Random Number Generation

5361 Homework 6

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1 Rejection Sampling

1.1 Find Distribution

Let f and g be two probability densities on $(0, \infty)$, such that

$$f(x) \propto q(x) = \sqrt{4+x}\ x^{\theta-1}\ e^{-x}$$

$$g(x) \propto g_1(x) = (2x^{\theta-1} + x^{\theta-\frac{1}{2}})e^{-x}$$

$$C\int_0^\infty g_1 x\ \mathrm{d}x = C\int_0^\infty (2x^{\theta-1} + x^{\theta-\frac{1}{2}})e^{-x}\ \mathrm{d}x = 1$$

Proof:

$$\begin{aligned}
\int_0^\infty g_1(x)\ \mathrm{d}x &= 2\int_0^\infty x^{\theta-1}e^{-x}\mathrm{d}x + \int_0^\infty x^{\theta-\frac{1}{2}}e^{-x}\mathrm{d}x \\
&= 2\Gamma(\theta)\int_0^\infty \frac{x^{\theta-1}}{\Gamma(\theta)}e^{-x}\mathrm{d}x + \Gamma(\theta+\frac{1}{2})\int_0^\infty \frac{x^{\theta-\frac{1}{2}}}{\Gamma(\theta+\frac{1}{2})}e^{-x}\mathrm{d}x \\
&= 2\Gamma(\theta) + \Gamma(\theta+\frac{1}{2})
\end{aligned}$$

So $C[2\Gamma(\theta) + \Gamma(\theta+\frac{1}{2})] = 1 \Rightarrow C = \frac{1}{2\Gamma(\theta)+\Gamma(\theta+\frac{1}{2})}$, and the density function of $g(x)$ is

$$\begin{aligned}
g(x) &= \frac{1}{2\Gamma(\theta)+\Gamma(\theta+\frac{1}{2})}(2x^{\theta-1} + x^{\theta-\frac{1}{2}})e^{-x} \\
&= \frac{1}{2\Gamma(\theta)+\Gamma(\theta+\frac{1}{2})}(2\Gamma(\theta)\frac{x^{\theta-1}e^{-x}}{\Gamma(\theta)} + \Gamma(\theta+\frac{1}{2})\frac{x^{\theta-\frac{1}{2}}e^{-x}}{\Gamma(\theta+\frac{1}{2})}) \\
&= \frac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+\frac{1}{2})}\mathrm{Gamma}(\theta,1) + \frac{\Gamma(\theta+\frac{1}{2})}{2\Gamma(\theta)+\Gamma(\theta+\frac{1}{2})}\mathrm{Gamma}(\theta+\frac{1}{2},1)
\end{aligned}$$

1.2 Kernel Density Estimation of g

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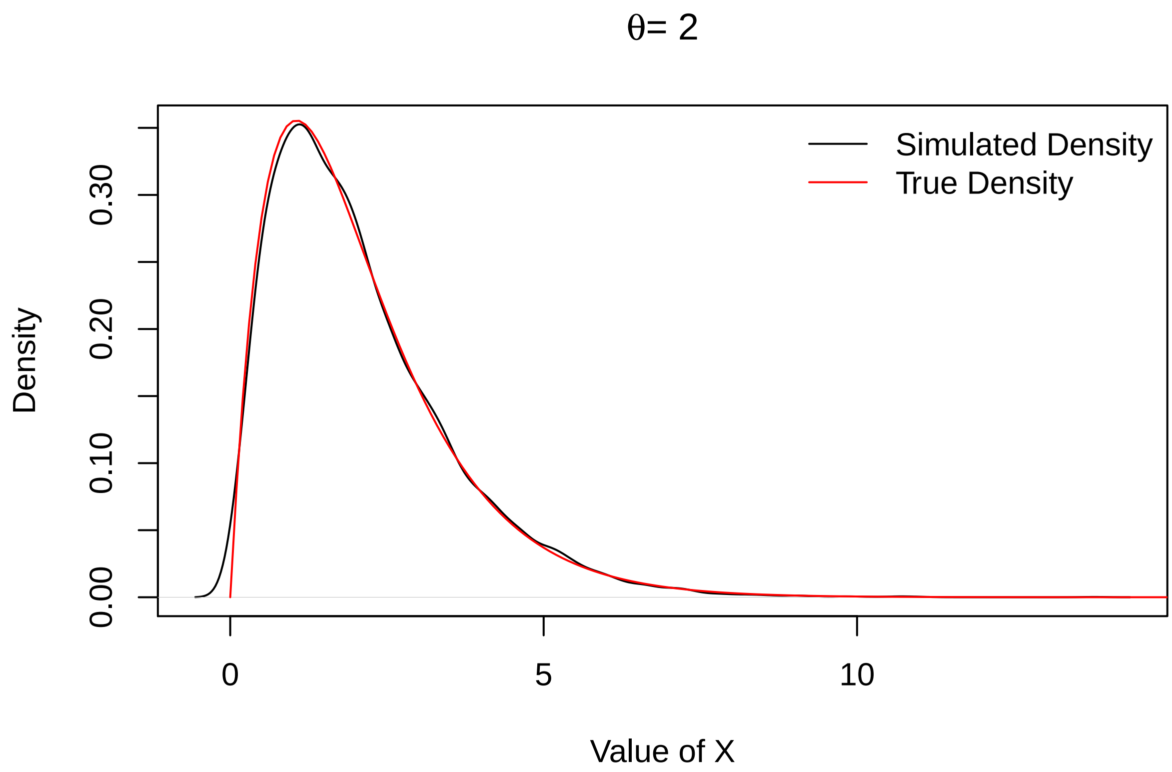
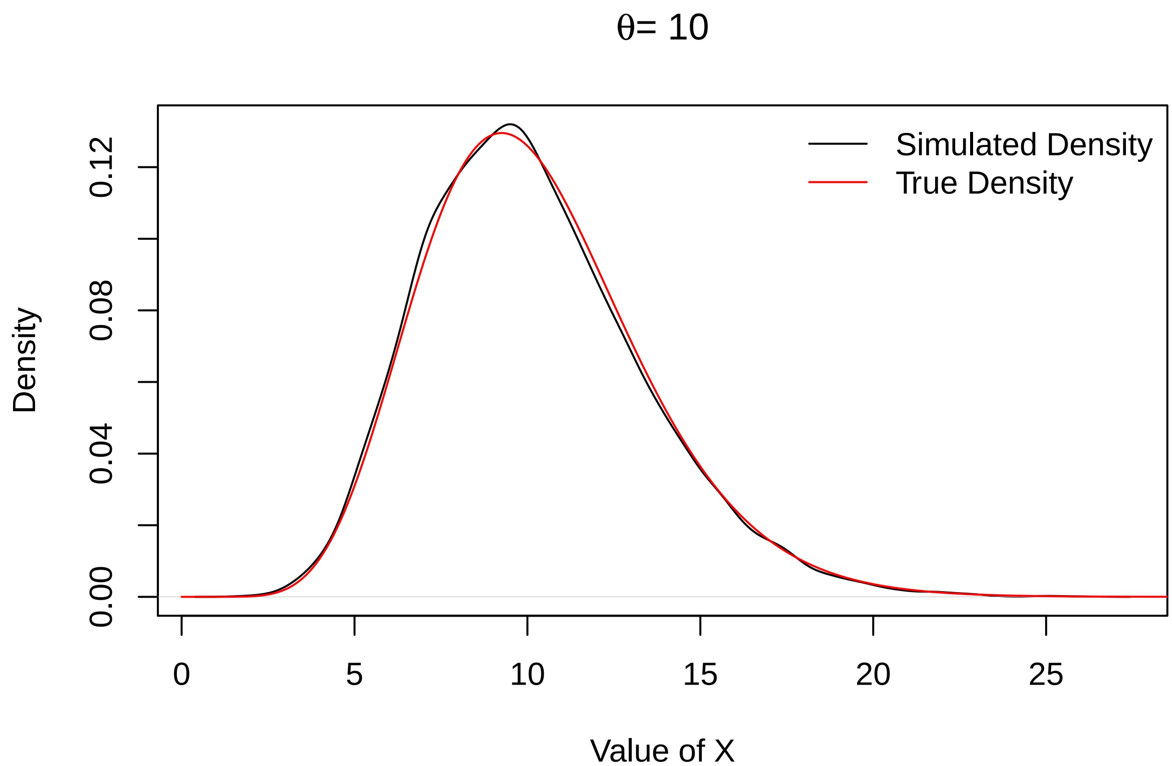
```
gdens <- function(n, theta, c1, c2){

g <- vector(length = n)
u <- runif(n, min=0, max=1)

for (i in 1:n) {
    if(u[i] > c2) {
      g1 <- rgamma(n = 1, shape = theta, scale = 1)
      g[i] <- c1*g1 + c2*g1
    }
    else {
      g2 <- rgamma(n = 1, shape = theta+1/2, scale = 1)
      g[i] <- c1*g2+c2*g2
    }
}
return(g)
}



### Theta = 1
n = 10000
theta = 2
c1 <- 2*gamma(theta)/(gamma(theta)+gamma(theta+1/2))
c2 <- 1-c1
x <- seq(0, 30, 0.1)
gd1 <- gdens(n = n, theta = theta, c1 = c1, c2 = c2)
gt1 <- c1*dgamma(x, shape = theta, scale = 1) + c2*dgamma(x, shape = theta+1/2, scale = 1)

plot(density(gd1), main = expression(paste(theta, "= 2")), xlab = "Value of X")
lines(x, gt1, col = "red")
legend("topright", box.lty = 0, legend = c("Simulated Density", "True Density"), col = c("black
```




```
### Theta = 10
n = 10000
theta = 10
c1 <- 2*gamma(theta)/(gamma(theta)+gamma(theta+1/2))
c2 <- 1-c1
x <- seq(0, 30, 0.1)
gd2 <- gdens(n = n, theta = theta, c1 = c1, c2 = c2)
gt2 <- c1*dgamma(x, shape = theta, scale = 1) + c2*dgamma(x, shape = theta+1/2, scale = 1)

plot(density(gd2), main = expression(paste(theta, "= 10")), xlab = "Value of X")
lines(x, gt2, col = "red")
legend("topright", box.lty = 0, legend = c("Simulated Density", "True Density"), col = c("black
```



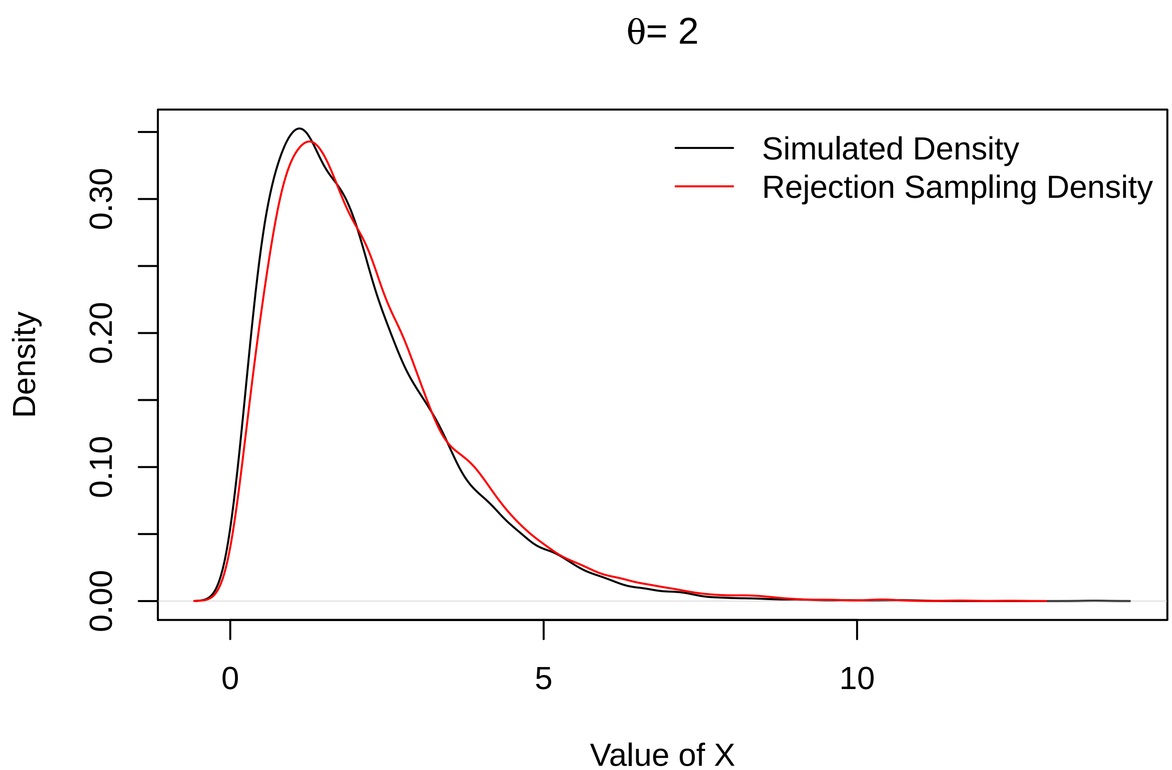
1.3 Kernel Density Estimation of Rejection Sampling

$$
\begin{aligned}
\alpha &= \sup_{x\geq 0} \frac{\sqrt{4+x}\; x^{\theta-1}\; e^{-x}}{C\left[(2x^{\theta-1}+x^{\theta-\frac{1}{2}})e^{-x}\right]} \\
&= \sup_{x\geq 0} \frac{\sqrt{4+x}}{C(2+\sqrt{x})} \\
&= \frac{1}{C}
\end{aligned}
$$

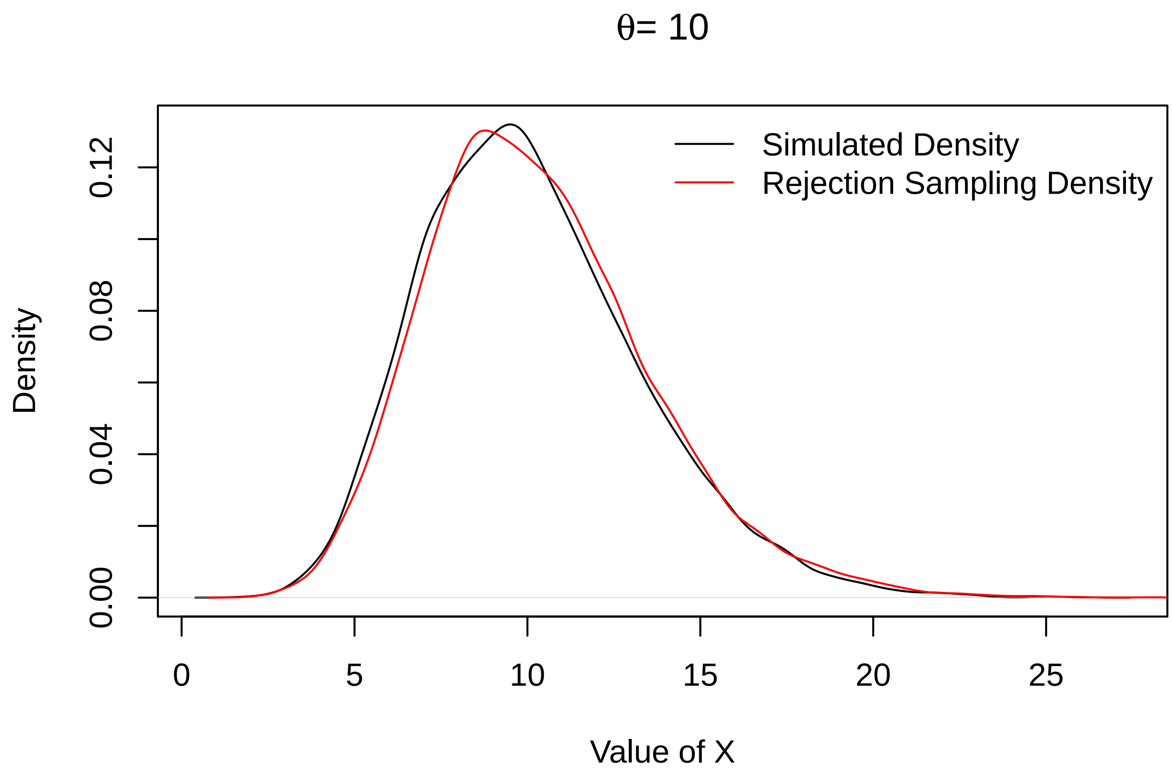
So let $\alpha = \frac{1}{C}$, $\frac{q(x)}{\alpha g(x)} = \frac{Cq(x)}{g(x)}$

```
reject.g <- function(n, theta, c1, c2){
  x_new <- vector(length = n)
  i <- 1
  while (i <= n) {
    u <- runif(1)
    temp <- gdens(1, theta, c1, c2)
    qx <- (4+temp)^0.5*temp^(theta-1)*exp(-temp)
    gx <- c1*dgamma(temp, shape = theta, scale = 1) + c2*dgamma(temp, shape = theta+1/2, scale
    qg <- (1/(gamma(theta)+gamma(theta+1/2)))*qx/gx
    if (u <= qg){
```

```
    x_new[i] <- temp
    i <- i + 1
  }
 }
 x_new
}

xv1 <- reject.g(n, theta = 2, c1, c2)
plot(density(gd1), main = expression(paste(theta, "= 2")), xlab = "Value of X")
lines(density(xv1), col = "red")
legend("topright", box.lty = 0, legend = c("Simulated Density", "Rejection Sampling Density"),
```




```
xv2 <- reject.g(n, theta = 10, c1, c2)
plot(density(gd2), main = expression(paste(theta, "= 10")), xlab = "Value of X")
lines(density(xv2), col = "red")
legend("topright", box.lty = 0, legend = c("Simulated Density", "Rejection Sampling Density"),
```



2 Mixture Proposal

Let f be a probability density on $(0, 1)$ such that

$$f(x) \propto \frac{x^{\theta-1}}{1+x^2} + \sqrt{2+x^2}(1-x)^{\beta-1}$$

where $0 < x < 1$.

2.1 Procedure Design

Since

$$q(x) = \frac{x^{\theta-1}}{1+x^2} + \sqrt{2+x^2}(1-x)^{\beta-1} \leq x^{\theta-1} + \sqrt{3}(1-x)^{\beta-1}$$

,

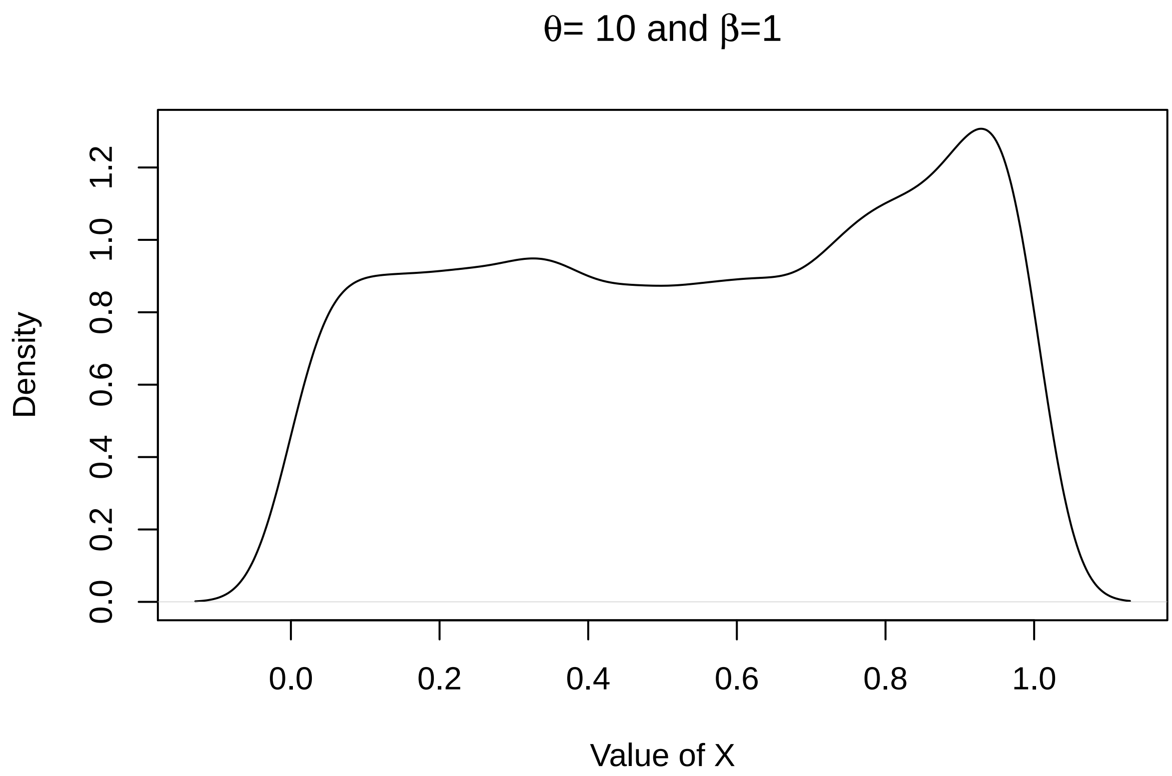
$$g(x) = C(x^{\theta-1} + \sqrt{3}(1-x)^{\beta-1})$$

$$\int_0^1 g(x)\ \mathrm{d}x = 1$$

$$\begin{aligned}
\int_0^1 C(x^{\theta-1} + \sqrt{3}(1-x)^{\beta-1})\ \mathrm{d}x &= C\left[\int_0^1 x^{\theta-1}\ \mathrm{d}x + \int_0^1 \sqrt{3}(1-x)^{\beta-1}\ \mathrm{d}x\right] \\
&= C\left[\mathrm{Beta}(\theta,1) + \sqrt{3}\mathrm{Beta}(1,\beta)\right] \\
&= 1
\end{aligned}$$

$$\Rightarrow C = \frac{1}{\mathrm{Beta}(\theta,1) + \sqrt{3}\mathrm{Beta}(1,\beta)}$$

```r
bdens <- function(n, theta, beta, p1, p2){

  b <- vector(length = n)
  u <- runif(n, min=0, max=1)

  for (i in 1:n) {
    if(u[i] > p2) {
      b1 <- rbeta(n = 1, shape1 = theta, shape2 = 1)
      b[i] <- p1*b1 + p2*b1
    }
    else {
      b2 <- rbeta(n = 1, shape1 = 1, shape2 = beta)
      b[i] <- p1*b2+p2*b2
    }
  }
  return(b)
}

n = 10000
theta = 10
beta = 1
p1 <- beta(theta, 1)/(beta(theta, 1)+beta(1, beta))
p2 <- 1-p1

x <- seq(0, 30, 0.1)
bd <- bdens(n = n, theta = theta, beta = beta, p1 = p1, p2 = p2)

plot(density(bd), main = expression(paste(theta, "= 10 and ", beta, "=1")), xlab = "Value of X'
```



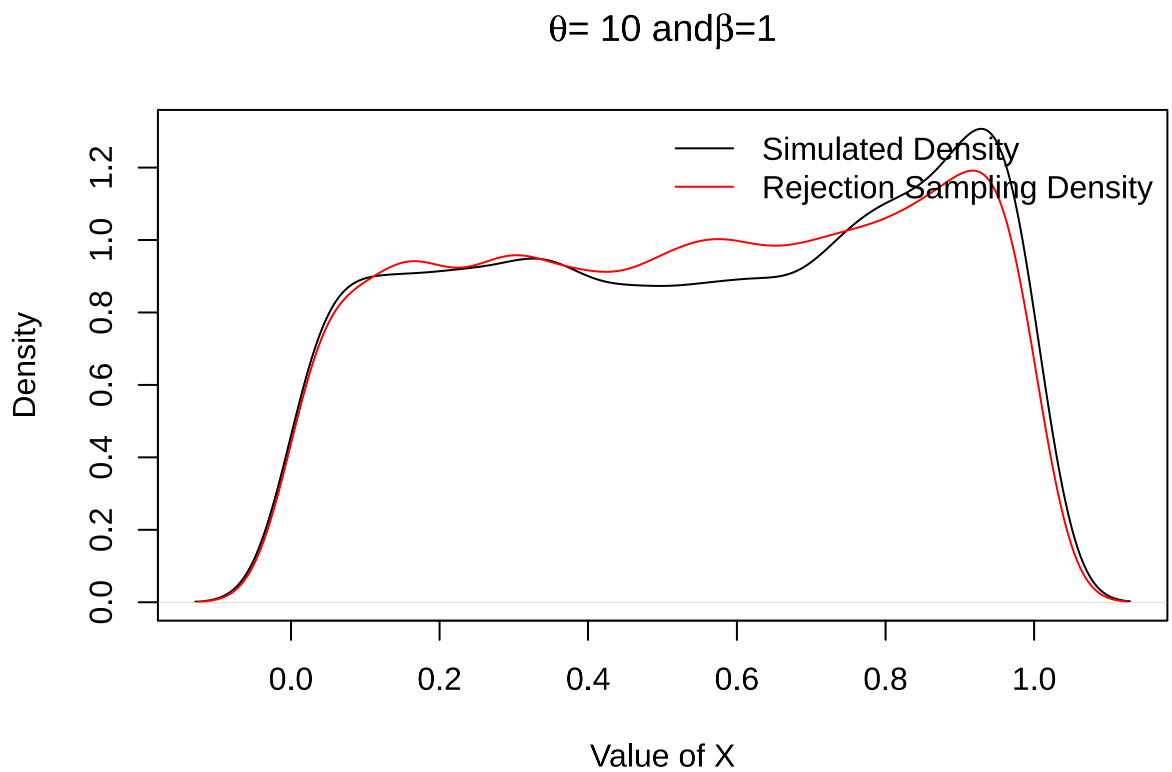
2.2 Rejection Sampling

Let $q_1 x = \frac{x^{\theta-1}}{1+x^2}$, $g_1 x = \frac{x^{\theta-1}}{\text{Beta}(\theta,1)}$,$q_2 x = \sqrt{2+x^2}(1-x)^{\beta-1}$, $g_2 x = \sqrt{3}\frac{(1-x)^{\beta-1}}{\text{Beta}(1,\beta)}$

$$\alpha_1 = \sup_{x\geq 0} \frac{q_1 x}{C_1 g_1 x} = \text{Beta}(\theta, 1)$$
$$\alpha_2 = \sup_{x\geq 0} \frac{q_2 x}{C_2 g_2 x} = \sqrt{3}\text{Beta}(1, \beta)$$

```
reject.b <- function(n, theta, beta, p1, p2){
  x_new1 <- vector(length = n)

  i <- 1
  while (i <= n) {
    u <- runif(1)
    u1 <- runif(1)
    if (u > p2) {
      temp <- rbeta(1, shape1 = theta, shape2 = 1)
      qx <- temp^(theta-1)/(1+temp^2)
      gx <- temp^(theta-1)/beta(theta, 1)
      qg <- qx/gx/ap1
      if (u1 <= qg){
```

```
        x_new1[i] <- temp
        i <- i + 1
      }
    }
    else {
      temp <- rbeta(1, shape1 = 1, shape2 = beta)
      qx <- (2+temp^2)^0.5*(1-temp)^(beta-1)
      gx <- (1-temp)^(beta-1)/beta(1, beta)
      qg <- qx/gx/ap2
      if (u1 <= qg){
        x_new1[i] <- temp
        i <- i + 1
      }
    }

  }
  x_new1
}

n <- 10000
ap1 <- beta(theta, 1)
ap2 <- 3^0.5*beta(1, beta)
p1 <- ap1/(ap1+ap2)
p2 <- 1-p1
xv3 <- reject.b(n, theta = 10, beta = 1, p1, p2)

plot(density(bd), main = expression(paste(theta, "= 10 and", beta, "=1")), xlab = "Value of X")
lines(density(xv3), col = "red")
legend("topright", box.lty = 0, legend = c("Simulated Density", "Rejection Sampling Density"),
```


θ= 10 andβ=1
Simulated Density
Rejection Sampling Density
Density
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0.0
0.2
0.4
0.6
0.8
1.0
Value of X