Last time:
problem solving technique
NB.
c.d.f.
Python (Stats . xxx . (pmf / cdf)

Today:
Finish Chapter 4 Poisson
Expectation.

preparation:
exponential Approx.
"Bootstrap"
Suppose we have a sample of size n.
Do a re-sample of size n with replacement
Some are resampled (≥ 1 times), some are not.
For a special individual from the original sample,
what is the chance that it is re-sampled ? (not)

Total # of re-sample $= n$ fixed
prob. of re-sample the special one $= \frac{1}{n}$

$\Rightarrow X :=$ # of times we re-sample the special one $\sim \text{Binom}(n, \frac{1}{n})$

$P(X>0) = 1 - P(X=0)$ (Left R.V.) follows (Right distribution)

$P(X=0) = (1-\frac{1}{n})^n$

For n large, this is e^{-1} (exponential approx.)

Fact 1: $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$ (by Def'n)

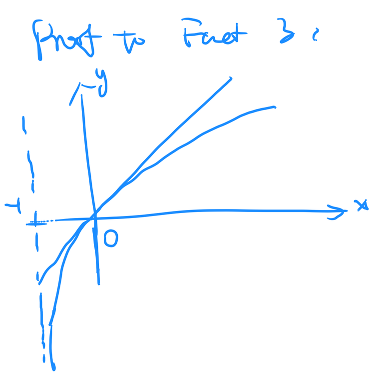
Fact 2: $\lim_{n\to\infty}(1-\frac{1}{n})^n = e^{-1}$ $\Leftrightarrow$ $\lim_{p\to 0}(1-p)^{\frac{1}{p}} = e^{-1}$

Fact 3. $\ln(1+x) \approx x$ for $|x|$ small

$1 \Rightarrow 2$: $e^{-1} = \lim_{n\to\infty}(1+\frac{1}{n})^{-n}$
$= \lim_{n\to\infty}\left(\left(\frac{n}{n+1}\right)\right)^n$
$= \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n$

$\lim_{n\to\infty}(1-\frac{1}{n})^n = \lim_{n\to\infty}(1-\frac{1}{n+1})^{n+1}$
$= \lim_{n\to\infty}\underbrace{(1-\frac{1}{n+1})^n}_{e^{-1}} \cdot \underbrace{(1-\frac{1}{n+1})}_{1} = e^{-1}$

Proof to Fact 3:



WTS: ① LHS(0) = RHS(0)
② LHS'(0) = RHS'(0)
① $\ln(1) = 0$ ✓
② $\frac{d}{dx}\ln(1+x)\Big|_{x=0} = \frac{1}{1+x}\Big|_{x=0} = 1$ ✓

Generally, for $Y \sim \text{Binom}(n,p)$, n large & p small
$np = \mu$.

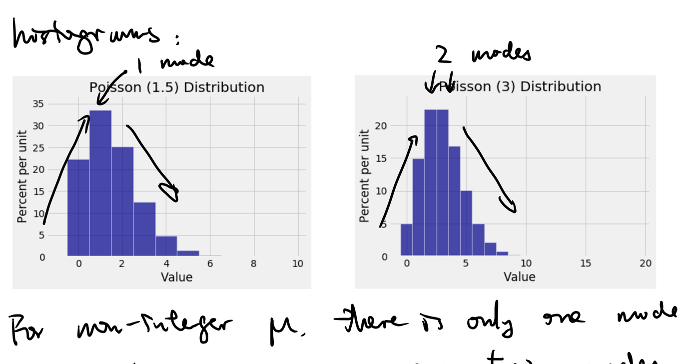
$P(Y=0) = (1-p)^n = \left(\underbrace{(1-p)^{\frac{1}{p}}}_{e^{-1} \text{ for } p \text{ small}}\right)^{np \ \mu} \approx e^{-\mu}$

$\Updownarrow$

Alt. solution:
$\ln P(Y=0) = n\ln(1-p) \approx n\cdot(-p) = -\mu$

Poisson Distr. $T \sim \text{Poisson}(\mu)$
p.m.f. $P(T=k) = e^{-\mu}\frac{\mu^k}{k!}$, $k = 0, 1, 2, \cdots$

histograms:



For non-integer μ, there is only one mode
For integer —— one two modes.

Check:
$\sum_{k=0}^{\infty} P(T=k) = \sum_{k=0}^{\infty} e^{-\mu}\frac{\mu^k}{k!}$
$= e^{-\mu}\sum_{k=0}^{\infty}\frac{\mu^k}{k!}$ Taylor Expansion of e^{μ} at $\mu=0$
$= e^{-\mu}\cdot e^{\mu} = 1$

Example.
large cases of USB drives $n = 1000$, $p = 0.0025$ $\Rightarrow np = 2.5$
each contains Poisson(2.5) defective ones., indpt
What is the chance that all of the next five cases contains more than 1 defective USB drive?

Consider one case $P(>1) = 1 - P(0) - P(1)$
$= 1 - e^{-2.5}\frac{2.5^0}{\underbrace{0!}_{=1}} - e^{-2.5}\frac{2.5^1}{1!}$

For 5 cases prob. $= (P(>1))^5$

$P(T=0) = e^{-\mu}$.

Property: Sum of indpt Poissons
$X \sim \text{Poisson}(\mu)$ $Y \sim \text{Poisson}(\lambda)$ indpt
then $S := X+Y \sim \text{Poisson}(\mu+\lambda)$

Example. $X_i \sim \text{Poi}(i)$, $i = 1, 2, 3$. indpt.
$P(X_1+X_2+X_3 \leq 10)$
$S := X_1+X_2+X_3 \sim \text{Poisson}(1+2+3)$
$P(S \leq 10)$.

(Poisson) Law of small numbers
Consider $X \sim \text{Binom}(n,p)$, n big, p small.
We have shown that $P(X=0) \approx P(T=0)$
for $T \sim \text{Poisson}(np)$.

$P(X=1) = \binom{n}{1}p(1-p)^{n-1}$
$P(T=1) = e^{-np}\cdot\frac{(np)^1}{1!}$

$\frac{P(X=1)}{P(T=1)} = \frac{\cancel{np}(1-p)^{n-1}}{e^{-np}\ \cancel{np}} = \underbrace{\frac{(1-p)^n}{e^{-np}}}_{\approx 1} \cdot \underbrace{\frac{1}{1-p}}_{\approx 1} \approx 1$

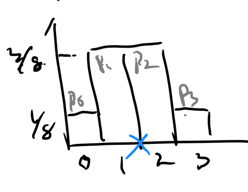
$P(X=0)$ $\approx P(T=0)$

One can check that
$P(X=k) \approx P(T=k)$

"When n is large and p is small, $\text{Binom}(n,p)$ is well-approximated by Poisson($\mu = np$)"
Binomial coefficient $\binom{n}{k}$ is time-consuming for n large'

Chapter 5.
"average"
Def'n For a discrete R.V. X, its expectation is
$\underbrace{\mathbb{E}X}_{\mathbb{E}X} = \sum_x x\,\underbrace{\mathbb{P}}_{P}(X=x)$
where $\sum_x$ is over Range of X

Example. 3 coin tosses.



$\mathbb{E}X = 0p_0 + 1p_1 + 2\cdot p_2 + 3\cdot p_3$ "weighted average"
$= 0\cdot\frac{1}{8} + 1\cdot\frac{3}{8} + 2\frac{3}{8} + 3\cdot\frac{1}{8}$
$= \frac{3+6+3}{8} = \frac{12}{8} = 1.5$

Notes:
① $\mathbb{E}X$ is a number, not random at all
② $\mathbb{E}X$ does not have to be a possible value
③ $\mathbb{E}X$ & X has the same unit measurement
④ Suppose we can repeat the experiment
$\mathbb{E}X$ is approx. the long-run average.
⑤ (physics interpretation) center of gravity

Example.
① Constant R.V. $\mathbb{E}c = c$
② Bernoulli $X \sim \text{Bern}(p)$. $\mathbb{E}X = 1\cdot p + 0\cdot(1-p) = p$.
1_A for $\mathbb{P}(A) = p$. "indicator R.V. of A"
$= \begin{cases} 1 & \text{if } A \text{ happens} \\ 0 & \text{o/w.} \end{cases}$
③ Discrete Uniform $X \sim \text{Unif}(\{1, 2, \cdots, n\})$
e.g. $n=6$ (roll a die once)
$\mathbb{E}X = \frac{1}{6}(1+2+3+\cdots+6) = \cancel{\frac{1}{6}}\cdot\frac{1}{2}\cdot\cancel{6}\cdot(6+1)$
$= \frac{7}{2} = 3.5$

④ Poisson $T \sim \text{Poisson}(\mu)$
$\mathbb{E}T = \sum_{t=0}^{\infty} t\left(e^{-\mu}\cdot\frac{\mu^t}{t!}\right)$
$= \sum_{t=1}^{\infty} e^{-\mu}\cdot\frac{\mu^t}{(t-1)!}$
$(s = t-1)$ $= \sum_{s=0}^{\infty} e^{-\mu}\frac{\mu^{s+1}}{s!}$
$= \mu\underbrace{\left(\sum_{s=0}^{\infty} e^{-\mu}\frac{\mu^s}{s!}\right)}_{\text{Sum of Poisson p.m.f}} = \mu$

Expectation of functions of R.V.s
Def'n For a R.V. X and a function g,
$\mathbb{E}g(X) = \sum_x g(x)\,\mathbb{P}(X=x)$
$\mathbb{E}X = \sum_x x\,\mathbb{P}(X=x)$ (special case when $g(x) = x$).

More R.V.s
joint distribution
Example: deal 2 cards from a standard deck
$X_1 = 1(\text{first is heart})$ $\mathbb{E}X_1 = \frac{13}{52}$
$X_2 = 1(\text{second is heart})$. $\mathbb{E}X_2 = \frac{13}{52}$

joint distr. table

$X_2 \backslash X_1$	0	1	
0	$p_{0,0} = \frac{39}{52}\times\frac{38}{51}$	$p_{1,0} = \frac{13}{52}\times\frac{39}{51}$	$\frac{39}{52}$
1	$p_{0,1} = \frac{39}{52}\times\frac{13}{51}$	$p_{1,1} = \frac{13}{52}\times\frac{12}{51}$	$\frac{13}{52}$
	$\frac{39}{52}$	$p_1 = \frac{13}{52}$	

e.g. $P(X_1+X_2=1)$
$= p_{1,0} + p_{0,1}$

marginal distr. i.e. distr. of a single R.V.
$P(X_1 = 0)$

$\mathbb{E}(\underbrace{X_1+X_2}_{=S}) = \sum_s s\cdot\mathbb{P}(S=s)$
$= 0\cdot\mathbb{P}(S=0) + 1\cdot\mathbb{P}(S=1) + 2\cdot\mathbb{P}(S=2)$
$= 1\cdot(p_{1,0}+p_{0,1}) + 2\cdot p_{1,1}$
$= 2\cdot p_{1,0} + 2\cdot p_{1,1}$
$= 2\cdot p_1 = 2\cdot\frac{13}{52}$

Additivity of expectation:
$\mathbb{E}(X+Y) = \mathbb{E}X + \mathbb{E}Y$
as long as $X+Y$ makes sense.