

Bootstrap Methods

Typically we use (asymptotic) theory to derive the sampling distribution of a statistic. From the sampling distribution, we can obtain the variance, construct confidence intervals, perform hypothesis tests, and more.

Challenge:

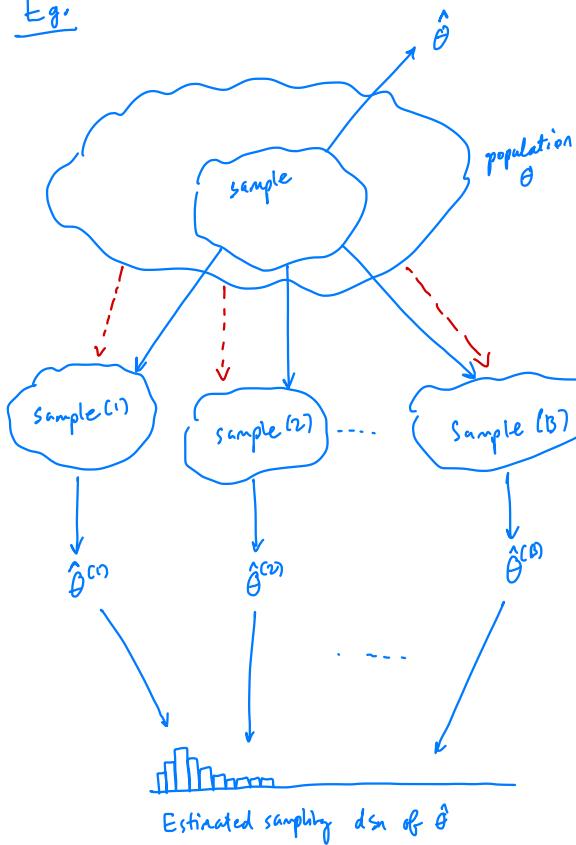
what if the sampling is impossible to obtain or asymptotic theory doesn't hold?

Basic idea of bootstrapping:

Use the data to approximate the sampling distribution of the statistic.

How? Estimate the sampling distribution by creating a large # of datasets that we might have seen and compute the statistic on each of these data sets.

Eg.



Goals:

estimate bias, se, CI's when

- ① There is doubt about whether distributional assumptions are met.
- ② There is doubt about whether asymptotic results are valid.
- ③ Theory to derive dsn is too hard.

In reality, we only have a sample, need to make $\text{sample}^{(1)}, \dots, \text{sample}^{(B)}$

“Bootstrap World” where the data analyst knows everything.

Idea: treat the sample Y_1, \dots, Y_n as the population.

E.g., we are interested in the variance of an estimator

→ In “bootstrap world” we can calculate the exact variance b/c we have access to the “population”

→ In practice, estimate variance by repeatedly sampling from the pseudo-population.

Real World

True population Y_1, Y_2, \dots w/ dsn F_θ

True pop. parameter θ

↓ sample

$Y_1, \dots, Y_n \Rightarrow \hat{\theta}(Y_1, \dots, Y_n)$ is estimator

$$MSE = E_F[(\hat{\theta} - \theta)^2]$$

If we don't have access to F (we don't),
we can't take this expectation.

If we had access to the population
could estimate MSE w/
{(we don't!)}
↓

$$\hat{MSE} = \frac{1}{rep} \sum_{i=1}^{rep} (\hat{\theta}_i - \theta)^2$$

Bootstrap World

Y_1, \dots, Y_n is population

$\hat{\theta}$ is true value of parameter

Since we have access to the population,

$$\hat{MSE}_{Boot} = \frac{1}{Boot Rep} \sum_{i=1}^{Boot Rep} (\hat{\theta}_i - \hat{\theta})^2$$

We hope $\hat{MSE}_{Boot} \approx MSE$.

1 Nonparametric Bootstrap

Let $\underline{Y} = (Y_1, \dots, Y_n) \sim F$ with pdf $f(y)$. Recall, the empirical cdf is defined as

$$F_n(y) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}(y_i \leq y) \quad y \in \mathbb{R}.$$



MLE of F and as $n \rightarrow \infty$, $F_n \rightarrow F$

Theoretical: Sample $\underline{Y} \sim F$, use Y_1, \dots, Y_n to compute F_n

Bootstrap: Sample $\underline{Y}^* \sim F_n$, use Y_1^*, \dots, Y_n^* to compute F_n^*

\nearrow of size n

The idea behind the nonparametric bootstrap is to sample many data sets from $F_n(y)$, which can be achieved by resampling from the data **with replacement**.

How many possible bootstrap samples? n^n

Are Y_1^*, \dots, Y_n^* independent?

$$P(Y_1^* = a, Y_2^* = b) = \underbrace{\frac{\sum_{i=1}^n \mathbb{I}(Y_i^* = a)}{n}}_{\text{of size } n} \cdot \underbrace{\frac{\sum_{i=1}^n \mathbb{I}(Y_i^* = b)}{n}}_{\text{of size } n} = P(Y_1^* = a) P(Y_2^* = b) \Rightarrow \text{yes}$$

Do we always want this?
No! More later

```
# observed data
x <- c(2, 2, 1, 1, 5, 4, 4, 3, 1, 2)

# create 10 bootstrap samples
x_star <- matrix(NA, nrow = length(x), ncol = 10)
for(i in 1:10) {
  x_star[, i] <- sample(x, length(x), replace = TRUE)
}
x_star
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
## [1,]     1     2     4     1     2     1     2     3     3     4
## [2,]     4     4     1     1     1     2     2     1     2     1
## [3,]     2     2     2     4     5     4     4     5     1     4
## [4,]     4     4     2     5     2     4     5     5     1     3
## [5,]     2     1     5     1     3     2     4     2     4     4
## [6,]     4     4     2     1     4     4     4     3     1     2
## [7,]     1     1     2     1     2     1     2     2     3     1
## [8,]     4     4     1     3     3     3     5     1     2     4
## [9,]     4     1     2     3     2     1     2     1     4     2
## [10,]    3     4     5     1     5     4     5     2     4     1
```

$\hat{\theta}$

```
# compare mean of the sample to the means of the bootstrap samples
mean(x)
```

```
## [1] 2.5
```

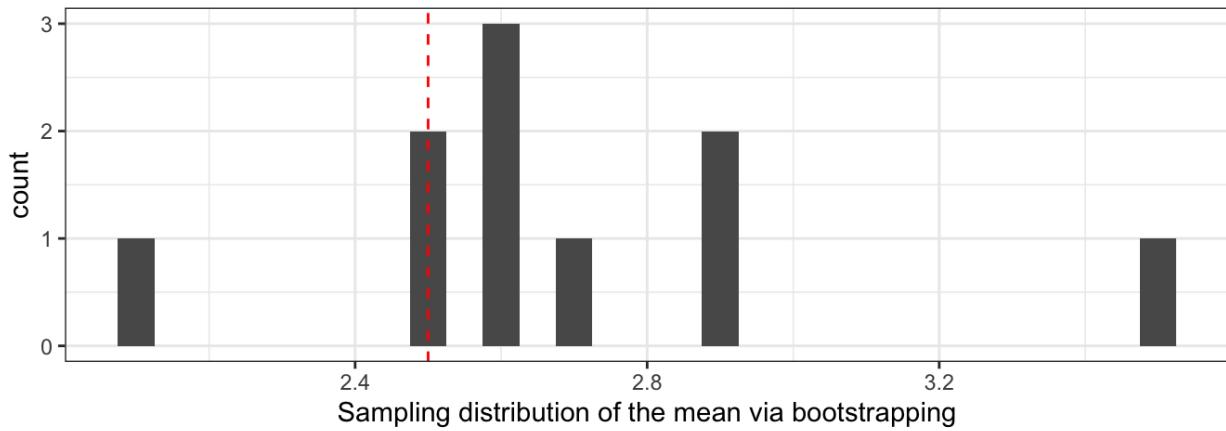
```
colMeans(x_star)
```

$\hat{\theta}_{\text{obs}} \approx \hat{\theta}^{(10)}$

```
## [1] 2.9 2.7 2.6 2.1 2.9 2.6 3.5 2.5 2.5 2.6
```

$\hat{\theta}_{\text{obs}} \approx \hat{\theta}^{(10)}$

```
ggplot() +
  geom_histogram(aes(colMeans(x_star)), binwidth = .05) +
  geom_vline(aes(xintercept = mean(x)), lty = 2, colour = "red") +
  xlab("Sampling distribution of the mean via bootstrapping")
```



1.1 Algorithm of NP bootstrap for iid data.

Goal: estimate the sampling distribution of a statistic based on observed data y_1, \dots, y_n .

Let θ be the parameter of interest and $\hat{\theta}$ be an estimator of θ . Then,

For $b=1, \dots, B$

① Sample $\hat{y}^{*(b)} = (\hat{y}_1^{*(b)}, \dots, \hat{y}_n^{*(b)})$ by sampling w/ replacement from the sample data
(i.e. sample from F_n)

② Compute $\hat{\theta}^{*(b)} = \hat{\theta}(\hat{y}^{*(b)})$
↑
estimate of θ based on b^{th} bootstrap sample.

Using $\hat{\theta}^{*(1)}, \dots, \hat{\theta}^{*(B)}$ we can

- estimate the sampling dist of $\hat{\theta}$ (histogram of $\hat{\theta}^{*(1)}, \dots, \hat{\theta}^{*(B)}$).

- estimate the SE of $\hat{\theta}$ (st. dev. of $\hat{\theta}^{*(1)}, \dots, \hat{\theta}^{*(B)}$).

- estimate a CI (many ways).

etc.

1.2 Justification for iid data

Suppose Y_1, \dots, Y_n are iid with $EY_1 = \mu \in \mathbb{R}$, $\text{Var}(Y_1) = \sigma^2 \in (0, \infty)$. Let's approximate the distribution of $T_n = \sqrt{n}(\bar{Y}_n - \mu)$ via the bootstrap.

Theorem: If Y_1, Y_2, \dots are iid with $\text{Var}(Y_1) = \sigma^2 \in (0, \infty)$, then

$$\sup_{y \in \mathbb{R}} |P(T_n \leq y) - P_*(T_n^* \leq y)| \equiv \Delta_n \rightarrow 0 \text{ as } n \rightarrow \infty \text{ almost surely (a.s.)}$$

Given $\underline{y} = \{y_1, \dots, y_n\}$ draw y_1^*, \dots, y_n^* bootstrap sample. Then,

$$\text{bootstrap probability. } \rightarrow P_*(y_i^* = y_i) = P(y_i^* = y_i | \underline{y}) = \frac{1}{n} \quad 1 \leq i \leq n$$

$$\text{The bootstrap version of } T_n \text{ is } T_n^* = \sqrt{n}(\bar{y}_n^* - E_* y_i^*) = \sqrt{n}(\bar{y}_n^* - \bar{y}_n)$$

$$\text{where } E_*[y_i^*] = E[y_i^* | \underline{y}] = \sum_{i=1}^n \frac{1}{n} y_i = \bar{y}_n \quad \text{also } E_*(\bar{y}_n^*) = E_*\left(\frac{1}{n} \sum_{i=1}^n y_i^*\right) = \frac{1}{n} \sum_{i=1}^n E_* y_i^* = \bar{y}_n$$

bootstrapping expected value exists b/c obs of y_1^*, \dots, y_n^* exist, but hard to compute directly b/c n^n bootstrap samples \Rightarrow use simulation to estimate

$$\text{Also, } P_*(T_n^* \leq y) = P(T_n^* \leq y | \underline{y}) \text{ approximates } P(T_n \leq y), y \in \mathbb{R} \text{ (Theorem).}$$

The proof of this theorem requires two facts:

- i. (Berry-Esseen Lemma) Let Y_1, \dots, Y_n be independent with $EY_i = 0$ and $E|Y_i|^3 < \infty$ for $i = 1, \dots, n$. Let $\sigma_n^2 = n\text{Var}(\bar{Y}_n) = n^{-1} \sum_{i=1}^n EY_i^2 > 0$. Then,

$$\sup_{y \in \mathbb{R}} \left| P\left(\frac{\sqrt{n}\bar{Y}_n}{\sigma_n} \leq y\right) - \Phi(y) \right| = \sup_{x \in \mathbb{R}} \left| P\left(\sqrt{n}\bar{Y}_n \leq x\right) - \Phi\left(\frac{x}{\sigma_n}\right) \right| \leq \frac{2.75}{n^{3/2} \sigma_n^3} \sum_{i=1}^n E|Y_i|^3.$$

M-Z

- ii. (Marcinkiewicz-Zygmund SLLN) Let X_i be a sequence of iid random variables with $E|X_i|^p < \infty$ for $p \in (0, 2)$. Then, for $S_n = \sum_{i=1}^n X_i$,

$$\frac{1}{n^{1/p}}(S_n - nc) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ almost surely (*)}$$

for any $c \in \mathbb{R}$ if $p \in (0, 1)$ and for $c = EX_1$ if $p \in [1, 2)$. If (*) holds for some $c \in \mathbb{R}$, then $E|X_1|^p < \infty$.

Specifically, we will use that if $\{Y_i\}$ are iid w/ $EY_i^2 < \infty$, then

$$\frac{1}{n^{3/2}} \sum_{i=1}^n |Y_i|^3 \rightarrow 0 \text{ as } n \rightarrow \infty \text{ a.s.}$$

Letting $X_i = |Y_i|^3$ because $E|X_i|^p = E|Y_i|^{3p} < \infty$ for $p = 2/3$ we may take $c=0$.

Proof:

$$\text{write } \sup_{y \in \mathbb{R}} |P(T_n \leq y) - P_*(T_n^* \leq y)| \leq \underbrace{\sup_{y \in \mathbb{R}} |P(T_n \leq y) - \Phi(y/\sigma)|}_{\tilde{\Delta}_n} + \underbrace{\sup_{y \in \mathbb{R}} |P_*(T_n^* \leq y) - \Phi(y/\sigma)|}_{\Delta_{n*}}$$

$\tilde{\Delta}_n \rightarrow 0$ by CLT since Y_1, \dots, Y_n iid, $EY_i^2 < \infty$.

Note that

$$\begin{aligned} \delta_{n*}^2 &\equiv n \text{Var}_*(\bar{Y}_n^*) = n \text{Var}_*\left(\frac{1}{n} \sum_{i=1}^n Y_i^*\right) = \frac{n}{n^2} \sum_{i=1}^n \text{Var}_* Y_i^* = \text{Var}_* Y_1^* \\ &= E_*(Y_1^*)^2 - [E_* Y_1^*]^2 \quad \text{where} \quad Y_1^* = \begin{cases} Y_1 & \text{u.p. } \frac{1}{n} \\ \vdots \\ Y_n & \text{u.p. } \frac{1}{n} \end{cases} \\ &= \frac{1}{n} \sum_{i=1}^n Y_i^2 - [\bar{Y}_n]^2 \end{aligned}$$

So, $\delta_{n*}^2 \rightarrow EY_1^2 - (EY_1)^2 = \sigma^2$ as $n \rightarrow \infty$ u.p. by SLLN since $EY_i^2 < \infty$.

By the Berry Esseen Lemma on $T_n^* = \sqrt{n}(\bar{Y}_n^* - E_* Y_1^*)$ and $|a-b| \leq 2 \max\{|a|, |b|\}$
 $\Rightarrow |a-b|^3 \leq 8 \max\{|a|^3, |b|^3\}$
 $\leq 8(|a|^3 + |b|^3)$

$$\begin{aligned} \sup_{y \in \mathbb{R}} |P_*(T_n^* \leq y) - \Phi\left(\frac{y}{\delta_{n*}}\right)| &\stackrel{\text{Berry Esseen}}{\leq} \frac{2.75}{n^{3/2} \delta_{n*}^3} n \underbrace{E_* |Y_1^* - E_* Y_1^*|^3}_{=\sum_{i=1}^n |Y_i - \bar{Y}_n|^3} \\ &= \frac{2.75}{n^{1/2} \delta_{n*}^3} \frac{1}{n} \sum_{i=1}^n |Y_i - \bar{Y}_n|^3 \\ &\leq \frac{2.75}{n^{1/2} \delta_{n*}^3} \frac{1}{n} \sum_{i=1}^n 8(|Y_i|^3 + |\bar{Y}_n|^3) \quad \text{as } n \rightarrow \infty \text{ u.p. 2.} \quad \text{since } \bar{Y}_n \rightarrow \mu \text{ u.p. 2.} \\ &= \frac{8(2.75)}{\delta_{n*}^3} \left(\boxed{\frac{|\bar{Y}_n|^3}{n^{1/2}}} + \underbrace{\frac{1}{n^{3/2}} \sum_{i=1}^n |Y_i|^3}_{\rightarrow 0 \text{ as } n \rightarrow \infty \text{ u.p. 1}} \right) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \text{ u.p. 1} \\ &\text{by M-Z SLLN (prev. page).} \end{aligned}$$

Finally, use $\delta_{n*}^2 \rightarrow \sigma^2$ u.p. 1 $\Rightarrow \Phi\left(\frac{y}{\delta_{n*}}\right) \rightarrow \Phi\left(\frac{y}{\sigma}\right)$ as $n \rightarrow \infty$ for any $y \in \mathbb{R}$ since $\Phi(\cdot)$ is continuous.

Thus $\sup_{y \in \mathbb{R}} \left| \Phi\left(\frac{y}{\delta_{n*}}\right) - \Phi\left(\frac{y}{\sigma}\right) \right| \rightarrow 0$ as $n \rightarrow \infty$ u.p. 1 (Polya's Theorem).

$$\text{So, } \Delta_{n*} = \sup_{y \in \mathbb{R}} |P_*(T_n^* \leq y) - \Phi\left(\frac{y}{\delta_{n*}}\right)| \leq \sup_{y \in \mathbb{R}} |P_*(T_n^* \leq y) - \Phi\left(\frac{y}{\sigma}\right)| + \sup_{y \in \mathbb{R}} \left| \Phi\left(\frac{y}{\delta_{n*}}\right) - \Phi\left(\frac{y}{\sigma}\right) \right| \rightarrow 0 \text{ u.p. 1} \quad \text{as } n \rightarrow \infty. //$$

1.3 Properties of Estimators

We can use the bootstrap to estimate different properties of estimators.

1.3.1 Standard Error

Recall $se(\hat{\theta}) = \sqrt{Var(\hat{\theta})}$. We can get a **bootstrap** estimate of the standard error:

$$\hat{se}(\hat{\theta}) = \sqrt{\frac{1}{B} \sum_{b=1}^B (\hat{\theta}^{*(b)} - \bar{\hat{\theta}}^*)^2} \quad \text{where} \quad \bar{\hat{\theta}}^* = \frac{1}{B} \sum_{b=1}^B \hat{\theta}^{(b)}$$

1.3.2 Bias

Recall $\text{bias}(\hat{\theta}) = E[\hat{\theta} - \theta] = E[\hat{\theta}] - \theta$. We can get a **bootstrap** estimate of the bias:

$$\hat{\text{bias}}(\hat{\theta}) = \bar{\hat{\theta}}^* - \hat{\theta}$$

↑ ↗
 computed based
 from Bootstrap on original data
 samples

Overall, we seek statistics with small se and small bias.

$$\underbrace{\text{MSE} = \text{Variance} + \text{Bias}^2}_{\text{MSE} = E[(\hat{\theta} - \theta)^2]} = E[(\hat{\theta} - \theta)^2].$$

\Rightarrow Bootstrap procedure to estimate the MSE:

- ① Compute $\hat{\theta}$ from original data $Y = (Y_1, \dots, Y_n)$
- ② Take B bootstrap samples of size n from data $\underbrace{Y^{*(1)}, \dots, Y^{*(B)}}$
- ③ Compute $\hat{\theta}^{*(b)}$ estimate of θ obtained from b^{th} BS sample
- ④ $\hat{\text{MSE}} = \frac{1}{B} \sum_{b=1}^B (\hat{\theta}^{*(b)} - \hat{\theta})^2$

1.4 Sample Size and # Bootstrap Samples

n = sample size & B = # bootstrap samples

If n is too small, or sample isn't representative of the population,

bootstrap results will be poor no matter how large B is.

Guidelines for B –

$B \approx 1000$ for estimating bias, se

$B \approx 2000$ for CI (depends on α : small $\alpha \Rightarrow \uparrow B$)

Best approach –

Repeat bootstrap w/ different seeds. If estimates very different, $\uparrow B$

Your Turn

In this example, we explore bootstrapping in the rare case where we know the values for the entire population. If you have all the data from the population, you don't need to bootstrap (or really, inference). It is useful to learn about bootstrapping by comparing to the truth in this example.

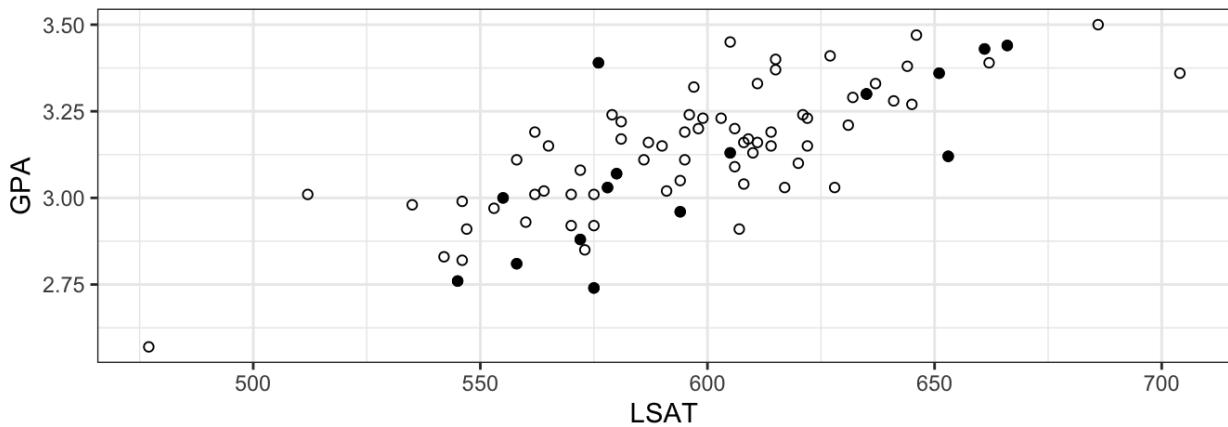
In the package `bootstrap` is contained the average LSAT and GPA for admission to the population of 82 USA Law schools (an old data set – there are now over 200 law schools). This package also contains a random sample of size $n = 15$ from this dataset.

```
library(bootstrap)

head(law)
  random sample of size n=15

##   LSAT   GPA
## 1 576 3.39
## 2 635 3.30
## 3 558 2.81
## 4 578 3.03
## 5 666 3.44
## 6 580 3.07
```

```
ggplot() +
  geom_point(aes(LSAT, GPA), data = law) +
  geom_point(aes(LSAT, GPA), data = law82, pch = 1)
    full population.
```



We will estimate the correlation $\theta = \rho(\text{LSAT}, \text{GPA})$ between these two variables and use a bootstrap to estimate the sample distribution of $\hat{\theta}$.

```
# sample correlation
cor(law$LSAT, law$GPA)
```

```
## [1] 0.7763745
```

```
# population correlation
cor(law82$LSAT, law82$GPA)
```

```
## [1] 0.7599979
```

```
# set up the bootstrap
B <- 200
n <- nrow(law)
r <- numeric(B) # storage for replicates.

for(b in B) {
  ## Your Turn: Do the bootstrap!
}
```

1. Plot the sample distribution of $\hat{\theta}$. Add vertical lines for the true value θ and the sample estimate $\hat{\theta}$.
2. Estimate $sd(\hat{\theta})$.
3. Estimate the bias of $\hat{\theta}$

1.5 Bootstrap CIs

We will look at ~~five~~^{four} different ways to create confidence intervals using the bootstrap and discuss which to use when.

1. Percentile Bootstrap CI
2. Basic Bootstrap CI
- ~~3. Standard Normal Bootstrap CI~~
4. Bootstrap t
5. Accelerated Bias-Corrected (BCa)
*//
adjusted for skewness*

Key ideas:

- ① When you say used "bootstrapping" to estimate a CI, need to say which one.
- ② (For now) whatever you are bootstrapping needs to be independent.
- ③ Bootstrapping is an attempt to simulate replication. (think about interpretation of CI).

1.5.1 Percentile Bootstrap CI (Probably the one you're thinking of).

Let $\hat{\theta}^{(1)}, \dots, \hat{\theta}^{(B)}$ be bootstrap replicates and let $\hat{\theta}_{\alpha/2}$ be the $\alpha/2$ quantile of $\hat{\theta}^{(1)}, \dots, \hat{\theta}^{(B)}$.

Then, the $100(1 - \alpha)\%$ Percentile Bootstrap CI for θ is

$$(\hat{\theta}_{\alpha/2}, \hat{\theta}_{1-\alpha/2})$$

\uparrow if bootstrap dsn is reasonably approximated by a cts, doesn't matter if we include end points.

In R, if `bootstrap.reps = c($\hat{\theta}^{(1)}, \dots, \hat{\theta}^{(B)}$)`, the percentile CI is
vector of bootstrap statistics.

```
quantile(bootstrap.reps, c(alpha/2, 1 - alpha/2))
```

Assumptions/usage

- Widely used because it's simple to implement and explain.
- drawback: CI's usually too narrow, leading to low coverage.
 \hookrightarrow more often when bias or skewness in bootstrap dsn.

Justification (Efron):

Assume the existence of an increasing transformation g s.t. $P[g(\hat{\theta}) - g(\theta) \leq x] = H(x)$

\Rightarrow "bootstrap world": $P_*[g(\hat{\theta}^*) - g(\hat{\theta}) \leq x] \approx H(x)$. (*).

Suppose g is known, then $P[g^{-1}(g(\hat{\theta}) - x) \leq \hat{\theta}] = H(x)$. Then set this probability to $1-\alpha$.

$P[g^{-1}(g(\hat{\theta}) - H'(1-\alpha)) \leq \hat{\theta}] = 1-\alpha \Rightarrow$ if g is known this is a one-sided CI w/ coverage prob α
 $(g^{-1}(g(\hat{\theta}) - H'(1-\alpha)), \infty)$. (upper bound version similar)

Goal: Show $\bar{g}(g(\hat{\theta}) - H'(1-\alpha))$ is estimated by the left endpoint of a percentile bootstrap interval $\hat{\theta}_\alpha$

$$\alpha = P_*(\hat{\theta}^* \leq \hat{\theta}_\alpha) = P_*(g(\hat{\theta}^*) \leq g(\hat{\theta}_\alpha))$$

$$= P_*(g(\hat{\theta}^*) - g(\hat{\theta}) \leq g(\hat{\theta}_\alpha) - g(\hat{\theta})) \approx H(g(\hat{\theta}_\alpha) - g(\hat{\theta})) \text{ by (*).}$$

$$H'(a) \approx g(\hat{\theta}_\alpha) - g(\hat{\theta}) \Rightarrow \bar{g}\left(\underbrace{H'(a) + g(\hat{\theta})}_{= H'(1-\alpha)}\right) \approx \hat{\theta}_\alpha$$

$$\underbrace{\bar{g}(g(\hat{\theta}) - H'(1-\alpha))}_{}$$

NOTE: we do not need to know g to calculate $\hat{\theta}_\alpha$!

1.5.2 Basic Bootstrap CI (Corrects for bias) based on residuals.

The $100(1 - \alpha)\%$ Basic Bootstrap CI for θ is

$$\left(\hat{\theta} - [\hat{\theta}_{1-\alpha/2} - \hat{\theta}], \hat{\theta} - [\hat{\theta}_{\alpha/2} - \hat{\theta}] \right) = \left(2\hat{\theta} - \hat{\theta}_{1-\alpha/2}, 2\hat{\theta} - \hat{\theta}_{\alpha/2} \right).$$

↑ $\hat{\theta}$
 estimate
 from
 original data

↑ $\hat{\theta}$
 1 - $\alpha/2$ quantile of bootstrap distribution

Why? Let $\varepsilon = \hat{\theta} - \theta$ and $\varepsilon_{1-\alpha/2}, \varepsilon_{\alpha/2}$ quantiles of the distribution of ε .

$$P(\varepsilon_{\alpha/2} < \hat{\theta} - \theta < \varepsilon_{1-\alpha/2}) = 1 - \alpha \Rightarrow (1 - \alpha)100\% \text{ CI is } (\hat{\theta} - \varepsilon_{1-\alpha/2}^*, \hat{\theta} - \varepsilon_{\alpha/2}^*)$$

Assumptions/usage

where $\varepsilon_{\alpha/2}^*$ is quantile of the bootstrap distribution of ε^*

* Corrects for bias, but slightly harder to explain.

1.5.3 Bootstrap t CI (Studentized Bootstrap)

Consider $Z = \frac{\hat{\theta} - E(\hat{\theta})}{\cdot \hat{se}(\hat{\theta})}$.

Even if the distribution of $\hat{\theta}$ is Normal and $\hat{\theta}$ is unbiased for θ , the Normal distribution is not exactly correct for z .

because we need to estimate $se(\hat{\theta}) \Rightarrow t^* = \frac{\hat{\theta} - E(\hat{\theta})}{\hat{se}(\hat{\theta})} \sim t_{n-1}$? NO

Additionally, the distribution of $\hat{se}(\hat{\theta})$ is unknown.

so we cannot claim $t^* \sim t_{n-1}$, even when $\hat{\theta} \sim \text{Normal}$!

\Rightarrow The bootstrap t interval does not use a Student t distribution as the reference distribution, instead we estimate the distribution of a “t type” statistic by resampling.

The $100(1 - \alpha)\%$ Bootstrap t CI is

$$(\hat{\theta} - t_{1-\alpha/2}^* \cdot \hat{se}(\hat{\theta}), \hat{\theta} + t_{\alpha/2}^* \cdot \hat{se}(\hat{\theta}))$$

\uparrow ↑
= quantile of the estimated based on bootstrap samples $\hat{\theta}^{*(1)}, \dots, \hat{\theta}^{*(B)}$

Overview bootstrap “t-type” statistic.

$$\text{t-type statistic: } t^{(1)} = \frac{\hat{\theta}^{(1)} - \hat{\theta}}{\hat{se}(\hat{\theta}^{(1)})}, \dots, t^{(B)} = \frac{\hat{\theta}^{(B)} - \hat{\theta}}{\hat{se}(\hat{\theta}^{(B)})}$$

$\underbrace{\quad}_{\text{= bootstrap estimate of } se(\hat{\theta}) \text{ based on the 1st bootstrap sample ??}}$ DOUBLE
BOOTSTRAP!

To estimate the “t style distribution” for $\hat{\theta}$,

① Compute $\hat{\theta}$

② For each replicate $b=1, \dots, B$

(a) Sample w/ replacement from y , $y^{(b)} = (y_1^{(b)}, \dots, y_n^{(b)})$

(b) Compute $\hat{\theta}^{(b)}$

(c) For each replicate $r=1, \dots, R$

i. sample w/ replacement from $y^{(b)}$, $y^{(b)(r)} = (y_1^{(b)(r)}, \dots, y_n^{(b)(r)})$

ii. Compute $\hat{\theta}^{(b)(r)}$

(d) Compute $\hat{se}(\hat{\theta}^{(b)}) = \text{sd}(\hat{\theta}^{(b)(1)}, \dots, \hat{\theta}^{(b)(R)})$

(e) Compute t-style statistic $t^{(b)} = \frac{\hat{\theta}^{(b)} - \hat{\theta}}{\hat{se}(\hat{\theta}^{(b)})}$

③ get quantiles $t_{\alpha/2}^*, t_{1-\alpha/2}^*$

④ Compute CI.

Assumptions/usage

- Computationally intensive
- Not doing anything for bias/skewness in bootstrap dsn.
- Need $\hat{\theta}$ independent of $\hat{S}^2(\hat{\theta})$.

This idea is based on "pivotal quantities" = a function of observations and parameters whose probability dsn does not depend on the parameters.

$$\text{e.g. } \frac{\bar{X} - \mu}{s/\sqrt{n}} \sim t_{n-1} \quad t \text{ doesn't depend on } \mu.$$

You could create other pivotal quantities, e.g.

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1} \quad \text{if } y_i \stackrel{iid}{\sim} N(\mu, \sigma^2).$$

$\Rightarrow \left(\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}, \frac{(n-1)s^2}{\chi^2_{\alpha/2}} \right)$ would be a $(1-\alpha)100\%$ CI for σ^2 .

Bootstrap version of this, do not assume $y_i \stackrel{iid}{\sim} N(\mu, \sigma^2)$.

We would now need to estimate a χ^2 -type quantile.

"bias-corrected"
↓
accelerated"

1.5.4 BCa CIs

Modified version of percentile intervals that adjusts for bias of estimator and skewness of the sampling distribution.

This method automatically selects a transformation so that the normality assumption holds.

Idea:

Assume there exist a monotonically increasing function g and constants a, b s.t.

$$\frac{g(\hat{\theta}) - g(\theta)}{1 + ag(\theta)} + b \sim N(0, 1) \text{ where } 1 + ag(\theta) > 0.$$

$$\begin{aligned} \text{Let } z_p \text{ be the } 100p^{\text{th}} \text{ percentile of the standard Normal} &\stackrel{\text{to est CI}}{\Rightarrow} -z_p < \frac{g(\hat{\theta}) - g(\theta)}{1 + ag(\theta)} + b < z_p \\ &\Leftrightarrow \frac{g(\hat{\theta}) + b - z_p}{1 - a(b - z_p)} < g(\theta) < \frac{g(\hat{\theta}) + b + z_p}{1 - a(b + z_p)} \\ &\text{is a } 100(1-\alpha)\% \text{ CI if } p = 1 - \alpha/2. \end{aligned}$$

Bootstrap version: $\frac{g(\hat{\theta}^*) - g(\hat{\theta})}{1 + ag(\hat{\theta})} + b \stackrel{\text{approx}}{\sim} N(0, 1).$

$$\Rightarrow P\left(g(\hat{\theta}) < \frac{g(\hat{\theta}^*) + b + z_p}{1 - a(b + z_p)}\right) = P\left(\frac{g(\hat{\theta}^*) - g(\hat{\theta})}{1 + ag(\hat{\theta})} + b < \frac{b + z_p}{1 - a(b + z_p)} + b\right) \stackrel{\text{approx}}{\sim} N(0, 1).$$

Thus the BCa interval for θ is $(z_L, z_U) = \left(\frac{b - z_p + b}{1 - a(b - z_p)}, \frac{b + z_p + b}{1 - a(b + z_p)}\right)$.

↑
there are different
quantiles of $N(0, 1)$.

e.g. If $b = 0.2$, $a = 0.01$, $\alpha = 0.05$, $z_L = -1.53$, $z_U = 2.41$

↑
6.3rd percentile of $N(0, 1)$ ↗ 99.2th percentile of $N(0, 1)$.

How to find a and b ? $\frac{g(\hat{\theta}^*) - g(\hat{\theta})}{1 + ag(\hat{\theta})} \stackrel{\text{approx}}{\sim} N(b, 1) \Rightarrow$

Let p_0 be the fraction of observations from bootstrap dsn s.t. $\hat{\theta}^{*(1)} < \hat{\theta}$. Since g is monotone, this is the same fraction of $\hat{\theta}^*$'s s.t. $g(\hat{\theta}^*) < g(\hat{\theta})$.

$\Rightarrow p_0 = P(Z < b)$ where $Z \sim N(0, 1)$ gives a way to approximate b . (If the bootstrap dsn has $\hat{\theta}$ as its median, then $b = 0$)

a corrects
for skewness!

Let $S_{-i} = \{y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n\}$ and let $\hat{\theta}_{-i}$ denote the estimate of θ based on S_{-i}

18

$$\alpha = \frac{\sum_{i=1}^n (\hat{\theta}_{(i)} - \hat{\theta}_{-i})^3}{6 \left[\sum_{i=1}^n (\hat{\theta}_{(i)} - \hat{\theta}_{-i})^2 \right]^{3/2}}$$

based on skewness estimate.

where $\hat{\theta}_{(i)}$ is the mean of $\hat{\theta}_{-i}$'s.
If $a=0$, don't
1 Nonparametric Bootstrap adjust for
skewness
 $\Rightarrow BC$ interval.

The BCa method uses bootstrapping to estimate the bias and ~~skewness~~ then modifies which percentiles are chosen to get the appropriate confidence limits for a given data set.

In summary,

Has better coverage in empirical studies, but harder to explain.

Your Turn

We will consider a telephone repair example from Hesterberg (2014). `Verizon` has repair times, with two groups, CLEC and ILEC, customers of the “Competitive” and “Incumbent” local exchange carrier.

```
library(resample) # package containing the data

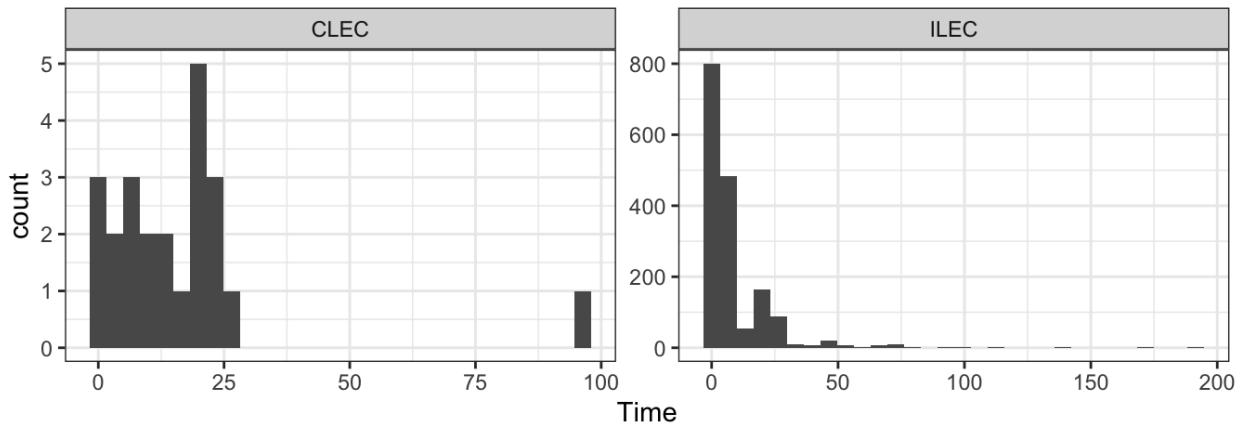
data(Verizon)
head(Verizon)
```

```
##      Time Group
## 1 17.50  ILEC
## 2  2.40  ILEC
## 3  0.00  ILEC
## 4  0.65  ILEC
## 5 22.23  ILEC
## 6  1.20  ILEC
```

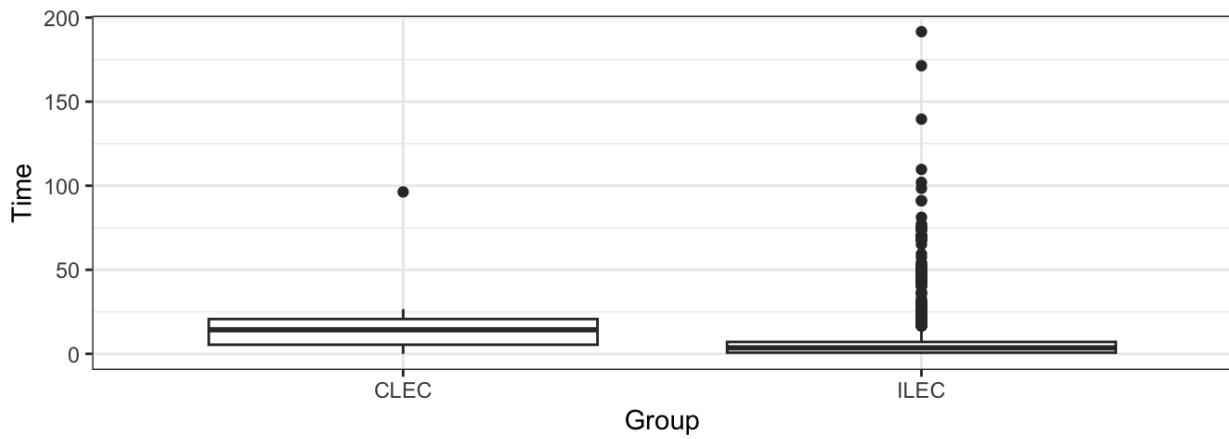
```
Verizon |>
  group_by(Group) |>
  summarize(mean = mean(Time), sd = sd(Time), min = min(Time), max =
    max(Time)) |>
  kable()
```

Group	mean	sd	min	max
CLEC	16.509130	19.50358	0	96.32
ILEC	8.411611	14.69004	0	191.60

```
ggplot(Verizon) +
  geom_histogram(aes(Time)) +
  facet_wrap(.~Group, scales = "free")
```



```
ggplot(Verizon) +
  geom_boxplot(aes(Group, Time))
```



1.6 Bootstrapping CIs

There are many bootstrapping packages in R, we will use the `boot` package. The function `boot` generates R resamples of the data and computes the desired statistic(s) for each sample. This function requires 3 arguments:

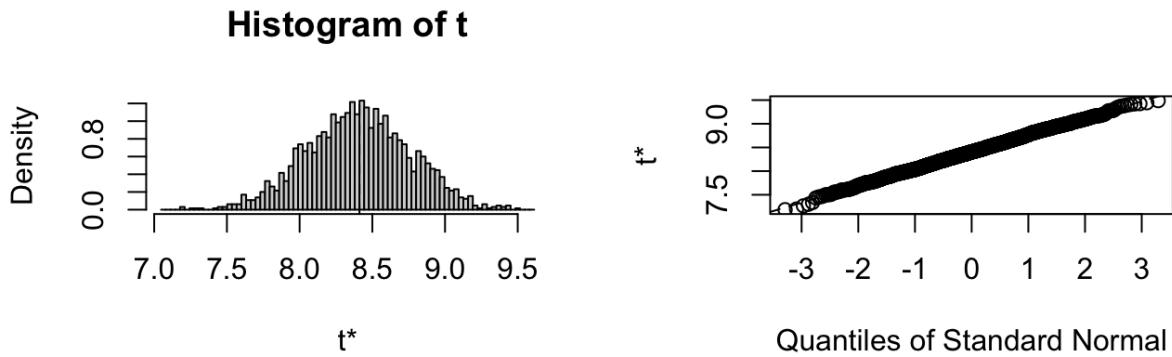
1. `data` = the data from the original sample (`data.frame` or `matrix`).
2. `statistic` = a function to compute the statistic from the data where the first argument is the data and the second argument is the indices of the observations in the bootstrap sample.
3. R = the number of bootstrap replicates.

```
library(boot) # package containing the bootstrap function

mean_func <- function(x, idx) {
  mean(x[idx])
}

ilec_times <- Verizon[Verizon$Group == "ILEC", ]$Time
boot.ilec <- boot(ilec_times, mean_func, 2000)

plot(boot.ilec)
```



If we want to get Bootstrap CIs, we can use the `boot.ci` function to generate the different nonparametric bootstrap confidence intervals.

```
boot.ci(boot.ilec, conf = .95, type = c("perc", "basic", "bca"))
```

```
## BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
## Based on 2000 bootstrap replicates
##
## CALL :
## boot.ci(boot.out = boot.ilec, conf = 0.95, type = c("perc",
## "basic",
##         "bca"))
##
## Intervals :
##   Level      Basic              Percentile          BCa
## 95%    ( 7.733,  9.110 )    ( 7.714,  9.091 )    ( 7.755,  9.125 )
## Calculations and Intervals on Original Scale
```

```
## we can do some of these on our own
## percentile
quantile(boot.ilec$t, c(.025, .975))
```

```
##      2.5%    97.5%
## 7.714075 9.084725
```

```
## basic
2*mean(ilec_times) - quantile(boot.ilec$t, c(.975, .025))
```

```
##      97.5%      2.5%
## 7.738496 9.109147
```

To get the studentized bootstrap CI, we need our statistic function to also return the variance of $\hat{\theta}$.

```
mean_var_func <- function(x, idx) {
  c(mean(x[idx]), var(x[idx])/length(idx))
}

boot.ilec_2 <- boot(ilec_times, mean_var_func, 2000)
boot.ci(boot.ilec_2, conf = .95, type = "stud")
```

```
## BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
## Based on 2000 bootstrap replicates
##
## CALL :
## boot.ci(boot.out = boot.ilec_2, conf = 0.95, type = "stud")
##
## Intervals :
## Level     Studentized
## 95%   ( 7.728, 9.183 )
## Calculations and Intervals on Original Scale
```

Which CI should we use?

1.7 Bootstrapping for the difference of two means

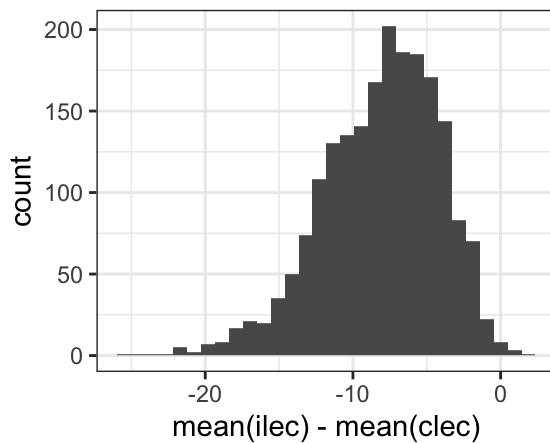
Given iid draws of size n and m from two populations, to compare the means of the two groups using the bootstrap,

The function `two.boot` in the `simpleboot` package is used to bootstrap the difference between univariate statistics. Use the bootstrap to compute the shape, bias, and bootstrap sample error for the samples from the `Verizon` data set of CLEC and ILEC customers.

```
library(simpleboot)

clec_times <- Verizon[Verizon$Group == "CLEC", ]$Time
diff_means.boot <- two.boot(ilec_times, clec_times, "mean", R = 2000)

ggplot() +
  geom_histogram(aes(diff_means.boot$t)) +
  xlab("mean(ilec) - mean(clec)")
```



```
# Your turn: estimate the bias and se of the sampling distribution
```

Which confidence intervals should we use?

```
# Your turn: get the chosen CI using boot.ci
```

Is there evidence that

$$H_0 : \mu_1 - \mu_2 = 0$$

$$H_a : \mu_1 - \mu_2 < 0$$

is rejected?