1.3 Likelihoods for Regression Models

We will start with linear regression and then talk about more general models.

↓
nonlinear
GLM

1.3.1 Linear Model

Consider the familiar linear model

$$Y_i = \boldsymbol{x}_i^\top \boldsymbol{\beta} + \epsilon_i, \qquad i = 1, \ldots, n,$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are known nonrandom vectors.

$$E[\varepsilon_i] = 0 \quad \text{and} \quad Var[\varepsilon_i] = \sigma^2$$

often estimate β by $\hat{\beta}_{OLS}$, which does not require a distribution for ε_i.

For likelihood-based estimation, We need a distribution for ε_i! Start w/ $\varepsilon_i \sim N(0, \sigma^2)$.

$$\Rightarrow L(\boldsymbol{\beta}, \sigma | \{Y_i, \boldsymbol{x}_i\}_{i=1}^n) = \prod_{i=1}^n \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right) \exp\left(-\frac{(Y_i - \underline{x}_i^T \underline{\beta})^2}{2\sigma^2}\right)$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (Y_i - \underline{x}_i^T \underline{\beta})^2\right)$$

take log, derivatives, set = 0, solve $\longrightarrow$

$$\hat{\underline{\beta}}_{MLE} = (X^T X)^{-1} X^T \underline{Y}$$ Same as $\hat{\beta}_{OLS}$!

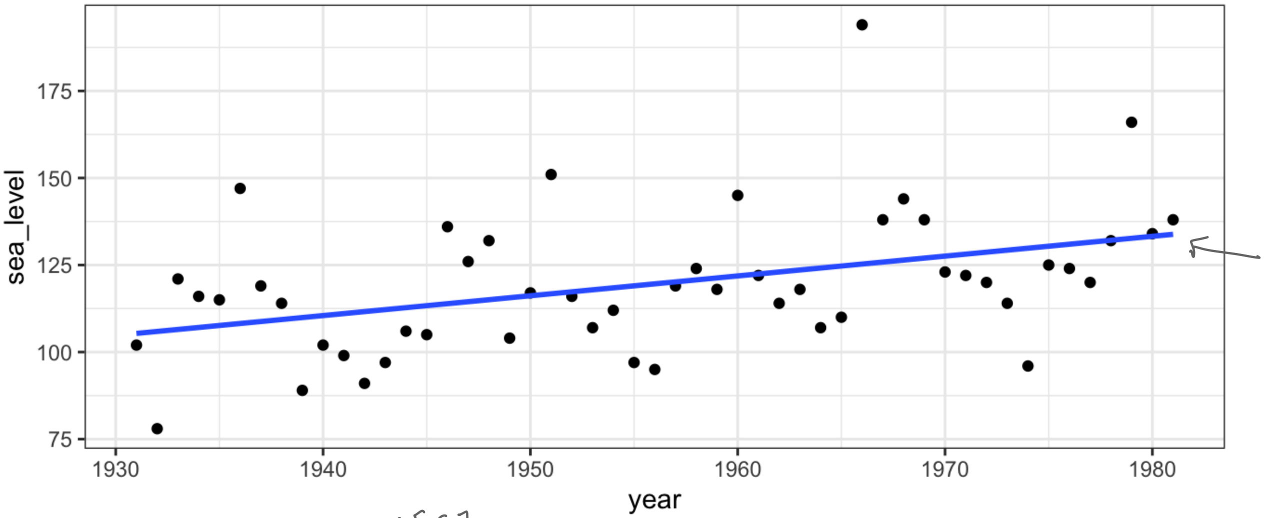
$$\hat{\sigma}^2_{MLE} = \frac{1}{n} \sum_{i=1}^n (Y_i - \underline{x}_i^T \underline{\beta})^2$$ (only asymptotically unbiased).

What do you do when ϵ_i are not Gaussian?

- transform data so ε_i look Gaussian.
- Use a different distribution for ε_i!

Example (Venice sea levels): The annual maximum sea levels in Venice for 1931–1981 are :

We know maxima are not Gaussian!



least squares line.

Approach 1: OLS $E[\varepsilon_i]=0$, $Var[\varepsilon_i]=\sigma^2$ No distributional assumption.

Approach 2: Assume $\varepsilon_i \sim$ Gumbel (extreme value dsn), use ML

$$f(y;\sigma) = \frac{1}{\sigma}\exp\left(-\frac{y}{\sigma}\right)\exp\left(-\exp\left(-\frac{y}{\sigma}\right)\right).$$

$$\Rightarrow L(\beta,\sigma \mid \{y_i, \underline{x}_i\}_{i=1}^n) = \prod_{i=1}^n f(y_i - \underline{x}_i^T\beta) = \prod_{i=1}^n \frac{1}{\sigma}\exp\left(-\frac{y_i-\underline{x}_i^T\beta}{\sigma}\right)\exp\left(-\exp\left(-\frac{y_i-\underline{x}_i^T\beta}{\sigma}\right)\right)$$

YOUR TURN: Fit both approaches to the venice data.

OLS

$\hat\beta_0 = 104.8$ $\hat\beta_1 = .56$ (SE .177)

MLE, GUMBEL

$\hat\beta_0 = 96.8$, $\hat\beta_1 = 0.56$ $\left(\begin{matrix}SE\\ .136\end{matrix}\right)$

OLS vs MLE? If EV model is correct, more efficient (note: standard errors).

β_0 difference: $\overset{\text{Gumbel}}{E}[\varepsilon_i] = 0.577\sigma = 0.577\hat\sigma_{MLE} = 0.577(14.5) \neq 0$

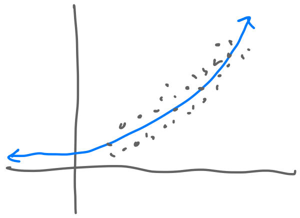
$96.8 + .577(14.5) = 105.1$

1.3.2 Additive Errors Nonlinear Model

Previous example had ① linear trend, ② Non-Gaussian errors.

Example: exponential growth model

$$g(x, \underline{\beta}) = \beta_0 \exp(\beta_1 x)$$



Non-linear additive model:

$$Y_i = g(\underline{x}_i, \underline{\beta}) + \varepsilon_i$$

Often interesten in $\varepsilon_i \sim N(0, \sigma^2)$ but $g(\underline{x}_i, \underline{\beta}) \neq \underline{x}_i^T \underline{\beta}$ $\Rightarrow$ ML required.

① non-liner trend, ② Gaussian errors.

1.3.3 Generalized Linear Models

↗ of response.

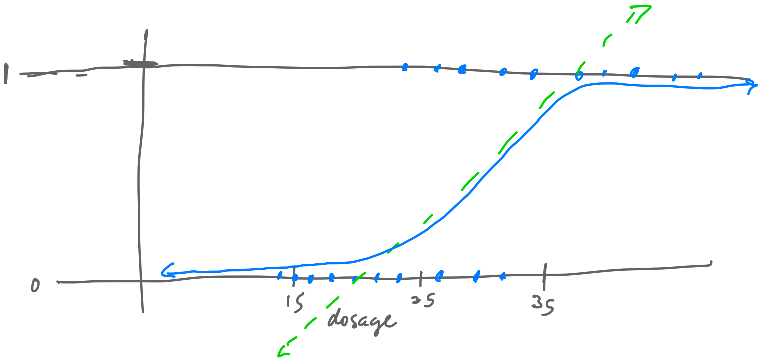
Regression: build a relationship between a parameter (mean) + covariates.

LM's: stochastic element is additive w/ mean.

GLM's: stochastic element is diffrent.

Imagine an experiment where individual mosquitos are given some dosage of pesticide. The response is whether the mosquito $\underbrace{\text{lives}}_{=0}$ or $\underbrace{\text{dies.}}_{=1}$ The data might look something like:

x (dosage)	y (0 = lives, 1 = dies)
15	0
17	0
18	1
20	0
21	1
⋮	⋮



Goal: Model the relationship between the predictor and response.

dosage dead/alive.

Sounds like regression!

Big difference: Y_i's are not continuous. They only take values of 0 or 1.

Question: What would a curve of best fit look like? Would we want a function that only takes values in $\{0, 1\}$.

It seems sensible to have a curve which takes values near 0 for low doses + near 1 for high doses and intermediate values for middle does.

what does this curve represent? Probability.

Refined Goal: Model relationship betwen predictor (dosage) + probability of success in response (mosquito dies).

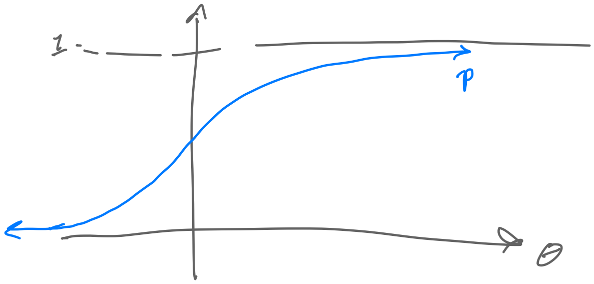
Let's build a sensible model. Note: We don't observe the probability.

Step 1: Find a function that behaves the way we want.

like the blue curve.

Consider the logistic function,

$$p = \frac{\exp(\theta)}{1+\exp(\theta)}.$$



As $\theta \to \infty$, $p \to 1$

$\theta \to -\infty$, $p \to 0$

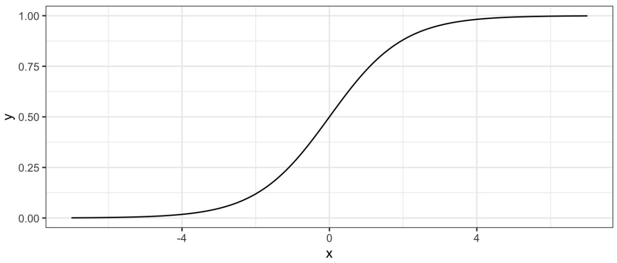
$\theta = 0$, $p = \frac{1}{2}$

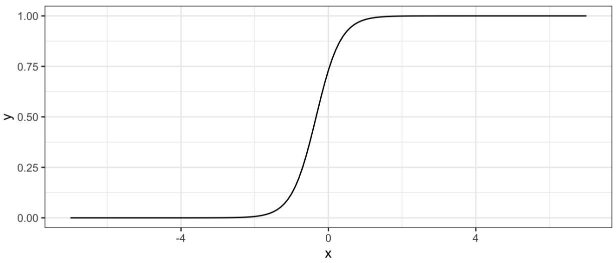
By changing θ, we can change location, slope, direction of this function.

Let $\theta = \beta_0 + \beta_1 x \Rightarrow p = \dfrac{\exp(\beta_0 + \beta_1 x)}{1+\exp(\beta_0+\beta_1 x)}.$

```r
# understanding the logistic function
# first, theta just equals x
x <- seq(-7, 7, .1)
theta <- x
y <- exp(theta)/(1 + exp(theta))
ggplot() + geom_line(aes(x, y))

# now, let theta be a linear function of x
theta <- 1 + 3*x
y <- exp(theta)/(1 + exp(theta))
ggplot() + geom_line(aes(x, y))
```





Now we can connect probabilities to covariate x!

We'd be done if we observed probabilities, but our response only takes values of 0 and 1.

Step 2: Build a stochastic mechanism to relate to a binary response.

Recall the Bernoulli distribution

$$Y = \begin{cases} 0 & \text{w.p. } 1-p \\ 1 & \text{w.p. } p. \end{cases}$$

Coin biased flip example w/ $p = 0.75$. Flip coin, you will observe 0 (tails) or 1 (heads).

Aside: We could instead think about binomial dsn, which counts # of successes for n trials.

$X = \sum_{i=1}^{n} Y_i$, $Y_i \overset{iid}{\sim} \text{Bern}(p)$. X takes values in $\{0, 1, \dots, n\}$.

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Step 3: Put Step 1 and Step 2 together.

$$Y_i \overset{ind}{\sim} \text{Bernoulli}(p_i) \quad + \quad p_i = \frac{\exp(\theta_i)}{1+\exp(\theta_i)}, \quad \theta_i = \beta_0 + \beta_1 x_i$$

Y_i: outcome of ith observation (observed).

p_i: prob of ith observation having success (unobserved).

Goal: estimate β_0 and β_1. Find the "best" estimates.

Fitting our model: Does OLS make sense? No.

What else can we do? Maximum likelihood!

↳ Find the parameters (βs) which make the density agree best w/ data we observed!

pmf of Bernoulli: $f(y_i; p_i) = p_i^{y_i}(1-p_i)^{1-y_i}$

> take y_i's to estimate p_i's.

Consider the likelihood contribution.

$$L_i(p_i|Y_i) = p_i^{Y_i}(1-p_i)^{1-Y_i} \qquad (Y_i\text{'s are } 0 \text{ or } 1).$$

So the log-likelihood contribution is

$$\ell_i(p_i) = Y_i \log p_i + (1-Y_i)\log(1-p_i) = \underbrace{\log(1-p_i) + Y_i \log\frac{p_i}{1-p_i}}_{(*)}$$

Recall, we said $\underbrace{p_i = \frac{\exp(\theta_i)}{1+\exp(\theta_i)}}$ was sensible.

Manipulating: $p_i + p_i\exp(\theta_i) = \exp(\theta_i)$

$$p_i = (1-p_i)\exp(\theta_i).$$

OR

$$\frac{p_i}{1-p_i} = \exp(\theta_i)$$

$$\log\left(\frac{p_i}{1-p_i}\right) = \theta_i \quad (1)$$

$$\underline{p_i}\exp(-\theta_i) = 1-p_i$$

$$\frac{\cancel{\exp(\theta_i)}}{1+\exp(\theta_i)}\exp(\cancel{-}\theta_i) = 1-p_i$$

$$\frac{1}{1+\exp(\theta_i)} = 1-p_i$$

$$-\log(1+\exp(\theta_i)) = \log(1-p_i) \quad (2).$$

Pluggin in (1) and (2) into (*).

Which gives us,

$$\ell_i(\theta_i) = -\log(1+\exp(\theta_i)) + \underbrace{Y_i\theta_i} \qquad (\text{now in terms of } \theta_i \text{ not } p_i)$$

notice now the term w/ the data is "nice" for MLE things. Why? Because log-likelihood + "sensible" function $p_i = \frac{\exp(\theta_i)}{1+\exp(\theta_i)}$ work well together.

<u>NOT</u> a coincidence.

So the log-likelihood is

$$\ell(\theta_1,\dots,\theta_n) = \sum_{i=1}^{n} \ell_i(\theta_i)$$

$$= \sum_{i=1}^{n}\left\{-\log(1+\exp(\theta_i)) + Y_i\theta_i\right\}$$

$$\Rightarrow \ell(\beta_0,\beta_1) = \sum_{i=1}^{n}\left\{-\log(1+\exp(\beta_0+\beta_1x_i)) + Y_i(\beta_0+\beta_1x_i)\right\}$$

To optimize? Must be done numerically.

```
## data on credit default
data("Default", package = "ISLR")
head(Default)
```

```
##   default student   balance    income
## 1      No      No  729.5265 44361.625
## 2      No     Yes  817.1804 12106.135
## 3      No      No 1073.5492 31767.139
## 4      No      No  529.2506 35704.494
## 5      No      No  785.6559 38463.496
## 6      No     Yes  919.5885  7491.559
```

```
## fit model with ML
m0 <- glm(default ~ balance, data = Default, family = binomial)
tidy(m0) |> kable()
```

optimizing likelihood numerically.

data y's are 0,1

broom package.

$\hat\beta_0, \hat\beta_1$ $se(\hat\beta_0), se(\hat\beta_1)$ $\frac{\hat\beta_i}{se(\hat\beta_i)}$.

term	estimate	std.error	statistic	p.value
(Intercept)	-10.6513306	0.3611574	-29.49221	0
balance	0.0054989	0.0002204	24.95309	0

$H_0: \beta_i = 0$ $i=1,2.$

$H_a: \beta_i \neq 0$

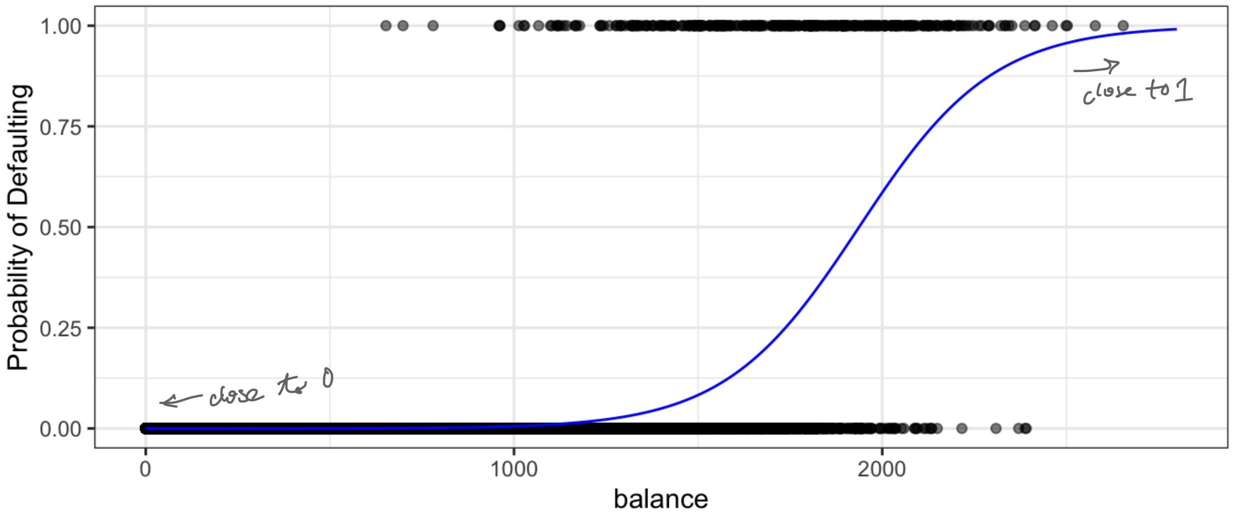
```
glance(m0) |> kable()
```

null.deviance	df.null	logLik	AIC	BIC	deviance	df.residual	nobs
2920.65	9999	-798.2258	1600.452	1614.872	1596.452	9998	10000

$\ell(\hat\beta_0, \hat\beta_1)$.

```
## plot the curve
x_new <- seq(0, 2800, length.out = 200)
theta <- m0$coefficients[1] + m0$coefficients[2]*x_new
p_hat <- exp(theta)/(1 + exp(theta))

ggplot() +
  geom_point(aes(balance, as.numeric(default) - 1), alpha = 0.5, data
        = Default) +
  geom_line(aes(x_new, p_hat), colour = "blue") +
  ylab("Probability of Defaulting")
```



never outside of [0,1] → valid probabilites!

In general, a GLM is three pieces:

link function.

	Ex: Logistic Regression	Explanation:
1. The random component probability dsn from exponential family.	$Y_i \sim \text{Binom}(p_i)$	decribe the generating mechanism of obsrved data
2. The systemic component A function relating the parameter of interest (mean!) to θ $E[Y] = g^{-1}(\mu)$	$p_i = \frac{\exp(\theta_i)}{1+\exp(\theta_i)} = g^{-1}(\theta_i)$. ↑ Note $Y_i \sim \text{Bern}(p_i)$ $E[Y_i] = p_i$	transforming linear relationship to be on a scale that makes sense for the parameter of interest "linking" linear relationship to mean.
3. A linear predictor $\theta = X\beta$.	$\theta_i = \underline{x}_i\beta$	describing how θ is a linear function of predictor variables.

Remarks:

① Standard formulation denotes function by g^{-1} : $p = g^{-1}(\theta) = \frac{\exp(\theta)}{1+\exp(\theta)}$.

$$\Rightarrow \theta = g(p) = \log\left(\frac{p}{1-p}\right).$$

② Parameter of interest is still the mean, just like linear regression.

③. Theoretical reasons for exponential family ... relationship btw/ param of interest + Variance.

→ for count data.

Example (Poisson regression):

① Poisson (λ).

③ $\theta = X\beta$

② $\lambda = g^{-1}(\theta) = \exp(\theta)$ $(\lambda > 0)$.

$\Downarrow$

$\theta = \log(\lambda)$.

Some background on choice of g.

Consider a general family of distributions:

subfamily of exponential family dsns that includes Binomial, Poisson, etc.

$$\log f(y_i;\theta_i,\phi) = \frac{y_i\theta_i - b(\theta_i)}{a_i(\phi)} + c(y_i,\phi).$$

$$f(y_i;\theta_i,\phi) = \exp\left\{\frac{y_i\theta_i - b(\theta_i)}{a_i(\phi)} + c(y_i,\phi)\right\}$$

$$= \exp\left\{\underbrace{\frac{y_i\theta_i}{a_i(\phi)} + c(y_i;\phi)}_{*} - \underbrace{\frac{b(\theta_i)}{a_i(\phi)}}_{**}\right\}.$$

recall exponential family w/ parameter $\underline{\theta} = (\theta_1,\dots,\theta_s)^T$ is of the form:

$$f(y;\underline{\theta}) = h(y)\exp\left\{\underbrace{\sum_{j=1}^{s} g_j(\underline{\theta})T_j(y)}_{*} - \underbrace{B(\underline{\theta})}_{**}\right\}$$

assumes $T_1(y_i) = y_i$

$g_1(\underline{\theta}) = \frac{\theta_i}{a_i(\phi_i)}$

subfamily of exponential family.

similar to single param exp family except dispersion term $a_i(\phi)$.

Example (Normal model): $E[y_i] = \mu_i$

$$f(y_i;\mu_i,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(y_i-\mu_i)^2}{2\sigma^2}\right).$$

$$\log f(y_i;\mu_i,\sigma) = \log\left(\frac{1}{\sqrt{2\pi}\sigma}\right) - \frac{(y_i-\mu_i)^2}{2\sigma^2}$$

$$= -\log\left(\sqrt{2\pi}\,\sigma\right) - \frac{y_i^2 - 2\mu_i y_i + \mu_i^3}{2\sigma^2}$$

$$= \boxed{\frac{y_i\mu_i - \frac{\mu_i^2}{2}}{\sigma^2}} - \log\left(\sqrt{2\pi}\sigma\right) - \frac{y_i^2}{2\sigma^2}$$

$\theta_i = \mu_i$

$a_i(\phi) = \sigma^2$

$b(\theta_i) = \frac{\mu_i^2}{2} = \frac{\theta_i^2}{2}$

$c(y_i,\phi) = -\log\left(\sqrt{2\pi}\sigma\right) - \frac{y_i^2}{2\sigma^2}$ (depends on σ^2, not μ_i).

We can learn something about this distribution by considering it's mean and variance. Because we don't have an explicit form of the density, we rely on two facts:

HW 2

1. $\mathrm{E}\left[\frac{\partial \log f(Y_i;\theta_i,\phi)}{\partial \theta_i}\right] = 0.$

These will come up again later (Thursday?).

when we talk about information matrix.

2. $\mathrm{E}\left[\frac{\partial^2 \log f(Y_i;\theta_i,\phi)}{\partial \theta_i^2}\right] + \mathrm{E}\left[\left(\frac{\partial \log f(Y_i;\theta_i,\phi)}{\partial \theta_i}\right)^2\right] = 0.$

For $\log f(y_i;\theta_i,\phi) = \frac{y_i\theta_i - b(\theta_i)}{a_i(\phi)} + c(y_i,\phi),$

Using ①:

$$\frac{\partial}{\partial \theta_i} \log f(y_i;\theta_i,\phi) = \frac{1}{a_i(\phi)}(y_i - b'(\theta_i)).$$

$$\Rightarrow E\left[\frac{1}{a_i(\phi)}(Y_i - b'(\theta_i))\right] \overset{\text{fact ①}}{=} 0 \quad \Rightarrow \quad b'(\theta_i) = E[Y_i]. \Rightarrow$$

information about the mean is contained in $b'(\theta_i)$.

E.g. Normal model

$$b(\theta_i) = \frac{\theta_i^2}{2} \Rightarrow b'(\theta_i) = \theta_i \underset{\text{from form}}{\equiv} \mu_i$$

Using ②:

$$\frac{\partial^2}{\partial \theta_i^2} \log f(y_i;\theta_i,\phi) = \frac{-b''(\theta_i)}{a_i(\phi)} \Rightarrow E\left[\frac{\partial^2}{\partial \theta_i} \log f(Y_i;\theta_i,\phi)\right] = \frac{-b''(\theta_i)}{a_i(\phi)}.$$

$$E\left[\left(\frac{\partial \log f(Y_i;\theta_i,\phi)}{\partial \theta_i}\right)^2\right] = E\left[\left(\frac{1}{a_i(\phi)}(Y_i - b'(\theta_i))\right)^2\right] = \frac{1}{a_i^2(\phi)} E\left[(Y_i - E[Y_i])^2\right]$$

$$\Rightarrow -\frac{b''(\theta_i)}{a_i(\phi)} + \frac{1}{a_i^2(\phi)} Var[Y_i] = 0 \Rightarrow Var[Y_i] = a_i(\phi)\, b''(\theta_i).$$

Thoughts:

- Variance can depend on i
- Variance $Var[Y_i]$ positive $\Rightarrow b''(\theta_i)$ positive $\forall$ values of θ_i

  $\Rightarrow b(\theta_i)$ strictly convex

  $b'(\theta_i)$ monotone increasing $\Rightarrow b'^{-1}$ exists.

Example (Bernoulli model):

$$f(y_i; p_i) = p_i^{y_i}(1-p_i)^{1-y_i}$$

$$\log f(y_i; p_i) = y_i \log p_i + (1-y_i)\log(1-p_i)$$

$$= \frac{y_i \boxed{\log \frac{p_i}{1-p_i}} - \boxed{[-\log(1-p_i).]}}{\boxed{1}} + \underline{0}$$

comparing to general form:

$$\log f(y_i; \theta_i, \phi) = \frac{y_i \boxed{\theta_i} - \boxed{b(\theta_i)}}{\boxed{a_i(\phi)}} + \underline{c(y_i, \phi)}.$$

$$\theta_i = \log\left(\frac{p_i}{1-p_i}\right) \Rightarrow p_i = \frac{\exp(\theta_i)}{1+\exp(\theta_i)}.$$

$$a_i(\theta) = 1$$

$$c(y_i, \phi) = 0$$

$$b(\theta_i) = -\log(1-p_i)$$

$$= -\log\left(1 - \frac{\exp(\theta_i)}{1+\exp(\theta_i)}\right)$$

$$= -\log\left(\frac{1}{1+\exp(\theta_i)}\right)$$

$$= \log(1+\exp(\theta_i)).$$

$$b'(\theta_i) = \frac{1}{1+\exp(\theta_i)} \cdot \exp(\theta_i) = p_i \overset{\checkmark}{=} E[Y_i]$$

$$a_i(\phi) b''(\theta_i) = \left(-(1+\exp(\theta_i))^{-2}\exp(\theta_i)\exp(\theta_i) + (1+\exp(\theta_i))^{-1}\exp(\theta_i)\right.$$

$$= \frac{(1+\exp(\theta_i))\exp(\theta_i) - \exp(\theta_i)\exp(\theta_i)}{1+\exp(\theta_i)}$$

$$= \frac{\exp(\theta_i)}{(1+\exp(\theta_i))^2} = \frac{\exp(\theta_i)}{1+\exp(\theta_i)} \cdot \frac{1}{1+\exp(\theta}$$

$$= p_i(1-p_i) \overset{\checkmark}{=} \mathrm{Var}(Y_i)$$

Finally, back to modelling. Our **goal** is to build a relationship between the mean of Y_i and covariates $\boldsymbol{x}_i$.

choose $\quad x_i^T\beta = g(E[Y_i])$

What to choose for g?

$$E[Y_i] = g^{-1}(x_i^T\beta)$$

We know $\quad E[Y_i] = b'(\theta_i)$

we know this exists!

$$\Rightarrow b'(\theta_i) = g^{-1}(x_i^T\beta) \qquad \text{OR} \qquad \theta_i = b'^{-1}\left(g^{-1}(x_i^T\beta)\right)$$

If we choose $\underbrace{g^{-1} = b'}_{\text{"canonical" or "natural" link function.}}$, then this will clean up nicely! $\theta_i = x_i^T\beta$.

$\Rightarrow$ log-likelihood is

$$\ell\left(\beta, \phi \mid \{Y_i, x_i\}_{i=1}^n\right) = \sum_{i=1}^n \left\{ \frac{Y_i x_i^T\beta - b(x_i^T\beta)}{a_i(\phi)} + c(Y_i, \phi) \right\}.$$

Example (Bernoulli model, cont'd):

$$b(\theta_i) = \log(1 + \exp(\theta_i))$$

$$b'(\theta_i) = \frac{\exp(\theta_i)}{1 + \exp(\theta_i)}.$$

let $g^{-1} = b' \Rightarrow$

$$b'(\theta_i) = \frac{\exp(x_i^T\beta)}{1 + \exp(x_i^T\beta)}$$

$$\begin{aligned} &\parallel \\ &E[Y_i] \\ &\parallel \\ &p_i \end{aligned}$$

(same as before).

1.4 Marginal and Conditional Likelihoods

Consider a model which has $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, where $\boldsymbol{\theta}_1$ are the parameters of interest and $\boldsymbol{\theta}_2$ are nuisance parameters.

not what we are interested in performing inference for.

When dimension of $\underline{\theta}_2$ is large, MLEs of $\underline{\theta}_1$ can be biased for small samples and inconsistent in large samples.

One way to improve estimation for $\boldsymbol{\theta}_1$ is to find a one-to-one transformation of the data $\boldsymbol{Y}$ to $(\boldsymbol{V}, \boldsymbol{W})$ such that either

$$f_{\underline{Y}}(\underline{y};\underline{\theta}_1,\underline{\theta}_2) = f_{\underline{W}|\underline{V}}(\underline{w}|\underline{v};\underline{\theta}_1,\underline{\theta}_2)\,\underline{f_{\underline{V}}(\underline{v};\underline{\theta}_1)}$$

" marginal "

alternative likelihoods

OR

$$f_{\underline{Y}}(\underline{y};\underline{\theta}_1,\underline{\theta}_2) = \underline{f_{\underline{W}|\underline{V}}(\underline{w}|\underline{v};\underline{\theta}_1)}\,f_{\underline{V}}(\underline{v};\underline{\theta}_1,\underline{\theta}_2)$$

" Conditional "

either way, looking to split density into a piece that doesn't depend on $\underline{\theta}_2$

nuisance parameters.

The key feature is that one component of each contains only the parameter of interest.

$\underline{\theta}_1$

Example (Neyman-Scott problem): Let $Y_{ij}, i = 1, \ldots, n, j = 1, 2$ be intependent normal random variables with possible different means μ_i but the same variance σ^2.

$$Y_{ij},\ \underbrace{i=1,\ldots,n}_{n \text{ groups}},\ \underbrace{j=1,2}_{\substack{\text{ONLY two}\\ \text{observations}\\ \text{per group}}} \overset{iid}{\sim} N(\underset{\substack{\uparrow\\ \text{group}\\ \text{mean}}}{\mu_i}, \sigma^2). \leftarrow \text{common variance.}$$

$$\underset{\sim}{\theta} = \underbrace{(\mu_1, \ldots, \mu_n, \sigma^2)^T}_{n+1 \text{ parameters}}$$

Our goal is to estimate σ^2. Should we be able to?

Yes: lots of groups!

No: only 2 obs per group

Usual asymptotic assumptions as n grows

Here as n grows, # of <u>groups</u> grows $\Rightarrow$ # parameters grows.

Following the usual arguments,

$$\hat{\mu}_{i,\text{MLE}} = \frac{Y_{i1} + Y_{i2}}{2}$$

$$\hat{\sigma}^2_{\text{MLE}} = \frac{1}{2n} \sum_{i=1}^{n} \sum_{j=1}^{2} (Y_{ij} - \hat{\mu}_{i,\text{MLE}})^2$$

$$= \frac{1}{2n} \sum_{i=1}^{n} \left\{ \left(Y_{i1} - \frac{Y_{i1} + Y_{i2}}{2}\right)^2 + \left(Y_{i2} - \frac{Y_{i1} + Y_{i2}}{2}\right)^2 \right\}$$

$$= \frac{1}{2n} \sum_{i=1}^{n} \left\{ \underbrace{\left(\frac{Y_{i1} - Y_{i2}}{2}\right)^2}_{} + \underbrace{\left(\frac{Y_{i2} - Y_{i1}}{2}\right)^2}_{\text{equivalent}} \right\}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{1}{4} (Y_{i1} - Y_{i2})^2$$

$$\mathrm{E}[\hat{\sigma}^2_{\mathrm{MLE}}] = E\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{4}\left(Y_{i_1}-Y_{i_2}\right)^2\right]$$

$$\text{(iid)} \quad = \frac{1}{4}E\left[\left[\left(Y_{i_1}-Y_{i_2}\right)^2\right]\right]$$

$$= \frac{1}{4}E\left[\left\{\left(Y_{i_1}-\mu_i\right)-\left(Y_{i_2}-\mu_i\right)\right\}^2\right]$$

$$= \frac{1}{4}E\left[\left(Y_{i_1}-\mu_i\right)^2-2\left(Y_{i_1}-\mu_i\right)\left(Y_{i_2}-\mu_i\right)+\left(Y_{i_2}-\mu_i\right)^2\right]$$

$$= \frac{1}{4}\left[\sigma^2-0+\sigma^2\right]$$

$$= \frac{\sigma^2}{2} \neq \sigma^2 \,!$$

So as $n \to \infty$, $\hat{\sigma}^2_{MLE} \xrightarrow{p} \frac{\sigma^2}{2}$ by WLLN!

This seems bad.

Happens because the # of nuisance parameters grows as n grows.

A reworking of the data seems more promising. Let,

$$V_i = \frac{Y_{i1}-Y_{i2}}{\sqrt{2}} \qquad \text{and} \qquad W_i = \frac{Y_{i1}+Y_{i2}}{\sqrt{2}}$$

Because Y_{ij} are Gaussian,

$V_i \sim N(0, \sigma^2)$ and $W_i \sim N(\sqrt{2}\mu_i, \sigma^2)$. Also, $V_i \perp W_i$!

$$\begin{bmatrix} V_i \\ W_i \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} Y_{i_1} \\ Y_{i_2} \end{bmatrix} \Rightarrow Var\left(\begin{bmatrix} V_i \\ W_i \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\sigma^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \sigma^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

consider the density of V & W:

$$f_{VW}(\underline{v}, \underline{w}; \sigma^2, \mu_1, \ldots, \mu_n) \overset{\text{ind}}{=} \underbrace{f_V(\underline{v}; \sigma^2)}_{\text{no nuisance parameters!}} f_W(w; \mu_1, \ldots, \mu_n, \sigma^2)$$

$$\Rightarrow \ell(\sigma \mid \underline{V}) = -n\log\sqrt{2\pi} - n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}V_i^2$$

$$\frac{\partial \ell(\sigma \mid \underline{V})}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}V_i^2 \Rightarrow \hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}V_i^2$$

A marginal likelihood approach is simple provided you can find a statistic V whose dsn is free of the nuisance parameter!

For conditional likelihoods, we can often exploit the existence of sufficient statistics for the nuisance parameters under the assumption that the parameter of interest is known.

let T_i be sufficient for a nuisance parameter (μ_i)

Then the condition distribution of the data ($\underline{Y}$) given $\underline{T} = (T_1, .., T_n)$ doesn't depend on the nuisance parameters.

$\Rightarrow$ We can look for the conditional dsn of data | $\underline{T}$

Example (Exponential Families): The structure of exponential families is such that it is often possible to exploit their properties to eliminated nuisance parameters. Let Y have a density of the form

$$f(y;\boldsymbol{\eta}) = h(y)\exp\left\{\sum_{i=1}^{s}\eta_i T_i(y) - A(\boldsymbol{\eta})\right\},$$

then

IF $\eta = (\underline{\theta}_1^T, \underline{\theta}_2^T)$,

(Th'm 2.6 pg 104)

sufficient statistics

$$f(y; \underline{\theta}_1, \underline{\theta}_2) = h(y) \exp\left\{ \sum \theta_{1i} W_i + \sum \theta_{2j} V_j - A(\underline{\theta}_1, \underline{\theta}_2) \right\}$$

Then the conditional dsn of $\underline{W}$ given $\underline{V}$ is exponential family of the form:

$$f(\underline{w} | \underline{v}; \underline{\theta}_1) = q(\underline{w}) \exp\left\{ \sum_{i=1}^{r} \theta_{1i} w_i - A_{\underline{v}}(\underline{\theta}_1) \right\}$$

a conditional density that doesn't depend on $\underline{\theta}_2$.

Thus, exponential families often provide an automatic procedure for finding $\boldsymbol{W}$ and $\boldsymbol{V}$.

Example (Logistic Regression): For binary Y_i, the standard logistics regression model is

$$P(Y_i = 1) = p_i(\boldsymbol{x}_i, \boldsymbol{\beta}) = \frac{\exp(\boldsymbol{x}_i^\top \boldsymbol{\beta})}{1 + \exp(\boldsymbol{x}_i^\top \boldsymbol{\beta})}$$

and the likelihood is

$$\begin{aligned}
L(\boldsymbol{\beta}|\boldsymbol{Y}, \boldsymbol{X}) &= \prod_{i=1}^{n} p_i(\underline{x}_i, \beta)^{Y_i} \{1 - p_i(\underline{x}_i, \beta)\}^{1-Y_i} \\
&= \prod_{i=1}^{n} \left\{ \frac{\exp(\underline{x}_i^\top \beta)}{1+\exp(\underline{x}_i^\top \beta)} \right\}^{Y_i} \left\{ \frac{1}{1+\exp(\underline{x}_i^\top \beta)} \right\}^{1-Y_i} \\
&= \frac{\exp(\sum_{i=1}^{n} Y_i (x_i^\top \beta)\}}{\prod_{i=1}^{n} (1+\exp(\underline{x}_i^\top \beta))} \\
&= c(X, \beta) \exp\left(\sum_{j=1}^{p} \beta_j \sum_{i=1}^{n} x_{ij} Y_i \right).
\end{aligned}$$

$\Rightarrow T_j = \sum_{i=1}^{n} x_{ij} Y_i \quad j = 1, .., p$ are sufficient for this exponential family model.

Suppose $\theta_1 = \beta_K$ is the parameter of interest and the others are nuisance parameters.

$\Rightarrow W_1 = T_K = \sum_{i=1}^{n} x_{iK} Y_i$ and $V = (T_1, \dots, T_{K-1}, T_{K+1}, \dots, T_p)^T$

$\frac{\text{joint}}{\text{marginal}}$

and the conditional density $P(T_K = t_K \mid T_1 = t_1, .., T_{K-1} = t_{K-1}, T_{K+1} = t_{K+1}, .., T_p = t_p)$

$$\vdots$$

$$= \frac{c(t_1, .., t_p) \exp(\beta_K t_K)}{\sum_u c(t_1, .., t_{K-1}, u, t_{K+1}, .., t_p) \exp(\beta_K u)}$$

only depends on β_K $\Rightarrow$ we can maximize this to get $\hat{\beta}_K$

There exists fast computational ways to compute.