Different Ways to Solve MLE for Cauchy Distribution

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1 Proof

Density function:

$$f(x;\theta) = \frac{1}{\pi[1+(x-\theta)^2]} \tag{1.1}$$

The likelihood funcion:

$$L(\theta) = \prod_{i=1}^{n} f(X_i;\theta) \tag{1.2}$$

The loglikelihood funcion:

$$l(\theta) = \ln L(\theta) = \sum_{i=1}^{n} \ln f(X_i;\theta) = -n\ln\pi - \sum_{i=1}^{n} \ln[1+(\theta-X_i)^2] \tag{1.3}$$

Compute the differential of loglikelihood funcion:

First derivative:

$$l'(\theta) = -2\sum_{i=1}^{n} \frac{\theta - X_i}{1+(\theta-X_i)^2} \tag{1.4}$$

Second derivative:

$$l''(\theta) = -2\sum_{i=1}^{n} \frac{1-(\theta-X_i)^2}{[1+(\theta-X_i)^2]^2} \tag{1.5}$$

Fisher information:

$$
\begin{aligned}
I_n(\theta) &= -E[(l''(\theta)] \\
&= 2E\left\{\sum_{i=1}^{n}\frac{1-(\theta-X_i)^2}{[1+(\theta-X_i)^2]^2}\right\} \\
&= 2nE\left\{\frac{1-(\theta-X)^2}{[1+(\theta-X)^2]^2}\right\} \\
&= 2n\int_{-\infty}^{\infty}\frac{1-(\theta-x)^2}{[1+(\theta-x)^2]^2}\frac{1}{\pi[1+(x-\theta)^2]}\mathrm{d}x \\
&= \frac{2n}{\pi}\int_{-\infty}^{\infty}\frac{1-(\theta-x)^2}{[1+(\theta-x)^2]^2}\frac{1}{1+(x-\theta)^2}\mathrm{d}x \\
\\
&= \frac{2n}{\pi}\int_{-\infty}^{\infty}\frac{1-x^2}{(1+x^2)^2}\frac{1}{1+x^2}\mathrm{d}x \\
&= \frac{2n}{\pi}\int_{-\infty}^{\infty}(\frac{x}{1+x^2})'\frac{1}{1+x^2}\mathrm{d}x \\
&= \frac{2n}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+x^2}\mathrm{d}(\frac{x}{1+x^2}) \\
&= \frac{2n}{\pi}\frac{x}{1+x^2}\frac{1}{1+x^2}\Big|_{-\infty}^{\infty} - \frac{2n}{\pi}\int_{-\infty}^{\infty}\frac{x}{1+x^2}\mathrm{d}(\frac{1}{1+x^2}) \\
&= \frac{4n}{\pi}\int_{-\infty}^{\infty}\frac{x^2}{(1+x^2)^3}\mathrm{d}x \\
&= \frac{4n}{\pi}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(\tan t)^2}{[(1+\tan t)^2]^3}\mathrm{d}\tan t \\
&= \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin t)^2(\cos t)^2\mathrm{d}t \\
&= \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\frac{1}{2}\sin 2t)^2\mathrm{d}t \\
&= \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1-\cos 4t}{8}\mathrm{d}t \\
&= \frac{n}{2}
\end{aligned}
\tag{1.6}
$$

2 Plot

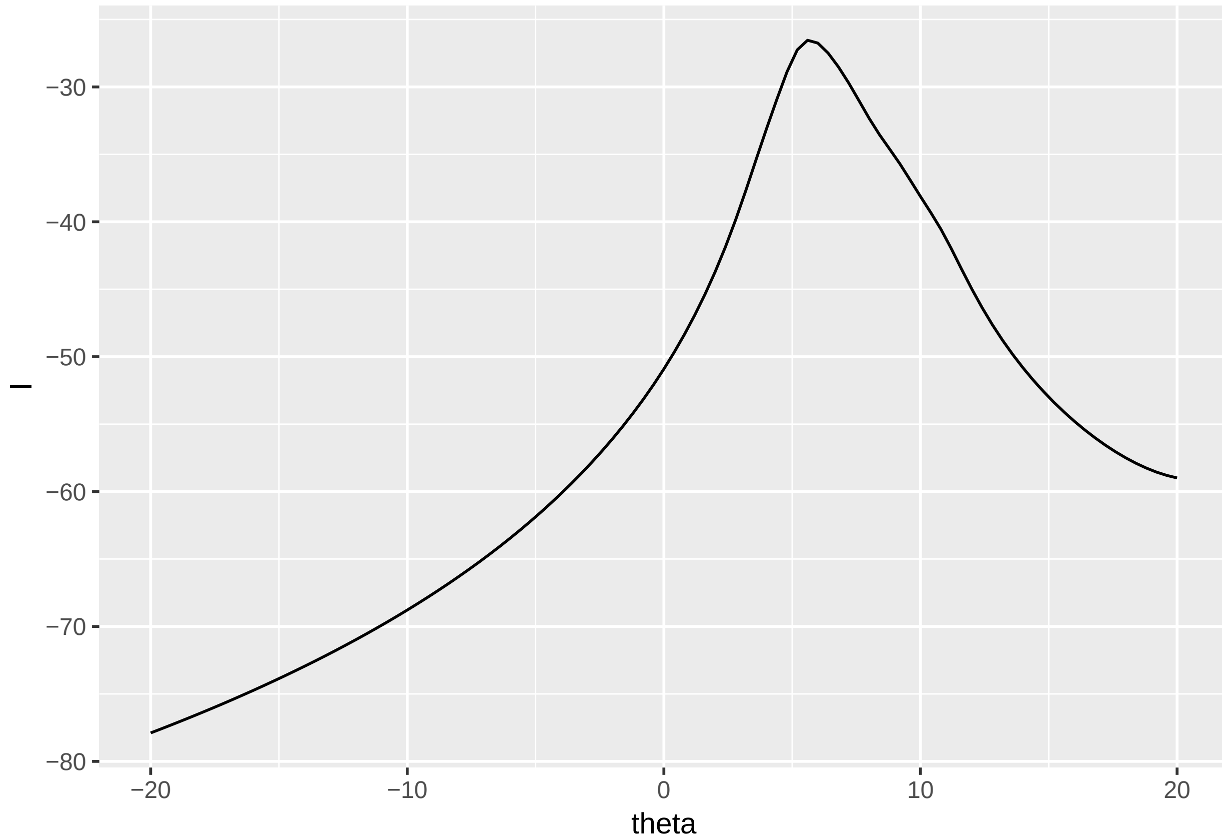
First, we need to define all functions above. And then We choose 10 random numbers according to density function above using “rcauchy”.

```
f=function(x,theta)1/(pi*(1+(x-theta)^2))

L=function(x,theta){
  prod=1;
  for (i in 1:length(x)){
    prod = prod*(1/(pi*(1+(x[i]-theta)^2)));
  }
  prod
}
```

```r
l=function(x,theta){log(L(x,theta))}

l1=function(x,theta){
  sum=0;
  for(i in 1:length(x)){
    sum=sum-2*(theta-x[i])/(1+(theta-x[i])^2)
  }
  sum
}

l2=function(x,theta){
  sum=0;
  for(i in 1:length(x)){
    sum=sum-2*(1-(theta-x[i])^2)/(1+(theta-x[i])^2)^2
  }
  sum
}

set.seed(20180909)
X=rcauchy(10,5)
```

Next, we use those 10 numbers to plot the figure of $l(\theta)$

```r
library("ggplot2")
ggplot(data.frame(x=c(-20,20)),aes(x=x)) +
  stat_function(fun=function(theta) l(X,theta)) +
  labs(x=expression("theta"),y="l")
```



3 Newton–Raphson method

According to the result, different initial value may cause different results which are totally different. However, some close number can lead to the same answer. For example, initial value 5,5.5,6,6.5 leads to the same root 5.685418.

```
i=seq(-10, 20, 0.5)
theta_N=matrix(0,1, length(i))
count_N=matrix(0,1, length(i))
for(k in 1:length(i)) {
  theta_N[k]=i[k]
  while (abs(l1(X,theta_N[k]))>0.001&&count_N[k]<10000) {
    temp=theta_N[k]-l1(X,theta_N[k])/l2(X,theta_N[k])
    theta_N[k]=temp
    count_N[k]=count_N[k]+1
  }
}
library(pander)
table_N=rbind(i,theta_N)
rownames(table_N)=c("i","root")
set.caption("Newton-Raphson method")
pander(table_N)
```

Table 1: Newton–Raphson method (continued below)

i	-10	-9.5	-9	-8.5	-8	-7.5	-7
root	-34927	-33867	-32807	-31744	-30681	-29615	-28548

Table 2: Table continues below

i	-6.5	-6	-5.5	-5	-4.5	-4	-3.5
root	-27480	-26409	-25336	-24262	-23185	-22105	-21024

Table 3: Table continues below

i	-3	-2.5	-2	-1.5	-1	-0.5	0
root	-39887	-37713	-35535	-33351	-31165	-28977	-26793

Table 4: Table continues below

i	0.5	1	1.5	2	2.5	3	3.5	4
root	-24623	-22487	-20436	-37213	-34868	-38758	30360	21.08

Table 5: Table continues below

i	4.5	5	5.5	6	6.5	7	7.5	8
root	19.38	5.685	5.685	5.685	5.685	25986	-30878	20.56

Table 6: Table continues below

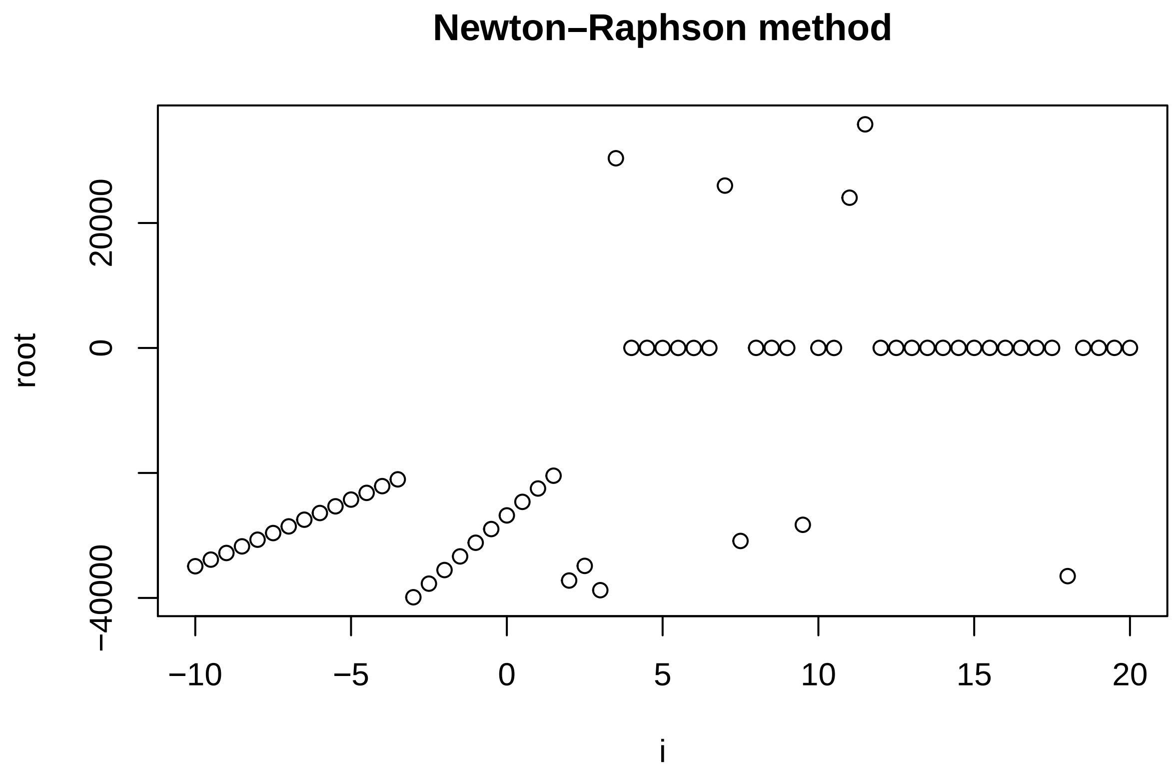
i	8.5	9	9.5	10	10.5	11	11.5	12
root	19.38	5.685	-28295	19.38	5.685	24047	35786	21.08

Table 7: Table continues below

i	12.5	13	13.5	14	14.5	15	15.5	16
root	21.08	21.08	19.38	19.38	19.38	20.56	21.08	21.08

i	16.5	17	17.5	18	18.5	19	19.5	20
root	20.56	20.56	20.56	-36510	21.08	21.08	21.08	21.08

```
plot(i,theta_N,xlab = NULL, ylab = "root",main="Newton-Raphson method")
```



4 Fixed-point iterations

From the results, we can see the smaller initial value causes more stable result.

```
i=seq(-10, 20, 0.5)
a1=1
a2=0.64
a3=0.25
theta1_F=theta2_F=theta3_F=matrix(0,1, length(i))
count1_F=count2_F=count3_F=matrix(0,1, length(i))
for(k in 1:length(i)) {
  theta1_F[k]=i[k]
  while (abs(l1(X,theta1_F[k]))>0.001&&count1_F[k]<10000) {
    temp=a1*l1(X,theta1_F[k])+theta1_F[k]
    count1_F[k]=count1_F[k]+1
    theta1_F[k]=temp
  }
}
for(k in 1:length(i)) {
  theta2_F[k]=i[k]
  while (abs(l1(X,theta2_F[k]))>0.001&&count2_F[k]<10000) {
    temp=a2*l1(X,theta2_F[k])+theta2_F[k]
    count2_F[k]=count2_F[k]+1
    theta2_F[k]=temp
```

```
  }
}
for(k in 1:length(i)) {
  theta3_F[k]=i[k]
  while (abs(l1(X,theta3_F[k]))>0.001&&count3_F[k]<10000) {
    temp=a3*l1(X,theta3_F[k])+theta3_F[k]
    count3_F[k]=count3_F[k]+1
    theta3_F[k]=temp
  }
}
library(pander)
table1_F=rbind(i,theta1_F)
table2_F=rbind(i,theta2_F)
table3_F=rbind(i,theta3_F)
rownames(table1_F)=c("i","root")
set.caption("Fixed-point iterations for a=1")
pander(table1_F)
```

Table 9: Fixed-point iterations for a=1 (continued below)

i	-10	-9.5	-9	-8.5	-8	-7.5	-7	-6.5
root	4.087	9.548	4.087	6.486	9.548	4.087	6.486	6.486

Table 10: Table continues below

i	-6	-5.5	-5	-4.5	-4	-3.5	-3	-2.5
root	9.548	4.087	4.087	6.486	9.548	9.548	6.486	4.087

Table 11: Table continues below

i	-2	-1.5	-1	-0.5	0	0.5	1	1.5
root	6.486	6.486	4.087	9.548	4.087	4.087	4.087	9.548

Table 12: Table continues below

i	2	2.5	3	3.5	4	4.5	5	5.5
root	6.486	6.486	6.486	9.548	9.548	9.548	9.548	6.486

Table 13: Table continues below

i	6	6.5	7	7.5	8	8.5	9	9.5
root	4.087	4.087	4.087	4.087	4.087	4.087	6.486	6.486

Table 14: Table continues below

i	10	10.5	11	11.5	12	12.5	13	13.5
root	6.486	6.486	6.486	6.486	4.087	9.548	9.548	9.548

Table 15: Table continues below

i	14	14.5	15	15.5	16	16.5	17	17.5
root	9.548	4.087	4.087	4.087	6.486	6.486	6.486	9.548

i	18	18.5	19	19.5	20
root	9.548	4.087	6.486	6.486	4.087

```
rownames(table2_F)=c("i","root")
set.caption("Fixed-point iterations for a=0.64")
pander(table2_F)
```

Table 17: Fixed-point iterations for a=0.64 (continued below)

i	-10	-9.5	-9	-8.5	-8	-7.5	-7	-6.5
root	5.443	7.589	4.964	5.161	6.118	5.352	6.179	7.183

Table 18: Table continues below

i	-6	-5.5	-5	-4.5	-4	-3.5	-3	-2.5
root	5.182	4.95	5.035	7.729	5.333	5.577	5.148	7.727

Table 19: Table continues below

i	-2	-1.5	-1	-0.5	0	0.5	1	1.5
root	7.452	5.174	5.067	5.435	7.729	7.051	7.236	7.061

Table 20: Table continues below

i	2	2.5	3	3.5	4	4.5	5	5.5
root	5.222	6.064	5.204	5.285	4.953	6.031	5.149	5.373

Table 21: Table continues below

i	6	6.5	7	7.5	8	8.5	9	9.5
root	6.514	6.574	7.066	7.729	5.143	5.217	4.99	5.138

Table 22: Table continues below

i	10	10.5	11	11.5	12	12.5	13	13.5	14
root	7.571	7.542	7.729	7.1	7.537	5.186	5.154	5.26	7.021

Table 23: Table continues below

i	14.5	15	15.5	16	16.5	17	17.5	18

Table 23: Table continues below

root	5.965	5.145	5.22	7.161	7.624	4.956	5.403	6.635

i	18.5	19	19.5	20
root	5.239	7.249	5.563	5.312

```
rownames(table3_F)=c("i","root")
set.caption("Fixed-point iterations for a=0.25")
pander(table3_F)
```

Table 25: Fixed-point iterations for a=0.25 (continued below)

i	-10	-9.5	-9	-8.5	-8	-7.5	-7	-6.5
root	5.685	5.686	5.686	5.685	5.685	5.686	5.685	5.685

Table 26: Table continues below

i	-6	-5.5	-5	-4.5	-4	-3.5	-3	-2.5
root	5.686	5.685	5.686	5.685	5.686	5.686	5.686	5.686

Table 27: Table continues below

i	-2	-1.5	-1	-0.5	0	0.5	1	1.5
root	5.685	5.686	5.685	5.685	5.685	5.686	5.686	5.686

Table 28: Table continues below

i	2	2.5	3	3.5	4	4.5	5	5.5
root	5.685	5.686	5.685	5.686	5.685	5.685	5.686	5.686

Table 29: Table continues below

i	6	6.5	7	7.5	8	8.5	9	9.5
root	5.686	5.685	5.686	5.686	5.685	5.686	5.686	5.685

Table 30: Table continues below

i	10	10.5	11	11.5	12	12.5	13	13.5
root	5.686	5.685	5.685	5.686	5.685	5.686	5.685	5.685

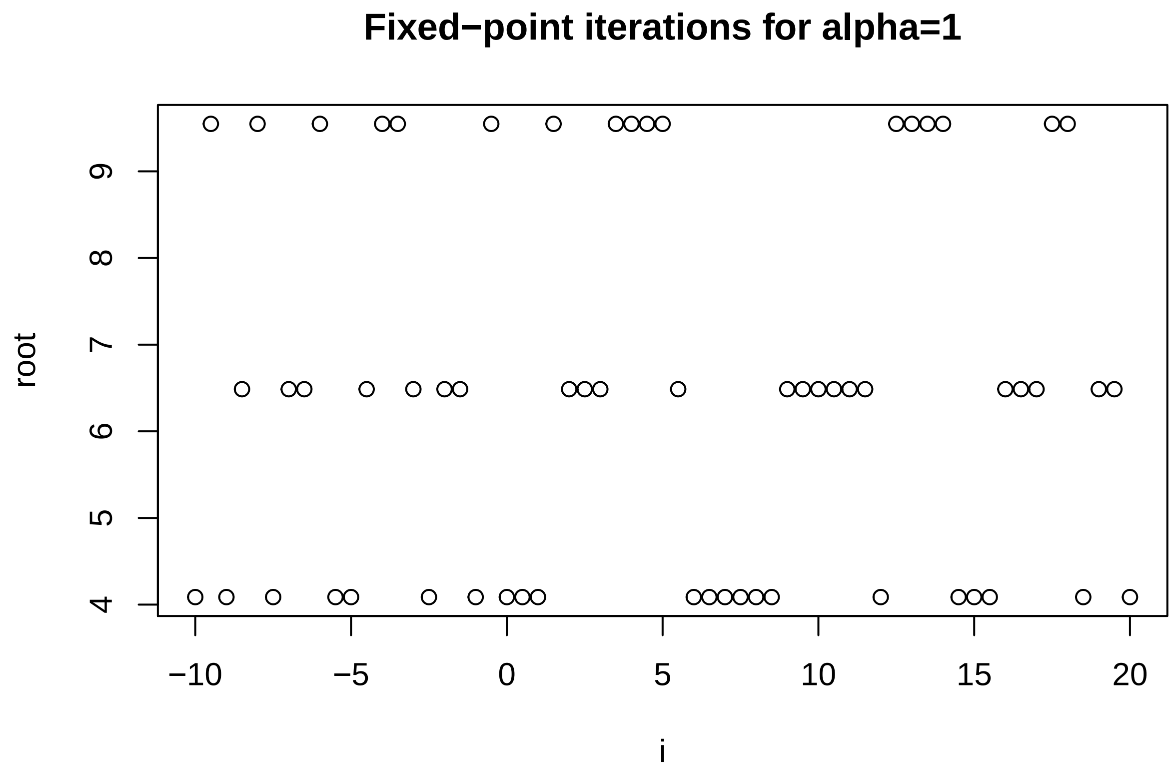
Table 31: Table continues below

i	14	14.5	15	15.5	16	16.5	17	17.5
root	5.685	5.685	5.685	5.685	5.686	5.685	5.685	5.686

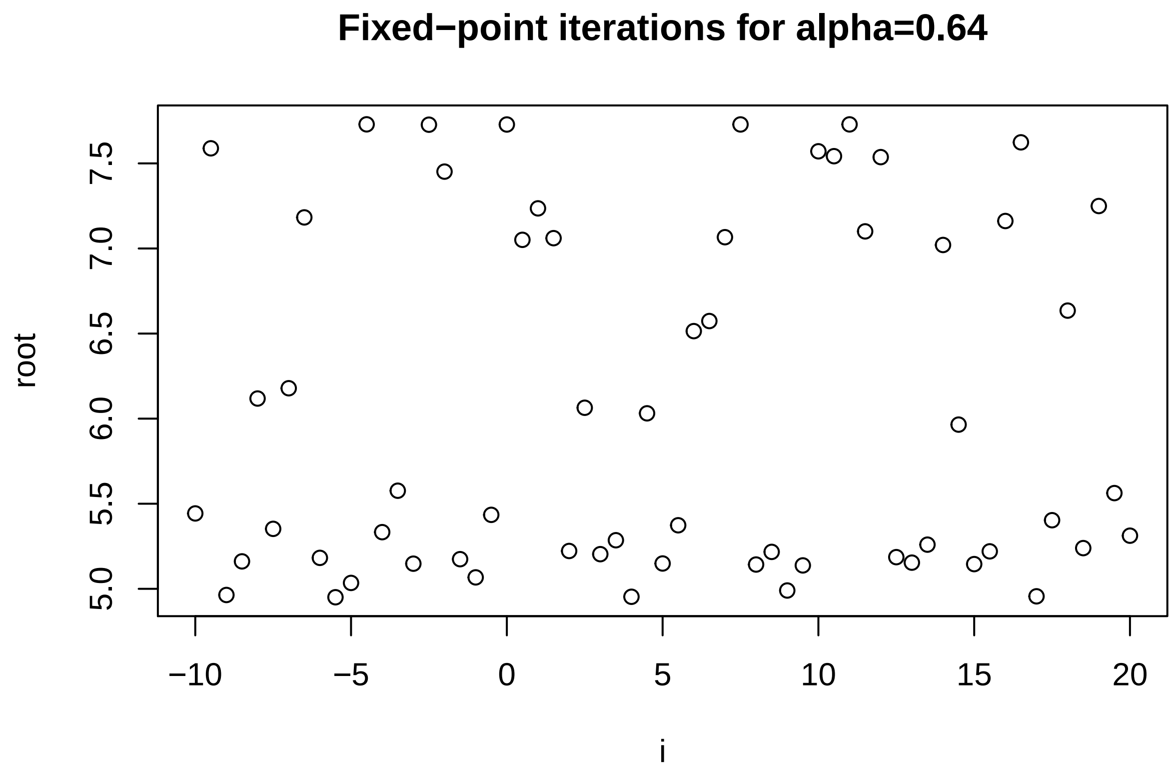
Table 31: Table continues below

i	18	18.5	19	19.5	20
root	5.686	5.685	5.685	5.686	5.685

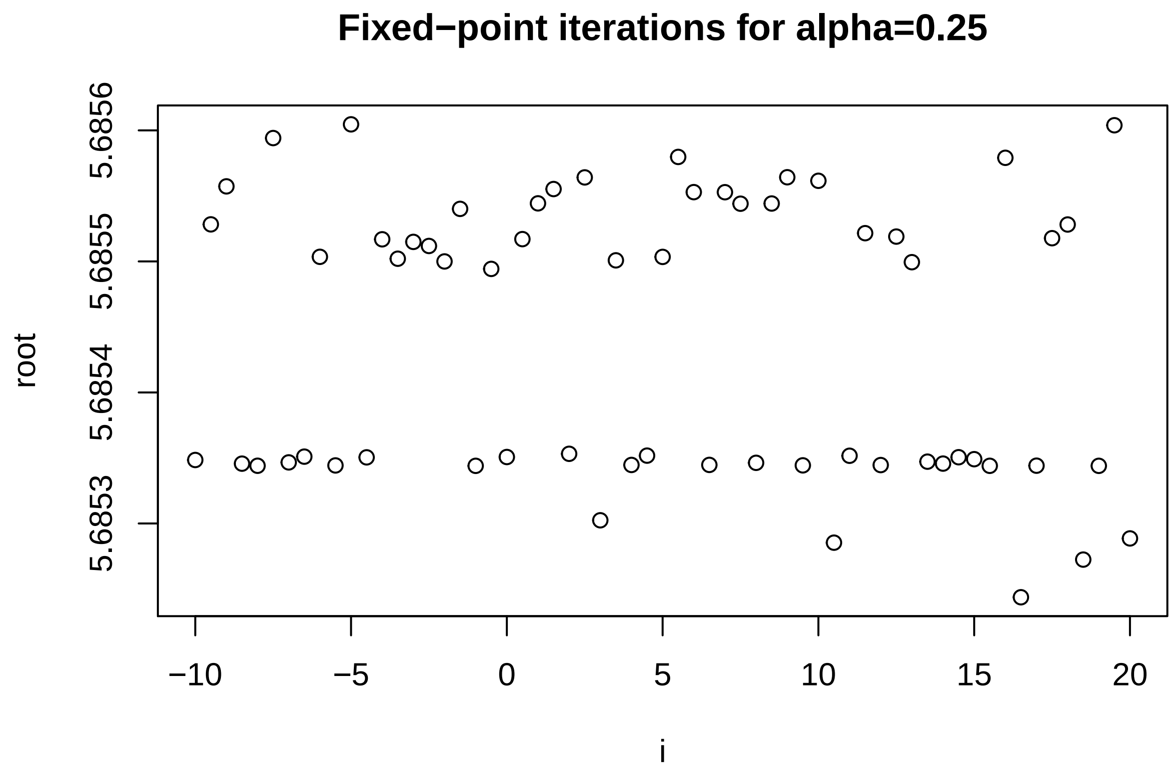
```
plot(i,theta1_F,xlab = NULL, ylab = "root",main="Fixed-point iterations for alpha=1")
```




```
plot(i,theta2_F,xlab = NULL, ylab = "root",main="Fixed-point iterations for alpha=0.64")
```




```r
plot(i,theta3_F,xlab = NULL, ylab = "root",main="Fixed-point iterations for alpha=0.25")
```



5 Fisher scoring

```
i=seq(-10, 20, 0.5)
theta_S=matrix(0,1, length(i))
count_S=matrix(0,1, length(i))
I=length(X)/2
for(k in 1:length(i)) {
  theta_S[k]=i[k]
  while (abs(l1(X,theta_S[k]))>0.001&&count_S[k]<10000) {
    temp=theta_S[k]+l1(X,theta_S[k])/I
    theta_S[k]=temp
    count_S[k]=count_S[k]+1
  }
}
library(pander)
table_S=rbind(i,theta_S)
rownames(table_S)=c("i","root")
set.caption("Fisher scoring")
pander(table_S)
```

Table 33: Fisher scoring (continued below)

i	-10	-9.5	-9	-8.5	-8	-7.5	-7	-6.5
root	5.685	5.685	5.685	5.685	5.685	5.685	5.685	5.685

Table 33: Fisher scoring (continued below)

Table 34: Table continues below

i	-6	-5.5	-5	-4.5	-4	-3.5	-3	-2.5
root	5.685	5.686	5.685	5.686	5.685	5.685	5.685	5.685

Table 35: Table continues below

i	-2	-1.5	-1	-0.5	0	0.5	1	1.5
root	5.685	5.685	5.685	5.685	5.685	5.685	5.685	5.685

Table 36: Table continues below

i	2	2.5	3	3.5	4	4.5	5	5.5
root	5.685	5.685	5.685	5.685	5.685	5.686	5.685	5.685

Table 37: Table continues below

i	6	6.5	7	7.5	8	8.5	9	9.5
root	5.685	5.685	5.686	5.685	5.685	5.686	5.686	5.685

Table 38: Table continues below

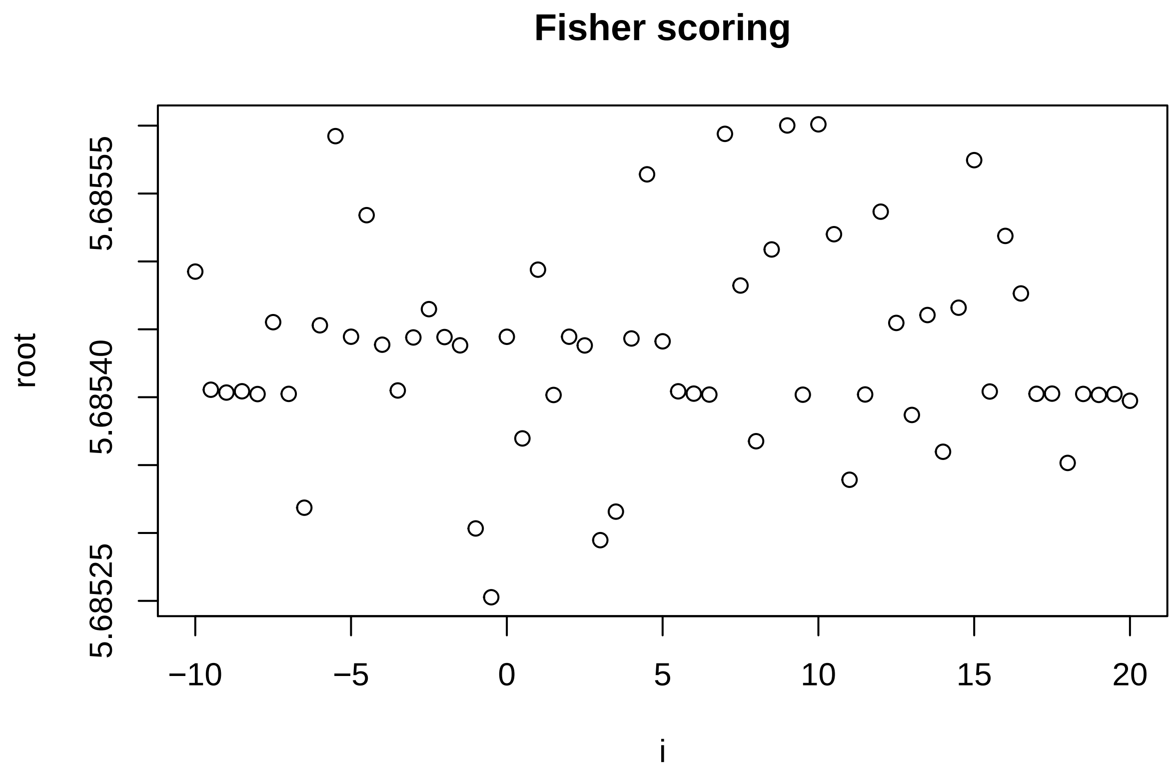
i	10	10.5	11	11.5	12	12.5	13	13.5
root	5.686	5.686	5.685	5.685	5.686	5.685	5.685	5.685

Table 39: Table continues below

i	14	14.5	15	15.5	16	16.5	17	17.5
root	5.685	5.685	5.686	5.685	5.686	5.685	5.685	5.685

i	18	18.5	19	19.5	20
root	5.685	5.685	5.685	5.685	5.685

```
plot(i,theta_S,xlab = NULL, ylab = "root",main="Fisher scoring")
```



6 Comment

From the table below, Fisher scoring seems to be the fastest. From the figures above, Fisher scoring seems to be the most stable.

```
c=rbind(i,count_N,count1_F,count2_F,count3_F,count_S)
rownames(c)=c("i","Newton Raphson","Fixed-point a=1","Fixed-point a=0.64","Fixed-point a=0.25","Fisher s
pander(c)
```

Table 41: Table continues below

i	-10	-9.5	-9	-8.5	-8	-7.5	-7
Newton Raphson	11	11	11	11	11	11	11
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.25	37	35	33	32	31	28	28
Fisher scoring	40	38	36	34	32	30	29

Table 42: Table continues below

i	-6.5	-6	-5.5	-5	-4.5	-4	-3.5
Newton Raphson	11	11	11	11	11	11	11
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000

Table 42: Table continues below

Fixed-point a=0.25	26	24	24	19	21	19	18
Fisher scoring	26	24	23	22	20	19	18

Table 43: Table continues below

i	-3	-2.5	-2	-1.5	-1	-0.5	0
Newton Raphson	12	12	12	12	12	12	12
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.25	17	16	15	14	14	13	13
Fisher scoring	16	14	14	13	11	10	10

Table 44: Table continues below

i	0.5	1	1.5	2	2.5	3	3.5
Newton Raphson	12	12	12	13	13	13	9
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.25	11	9	10	10	9	7	8
Fisher scoring	8	7	8	7	6	5	4

Table 45: Table continues below

i	4	4.5	5	5.5	6	6.5	7
Newton Raphson	10000	10000	4	2	3	6	11
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.25	9	8	7	7	6	8	8
Fisher scoring	5	4	4	4	4	5	5

Table 46: Table continues below

i	7.5	8	8.5	9	9.5	10	10.5
Newton Raphson	13	10000	10000	5	13	10000	4
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.25	8	10	10	8	12	12	11
Fisher scoring	6	6	7	8	10	10	11

Table 47: Table continues below

i	11	11.5	12	12.5	13	13.5	14
Newton Raphson	13	11	10000	10000	10000	10000	10000
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.25	14	10	15	15	15	17	18

Table 47: Table continues below

Fisher scoring	10	13	13	14	14	15	16

Table 48: Table continues below

i	14.5	15	15.5	16	16.5	17	17.5
Newton Raphson	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=1	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000	10000	10000
Fixed-point a=0.25	19	20	21	21	22	25	26
Fisher scoring	18	19	21	22	23	26	28

i	18	18.5	19	19.5	20
Newton Raphson	15	10000	10000	10000	10000
Fixed-point a=1	10000	10000	10000	10000	10000
Fixed-point a=0.64	10000	10000	10000	10000	10000
Fixed-point a=0.25	26	29	33	35	39
Fisher scoring	29	33	36	40	44