Random Sampling

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5.2.1

1.

$$
\begin{aligned}
\int_0^\infty g(x)dx &= \int_0^\infty (2x^{\theta-1} + x^{\theta-1/2})e^{-x}dx && (1)\\
&= 2\int_0^\infty x^{\theta-1}e^{-x}dx + \int_0^\infty x^{\theta-1/2}e^{-x}dx && (2)\\
&= 2\Gamma(\theta) + \Gamma(\theta+1/2) && (3)
\end{aligned}
$$

Since, $C = \frac{1}{\int_0^\infty g(x)dx}$, then $C = \frac{1}{2\Gamma(\theta)+\Gamma(\theta+1/2)}$.

Plug C into g(x), we have:

$$g(x) = \frac{1}{2\Gamma(\theta)+\Gamma(\theta+1/2)}(2x^{\theta-1} + x^{\theta-1/2})e^{-x}$$

$$g(x) = \frac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+1/2)}\frac{2x^{\theta-1}e^{-x}}{\Gamma(\theta)} + \frac{\Gamma(\theta+1/2)}{2\Gamma(\theta)+\Gamma(\theta+1/2)}\frac{x^{\theta-1/2}e^{-x}}{\Gamma(\theta+1/2)}$$

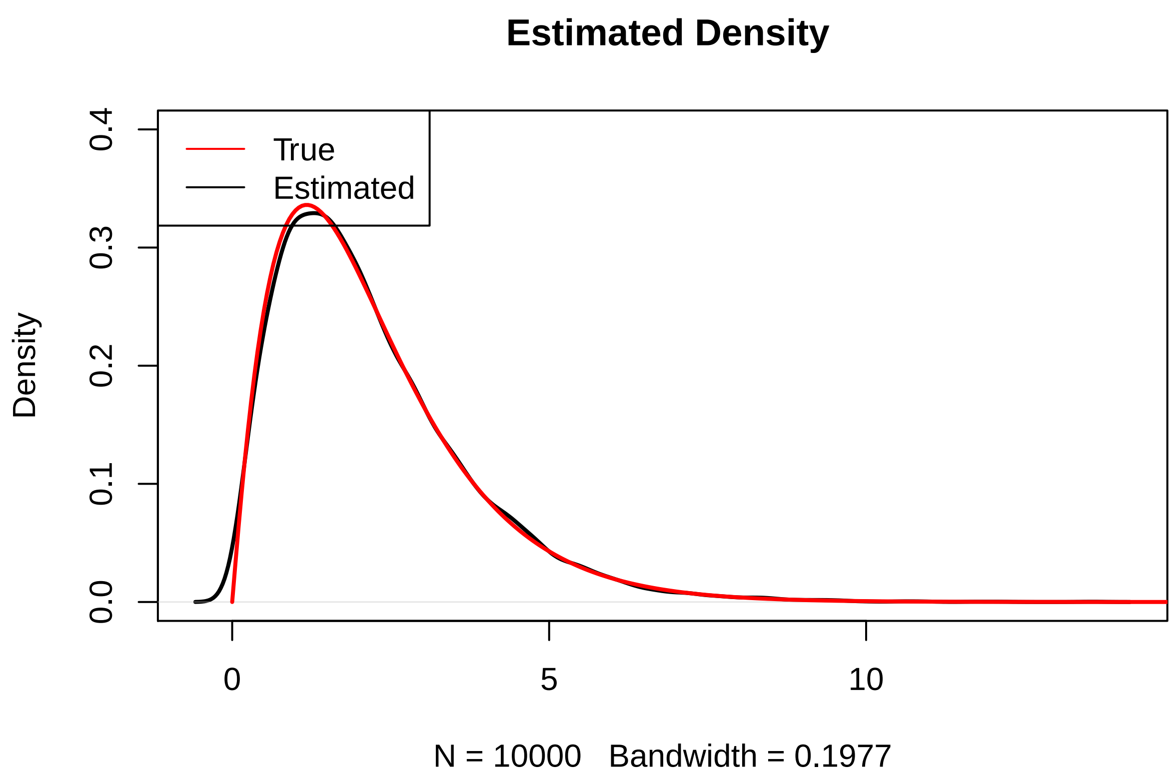
Thus, g(x) is a mixture of Gamma(θ,1) and Gamma($\theta + 0.5$,1). Their weight are $\frac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+1/2)}$ and $\frac{\Gamma(\theta+1/2)}{2\Gamma(\theta)+\Gamma(\theta+1/2)}$ respectively.

2.

Let w and 1-w be the weight of these two component Gamma distribution respectively. After randomly generating 10000 numbers from U(0,1), if U<w, sample X~Gamma(θ,1), otherwise sample X~Gamma($\theta + 0.5$,1).

```
theta <- 2
w <- 2*gamma(theta) / (2*gamma(theta) + gamma(theta + 0.5))
U <- runif(10000,min = 0,max = 1)
sample <- rep(0,10000)
for(i in 1:10000){
  if(U[i] < w){
    sample[i] <- rgamma(1,theta,1)
  }
  else{
    sample[i] <- rgamma(1,theta + 0.5,1)
  }
```

```
}
plot(density(sample), main = " Estimated Density",ylim = c(0,.4),lwd = 2)
x <- seq(0,20,0.05)
lines(x,w*dgamma(x,theta, 1) + (1-w) * dgamma(x,theta + 0.5, 1),col = "red",lwd = 2)
legend("topleft", c("True","Estimated"), col = c("red","black"),cex = 1,lwd = 1)
```

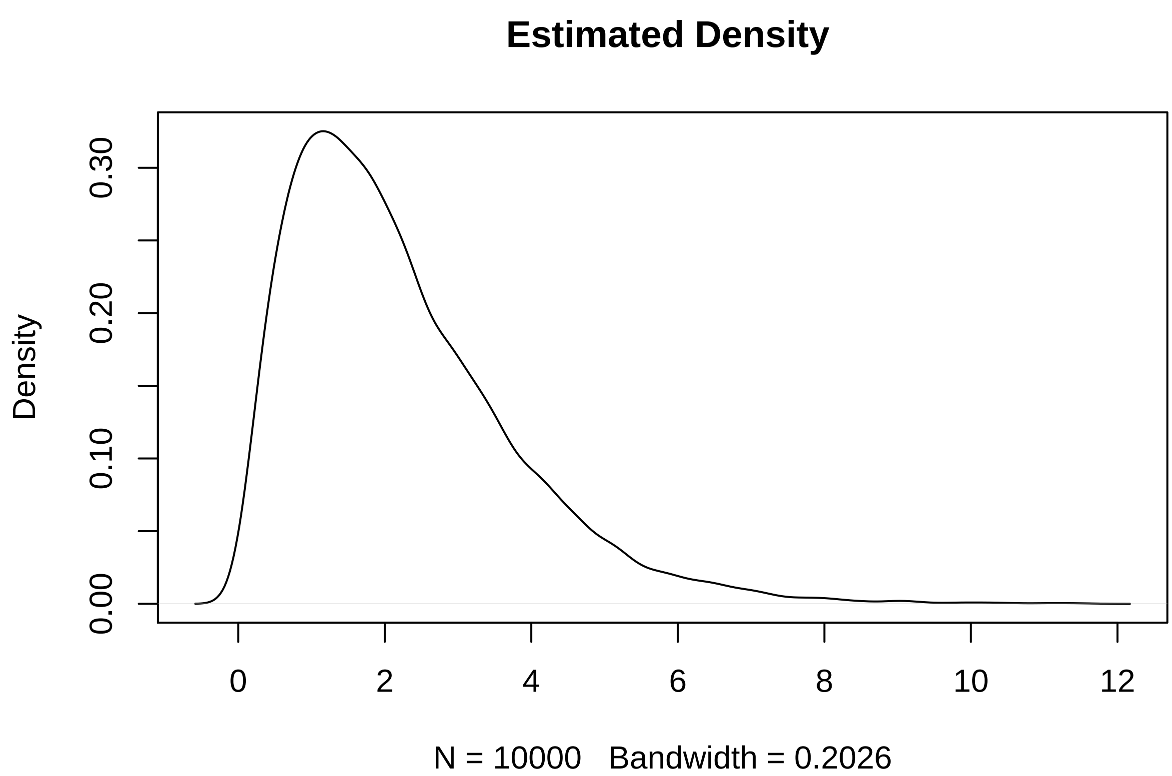


3.

First we know f(x) is proportional to $q(x) = \sqrt{x+4}x^{\theta-1}e^{-x}$, and g(x) is the "instrumental" density, $g(x) = \frac{1}{2\Gamma(\theta)+\Gamma(\theta+1/2)}(2x^{\theta-1} + x^{\theta-1/2})e^{-x}$. Then all we need to determine α, $\alpha = sup\frac{q(x)}{g(x)}$. After calculating the maximum of the ratio, we have $\alpha = \frac{\sqrt{2}*(2\Gamma(\theta)+\Gamma(\theta+1/2))}{2}$. Finally we will use the same method as above by comparing U and $\frac{q(x)}{\alpha g(x)}$.

```
theta <- 2
w <- 2*gamma(theta) / (2*gamma(theta) + gamma(theta + 0.5))
C <- 1 / (2 * gamma(theta) + gamma(theta + 0.5))
alpha <- sqrt(2) /(2*C)
sample <- rep(0,10000)
for (i in 1:10000) {
  U1 <- runif(1,0,1)
  if(U1 < w){
```

```r
    X <- rgamma(1,theta,1)
  }
  else{
    X <- rgamma(1,theta + 0.5,1)
  }
  U2 <- runif(1,0,1)
  g <- C * (2 * X ^ (theta - 1) + X ^ (theta - 0.5)) * exp(-1 * X)
  q <- sqrt(4 + X) * X^(theta - 1) * exp(-1 * X)
  r <- q/(alpha*g)
  if(U2<=r){
    sample[i] <- X
    i <- i+1
  }
}
plot(density(sample), main = " Estimated Density")
```



5.2.2

1.

As we all know, density of Beta distribution is

$$\frac{X^(\alpha - 1)}{B(\alpha, \beta)}(1 - X)^(\beta - 1)$$

Now we are given that f(x) is proportional to

$$q(x) = \frac{x^{\theta-1}}{1+x^2} + \sqrt{2+x^2}(1-x)^{\beta-1}$$

q(x) contains two key components of Beta distribution, so I will set

$$g(x) = w * Be(\theta, 1) + (1 - w) * Be(1, \beta)$$

Since $q(x) \leq x^{\theta-1} + 2(1-x)^{\beta-1}$, then $g(x) = C(x^{\theta-1} + 2(1-x)^{\beta-1})$. Now we need to calculte C.

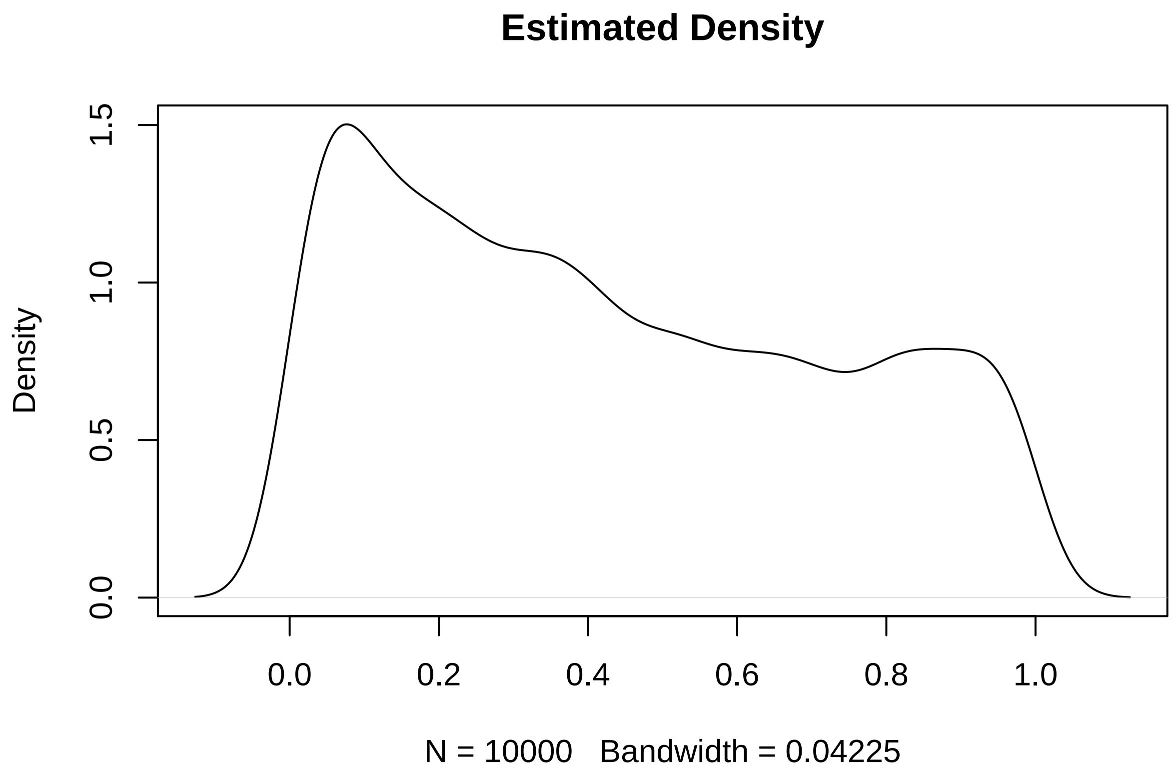
$$\begin{aligned}\int_0^1 (x^{\theta-1} + 2(1-x)^{\beta-1})dx &= \int_0^1 x^{\theta-1}dx + \int_0^1 2(1-x)^{\beta-1}dx \\ &= B(\theta, 1) + 2B(1, \beta)\end{aligned}$$

Thus, we have $C = \frac{1}{B(\theta,1)+2B(1,\beta)}$.

$$\begin{aligned}g(x) &= C(x^{\theta-1} + 2(1-x)^{\beta-1}) \\ &= \frac{1}{B(\theta,1)+2B(1,\beta)}(x^{\theta-1} + 2(1-x)^{\beta-1}) \\ &= \frac{B(\theta,1)}{B(\theta,1)+2B(1,\beta)}\frac{x^{\theta-1}}{B(\theta,1)} + \frac{2B(1,\beta)}{B(\theta,1)+2B(1,\beta)}\frac{(1-x)^{\beta-1}}{B(1,\beta)}\end{aligned}$$

Hence, $w = \frac{B(\theta,1)}{B(\theta,1)+2B(1,\beta)}$.

```
rng_f <- function(theta,beta){
  U <- runif(10000,0,1)
  w <- beta(theta,1)/(beta(theta,1) + 2*beta(1,beta))
  sample <- rep(0,10000)
  for (i in 1:10000) {
    if(U[i]<w){sample[i] <- rbeta(1,theta,1)}
    else{sample[i] <- rbeta(1,1,beta)}
  }
  return(sample)
}
s_f <- rng_f(2,3) #We assume theta=2, beta=3
plot(density(s_f),main = "Estimated Density")
```



2.

Now, we have f(x) is proportional to [q1(x)+q2(x)], where $q1(x) = \frac{x^{\theta-1}}{1+x^2}$ and $q2(x) = \sqrt{2+x^2}(1-x)^{\beta-1}$. As how we derive g(x) above, similarly, we can easily get:

$$g1(x) = Be(\theta, 1), \alpha 1 = B(\theta, 1)$$

$$g2(x) = Be(1, \beta), \alpha 2 = 2B(1, \beta)$$

```
samplef <- rep(0,10000)
theta <- 2
beta <- 3
alpha_1 <- beta(theta,1)
alpha_2 <- 2*beta(1,beta)
w <- alpha_1 / (alpha_1 + alpha_2)
i <- 1
while(i <= 10000){
  U <- runif(1)
  if(U < w){
    X <- rbeta(1,theta,1)
    U1 <- runif(1)
    q <- X^(theta-1) / (1 + X^2)
```

```
    g <- X^(theta-1) / beta(theta,1)
    r <- q / (alpha_1 * g)
  }
  else{
    X <- rbeta(1,1,beta)
    U2 <- runif(1)
    q <- sqrt(2 + X^2)*(1 - X)^(beta - 1)
    g <- (1 - X)^(beta - 1) / beta(1, beta)
    r <- q / (alpha_2 * g)
  }
  if ( U > r ) { next }
  samplef[i] <- X
  i <- i+1
}
plot(density(samplef), main = "Estimated Density")
```