the other two methods. Newton-Raphson methods converges fastest but is the most unstably one. Fisher-Scoring converges slower than Newton, but is very stable and accuracy, after refining with Newton-Raphson methods. Also we can see that if we use fisher scoring root to be the initial value of Newton-Raphson method, it will converge very fast.