MLE of Location Parameter of a Cauchy Distribution by Using Different Algorithms

HW 3 of STAT 5361 Statistical Computing

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Chapter 1

Proofs and Loglikelihood Function Plot against θ

1.1 Proofs

The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\pi[1+(X_i-\theta)^2]}$$

Hence, the log likelihood fucntion is

$$l(\theta) = \log L(\theta) = \log \prod_{i=1}^{n} \frac{1}{\pi[1+(X_i-\theta)^2]} = -n\log\pi - \sum_{i=1}^{n} \log[1+(\theta-X_i)^2] \tag{1.1}$$

Further

$$l'(\theta) = -\sum_{i=1}^{n} \frac{d}{d\theta}\log[1+(\theta-X_i)^2] = -2\sum_{i=1}^{n} \frac{\theta-X_i}{1+(\theta-X_i)^2} \tag{1.2}$$

Compute the second derivative according to $l'(\theta)$

$$l''(\theta) = -2\sum_{i=1}^{n} \frac{d}{d\theta}\frac{\theta-X_i}{1+(\theta-X_i)^2} = -2\sum_{i=1}^{n} \frac{1-(\theta-X_i)^2}{[1+(\theta-X_i)^2]^2} \tag{1.3}$$

Therefore, the Fisher information is

$$\begin{aligned}
I_n(\theta) &= -E_X[l''(\theta)] \\
&= 2nE_X\left[\frac{1-(\theta-X)^2}{[1+(\theta-X)^2]^2}\right] \\
&= 2n\int_{-\infty}^{\infty} \frac{1-(x-\theta)^2}{[1+(x-\theta)^2]^2}\frac{1}{\pi[1+(x-\theta)^2]}dx \\
&= \frac{2n}{\pi}\int_{-\infty}^{\infty} \frac{1-x^2}{(1+x^2)^2}\frac{1}{1+x^2}dx = \frac{2n}{\pi}\int_{-\infty}^{\infty}\left(\frac{x}{1+x^2}\right)'\frac{1}{1+x^2}dx = \frac{2n}{\pi}\left[\frac{x}{(1+x^2)^2}\bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{2x^2}{(1+x^2)^3}\right] \\
&= \frac{4n}{\pi}\int_{-\infty}^{\infty}\frac{x^2}{(1+x^2)^3}dx = \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\tan^2\alpha}{(1+\tan^2\alpha)^3}\sec^2\alpha d\alpha = \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1-\cos 4\alpha}{8}d\alpha = \frac{4n}{\pi}\cdot\frac{\pi}{8} \\
&= \frac{n}{2}
\end{aligned} \tag{1.4}$$

1.2 Loglikelihood Function Plot against θ

The following plot is the curve of log likelihood function

```
set.seed(20180909)
sample <- rcauchy(10, 5)
```

Chapter 2

Newton-Raphson Method

Chapter 3

Fixed-Point Iterations

Chapter 4

Fisher Scoring Method and Newton-Raphson Method

Chapter 5

Comments