HW6 - Exercise6

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Contents

1 Rejection sampling

1.1 Find the value of the normalizing constant. Show that g is a mixture of Gamma distributions. Identify the component distributions and their weights in the mixture.

$$
\begin{aligned}
C\int_0^\infty (2x^{\theta-1}+x^{\theta-1/2})e^{-x}dx &= C(\int_0^\infty 2x^{\theta-1}e^{-x}dx+\int_0^\infty x^{\theta-1/2}e^{-x}dx) &(1)\\
&= C(2\Gamma(\theta)+\Gamma(\theta+\frac{1}{2})) &(2)\\
&= 1 &(3)
\end{aligned}
$$

Thus, $C = \dfrac{1}{2\Gamma(\theta)+\Gamma(\theta+\frac{1}{2})}$

Then, we know that

$$g(x) = \frac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+\frac{1}{2})}Gamma(\theta,1)+\frac{\Gamma(\theta+\frac{1}{2})}{2\Gamma(\theta)+\Gamma(\theta+\frac{1}{2})}Gamma(\theta+\frac{1}{2},1)$$

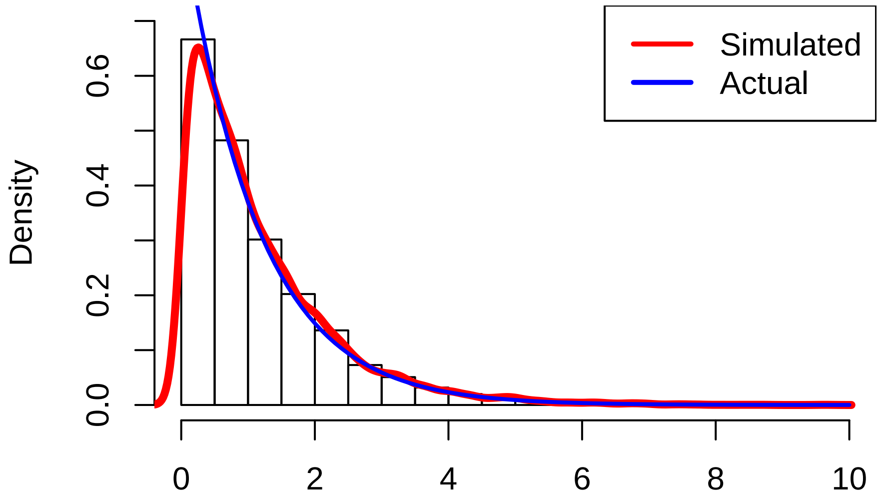
which is the mixture gamma distribution. The component distributions are $Gamma(\theta,1)$ and $Gamma(\theta+\frac{1}{2},1)$. The weights are $\dfrac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+\frac{1}{2})}$ and $\dfrac{\Gamma(\theta+\frac{1}{2})}{2\Gamma(\theta)+\Gamma(\theta+\frac{1}{2})}$.

1.2 Design a procedure (pseudo-code) to sample from g; implement it in an R function

Algorithm 1 procedure to sample from g

function (n, θ) ▷ n : sample size, θ : parameter
  u : Draw samples with size n from standard uniform distribution
  for i in 1:n **do** ▷ Loop by data size
    if $u < \frac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+\frac{1}{2})}$ **then**
      x_i : Draw a sample from $Gamma(\theta, 1)$
    else
      x_i : Draw a sample from $Gamma(\theta+\frac{1}{2}, 1)$
    end if
  end for
  return x : sample with size n
end function

```
sampling = function(n, theta)
{
  u = runif(n)
  Num = sum( u < 2*gamma(theta)/( 2*gamma(theta) + gamma(theta + 0.5) ) )
  sample = c(rgamma(Num, theta, 1), rgamma(n-Num, theta+0.5, 1))
  return(sample)
}
theta = 1
integrand = function(x)
{
  sqrt(4+x)*x^(theta-1)*exp(-x)
}
integral = integrate(integrand, 0, Inf)
aa = sampling(10000, theta)
hist(aa,freq=F,ylim = c(0,0.7),xlab='',main='')
lines(density(aa),col="red",lwd=4)
curve(sqrt(4+x)*x^(theta-1)*exp(-x)/as.numeric(integral[1]), add = T,col="blue",lwd=2)
legend('topright', legend=c("Simulated","Actual"),lty=c(1,1),
       lwd=c(2.5,2.5),col=c('red','blue'))
```



The figure is the result of $\theta = 1$ and n=10000.

1.3 Design a procedure (pseudo-code) to use rejection sampling to sample from f using g as the instrumental distribution.

First of all, we need to choose α such that

$$q(x) \le \alpha g(x)$$

where $q(x) = \sqrt{4+x}x^{\theta-1}e^{-x}$

Since $\sqrt{4+x} \le \sqrt{4} + \sqrt{x}$,

$$q(x) = \sqrt{4+x}x^{\theta-1}e^{-x} \le (2x^{\theta-1} + x^{\theta-1/2})e^{-x} = \frac{1}{C}g(x)$$

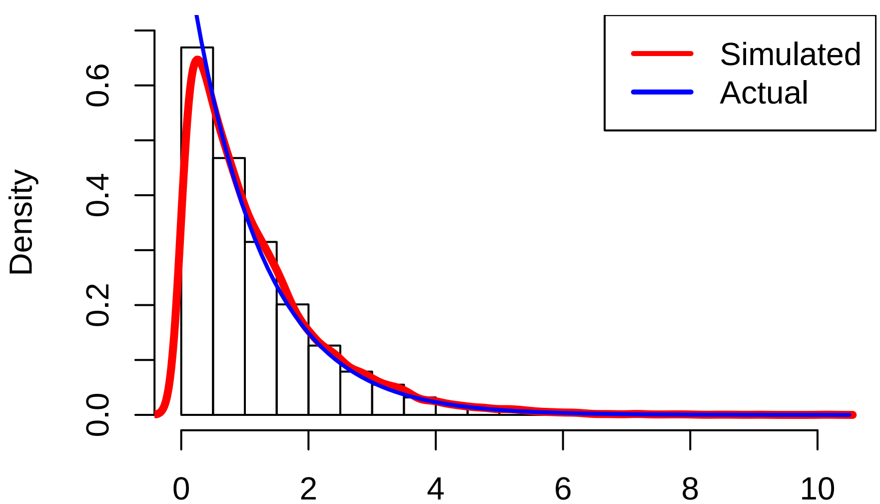
Then, α could be $\frac{1}{C} = 2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})$

Thus, we can choose $\alpha = \frac{1}{C}$ which satisfies the condition $q(x) \le \alpha g(x)$.

Algorithm 2 procedure to sample from f

1: **function** (n, θ) $\triangleright$ n : sample size, θ : parameter
2: u : Draw one sample from standard uniform distribution
3: x : Draw one sample from mixture dist. based on $g(x)$
4: **if** $u > \dfrac{q(x)}{\alpha g(x)}$ **then**
5: Go step 2-3
6: **else**
7: return x
8: **end if**
9: Repeat the steps until x's size is n
10: **return** x : sample with size n
11: **end function**

```r
sampling = function(n, theta)
{
   u = runif(n)
   Num = sum( u < 2*gamma(theta)/( 2*gamma(theta) + gamma(theta + 0.5) ) )
   sample = c(rgamma(Num, theta, 1), rgamma(n-Num, theta+0.5, 1))
   return(sample)
}
algorithm2 = function(n, theta)
{
   sample = c()
   while( length(sample) <= n )
   {
      u = runif(1); x = sampling(1, theta)
      if(u <= sqrt( x + 4)/(sqrt(x)))
         sample= c(sample, x)
   }
   return(sample)
}
theta = 1
integrand = function(x)
{
   sqrt(4+x)*x^(theta-1)*exp(-x)
}
integral = integrate(integrand, 0, Inf)
aa = algorithm2(10000, theta)
hist(aa,freq=F,ylim = c(0,0.7),xlab='',main='')
lines(density(aa),col="red",lwd=4)
curve(sqrt(4+x)*x^(theta-1)*exp(-x)/as.numeric(integral[1]),add = T,col="blue",lwd=2)
legend('topright', legend=c("Simulated","Actual"),lty=c(1,1),
       lwd=c(2.5,2.5),col=c('red','blue'))
```



The figure is the result of $\theta = 1$ and n=10000.

2 Mixture Proposal

2.1 Design a procedure (pseudo-code) to sample from f using a mixture of Beta distributions as the instrumental density.

First of all, we know that

$$\frac{x^{\theta-1}}{1+x^2} + \sqrt{2+x^2}(1-x)^{\beta-1} \le x^{\theta-1} + 2(1-x)^{\beta-1}$$

Denote that $q(x) = \frac{x^{\theta-1}}{1+x^2} + \sqrt{2+x^2}(1-x)^{\beta-1}$ and $g(x) \propto x^{\theta-1} + 2(1-x)^{\beta-1}$

Then,

$g(x) = \frac{\mathbf{B}(\theta,1)}{\mathbf{B}(\theta,1)+2\mathbf{B}(1,\beta)} Beta(\theta, 1) + \frac{2\mathbf{B}(1,\beta)}{\mathbf{B}(\theta,1)+2\mathbf{B}(1,\beta)} Beta(1, \beta)$ Thus, we get that

$$p_1 = \frac{\mathbf{B}(\theta,1)}{\mathbf{B}(\theta,1)+2\mathbf{B}(1,\beta)}$$

$$p_2 = \frac{2\mathbf{B}(1,\beta)}{\mathbf{B}(\theta,1)+2\mathbf{B}(1,\beta)}$$

$$g_1 = Beta(\theta, 1)$$
$$g_2 = Beta(1, \beta)$$

Let $\alpha = \mathbf{B}(\theta, 1) + 2\mathbf{B}(1, \beta)$.

Then,

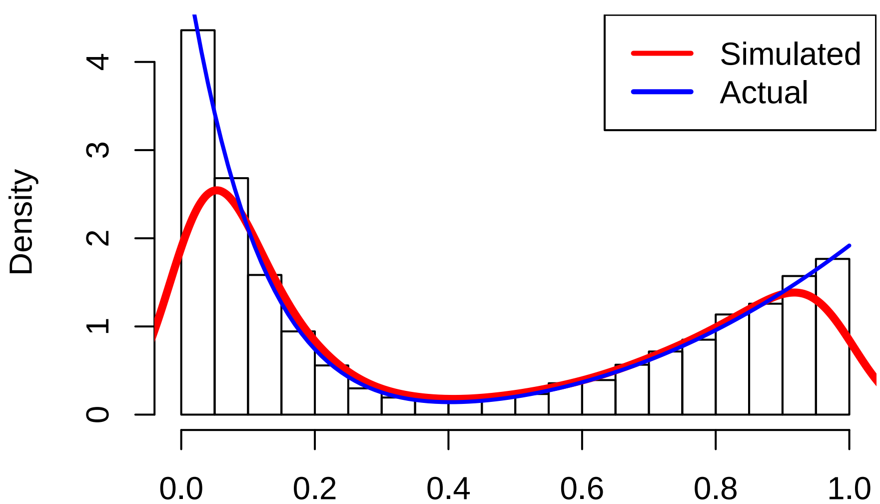
$$q(x) \le \alpha g(x)$$

Algorithm 3 procedure to sample from f

1: **function** (n, θ) ▷ n : sample size, θ : parameter
2: u : Draw one sample from standard uniform distribution
3: x : Draw one sample from mixture dist. based on $g(x)$
4: **if** $u > \frac{q(x)}{\alpha g(x)}$ **then**
5: Go step 2-3
6: **else**
7: return x
8: **end if**
9: Repeat the steps until x's size is n
10: **return** x : sample with size n
11: **end function**

```
sampling1 = function(n, theta, beta)
{
   u = runif(n)
   Num = sum( u < beta(theta, 1)/( 2*beta(1, beta) + beta(theta, 1) ) )
   sample = c(rbeta(Num, theta, 1), rbeta(n-Num, 1, beta))
```

```
  return(sample)
}
algorithm3 = function(n, theta, beta)
{
  sample = c()
  while( length(sample) <= n )
  {
    u = runif(1); x = sampling1(1, theta, beta)
    q = (x^(theta - 1)/(1+x^2)) + sqrt(2+x^2)*(1-x)^(beta-1)
    g = x^(theta -1) + 2*(1-x)^(beta-1)
    if(u <= q/g)
      sample= c(sample, x)
  }
  return(sample)
}
theta = 5; beta = 10
integrand = function(x)
{
  (x^(theta - 1)/(1+x^2)) + sqrt(2+x^2)*(1-x)^(beta-1)
}
integral = integrate(integrand, 0, 1)
aa = algorithm3(10000, theta, beta)
hist(aa,freq=F,xlab='',main='')
lines(density(aa),col="red",lwd=4)
curve( ((x^(theta - 1)/(1+x^2)) + sqrt(2+x^2)*(1-x)^(beta-1))/as.numeric(integral[1]),
       add = T,col="blue",lwd=2)
legend('topright', legend=c("Simulated","Actual"),lty=c(1,1),
       lwd=c(2.5,2.5),col=c('red','blue'))
```



The figure is the result of $(\theta = 5, \beta = 10)$ and n=10000.

2.2 Design a procedure (pseudo-code) to do this; implement it with an R function.

$$\frac{x^{\theta-1}}{1+x^2}, \quad \sqrt{2+x^2}(1-x)^{\beta-1}$$

We can see that

$$\frac{x^{\theta-1}}{1+x^2} \le x^{\theta-1}$$

$$\sqrt{2+x^2}(1-x)^{\beta-1} \le 2(1-x)^{\beta-1}$$

Then,

$$q_1 = \frac{x^{\theta-1}}{1+x^2},\ g_1 = Beta(\theta, 1),\ \alpha_1 = \mathbf{B}(\theta, 1)$$

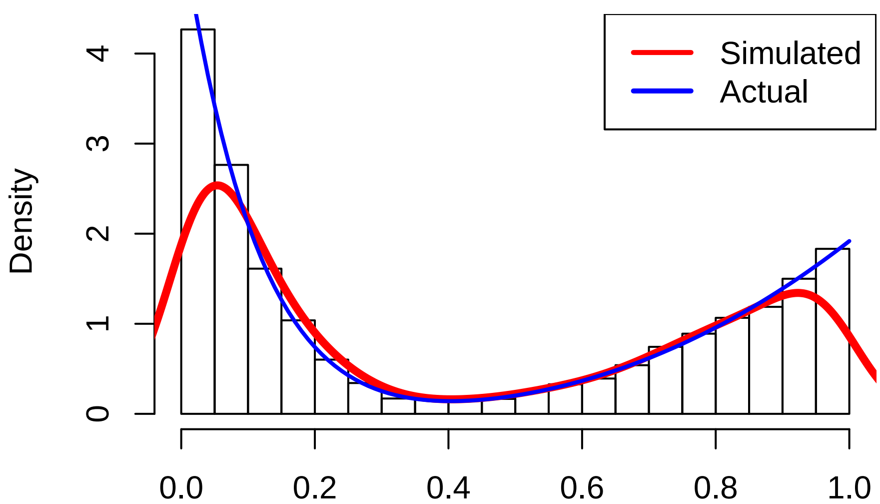
$$q_2 = \sqrt{2+x^2}(1-x)^{\beta-1},\ g_2 = Beta(1, \beta),\ \alpha_2 = 2\mathbf{B}(1, \beta)$$

Algorithm 4 procedure to sample from f

```
function (n, θ)                                    ▷ n : sample size, θ : parameter
    Sample k from {1,2}
    u : Draw one sample from standard uniform distribution
    x : Draw one sample from dist. based on g(x)_k
    if u > q_k(x) / (α_k g_k(x)) then
        Go step 2-3
    else
        return x
    end if
    Repeat the steps until x's size is n
    return x : sample with size n
end function
```

```r
algorithm4 = function(n, theta, beta)
{
   sample = c()
   while( length(sample) <= n )
   {
      k =sample(1:2,1)
      u = runif(1);
      if(k == 1)
      {
         x = rbeta(1,theta, 1)
         q = (x^(theta - 1)/(1+x^2))
         g = x^(theta -1)
         if(u <= q/g)
            sample= c(sample, x)
      }
      else {
```

```
        x = rbeta(1, 1, beta)
        q = sqrt(2+x^2)*(1-x)^(beta-1)
        g = 2*(1-x)^(beta-1)
        if(u <= q/g)
           sample= c(sample, x)
        }
   }
   return(sample)
}
theta = 5; beta = 10
integrand = function(x)
{
   (x^(theta - 1)/(1+x^2)) + sqrt(2+x^2)*(1-x)^(beta-1)
}
integral = integrate(integrand, 0, 1)
aa = algorithm4(10000, theta, beta)
hist(aa,freq=F,xlab='',main='')
lines(density(aa),col="red",lwd=4)
curve( ((x^(theta - 1)/(1+x^2)) + sqrt(2+x^2)*(1-x)^(beta-1))/as.numeric(integral[1]),
       add = T,col="blue",lwd=2)
legend('topright', legend=c("Simulated","Actual"),lty=c(1,1),
       lwd=c(2.5,2.5),col=c('red','blue'))
```



The figure is the result of $(\theta = 5, \beta = 10)$ and n=10000.