A Tutorial on Logistic Regression

Ying So, SAS Institute Inc., Cary, NC

ABSTRACT

Many procedures in SAS/STAT® can be used to perform logistic regression analysis: CATMOD, GENMOD,LOGISTIC, and PROBIT. Each procedure has special features that make it useful for certain applications. For most applications, PROC LOGISTIC is the preferred choice. It fits binary response or proportional odds models, provides various model-selection methods to identify important prognostic variables from a large number of candidate variables, and computes regression diagnostic statistics. This tutorial discusses some of the problems users encountered when they used the LOGISTIC procedure.

INTRODUCTION

PROC LOGISTIC can be used to analyze binary response as well as ordinal response data.

Binary Response

The response, Y, of a subject can take one of two possible values, denoted by 1 and 2 (for example, Y=1 if a disease is present; otherwise Y=2). Let $\mathbf{x} = (x_1, \ldots, x_k)'$ be the vector of explanatory variables. The logistic regression model is used to explain the effects of the explanatory variables on the binary response.

$$\text{logit}\{\Pr(Y = 1|\mathbf{x})\} = \log\left\{\frac{\Pr(Y = 1|\mathbf{x})}{1 - \Pr(Y = 1|\mathbf{x})}\right\} = \beta_0 + \mathbf{x}'\boldsymbol{\beta}$$

where β_0 is the intercept parameter, and $\boldsymbol{\beta}$ is the vector of slope parameters (Hosmer and Lameshow, 1989).

Ordinal Response

The response, Y, of a subject can take one of m ordinal values, denoted by $1, 2, \ldots, m$. PROC LOGISTIC fits the following cumulative logit* model:

$$\text{logit}\{\Pr(Y \leq r|\mathbf{x})\} = \alpha_r + \mathbf{x}'\boldsymbol{\beta} \qquad 1 \leq r < m$$

where $\alpha_1, \ldots, \alpha_{m-1}$ are (m-1) intercept parameters. This model is also called the proportional odds model because the odds of making response $\leq r$ are exp($\boldsymbol{\beta}'(\mathbf{x}_1 - \mathbf{x}_2)$) times higher at $\mathbf{x} = \mathbf{x}_1$ than at $\mathbf{x} = \mathbf{x}_2$ (Agresti, 1990).

This ordinal model is especially appropriate if the ordinal nature of the response is due to methodological limitations in collecting the data in which the researchers are forced to

*logit of the cumulative probabilities

lump together and identify various portions of an otherwise continuous variable. Let T be the underlying continuous variable and suppose that

$$Y = r \quad \text{if} \quad \gamma_{r-1} < T \leq \gamma_r$$

for some $-\infty = \gamma_0 < \gamma_1 < \ldots < \gamma_m = \infty$. Let $x_0 = 1$. Consider the regression model

$$T = \sum_{i=0}^{k} \beta_i^* x_i + e$$

where $\beta_0^*, \beta_1^*, \ldots, \beta_{m-1}^*$ are regression parameters and e is the error term with a logistic distribution F. Then

$$\Pr(Y \leq r) = \Pr(T \leq \gamma_r) = F(\gamma_r - \sum_{i=0}^{k} \beta_i^* x_i)$$

or

$$\text{logit}\{\Pr(Y \leq r|\mathbf{x})\} = \gamma_r - \sum_{i=0}^{k} \beta_i^* x_i$$

This is equivalent to the proportional odds model given earlier.

INFINITE PARAMETERS

The term *infinite parameters* refers to the situation when the likelihood equation does not have a finite solution (or in other words, the maximum likelihood estimate does not exist). The existence of maximum likelihood estimates for the logistic model depends on the configurations of the sample points in the observation space (Albert and Anderson, 1984, and Santner and Duffy, 1985). There are three mutually exclusive and exhaustive categories: complete separation, quasicomplete separation, and overlap.

Consider a binary response model. Let Y_i be the response of the ith subject and let $\mathbf{x}_i = (1, x_{i1}, \ldots, x_{ik})'$ be the vector of explanatory variables (including the constant 1).

Complete Separation

There is a complete separation of data points if there exists a vector $\mathbf{b}$ that correctly allocates all observations to their response groups; that is,

$$\begin{cases} \mathbf{b}'\mathbf{x}_i > 0 & Y_i = 1 \\ \mathbf{b}'\mathbf{x}_i < 0 & Y_i = 2 \end{cases}$$

The maximum likelihood estimate does not exist. The loglikelihood goes to 0 as iteration increases.

The following example illustrates such situation. Consider the data set DATA1 (Table 1) with 10 observations. Y is the response and x_1 and x_2 are two explanatory variables.

Table 1. Complete Separation Data (DATA1)

Observation	Y	x_1	x_2
1	1	29	62
2	1	30	83
3	1	31	74
4	1	31	88
5	1	32	68
6	2	29	41
7	2	30	44
8	2	31	21
9	2	32	50
10	2	33	33

Figure 1 shows that the vector $\mathbf{b} = (6, -2, 1)'$ completely separates the observations into their response groups; that is, all observations of the same response lie on the same side of the line $x_2 = 2x_1 - 6$.
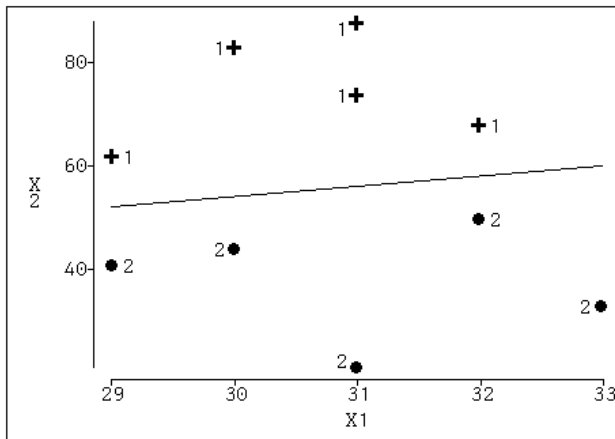

Figure 1. Scatterplot of Sample Points in DATA1

The iterative history of fitting a logistic regression model to the given data is shown in Output 1. Note that the negative loglikehood decreases to 0 --- a perfect fit.

Quasicomplete Separation

If the data are not completely separated and there exists a vector $\mathbf{b}$ such that

$$\begin{cases} \mathbf{b}'\mathbf{x}_i \geq 0 & Y_i = 1 \\ \mathbf{b}'\mathbf{x}_i \leq 0 & Y_i = 2 \end{cases}$$

with equality holds for at least one subject in each response group, there is a quasicomplete separation. The maximum likelihood estimate does not exist. The loglikelihood does not diminish to 0 as in the case of complete separation, but the dispersion matrix becomes unbound.

Output 1. Partial LOGISTIC Printout for DATA1

```
                 Maximum Likelihood Iterative Phase

Iter Step        -2 Log L     INTERCPT          X1          X2

  0 INITIAL     13.862944            0            0            0
  1 IRLS         4.691312    -2.813220    -0.062042     0.083761
  2 IRLS         2.280691    -2.773158    -0.187259     0.150942
  3 IRLS         0.964403    -0.425345    -0.423977     0.238202
  4 IRLS         0.361717     2.114730    -0.692202     0.339763
  5 IRLS         0.133505     4.250789    -0.950753     0.443518
  6 IRLS         0.049378     6.201510    -1.203505     0.547490
  7 IRLS         0.018287     8.079876    -1.454499     0.651812
  8 IRLS         0.006774     9.925139    -1.705284     0.756610
  9 IRLS         0.002509    11.748893    -1.956323     0.861916
 10 IRLS         0.000929    13.552650    -2.207666     0.967727
 11 IRLS         0.000344    15.334133    -2.459215     1.074024
 12 IRLS         0.000127    17.089516    -2.710811     1.180784
 13 IRLS      0.000047030    18.814237    -2.962266     1.287983
 14 IRLS      0.000017384    20.503310    -3.213375     1.395594
 15 IRLS      0.000006423    22.151492    -3.463924     1.503590
 16 IRLS      0.000002372    23.753408    -3.713693     1.611943
 17 IRLS      0.000000876    25.303703    -3.962463     1.720626
 18 IRLS      0.000000323    26.797224    -4.210021     1.829610
 19 IRLS      0.000000119    28.229241    -4.456170     1.938869
 20 IRLS      4.3956397E-8    29.595692    -4.700735     2.048377
 21 IRLS      1.620409E-8    30.893457    -4.943572     2.158109
 22 IRLS      5.9717453E-9    32.120599    -5.184576     2.268042
 23 IRLS      2.2002107E-9    33.276570    -5.423689     2.378153
 24 IRLS      8.10449E-10    34.362317    -5.660901     2.488421
 25 IRLS      2.984679E-10    35.380281    -5.896252     2.598826

WARNING: Convergence was not attained in 25 iterations.
         Iteration control is available with the MAXITER and the
         CONVERGE options on the MODEL statement.
```

You can modify DATA1 to create a situation of quasicomplete separation, for instance, change $x_2 = 44$ to $x_2 = 64$ in observation 6. Let the modified data set be DATA2. With $\mathbf{b} = (-4, -2, 1)'$, the equality holds for observations 1, 5, and 7, and the rest of the observations are separated into their response groups (Figure 2). It is easy to see that there is no straight line that can completely separate the two response groups.
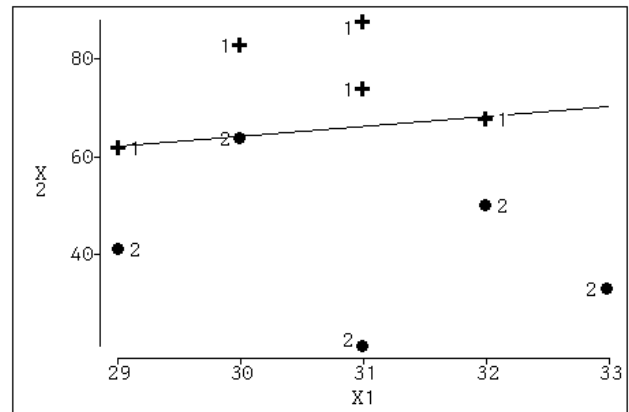

Figure 2. Scatterplot of Sample Points in DATA2

The parameter estimates during the iterative phase are displayed in Output 2 and the dispersion matrices for iterations 0, 5, 10, 15, and 25 are shown in Output 3. The log-likelihood approaches a nonzero constant . The seemingly large variances of pseudoestimates are typical of a quasicomplete separation of data.

Output 2. Partial LOGISTIC Printout for DATA2

```
              Maximum Likelihood Iterative Phase

Iter Step      -2 Log L     INTERCPT          X1          X2

  0 INITIAL   13.862944            0           0           0
  1 IRLS       6.428374    -4.638506    0.003387    0.077640
  2 IRLS       4.856439    -7.539932   -0.011060    0.131865
  3 IRLS       4.190154    -9.533783   -0.066638    0.190242
  4 IRLS       3.912968   -11.081432   -0.146953    0.252400
  5 IRLS       3.800380   -11.670780   -0.265281    0.316912
  6 IRLS       3.751126   -11.666819   -0.417929    0.388135
  7 IRLS       3.727865   -11.697310   -0.597641    0.472639
  8 IRLS       3.716764   -11.923095   -0.806371    0.573891
  9 IRLS       3.711850   -12.316216   -1.038254    0.688687
 10 IRLS       3.709877   -12.788230   -1.281868    0.810247
 11 IRLS       3.709130   -13.282112   -1.529890    0.934224
 12 IRLS       3.708852   -13.780722   -1.779320    1.058935
 13 IRLS       3.708750   -14.280378   -2.029162    1.183855
 14 IRLS       3.708712   -14.780288   -2.279118    1.308833
 15 IRLS       3.708698   -15.280265   -2.529107    1.433827
 16 IRLS       3.708693   -15.780258   -2.779104    1.558826
 17 IRLS       3.708691   -16.280257   -3.029103    1.683825
 18 IRLS       3.708691   -16.780256   -3.279103    1.808825
 19 IRLS       3.708690   -17.280256   -3.529103    1.933825
 20 IRLS       3.708690   -17.780256   -3.779103    2.058825
 21 IRLS       3.708690   -18.280256   -4.029102    2.183825
 22 IRLS       3.708690   -18.780255   -4.279102    2.308825
 23 IRLS       3.708690   -19.280256   -4.529102    2.433825
 24 IRLS       3.708690   -19.780257   -4.779103    2.558825
 25 IRLS       3.708690   -20.280250   -5.029099    2.683824

WARNING: Convergence was not attained in 25 iterations.
         Iteration control is available with the MAXITER and the
         CONVERGE options on the MODEL statement.
```

Output 3. Dispersion Matrices on Selected Iterations (DATA2)

```
              Iter= 0      -2 Log L = 13.862944

Variable     INTERCPT           Z1          Z2      ESTIMATE

INTERCPT 269.05188212   -8.42405441  -0.157380245   0
Z1        -8.42405441  0.2673239797  0.0032615725   0
Z2       -0.157380245  0.0032615725  0.0009747228   0


              Iter=5       -2 Log L = 3.800380

Variable     INTERCPT           Z1          Z2      ESTIMATE

INTERCPT 985.12006548  -29.47104673  -1.460819309  -11.670780
Z1       -29.47104673  1.4922999204  -0.242120428   -0.265281
Z2       -1.460819309  -0.242120428  0.1363093424    0.316912


              Iter= 10     -2 Log L = 3.709877

Variable     INTERCPT           Z1          Z2      ESTIMATE

INTERCPT 1391.583624    169.160036  -100.9268654  -12.788230
Z1        169.160036   105.7305273  -52.20138038   -1.281868
Z2       -100.9268654  -52.20138038  26.043666498    0.810247


              Iter= 15     -2 Log L = 3.708698

Variable     INTERCPT           Z1          Z2      ESTIMATE

INTERCPT 62940.299541  30943.762505  -15488.22021  -15.280265
Z1       30943.762505  15493.136539  -7745.900995   -2.529107
Z2       -15488.22021  -7745.900995  3872.8917604    1.433827


              Iter=20      -2 Log L = 3.708690

Variable     INTERCPT           Z1          Z2      ESTIMATE

INTERCPT 9197536.1382  4598241.6822   -2299137.18  -17.780256
Z1       4598241.6822  2299142.0966  -1149570.381   -3.779103
Z2        -2299137.18  -1149570.381  574785.13177    2.058825


              Iter=25      -2 Log L = 3.708690

Variable     INTERCPT           Z1          Z2      ESTIMATE

INTERCPT 502111231.75  251055089.49  -125527561.1  -20.280250
Z1       251055089.49    125527566  -62763782.33   -5.029099
Z2       -125527561.1  -62763782.33  31381891.107    2.683824
```

Overlap

If neither complete nor quasicomplete separation exists in the sample points, there is an overlap of sample points. The maximum likelihood estimate exists and is unique.
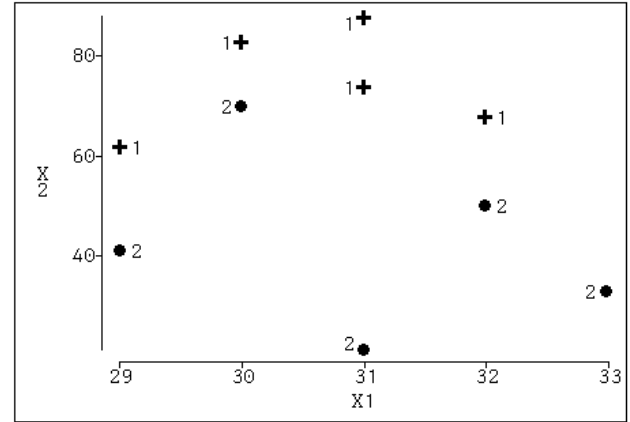

Figure 3. Scatterplot of Sample Points in DATA3

If you change $x_2 = 44$ to $x_2 = 74$ in observation 6 of DATA1, the modified data set (DATA3) has overlapped sample points. A scatterplot of the sample points in DATA3 is shown in Figure 3. For every straight line on the drawn on the plot, there is always a sample point from each response group on same side of the line. The maximum likelihood estimates are finite (Output 4).

Output 4. PROC LOGISTIC Printout for DATA3

```
              Maximum Likelihood Iterative Phase

Iter Step      -2 Log L     INTERCPT          X1          X2

  0 INITIAL   13.862944            0           0           0
  1 IRLS       7.192759    -4.665775    0.011192    0.073238
  2 IRLS       6.110729    -7.383116    0.010621    0.116549
  3 IRLS       5.847544    -8.760124   -0.013538    0.148942
  4 IRLS       5.816454    -9.185086   -0.033276    0.164399
  5 IRLS       5.815754    -9.228343   -0.037848    0.167125
  6 IRLS       5.815754    -9.228973   -0.037987    0.167197
  7 IRLS       5.815754    -9.228973   -0.037987    0.167197

Last Change in -2 Log L: 2.282619E-13


              Last Evaluation of Gradient

        INTERCPT             X1              X2

     -1.109604E-7     -3.519319E-6     -3.163568E-6
```

Empirical Approach to Detect Separation

Complete separation and quasicomplete separation are problems typical for small sample. Although complete separation can occur with any type of data, quasicomplete separation is not likely with truly continuous data.

At the jth iteration, let $\mathbf{b}^j$ be the vector of pseudoestimates. The probability of correct allocation based on $\mathbf{b}^j$ is given by

$$
\begin{cases}
\dfrac{\exp(\mathbf{x}'\mathbf{b}^j)}{1+\exp(\mathbf{x}'\mathbf{b}^j)} & Y = 1 \\[2ex]
\dfrac{1}{1+\exp(\mathbf{x}'\mathbf{b}^j)} & Y = 2
\end{cases}
$$

3

- Stop at the iteration when the probability of correct allocation is 1 for all observations. There is a complete separation of data points. For DATA1, correct allocation of all data points is achieved at iteration 13 (Table 2).
- At each iteration, look for the observation with the largest probability of correct allocation. If this probability has become extremely close to 1, and any diagonal element of the dispersion matrix becomes very large, stop the iteration. It is very likely there is a quasicomplete separation in the data set. Table 3 displays the maximum probability of correct allocation for DATA2. The dispersion matrix should be examined after the 5th iteration.

Table 2.   Percentage of Correct Allocation (DATA1)

j	-2 Log L	b_0^j	b_1^j	b_2^j	% of Correct Allocation
0	13.8629	0.0000	0.00000	0.00000	0
1	4.6913	-2.8132	-0.06204	0.08376	0
2	2.2807	-2.7732	-0.18726	0.15094	0
3	0.9644	-0.4253	-0.42398	0.23820	0
4	0.3617	2.1147	-0.69220	0.33976	10
5	0.1335	4.2508	-0.95075	0.44352	40
6	0.0494	6.2015	-1.20351	0.54749	40
7	0.0183	8.0799	-1.45450	0.65181	40
8	0.0068	9.9251	-1.70528	0.75661	50
9	0.0025	11.7489	-1.95632	0.86192	50
10	0.0009	13.5527	-2.20767	0.96773	50
11	0.0003	15.3341	-2.45922	1.07402	60
12	0.0001	17.0895	-2.71081	1.18078	80
13	0.0000	18.8142	-2.96227	1.28798	100
14	0.0000	20.5033	-3.21338	1.39559	100
15	0.0000	22.1515	-3.46392	1.50359	100

Table 3.   Maximum Probability of Correct Allocation (DATA2)

j	-2 Log L	b_0^j	b_1^j	b_2^j	Maximum Probability
0	13.8629	0.0000	0.00000	0.00000	0.50000
1	6.4284	-4.6385	0.00339	0.07764	0.87703
2	4.8564	-7.5399	-0.01106	0.13187	0.97217
3	4.1902	-9.5338	-0.06664	0.19024	0.99574
4	3.9130	-11.0814	-0.14695	0.25240	0.99950
5	3.8004	-11.6708	-0.26528	0.31691	0.99995
6	3.7511	-11.6668	-0.41793	0.38814	1.00000
7	3.7279	-11.6973	-0.59764	0.47264	1.00000
8	3.7168	-11.9231	-0.80637	0.57389	1.00000
9	3.7119	-12.3162	-1.03825	0.68869	1.00000
10	3.7099	-12.7882	-1.28187	0.81025	1.00000
11	3.7091	-13.2821	-1.52989	0.93422	1.00000
12	3.7089	-13.7807	-1.77932	1.05894	1.00000
13	3.7088	-14.2804	-2.02916	1.18386	1.00000
14	3.7087	-14.7803	-2.27912	1.30883	1.00000
15	3.7087	-15.2803	-2.52911	1.43383	1.00000

ORDERING OF THE BINARY RESPONSE LEVELS

If the binary response is 0 and 1, PROC LOGISTIC, by default, models the probability of 0 instead of 1; that is,

$$\log\left\{ \frac{\Pr(Y = 0|\mathbf{x})}{\Pr(Y = 1|\mathbf{x})} \right\} = \beta_0 + \mathbf{x}'\boldsymbol{\beta}$$

This is consistent with the cumulative logit model, though this may not always be desirable because 1 is often used to denote the response of the event of interest. Consider the following logistic regression example. Y is the response variable with value 1 if the disease is present and 0 otherwise. EXPOSURE is the only explanatory variable with value 1 if the subject is exposed and 0 otherwise.

```
data disease;
   input y exposure freq;
   cards;
   1 0 10
   1 1 40
   0 0 45
   0 1  5
   ;
   run;
proc logistic data=disease;
   model y=exposure;
   freq freq;
   run;
```

Output 5.   Logistic Regression of Disease on Exposure

```
                   Response Profile

             Ordered
              Value        Y      Count

                1          0         50
                2          1         50


           Criteria for Assessing Model Fit

                      Intercept
           Intercept     and
Criterion    Only     Covariates  Chi-Square for Covariates

AIC         140.629     87.550         .
SC          143.235     92.761         .
-2 LOG L    138.629     83.550     55.079 with 1 DF (p=0.0001)
Score          .           .       49.495 with 1 DF (p=0.0001)


        Analysis of Maximum Likelihood Estimates

               Parameter   Standard     Wald        Pr >
Variable   DF   Estimate     Error    Chi-Square  Chi-Square

INTERCPT    1     1.5041     0.3496     18.5093     0.0001
EXPOSURE    1    -3.5835     0.5893     36.9839     0.0001

        Analysis of Maximum
       Likelihood Estimates

           Standardized       Odds
Variable     Estimate        Ratio

INTERCPT          .          4.500
EXPOSURE      -0.987849      0.028
```

Results of the analysis are displayed in Output 5. Since the coefficient for EXPOSURE is negative, as EXPOSURE

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changes from 0 to 1, the probability of "no disease" decreases. This is a less direct way of saying that the probability of "disease" increases with EXPOSURE.

Since

$$\log\left\{\frac{\Pr(Y=1|\mathbf{x})}{\Pr(Y=0|\mathbf{x})}\right\} = -\log\left\{\frac{\Pr(Y=0|\mathbf{x})}{\Pr(Y=1|\mathbf{x})}\right\}$$

the probability of response 1 is given by

$$\log\left\{\frac{\Pr(Y=1|\mathbf{x})}{1-\Pr(Y=1|\mathbf{x})}\right\} = -\beta_0 - \mathbf{x}'\beta$$

That is, the regression coefficients for modeling the probability of 1 will have the same magnitude but opposite sign as those of modeling the probability of 0. In order to have a more direct interpretation of the regression coefficient, it is desirable to model the probability of the event of interest. In the LOGISTIC procedure, the response levels are sorted according to the ORDER= option (the Response Profiles table lists the ordering of the responses). PROC LOGISTIC then models the probability of the response that corresponds to the lower ordered value.

Note that the first observation in the given input data has response 1. By using the option ORDER=DATA, the response 1 will have ordered value 1 and response 0 will have ordered value 2. As such the probability modeled is the probability of response 1. There are several other ways that you can reverse the response level ordering in the given example (Schlotzhauer, 1993).

- The simplest method, available in Release 6.07 TS301 and later, uses the option DESCENDING. Specify the DESCENDING option on the PROC LOGISTIC statement to reverse the ordering of Y.

```
proc logistic data=disease descending;
   model y=exposure;
   freq freq;
   run;
```

- Assign a format to Y such that the first formatted value (when the formatted values are put in sorted order) corresponds to the presence of the disease. For this example, Y=0 could be assigned the formatted value 'no disease' and Y=1 could be assigned the formatted value 'disease'.

```
proc format;
   value disfmt 1='disease'
                0='no disease';
   run;
proc logistic data=disease;
   model y=exposure;
   freq freq;
   format y disfmt.;
   run;
```

- Create a new variable to replace Y as the response variable in the MODEL statement such that observation Y=1 takes on the smaller value of the new variable.

```
data disease2;
   set disease;
   if y=0 then y1='no disease';
   else 'disease';
   run;
```

```
proc logistic data=disease2;
   model y1=exposure;
   freq freq;
   run;
```

- Create a new variable (N, for example) with constant value 1 for each observation. Use the *event/trial* MODEL statement syntax with Y as the *event* variable and N as the *trial* variable.

```
data disease3;
   set disease;
   n=1;
   run;
proc logistic data=disease;
   model y/n=exposure;
   freq freq;
   run;
```

OTHER LOGISTIC REGRESSION APPLICATIONS

There are many logistic regression models that are not of the standard form as given earlier (Agresti, 1990, and Strauss, 1992). For some of them you could "trick" PROC LOGISTIC to do the estimation, for others you may have to resort to other means. The following sections discuss some of the models that are often inquired by SAS users.

Conditional Logistic Regression

Conditional logistic regression is useful in investigating the relationship between an outcome and a set of prognostic factors in a matched case-control studies, the outcome being whether the subject is a case or a control. When there is one case and one control in a matched set, the matching is 1:1. 1:n matching refers to the situation when there is one case and a varying number of controls in a matched set. For the ith set, let $\mathbf{u}_i$ the covariate vector for the case and let $\mathbf{v}_{i1}, \ldots, \mathbf{v}_{in_i}$ be the covariate vectors for the n_i controls. The likelihood for the *N* matched sets is given by

$$\mathcal{L}(\beta) = \prod_{i=1}^{N} \frac{\exp(\mathbf{u}_i'\beta)}{\sum_{j=1}^{n_i} \exp(\mathbf{v}_{ij}'\beta)}$$

For the 1-1 matching, the likelihood is reduced to

$$\mathcal{L}(\beta) = \prod_{i=1}^{N} \frac{\exp(\mathbf{u}_i'\beta)}{\exp(\mathbf{u}_i'\beta) + \exp(\mathbf{v}_{i1}'\beta)}$$

By dividing the numerator and the denominator by $\exp(\mathbf{v}_{i1}'\beta)$, one obtains

$$\mathcal{L}(\beta) = \prod_{i=1}^{N} \frac{\exp((\mathbf{u}_i - \mathbf{v}_{i1})'\beta)}{1 + \exp((\mathbf{u}_i - \mathbf{v}_{i1})'\beta)}$$

Thus the likelihood is identical to that of the binary logistic model with $\mathbf{d}_i = \mathbf{u}_i - \mathbf{v}_{i1}$ as covariates, no intercept, and a constant response. Therefore, you can "trick" PROC LOGISTIC to perform the conditional logistic regression for 1-1 matching (See Example 5 of the LOGISTIC documentation). For 1:n matching, it is more convenient to use PROC PHREG (see Example 3 of the PHREG documentation).

Bradley-Terry Model for Paired Comparison

The Bradley-Terry Model is useful in establishing the overall ranking of n items through paired comparisons. For instance, it is difficult for a panelist to rate all 9 brands of beer at the same occasion; rather it is preferable to compare the brands in a pairwise manner. For a given pair of products, the panelist would state his preference after tasting them at the same occasion. Let $\beta_1, \beta_2, \ldots, \beta_n$ be regression coefficients associated with the n items $I_1, \ldots, I_n$, respectively. The probability that I_i is preferred to I_j is

$$
\begin{aligned}
\pi_{ij} &= \frac{\exp(\beta_i)}{\exp(\beta_i) + \exp(\beta_j)} \\
&= \frac{\exp(\beta_i - \beta_j)}{1 + \exp(\beta_i - \beta_j)}
\end{aligned}
$$

and, therefore, the likelihood function for the paired comparison model is

$$
\mathcal{L}(\beta_1, \ldots, \beta_n) = \prod_{(i,j) \in A} \pi_{ij}
$$

where A is the sample collection of all the test pairs. For the lth pair of comparison, if I_i is preferable to I_j, let the vector $\mathbf{d}_l = (d_{l1}, \ldots, d_{ln})$ be such that

$$
d_{lk} = \begin{cases} 1 & k = i \\ -1 & k = j \\ 0 & \text{otherwise} \end{cases}
$$

The likelihood for the Bradley-Terry model is identical to the binary logistic model with $\mathbf{d}_l$ as covariates, no intercept, and a constant response.

Multinormial Logit Choice Model

The multinormial logit model is useful in investigating consumer choice behavior and has become increasingly popular in marketing research. Let C be a set of n choices, denoted by $\{1, 2, \ldots, n\}$. A subject is present with alternatives in C and is asked to choose the most preferred alternative. Let $\mathbf{x}_i$ be a covariate vector associated with the alternative i. The multinomial logit model for the choice probabilities is given by

$$
\Pr(i|\mathbf{C}) = \frac{\exp(\mathbf{x}_i'\beta)}{\sum_{j=1}^{n} \exp(\mathbf{x}_j'\beta)}
$$

where β is a vector of unknown regression parameters. It is difficult to use PROC LOGISTIC to fit such a model. Instead, by defining a proper time and a proper censoring variable, you can trick PROC PHREG to provide the maximum likelihood estimates of the parameters. For details on using PROC PHREG to analyse discrete choice studies, write to Warren Kuhfeld at SAS Institute Inc. (email: saswfk@unx.sas.com) for a copy of the article "Multinormial Logit, Discrete Choice Model."

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