< STAT-5361 > HW#5-Exercises 4.8.1

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1 Exercises 4.8.1

1.1 1.(a)

$$
\begin{aligned}
Q(\Psi|\Psi^{(k)}) &= E[\ln L(\Psi|x_i, y_i, z)|x_i, y_i, \Psi^{(k)}] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}(\log \pi_j + \log \phi(y_i - x_i^T\beta_j; 0, \sigma^2))
\end{aligned}
$$

$$
\sum_{j=1}^{m} \pi_j = 1
$$

$$
L(\pi_1, ..., \pi_m, \lambda) = Q(\Psi|\Psi(k)) - \lambda(\sum_{j=1}^{m} \pi_j - 1) = 0
$$

$$
L'_{\pi_j} = 0 L'_{\lambda} = 0, (j = 1, 2, ..., m)
$$

$$
\sum_{i=1}^{n} p_{ij}^{(k+1)} \frac{1}{\pi_j} - \lambda = 0, (j = 1, 2, ..., m)
$$

$$
\pi_j = \frac{\sum_{i=1}^{n} p_{ij}^{(k+1)}}{\lambda}
$$

$$
\sum_{j=1}^{m} \pi_j = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}}{\lambda} = \frac{n}{\lambda} = 1
$$

$$
\therefore \lambda = n
$$

$$
\pi_j = \frac{\sum_{i=1}^{n} p_{ij}^{(k+1)}}{n}
$$

1.2 1.(b)

$$
\begin{aligned}
Q(\Psi|\Psi^{(k)}) &= \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}(\log \pi_j + \log(\frac{1}{\sqrt{2\pi}\sigma} exp[-\frac{(y_i - x_i^T\beta_j)^2}{2\sigma^2}])) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}(\log \pi_j + \log(\frac{1}{\sqrt{2\pi}\sigma}) + [-\frac{(y_i - x_i^T\beta_j)^2}{2\sigma^2}])
\end{aligned}
$$

It's the sum of m quadratic forms, where each form includes a single β_j for every j.

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}[-\frac{(y_i - x_i^T\beta_j)^2}{2\sigma^2}] &= -\sum_{i=1}^{n} p_{ij}^{(k+1)}[\frac{(y_i - x_i^T\beta_j)^2}{2\sigma^2}] \\
&= \sum_{i=1}^{n} p_{ij}^{(k+1)}[x_i^T(\frac{y_i}{x_i^T} - \beta_j)]^2 \\
&= \sum_{i=1}^{n} p_{ij}^{(k+1)} x_i x_i^T(\frac{y_i}{x_i^T} - \beta_j)^2
\end{aligned}
$$

$$
\beta_j = \sum_{i=1}^{n} p_{ij} x_i x_i^T \frac{y_i}{x_i^T} = \sum_{i=1}^{n} p_{ij} x_i y_i^T
$$

1.3 1.(c)

$$\sigma^{2(k+1)} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}(y_i - x_i^T\beta_j^{(k+1)})^2}{\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}}$$
$$= \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}(y_i - x_i^T\beta_j^{(k+1)})^2}{n}$$

1.4 2.

```
regmix_em = function(y, xmat, initial.pi , initial.beta, initial.sigma,
                     control = list(maxit = 500, tol = 1e-5))
{
  updated.pi <- curr.pi <- initial.pi
  updated.beta<- curr.beta <- initial.beta
  updated.sigma <- curr.sigma <- initial.sigma
  n = length(y)
  m = length(initial.pi)
  xmat = as.matrix(xmat)
  pb = matrix(0,length(y),length(initial.pi))
  for(k in 1:control$maxit)
  {
    for(i in 1:m)
      pb[,i] = curr.pi[i] * dnorm(y - xmat %*% curr.beta[,i],0,curr.sigma)
    updated.p =  pb/rowSums(pb)
    updated.pi = colMeans(updated.p)

    for(j in 1:m)
    updated.beta[,j] = solve(t(xmat * updated.p[,j]) %*% xmat ) %*%
      t(xmat * updated.p[,j] ) %*% y
    updated.sigma = sqrt(sum(updated.p[,1] * (y - xmat %*% updated.beta[,1])^2 +
                       updated.p[,2] * (y - xmat %*% updated.beta[,2])^2 +
                       updated.p[,3] * (y - xmat %*% updated.beta[,3])^2)/n)

    if( sum(abs(curr.pi - updated.pi)) + sum(abs(curr.beta - updated.beta)) +
        sum(abs(curr.sigma - updated.sigma)) < control$tol ) break
    curr.pi = updated.pi
    curr.beta = updated.beta
    curr.sigma = updated.sigma
  }
  return(list(pi = curr.pi, beta = curr.beta, sigma = curr.sigma, iter = k))
}
```

1.5 3

```
regmix_sim <- function(n, pi, beta, sigma) {
  K <- ncol(beta)
  p <- NROW(beta)
  xmat <- matrix(rnorm(n * p), n, p) # normal covaraites
  error <- matrix(rnorm(n * K, sd = sigma), n, K)
  ymat <- xmat %*% beta + error # n by K matrix
  ind <- t(rmultinom(n, size = 1, prob = pi))
  y <- rowSums(ymat * ind)
  data.frame(y, xmat)
}
n <- 400
pi <- c(.3, .4, .3)
bet <- matrix(c( 1,  1,  1,
                -1, -1, -1), 2, 3)
sig <- 1
set.seed(1205)
dat <- regmix_sim(n, pi, bet, sig)
regmix_em(y = dat[,1], xmat = dat[,-1], initial.pi = pi / pi / length(pi),
          initial.beta = bet * 0, initial.sigma = sig/sig,
          control = list(maxit = 500, tol = 1e-10))
```

```
## $pi
## [1] 0.3333333 0.3333333 0.3333333
##
## $beta
##            [,1]       [,2]       [,3]
## [1,]  0.3335660  0.3335660  0.3335660
## [2,] -0.4754645 -0.4754645 -0.4754645
##
## $sigma
## [1] 1.732492
##
## $iter
## [1] 2
```

```
regmix_em(y = dat[,1], xmat = dat[,-1], initial.pi = pi / pi / length(pi),
          initial.beta = matrix(-2:3,2,3), initial.sigma = sig/sig,
          control = list(maxit = 500, tol = 1e-10))
```

```
## $pi
## [1] 0.3453959 0.3858238 0.2687802
##
## $beta
##            [,1]       [,2]       [,3]
```

```
## [1,] -0.9136895 0.8796621  0.9911949
## [2,] -1.1990373 0.9341928 -1.2424624
##
## $sigma
## [1] 1.023598
##
## $iter
## [1] 131
```