Statistical Computing Homework 4

Jieying Jiao

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Abstract

This is Jieying Jiao's homework 4 for statistical computing, fall 2018.

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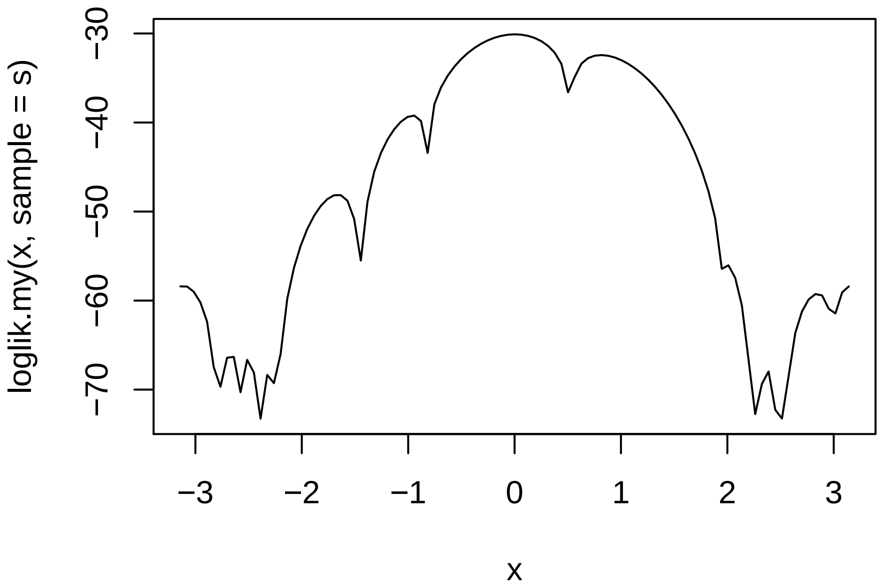
1 Exercise 3.3.2

```r
##' define the loglikelihood function
loglik.my0 <- function(theta, sample) {
  n <- length(sample)
  if (sum(sample >=0 & sample <= 2*pi) < n) {
    print("sample is out of range")
  } else if(theta < -pi | theta > pi) {
    print("theta is out of range")
  } else {
   l <- sum(log(1-cos(sample-theta))) - n * log(2*pi)
   return(l)
  }
}
loglik.my <- function(theta, sample) {
  l <- sapply(theta, FUN = loglik.my0, sample = sample)
  l
}

s <- c(3.91, 4.85, 2.28, 4.06, 3.70, 4.04, 5.46, 3.53, 2.28, 1.96,
       2.53, 3.88, 2.22, 3.47, 4.82, 2.46, 2.99, 2.54, 0.52)
curve(loglik.my(x, sample = s), -pi, pi)
```



$$
\begin{aligned}
E(X|\theta) &= \int_0^{2\pi} x \frac{1-\cos(x-\theta)}{2\pi} dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} x dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos(x-\theta) dx \\
&= \frac{1}{2\pi} \times \frac{1}{2}\Big|_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} x d\sin(x-\theta) \\
&= \pi - \frac{1}{2\pi}\left[x\sin(x-\theta)|_0^{2\pi} - \int_0^{2\pi} \sin(x-\theta) dx \right] \\
&= \pi - \frac{1}{2\pi}\left[2\pi\sin(2\pi-\theta) + \cos(x-\theta)\ |_0^{2\pi} \right] \\
&= \pi - \frac{1}{2\pi}\left[-2\pi\sin(\theta) \right] \\
&= \pi + \sin(\theta) \\
&= \bar{X}_n
\end{aligned}
$$

$$\Rightarrow \tilde{\theta}_n = \arcsin(\bar{X}_n - \pi)$$

```r
library(pracma)
theta_0 <- asin(mean(s) - pi)
##' define derivitive of log-likelihood function
dev.loglik0 <- function(theta, sample) {
  dev.l <- sum(sin(theta-sample)/(1-cos(theta-sample)))
  dev.l
}
dev.loglik <- function(theta, sample) {
  dev.l <- sapply(theta, FUN = dev.loglik0, sample = sample)
}
```

```
x1 <- newtonRaphson(fun = dev.loglik, x0 = theta_0, sample = s)$root
x2 <- newtonRaphson(fun = dev.loglik, x0 = -2.7, sample = s)$root
x3 <- newtonRaphson(fun = dev.loglik, x0 = 2.7, sample = s)$root
x1
```

```
## [1] 0.003118157
```

```
x2
```

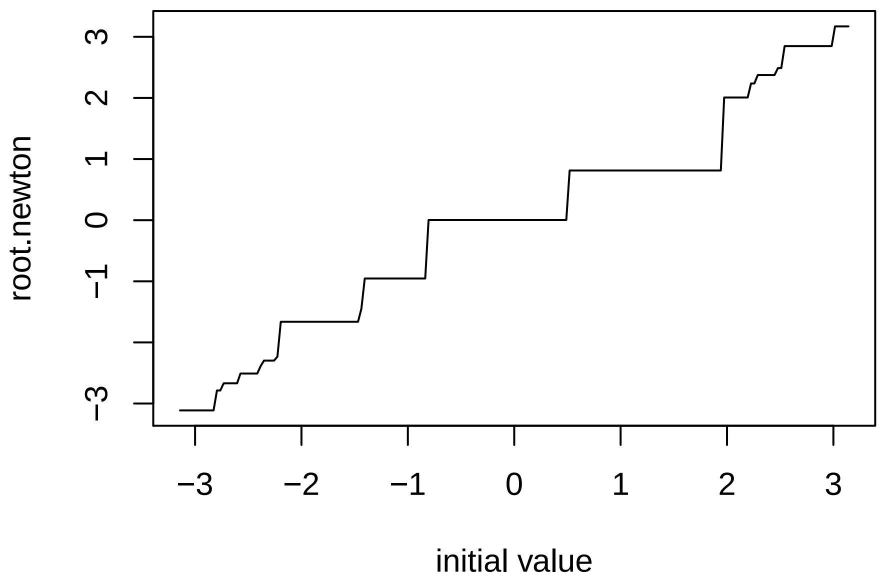
```
## [1] -2.668857
```

```
x3
```

```
## [1] 2.848415
```

Newton-Raphson method gives 0.0031 as MLE when MOM 0.095 is initial value, -2.669 as MLE when -2.7 is initial value, 2.848 as MLE when 2.7 is initial value.

```
n <- 200
init <- seq(-pi, pi, length.out = n)
root.newton <- rep(0, n)
for (i in 1:n) root.newton[i] <- newtonRaphson(fun = dev.loglik, x0 = init[i],
                                               sample = s)$root
plot(init, root.newton, type = "l", xlab = "initial value")
```



2 Exercise 3.3.3

```
library(graphics)
library(Matrix)
```

```
##
## Attaching package: 'Matrix'
```

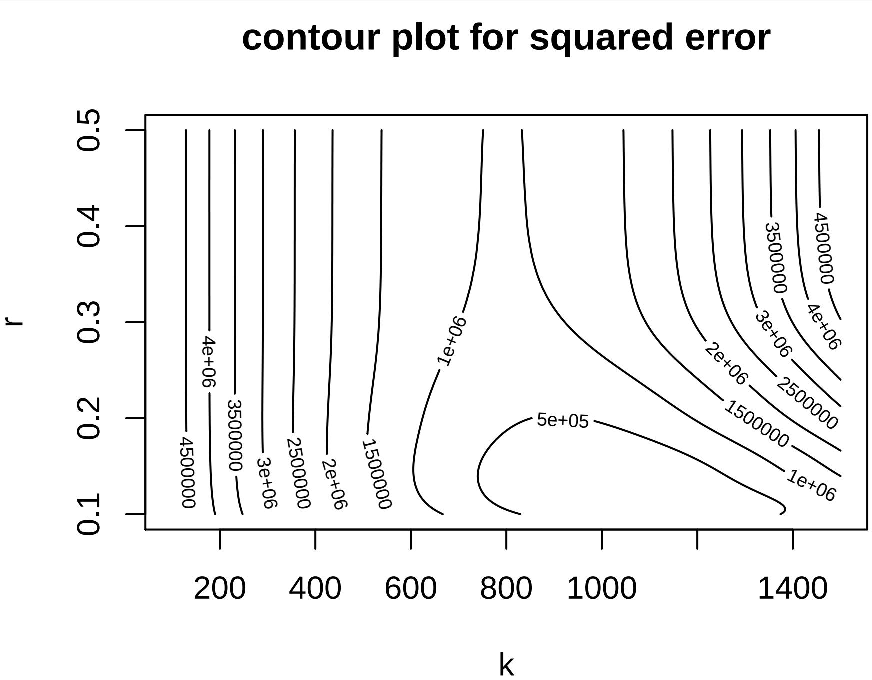
```
## The following objects are masked from 'package:pracma':
##
##     expm, lu, tril, triu
```

```r
beetles <- data.frame(
    days    = c(0,  8,  28,  41,  63,  69,   97, 117,  135,  154),
    beetles = c(2, 47, 192, 256, 768, 896, 1120, 896, 1184, 1024))

##' define the sum of squared errors function
sqerr <- function(k, r) {
  s <- matrix(0, nrow = length(k), ncol = length(r))
  for (i in 1:length(k)) {
    for (j in 1:length(r)) {
      s[i, j] <- sum((beetles$beetles - k[i] * beetles$beetles[1] /
              (beetles$beetles[1] + (k[i] - beetles$beetles[1]) *
                 exp(-r[j]*beetles$days)))^2)
    }
  }
  s
}

##' define z function
z.vec <- function(k, r) {
  n <- length(beetles$days)
  z <- rep(0, n)
  for (i in 1:n) {
   z[i] <- beetles$beetles[i] - k*beetles$beetles[1] /
     (beetles$beetles[1] + (k - beetles$beetles[1])*exp(-r*beetles$days[i]))
  }
  return(z)
}

##' define A matrix
A.mat <- function(k, r) {
  n <- length(beetles$days)
  A <- matrix(0, nrow = n, ncol = 2)
  for (i in 1:n) {
    A[i, 1] <- beetles$beetles[1]^2 * (1-exp(-r*beetles$days[i])) /
      (beetles$beetles[1] + (k-beetles$beetles[1])*exp(-r*beetles$days[i]))^2

    A[i, 2] <- beetles$beetles[1]*k*beetles$days[i]*(k-beetles$beetles[1]) *
      exp(-r*beetles$days[i]) / (beetles$beetles[1]+(k-beetles$beetles[1]) *
                                   exp(-r*beetles$days[i]))^2
  }
  return(A)
}

gaussNewton.beetles <- function(para0, z.vec, A.mat, maxiter = 100,
```

```
                                tol = .Machine$double.eps^0.5) {
  para <- para0
  for (i in 1:maxiter) {
    Amat <- A.mat(para[1], para[2])
    zvec <- z.vec(para[1], para[2])
    para.new <- para + solve(t(Amat) %*% Amat + 0.0001*diag(nrow = 2)) %*%
      t(Amat) %*% zvec
    if (sum(abs(para.new - para)) < tol) break
    para <- para.new
  }
  if (i == maxiter) warning("maximum iteration has reached")
  return(list(root = para, niter = i))
}

fit <- gaussNewton.beetles(para0 = c(1200, 0.1), z.vec = z.vec, A.mat = A.mat)
fit$root
```

```
##              [,1]
## [1,] 1049.4072443
## [2,]    0.1182684
```

```
k <- seq(100, 1500, by = 10)
r <- seq(0.1, 0.5, by = 0.001)
z <- sqerr(k, r)
contour(k, r, z, xlab = "k", ylab = "r", main = "contour plot for squared error")
```



Estimation using Gauss-Newton method is $\hat{k} = 1049$, $\hat{r} = 0.12$.

```
##' define log-likelihood function
loglikeli <- function(para) {
```

```r
  k <- para[1]
  r <- para[2]
  sigma2 <- para[3]
  l <- sum(-log(2*pi*sigma2)/2 - (log(beetles$beetles) - log(k) -
                                    log(beetles$beetles[1]) +
                                    log(beetles$beetles[1] +
                                          (k-beetles$beetles[1]) *
                                          exp(-r*beetles$days))^2)/2/sigma2)
  return(l)
}

##' define gradiant of loglikelihood function
grad.my <- function(para) {
  k <- para[1]
  r <- para[2]
  sigma2 <- para[3]
  g <- rep(0, 3)
  g[1] <- sum(-2*(log(beetles$beetles)-log(k)-log(beetles$beetles[1])+
                    log(beetles$beetles[1]+(k-beetles$beetles[1])*
                          exp(-r*beetles$days)))*
                (-1/k+exp(-r*beetles$days)/
                   (beetles$beetles[1]+
                      (k-beetles$beetles[1])*exp(-r*beetles$days)))/2/sigma2)
  g[2] <- sum(2*(log(beetles$beetles)-log(k)-log(beetles$beetles[1])+
                   log(beetles$beetles[1]+(k-beetles$beetles[1])*
                         exp(-r*beetles$days)))*
                beetles$days*(k-beetles$beetles[1])*exp(-r*beetles$days)/
                (beetles$beetles[1]+(k-beetles$beetles[1])*exp(-r*beetles$days))/
                2/sigma2)
  g[3] <- sum(-1/2/sigma2+(log(beetles$beetles)-log(k)-log(beetles$beetles[1])+
                             log(beetles$beetles[1]+(k-beetles$beetles[1])*
                                   exp(-r*beetles$days)))^2/2/sigma2^2)
  return(g)
}

fit <- constrOptim(theta = c(10, 0.1, 1), f = loglikeli, grad = grad.my,
                   ui = diag(1, 3), ci = rep(0, 3),
                   control = list(fnscale = -1), hessian = TRUE)
fit$par
```

```
## [1] 103.983292  12.124656   2.913935
```

```r
fit$convergence
```

```
## [1] 0
```

```r
Diag(-solve(fit$hessian))
```

```
## [1] 2.707176e+03 1.212466e+05 2.841633e+00
```

Using BFGS methods, set constrain to make k, r, $sigma^2$ nonnegative, we get the MLE estimates $\hat{k} = 103.98$, $\hat{r} = 12.12$, $\hat{\sigma}^2 = 2.91$. Using inverse of negative hessian, we have these estimates' variance to be 2.7×10^3, 1.2×10^5, 2.8 respectively.

But there's problem that results will change dramatically with initial values. I just tried several different initial values, get the estimates and compare their corresponding loglikelihood. I just choose the one with maximum loglikelihood.