Homework 4

Travis Nestor

3.3.2

Given

$$f(x;\theta) = \frac{1 - cos(x - \theta)}{2\pi}, 0 \leq x \leq 2\pi, \theta \in (-\pi, \pi)$$

and random distribution

```
x <- c(3.91, 4.85, 2.28, 4.06, 3.70, 4.04, 5.46, 3.53, 2.28, 1.96, 2.53, 3.88, 2.22, 3.47, 4.82, 2.46, 1
```

3.3.2.1) Find Log likelihood of θ based on sample and plot between $-\pi$ and π

$$L(\theta) = \Pi_{i=1}^{n}(\frac{1 - cos(x_i - \theta)}{2\pi})$$

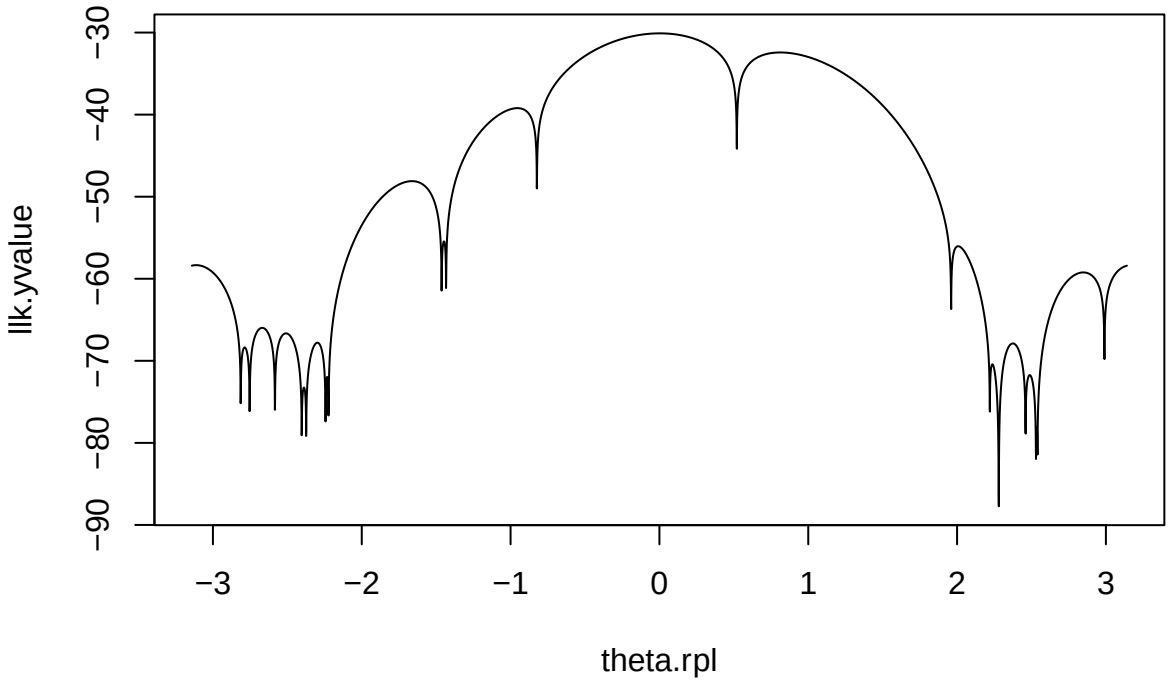
$$l(\theta) = log[\Pi_{i=1}^{n}(\frac{1 - cos(x_i - \theta)}{2\pi})]$$

$$l(\theta) = log\Pi_{i=1}^{n}([1 - cos(x_i - \theta)]) - log[(2\pi)^n]$$

```
llk.theta <- function(x,theta){
x <- c(3.91, 4.85, 2.28, 4.06, 3.70, 4.04, 5.46, 3.53, 2.28, 1.96, 2.53, 3.88, 2.22, 3.47, 4.82, 2.46, 1
 log(prod((1 - cos(x - theta))/ (2*pi)))
}

theta.rpl <- seq(-pi, pi, 1e-3)     #Create large number of theta values to test

llk.yvalue <- sapply(theta.rpl, FUN=function(theta.rpl) llk.theta(x, theta.rpl))
plot(theta.rpl, llk.yvalue, type = "l")
```



3.3.2.2) Find method of moments estimator of θ, i.e. find θ where $E[X|\theta] = X_n$, where X_n is the sample mean.

First find $E[X|\theta]$

$$E[X|\theta] = \sum_x x_i \frac{1 - cos(x_i - \theta)}{2\pi}$$

```
x <- c(3.91, 4.85, 2.28, 4.06, 3.70, 4.04, 5.46, 3.53, 2.28, 1.96, 2.53, 3.88, 2.22, 3.47, 4.82, 2.46, 2
mean(x)
```

```
## [1] 3.236842
```

$$\Rightarrow \sum_x x_i \frac{1 - cos(x_i - \theta)}{2\pi} = 3.236842$$

$$= \sum_x x_i(1 - cos(x_i - \theta)) = 3.236842 * 2\pi$$

$$= \sum_x x_i - \sum_x x_i * cos(x_i - \theta)$$

$$= 61.5 - \sum_x x_i * cos(x_i - \theta)$$

$$= \sum_x x_i * cos(x_i - \theta) = 41.16232$$

```
exval <- function(theta) {

    x <- c(3.91, 4.85, 2.28, 4.06, 3.70, 4.04, 5.46, 3.53, 2.28, 1.96, 2.53, 3.88, 2.22, 3.47, 4.82, 2.4

sum(x*(1 - cos(x - theta))) - 41.16232
}
uniroot(exval, lower = -10, upper = 10)$root
```

```
## [1] 2.755224
```

$$\Rightarrow \theta_n = 2.755224$$

3.3.2.3) Find MLE using Newton_raphson

$$l(\theta) = \sum(ln(1 - cos(x_i - \theta))) - ln(2\pi)$$

```
x <- c(3.91, 4.85, 2.28, 4.06, 3.70, 4.04, 5.46, 3.53, 2.28, 1.96, 2.53, 3.88, 2.22, 3.47, 4.82, 2.46, 1
llk.nr <- function(theta) {
    sum(log(1 - cos(x - theta))) - log(2*pi)
}

llk.nrprime <- function(theta) {
  sum((sin(x - theta)) / (1 - cos(x - theta)))
}

llk.nrprime2 <- function(theta) {
    sum(
    (cos(x - theta) * (1 - cos(x - theta)) - (sin(x - theta))^2 )
    / (1 - cos(x - theta))^2
    )

}

newton <- function(llkprime, llkprime2, theta0, n = 1000, tol = 1e-7){
  k <- n
  for (i in 1:n) {
  theta1 <- theta0 - (llkprime(theta0) / llkprime2(theta0))
  k[i] <- theta1
  if (abs(theta1 - theta0) < tol){
    root.nr <- tail(k, n=1)
    res <- list(root.nr)
    return(res)
  }
     theta0 <- theta1
  }
  if (i==n)
  return(c(theta0= i, root = theta1))

}

newton(llk.nrprime, llk.nrprime2, theta0=2.755)
```

```
## [[1]]
## [1] 2.53
```

$$\Rightarrow \theta_{MLE} = 2.53$$

3.3.2.4)

```
newton(llk.nrprime, llk.nrprime2, theta0=-2.7)
```

```
## [[1]]
## [1] -2.753185
```

```
newton(llk.nrprime, llk.nrprime2, theta0=2.7)
```

```
## [[1]]
## [1] 2.53
```

3.3.2.5)

```
theta.nr <- seq(-3.14,3.14,.0314)
newton.nr <- function(y){
  newton(llk.nrprime, llk.nrprime2, theta0=y, tol=1e-4, n=1000)
}
sapply(theta.nr, newton.nr)
```

```
## [[1]]
## [1] -3.293185
##
## [[2]]
## [1] -2.813185
##
## [[3]]
## [1] -2.813185
##
## [[4]]
## [1] -2.813185
##
## [[5]]
## [1] -2.813185
##
## [[6]]
## [1] -2.813185
##
## [[7]]
## [1] -2.813185
##
## [[8]]
## [1] -2.813185
##
## [[9]]
## [1] -2.813185
##
## [[10]]
## [1] -2.813185
##
## [[11]]
## [1] -2.813185
##
## [[12]]
```

```
## [1] -2.813185
##
## [[13]]
## [1] -2.753185
##
## [[14]]
## [1] -2.753185
##
## [[15]]
## [1] -2.753185
##
## [[16]]
## [1] -2.753185
##
## [[17]]
## [1] -2.583185
##
## [[18]]
## [1] -2.583185
##
## [[19]]
## [1] -2.583185
##
## [[20]]
## [1] -2.583185
##
## [[21]]
## [1] -2.583185
##
## [[22]]
## [1] -2.403185
##
## [[23]]
## [1] -2.403185
##
## [[24]]
## [1] -2.403185
##
## [[25]]
## [1] -2.373185
##
## [[26]]
## [1] -2.373185
##
## [[27]]
## [1] -2.373185
##
## [[28]]
## [1] -2.243185
##
## [[29]]
## [1] -2.243185
##
## [[30]]
```

```
## [1] -2.223185
##
## [[31]]
## [1] -2.243185
##
## [[32]]
## [1] -2.243185
##
## [[33]]
## [1] -2.243185
##
## [[34]]
## [1] -2.243185
##
## [[35]]
## [1] -2.243185
##
## [[36]]
## [1] -2.243185
##
## [[37]]
## [1] -2.243185
##
## [[38]]
## [1] -2.223185
##
## [[39]]
## [1] -2.243185
##
## [[40]]
## [1] -2.243185
##
## [[41]]
## [1] -2.243185
##
## [[42]]
## [1] -2.243185
##
## [[43]]
## [1] -2.243185
##
## [[44]]
## [1] -2.243185
##
## [[45]]
## [1] -2.243185
##
## [[46]]
## [1] -2.223186
##
## [[47]]
## [1] -2.223185
##
## [[48]]
```

```
## [1] -2.243185
##
## [[49]]
## [1] -1.463185
##
## [[50]]
## [1] -1.463185
##
## [[51]]
## [1] -1.463185
##
## [[52]]
## [1] -1.463185
##
## [[53]]
## [1] -1.463185
##
## [[54]]
## [1] -1.463185
##
## [[55]]
## [1] -1.433185
##
## [[56]]
## [1] -1.433185
##
## [[57]]
## [1] -1.463185
##
## [[58]]
## [1] -1.463185
##
## [[59]]
## [1] -1.463185
##
## [[60]]
## [1] -1.463185
##
## [[61]]
## [1] -1.463185
##
## [[62]]
## [1] -1.433185
##
## [[63]]
## [1] -1.433185
##
## [[64]]
## [1] -1.463185
##
## [[65]]
## [1] -1.463185
##
## [[66]]
```

```
## [1] -1.463185
##
## [[67]]
## [1] -1.433185
##
## [[68]]
## [1] -1.463185
##
## [[69]]
## [1] -1.433185
##
## [[70]]
## [1] -1.463185
##
## [[71]]
## [1] -0.8231853
##
## [[72]]
## [1] -0.8231853
##
## [[73]]
## [1] -0.8231853
##
## [[74]]
## [1] -0.8231854
##
## [[75]]
## [1] -0.8231853
##
## [[76]]
## [1] -0.8231853
##
## [[77]]
## [1] -0.8231853
##
## [[78]]
## [1] -0.8231853
##
## [[79]]
## [1] -0.8231853
##
## [[80]]
## [1] -0.8231854
##
## [[81]]
## [1] -0.8231853
##
## [[82]]
## [1] -0.8231853
##
## [[83]]
## [1] -0.8231853
##
## [[84]]
```

```
## [1] -0.8231853
##
## [[85]]
## [1] -0.8231853
##
## [[86]]
## [1] -0.8231853
##
## [[87]]
## [1] -0.8231853
##
## [[88]]
## [1] -0.8231853
##
## [[89]]
## [1] -0.8231853
##
## [[90]]
## [1] -0.8231853
##
## [[91]]
## [1] -0.8231853
##
## [[92]]
## [1] -0.8231853
##
## [[93]]
## [1] -0.8231853
##
## [[94]]
## [1] -0.8231853
##
## [[95]]
## [1] -0.8231853
##
## [[96]]
## [1] -0.8231853
##
## [[97]]
## [1] -0.8231853
##
## [[98]]
## [1] -0.8231853
##
## [[99]]
## [1] -0.8231853
##
## [[100]]
## [1] -0.8231853
##
## [[101]]
## [1] -0.8231853
##
## [[102]]
```

```
## [1] 0.52
##
## [[103]]
## [1] 0.52
##
## [[104]]
## [1] 0.52
##
## [[105]]
## [1] 0.52
##
## [[106]]
## [1] 0.52
##
## [[107]]
## [1] 0.52
##
## [[108]]
## [1] 0.52
##
## [[109]]
## [1] 0.52
##
## [[110]]
## [1] 0.52
##
## [[111]]
## [1] 0.52
##
## [[112]]
## [1] 0.52
##
## [[113]]
## [1] 0.52
##
## [[114]]
## [1] 0.52
##
## [[115]]
## [1] 0.52
##
## [[116]]
## [1] 0.52
##
## [[117]]
## [1] 0.52
##
## [[118]]
## [1] 0.52
##
## [[119]]
## [1] 0.52
##
## [[120]]
```

```
## [1] 0.52
##
## [[121]]
## [1] 0.52
##
## [[122]]
## [1] 0.52
##
## [[123]]
## [1] 0.52
##
## [[124]]
## [1] 0.52
##
## [[125]]
## [1] 0.52
##
## [[126]]
## [1] 0.52
##
## [[127]]
## [1] 1.96
##
## [[128]]
## [1] 2.22
##
## [[129]]
## [1] 1.96
##
## [[130]]
## [1] 2.22
##
## [[131]]
## [1] 1.96
##
## [[132]]
## [1] 2.28
##
## [[133]]
## [1] 2.22
##
## [[134]]
## [1] 2.22
##
## [[135]]
## [1] 2.22
##
## [[136]]
## [1] 1.96
##
## [[137]]
## [1] 2.28
##
## [[138]]
```

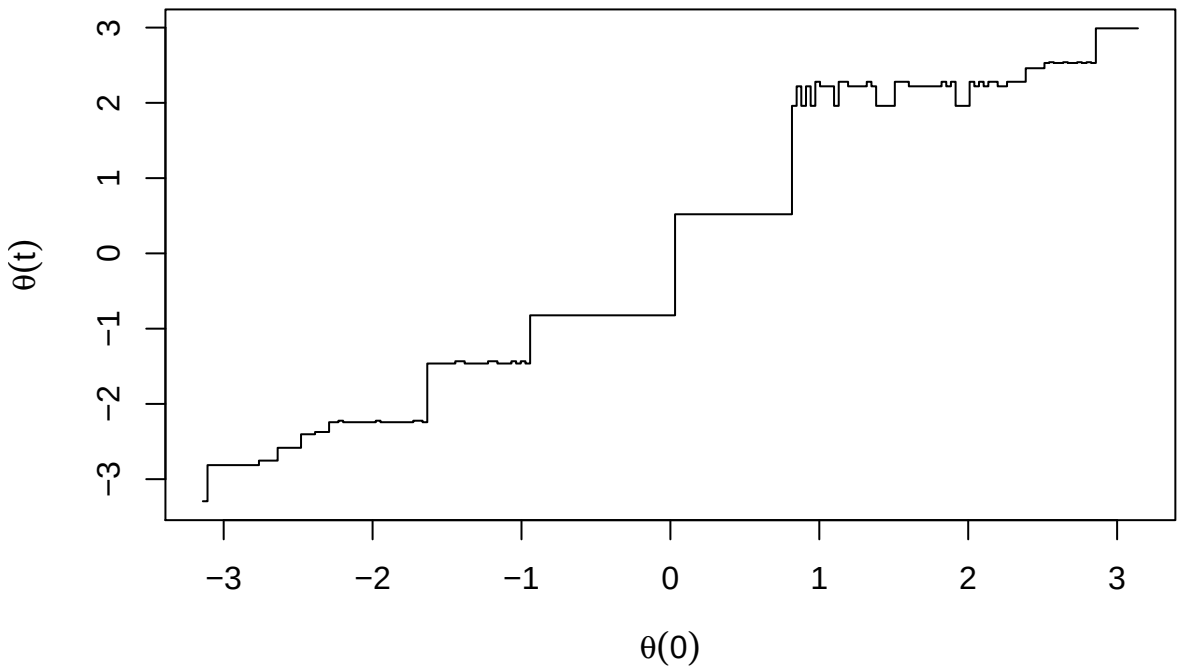
```
## [1] 2.28
##
## [[139]]
## [1] 2.22
##
## [[140]]
## [1] 2.22
##
## [[141]]
## [1] 2.22
##
## [[142]]
## [1] 2.22
##
## [[143]]
## [1] 2.28
##
## [[144]]
## [1] 2.22
##
## [[145]]
## [1] 1.96
##
## [[146]]
## [1] 1.96
##
## [[147]]
## [1] 1.96
##
## [[148]]
## [1] 1.96
##
## [[149]]
## [1] 2.28
##
## [[150]]
## [1] 2.28
##
## [[151]]
## [1] 2.28
##
## [[152]]
## [1] 2.22
##
## [[153]]
## [1] 2.22
##
## [[154]]
## [1] 2.22
##
## [[155]]
## [1] 2.22
##
## [[156]]
```

```
## [1] 2.22
##
## [[157]]
## [1] 2.22
##
## [[158]]
## [1] 2.22
##
## [[159]]
## [1] 2.28
##
## [[160]]
## [1] 2.22
##
## [[161]]
## [1] 2.28
##
## [[162]]
## [1] 1.96
##
## [[163]]
## [1] 1.96
##
## [[164]]
## [1] 1.96
##
## [[165]]
## [1] 2.28
##
## [[166]]
## [1] 2.22
##
## [[167]]
## [1] 2.28
##
## [[168]]
## [1] 2.22
##
## [[169]]
## [1] 2.28
##
## [[170]]
## [1] 2.28
##
## [[171]]
## [1] 2.22
##
## [[172]]
## [1] 2.22
##
## [[173]]
## [1] 2.28
##
## [[174]]
```

```
## [1] 2.28
##
## [[175]]
## [1] 2.28
##
## [[176]]
## [1] 2.28
##
## [[177]]
## [1] 2.46
##
## [[178]]
## [1] 2.46
##
## [[179]]
## [1] 2.46
##
## [[180]]
## [1] 2.46
##
## [[181]]
## [1] 2.53
##
## [[182]]
## [1] 2.539999
##
## [[183]]
## [1] 2.53
##
## [[184]]
## [1] 2.53
##
## [[185]]
## [1] 2.54
##
## [[186]]
## [1] 2.53
##
## [[187]]
## [1] 2.53
##
## [[188]]
## [1] 2.54
##
## [[189]]
## [1] 2.53
##
## [[190]]
## [1] 2.54
##
## [[191]]
## [1] 2.53
##
## [[192]]
```

```
## [1] 2.99
##
## [[193]]
## [1] 2.99
##
## [[194]]
## [1] 2.99
##
## [[195]]
## [1] 2.99
##
## [[196]]
## [1] 2.99
##
## [[197]]
## [1] 2.99
##
## [[198]]
## [1] 2.99
##
## [[199]]
## [1] 2.99
##
## [[200]]
## [1] 2.99
##
## [[201]]
## [1] 2.99
```

```
plot(theta.nr, sapply(theta.nr,newton.nr), type = "s", ylab=expression(theta(t)), xlab = expression(the
```



3.3.3.1)

```
beetles <- data.frame(
    days    = c(0,  8,  28,  41,  63,  69,   97, 117,  135,  154),
    beetles = c(2, 47, 192, 256, 768, 896, 1120, 896, 1184, 1024))



n0 <- 2
nt     <- function(x) {
    beetles$beetles - (2 * x[1]) / (2 + (x[1] - 2) * exp(-x[2]*beetles$days))
}

library(pracma)
```

```
## Warning: package 'pracma' was built under R version 3.4.4
```

```
##
## Attaching package: 'pracma'
```

```
## The following object is masked _by_ '.GlobalEnv':
##
##     newton
```

```
gaussNewton(c(100, 1), nt)
```

```
## $xs
## [1] 1049.4072453    0.1182684
##
## $fs
## [1] 73419.7
##
## $niter
## [1] 10
##
## $relerr
## [1] 1.455192e-11
```

```
sumse <- function(r, k){
  n0 <- 2
  sum(
    (beetles$beetles - (n0 * k) / (n0 + (k - n0) * exp(-r * beetles$days)))^2
  )
}

r <- seq(0, 1, .01)
k <- seq(0, 1500, 15)

z <- outer(r, k, Vectorize(sumse))
contour(k, r, z, xlab="k", ylab="r")
```