HW4

Qi Qi

9/25/2018

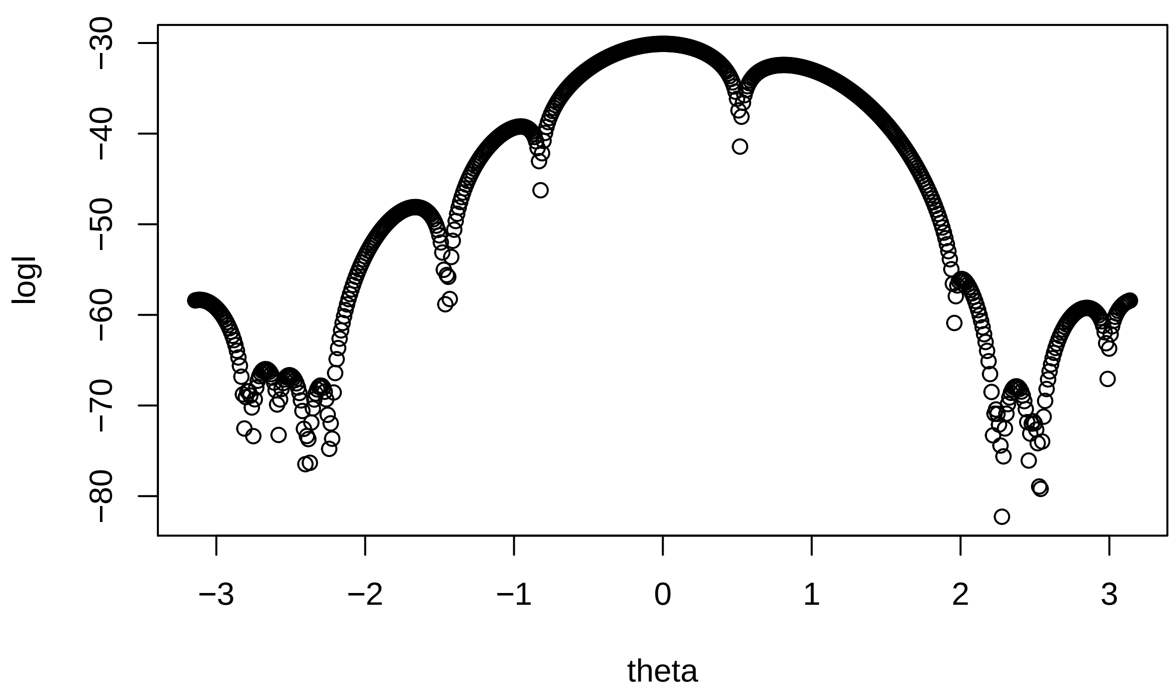
Exercise 3.3.2

1.

$$f(x;\theta) = \frac{1 - cos(x-\theta)}{2\pi}$$

$$\ell = \sum_{i=1}^{n} log f(x_i;\theta) = \sum_{i=1}^{n} log(1 - cos(x_i - \theta)) - nlog(2\pi)$$

```
x <- c(3.91, 4.85, 2.28, 4.06, 3.70, 4.04, 5.46, 3.53, 2.28, 1.96,
       2.53, 3.88, 2.22, 3.47, 4.82, 2.46, 2.99, 2.54, 0.52)
logL <- function(theta) {
  sum(log(1 - cos(x - theta))-log(2 * pi))
}
theta <- seq(-pi , pi , by = .01)
logl <- sapply(theta, logL)
plot(theta, logl)
```



2.

$$
\begin{aligned}
E(X|\theta) &= \int_0^{2\pi} \frac{x(1-\cos(x-\theta))}{2\pi}dx = \int_0^{2\pi} \frac{x}{2\pi}dx - \int_0^{2\pi} \frac{x\cos(x-\theta)}{2\pi}dx \\
&= \frac{x^2}{4\pi}|_0^{2\pi} - \frac{x}{2\pi}\sin(x-\theta)|_0^{2\pi} + \int_0^{2\pi} \frac{\sin(x-\theta)}{2\pi}dx \\
&= \pi - \sin(-\theta) - \frac{\cos(x-\theta)}{2\pi}|_0^{2\pi} = \pi + \sin\theta \\
&\Rightarrow \bar{X}_n = \pi + \sin\tilde{\theta}_n \ \Rightarrow \tilde{\theta}_n = \arcsin(\bar{X}_n - \pi)
\end{aligned}
$$

3.

$$
\ell' = \sum_{i=1}^{n} \frac{\sin(\theta - x_i)}{1-\cos(\theta - x_i)}
$$

$$
\begin{aligned}
\ell'' &= \sum_{i=1}^{n} \frac{\cos(\theta - x_i)[1-\cos(\theta-x_i)] - [\sin(\theta-x_i)]^2}{[1-\cos(\theta-x_i)]^2} \\
&= \sum_{i=1}^{n} \frac{\cos(\theta-x_i)-1}{[1-\cos(\theta-x_i)]^2} = \sum_{i=1}^{n} \frac{1}{\cos(\theta-x_i)-1}
\end{aligned}
$$

```
init <- asin(mean(x) - pi)
g <- function(theta){
  sum(sin(theta - x) / (1 - cos(theta - x)))
}
dg <- function(theta){
  sum(1 / (cos(theta - x) - 1))
}
newton<- function(fun, dfun, x0, eps, maxit){
  for (i in 1:maxit) {
    x1 <- x0 - fun(x0) / dfun(x0)
    if (abs(x1 - x0) < eps | abs(fun(x1)) < eps)
      return(x1)
    x0 <- x1
  }
  return(NA)
}
newton(g, dg, init, 1e-6, 1000)
```

```
## [1] 0.003118157
```

The MLE of θ is 0.0031 with initial value $\tilde{\theta}_n$.

4.

```
newton(g, dg, 2.7, 1e-6, 1000)
```

```
## [1] 2.848415
```

```
newton(g, dg, -2.7, 1e-6, 1000)
```

```
## [1] -2.668857
```

The MLE for θ is 2.8484 with initial value $\theta_0 = 2.7$; MLE for θ is -2.6689 starting at $\theta_0 = -2.7$.

5.

These MLE's could be classified into 197 groups due to very slight differences. So, I round MLE to 6 decimal places. Since there are at most 46 initial values in one group, I report the minimum and maximum of initial values in each group instead of listing all values in each group.

```
init <- seq(-pi, pi, length.out = 200)
mle <- sapply(init, function(x) newton(g, dg, x, 1e-6, 1000))
apprmle <- round(mle, digits = 6)
tab <- as.data.frame(cbind(init, apprmle))
aggregate(init ~ apprmle, data = tab, FUN = function(x) c(min = min(x), maxi = max(x) ))
```

```
##      apprmle   init.min  init.maxi
## 1  -3.112471 -3.1415927 -2.8258547
## 2  -2.786557 -2.7942809 -2.7627071
## 3  -2.668857 -2.7311333 -2.6048381
## 4  -2.509356 -2.5732643 -2.4153954
## 5  -2.388267 -2.3838216 -2.3838216
## 6  -2.297926 -2.3522478 -2.2575264
## 7  -2.232192 -2.2259526 -2.2259526
## 8  -1.662712 -2.1943788 -1.4681815
## 9  -1.447503 -1.4366077 -1.4366077
## 10 -0.954406 -1.4050339 -0.8367056
## 11  0.003118 -0.8051318  0.4893938
## 12  0.812637  0.5209676  1.9417884
## 13  2.007223  1.9733622  2.1943788
## 14  2.237013  2.2259526  2.2575264
## 15  2.374712  2.2891002  2.4469692
## 16  2.488450  2.4785429  2.5101167
## 17  2.848415  2.5416905  2.9837237
## 18  3.170715  3.0152975  3.1415927
```

Exercise 3.3.3

1.

```
beetles <- data.frame(
    days    = c(0,  8,  28,  41,  63,  69,   97, 117,  135,  154),
    beetles = c(2, 47, 192, 256, 768, 896, 1120, 896, 1184, 1024))

growth <- nls(beetles ~ K * beetles[which(days == 0)] / (beetles[which(days == 0)] + (K - beetles[which
```

```
## 164487.9 :     0.15 1000.00
## 86407.24 :     0.1275481 1005.5576477
## 73936.64 :     0.1197121 1039.3710445
## 73430.59 :     0.1185102 1048.3579593
## 73420.01 :     0.1183094 1049.2521408
## 73419.71 :     0.1182753 1049.3815481
## 73419.7 :     0.1182696 1049.4029151
## 73419.7 :     0.1182686 1049.4065121
```

```r
summary(growth)
```

```
##
## Formula: beetles ~ K * beetles[which(days == 0)]/(beetles[which(days ==
##     0)] + (K - beetles[which(days == 0)]) * exp(-r * days))
##
## Parameters:
##    Estimate Std. Error t value Pr(>|t|)
## r 1.183e-01  6.533e-03   18.10 8.90e-08 ***
## K 1.049e+03  4.717e+01   22.25 1.76e-08 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 95.8 on 8 degrees of freedom
##
## Number of iterations to convergence: 7
## Achieved convergence tolerance: 8.971e-06
```

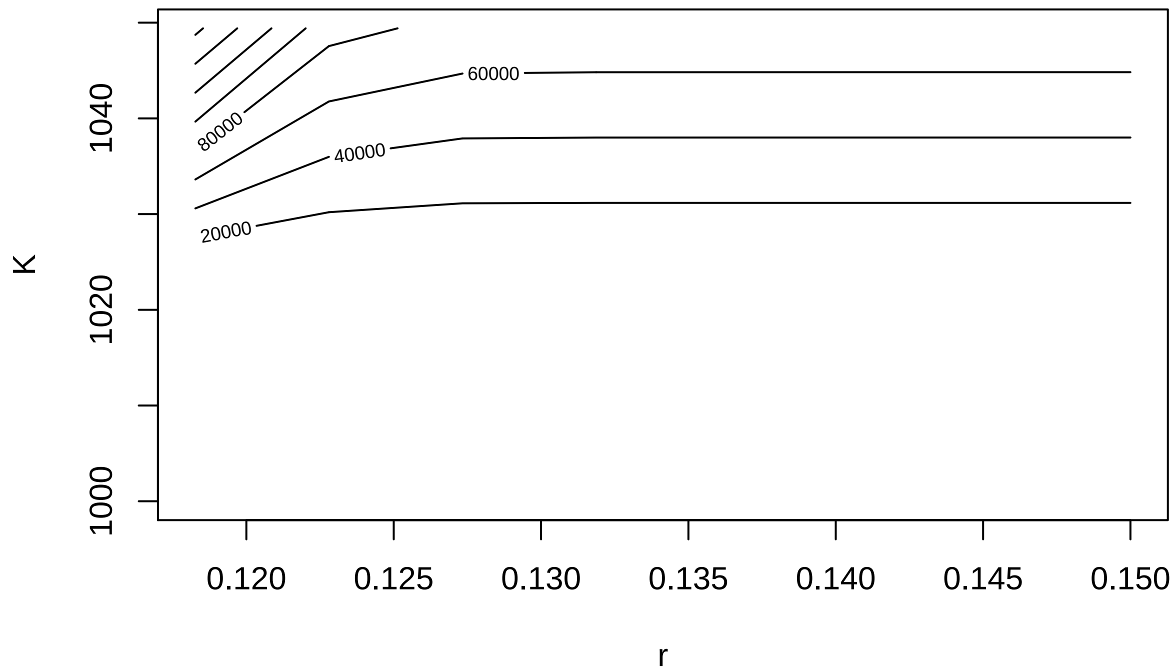
2.

I set r and K as the values in the trace of previous iteration. The sum squared errors are also obtained from previous iteration.

```r
r <- c(0.15, 0.1275481, 0.1197121, 0.1185102, 0.1183094, 0.1182753, 0.1182696, 0.1182686)
K <- c(1000, 1005.5576477, 1039.3710445, 1048.3579593, 1049.2521408, 1049.3815481, 1049.4029151, 1049.40
error <- c(164487.9, 86407.24, 73936.64, 73430.59, 73420.01, 73419.71, 73419.7, 73419.7)

z <-  cbind(r, K, error)
x <-  seq(min(r), max(r), length.out = nrow(z))
y <-  seq(min(K), max(K), length.out = ncol(z))

contour(x, y, z, xlab = "r", ylab = "K")
```



3.

$$f(N_t) = \frac{1}{N_t\sigma\sqrt{2\pi}}\exp(-\frac{(\log N_t - \log(\frac{N_0}{N_0+(K-N_0)e^{-rt}}))^2}{2\sigma^2})$$

$$\ell = \sum_t \log f(N_t) = \sum_t[-\log(N_t\sigma) - \log(\sqrt{2\pi}) - \frac{1}{2\sigma^2}(\log N_t - \log(\frac{N_0}{N_0+(K-N_0)e^{-rt}}))^2]$$

$$\frac{\partial \ell}{\partial r} = \sum_t \frac{(\log N_t - \log\frac{N_0}{N_0+(K-N_0)e^{-rt}})(K-N_0)e^{-rt}t}{\sigma^2(N_0+(K-N_0)e^{-rt})}$$

$$\frac{\partial \ell}{\partial K} = \sum_t -\frac{(\log N_t - \log\frac{N_0}{N_0+(K-N_0)e^{-rt}})e^{-rt}}{\sigma^2(N_0+(K-N_0)e^{-rt})}$$

$$\frac{\partial \ell}{\partial \sigma^2} = \sum_t -\frac{1}{2\sigma^2} + \frac{\log N_t - \log\frac{N_0}{N_0+(K-N_0)e^{-rt}}}{2\sigma^4}$$

```
N <- beetles$beetles
t <- beetles$days
N0 <- N[t==0]
l <- function(theta){
  sum(- log(N) -log(theta[3]) / 2 - log(2 * pi)/2 - (log(N) - log(N0 / (N0 + (theta[2] - N0) * exp(- the
}
```

```
dl <- function(theta){
  c(sum((log(N) - log(N0 / (N0 + (theta[2] - N0) * exp(- theta[1] * t)))) * (theta[2] - N0) * exp(-theta
    sum((log(N)-log(N0 / (N0 + (theta[2] - N0) * exp(- theta[1] * t)))) * exp(-theta[1] * t) / (theta[3]
    sum(-1 / (2 * theta[3]) + (log(N)-log(N0 / (N0 + (theta[2] - N0) * exp(- theta[1] * t)))) / (2 * the
}


fit <- optim(par = c(.1, 10, 3), l, dl, method = "L-BFGS-B", lower = c(0, 0, 0), control = list(fnscale
fit$par
```

```
## [1]  1.603498 22.121291  6.938461
```

```
fit$value
```

```
## [1] -101.921
```

```
fit$convergence
```

```
## [1] 0
```

```
diag(-solve(fit$hessian))
```

```
## [1] 1042.59925 1674.85599   14.20093
```

I use "L-BFGS-B"method and set lower bound of parameters since they should be non-negative. Then MLE's are $\hat{r} = 1.603498$, $\hat{K} = 22.121291$, and $\hat{\sigma}^2 = 6.938461$. Their variance are $var(\hat{r}) = 042.59925$, $var(\hat{K}) = 1674.85599$, and $var(\hat{\sigma}^2) = 14.20093$.