

HW6

Qi Qi

10/19/2018

Exercise 5.2.1

•

$$\begin{aligned} C \int_0^\infty (2x^{\theta-1} + x^{\theta-1/2})e^{-x} dx &= 2C \int_0^\infty x^{\theta-1} e^{-x} dx + C \int_0^\infty x^{\theta-1/2} e^{-x} dx \\ &= 2C\Gamma(\theta) \int_0^\infty \frac{x^{\theta-1} e^{-x}}{\Gamma(\theta)} dx + C\Gamma(\theta + 1/2) \int_0^\infty \frac{x^{\theta-1/2} e^{-x}}{\Gamma(\theta + 1/2)} dx \\ &= 2C\Gamma(\theta) + C\Gamma(\theta + 1/2) = 1 \Rightarrow C = \frac{1}{2\Gamma(\theta) + \Gamma(\theta + 1/2)} \\ g(x) &= \frac{2\Gamma(\theta)}{2\Gamma(\theta) + \Gamma(\theta + 1/2)} x^{\theta-1} e^{-x} + \frac{\Gamma(\theta + 1/2)}{\Gamma(\theta) + \Gamma(\theta + 1/2)} x^{\theta-1/2} e^{-x} \end{aligned}$$

Thus, g is a mixture of $\text{Gamma}(\theta, 1)$ and $\text{Gamma}(\theta + 1/2, 1)$ with weight $\frac{2\Gamma(\theta)}{2\Gamma(\theta) + \Gamma(\theta + 1/2)}$ and $\frac{\Gamma(\theta + 1/2)}{2\Gamma(\theta) + \Gamma(\theta + 1/2)}$.

•

Step 1: sample U from $U(0, 1)$.

Step 2: if $U < \frac{2\Gamma(\theta)}{2\Gamma(\theta) + \Gamma(\theta + 1/2)}$ then sample X from $\text{Gamma}(\theta, 1)$; if $U \geq \frac{2\Gamma(\theta)}{2\Gamma(\theta) + \Gamma(\theta + 1/2)}$ then sample X from $\text{Gamma}(\theta + 1/2, 1)$.

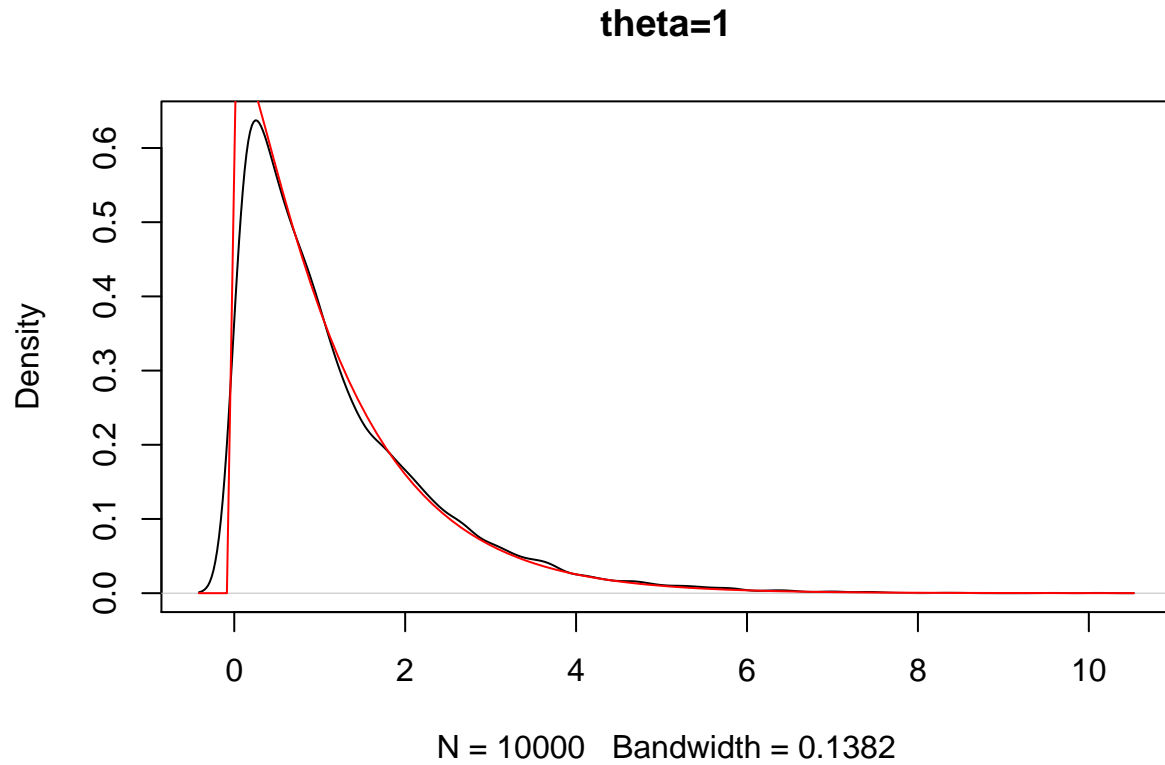
Step 3: return X .

```
rg <- function(theta, n){
  X <- NA
  for (i in 1:n){
    u <- runif(1,0,1)
    if (u < 2 * gamma(theta) / (2 * gamma(theta) + gamma(theta + 1/2))){
      X[i] <- rgamma(1, shape = theta, scale = 1)
    }else{
      X[i] <- rgamma(1, shape = theta + 1 / 2, scale = 1)
    }
  }
  return(X)
}

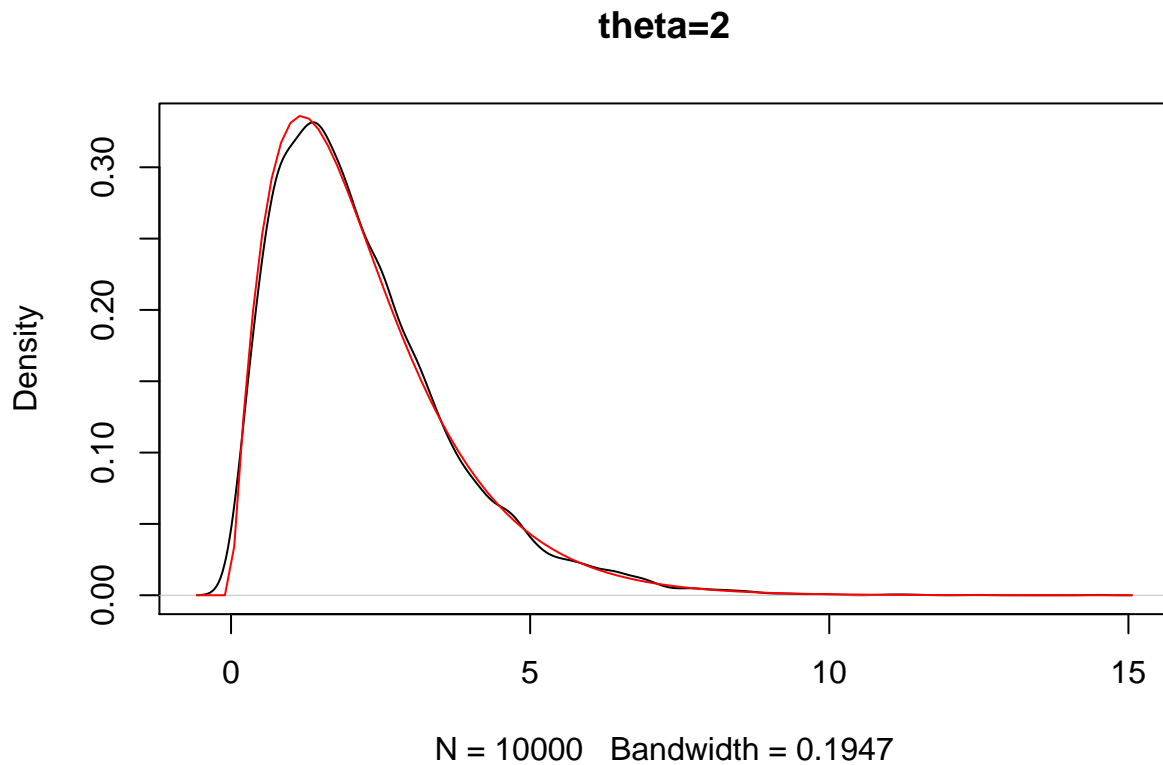
n <- 10000
theta1 <- 1
x1 <- rg(theta1, n)

dg <- function(x, theta){
  y <- 2 * gamma(theta) / (2 * gamma(theta) + gamma(theta + 1/2)) * dgamma(x, shape = theta, scale = 1) +
  gamma(theta + 1/2) / (2 * gamma(theta) + gamma(theta + 1/2)) * dgamma(x, shape = theta + 1/2, scale = 1)
  return(y)
}
```

```
plot(density(x1), main = "theta=1")  
curve(dg(x, theta1), add = TRUE, col = "red")
```



```
theta2 <- 2  
x2 <- rg(theta2, n)  
  
plot(density(x2), main = "theta=2")  
curve(dg(x, theta2), add = TRUE, col = "red")
```



•

$$q(x) = \sqrt{4+x} x^{\theta-1} e^{-x}$$

$$\frac{q(x)}{g(x)} = \frac{\sqrt{4+x}}{C(2+\sqrt{x})} \leq \frac{2+\sqrt{x}}{C(2+\sqrt{x})} = \frac{1}{C}$$

So, we can let $\alpha = \frac{1}{C}$.

$$\frac{q(x)}{\alpha g(x)} = \frac{\sqrt{4+x} x^{\theta-1} e^{-x}}{(2x^{\theta-1} + x^{\theta-1/2}) e^{-x}} = \frac{\sqrt{4+x}}{2+\sqrt{x}}$$

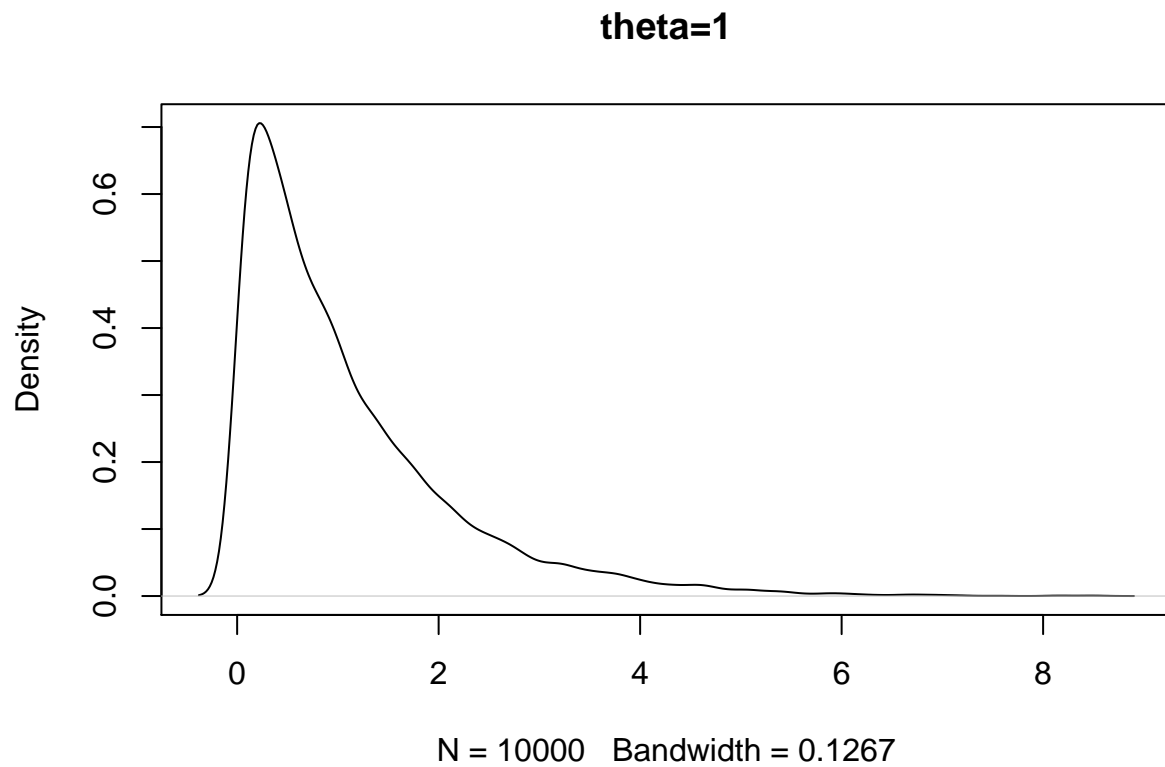
Step 1: sample $X \sim g$ and $U \sim U(0, 1)$.

Step 2: if $U > \frac{q(x)}{\alpha g(x)}$, then go to step 1; else, return X .

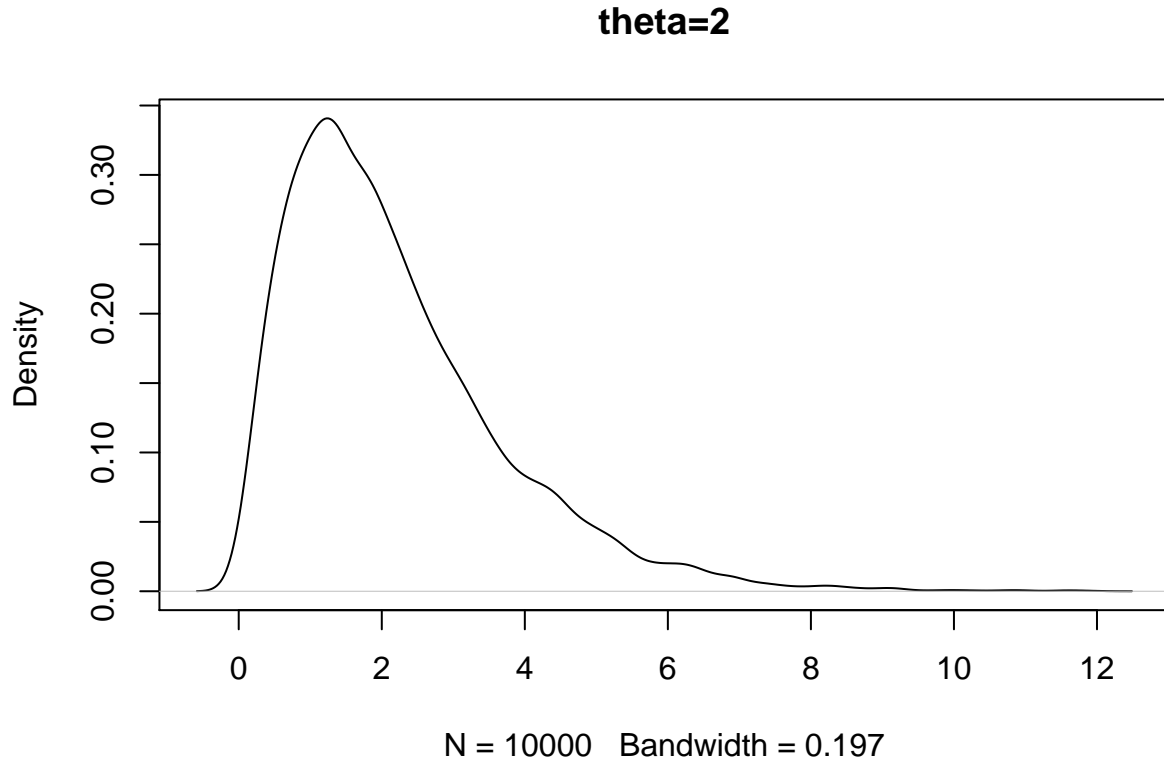
```
rf <- function(theta, n){
  y <- NA
  i <- 0
  while (i <= n){
    u <- runif(1, 0, 1)
    x <- rg(theta, 1)
    if (u <= sqrt(4 + x) / (2 + sqrt(x))){
      y[i] <- x
      i <- i + 1
    }
  }
}
```

```
}  
  return(y)  
}
```

```
theta1 <- 1  
x1 <- rf(theta1, 10000)  
plot(density(x1), main = "theta=1")
```



```
theta2 <- 2  
x2 <- rf(theta2, 10000)  
plot(density(x2), main = "theta=2")
```



Exercise 5.2.2

•

$$g_1(x) = \frac{x^{\theta-1}}{\text{Beta}(\theta, 1)} = \frac{x^{\theta-1}}{\theta}$$

$$g_2(x) = \frac{(1-x)^{\beta-1}}{\text{Beta}(1, \beta)} = \frac{(1-x)^{\beta-1}}{\beta}$$

$$p_1 = \frac{\theta}{\theta + \beta}, \quad p_2 = \frac{\beta}{\theta + \beta}$$

Then,

$$g(x) = \frac{x^{\theta-1} + (1-x)^{\beta-1}}{\theta + \beta}$$

$$\frac{q(x)}{g(x)} = \frac{\frac{x^{\theta-1}}{1+x^2} + \sqrt{2+x^2}(1-x)^{\beta-1}}{\frac{x^{\theta-1} + (1-x)^{\beta-1}}{\theta + \beta}}$$

$$\leq \frac{\frac{x^{\theta-1}}{1+x^2} + \frac{(1-x)^{\beta-1}}{1+x^2}}{\frac{x^{\theta-1} + (1-x)^{\beta-1}}{\theta + \beta}} = \frac{\theta + \beta}{1 + x^2} \leq \theta + \beta$$

Let $\alpha = \theta + \beta$,

$$\frac{q(x)}{\alpha g(x)} = \frac{\frac{x^{\theta-1}}{1+x^2} + \sqrt{2+x^2}(1-x)^{\beta-1}}{x^{\theta-1} + (1-x)^{\beta-1}}$$

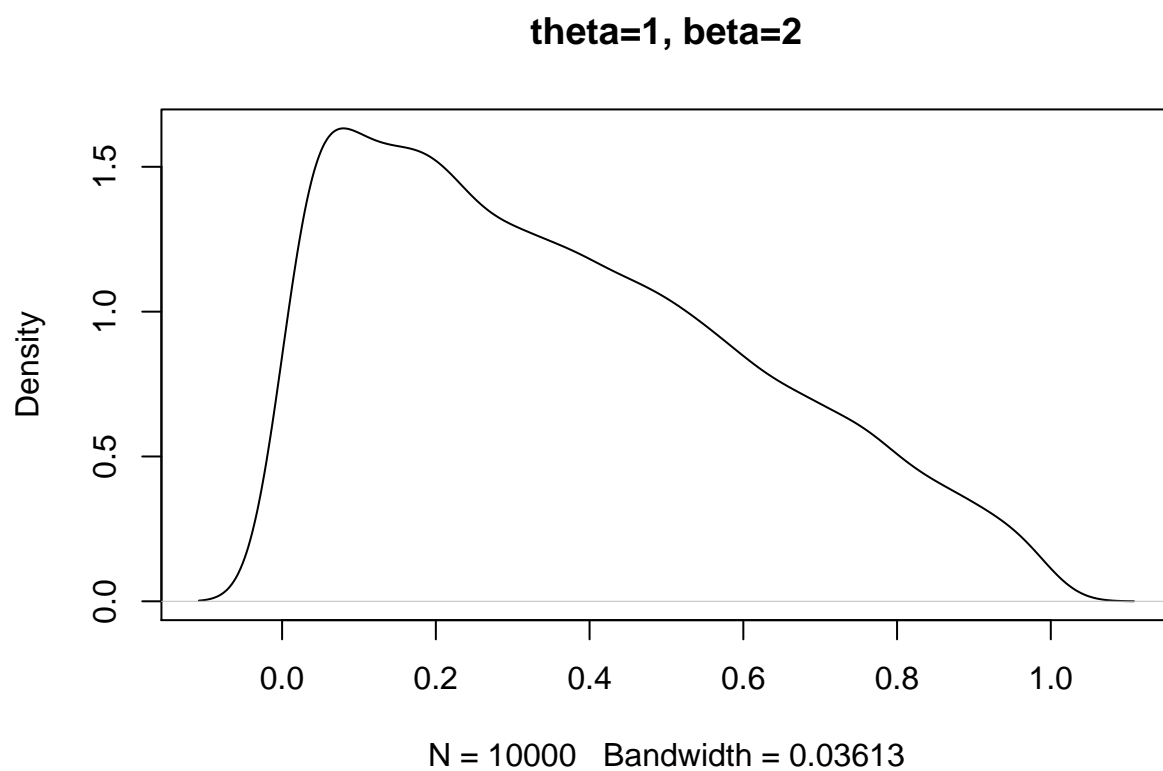
Step 1: sample $X \sim g$ and $U \sim U(0,1)$.

Step 2: if $U > \frac{q(x)}{\alpha g(x)}$, then go to step 1; else, return X .

```
rg <- function(theta, beta, n){
  X <- NA
  for (i in 1:n){
    u <- runif(1,0,1)
    if (u < theta / (theta + beta)){
      X[i] <- rbeta(1, shape1 = theta, shape2 = 1)
    }else{
      X[i] <- rbeta(1, shape1 = 1, shape2 = beta)
    }
  }
  return(X)
}

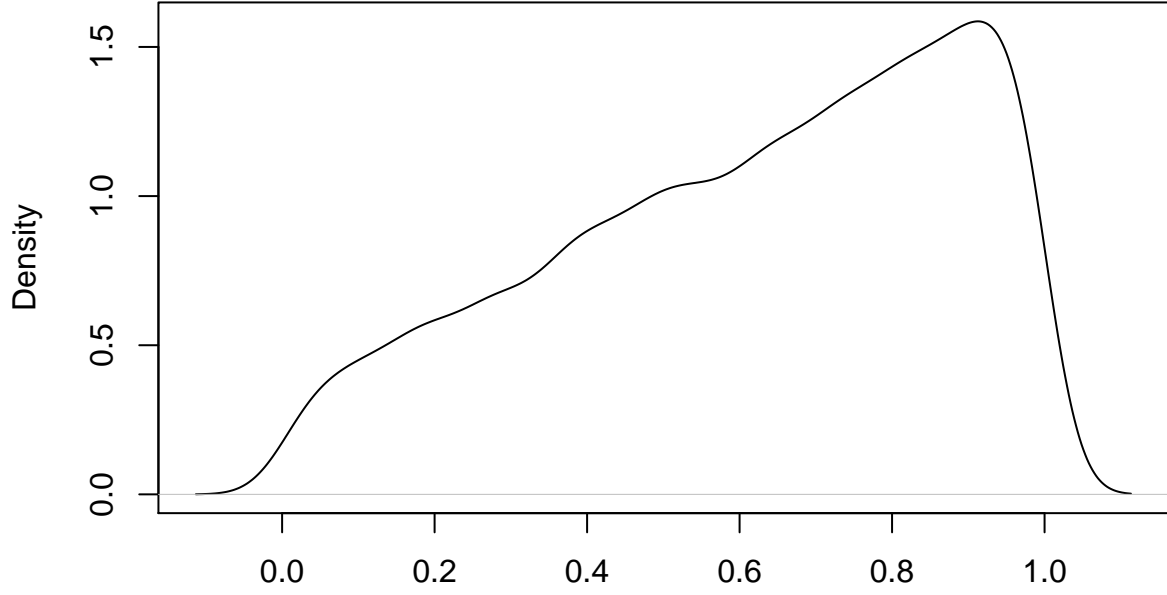
rf <- function(theta, beta, n){
  y <- NA
  i <- 0
  while (i <= n){
    u <- runif(1, 0, 1)
    x <- rg(theta, beta, 1)
    if (u <= (x ^ (theta - 1) / (1 + x ^ 2) + sqrt(2 + x ^ 2) * (1 - x) ^ (beta - 1)) / (x ^ (theta - 1) + (1 - x) ^ (beta - 1))){
      y[i] <- x
      i <- i + 1
    }
  }
  return(y)
}

theta1 <- 1
beta1 <- 2
x1 <- rf(theta1, beta1, 10000)
plot(density(x1), main = "theta=1, beta=2")
```



```
theta2 <- 2  
beta2 <- 1  
x2 <- rf(theta2, beta2, 10000)  
plot(density(x2), main = "theta=2, beta=1")
```

theta=2, beta=1



N = 10000 Bandwidth = 0.03779

•

$$q_1(x) = \frac{x^{\theta-1}}{1+x^2}$$

$$g_1(x) = \frac{x^{\theta-1}}{\text{Beta}(\theta, 1)} = \frac{x^{\theta-1}}{\theta}$$

$$\frac{q_1(x)}{g_1(x)} = \frac{\theta}{1+x^2} \leq \theta = \alpha_1$$

$$\frac{q_1(x)}{\alpha_1 g_1(x)} = \frac{1}{1+x^2}$$

$$q_2(x) = \sqrt{2+x^2}(1-x)^{\beta-1}$$

$$g_2(x) = \frac{(1-x)^{\beta-1}}{\text{Beta}(1, \beta)} = \frac{(1-x)^{\beta-1}}{\beta}$$

$$\frac{q_2(x)}{g_2(x)} = \beta\sqrt{2+x^2} \leq \beta\sqrt{3} = \alpha_2$$

$$\frac{q_2(x)}{\alpha_2 g_2(x)} = \sqrt{\frac{2+x^2}{3}}$$

Step 1: sample k from $\{1, 2\}$ with probability $\frac{\theta}{\theta+\beta\sqrt{3}}$ and $\frac{\beta\sqrt{3}}{\theta+\beta\sqrt{3}}$.

Step 2: sample $X \sim g_k$ and $U \sim U(0, 1)$

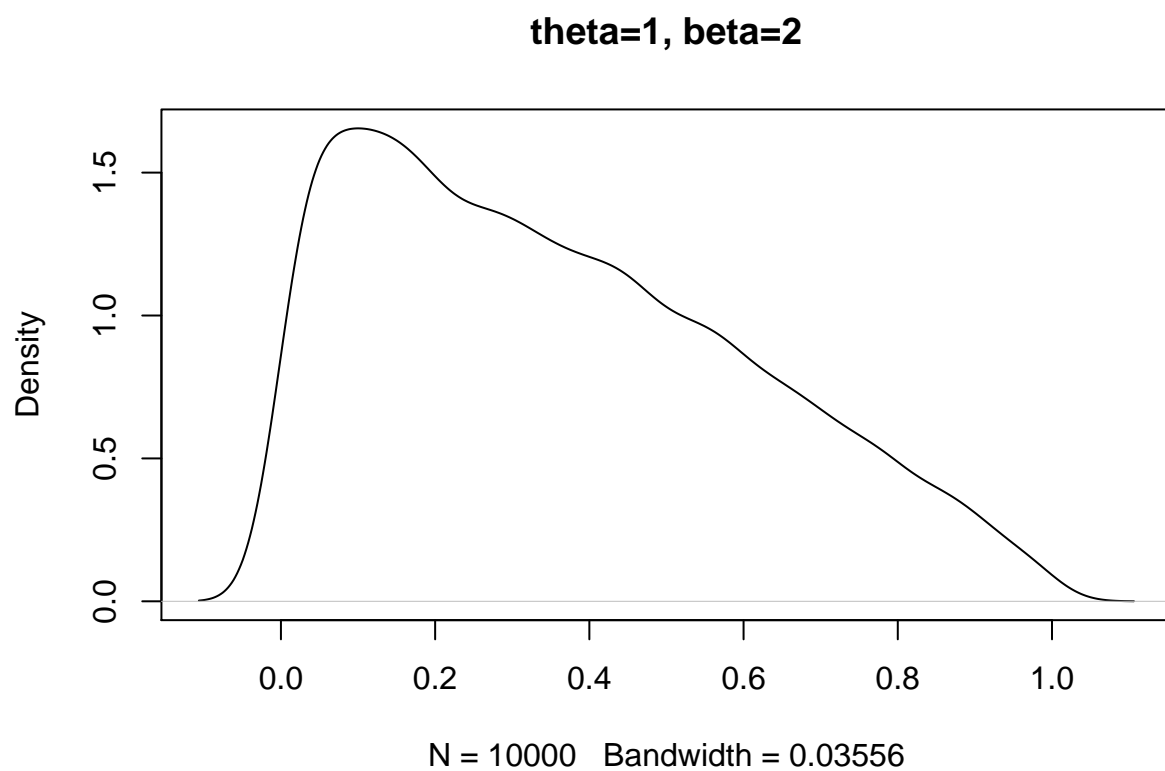
Step 3: if $U > \frac{q_k(x)}{\alpha_k g_k(x)}$ then go to step 1; else, return X .


```

rf <- function(theta, beta, n){
  i <- 0
  y <- NA
  while (i <= n){
    p <- runif(1, 0, 1)
    if (p < theta / (theta + beta * sqrt(3))){
      u <- runif(1, 0, 1)
      x <- rbeta(1, shape1 = theta, shape2 = 1)
      if (u <= 1 / (1 + x ^ 2)){
        y[i] <- x
        i <- i + 1
      }
    }else{
      u <- runif(1, 0, 1)
      x <- rbeta(1, shape1 = 1, shape2 = beta)
      if (u <= sqrt( (2 + x ^ 2) / 3)) {
        y[i] <- x
        i <- i + 1
      }
    }
  }
  return (y)
}

theta1 <- 1
beta1 <- 2
x1 <- rf(theta1, beta1, 10000)
plot(density(x1), main = "theta=1, beta=2")

```



```
theta2 <- 2  
beta2 <- 1  
x2 <- rf(theta2, beta2, 10000)  
plot(density(x2), main = "theta=2, beta=1")
```

theta=2, beta=1

