Homework 6

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5.2.1) Rejection Sampling

Let f and g be two probability densities on $(0, \infty)$ such that

$$f(x) \propto \sqrt{4+x}x^{\theta-1}e^{-x}$$

$$g(x) \propto (2x^{\theta-1} + x^{\theta-\frac{1}{2}})e^{-x}$$

Find the value of normalizing constant, C, such that

$$C\int_0^\infty (2x^{\theta-1} + x^{\theta-\frac{1}{2}})e^{-x}dx = 1$$

.

First we show g is a mixture of gamma distirbutions, by separating g we get:

$$2\int_0^\infty (x^{\theta-1})(e^{-x})dx + \int_0^\infty (x^{\theta-\frac{1}{2}})(e^{-x})dx$$

$$= 2\Gamma(\theta) + \Gamma(\theta + 1/2)$$

Subsituting these terms into our equation

$$\Rightarrow C[2\Gamma(\theta) + \Gamma(\theta + 1/2)] = 1$$

$$\Rightarrow C = \frac{1}{2\Gamma(\theta) + \Gamma(\theta + 1/2)}$$

Therefore

$$g(x) = \frac{1}{2\Gamma(\theta) + \Gamma(\theta + 1/2)}(2x^{\theta-1})(e^{-x}) + \frac{1}{2\Gamma(\theta) + \Gamma(\theta + 1/2)}(x^{\theta-1/2})(e^{-x})$$

By Multiplying each term by $\frac{\Gamma(\theta)}{\Gamma(\theta)}$ and $\frac{\Gamma(\theta+1/2)}{\Gamma(\theta+1/2)}$ respectively, we get the mixture of distributions:

$$g(x) = \frac{\Gamma(\theta)}{2\Gamma(\theta) + \Gamma(\theta + 1/2)}(2x^{\theta-1})(e^{-x})\frac{1}{\Gamma(\theta)} + \frac{\Gamma(\theta + 1/2)}{2\Gamma(\theta) + \Gamma(\theta + 1/2)}(x^{\theta-1/2})(e^{-x})\frac{1}{\Gamma(\theta + 1/2)}$$

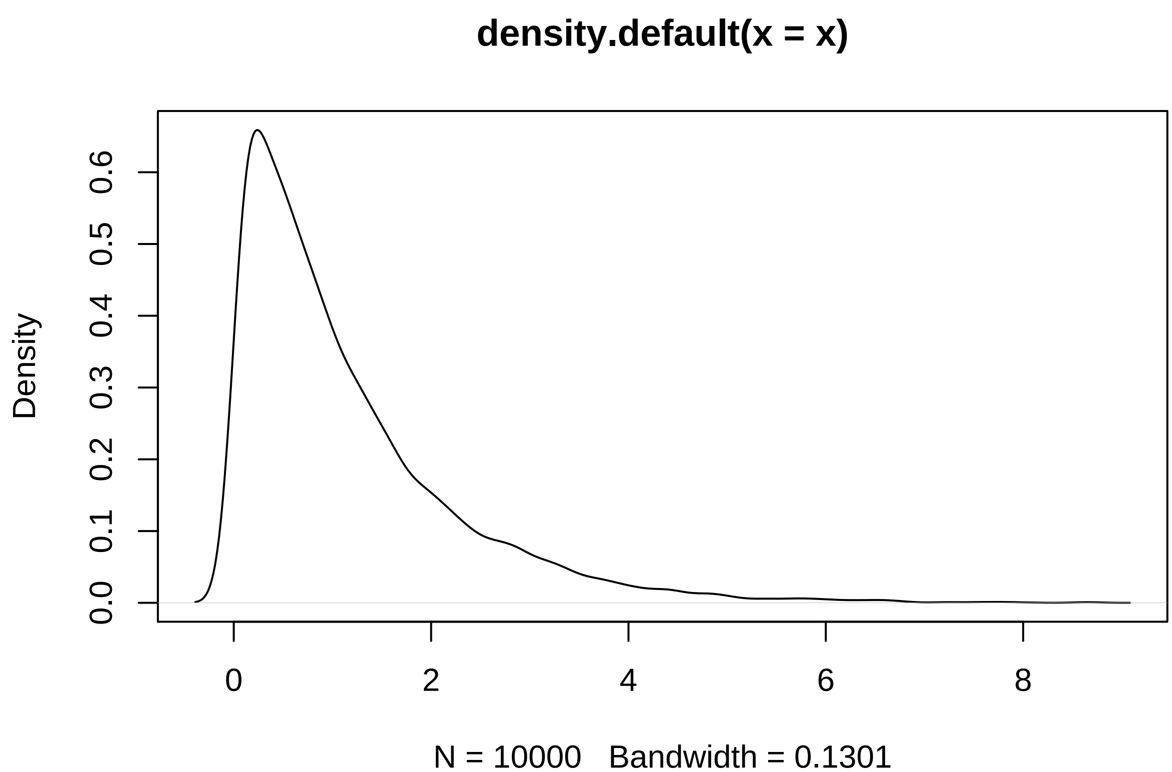
Therefore the component distributions are $\Gamma(\theta, 1)$ and $\Gamma(\theta + 1/2, 1)$ And corresponding weights: $\frac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+1/2)}$ and $\frac{\Gamma(\theta+1/2)}{2\Gamma(\theta)+\Gamma(\theta+1/2)}$

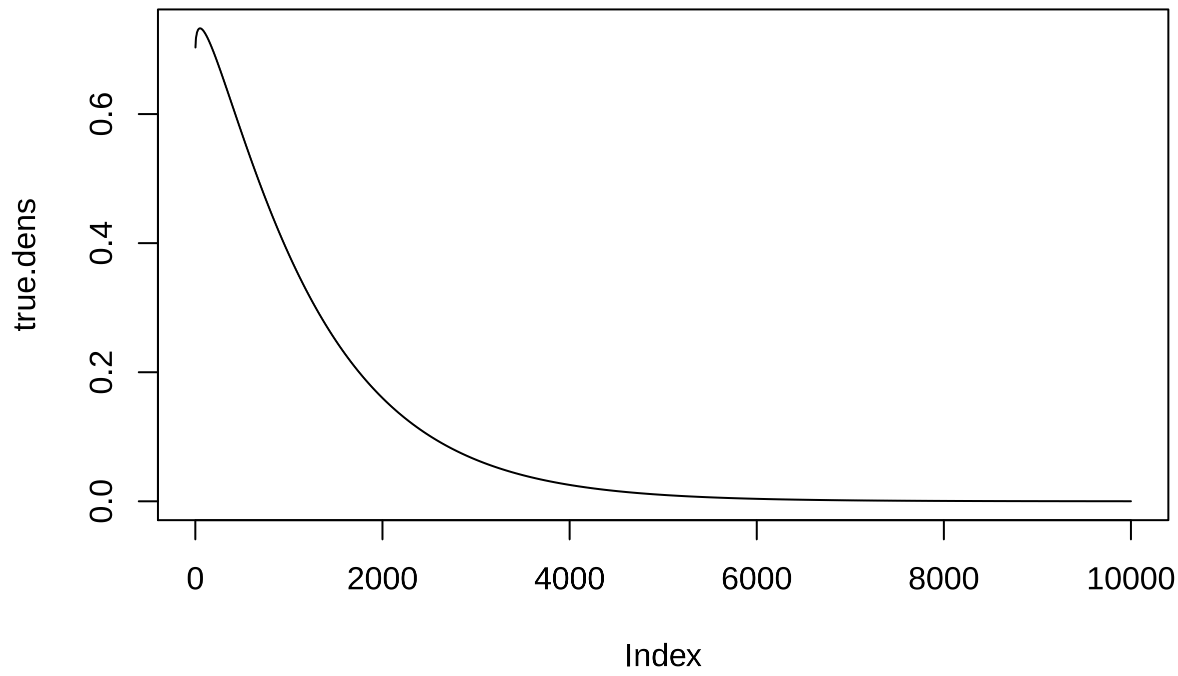
To sample from this code, we need to sample from a uniform distribution, comparing whether U is less than the weight of each Gamma distribution component

```
gamma.sample <- function(n){
    n <- 10000
    U <- runif(n)
    x <- rep(NA,n)
    theta <- 1                              #arbitrary theta value
    C <- 1 / (2 * gamma(theta) + gamma(theta + 0.5))       # Normalizing Constant
    weight.g1 <- (2 * gamma(theta)) * C
    weight.g2 <- (gamma(theta + 0.5)) * C
```

```
    hh <- seq(.001, 10, .001)
    true.dens <- weight.g1*dgamma(hh,1,1) + weight.g2*dgamma(hh,1.5,1)

    for (i in 1:n){
      if (U[i] < weight.g1) {
        x[i] <- rgamma(1, theta, 1)
      }
      else {
        x[i] <- rgamma(1, theta + 0.5, 1)
      }
    }
    aa <- plot(density(x))
    z <- plot(true.dens, type = "l")
    lines(aa,z)
}
gamma.sample(10000)
```

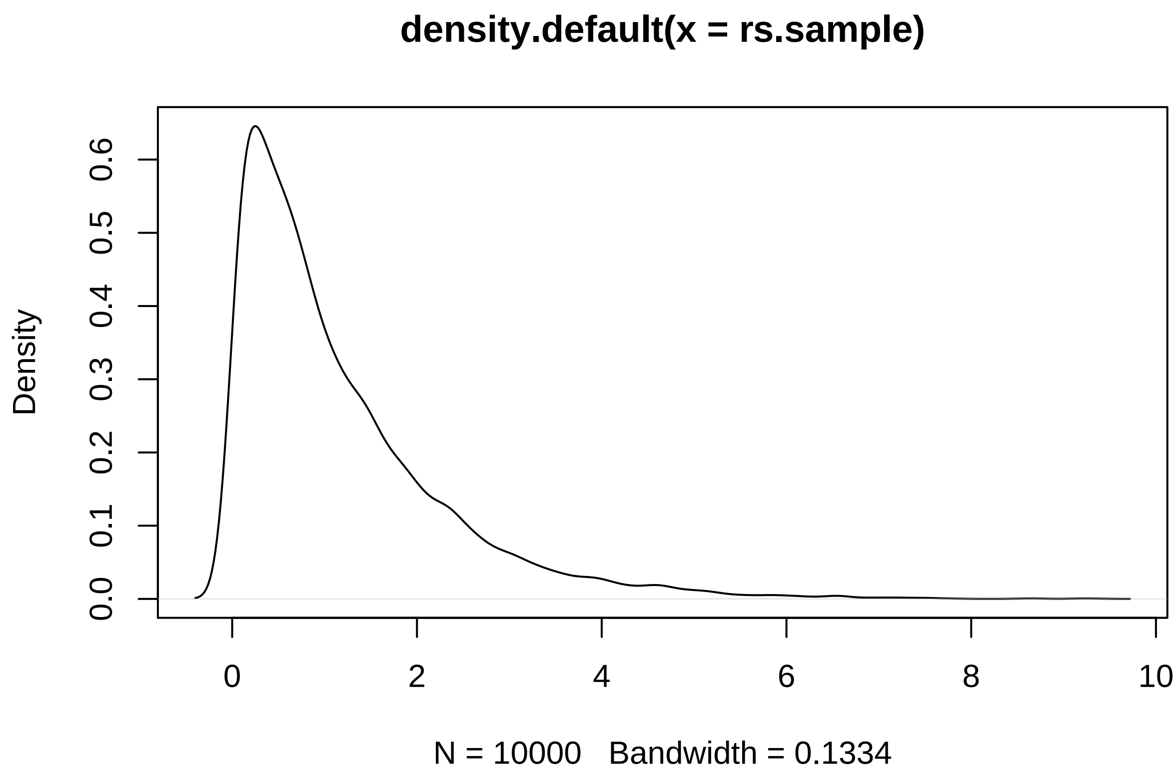




Rejection Sampling Method

```
n <- 10000
theta <- 1
rs.sample <- rep(NA, n)
C <- 1 / (2 * gamma(theta) + gamma(theta + 0.5))
weight.f1 <- C * (2 * gamma(theta))
alpha <- sqrt(2) / (2*C)

for (i in 1:n){
  U <- runif(1)
  if (U < weight.f1){
    x <- rgamma(1, theta, 1)
  }
  else{
    x <- rgamma(1, theta + 0.5, 1)
  }
  UU <- runif(1)
  f.x <- sqrt(4 + x) * x^(theta - 1) *exp(-x)
  g.x <- C * (2 *x ^ (theta - 1) + x^(theta - 0.5)) * exp(-x)
  z <- f.x / (alpha * g.x)
  if (U > z) {
    i <- i + 1; next
  }
  else rs.sample[i] <- x
}
plot(density(rs.sample))
```



5.2.2) Mixture Proposal

Given probability density f:

$$f(x) \propto \frac{x^{\theta-1}}{1+x^2} + \sqrt{2+x^2}(1-x)^{\beta-1}$$

Sample from f as a mixture of beta distributions Let $0 < x < 1$, since

$$\frac{x^{\theta-1}}{1+x^2} \leq x^{\theta-1}$$

and

$$\sqrt{2+x^2}(1-x)^{\beta-1} \leq (\sqrt{2}+x)(1-x)^{\beta-1}$$

we can say

$$Cf(x) \leq q(x) = x^{\theta-1} + (\sqrt{3})(1-x)^{\beta-1}$$

where $\sqrt{3}$ is derrived from the max of $\sqrt{2+x^2}$ given x's domain.

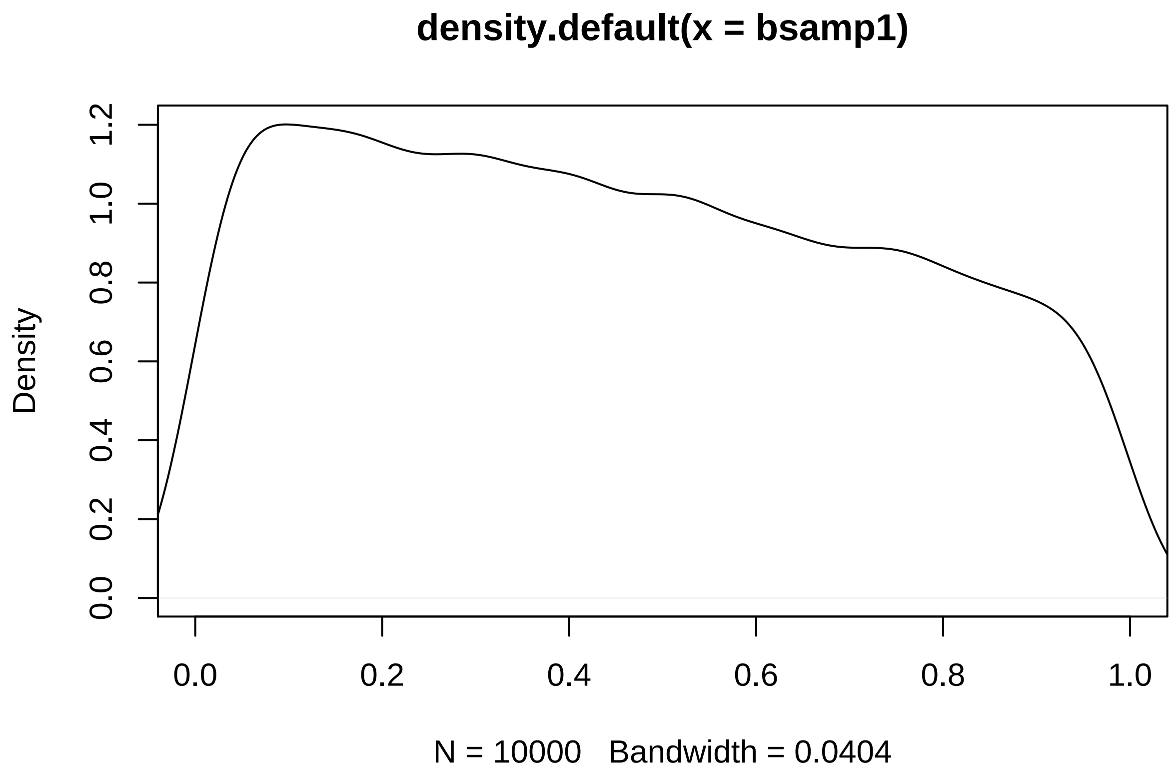
Since beta distibutions are of the form, $p(y|a,b) = \frac{y^{a-1}(1-y)^{b-1}}{B(a,b)}$, we get distibutions $beta(\theta, 1)$ and $beta(1, \beta)$ with weights $w_1 = (1/\theta) * C$ and $w_2 = C * (\sqrt{3}/\beta)$

```
beta.samp <- function(n, theta, beta0){
      bsamp <- rep(NA, n)
      w1 <- beta(theta, 1) / (beta(theta, 1) + sqrt(3)* beta(1, beta0))
      for (i in 1:n){
        U <- runif(1)
        if (U < w1){bsamp[i] <- rbeta(1, theta, 1)}
```

```r
        else {bsamp[i] <- rbeta(1,1, beta0)}
      }
    return(list(samp = bsamp,w1= w1, w2 =1-w1))
      }

n <- 10000
theta <- 2
beta0 <- 2
bsamp1 <- rep(0, n)
C <- 1 / (theta) + sqrt(3) / beta0
for (i in 1:n){
  U <- 10000
  upper <- 1
  lower <- 0.5
  while (U > C){
    U <- runif(1)
    b <- beta.samp(1, theta, beta0)
    x <- b$samp
    bsamp1[i] <- x
    upper <- (x^(theta - 1) / (1 + x^2) + sqrt(2+x^2)*(1-x)^(beta0-1))
    lower <- b$w1*dbeta(x, theta, 1) + b$w2*dbeta(x,1,beta0)
  }
}

gg <- function(x) {x ^(theta-1) / (1+x^2) + sqrt(2+x^2) * (1-x)^(beta0-1)}
dens.beta <- integrate(gg, lower=0, upper=1)$value
plot(density(bsamp1), xlim=c(0,1))
```



Sampleing Separately

```
beta.sep <- function(n, beta, theta){
  b <- rep(NA, n)
  r <- rep(NA, n)
  U <- runif(n)
  wt <- 1/theta + sqrt(3) / beta
  while (U > wt)
}
```