Ex6.2

Guanting Wei

Oct.17.2018

1. Define instrumental density

$$f(x) \propto h(x) = \frac{x^{\theta-1}}{1+x^2} + \sqrt{2+x^2}(1-x)^{\beta-1}, x \in (0,1)$$

$$h(x) \leq q(x) = x^{\theta-1} + \sqrt{3}(1-x)^{\beta-1}$$

$$g_1(x) = g_1(x;\theta,1) = \frac{x^{\theta-1}}{B(\theta,1)} = \theta x^{\theta-1}$$

$$g_2(x) = g_2(x;1,\beta) = \frac{(1-x)^{\beta-1}}{B(1,\beta)} = \beta(1-x)^{\beta-1}$$

$$g(x) = Cq(x) = C(x^{\theta-1} + \sqrt{3}(1-x)^{\beta-1}) = C(\frac{1}{\theta}\theta x^{\theta-1} + \frac{\sqrt{3}}{\beta}\beta(1-x)^{\beta-1})$$

$$C = \frac{\theta\beta}{\beta+\sqrt{3}\theta}$$

$$p_1 = \frac{C}{\theta} = \frac{\beta}{\beta+\sqrt{3}\theta}, p_2 = \frac{C}{\beta} = \frac{\theta}{\beta+\sqrt{3}\theta}$$

Let $M = \frac{1}{C}$ $h(x) \leq Mg(x) = q(x)$

2. Sampling

2.1 Pseudo-code

Algorithm 1 Sampling from $f(x)$

1: **procedure**
2: $P \sim U(0,1)$
3: **if** $P \leq p_1$ **then**
4: $X \sim g_1(x)$
5: **else**
6: $X \sim g_2(x)$
7: **end if**
8: $P' \sim U(0,1)$
9: **if** $P' \leq \frac{h(x)}{Mg(x)} = \frac{h(x)}{q(x)}$ **then**
10: **return** X
11: **else**
12: go back to 2
13: **end if**
14: **end procedure**

2.2Code

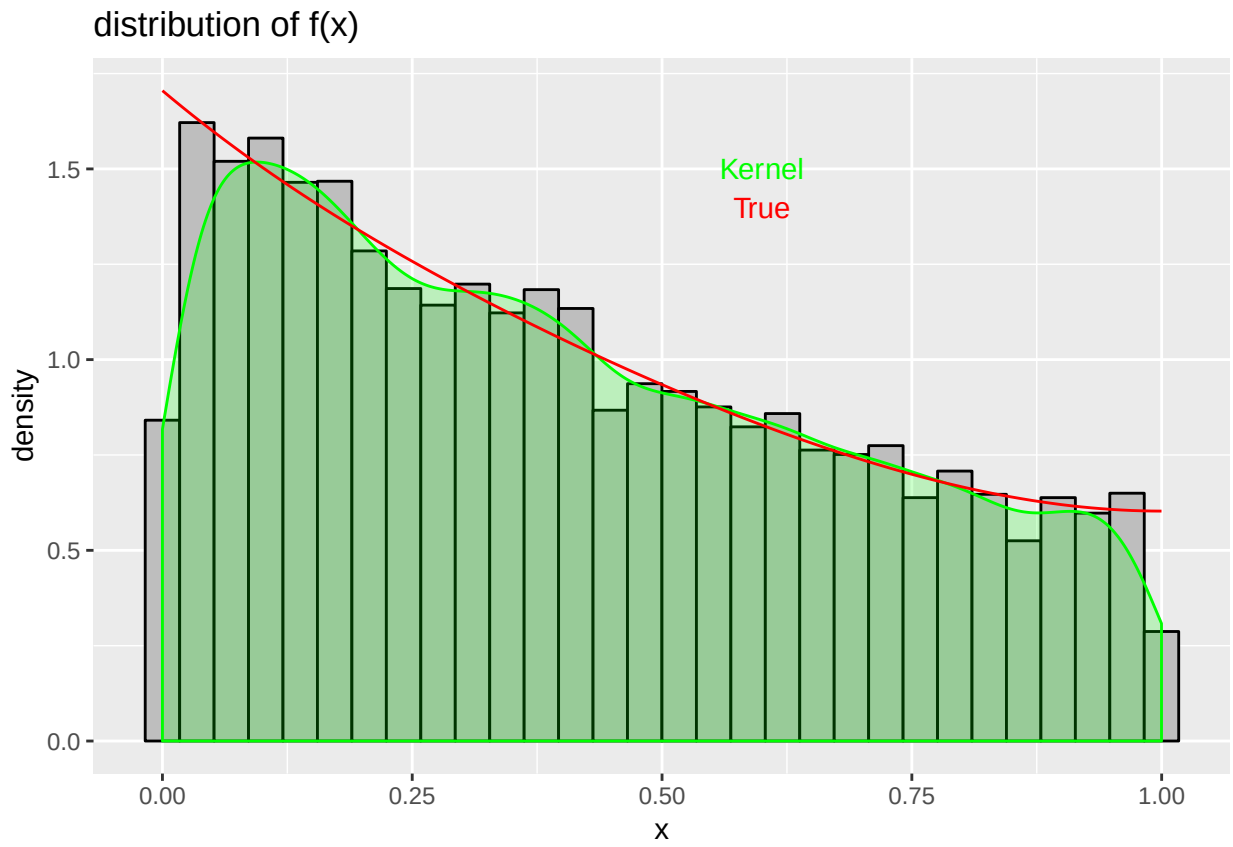
```
library(abind)
sample_g=function(n,a,b,weight){
  x_g=rep(0,n)
  p1=runif(n,0,1)
  g1=rbeta(n,a[1],b[1])
  g2=rbeta(n,a[2],b[2])
  x_g=abind(g1[p1<=weight],g2[p1>weight])
  x_g
}

h=function(x,a,b){
  x^(a[1]-1)/(1+x^2)+sqrt(2+x^2)*(1-x)^(b[2]-1)
}

f=function(x,a,b){
  h(x,a,b)/integrate(function(x)h(x,a,b),0,1)$value
}

q=function(x,a,b){
  x^(a[1]-1)+sqrt(3)*(1-x)^(b[2]-1)
}

sample_f=function(n,a,b,weight){
  x_f=rep(0,n)
  count=1
  while(count<=n){
    x_g=sample_g(1,a,b,weight)
    p2=runif(1,0,1)
    if(p2<=h(x_g,a,b)/q(x_g,a,b)){
      x_f[count]=x_g
      count=count+1
    }
  }
  x_f
}
```

Let $(\theta, \beta) = (2, 3)$

```
n=10000
theta=2
beta=3
a=c(theta,1)
b=c(1,beta)
weight=beta/(beta+sqrt(3)*theta)
x=sample_f(n,a,b,weight)
library(ggplot2)
ggplot(data.frame(x=x),aes(x=x))+
  xlab("x")+
  ylab("density")+
  ggtitle("distribution of f(x)")+
  geom_histogram(aes(y=..density..),fill="gray", colour="black")+
  geom_density(fill="green",colour="green" ,alpha=0.2)+
```

```
  stat_function(fun=function(x) f(x,a,b),color="red")+
  annotate("text", x=0.6, y=1.5, label="Kernel",color="green")+
  annotate("text", x=0.6, y=1.4, label="True",color="red")
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```



3. Dealing With the Two Components

$$h_1(x) = \frac{x^{\theta-1}}{1+x^2} \qquad h_2(x) = \sqrt{2+x^2}(1-x)^{\beta-1}, x \in (0,1)$$

$$g_1(x) = \theta x^{\theta-1} \qquad g_2(x) = \beta(1-x)^{\beta-1}, x \in (0,1)$$

$$M_1 = \frac{1}{\theta} \qquad M_2 = \frac{\sqrt{3}}{\beta}$$

$$f(x) \propto h_1(x) + h_2(x)$$

$$h_1(x) \leq M_1 g_1(x)$$

$$h_2(x) \leq M_2 g_2(x)$$

3.1 Pseudo-code

Algorithm 2 Sampling from $f(x)$

procedure
  $P \sim U(0,1)$
  if $P \leq \frac{M_1}{M_1+M_2}$ **then**
    $X \sim g_1(x)$
    $P_1 \sim U(0,1)$
    if $P_1 \leq \frac{h_1(x)}{M_1 g_1(x)}$ **then return** X
    elsego back to 2
      $X \sim g_2(x)$
    end if
  else
    $X \sim g_2(x)$
    $P_2 \sim U(0,1)$
    if $P_2 \leq \frac{h_2(x)}{M_2 g_2(x)}$ **then return** X
    else
      go back to 2
    end if
  end if
end procedure

3.2 Code

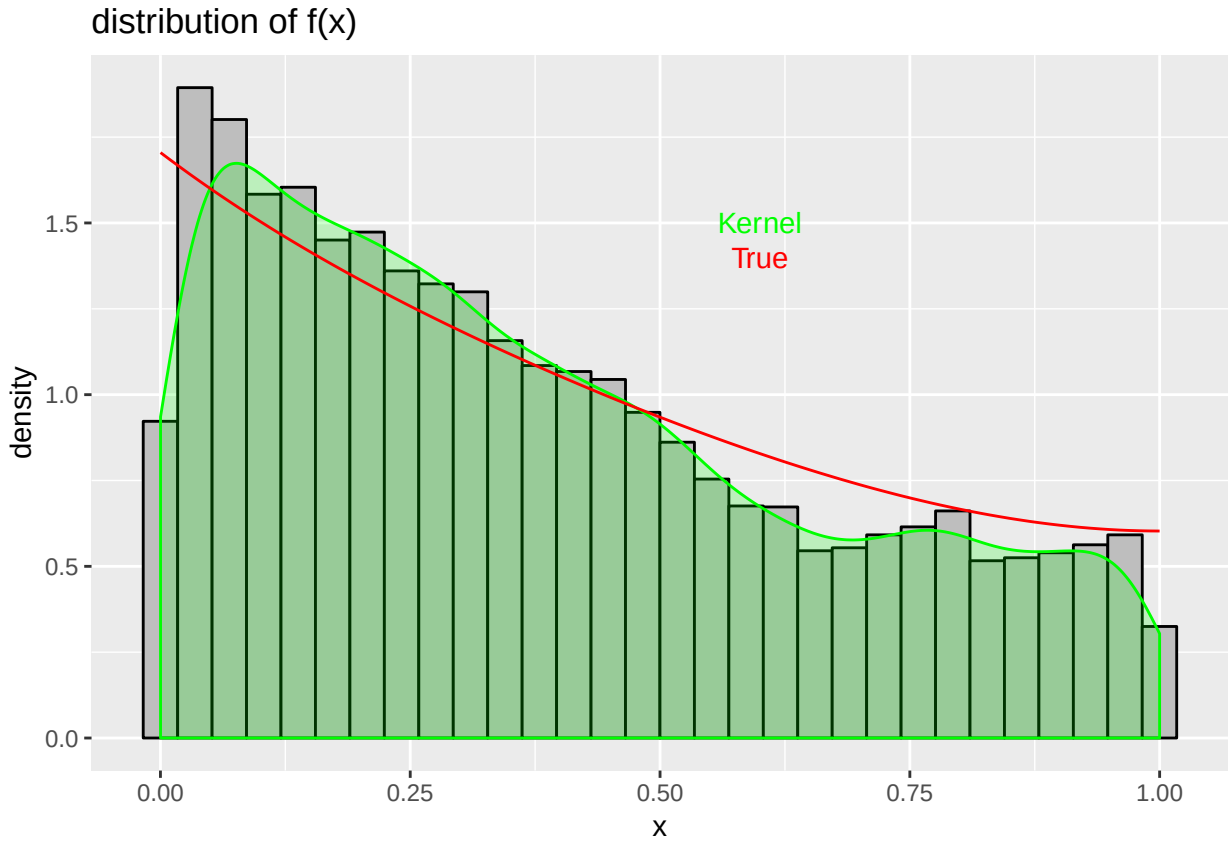
```
h1=function(x,theta){
  x^(theta-1)/(1+x^2)
}
h2=function(x,beta){
  sqrt(2+x)*(1-x)^(beta-1)
}
g1=function(x,theta){
  theta*x^(theta-1)
}
g2=function(x,beta){
  beta*(1-x)^(beta-1)
}
f=function(x,a,b){
  h(x,a,b)/integrate(function(x)h(x,a,b),0,1)$value
}
sample_f=function(n,a,b){
  x_f=rep(0,n)
  M1=1/a[1]
  M2=sqrt(3)/b[2]
  weight=M1/(M1+M2)
  count=1
  while(count<=n){
```

```
    p=runif(1,0,1)
    if(p<=weight){
      x=rbeta(1,a[1],b[1])
      p1=runif(1,0,1)
      if(p1<=h1(x,a[1])/(M1*g1(x,a[1]))){
        x_f[count]=x
        count=count+1
      }
    }
    else{
      x=rbeta(1,a[2],b[2])
      p2=runif(1,0,1)
      if(p2<=h2(x,b[2])/(M1*g1(x,b[2]))){
        x_f[count]=x
        count=count+1
      }
    }
  }
  x_f
}
```

Let $\theta = 2$,$\beta = 3$, same as above.

```
theta=2
beta=3
a=c(theta,1)
b=c(1,beta)
x=sample_f(n,a,b)
library(ggplot2)
ggplot(data.frame(x=x),aes(x=x))+
  xlab("x")+
  ylab("density")+
  ggtitle("distribution of f(x)")+
  geom_histogram(aes(y=..density..),fill="gray", colour="black")+
  geom_density(fill="green",colour="green" ,alpha=0.2)+
  stat_function(fun=function(x) f(x,a,b),color="red")+
  annotate("text", x=0.6, y=1.5, label="Kernel",color="green")+
  annotate("text", x=0.6, y=1.4, label="True",color="red")
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```


distribution of f(x)
Kernel
True
density
1.5
1.0
0.5
0.0
0.00
0.25
0.50
0.75
1.00
x