HW6

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Exercise 5.2.1

- $$
\begin{aligned}
C\int_0^\infty (2x^{\theta-1}+x^{\theta-1/2})e^{-x}dx &= 2C\int_0^\infty x^{\theta-1}e^{-x}dx + C\int_0^\infty x^{\theta-1/2}e^{-x}dx \\
&= 2C\Gamma(\theta)\int_0^\infty \frac{x^{\theta-1}e^{-x}}{\Gamma(\theta)}dx + C\Gamma(\theta+1/2)\int_0^\infty \frac{x^{\theta-1/2}e^{-x}}{\Gamma(\theta+1/2)}dx \\
&= 2C\Gamma(\theta)+C\Gamma(\theta+1/2) = 1 \Rightarrow C = \frac{1}{2\Gamma(\theta)+\Gamma(\theta+1/2)} \\
g(x) &= \frac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+1/2)}x^{\theta-1}e^{-x} + \frac{\Gamma(\theta+1/2)}{\Gamma(\theta)+\Gamma(\theta+1/2)}x^{\theta-1/2}e^{-x}
\end{aligned}
$$

  Thus, g is a mixture of $Gamma(\theta,1)$ and $Gamma(\theta+1/2,1)$ with weight $\frac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+1/2)}$ and $\frac{\Gamma(\theta+1/2)}{2\Gamma(\theta)+\Gamma(\theta+1/2)}$.

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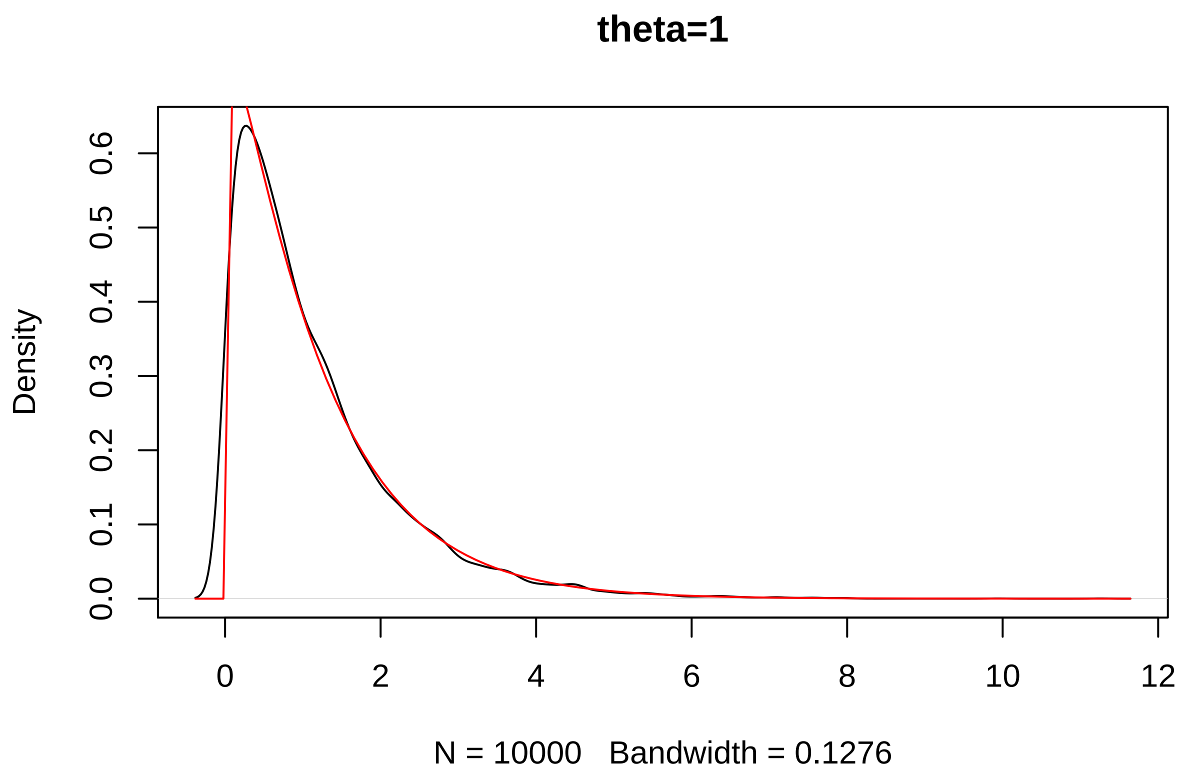
Step 1: sample U from $U(0,1)$.

Step 2: if $U < \frac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+1/2)}$ then sample X from $Gamma(\theta,1)$; if $U \geq \frac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+1/2)}$ then sample X from $Gamma(\theta+1/2,1)$.

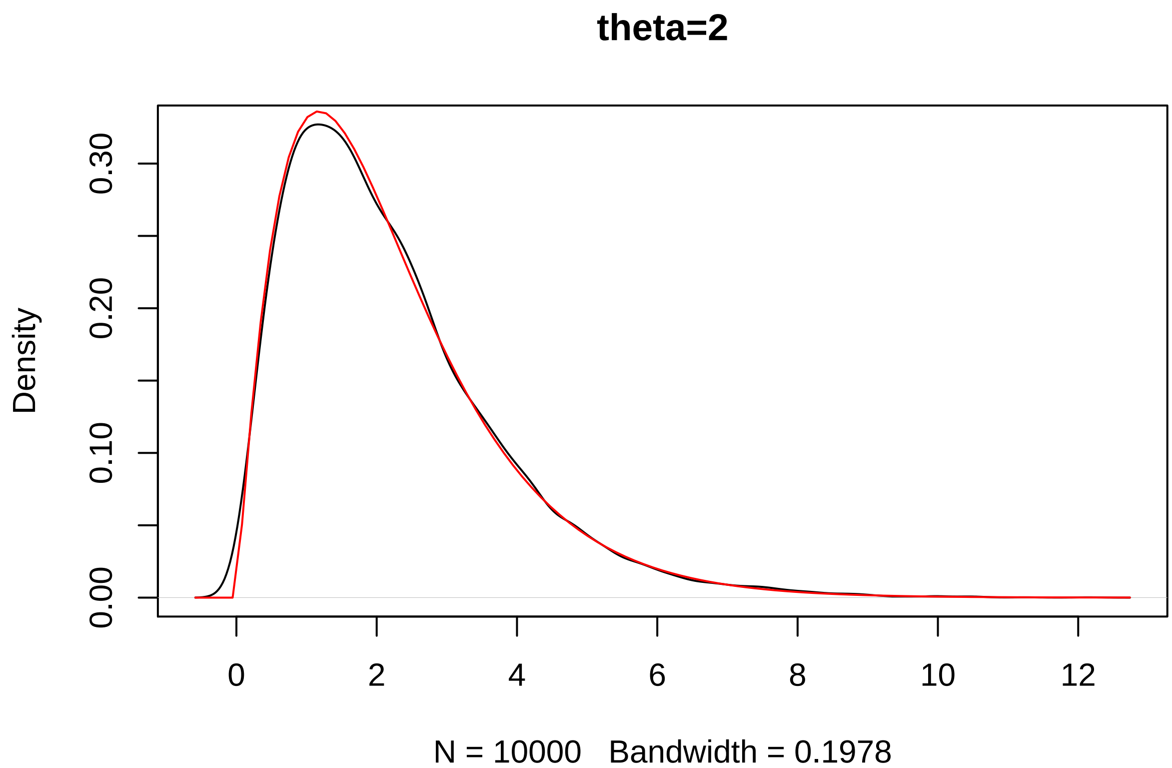
Step 3: return X.

```
rg <- function(theta, n){
  X <- NA
  for (i in 1:n){
    u <- runif(1,0,1)
    if (u < 2 * gamma(theta) / (2 * gamma(theta) + gamma(theta + 1/2))){
      X[i] <- rgamma(1, shape = theta, scale = 1)
    }else{
      X[i] <- rgamma(1, shape = theta + 1 / 2, scale = 1)
    }
  }
  return(X)
}

n <- 10000
theta1 <- 1
x1 <- rg(theta1, n)

dg <- function(x, theta){
  y <-2 * gamma(theta) / (2 * gamma(theta) + gamma(theta + 1/2)) * dgamma(x, shape = theta, scale = 1)
  return(y)
}
```

```r
plot(density(x1), main = "theta=1")
curve(dg(x, theta1), add = TRUE, col = "red")
```




```r
theta2 <- 2
x2 <- rg(theta2, n)

plot(density(x2), main = "theta=2")
curve(dg(x, theta2), add = TRUE, col = "red")
```



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$$q(x) = \sqrt{4+x}x^{\theta-1}e^{-x}$$

$$\frac{q(x)}{g(x)} = \frac{\sqrt{4+x}}{C(2+\sqrt{x})} \leq \frac{2+\sqrt{x}}{C(2+\sqrt{x})} = \frac{1}{C}$$

So, we can let $\alpha = \frac{1}{C}$.

$$\frac{q(x)}{\alpha g(x)} = \frac{\sqrt{4+x}x^{\theta-1}e^{-x}}{(2x^{\theta-1}+x^{\theta-1/2})e^{-x}} = \frac{\sqrt{4+x}}{2+\sqrt{x}}$$

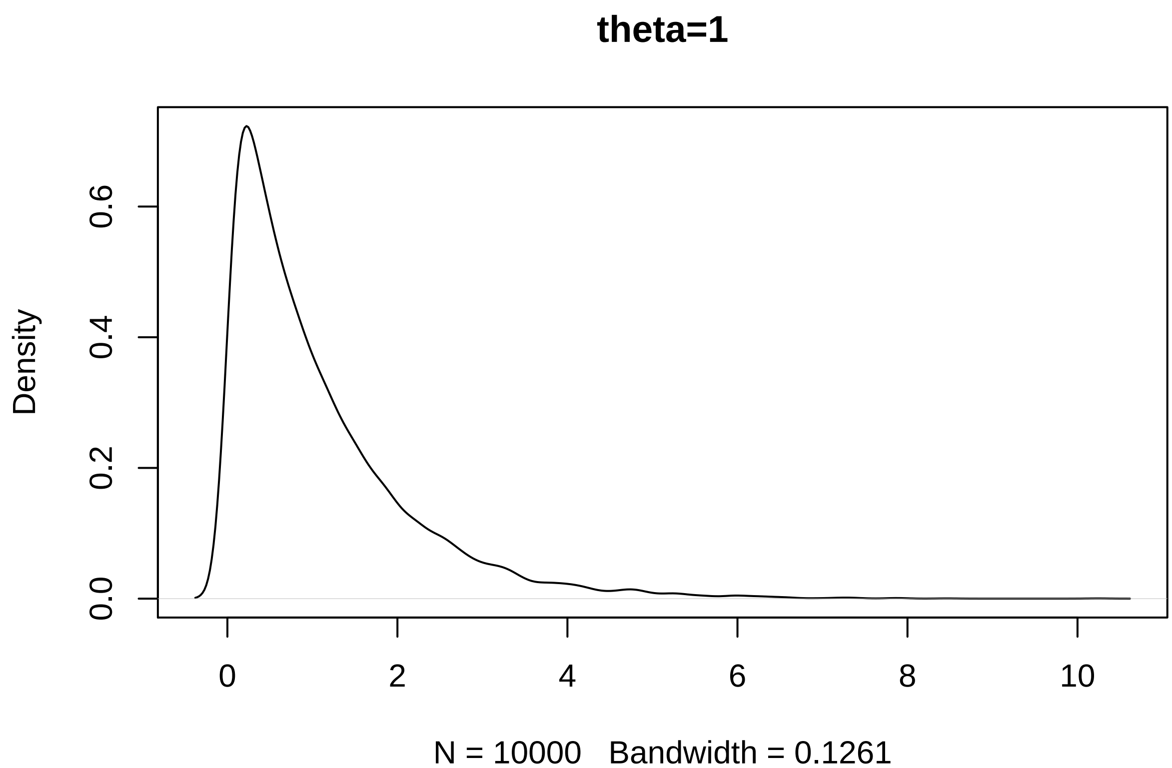
Step 1: sample $X \sim g$ and $U \sim U(0,1)$.

Step 2: if $U > \frac{q(x)}{\alpha g(x)}$, then go to step 1; else, return X.

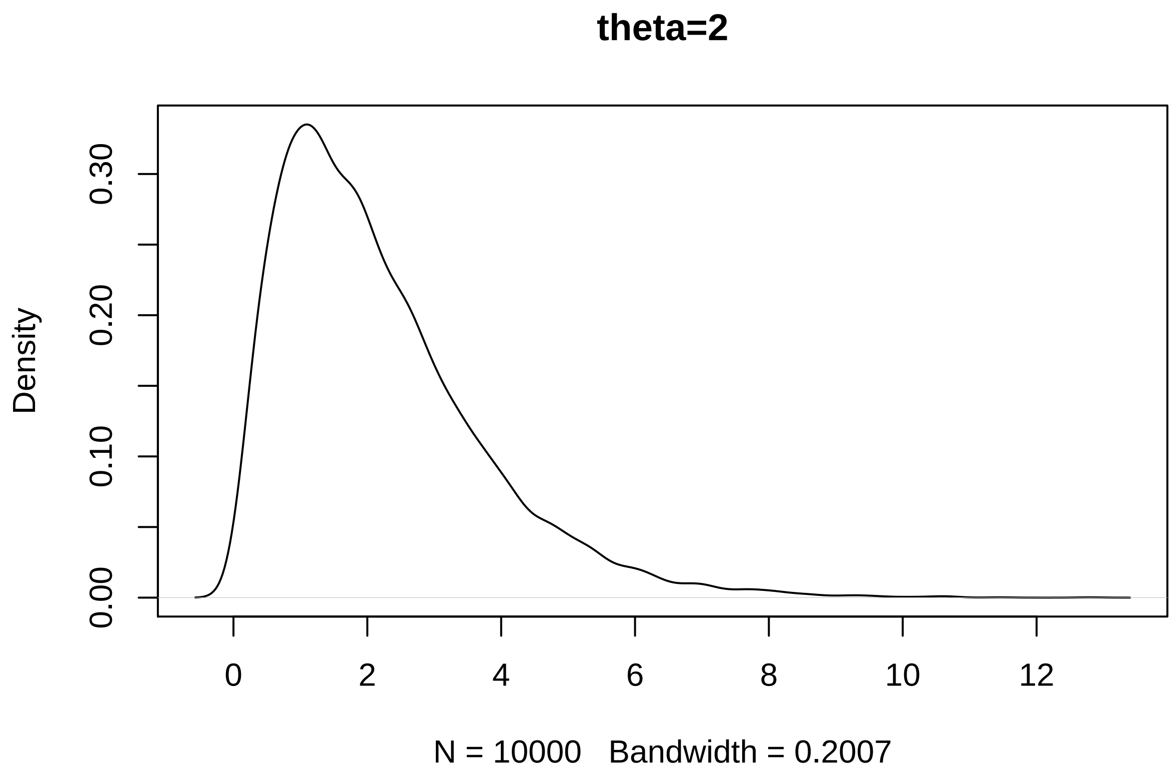
```
rf <- function(theta, n){
  y <- NA
  i <- 0
  while (i <= n){
     u <- runif(1, 0, 1)
     x <- rg(theta, 1)
     if (u <= sqrt(4 + x) / (2 + sqrt(x))){
       y[i] <- x
       i <- i + 1
     }
```

```r
  }
 return(y)
}


theta1 <- 1
x1 <- rf(theta1, 10000)
plot(density(x1), main = "theta=1")
```




```r
theta2 <- 2
x2 <- rf(theta2, 10000)
plot(density(x2), main = "theta=2")
```



Exercise 5.2.2

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$$g_1(x) = \frac{x^{\theta-1}}{Beta(\theta,1)} = \frac{x^{\theta-1}}{\theta}$$

$$g_2(x) = \frac{(1-x)^{\beta-1}}{Beta(1,\beta)} = \frac{(1-x)^{\beta-1}}{\beta}$$

$$p_1 = \frac{\theta}{\theta+\beta},\ p_2 = \frac{\beta}{\theta+\beta}$$

Then,

$$g(x) = \frac{x^{\theta-1} + (1-x)^{\beta-1}}{\theta+\beta}$$

$$\frac{q(x)}{g(x)} = \frac{\frac{x^{\theta-1}}{1+x^2} + \sqrt{2+x^2}(1-x)^{\beta-1}}{\frac{x^{\theta-1}+(1-x)^{\beta-1}}{\theta+\beta}}$$

$$\leq \frac{\frac{x^{\theta-1}}{1+x^2} + \frac{(1-x)^{\beta-1}}{1+x^2}}{\frac{x^{\theta-1}+(1-x)^{\beta-1}}{\theta+\beta}} = \frac{\theta+\beta}{1+x^2} \leq \theta+\beta$$

Let $\alpha = \theta + \beta$,

$$\frac{q(x)}{\alpha g(x)} = \frac{\frac{x^{\theta-1}}{1+x^2} + \sqrt{2+x^2}(1-x)^{\beta-1}}{x^{\theta-1} + (1-x)^{\beta-1}}$$

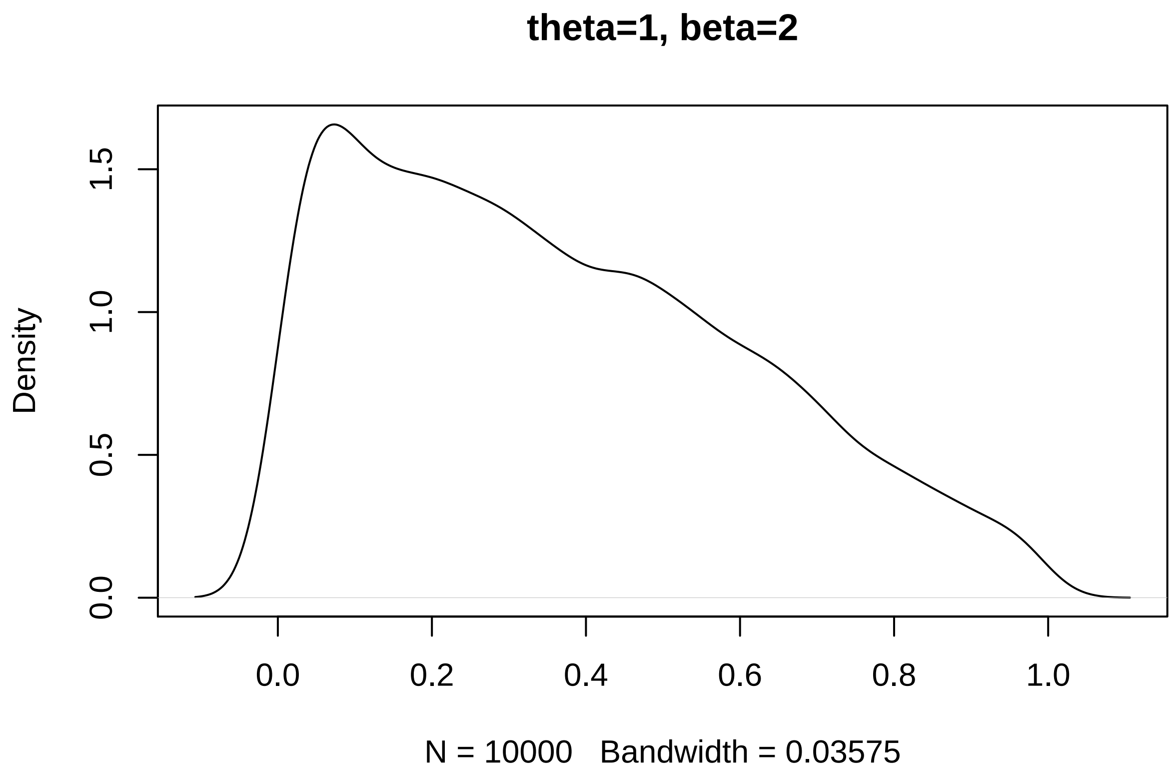
Step 1: sample $X \sim g$ and $U \sim U(0,1)$.

Step 2: if $U > \frac{q(x)}{\alpha g(x)}$, then go to step 1; else, return X.

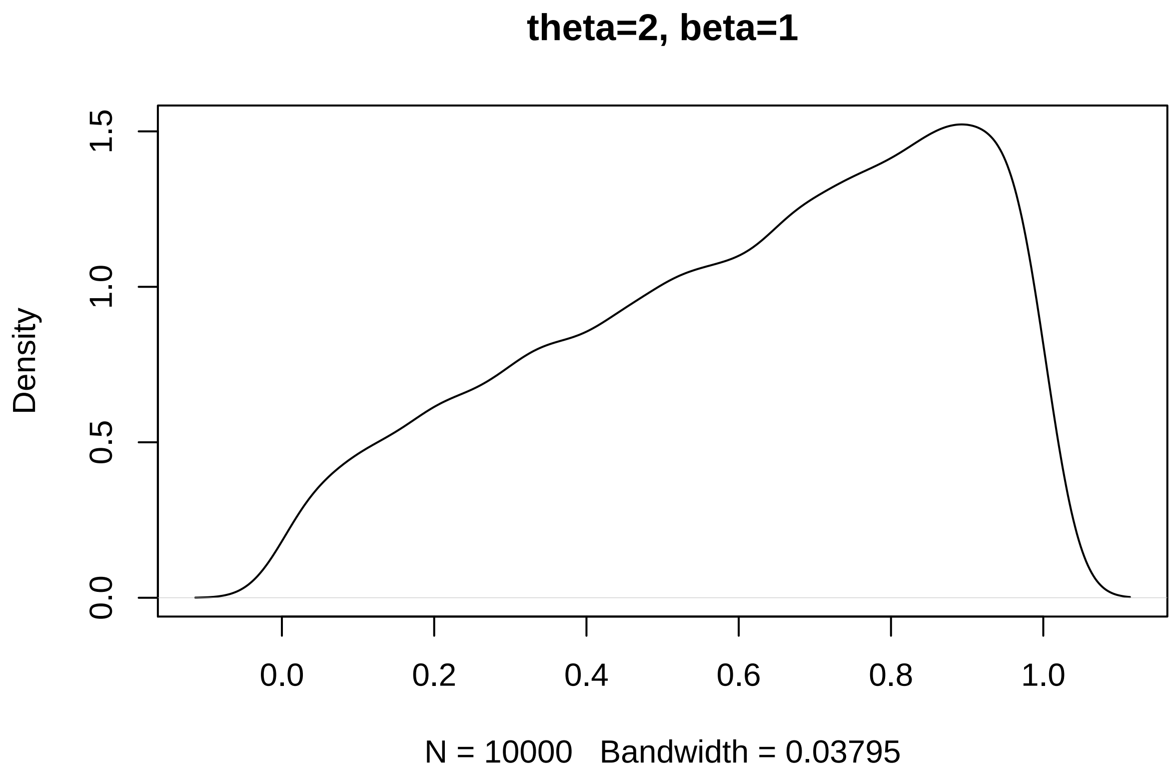
```
rg <- function(theta, beta, n){
  X <- NA
  for (i in 1:n){
    u <- runif(1,0,1)
    if (u < theta / (theta + beta)){
      X[i] <- rbeta(1, shape1 = theta, shape2 = 1)
    }else{
      X[i] <- rbeta(1, shape1 = 1, shape2 = beta)
    }
  }
  return(X)
}

rf <- function(theta, beta, n){
  y <- NA
  i <- 0
  while (i <= n){
     u <- runif(1, 0, 1)
     x <- rg(theta, beta, 1)
     if (u <= (x ^ (theta - 1) / (1 + x ^ 2) + sqrt(2 + x ^ 2) * (1 - x) ^ (beta - 1)) / (x ^ (theta - 
       y[i] <- x
       i <- i + 1
     }
  }
 return(y)
}

theta1  <- 1
beta1 <- 2
x1 <- rf(theta1, beta1, 10000)
plot(density(x1), main = "theta=1, beta=2")
```




```
theta2  <- 2
beta2 <- 1
x2 <- rf(theta2, beta2, 10000)
plot(density(x2), main = "theta=2, beta=1")
```



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$$q_1(x) = \frac{x^{\theta-1}}{1+x^2}$$
$$g_1(x) = \frac{x^{\theta-1}}{Beta(\theta,1)} = \frac{x^{\theta-1}}{\theta}$$
$$\frac{q_1(x)}{g_1(x)} = \frac{\theta}{1+x^2} \leq \theta = \alpha_1$$
$$\frac{q_1(x)}{\alpha_1 g_1(x)} = \frac{1}{1+x^2}$$
$$q_2(x) = \sqrt{2+x^2}(1-x)^{\beta-1}$$
$$g_2(x) = \frac{(1-x)^{\beta-1}}{Beta(1,\beta)} = \frac{(1-x)^{\beta-1}}{\beta}$$
$$\frac{q_2(x)}{g_2(x)} = \beta\sqrt{2+x^2} \leq \beta\sqrt{3} = \alpha_2$$
$$\frac{q_2(x)}{\alpha_2 g_2(x)} = \sqrt{\frac{2+x^2}{3}}$$

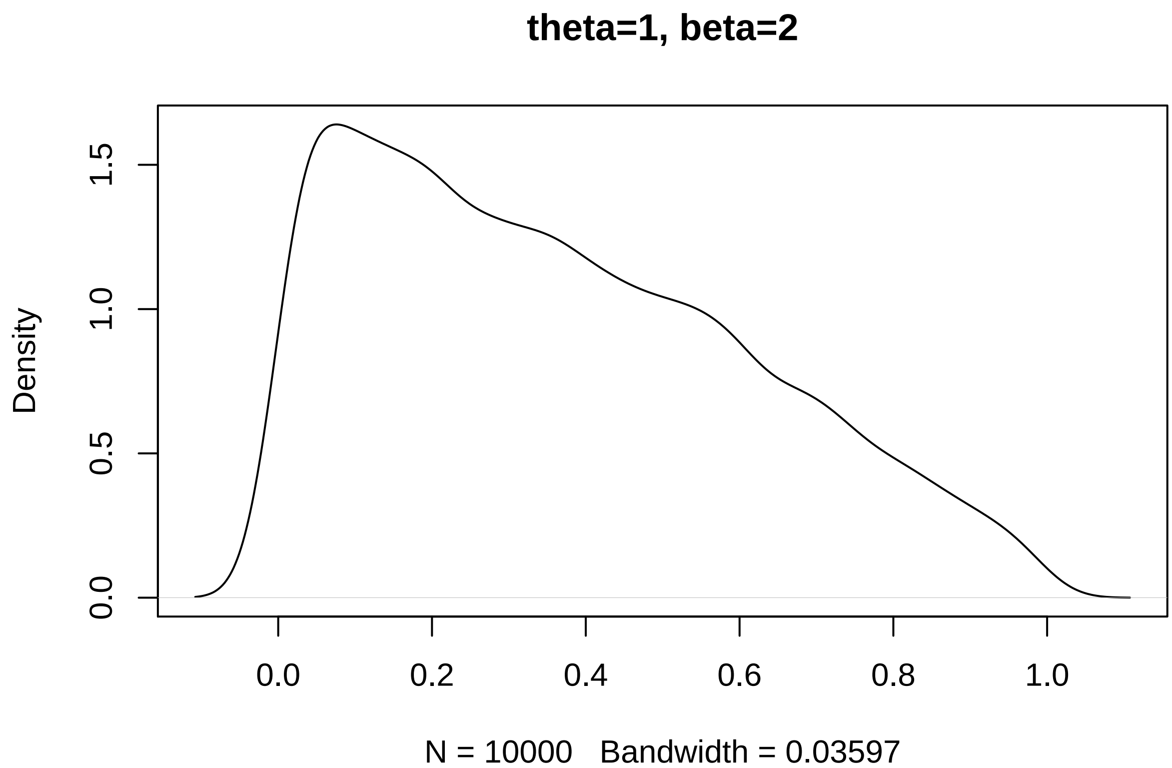
Step 1: sample k from $\{1,2\}$ with probability $\frac{\theta}{\theta+\beta\sqrt{3}}$ and $\frac{\beta\sqrt{3}}{\theta+\beta\sqrt{3}}$.

Step 2: sample $X \sim g_k$ and $U \sim U(0,1)$

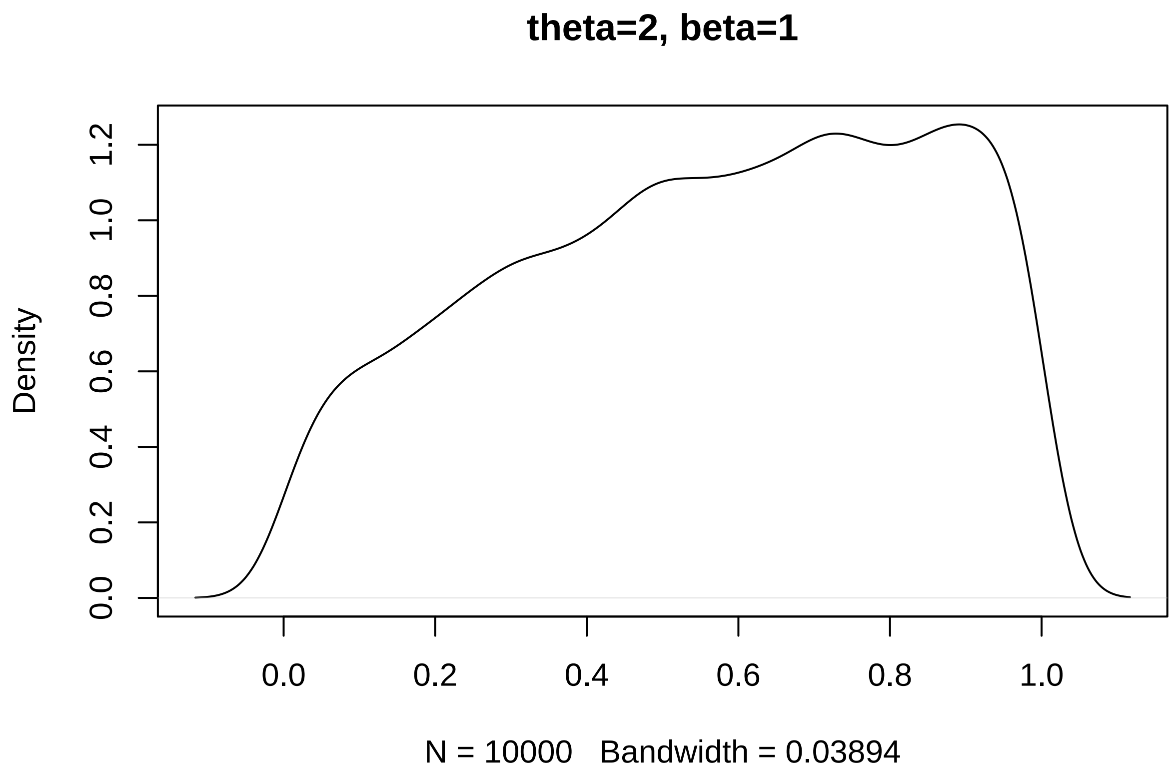
Step 3: if $U > \frac{q_k(x)}{\alpha_k g_k(x)}$ then go to step 1; else, return X.

```r
rf <- function(theta, beta, n){
  i <- 0
  y <- NA
  while (i <= n){
    p <- runif(1, 0, 1)
    if (p < theta / (theta + beta * sqrt(3))){
      u <- runif(1, 0, 1)
      x <- rbeta(1, shape1 = theta, shape2 = 1)
      if (u <= 1 / (1 + x ^ 2)){
        y[i] <- x
        i <- i + 1
      }
    }else{
      u <- runif(1, 0, 1)
      x <- rbeta(1, shape1 = 1, shape2 = beta)
      if (u <= sqrt( (2 + x ^ 2) / 3)) {
        y[i] <- x
        i <- i + 1
      }
    }
  }
  return (y)
}

theta1  <- 1
beta1 <- 2
x1 <- rf(theta1, beta1, 10000)
plot(density(x1), main = "theta=1, beta=2")
```




```
theta2  <- 2
beta2 <- 1
x2 <- rf(theta2, beta2, 10000)
plot(density(x2), main = "theta=2, beta=1")
```


theta=2, beta=1
Density
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0.0
0.2
0.4
0.6
0.8
1.0
N = 10000 Bandwidth = 0.03894