Rejection sampling

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Abstract

In this assignment we will express how to use the rejection sampling to sample from some distributions using what we've learned.

1 Rejection sampling

Let f and g be two probability densities on $(0, \infty)$, such that

$$f(x) \propto \sqrt{4+x}\, x^{\theta-1} e^{-x}, \quad g(x) \propto (2x^{\theta-1} + x^{\theta-1/2}) e^{-x}, \quad x > 0.$$

- Find the value of the normalizing constant for g, i.e., the constant C such that

  $$C \int_0^\infty (2x^{\theta-1} + x^{\theta-1/2}) e^{-x} dx = 1.$$

  Show that g is a mixture of Gamma distributions. Identify the component distributions and their weights in the mixture.
- Design a procedure (pseudo-code) to sample from g; implement it in an R function; draw a sample of size $n = 10,000$ using your function for at least one θ value; plot the kernel density estimation of g from your sample and the true density in one figure.
- Design a procedure (pseudo-code) to use rejection sampling to sample from f using g as the instrumental distribution. Overlay the estimated kernel density of a random sample generated by your procedure and f.

2 Find the value of the normalizing constant

In this question, we need to calculate the normalizing constant for g

$$\begin{aligned}
&\int_0^\infty (2x^{\theta-1} + x^{\theta-1/2}) e^{-x} dx \\
&= \int_0^\infty 2x^{\theta-1} e^{-x} dx + \int_0^\infty x^{\theta-1/2} e^{-x} dx \\
&= 2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})
\end{aligned}$$

so that we can get

$$C = \frac{1}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})}$$

next we get the full expression of g:

$$g(x) = \frac{1}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})} (2x^{\theta-1} + x^{\theta-1/2}) e^{-x}$$

we can change it to another expression:

$$g(x) = \frac{2\Gamma(\theta)}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})} \frac{x^{\theta-1} e^{-x}}{\Gamma(\theta)} + \frac{\Gamma(\theta + \frac{1}{2})}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})} \frac{x^{(\theta+\frac{1}{2})-1} e^{-x}}{\Gamma(\theta + \frac{1}{2})}$$

and that g is a mixture of Gamma distributions,the component distribution is $\Gamma(\theta, 1)$ with weight $w1$, and $\Gamma(\theta + \frac{1}{2}, 1)$ with weight $w2$:

$$w1 = \frac{2\Gamma(\theta)}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})}, w2 = \frac{\Gamma(\theta + \frac{1}{2})}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})}$$

3 Procedure, sample from g

Pseudocode is an informal high-level description of the operating principle of a computer program or other algorithm.

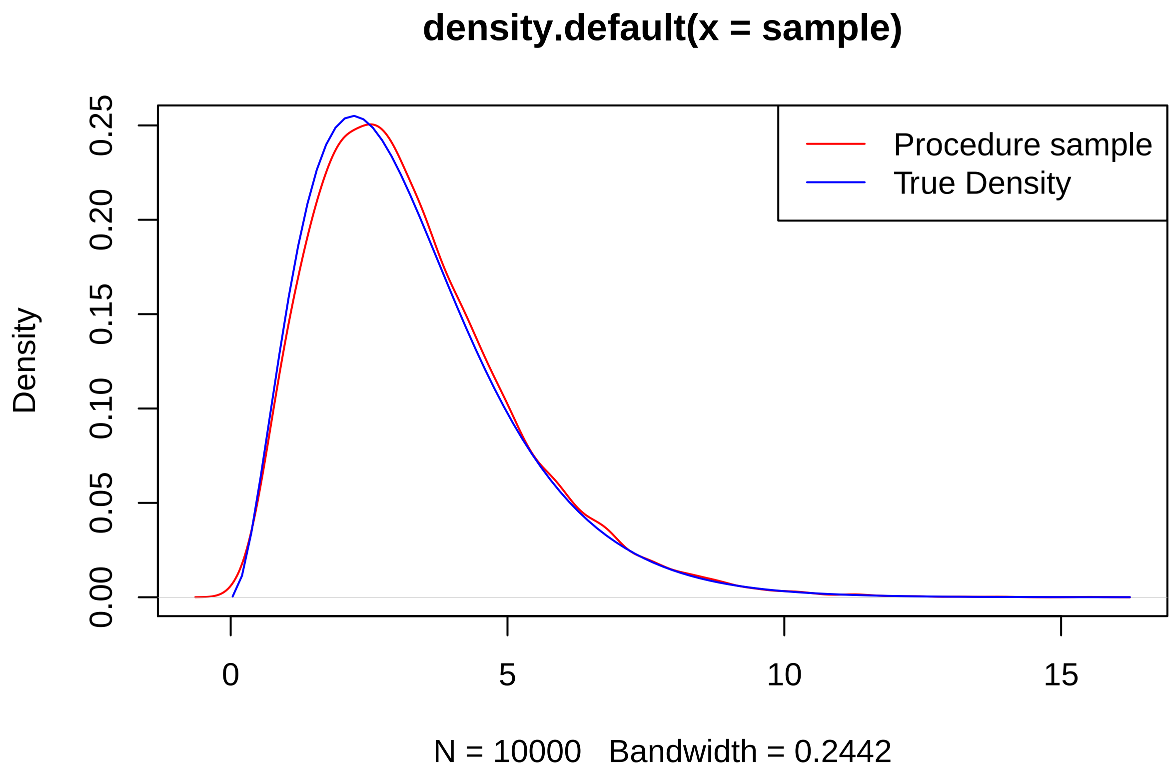
```
gprocedure <- function(size, theta) {

  weight1 = 2 * gamma(theta) / (2 * gamma(theta) + gamma(theta + (1/2)))
  u = runif(size,0,1)
  sample = double(size)

  for (i in 1:size) {
    if (u[i] > weight1) {
      sample[i] <- rgamma(1, shape = theta, rate = 1)
    }
    else {
      sample[i] <- rgamma(1, shape = theta + (1/2), rate = 1)
    }
  }
  return(sample)
}
gfunc <- function(theta, x) {
  g= 1/(2*gamma(theta) + gamma(theta + (1/2)))*(2*x^(theta-1) +  x^(theta-1/2))*exp(-x)
}
```

next let's assume $\theta = 3$, and $size = 10000$

```
theta <- 3
size <-10000
sample <- gprocedure(size, theta)
plot(density(sample), col = "red")
curve(gfunc(theta = 3,x), add = TRUE, col = "blue")
legend("topright", legend = c("Procedure sample", "True Density"),
      col = c("red","blue"), lty = c(1,1))
```



4 Rejection sampling, sample from f

In this section, we will use g to estimate the distribution of f , and $f(x) \propto q(x) = \sqrt{4+x}x^{\theta-1}e^{-x}$, and

$$\begin{aligned}
\alpha &= \sup_{x>0} \frac{q(x)}{g(x)} \\
&= \sup_{x>0} \frac{\sqrt{4+x}x^{\theta-1}e^{-x}}{C(2x^{\theta-1}+x^{\theta-\frac{1}{2}})e^{-x}} \\
&= \sup_{x>0} \frac{\sqrt{4+x}}{C\sqrt{x}} \\
&= \frac{1}{C}
\end{aligned}$$

which means $q(x) = \sqrt{4+x}x^{\theta-1}e^{-x} \leqslant \alpha g(x) = \frac{1}{C}g(x) = (2x^{\theta-1}+x^{\theta-\frac{1}{2}})e^{-x}$

```
fprocedure <- function(size, theta) {
  iter <- 0
  sample <- double(0)
  while(iter < size) {
    g <- gprocedure(1, theta)
    u <- runif(1,0,1)
    q <- sqrt(4 + g)*((g)^(theta-1))*exp(-1*g)
    r <- q/(2*g^(theta-1) +  g^(theta-1/2))*exp(-g)
    if (u <= r) {
```

```
      sample <- append(sample, g)
      iter <- iter + 1
    }
  }
  return(sample)
}

theta <- 3
size <-10000
sample <- fprocedure(size, theta)
plot(density(sample), col = "red")
```