Mixture Proposal

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Abstract

In this assignment we will use the a mixture of the beta distribution as the instrumental density to sample from a function.

1 Mixture Proposal

Let f be a probability density on $(0,1)$ such that

$$f(x) \propto \frac{x^{\theta-1}}{1+x^2} + \sqrt{2+x^2}(1-x)^{\beta-1}, \quad 0 < x < 1.$$

- Design a procedure (pseudo-code) to sample from f using a mixture of Beta distributions as the instrumental density. That is, the instrumental density should have the form

  $$\sum_{k=1}^{m} p_k g_k(x),$$

  where p_k are weights and g_k are densities of Beta distributions. Specify your choice of the mixture. Implement your algorithm in an R function. Graph the estimated density of a random sample of $n = 10,000$ generated by your procedure and f for at least one (θ, β).
- As shown in class, $f(x)$ can also be sampled using rejection sampling, by dealing with the two components

  $$\frac{x^{\theta-1}}{1+x^2}, \quad \sqrt{2+x^2}(1-x)^{\beta-1}$$

  separately using individual Beta distributions. Design a procedure (pseudo-code) to do this; implement it with an R function; overlay the estimated density of a random sample of size $n = 10,000$ generated by your procedure and f.

2 Procedue, sample from f

let's define:

$$q(x) = \frac{x^{\theta-1}}{1+x^2} + \sqrt{2+x^2}(1-x)^{\beta-1}$$

since $1 + x^2 >= 1$ and $\sqrt{2+x^2} \leq \sqrt{2+1} = \sqrt{3}$, we can say:

$$q(x) \leq x^{\theta-1} + \sqrt{3}(1-x)^{\beta-1}$$

then define $g(x)$ as a density function as:

$$g(x) = C(x^{\theta-1} + \sqrt{3}(1-x)^{\beta-1})$$

In order to calculate C:

$$\begin{aligned}
&\int_0^1 (x^{\theta-1} + \sqrt{3}(1-x)^{\beta-1})dx \\
&= \int_0^1 x^{\theta-1}dx + \int_0^1 \sqrt{3}(1-x)^{\beta-1}dx \\
&= B(\theta,1) + \sqrt{3}B(1,\beta)
\end{aligned}$$

so we can get C:

$$C = \frac{1}{B(\theta,1) + \sqrt{3}B(1,\beta)}$$

and $g(x)$:

$$g(x) = \frac{1}{B(\theta,1) + \sqrt{3}B(1,\beta)}(x^{\theta-1} + \sqrt{3}(1-x)^{\beta-1})$$

change it to another expression:

$$g(x) = \frac{B(\theta,1)}{B(\theta,1) + \sqrt{3}B(1,\beta)}\frac{x^{\theta-1}}{B(\theta,1)} + \frac{\sqrt{3}B(1,\beta)}{B(\theta,1) + \sqrt{3}B(1,\beta)}\frac{(1-x)^{\beta-1}}{B(1,\beta)}$$

we can find the weights below:

$$p1 = \frac{B(\theta,1)}{B(\theta,1) + \sqrt{3}B(1,\beta)}$$

and

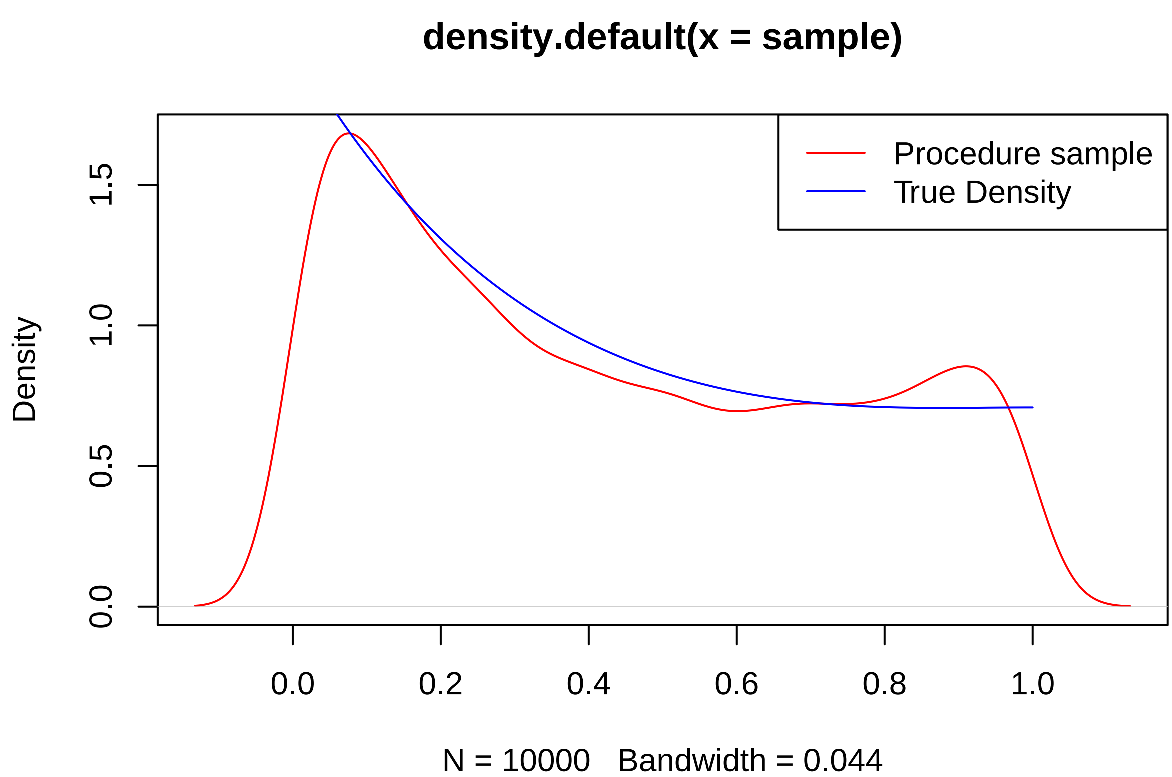
$$p2 = \frac{\sqrt{3}B(1,\beta)}{B(\theta,1) + \sqrt{3}B(1,\beta)}$$

```
fprocedure <- function(size, theta, beta) {

  weight1 = beta(theta,1)/(beta(theta,1) + sqrt(3)*beta(1,beta))
  u = runif(size,0,1)
  sample = double(size)

  for (i in 1:size) {
    if (u[i] > weight1) {
      sample[i] <- rbeta(1, shape1 = theta, shape2 = 1)
    }
    else {
      sample[i] <- rbeta(1, shape1 = 1, shape2 = beta)
    }
  }
  return(sample)
}
ffunc <- function(x, theta, beta){
  x ^ (theta - 1)/(1 + x^2) + sqrt(2 + x^2) * (1 - x)^(beta - 1)
}
```

next let's assume $\theta = 2$, $\beta = 2$, and $size = 10000$

```
theta <- 2
beta <- 4
size <-10000
sample <- fprocedure(size, theta, beta)
plot(density(sample), col = "red")
```

```r
curve(ffunc(x,theta = 2,beta =4)/0.70587, xlim = c(0,1), add = TRUE, col = "blue")
legend("topright", legend = c("Procedure sample", "True Density"),
       col = c("red","blue"), lty = c(1,1))
```



3 Rejection sampling, sample from f

In this section we will use two components to sample from f,

$$q_1(x) = \frac{x^{\theta-1}}{1+x^2}$$

$$q_2(x) = \sqrt{2+x^2}(1-x)^{\beta-1}$$

we can get:

$$q_1(x) = \frac{x^{\theta-1}}{1+x^2} \leq B(\theta,1) \cdot \frac{x^{\theta-1}}{B(\theta,1)} = \alpha_1 \cdot g_1(x)$$

also:

$$q_2(x) = \sqrt{2+x^2}(1-x)^{\beta-1} \leq \sqrt{3}(1-x)^{\beta-1} = \sqrt{3}B(1,\beta) \cdot \frac{(1-x)^{\beta-1}}{B(1,\beta)} = \alpha_2 \cdot g_2(x)$$

```r
frejection <- function(size, theta, beta) {
  iter <- 0
  weight1 = beta(theta,1)/(beta(theta,1) + sqrt(3)*beta(1,beta))
  sample = double(0)
```

```
  while (iter < size) {

   u <- runif(1, 0, 1)

    if(u < weight1){
      x <- rbeta(1, shape1 = theta, shape2 = 1)
      u1 <- runif(1, 0, 1)

      if(u1 <= 1/(1 + x^2)){
        sample <- append(sample, x)
        iter <- iter + 1
      }
    }
    else{
      x <- rbeta(1, shape1 = 1, shape2 = beta)
      u2 <- runif(1, 0, 1)

      if(u2 <= sqrt(2 + x^2)/sqrt(3)){
        sample <- append(sample, x)
        iter <- iter + 1
      }
    }
  }
  return(sample)
}
ffunc <- function(x, theta, beta){
  x ^ (theta - 1)/(1 + x^2) + sqrt(2 + x^2) * (1 - x)^(beta - 1)
}
```

and sample it

```
theta <- 2
beta <- 4
size <-10000
sample <- frejection(size, theta, beta)
plot(density(sample), col = "red")
curve(ffunc(x,theta = 2,beta =4)/0.70587, xlim = c(0,1), add = TRUE, col = "blue")
legend("topright", legend = c("Procedure sample", "True Density"),
      col = c("red","blue"), lty = c(1,1))
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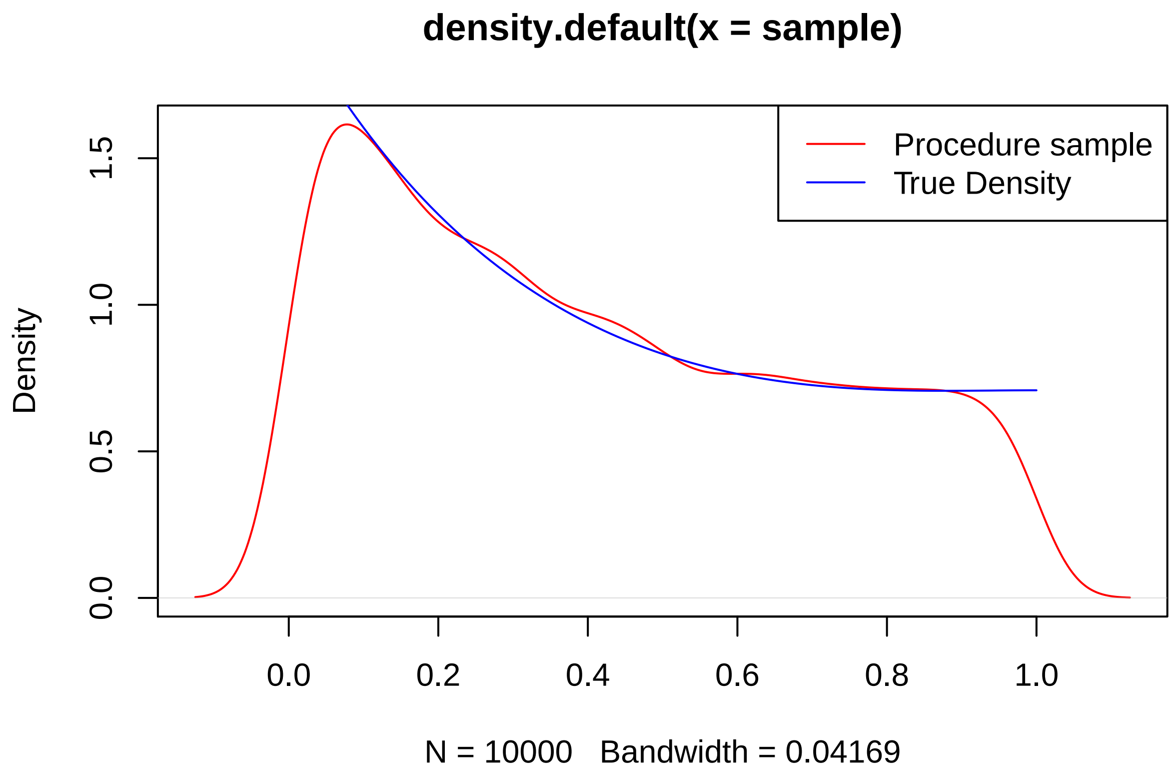
density.default(x = sample)
Procedure sample
True Density
Density
0.0
0.5
1.0
1.5
0.0
0.2
0.4
0.6
0.8
1.0
N = 10000   Bandwidth = 0.04169