Question3_2

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Abstract

In this assignment, we use the probability density function to find the log-likelihood function and the method-of-moments estimator of θ. Then we will discuss the influence of different initial value when using Newton-Raphson method to estimate the Maximum Likelihood Estimation for θ.

1 Many local maxima

The probability density function with parameter θ:

$$f(x;\theta) = \frac{1-\cos(x-\theta)}{2\pi}, \quad 0 \le x \le 2\pi, \quad \theta \in (-\pi, \pi).$$

And the random sample from the distribution is

$x_i = 3.91, 4.85, 2.28, 4.06, 3.70, 4.04, 5.46, 3.53, 2.28, 1.96, 2.53, 3.88, 2.22, 3.47, 4.82, 2.46, 2.99, 2.54, 0.52$

- Find the the log-likelihood function of θ based on the sample and plot it between $-\pi$ and π.
- Find the method-of-moments estimator of θ. That is, the estimator $\tilde{\theta}_n$ is value of θ with

$$E(X \mid \theta) = \bar{X}_n,$$

  where $\bar{X}_n$ is the sample mean. This means you have to first find the expression for $E(X \mid \theta)$.
- Find the MLE for θ using the Newton–Raphson method initial value $\theta_0 = \tilde{\theta}_n$.
- What solutions do you find when you start at $\theta_0 = -2.7$ and $\theta_0 = 2.7$?
- Repeat the above using 200 equally spaced starting values between $-\pi$ and π. Partition the values into sets of attraction. That is, divide the set of starting values into separate groups, with each group corresponding to a separate unique outcome of the optimization.

2 Log-likelihood function

The log-likelihood function with parameter θ and x:

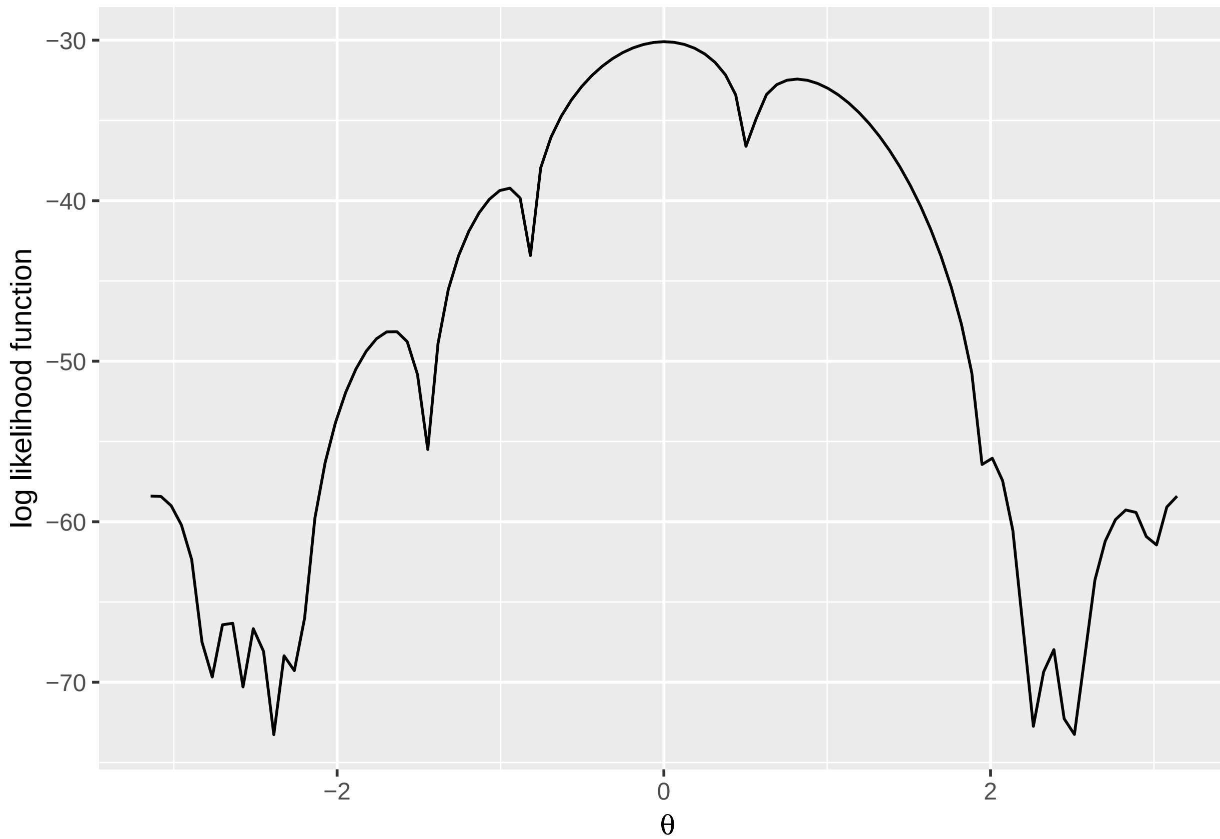
$$\ell(\theta) = \sum_{i=1}^{n} ln(1-\cos(x-\theta)) - nln2\pi$$

then plot it between $-\pi$ and π:

```
X = c(3.91, 4.85, 2.28, 4.06, 3.70, 4.04, 5.46, 3.53, 2.28, 1.96,
      2.53, 3.88, 2.22, 3.47, 4.82, 2.46, 2.99, 2.54, 0.52)
logfunction <- function(x) {                         #calculate the log likelihood function
  logfunction <- 0
  for (i in 1:length(X)){
    logfunction <- logfunction+log(1-cos(X[i]-x))-log(2*pi)
  }
  logfunction
```

```r
}

library("ggplot2")                                          #plot the function against theta
ggplot(data.frame(x= c(-pi,pi)),aes(x = x))+
         stat_function(fun = function(x) logfunction(x))+
         labs(x=expression(~theta),y="log likelihood function")
```



3 Method-of-moments estimator

The expression for $E(X \mid \theta)$ is:

$$
\begin{aligned}
E(X|\theta) &= \int_0^{2\pi} x \frac{1-\cos(x-\theta)}{2\pi} dx \\
&= \frac{1}{2\pi} \left[\int_0^{2\pi} x dx - \int_0^{2\pi} x \cos(x-\theta) dx \right] \\
&= \frac{1}{2\pi} \left[\frac{1}{2} x^2 \Big|_0^{2\pi} - x sin(x-\theta) \Big|_0^{2\pi} \right] \\
&= \frac{1}{2\pi} \left(2\pi^2 + 2\pi \sin\theta \right) \\
&= \pi + \sin\theta
\end{aligned}
$$

And $\bar{X}_n$ is 3.236842, so we can get θ:

$$\begin{aligned}\theta &= \arcsin(mean(x)-\pi) \quad or \quad \pi-\arcsin(mean(x)-\pi)\\ &= 0.095394 \quad or \quad 3.046199\end{aligned}$$

4 Newton–Raphson method

Find the MLE for θ using the Newton–Raphson method initial value $\theta_0 = \tilde{\theta}_n$. In this question, we will calculate$\ell'(\theta), \ell''(\theta)$ as follow:

$$\ell'(\theta) = \sum_{i=1}^{n} \frac{-sin(x-\theta)}{1-cos(x-\theta)}$$
$$\ell"(\theta) = \sum_{i=1}^{n} \frac{1}{cos(x-\theta)-1}$$

Newton–Raphson method:

```
library(pracma)
derivative_1 <- function(x) {          #calculate the derivative of log likelihood function
  dev1 <- 0
  for (i in 1:length(X)){
    dev1 <- dev1-sin(X[i]-x)/(1-cos(X[i]-x))
  }
  dev1
}
derivative_2 <- function(x) {  #calculate the second derivative of log likelihood function
  dev2 <- 0
  for (i in 1:length(X)){
    dev2 <- dev2+1/(cos(X[i]-x)-1)
  }
  dev2
}
x_initial <- 0.095394
result1 <- newtonRaphson(derivative_1,x_initial,dfun = derivative_2)
x_initial <- 3.046199
result2 <- newtonRaphson(derivative_1,x_initial,dfun = derivative_2)
```

If we start with $\theta = 0.095394$ we will get $root = 0.003118157$, when we start with $\theta = 3.046199$ the root is 3.170715.

5 Solutions find using selected $\theta = \pm 2.7$

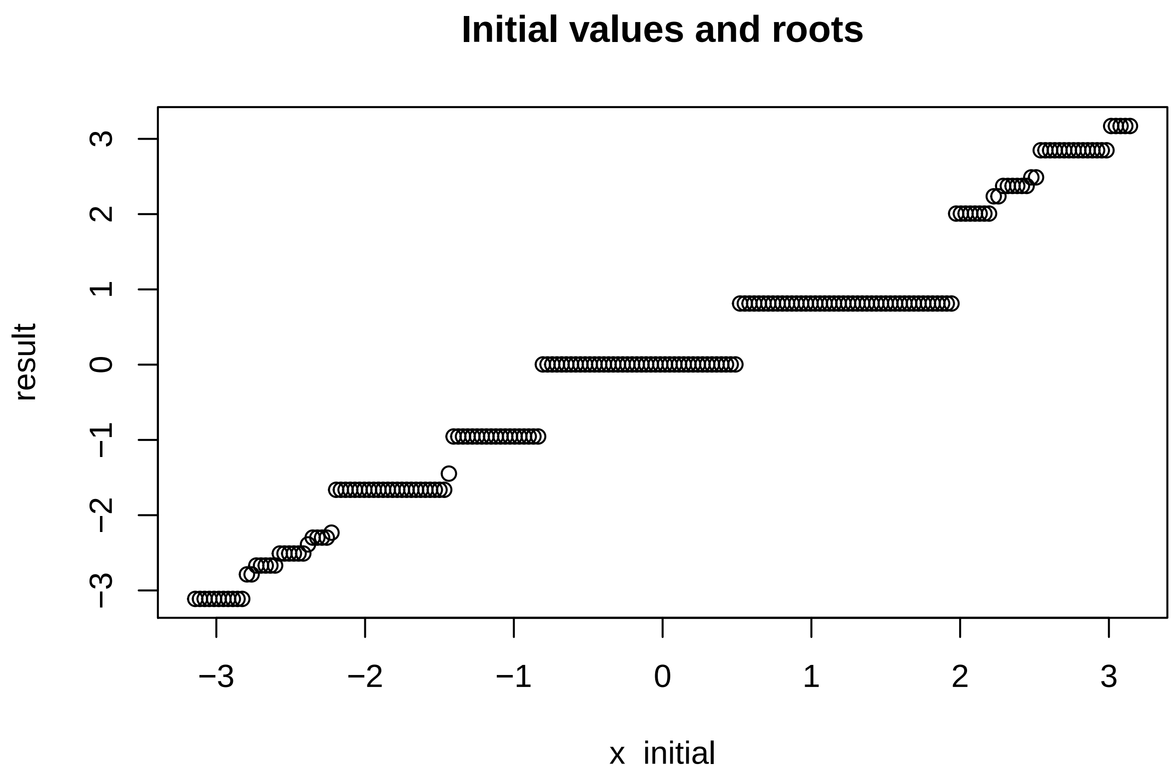
We can just change the initial vaule of θ to find the root:

```
x_initial <- 2.7
result3 <- newtonRaphson(derivative_1,x_initial,dfun = derivative_2)
x_initial <- -2.7
result4 <- newtonRaphson(derivative_1,x_initial,dfun = derivative_2)
```

If we start with $\theta = 2.7$ we will get $root = 2.848415$, when we start with $\theta = -2.7$ the root is -2.668857.

6 Solutions using θ between $-\pi$ and π

```
library("pander")
options(digits = 5)
x_initial <- seq(-pi,pi,length = 200)
result <- matrix(0,1,length(x_initial))
for (i in 1:length(x_initial)){
  result[i] <- newtonRaphson(derivative_1,x_initial[i],dfun = derivative_2)
}
plot(x_initial, result, title("Initial values and roots"))
```




```
table = rbind(x_initial,result)
set.caption("Initial values and roots")
pander(table,split.table = 100,style = 'rmarkdown')
```

Table 1: Initial values and roots (continued below)

x_initial	-3.1416	-3.11	-3.0784	-3.0469	-3.0153	-2.9837	-2.9521
result	-3.1125	-3.1125	-3.1125	-3.1125	-3.1125	-3.1125	-3.1125

Table 2: Table continues below

x__initial	-2.9206	-2.889	-2.8574	-2.8259	-2.7943	-2.7627	-2.7311
result	-3.1125	-3.1125	-3.1125	-3.1125	-2.7866	-2.7866	-2.6689

Table 3: Table continues below

x__initial	-2.6996	-2.668	-2.6364	-2.6048	-2.5733	-2.5417	-2.5101
result	-2.6689	-2.6689	-2.6689	-2.6689	-2.5094	-2.5094	-2.5094

Table 4: Table continues below

x__initial	-2.4785	-2.447	-2.4154	-2.3838	-2.3522	-2.3207	-2.2891
result	-2.5094	-2.5094	-2.5094	-2.3883	-2.2979	-2.2979	-2.2979

Table 5: Table continues below

x__initial	-2.2575	-2.226	-2.1944	-2.1628	-2.1312	-2.0997	-2.0681
result	-2.2979	-2.2322	-1.6627	-1.6627	-1.6627	-1.6627	-1.6627

Table 6: Table continues below

x__initial	-2.0365	-2.0049	-1.9734	-1.9418	-1.9102	-1.8786	-1.8471
result	-1.6627	-1.6627	-1.6627	-1.6627	-1.6627	-1.6627	-1.6627

Table 7: Table continues below

x__initial	-1.8155	-1.7839	-1.7523	-1.7208	-1.6892	-1.6576	-1.6261
result	-1.6627	-1.6627	-1.6627	-1.6627	-1.6627	-1.6627	-1.6627

Table 8: Table continues below

x__initial	-1.5945	-1.5629	-1.5313	-1.4998	-1.4682	-1.4366	-1.405
result	-1.6627	-1.6627	-1.6627	-1.6627	-1.6627	-1.4475	-0.95441

Table 9: Table continues below

x__initial	-1.3735	-1.3419	-1.3103	-1.2787	-1.2472	-1.2156	-1.184
result	-0.95441	-0.95441	-0.95441	-0.95441	-0.95441	-0.95441	-0.95441

Table 10: Table continues below

x__initial	-1.1524	-1.1209	-1.0893	-1.0577	-1.0261	-0.99457	-0.963
result	-0.95441	-0.95441	-0.95441	-0.95441	-0.95441	-0.95441	-0.95441

Table 11: Table continues below

x__initial	-0.93143	-0.89985	-0.86828	-0.83671	-0.80513	-0.77356
result	-0.95441	-0.95441	-0.95441	-0.95441	0.0031182	0.0031182

Table 12: Table continues below

x__initial	-0.74198	-0.71041	-0.67884	-0.64726	-0.61569	-0.58412
result	0.0031182	0.0031182	0.0031182	0.0031182	0.0031182	0.0031182

Table 13: Table continues below

x__initial	-0.55254	-0.52097	-0.48939	-0.45782	-0.42625	-0.39467
result	0.0031182	0.0031182	0.0031182	0.0031182	0.0031182	0.0031182

Table 14: Table continues below

x__initial	-0.3631	-0.33152	-0.29995	-0.26838	-0.2368	-0.20523
result	0.0031182	0.0031182	0.0031182	0.0031182	0.0031182	0.0031182

Table 15: Table continues below

x__initial	-0.17366	-0.14208	-0.11051	-0.078934	-0.047361	-0.015787
result	0.0031182	0.0031182	0.0031182	0.0031182	0.0031182	0.0031182

Table 16: Table continues below

x__initial	0.015787	0.047361	0.078934	0.11051	0.14208	0.17366
result	0.0031182	0.0031182	0.0031182	0.0031182	0.0031182	0.0031182

Table 17: Table continues below

x__initial	0.20523	0.2368	0.26838	0.29995	0.33152	0.3631
result	0.0031182	0.0031182	0.0031182	0.0031182	0.0031182	0.0031182

Table 18: Table continues below

x__initial	0.39467	0.42625	0.45782	0.48939	0.52097	0.55254
result	0.0031182	0.0031182	0.0031182	0.0031182	0.81264	0.81264

Table 19: Table continues below

x__initial	0.58412	0.61569	0.64726	0.67884	0.71041	0.74198	0.77356
result	0.81264	0.81264	0.81264	0.81264	0.81264	0.81264	0.81264

Table 20: Table continues below

x__initial	0.80513	0.83671	0.86828	0.89985	0.93143	0.963	0.99457
result	0.81264	0.81264	0.81264	0.81264	0.81264	0.81264	0.81264

Table 21: Table continues below

x__initial	1.0261	1.0577	1.0893	1.1209	1.1524	1.184	1.2156
result	0.81264	0.81264	0.81264	0.81264	0.81264	0.81264	0.81264

Table 22: Table continues below

x__initial	1.2472	1.2787	1.3103	1.3419	1.3735	1.405	1.4366
result	0.81264	0.81264	0.81264	0.81264	0.81264	0.81264	0.81264

Table 23: Table continues below

x__initial	1.4682	1.4998	1.5313	1.5629	1.5945	1.6261	1.6576
result	0.81264	0.81264	0.81264	0.81264	0.81264	0.81264	0.81264

Table 24: Table continues below

x__initial	1.6892	1.7208	1.7523	1.7839	1.8155	1.8471	1.8786
result	0.81264	0.81264	0.81264	0.81264	0.81264	0.81264	0.81264

Table 25: Table continues below

x__initial	1.9102	1.9418	1.9734	2.0049	2.0365	2.0681	2.0997	2.1312
result	0.81264	0.81264	2.0072	2.0072	2.0072	2.0072	2.0072	2.0072

Table 26: Table continues below

x__initial	2.1628	2.1944	2.226	2.2575	2.2891	2.3207	2.3522	2.3838
result	2.0072	2.0072	2.237	2.237	2.3747	2.3747	2.3747	2.3747

Table 27: Table continues below

x__initial	2.4154	2.447	2.4785	2.5101	2.5417	2.5733	2.6048	2.6364
result	2.3747	2.3747	2.4884	2.4884	2.8484	2.8484	2.8484	2.8484

Table 28: Table continues below

x__initial	2.668	2.6996	2.7311	2.7627	2.7943	2.8259	2.8574	2.889
result	2.8484	2.8484	2.8484	2.8484	2.8484	2.8484	2.8484	2.8484

x__initial	2.9206	2.9521	2.9837	3.0153	3.0469	3.0784	3.11	3.1416
result	2.8484	2.8484	2.8484	3.1707	3.1707	3.1707	3.1707	3.1707

So we have get the table of initial value and root, we can see there are several groups of the results, they are

```r
pander(table1)
```

Table 30: Table continues below

roots	-3.112	-2.787	-2.669	-2.509	-2.388	-2.298	-2.232
freq	11	2	5	6	1	4	1

Table 31: Table continues below

roots	-1.663	-1.448	-0.9544	-0.003118	-0.8126	2.007	2.237
freq	24	1	19	42	46	8	2

roots	2.375	2.488	2.848	3.171
freq	6	2	15	5