Last time

prob. of intersection of events

multiplication rule

SRS – w/o. repl. symmetry

conditional prob. and Bayes' rule

forward conditioning – likelihood

backward 〃 – posterior / inference

tree diagram

This time

Independence (of events)

Example : roll a fair, 6-sided die twice

P(at least one six)

$= \frac{6}{36} + \frac{6}{36} - \boxed{\frac{1}{36}}$

P(both are sixes) $= \frac{1}{36}$

$= P(\text{first is } 6)\, P(\text{second is } 6 \mid \text{first is } 6)$

$= \frac{1}{6} \times \frac{1}{6}$ is due to the fact that the first roll has nothing to do with the second.

Definition. We say two events are indpt. independent. if the information that one occurs does not change the prob. that the other occurs.

$P(\underset{\text{the other}}{B} \mid \underset{\text{one}}{A}) = \underset{\text{the other}}{P(B)}$

mult. rule

For events A, B, $P(AB) = P(A)P(B \mid A)$

For events A, B indpt. $P(AB) = P(A)P(B)$. simplified mult. rule

A & B exchangeable

A indpt. of B ⟺ B indpt. of A.

Example (of not indpt.)

deal two cards from a standard deck of 52 cards w/o. repl.

$P(H_1H_2) = P(H_1)P(H_2 \mid H_1)$

$= \frac{13}{52} \times \frac{12}{51}$

$P(H_2) = \frac{13}{52} \neq P(H_2 \mid H_1)$

H_1 is not indpt. of H_2

or H_1 is dependent of H_2.

Reading : stories in textbook about misuse of independence

Chapter 3. Integer-valued Random Variable

"Random counts"

# of buses pass a bus stop during a day

# of bananas consumed per week in a canteen.

# of sixes in 5 rolls of a fair 6-sided die

many more.

* formal definition of R.V.

a function maps an outcome to a number

$X : \Omega \to \mathbb{Z}$

$\omega \mapsto X(\omega)$

informal definition.

Examples (we discussed).

① Coin tossing $<^{H}_{T}$ $p = \frac{1}{2}$ or not

② dice rolling multiple outcomes from a single roll.

③ cards dealing (usually) w/o. repl. SRS.

Coin tossing model

H – success

T – failure

each toss – a trial

Reading : Chapter 3.1

Toss once.

outcome space $\Omega = \{H, T\}$.

$P(H) = p$. $P(T) = 1-p$.

"Bernoulli distribution" (Bern(p))

say : a R.V. has some distr. distribution.

toss a fair coin twice. $X_1 = 1^{st}$ toss

$X_2 = 2^{nd}$ toss

Can we say $X_1 = X_2$? No.

Correct: $X_1 \overset{d}{=} X_2$ X_1 is equal in distr. to X_2.

Uniform distribution

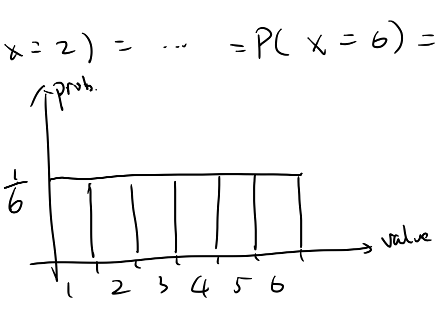
Comes from dice rolling.

X = # shown from a single roll of a fair, 6-sided die

outcome space $\Omega = \{1, 2, \ldots, 6\}$.

$P(X=1) = P(X=2) = \cdots = P(X=6) = \frac{1}{6}$

"Histogram"



Example: # of H's in 3 tosses.

outcome space $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

assumption of equally likely outcomes.

$P(HHH) = \cdots = \frac{1}{8}$.

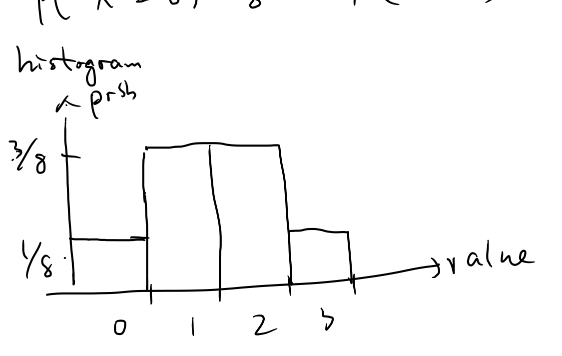
Distr. table

# of H's	0	1	2	3
# of outcomes	1	3	3	1
prob.	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

prob. mass function (p.m.f / pmf)

$P(X=0) = \frac{1}{8}$, $P(X=1) = \frac{3}{8}$, ...

histogram



Some important (non-random) counts

① full permutation of n items $\{1, 2, \ldots, n\}$

$n!$

② permutation of k items from a total of n items

i.e. select k items w/ in order

$n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}$

③ # of ways to select k items from n items without order (at the same time)

Similar to ②, but a lot of duplication

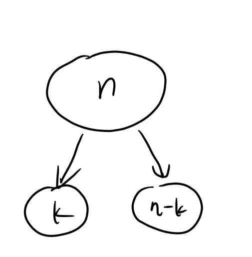
Suppose there are $\binom{n}{k}$ ways to do so. needs to choose k.

To recover the order, we need a full permutation of them k items.

$\binom{n}{k} k! = n \cdot (n-1) \cdots (n-k+1)$

$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$

$= \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$



Also known as binomial coefficient.

because of binomial expansion of $(a+b)^n$

n=0 1

n=1 a+b

n=2 $a^2+2ab+b^2$

n=3 $a^3+3a^2b+3ab^2+b^3$

1

1 1

1 2 1

1 3 3 1

"Pascal's triangle"

Binomial distr.

- repeated trials (identically distributed trials)
- independent trials
- total # of trials = n, a fixed number.

Example roll a die 5 times. treat this as a coin with $p = \frac{1}{6}$

P(exactly 2 sixes) P(two H's among 5 tosses)

in setup of coin-tossing

H = six.

T = non-six

$P(HHTTT) = \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3$

$P(HTHTT) = \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} = \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3$

How many such sequences?

# of ways to have exactly 2 H's

= # of ways to select 2 numbers from $\{1, 2, \ldots, 5\}$

$= \binom{5}{2} = \frac{5 \times 4}{2 \times 1} = 10$

$P(\text{exactly 2 sixes}) = \binom{5}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3$

Generally, n tosses of a p-coin. $(P(H) = p)$

$P(\text{exactly } k \text{ H's}) = \binom{n}{k} p^k (1-p)^{n-k}$ $k = 0, 1, 2, \ldots, n$ range of this R.V.

Denote as $X \sim \text{Binom}(n, p)$. $P(X = k) =$

$P(X=0) = \binom{n}{0} p^0 (1-p)^{n-0} = (1-p)^n$

$P(X=n) = \binom{n}{n} p^n (1-p)^{n-n} = p^n$

Geometric distribution

Defn: X = # of trials needed for the first success.

TTTTTH 6 sometimes also 5 (excluding the success)

$P(X = k) = (1-p)^{k-1} p$ $k = 1, 2, 3, \ldots$

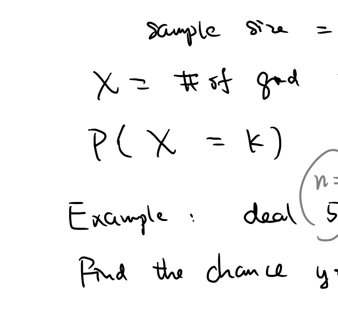
Check that this is a distr.

$\sum_{k=1}^{\infty} P(X=k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p$

$(l = k-1) \quad = p \sum_{l=0}^{\infty} (1-p)^l$ geometric sequence with common ratio $(1-p)$

$= \frac{1}{1-(1-p)} \cdot p = 1$.

histogram



Hypergeometric distn SRS

setup. population = N # of 'good' individuals = G

= target

of 'bad' individuals = B = N − G

sample size = n. (w/o. repl.)

X = # of good samples

$P(X = k)$

Example : deal 5 cards from a deck $n=5$ ⟹ N = 52

Find the chance you get exactly 2 Aces. ⟹ G = 4

B = 48

X = # of Aces dealt

$P(X = 2)$

Analogue :

$P(GGBBB) = \frac{4}{52} \times \frac{3}{51} \times \frac{48}{50} \times \frac{47}{49} \times \frac{46}{48}$

$P(GBGBB) = \frac{4}{52} \times \frac{48}{51} \times \frac{3}{50} \times \frac{47}{49} \times \frac{46}{48}$

the same!

$P(X=2) = \binom{5}{2} P(GGBBB)$

Alt. calculation. do as a whole : draw 5 cards at once.

outcome spaces $\Omega = \{$all choices of 5 cards from the deck$\}$

Assumption of equally likely outcomes holds.

total # of ways $= |\Omega| = \binom{52}{5}$

# of target ways = # of ways to have 2 Aces & 3 others. ← Not a prob. question!

= (# of ways to have 2 Aces) × (# of ways to have 3 non-Aces)

$= \binom{4}{2} \times \binom{48}{3}$

$P(X=2) = \frac{\text{\# of target ways}}{\text{total \# of ways}} = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$

Generally, for N, G, n

$P(X = g) = \frac{\binom{G}{g}\binom{B}{b}}{\binom{N}{n}}$ $\binom{B}{b} = \binom{N-G}{n-g}$.

Histogram