Ex6.1

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1. Find function g

We have

$$g(x) \propto (2x^{\theta-1} + x^{\theta-\frac{1}{2}})e^{-x}$$

So

$$g(x) = C(2x^{\theta-1} + x^{\theta-\frac{1}{2}})e^{-x}$$

Because g is a mixture of gamma distribution, let's consider gamma distribution.

$$f(x, \beta, \alpha) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

According the formation of function g, we let $\beta = 1$, $\alpha = \theta \quad and \quad \theta + \frac{1}{2}$. The component distributions are as follow.

$$f(x, 1, \theta) = \frac{1}{\Gamma(\theta)} x^{\theta-1} e^{-x}$$

$$f(x, 1, \theta + \frac{1}{2}) = \frac{1}{\Gamma(\theta + \frac{1}{2})} x^{\theta-\frac{1}{2}} e^{-x}$$

Due to the Properties of density functions, we have

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$$

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x, 1, \theta)\mathrm{d}x &= \int_{0}^{\infty} \frac{1}{\Gamma(\theta)} x^{\theta-1} e^{-x}\mathrm{d}x = 1 \\
\int_{-\infty}^{\infty} f(x, 1, \theta + \frac{1}{2})\mathrm{d}x &= \int_{0}^{\infty} \frac{1}{\Gamma(\theta + \frac{1}{2})} x^{\theta-\frac{1}{2}} e^{-x}\mathrm{d}x = 1 \\
\int_{0}^{\infty} x^{\theta-1} e^{-x}\mathrm{d}x &= \Gamma(\theta) \\
\int_{0}^{\infty} x^{\theta-\frac{1}{2}} e^{-x}\mathrm{d}x &= \Gamma(\theta + \frac{1}{2}) \\
\int_{0}^{\infty} g(x)\mathrm{d}x &= C\int_{0}^{\infty} (2x^{\theta-1} + x^{\theta-\frac{1}{2}})e^{-x}\mathrm{d}x \\
&= C[\int_{0}^{\infty} 2x^{\theta-1}e^{-x}\mathrm{d}x + \int_{0}^{\infty} x^{\theta-\frac{1}{2}}e^{-x}\mathrm{d}x] \\
&= C[2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})] = 1 \\
&\Rightarrow C = \frac{1}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})}
\end{aligned}$$

So

$$g(x) = \frac{2\Gamma(\theta)}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})} \frac{x^{\theta-1}e^{-x}}{\Gamma(\theta)} + \frac{\Gamma(\theta + \frac{1}{2})}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})} \frac{x^{\theta-\frac{1}{2}}e^{-x}}{\Gamma(\theta + \frac{1}{2})}$$

Weights are $\frac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+\frac{1}{2})}$ for $f(x, 1, \theta)$ and $\frac{\Gamma(\theta+\frac{1}{2})}{2\Gamma(\theta)+\Gamma(\theta+\frac{1}{2})}$ for $f(x, 1, \theta + \frac{1}{2})$

2. Sample

2.1 Pseudo-code

Algorithm 1 Sampling from $g(x)$

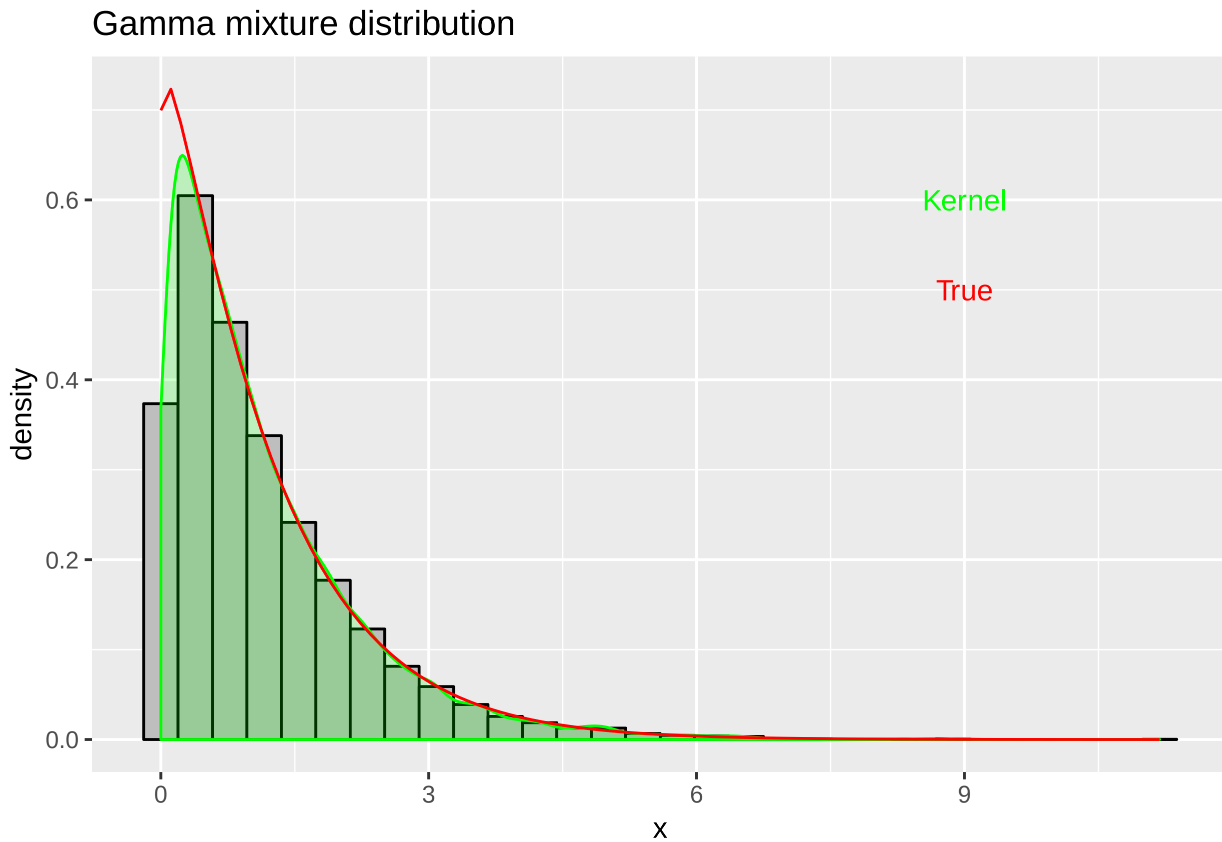
1: **procedure**
2: $\quad P \sim U(0,1)$
3: $\quad$ **if** $P < \frac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+\frac{1}{2})}$ **then**
4: $\quad\quad X \sim (\theta, 1)$
5: $\quad$ **else**
6: $\quad\quad X \sim (\theta+\frac{1}{2}, 1)$
7: $\quad$ **end if**
8: $\quad$ **return** X
9: **end procedure**

2.2 Code

```
library(abind)
g=function(x,theta){
  (2*x^(theta-1)+x^(theta-0.5))*exp(-x)/(2*gamma(theta)+gamma(theta+0.5))
}

sample_g=function(n,alpha,beta,weight){
  x_g=array(0,n)
  p1=runif(n,0,1)
  f1=rgamma(n,alpha[1],beta[1])
  f2=rgamma(n,alpha[2],beta[2])
  x_g=abind(f1[p1<weight],f2[p1>=weight])
}
n=10000
theta=1
alpha=c(theta,theta+0.5)
beta=c(1,1)
weight=2*gamma(theta)/(2*gamma(theta)+gamma(theta+0.5))
x=sample_g(n,alpha,beta,weight)
library(ggplot2)
ggplot(data.frame(x=x),aes(x=x))+
  xlab("x")+
  ylab("density")+
  ggtitle("Gamma mixture distribution")+
  geom_histogram(aes(y=..density..),fill="gray", colour="black")+
  geom_density(fill="green",colour="green" ,alpha=0.2)+
  stat_function(fun=function(x) g(x,theta),color="red")+
  annotate("text", x=9, y=0.6, label="Kernel",color="green")+
  annotate("text", x=9, y=0.5, label="True",color="red")
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```



3. Rejection Sampling

$$f(x) \propto h(x) = \sqrt{x+4}x^{\theta-1}e^{-x}$$

$$E = \int_0^\infty h(x)\mathrm{d}x = \int_0^\infty \sqrt{x+4}x^{\theta-1}e^{-x}\mathrm{d}x$$

$$f(x) = \frac{h(x)}{E}$$

$$f(x) \leq Mg(x) \Rightarrow M = max(\frac{f(x)}{g(x)})$$

$$\frac{f(x)}{g(x)} = \frac{\sqrt{x+4}x^{\theta-1}e^{-x}}{CE(2x^{\theta-1}+x^{\theta-\frac{1}{2}})e^{-x}} = \frac{\sqrt{x+4}}{CE(2+\sqrt{x})}$$

In order to calculate $max(\frac{f(x)}{g(x)})$, let $q(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x+4}}{CE(2+\sqrt{x})}$

$$q'(x) = \frac{1}{CE}\frac{\frac{1}{2\sqrt{x+4}}(2+\sqrt{x}) - \frac{1}{2\sqrt{x}}\sqrt{x+4}}{(2+\sqrt{x})^2} = \frac{1}{CE}\frac{2\sqrt{x}-4}{2\sqrt{x+4}\sqrt{x}(2+\sqrt{x})^2}$$

We can know that when $x < 4$,$q'(x) < 0$;when $x > 4$,$q'(x) > 0$. So max(q(x)) can be q(0) or q(∞). $q(0) = \frac{1}{CE} = q(\infty)$.

$$M = max(\frac{f(x)}{g(x)}) = \frac{1}{CE}$$

3.1 Pseudo-code

Algorithm 2 Sampling from $f(x)$

```
procedure
    Sample P ~ U(0,1)
    if P ≤ f(x)/(Mg(x)) = √(x+4)/(2+√x) then
        return X
    else
        go back to 2
    end if
end procedure
```

$$P \leq \frac{f(x)}{Mg(x)} = \frac{\sqrt{x+4}}{(2+\sqrt{x})}$$

3.2 Code

```
sample_f=function(n,alpha,beta,weight){
  x_f=rep(0,n)
  count=1
  while(count<=n){
  x_g=sample_g(1,alpha,beta,weight)
  p2=runif(1,0,1)
  if(p2<=sqrt(x_g+4)/(2+sqrt(x_g))){
    x_f[count]=x_g
    count=count+1
  }
  }
  x_f
}

h=function(x,theta){
  sqrt(x+4)*x^(theta-1)*exp(-x)
}

f=function(x,theta){
  h(x,theta)/integrate(function(x)h(x,theta),0,Inf)$value
}

n=10000
theta=1
alpha=c(theta,theta+0.5)
beta=c(1,1)
weight=2*gamma(theta)/(2*gamma(theta)+gamma(theta+0.5))
x_f=sample_f(n,alpha,beta,weight)
```

```
ggplot(data.frame(x=x_f),aes(x=x))+
  xlab("x")+
  ylab("density")+
  ggtitle("distribution of f(x)")+
  geom_histogram(aes(y=..density..),fill="gray", colour="black")+
  geom_density(fill="green",colour="green" ,alpha=0.2)+
  stat_function(fun=function(x) g(x,theta),color="red")+
  annotate("text", x=7.5, y=0.5, label="Kernel",color="green")+
```

```r
annotate("text", x=7.5, y=0.4, label="True",color="red")
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```