

Random Sampling

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5.2.1

1.

$$\int_0^\infty g(x)dx = \int_0^\infty (2x^{\theta-1} + x^{\theta-1/2})e^{-x}dx \quad (1)$$

$$= 2 \int_0^\infty x^{\theta-1}e^{-x}dx + \int_0^\infty x^{\theta-1/2}e^{-x}dx \quad (2)$$

$$= 2\Gamma(\theta) + \Gamma(\theta + 1/2) \quad (3)$$

Since, $C = \frac{1}{\int_0^\infty g(x)dx}$, then $C = \frac{1}{2\Gamma(\theta) + \Gamma(\theta + 1/2)}$.

Plug C into g(x), we have:

$$g(x) = \frac{1}{2\Gamma(\theta) + \Gamma(\theta + 1/2)} (2x^{\theta-1} + x^{\theta-1/2})e^{-x}$$

$$g(x) = \frac{2\Gamma(\theta)}{2\Gamma(\theta) + \Gamma(\theta + 1/2)} \frac{2x^{\theta-1}e^{-x}}{\Gamma(\theta)} + \frac{\Gamma(\theta + 1/2)}{2\Gamma(\theta) + \Gamma(\theta + 1/2)} \frac{x^{\theta-1/2}e^{-x}}{\Gamma(\theta + 1/2)}$$

Thus, g(x) is a mixture of Gamma($\theta, 1$) and Gamma($\theta + 0.5, 1$). Their weight are $\frac{2\Gamma(\theta)}{2\Gamma(\theta) + \Gamma(\theta + 1/2)}$ and $\frac{\Gamma(\theta + 1/2)}{2\Gamma(\theta) + \Gamma(\theta + 1/2)}$ respectively.

2.

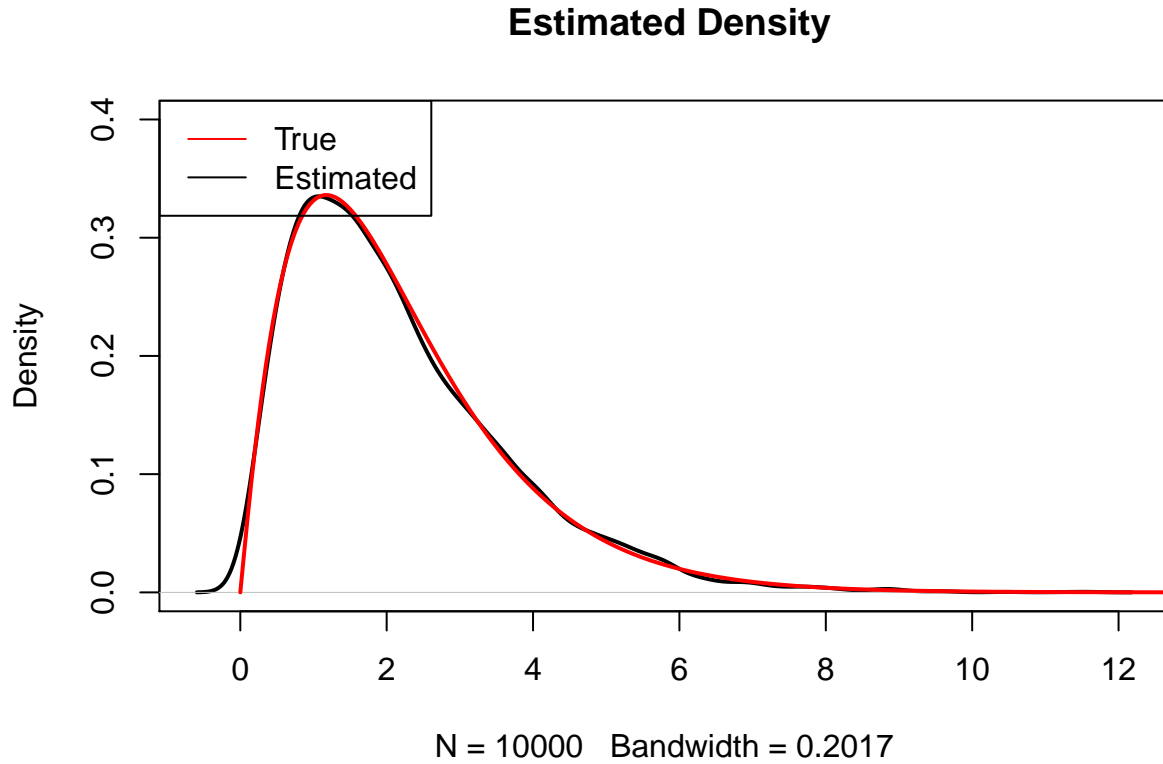
Let w and 1-w be the weight of these two component Gamma distribution respectively. After randomly generating 10000 numbers from U(0,1), if $U < w$, sample $X \sim \text{Gamma}(\theta, 1)$, otherwise sample $X \sim \text{Gamma}(\theta + 0.5, 1)$.

```
theta <- 2
w <- 2*gamma(theta) / (2*gamma(theta) + gamma(theta + 0.5))
U <- runif(10000,min = 0,max = 1)
sample <- rep(0,10000)
for(i in 1:10000){
  if(U[i] < w){
    sample[i] <- rgamma(1,theta,1)
  }
  else{
    sample[i] <- rgamma(1,theta + 0.5,1)
  }
}
```

```

}
plot(density(sample), main = " Estimated Density",ylim = c(0,.4),lwd = 2)
x <- seq(0,20,0.05)
lines(x,w*dgamma(x,theta, 1) + (1-w) * dgamma(x,theta + 0.5, 1),col = "red",lwd = 2)
legend("topleft", c("True","Estimated"), col = c("red","black"),cex = 1,lwd = 1)

```



3.

First we know $f(x)$ is proportional to $q(x) = \sqrt{x + 4}x^{\theta-1}e^{-x}$, and $g(x)$ is the “instrumental” density, $g(x) = \frac{1}{2\Gamma(\theta)+\Gamma(\theta+1/2)}(2x^{\theta-1} + x^{\theta-1/2})e^{-x}$. Then all we need to determine α , $\alpha = \sup \frac{q(x)}{g(x)}$. After calculating the maximum of the ratio, we have $\alpha = \frac{\sqrt{2}*(2\Gamma(\theta)+\Gamma(\theta+1/2))}{2}$. Finally we will use the same method as above by comparing U and $\frac{q(x)}{\alpha g(x)}$.

```

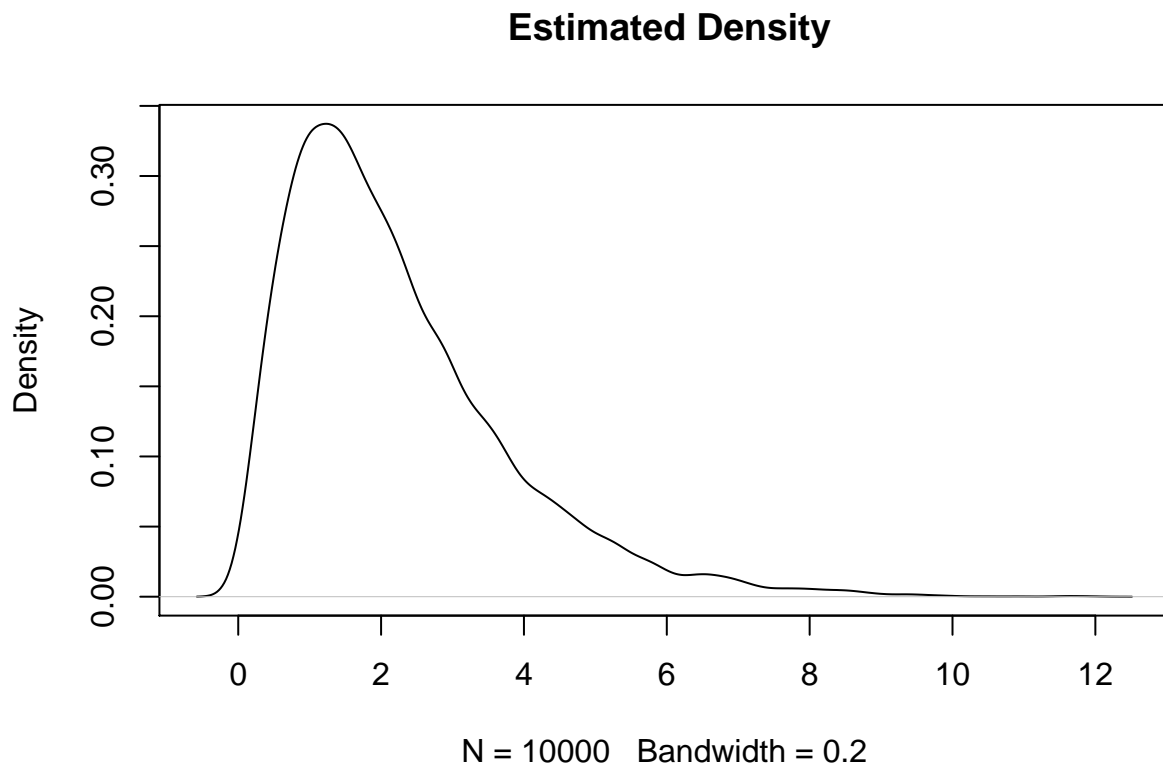
theta <- 2
w <- 2*gamma(theta) / (2*gamma(theta) + gamma(theta + 0.5))
C <- 1 / (2 * gamma(theta) + gamma(theta + 0.5))
alpha <- sqrt(2) / (2*C)
sample <- rep(0,10000)
for (i in 1:10000) {
  U1 <- runif(1,0,1)
  if(U1 < w){

```

```

X <- rgamma(1,theta,1)
}
else{
  X <- rgamma(1,theta + 0.5,1)
}
U2 <- runif(1,0,1)
g <- C * (2 * X ^ (theta - 1) + X ^ (theta - 0.5)) * exp(-1 * X)
q <- sqrt(4 + X) * X^(theta - 1) * exp(-1 * X)
r <- q/(alpha*g)
if(U2<=r){
  sample[i] <- X
  i <- i+1
}
}
plot(density(sample), main = " Estimated Density")

```



5.2.2

1.

As we all know, density of Beta distribution is

$$\frac{X^{\alpha-1}}{B(\alpha, \beta)}(1-X)^{\beta-1}$$

Now we are given that $f(x)$ is proportional to

$$q(x) = \frac{x^{\theta-1}}{1+x^2} + \sqrt{2+x^2}(1-x)^{\beta-1}$$

$q(x)$ contains two key components of Beta distribution, so I will set

$$g(x) = w * Be(\theta, 1) + (1-w) * Be(1, \beta)$$

Since $q(x) \leq x^{\theta-1} + 2(1-x)^{\beta-1}$, then $g(x) = C(x^{\theta-1} + 2(1-x)^{\beta-1})$. Now we need to calculate C.

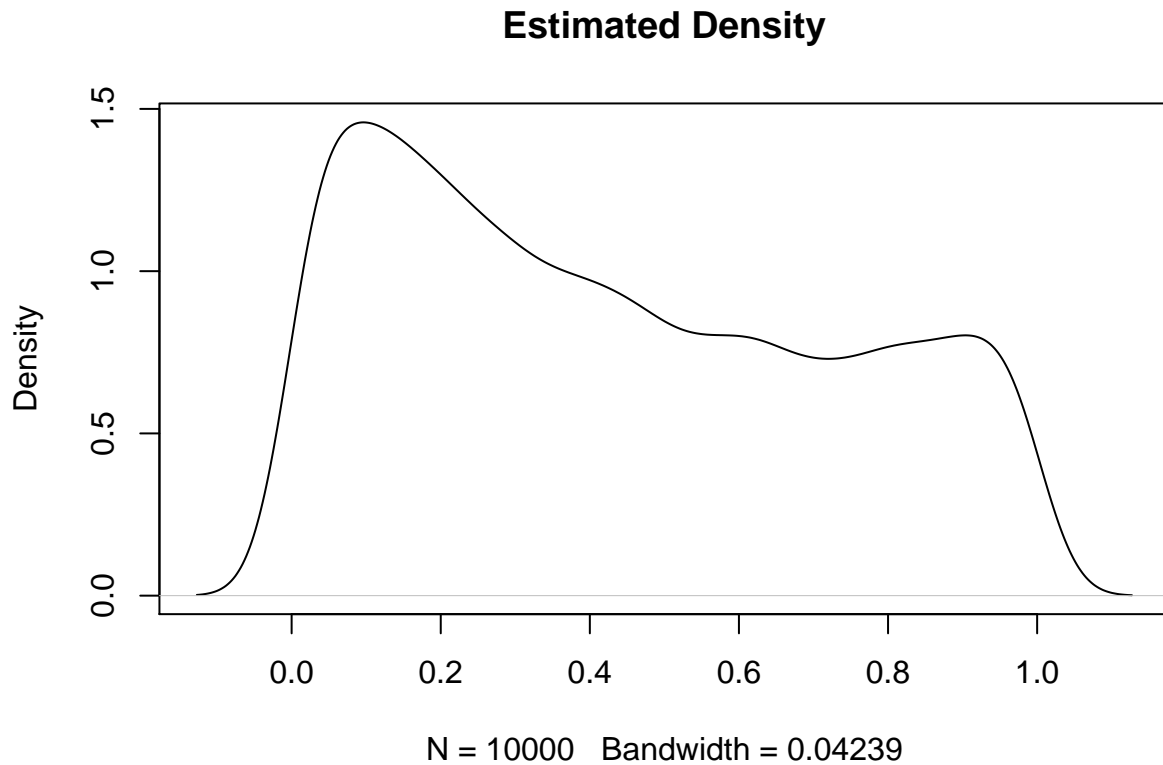
$$\begin{aligned} \int_0^1 (x^{\theta-1} + 2(1-x)^{\beta-1}) dx &= \int_0^1 x^{\theta-1} dx + \int_0^1 2(1-x)^{\beta-1} dx \\ &= B(\theta, 1) + 2B(1, \beta) \end{aligned}$$

Thus, we have $C = \frac{1}{B(\theta, 1) + 2B(1, \beta)}$.

$$\begin{aligned} g(x) &= C(x^{\theta-1} + 2(1-x)^{\beta-1}) \\ &= \frac{1}{B(\theta, 1) + 2B(1, \beta)}(x^{\theta-1} + 2(1-x)^{\beta-1}) \\ &= \frac{B(\theta, 1)}{B(\theta, 1) + 2B(1, \beta)} \frac{x^{\theta-1}}{B(\theta, 1)} + \frac{2B(1, \beta)}{B(\theta, 1) + 2B(1, \beta)} \frac{(1-x)^{\beta-1}}{B(1, \beta)} \end{aligned}$$

Hence, $w = \frac{B(\theta, 1)}{B(\theta, 1) + 2B(1, \beta)}$.

```
rng_f <- function(theta,beta){
  U <- runif(10000,0,1)
  w <- beta(theta,1)/(beta(theta,1) + 2*beta(1,beta))
  sample <- rep(0,10000)
  for (i in 1:10000) {
    if(U[i]<w){sample[i] <- rbeta(1,theta,1)}
    else{sample[i] <- rbeta(1,1,beta)}
  }
  return(sample)
}
s_f <- rng_f(2,3) #We assume theta=2, beta=3
plot(density(s_f),main = "Estimated Density")
```



2.

Now, we have $f(x)$ is proportional to $[q1(x)+q2(x)]$, where $q1(x) = \frac{x^{\theta-1}}{1+x^2}$ and $q2(x) = \sqrt{2+x^2}(1-x)^{\beta-1}$. As how we derive $g(x)$ above, similarly, we can easily get:

$$g1(x) = Be(\theta, 1), \alpha1 = B(\theta, 1)$$

$$g2(x) = Be(1, \beta), \alpha2 = 2B(1, \beta)$$

```
samplef <- rep(0,10000)
theta <- 2
beta <- 3
alpha_1 <- beta(theta,1)
alpha_2 <- 2*beta(1,beta)
w <- alpha_1 / (alpha_1 + alpha_2)
i <- 1
while(i <= 10000){
  U <- runif(1)
  if(U < w){
    X <- rbeta(1,theta,1)
    U1 <- runif(1)
    q <- X^(theta-1) / (1 + X^2)
```

```

    g <- X^(theta-1) / beta(theta,1)
    r <- q / (alpha_1 * g)
  }
  else{
    X <- rbeta(1,1,beta)
    U2 <- runif(1)
    q <- sqrt(2 + X^2)*(1 - X)^(beta - 1)
    g <- (1 - X)^(beta - 1) / beta(1, beta)
    r <- q / (alpha_2 * g)
  }
  if ( U > r ) { next }
  samplef[i] <- X
  i <- i+1
}
plot(density(samplef), main = "Estimated Density")
curve(x^(theta-1)/(1+x^2)+sqrt(2+x^2)*(1-x)^(beta-1)/0.706, from=0, to=1, add=TRUE, col="red")

```

