Many Local Maxima

Guanting Wei

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1. Formula

Density function:

$$f(x;\theta) = \frac{1-\cos(x-\theta)}{2\pi}, \\ 0 \le x \le 2\pi, \theta \in (-\pi,\pi) \tag{1.1}$$

The likelihood funcion:

$$L(\theta) = \prod_{i=1}^{n} f(X_i;\theta) \\ 0 \le x \le 2\pi, \theta \in (-\pi,\pi) \tag{1.2}$$

The loglikelihood funcion:

$$l(\theta) = \ln L(\theta) = \sum_{i=1}^{n} \ln f(X_i;\theta) = \sum_{i=1}^{n} \ln[\frac{1-\cos(x-\theta)}{2\pi}] \\ 0 \le x \le 2\pi, \theta \in (-\pi,\pi) \tag{1.3}$$

Compute the differential of loglikelihood funcion:

First derivative:

$$l'(\theta) = \sum_{i=1}^{n} \frac{\sin(X_i-\theta)}{1-\cos(X_i-\theta)} \\ 0 \le x \le 2\pi, \theta \in (-\pi,\pi) \tag{1.4}$$

Second derivative:

$$l''(\theta) = \sum_{i=1}^{n} \frac{1}{[1-\cos(X_i-\theta)]^2} \\ 0 \le x \le 2\pi, \theta \in (-\pi,\pi) \tag{1.5}$$

2. Plot The Log-likelihood Function

Sample:

```
X <- c(3.91, 4.85, 2.28, 4.06, 3.70, 4.04, 5.46, 3.53, 2.28, 1.96,
       2.53, 3.88, 2.22, 3.47, 4.82, 2.46, 2.99, 2.54, 0.52)
```

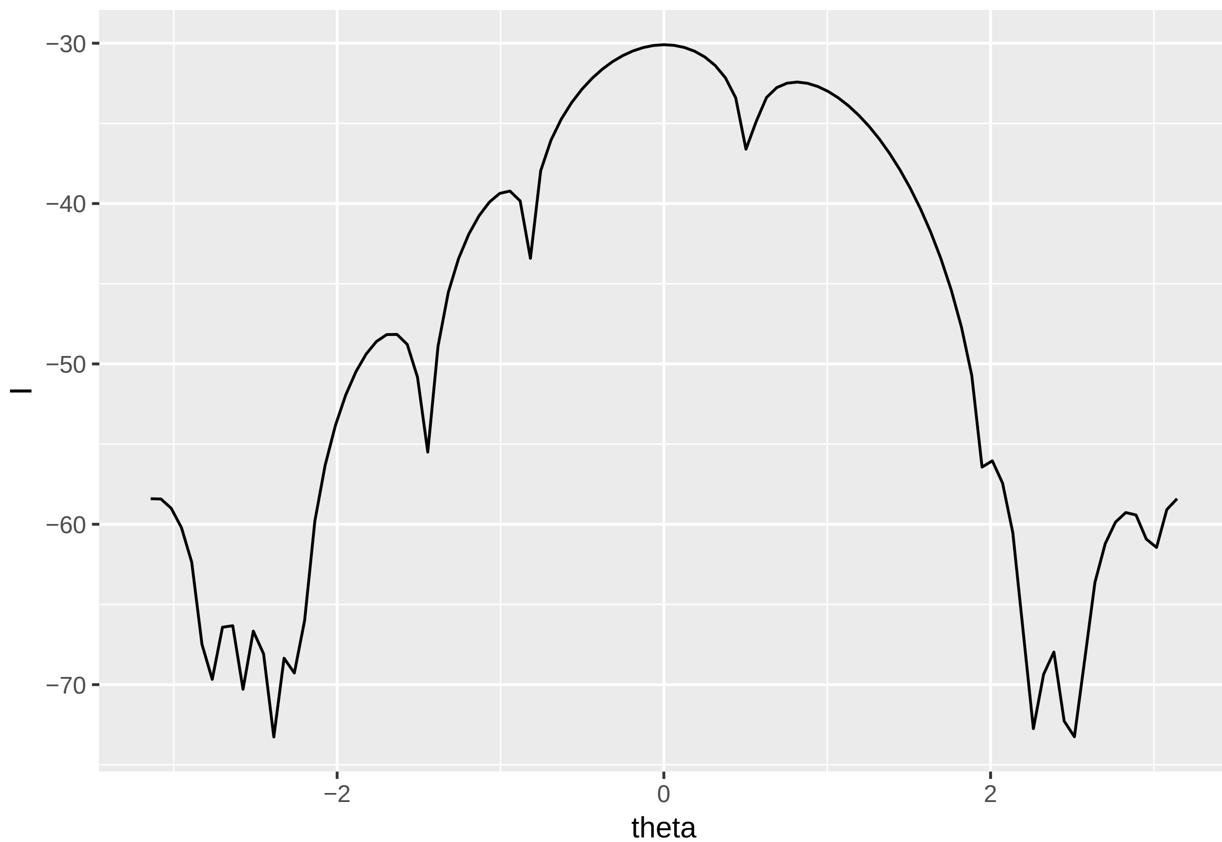
Define loglikelihood function and its differential function

```
f=function(x,theta)(1-cos(x-theta))/(2*pi)

L=function(x,theta){
  prod=1;
  for (i in 1:length(x)){
```

```r
    prod = prod*((1-cos(x[i]-theta))/(2*pi));
  }
  prod
}

l=function(x,theta){log(L(x,theta))}

l1=function(x,theta){
  sum=0;
  for(i in 1:length(x)){
    sum=sum+sin(x[i]-theta)/(1-cos(x[i]-theta))
  }
  sum
}

l2=function(x,theta){
  sum=0;
  for(i in 1:length(x)){
    sum=sum+1/(1-cos(x[i]-theta))^2
  }
  sum
}
```

plot

```r
library("ggplot2")
ggplot(data.frame(x=c(-pi,pi)),aes(x=x)) +
  stat_function(fun=function(theta) l(X,theta)) +
  labs(x=expression("theta"),y="l")
```



3. Method-of-Moments

$$
\begin{aligned}
E(X|\theta) &= \int_0^{2\pi} x f(x;\theta)\mathrm{d}x \\
&= \int_0^{2\pi} x \frac{1-\cos(x-\theta)}{2\pi}\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} x\mathrm{d}[x-\sin(x-\theta)] \\
&= \frac{1}{2\pi}\left\{ x[x-\sin(x-\theta)]\bigg|_0^{2\pi} - \int_0^{2\pi} x-\sin(x-\theta)\mathrm{d}x \right\} \\
&= \pi + \sin(\theta)
\end{aligned}
$$

```
E=function(theta){pi+sin(theta)}
X_n=mean(X)
theta_n=c(0,0)
theta_n[1]=asin(mean(X)-pi)
theta_n[2]=pi-theta_n[1]
theta_n[1]
```

```
## [1] 0.09539407
```

```
theta_n[2]
```

```
## [1] 3.046199
```

So we can get$\tilde{\theta_n} = 0.09539407 \ or \ 3.046199$

4. MLE

```
r1=c(0,0)
count1=matrix(0,1, length(theta_n))
for(i in 1:length(theta_n)) {
  r1[i]=theta_n[i]
  while (abs(l1(X,r1[i]))>.Machine$double.eps&&count1[i]<10000) {
    temp=r1[i]-l1(X,r1[i])/l2(X,r1[i])
    r1[i]=temp
    count1[i]=count1[i]+1
  }
}
table1=rbind(theta_n,r1)
rownames(table1)=c('theta_0',"root")
library(pander)
set.caption("theta_0 and roots")
pander(table1)
```

Table 1: theta_0 and roots

theta_0	0.09539	3.046
root	0.003118	3.171

5. Different Initial θ

```
theta_0=c(-2.7,2.7)
r2=c(0,0)
count2=matrix(0,1, length(theta_0))
for(i in 1:length(theta_0)) {
  r2[i]=theta_0[i]
  while (abs(l1(X,r2[i]))>0.00000001&&count2[i]<10000) {
    temp=r2[i]-l1(X,r2[i])/l2(X,r2[i])
    r2[i]=temp
    count2[i]=count2[i]+1
  }
}
table2=rbind(theta_0,r2)
rownames(table2)=c('theta_0',"root")
library(pander)
set.caption("theta_0 and roots")
pander(table2)
```

Table 2: theta_0 and roots

theta_0	-2.7	2.7
root	-2.669	2.848

6. Initial θ from $-\pi$ to π

```r
i=seq(-pi, pi, length=200)
r=matrix(0,1, length(i))
count=matrix(0,1, length(i))
for(k in 1:length(i)) {
  r[k]=i[k]
  while (abs(l1(X,r[k]))>0.00000001&&count[k]<10000) {
    temp=r[k]-l1(X,r[k])/l2(X,r[k])
    r[k]=temp
    count[k]=count[k]+1
  }
}
library(pander)
table3=rbind(i,r)
rownames(table3)=c("i","root")
set.caption("200 equally spaced initial thetas from -pi to pi")
pander(table3)
```

Table 3: 200 equally spaced initial thetas from -pi to pi (continued below)

i	-3.142	-3.11	-3.078	-3.047	-3.015	-2.984	-2.952
root	-3.112	-3.112	-3.112	-3.112	-3.112	-3.112	-3.112

Table 4: Table continues below

i	-2.921	-2.889	-2.857	-2.826	-2.794	-2.763	-2.731
root	-3.112	-3.112	-3.112	-3.112	-2.787	-2.772	-2.669

Table 5: Table continues below

i	-2.7	-2.668	-2.636	-2.605	-2.573	-2.542	-2.51
root	-2.669	-2.669	-2.669	-2.669	-2.549	-2.509	-2.509

Table 6: Table continues below

i	-2.479	-2.447	-2.415	-2.384	-2.352	-2.321	-2.289
root	-2.509	-2.509	-2.509	-2.386	-2.298	-2.298	-2.298

Table 7: Table continues below

i	-2.258	-2.226	-2.194	-2.163	-2.131	-2.1	-2.068
root	-2.298	-2.226	-1.663	-1.663	-1.663	-1.663	-1.663

Table 8: Table continues below

Table 8: Table continues below

i	-2.037	-2.005	-1.973	-1.942	-1.91	-1.879	-1.847
root	-1.663	-1.663	-1.663	-1.663	-1.663	-1.663	-1.663

Table 9: Table continues below

i	-1.815	-1.784	-1.752	-1.721	-1.689	-1.658	-1.626
root	-1.663	-1.663	-1.663	-1.663	-1.663	-1.663	-1.663

Table 10: Table continues below

i	-1.594	-1.563	-1.531	-1.5	-1.468	-1.437	-1.405
root	-1.663	-1.663	-1.663	-1.663	-1.469	-1.437	-0.9544

Table 11: Table continues below

i	-1.373	-1.342	-1.31	-1.279	-1.247	-1.216	-1.184
root	-0.9544	-0.9544	-0.9544	-0.9544	-0.9544	-0.9544	-0.9544

Table 12: Table continues below

i	-1.152	-1.121	-1.089	-1.058	-1.026	-0.9946	-0.963
root	-0.9544	-0.9544	-0.9544	-0.9544	-0.9544	-0.9544	-0.9544

Table 13: Table continues below

i	-0.9314	-0.8999	-0.8683	-0.8367	-0.8051	-0.7736
root	-0.9544	-0.9544	-0.9544	-0.9544	0.003118	0.003118

Table 14: Table continues below

i	-0.742	-0.7104	-0.6788	-0.6473	-0.6157	-0.5841
root	0.003118	0.003118	0.003118	0.003118	0.003118	0.003118

Table 15: Table continues below

i	-0.5525	-0.521	-0.4894	-0.4578	-0.4262	-0.3947
root	0.003118	0.003118	0.003118	0.003118	0.003118	0.003118

Table 16: Table continues below

i	-0.3631	-0.3315	-0.3	-0.2684	-0.2368	-0.2052
root	0.003118	0.003118	0.003118	0.003118	0.003118	0.003118

Table 17: Table continues below

i	-0.1737	-0.1421	-0.1105	-0.07893	-0.04736	-0.01579
root	0.003118	0.003118	0.003118	0.003118	0.003118	0.003118

Table 18: Table continues below

i	0.01579	0.04736	0.07893	0.1105	0.1421	0.1737
root	0.003118	0.003118	0.003118	0.003118	0.003118	0.003118

Table 19: Table continues below

i	0.2052	0.2368	0.2684	0.3	0.3315	0.3631
root	0.003118	0.003118	0.003118	0.003118	0.003118	0.003118

Table 20: Table continues below

i	0.3947	0.4262	0.4578	0.4894	0.521	0.5525	0.5841
root	0.003118	0.003118	0.003118	0.003118	0.521	0.8126	0.8126

Table 21: Table continues below

i	0.6157	0.6473	0.6788	0.7104	0.742	0.7736	0.8051
root	0.8126	0.8126	0.8126	0.8126	0.8126	0.8126	0.8126

Table 22: Table continues below

i	0.8367	0.8683	0.8999	0.9314	0.963	0.9946	1.026
root	0.8126	0.8126	0.8126	0.8126	0.8126	0.8126	0.8126

Table 23: Table continues below

i	1.058	1.089	1.121	1.152	1.184	1.216	1.247
root	0.8126	0.8126	0.8126	0.8126	0.8126	0.8126	0.8126

Table 24: Table continues below

i	1.279	1.31	1.342	1.373	1.405	1.437	1.468
root	0.8126	0.8126	0.8126	0.8126	0.8126	0.8126	0.8126

Table 25: Table continues below

i	1.5	1.531	1.563	1.594	1.626	1.658	1.689
root	0.8126	0.8126	0.8126	0.8126	0.8126	0.8126	0.8126

Table 26: Table continues below

i	1.721	1.752	1.784	1.815	1.847	1.879	1.91
root	0.8126	0.8126	0.8126	0.8126	0.8126	0.8126	0.8126

Table 27: Table continues below

i	1.942	1.973	2.005	2.037	2.068	2.1	2.131	2.163
root	0.8126	2.005	2.007	2.007	2.007	2.007	2.007	2.007

Table 28: Table continues below

i	2.194	2.226	2.258	2.289	2.321	2.352	2.384	2.415
root	2.007	2.227	2.238	2.301	2.375	2.375	2.375	2.375

Table 29: Table continues below

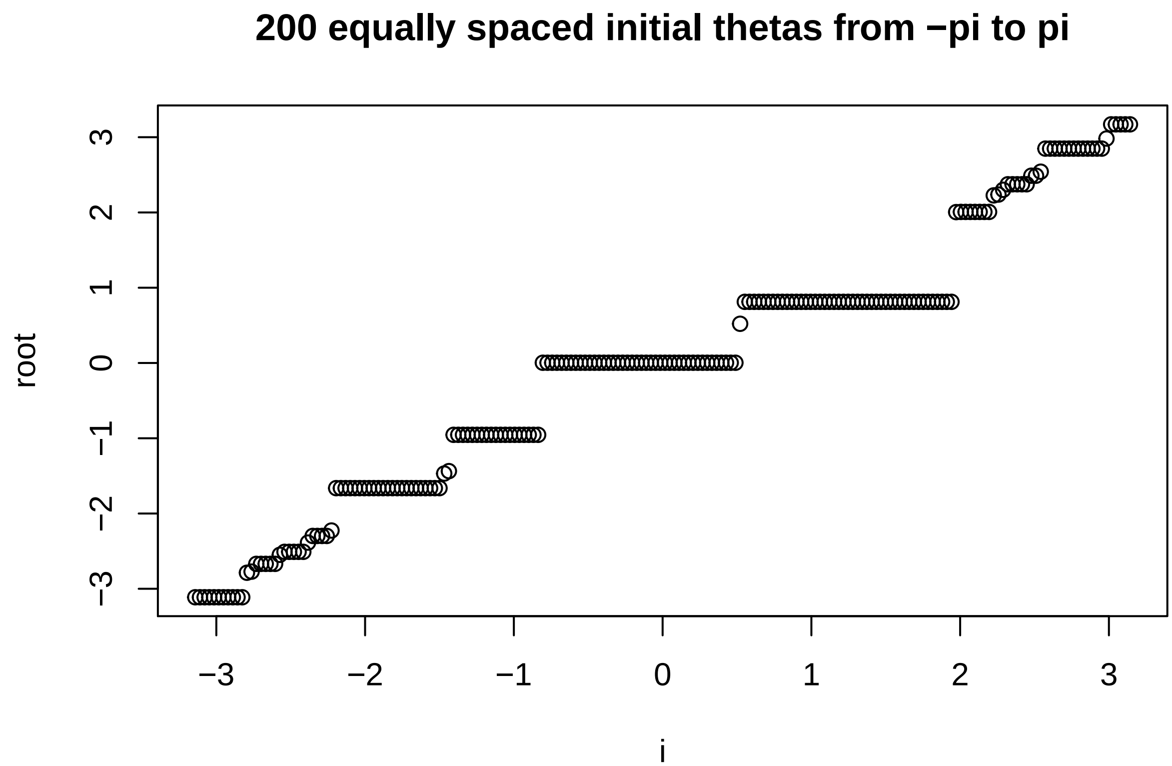
i	2.447	2.479	2.51	2.542	2.573	2.605	2.636	2.668
root	2.375	2.488	2.489	2.542	2.848	2.848	2.848	2.848

Table 30: Table continues below

i	2.7	2.731	2.763	2.794	2.826	2.857	2.889	2.921
root	2.848	2.848	2.848	2.848	2.848	2.848	2.848	2.848

i	2.952	2.984	3.015	3.047	3.078	3.11	3.142
root	2.848	2.982	3.171	3.171	3.171	3.171	3.171

```
plot(i,r,xlab = NULL, ylab = "root",main="200 equally spaced initial thetas from -pi to pi")
```



Count the roots

```
r_simp=round(r,4)
pander(table(r_simp))
```

Table 32: Table continues below

-3.1125	-2.7868	-2.7723	-2.6689	-2.549	-2.5094	-2.3863	-2.2979
11	1	1	5	1	5	1	4

Table 33: Table continues below

-2.226	-1.6627	-1.4691	-1.4368	-0.9544	0.0031	0.521	0.8126
1	23	1	1	19	42	1	45

Table 34: Table continues below

2.0047	2.0072	2.2268	2.2379	2.301	2.3747	2.4883	2.4885	2.5417
1	7	1	1	1	5	1	1	1

2.8484	2.982	3.1707
13	1	5

2.8484	2.982	3.1707

So the partion can be [1:11] [12] [13] [14:18] [19] [20:24] [25] [26:29] [30] [31:53] [54] [55] [56:74] [75:116] [117] [118 162] [163] [164:170] [171] [172] [173] [174:178] [179] [180] [181] [182:194] [195] [196:200]