< STAT-5361 > HW#4-Exercises 3.3

Hee-Seung, Kim

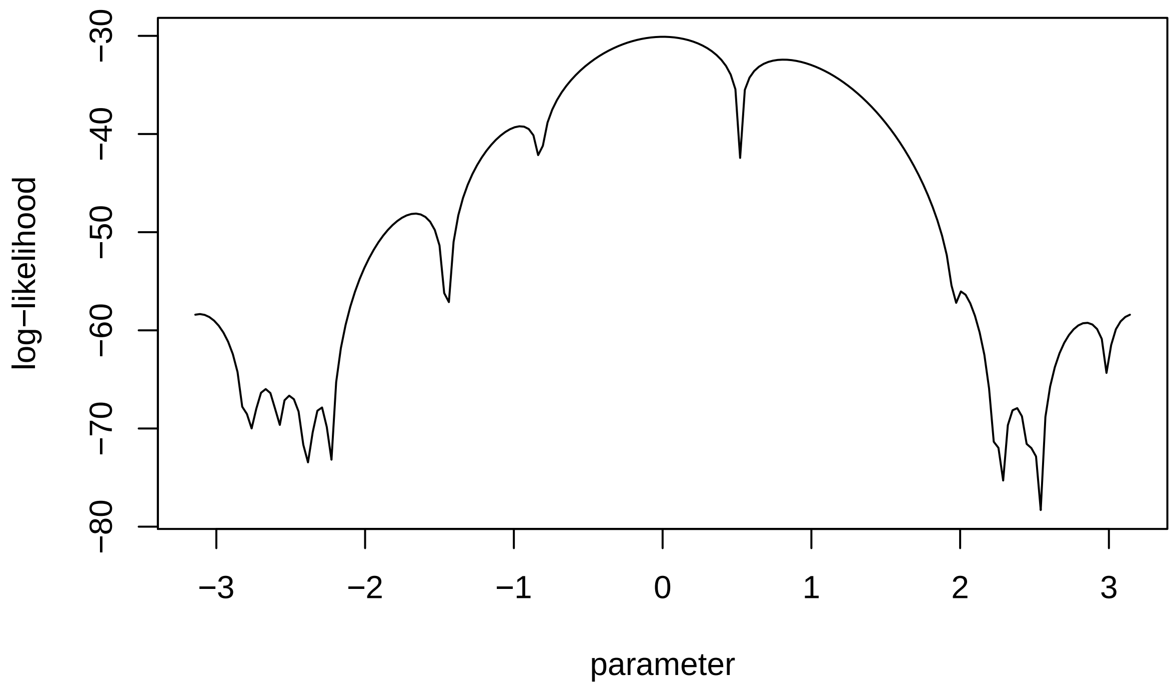
September 28, 2018

Contents

1 Exercises 3.3.2

1.1 (a)

```
data2<- c(3.91,4.85,2.28,4.06,3.7,4.04,5.46,3.53,2.28,1.96,2.53,3.88,2.22,3.47,
          4.82,2.46,2.99,2.54,0.52);
a.value = seq(-pi, pi, length.out = 200);

m <- length(a.value)
y <- rep(0,m)

for (i in 1:m) {
  y[i] <- -length(data2)*log(2*pi) + sum(log(1-cos(data2-a.value[i])))
}
plot(a.value, y, xlab="parameter", ylab="log-likelihood", type="l");
```



1.2 (b)

$$
\begin{aligned}
E[x|\theta] &= \int_0^{2\pi} x\frac{1-cos(x-\theta)}{2\pi}dx \\
&= \frac{x^2}{4\pi}|_0^{2\pi} - \frac{xsin(x-\theta)}{2\pi}|_0^{2\pi} + \int_0^{2\pi}\frac{sin(x-\theta)}{2\pi}dx \\
&= \pi - sin(2\pi-\theta) - \frac{cos(x-\theta)}{2\pi}|_0^{2\pi} \\
&= \pi - sin(2\pi-\theta) - \frac{cos(2\pi-\theta)}{2\pi} + \frac{cos\theta}{2\pi} \\
&= \pi - sin(2\pi-\theta) = \pi + sin(\theta moment)
\end{aligned}
$$

$$
\therefore \theta moment = sin^{-1}(E[x|\theta]-\pi) = 0.09539
$$

1.3 (c)-(d)

```r
data2 <- c(3.91,4.85,2.28,4.06,3.7,4.04,5.46,3.53,2.28,1.96,2.53,3.88,2.22,3.47,
           4.82,2.46,2.99,2.54,0.52);
mme <- asin(sum(data2)/length(data2)-pi)

mle.trig <- function(sample, int.value,  lower=-10000, upper=10000) {
  n = length(sample);
  neg.log.lik <- function(a) {
    return( n*log(2*pi) - sum(log(1-cos(sample-a))));
  }

  minus.score <- function(a) {
    return( sum (sin(sample-a) / (1-cos(sample-a))  ));
  }

  hess.f <- function(a) {
    return( matrix(sum (1/ (1-cos(sample-a))  ),nrow=1));
  }
  ## MLE
  a.est = nlminb(start = int.value, neg.log.lik, gradient=minus.score,
                hessian = hess.f, lower=lower,upper=upper);
  return(a.est);
}

trig_MME <- mle.trig(data2,mme)
trig_Z1 <- mle.trig(data2,-2.7)
trig_Z2 <- mle.trig(data2,2.7)

trig_par <- c(trig_MME$par,trig_Z1$par,trig_Z2$par)
trig_obj <- -1*c(trig_MME$objective,trig_Z1$objective,trig_Z2$objective)
```

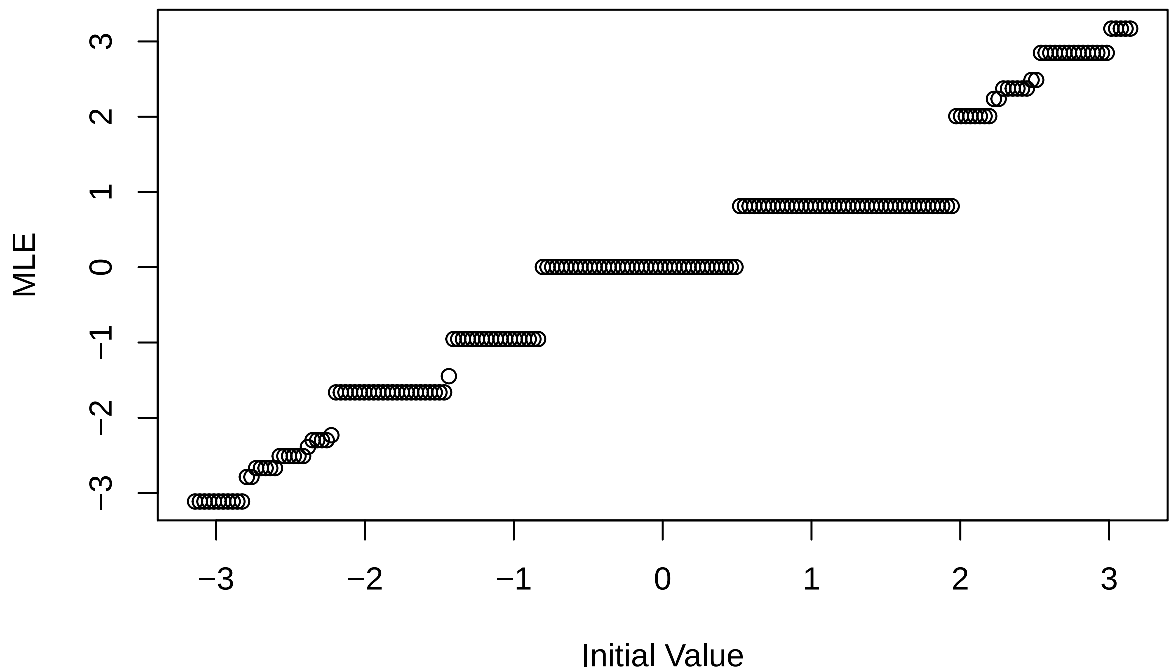
```
trig_iter <- c(trig_MME$iteration,trig_Z1$iteration,trig_Z2$iteration)

TRIG <- rbind(trig_par,trig_obj,trig_iter)
TRIG
```

```
##                     [,1]       [,2]       [,3]
## trig_par    0.003118157  -2.668857   2.848415
## trig_obj  -30.094933339 -65.981830 -59.227752
## trig_iter   4.000000000   4.000000   5.000000
```

With $\theta moment = 0.05939$, the golbal maximum is reached, and the MLE for θ is found to be 0.003118. With -2.7 and 2.7 as starting points, the MLE for θ is found to be -2.66886 and 2.848415 respectively.

1.4 (e)

```
mle_trigs <- rep(0,200)
for (i in 1:200) {
  mle_trigs[i] <- mle.trig(data2,a.value[i])$par
}

plot(a.value,mle_trigs,xlab="Initial Value", ylab="MLE")
```




```
mle_trigs <- floor(mle_trigs*1000000)/1000000

SetAtt <- cbind(aggregate(a.value, list(mle_trigs), min),aggregate(a.value,
                list(mle_trigs), max)[,2])
colnames(SetAtt) <- c("Unique Outcome","From","To")
a.value
```

```
##   [1] -3.14159265 -3.11001886 -3.07844506 -3.04687127 -3.01529747
##   [6] -2.98372368 -2.95214988 -2.92057608 -2.88900229 -2.85742849
##  [11] -2.82585470 -2.79428090 -2.76270711 -2.73113331 -2.69955952
##  [16] -2.66798572 -2.63641193 -2.60483813 -2.57326433 -2.54169054
##  [21] -2.51011674 -2.47854295 -2.44696915 -2.41539536 -2.38382156
##  [26] -2.35224777 -2.32067397 -2.28910017 -2.25752638 -2.22595258
##  [31] -2.19437879 -2.16280499 -2.13123120 -2.09965740 -2.06808361
##  [36] -2.03650981 -2.00493602 -1.97336222 -1.94178842 -1.91021463
##  [41] -1.87864083 -1.84706704 -1.81549324 -1.78391945 -1.75234565
##  [46] -1.72077186 -1.68919806 -1.65762426 -1.62605047 -1.59447667
##  [51] -1.56290288 -1.53132908 -1.49975529 -1.46818149 -1.43660770
##  [56] -1.40503390 -1.37346010 -1.34188631 -1.31031251 -1.27873872
##  [61] -1.24716492 -1.21559113 -1.18401733 -1.15244354 -1.12086974
##  [66] -1.08929595 -1.05772215 -1.02614835 -0.99457456 -0.96300076
##  [71] -0.93142697 -0.89985317 -0.86827938 -0.83670558 -0.80513179
```

```
##  [76] -0.77355799 -0.74198419 -0.71041040 -0.67883660 -0.64726281
##  [81] -0.61568901 -0.58411522 -0.55254142 -0.52096763 -0.48939383
##  [86] -0.45782003 -0.42624624 -0.39467244 -0.36309865 -0.33152485
##  [91] -0.29995106 -0.26837726 -0.23680347 -0.20522967 -0.17365588
##  [96] -0.14208208 -0.11050828 -0.07893449 -0.04736069 -0.01578690
## [101]  0.01578690  0.04736069  0.07893449  0.11050828  0.14208208
## [106]  0.17365588  0.20522967  0.23680347  0.26837726  0.29995106
## [111]  0.33152485  0.36309865  0.39467244  0.42624624  0.45782003
## [116]  0.48939383  0.52096763  0.55254142  0.58411522  0.61568901
## [121]  0.64726281  0.67883660  0.71041040  0.74198419  0.77355799
## [126]  0.80513179  0.83670558  0.86827938  0.89985317  0.93142697
## [131]  0.96300076  0.99457456  1.02614835  1.05772215  1.08929595
## [136]  1.12086974  1.15244354  1.18401733  1.21559113  1.24716492
## [141]  1.27873872  1.31031251  1.34188631  1.37346010  1.40503390
## [146]  1.43660770  1.46818149  1.49975529  1.53132908  1.56290288
## [151]  1.59447667  1.62605047  1.65762426  1.68919806  1.72077186
## [156]  1.75234565  1.78391945  1.81549324  1.84706704  1.87864083
## [161]  1.91021463  1.94178842  1.97336222  2.00493602  2.03650981
## [166]  2.06808361  2.09965740  2.13123120  2.16280499  2.19437879
## [171]  2.22595258  2.25752638  2.28910017  2.32067397  2.35224777
## [176]  2.38382156  2.41539536  2.44696915  2.47854295  2.51011674
## [181]  2.54169054  2.57326433  2.60483813  2.63641193  2.66798572
## [186]  2.69955952  2.73113331  2.76270711  2.79428090  2.82585470
## [191]  2.85742849  2.88900229  2.92057608  2.95214988  2.98372368
## [196]  3.01529747  3.04687127  3.07844506  3.11001886  3.14159265
```

For 200 equally spaced starting points bwtween $-\pi$ and π, we have grouped the points with each group corresponding to each unique outcome of the optimization. There are total 18 groups with 18 different outcomes of the optimization.

---

2 Exercises 3.3.3

2.1 (a)

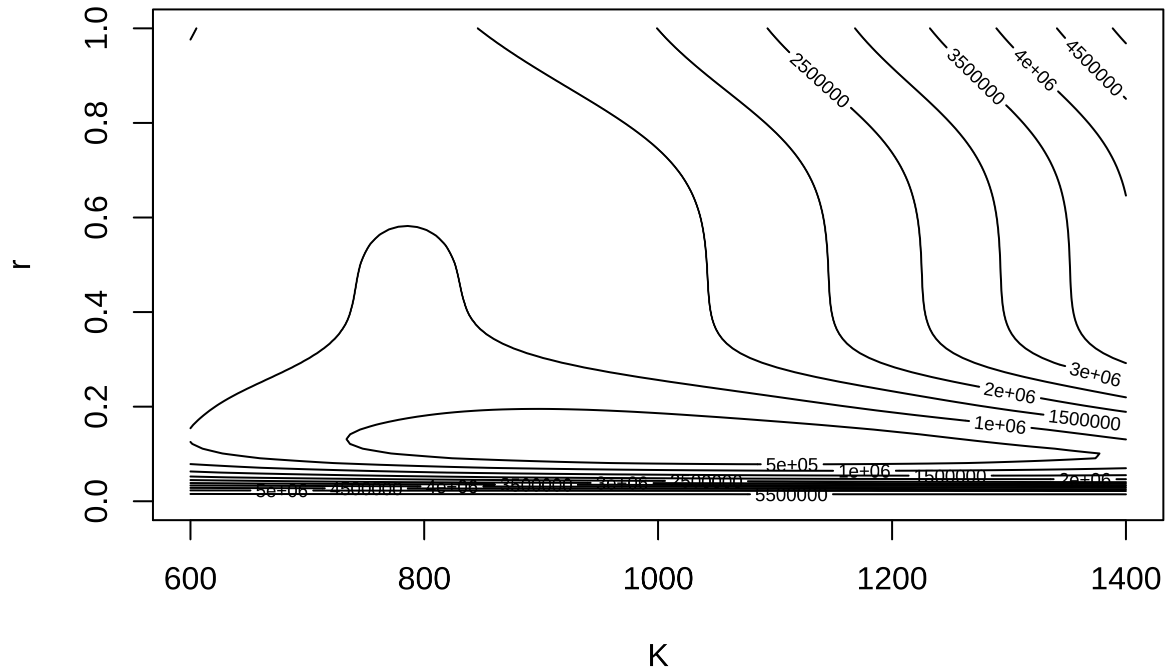
```r
t <- c(0,8,28,41,63,69,97,117,135,154)
x <- c(2,47,192,256,768,896,1120,896,1184,1024)
f <- expression(2*K/(2+(K-2)*exp(-r*t)))

df <- function(K,r,t){
  dfk <- D(f,"K")
  dfr <- D(f,"r")
  K <- K
  r <- r
  t <- t
  a <- eval(dfk)
  b <- eval(dfr)
  c <- array(c(a,b),c(1,2))
  return(c)
}

Df <- function(K,r){
  a <- K
  b <- r
  m <- df(a,b,t[1])
  for(i in 2:10){
    c <- df(a,b,t[i])
    m <- rbind(m,c)
  }
  return(m)
}

Z <- function(K,r){
  a <- c()
  for(i in 1:10){
    a[i] <- x[i] - 2*K/(2+(K-2)*exp(-r*t[i]))
  }
  m <- array(a,c(10,1))
  return(m)
}

theta <- matrix(c(1200,0.2),nrow=2)
delta <- matrix(c(1,1),nrow=2)

while(crossprod(delta,delta)>=0.001){
  theta1 <- theta
  a <- Df(theta[1,1],theta[2,1])
```

```r
  z <- Z(theta[1,1],theta[2,1])
  theta <- theta + solve(t(a)%*%a)%*%t(a)%*%z
  delta <- theta - theta1
}

a.est <- theta
print(a.est)
```

```
##              [,1]
## [1,] 1049.4038970
## [2,]    0.1182693
```

2.2 (b)

```r
f <- function(K,r){
  return(sum((x-2*K/(2+(K-2)*exp(-r*t)))^2))
}

z <- matrix(0,100,100,byrow=T)
for (i in 1:100){
  for (j in 1:100){
    K <- 600 + 8*j
    r <- 0 + 0.01*i
    z[j,i] <- f(K,r)
  }
}
K <- seq(600,1400,length.out = 100)
r <- seq(0,1,length.out = 100)
contour(K,r,z,xlab="K",ylab="r")
```



2.3 (c)

```r
t <- c(0,8,28,41,63,69,97,117,135,154)
x <- c(2,47,192,256,768,896,1120,896,1184,1024)
f <- expression(2*K/(2+(K-2)*exp(-r*t)))

df <- function(K,r,t){
  dfk <- D(f,"K")
  dfr <- D(f,"r")
  K <- K
  r <- r
  t <- t
  a <- eval(dfk)
  b <- eval(dfr)
  c <- array(c(a,b),c(1,2))
  return(c)
}

Df <- function(K,r){
  a <- K
  b <- r
```

```
  m <- df(a,b,t[1])
  for(i in 2:10){
    c <- df(a,b,t[i])
    m <- rbind(m,c)
  }
  return(m)
}

Z <- function(K,r){
  a <- c()
  for(i in 1:10){
    a[i] <- x[i] - 2*K/(2+(K-2)*exp(-r*t[i]))
  }
  m <- array(a,c(10,1))
  return(m)
}

theta <- matrix(c(1200,0.2),nrow=2)
delta <- matrix(c(1,1),nrow=2)

while(crossprod(delta,delta)>=0.001){
  theta1 <- theta
  a <- Df(theta[1,1],theta[2,1])
  z <- Z(theta[1,1],theta[2,1])
  theta <- theta + solve(t(a)%*%a)%*%t(a)%*%z
  delta <- theta - theta1
}

a.est <- theta
print(a.est)
```

```
##               [,1]
## [1,] 1049.4038970
## [2,]    0.1182693
```

```
l <- expression(log(1/(sqrt(2*pi)*sigema))-
                  (log((2*2+2*(K-2)*exp(-r*0))/(2*K)))^2/(2*sigema^2)+
                  log(1/(sqrt(2*pi)*sigema))-
                  (log((2*47+47*(K-2)*exp(-r*8))/(2*K)))^2/(2*sigema^2)+
                  log(1/(sqrt(2*pi)*sigema))-
                  (log((2*192+192*(K-2)*exp(-r*28))/(2*K)))^2/(2*sigema^2)+
                  log(1/(sqrt(2*pi)*sigema))-
                  (log((2*256+256*(K-2)*exp(-r*41))/(2*K)))^2/(2*sigema^2)+
                  log(1/(sqrt(2*pi)*sigema))-
                  (log((2*768+768*(K-2)*exp(-r*63))/(2*K)))^2/(2*sigema^2)+
                  log(1/(sqrt(2*pi)*sigema))-
                  (log((2*896+896*(K-2)*exp(-r*69))/(2*K)))^2/(2*sigema^2)+
```

```
                    log(1/(sqrt(2*pi)*sigema))-
                    (log((2*1120+1120*(K-2)*exp(-r*97))/(2*K)))^2/(2*sigema^2)+
                    log(1/(sqrt(2*pi)*sigema))-
                    (log((2*896+896*(K-2)*exp(-r*117))/(2*K)))^2/(2*sigema^2)+
                    log(1/(sqrt(2*pi)*sigema))-
                    (log((2*1185+1184*(K-2)*exp(-r*135))/(2*K)))^2/(2*sigema^2)+
                    log(1/(sqrt(2*pi)*sigema))-
                    (log((2*1024+1024*(K-2)*exp(-r*154))/(2*K)))^2/(2*sigema^2))

dl <- function(beita){
  dlk <- D(l,"K")
  dlr <- D(l,"r")
  dlsigema <- D(l,"sigema")
  K <- beita[1]
  r <- beita[2]
  sigema <- beita[3]
  a <- eval(dlk)
  b <- eval(dlr)
  c <- eval(dlsigema)
  return(c(a,b,c))
}

ddl <- function(beita){
  dlkk <- D(D(l,"K"),"K")
  dlkr <- D(D(l,"K"),"r")
  dlksigema <- D(D(l,"K"),"sigema")
  dlrr <- D(D(l,"r"),"r")
  dlrsigema <- D(D(l,"r"),"sigema")
  dlsigema2 <- D(D(l,"sigema"),"sigema")
  K <- beita[1]
  r <- beita[2]
  sigema <- beita[3]
  a <- c(eval(dlkk),eval(dlkr),eval(dlksigema),eval(dlkr),eval(dlrr),
         eval(dlrsigema),eval(dlksigema),eval(dlrsigema),eval(dlsigema2))
  m <- matrix(a,byrow=TRUE,nrow=3)
  return(m)
}

a <- matrix(c(1200,0.2,0.5),nrow=3)
delta <- matrix(c(1,1,1),nrow=3)
while(crossprod(delta,delta)>=0.001){
  b <- a
  c <- matrix(dl(a),nrow=3)
  d <- solve(ddl(a))
  a <- a - d%*%c
  delta <- a - b
}
```

```r
a
```

```
##              [,1]
## [1,] 820.5349872
## [2,]   0.1926176
## [3,]   0.6441323
```

```r
solve(-ddl(a))
```

```
##               [,1]          [,2]          [,3]
## [1,]  6.248530e+04 -9.054089e+00  1.051866e-07
## [2,] -9.054089e+00  3.974068e-03 -5.353628e-11
## [3,]  1.051866e-07 -5.353628e-11  2.074532e-02
```