Different Ways to Solve MLE for Cauchy Distribution

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1 Proof

Density function:

$$f(x;\theta) = \frac{1}{\pi[1+(x-\theta)^2]} \tag{1.1}$$

The likelihood funcion:

$$L(\theta) = \prod_{i=1}^{n} f(X_i;\theta) \tag{1.2}$$

The loglikelihood funcion:

$$l(\theta) = \ln L(\theta) = \sum_{i=1}^{n} \ln f(X_i;\theta) = -n\ln\pi - \sum_{i=1}^{n} \ln[1+(\theta-X_i)^2] \tag{1.3}$$

Compute the differential of loglikelihood funcion:

First derivative:

$$l'(\theta) = -2\sum_{i=1}^{n} \frac{\theta-X_i}{1+(\theta-X_i)^2} \tag{1.4}$$

Second derivative:

$$l''(\theta) = -2\sum_{i=1}^{n} \frac{1-(\theta-X_i)^2}{[1+(\theta-X_i)^2]^2} \tag{1.5}$$

Fisher information:

$$
\begin{aligned}
I_n(\theta) &= -E[(l''(\theta)] \\
&= 2E\left\{\sum_{i=1}^{n} \frac{1-(\theta-X_i)^2}{[1+(\theta-X_i)^2]^2}\right\} \\
&= 2nE\left\{\frac{1-(\theta-X)^2}{[1+(\theta-X)^2]^2}\right\} \\
&= 2n\int_{-\infty}^{\infty} \frac{1-(\theta-x)^2}{[1+(\theta-x)^2]^2}\frac{1}{\pi[1+(x-\theta)^2]}\mathrm{d}x \\
&= \frac{2n}{\pi}\int_{-\infty}^{\infty} \frac{1-(\theta-x)^2}{[1+(\theta-x)^2]^2}\frac{1}{1+(x-\theta)^2}\mathrm{d}x \\
\\
&= \frac{2n}{\pi}\int_{-\infty}^{\infty} \frac{1-x^2}{(1+x^2)^2}\frac{1}{1+x^2}\mathrm{d}x \\
&= \frac{2n}{\pi}\int_{-\infty}^{\infty} (\frac{x}{1+x^2})'\frac{1}{1+x^2}\mathrm{d}x \\
&= \frac{2n}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}(\frac{x}{1+x^2}) \\
&= \frac{2n}{\pi}\frac{x}{1+x^2}\frac{1}{1+x^2}\Big|_{-\infty}^{\infty} - \frac{2n}{\pi}\int_{-\infty}^{\infty}\frac{x}{1+x^2}\mathrm{d}(\frac{1}{1+x^2}) \\
&= \frac{4n}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3}\mathrm{d}x \\
&= \frac{4n}{\pi}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\tan t)^2}{[(1+\tan t)^2]^3}\mathrm{d}\tan t \\
&= \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^2(\cos t)^2\mathrm{d}t \\
&= \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\frac{1}{2}\sin 2t)^2\mathrm{d}t \\
&= \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos 4t}{8}\mathrm{d}t \\
&= \frac{n}{2}
\end{aligned}
\tag{1.6}
$$

2 Plot

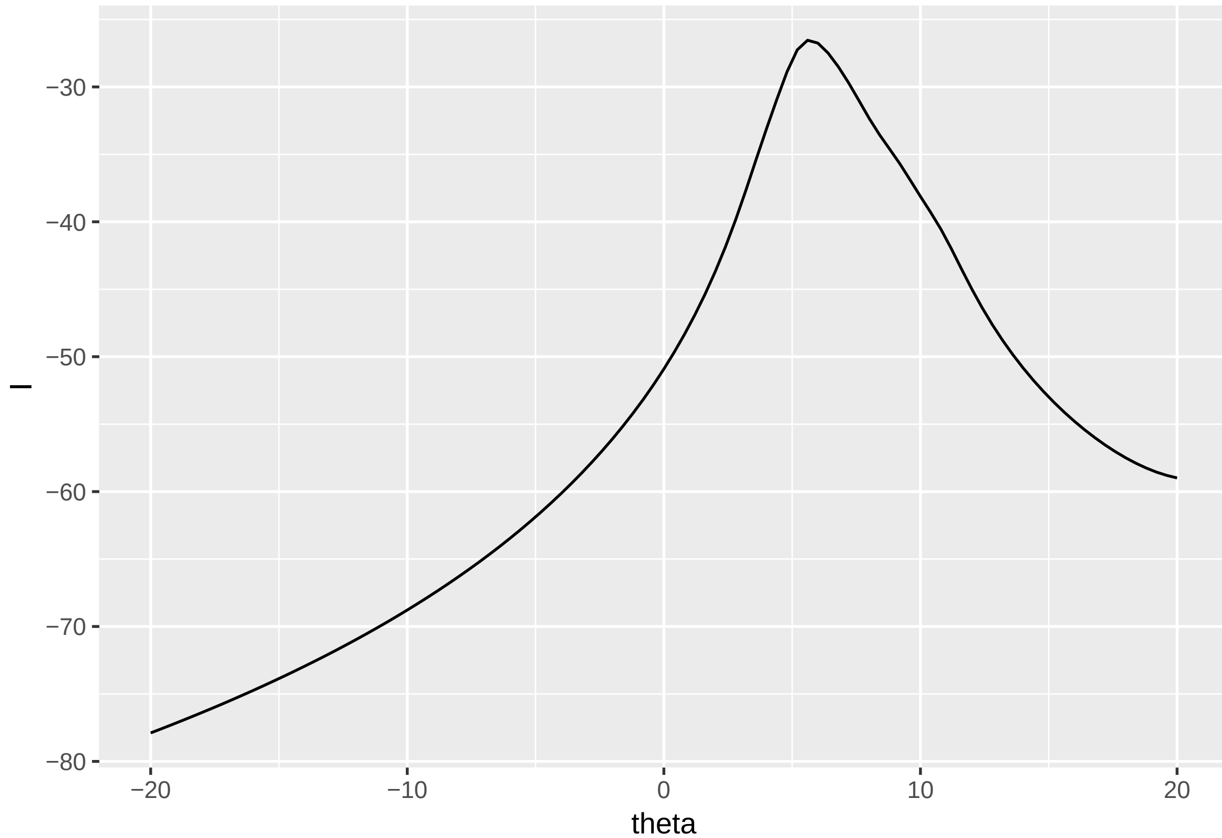
First, we need to define all functions above. And then We choose 10 random numbers according to density function above using “rcauchy”.

```
f=function(x,theta)1/(pi*(1+(x-theta)^2))

L=function(x,theta){
  prod=1;
  for (i in 1:length(x)){
    prod = prod*(1/(pi*(1+(x[i]-theta)^2)));
  }
  prod
}
```

```
l=function(x,theta){log(L(x,theta))}

l1=function(x,theta){
  sum=0;
  for(i in 1:length(x)){
    sum=sum-2*(theta-x[i])/(1+(theta-x[i])^2)
  }
  sum
}

l2=function(x,theta){
  sum=0;
  for(i in 1:length(x)){
    sum=sum-2*(1-(theta-x[i])^2)/(1+(theta-x[i])^2)^2
  }
  sum
}

set.seed(20180909)
X=rcauchy(10,5)
```

Next, we use those 10 numbers to plot the figure of $l(\theta)$

```
library("ggplot2")
ggplot(data.frame(x=c(-20,20)),aes(x=x)) +
  stat_function(fun=function(theta) l(X,theta)) +
  labs(x=expression("theta"),y="l")
```



3 Newton–Raphson method

According to the result, different initial value may cause different results which are totally different. However, some close number can lead to the same answer. For example, initial value 5,5.5,6,6.5 leads to the same root 5.685418.

```
i=seq(-10, 20, 0.5)
theta_N=matrix(0,1, length(i))
count_N=matrix(0,1, length(i))
for(k in 1:length(i)) {
  theta_N[k]=i[k]
  while (abs(l1(X,theta_N[k]))>0.001&&count_N[k]<10000) {
    temp=theta_N[k]-l1(X,theta_N[k])/l2(X,theta_N[k])
    theta_N[k]=temp
    count_N[k]=count_N[k]+1
  }
}
library(pander)
table_N=rbind(i,theta_N)
rownames(table_N)=c("i","root")
set.caption("Newton-Raphson method")
pander(table_N)
```

Table 1: Newton–Raphson method (continued below)

i	-10	-9.5	-9	-8.5	-8	-7.5	-7
root	-34927	-33867	-32807	-31744	-30681	-29615	-28548

Table 2: Table continues below

i	-6.5	-6	-5.5	-5	-4.5	-4	-3.5
root	-27480	-26409	-25336	-24262	-23185	-22105	-21024

Table 3: Table continues below

i	-3	-2.5	-2	-1.5	-1	-0.5	0
root	-39887	-37713	-35535	-33351	-31165	-28977	-26793

Table 4: Table continues below

i	0.5	1	1.5	2	2.5	3	3.5	4
root	-24623	-22487	-20436	-37213	-34868	-38758	30360	21.08

Table 5: Table continues below

i	4.5	5	5.5	6	6.5	7	7.5	8
root	19.38	5.685	5.685	5.685	5.685	25986	-30878	20.56

Table 6: Table continues below

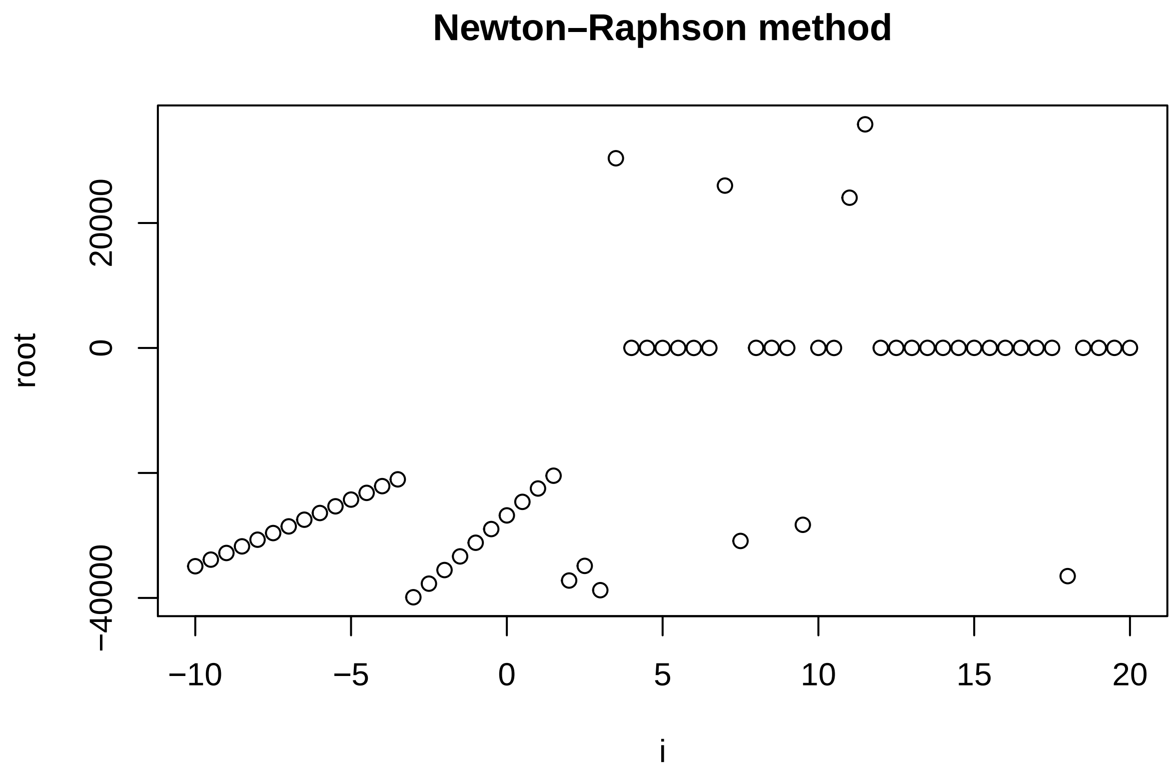
i	8.5	9	9.5	10	10.5	11	11.5	12
root	19.38	5.685	-28295	19.38	5.685	24047	35786	21.08

Table 7: Table continues below

i	12.5	13	13.5	14	14.5	15	15.5	16
root	21.08	21.08	19.38	19.38	19.38	20.56	21.08	21.08

i	16.5	17	17.5	18	18.5	19	19.5	20
root	20.56	20.56	20.56	-36510	21.08	21.08	21.08	21.08

```r
plot(i,theta_N,xlab = NULL, ylab = "root",main="Newton-Raphson method")
```



4 Fixed-point iterations

From the results, we can see the smaller initial value causes more stable result.

```
i=seq(-10, 20, 0.5)
a1=1
a2=0.64
a3=0.25
theta1_F=theta2_F=theta3_F=matrix(0,1, length(i))
count1_F=count2_F=count3_F=matrix(0,1, length(i))
for(k in 1:length(i)) {
  theta1_F[k]=i[k]
  while (abs(l1(X,theta1_F[k]))>0.001&&count1_F[k]<10000) {
    temp=a1*l1(X,theta1_F[k])+theta1_F[k]
    count1_F[k]=count1_F[k]+1
    theta1_F[k]=temp
  }
}
for(k in 1:length(i)) {
  theta2_F[k]=i[k]
  while (abs(l1(X,theta2_F[k]))>0.001&&count2_F[k]<10000) {
    temp=a2*l1(X,theta2_F[k])+theta2_F[k]
    count2_F[k]=count2_F[k]+1
    theta2_F[k]=temp
```

```
  }
}
for(k in 1:length(i)) {
  theta3_F[k]=i[k]
  while (abs(l1(X,theta3_F[k]))>0.001&&count3_F[k]<10000) {
    temp=a3*l1(X,theta3_F[k])+theta3_F[k]
    count3_F[k]=count3_F[k]+1
    theta3_F[k]=temp
  }
}
library(pander)
table_F=rbind(i,theta1_F,theta2_F,theta3_F)
rownames(table_F)=c("i","root for a=1","root for a=0.64","root for a=0.25")
set.caption("Fixed-point iterations")
pander(table_F)
```

Table 9: Fixed-point iterations (continued below)

i	-10	-9.5	-9	-8.5	-8	-7.5	-7
root for a=1	4.087	9.548	4.087	6.486	9.548	4.087	6.486
root for a=0.64	5.443	7.589	4.964	5.161	6.118	5.352	6.179
root for a=0.25	5.685	5.686	5.686	5.685	5.685	5.686	5.685

Table 10: Table continues below

i	-6.5	-6	-5.5	-5	-4.5	-4	-3.5
root for a=1	6.486	9.548	4.087	4.087	6.486	9.548	9.548
root for a=0.64	7.183	5.182	4.95	5.035	7.729	5.333	5.577
root for a=0.25	5.685	5.686	5.685	5.686	5.685	5.686	5.686

Table 11: Table continues below

i	-3	-2.5	-2	-1.5	-1	-0.5	0
root for a=1	6.486	4.087	6.486	6.486	4.087	9.548	4.087
root for a=0.64	5.148	7.727	7.452	5.174	5.067	5.435	7.729
root for a=0.25	5.686	5.686	5.685	5.686	5.685	5.685	5.685

Table 12: Table continues below

i	0.5	1	1.5	2	2.5	3	3.5
root for a=1	4.087	4.087	9.548	6.486	6.486	6.486	9.548
root for a=0.64	7.051	7.236	7.061	5.222	6.064	5.204	5.285
root for a=0.25	5.686	5.686	5.686	5.685	5.686	5.685	5.686

Table 13: Table continues below

i	4	4.5	5	5.5	6	6.5	7
root for a=1	9.548	9.548	9.548	6.486	4.087	4.087	4.087
root for a=0.64	4.953	6.031	5.149	5.373	6.514	6.574	7.066

Table 13: Table continues below

root for a=0.25	5.685	5.685	5.686	5.686	5.686	5.685	5.686

Table 14: Table continues below

i	7.5	8	8.5	9	9.5	10	10.5
root for a=1	4.087	4.087	4.087	6.486	6.486	6.486	6.486
root for a=0.64	7.729	5.143	5.217	4.99	5.138	7.571	7.542
root for a=0.25	5.686	5.685	5.686	5.686	5.685	5.686	5.685

Table 15: Table continues below

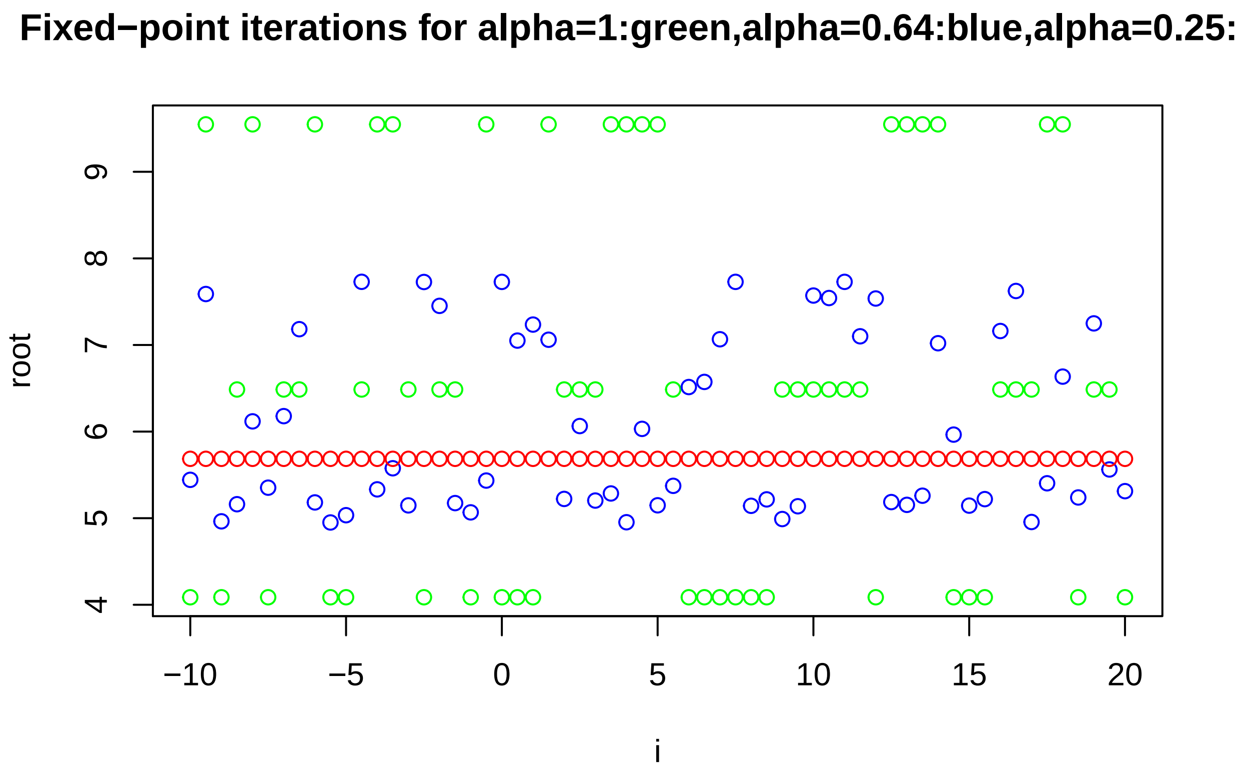
i	11	11.5	12	12.5	13	13.5	14
root for a=1	6.486	6.486	4.087	9.548	9.548	9.548	9.548
root for a=0.64	7.729	7.1	7.537	5.186	5.154	5.26	7.021
root for a=0.25	5.685	5.686	5.685	5.686	5.685	5.685	5.685

Table 16: Table continues below

i	14.5	15	15.5	16	16.5	17	17.5
root for a=1	4.087	4.087	4.087	6.486	6.486	6.486	9.548
root for a=0.64	5.965	5.145	5.22	7.161	7.624	4.956	5.403
root for a=0.25	5.685	5.685	5.685	5.686	5.685	5.685	5.686

i	18	18.5	19	19.5	20
root for a=1	9.548	4.087	6.486	6.486	4.087
root for a=0.64	6.635	5.239	7.249	5.563	5.312
root for a=0.25	5.686	5.685	5.685	5.686	5.685

```
plot(i,theta1_F,col="green",xlab = NULL, ylab = "root",main="Fixed-point iterations for alpha=1:green,al
points(i,theta2_F,col="blue")
points(i,theta3_F,col="red")
```



5 Fisher scoring

```
i=seq(-10, 20, 0.5)
theta_S=matrix(0,1, length(i))
count_S=matrix(0,1, length(i))
I=length(X)/2
for(k in 1:length(i)) {
  theta_S[k]=i[k]
  while (abs(l1(X,theta_S[k]))>0.001&&count_S[k]<10000) {
    temp=theta_S[k]+l1(X,theta_S[k])/I
    theta_S[k]=temp
    count_S[k]=count_S[k]+1
  }
}
library(pander)
table_S=rbind(i,theta_S)
rownames(table_S)=c("i","root")
set.caption("Fisher scoring")
pander(table_S)
```

Table 18: Fisher scoring (continued below)

i	-10	-9.5	-9	-8.5	-8	-7.5	-7	-6.5
root	5.685	5.685	5.685	5.685	5.685	5.685	5.685	5.685

Table 18: Fisher scoring (continued below)

Table 19: Table continues below

i	-6	-5.5	-5	-4.5	-4	-3.5	-3	-2.5
root	5.685	5.686	5.685	5.686	5.685	5.685	5.685	5.685

Table 20: Table continues below

i	-2	-1.5	-1	-0.5	0	0.5	1	1.5
root	5.685	5.685	5.685	5.685	5.685	5.685	5.685	5.685

Table 21: Table continues below

i	2	2.5	3	3.5	4	4.5	5	5.5
root	5.685	5.685	5.685	5.685	5.685	5.686	5.685	5.685

Table 22: Table continues below

i	6	6.5	7	7.5	8	8.5	9	9.5
root	5.685	5.685	5.686	5.685	5.685	5.686	5.686	5.685

Table 23: Table continues below

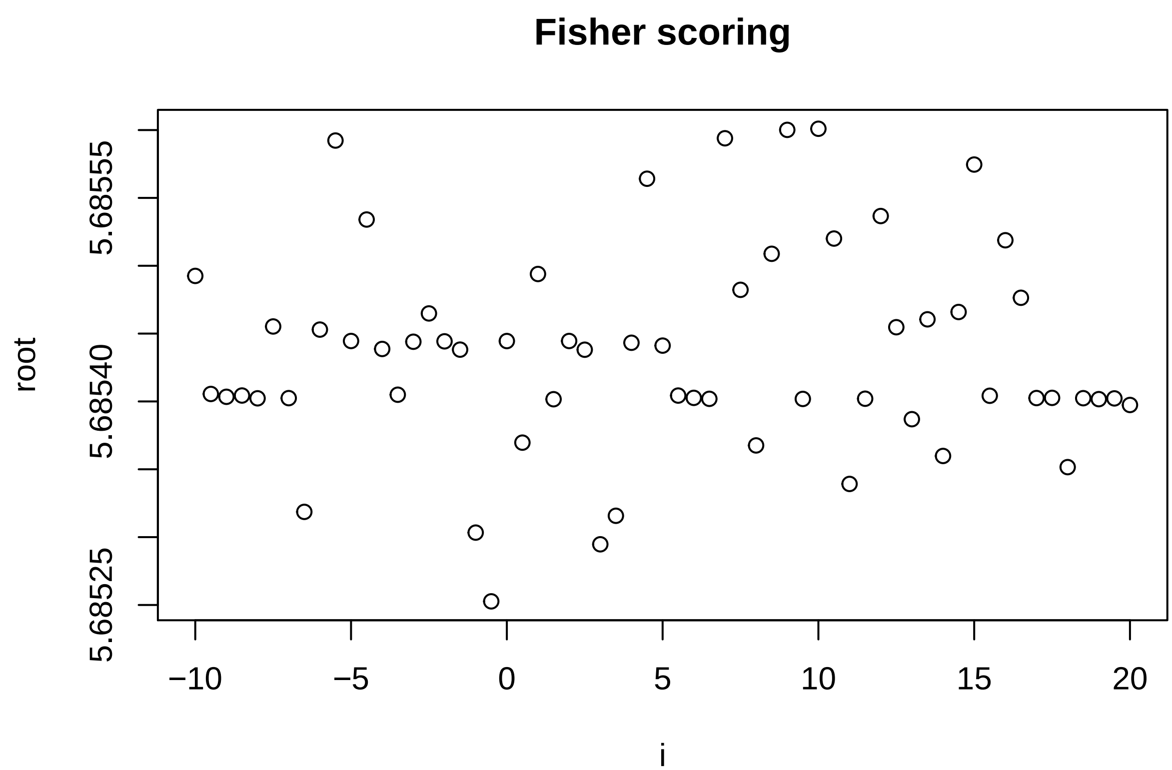
i	10	10.5	11	11.5	12	12.5	13	13.5
root	5.686	5.686	5.685	5.685	5.686	5.685	5.685	5.685

Table 24: Table continues below

i	14	14.5	15	15.5	16	16.5	17	17.5
root	5.685	5.685	5.686	5.685	5.686	5.685	5.685	5.685

i	18	18.5	19	19.5	20
root	5.685	5.685	5.685	5.685	5.685

```
plot(i,theta_S,xlab = NULL, ylab = "root",main="Fisher scoring")
```



6 Comment

From the table below, Fisher scoring seems to be the fastest. From the figures above, Fisher scoring seems to be the most stable.

```
c=rbind(i,count_N,count1_F,count2_F,count3_F,count_S)
rownames(c)=c("i","Newton Raphson","Fixed-point a=1","Fixed-point a=0.64","Fixed-point a=0.25","Fisher s
pander(c)
```

Table 26: Table continues below

i	-10	-9.5	-9	-8.5	-8	-7.5	-7
Newton Raphson	11	11	11	11	11	11	11
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.25	37	35	33	32	31	28	28
Fisher scoring	40	38	36	34	32	30	29

Table 27: Table continues below

i	-6.5	-6	-5.5	-5	-4.5	-4	-3.5
Newton Raphson	11	11	11	11	11	11	11
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000

Table 27: Table continues below

Fixed-point a=0.25	26	24	24	19	21	19	18
Fisher scoring	26	24	23	22	20	19	18

Table 28: Table continues below

i	-3	-2.5	-2	-1.5	-1	-0.5	0
Newton Raphson	12	12	12	12	12	12	12
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.25	17	16	15	14	14	13	13
Fisher scoring	16	14	14	13	11	10	10

Table 29: Table continues below

i	0.5	1	1.5	2	2.5	3	3.5
Newton Raphson	12	12	12	13	13	13	9
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.25	11	9	10	10	9	7	8
Fisher scoring	8	7	8	7	6	5	4

Table 30: Table continues below

i	4	4.5	5	5.5	6	6.5	7
Newton Raphson	10000	10000	4	2	3	6	11
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.25	9	8	7	7	6	8	8
Fisher scoring	5	4	4	4	4	5	5

Table 31: Table continues below

i	7.5	8	8.5	9	9.5	10	10.5
Newton Raphson	13	10000	10000	5	13	10000	4
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.25	8	10	10	8	12	12	11
Fisher scoring	6	6	7	8	10	10	11

Table 32: Table continues below

i	11	11.5	12	12.5	13	13.5	14
Newton Raphson	13	11	10000	10000	10000	10000	10000
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.25	14	10	15	15	15	17	18

Table 32: Table continues below

Fisher scoring	10	13	13	14	14	15	16

Table 33: Table continues below

i	14.5	15	15.5	16	16.5	17	17.5
Newton Raphson	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.25	19	20	21	21	22	25	26
Fisher scoring	18	19	21	22	23	26	28

i	18	18.5	19	19.5	20
Newton Raphson	15	10000	10000	10000	10000
Fixed-point a=1	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000
Fixed-point a=0.25	26	29	33	35	39
Fisher scoring	29	33	36	40	44