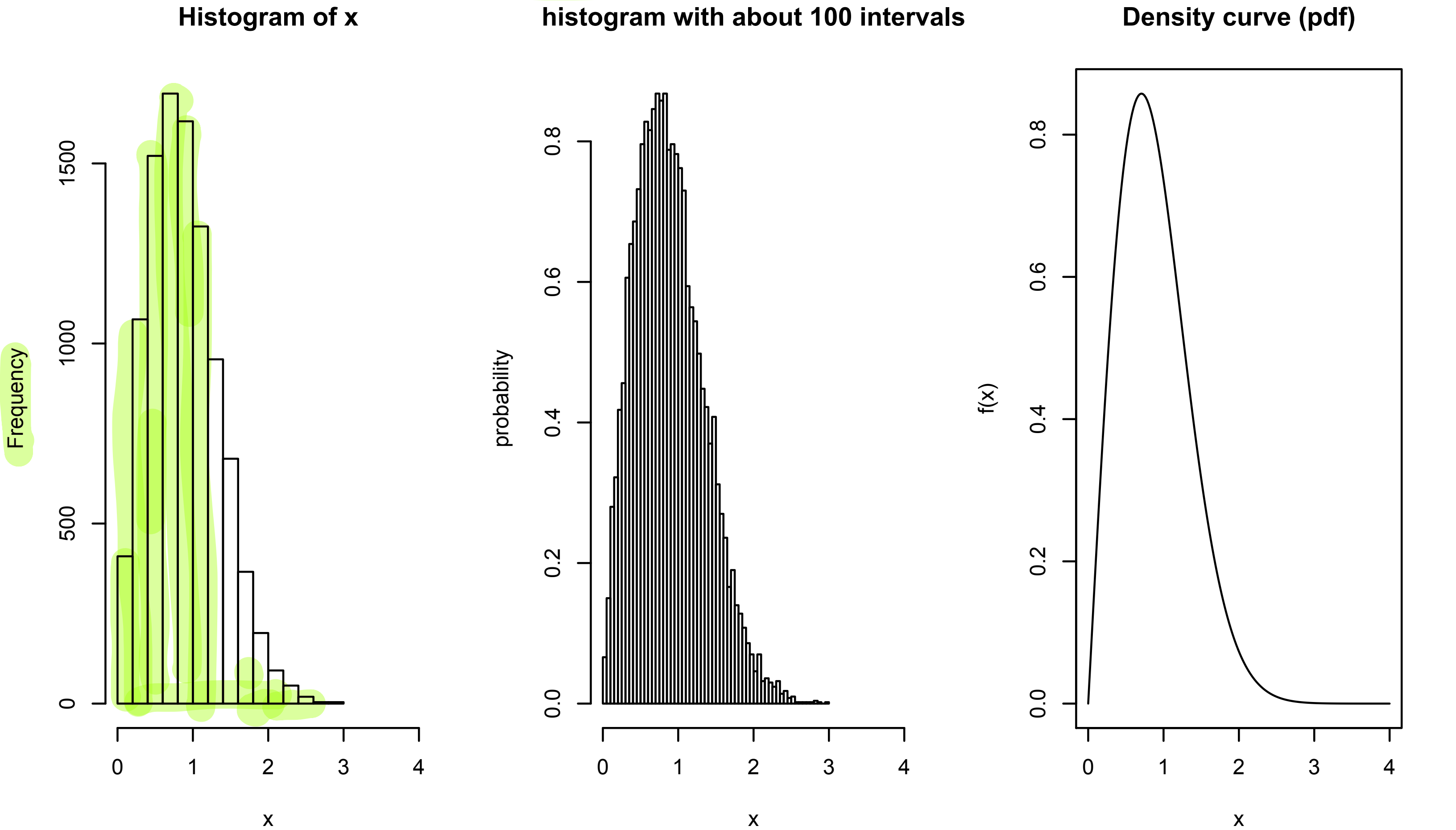
PROBABILITY DENSITY FUNCTIONS

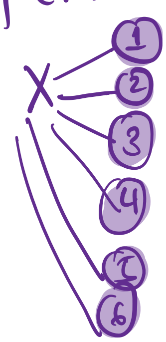


Lecture 24: Chapter 10

So far we have talked about DISCRETE dens Random variable with discrete mass functions are restricted to take particular values on $\mathbb{R}$, or in an interval

For example let X be the # of spots when we roll a fair die. $X = 1, 2, 3, 4, 5$ or 6 w.p. $\frac{1}{6}$ each

$f(x)$ = p.m.f. of X & $f(x) = \begin{cases} \frac{1}{6}, & x = 1,2,3,4,5,6 \\ 0, & \text{o/w} \end{cases}$



mass of 1 is divided into 6 equal portions

Instead of restricting the r.v. we might allow it to take any value in some interval.

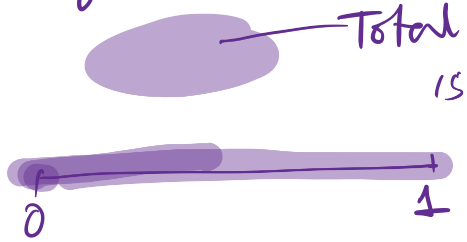
For example, let X be a random # b/w 0 & 1

Instead of a p.m.f., we will define a prob. DENSITY function. So imagine that the total prob. mass of 1 is smeared over the interval.

Such r.v. that can take any values in an interval on the real line are called
CONTINUOUS RANDOM VARIABLES

Their probabilities are specified by functions called probability density functions & the cdf.

Say X takes any value in $(0,1)$



Total prob mass of 1 is smeared across the interval $(0,1)$ so all subintervals of the same length are equally likely.

$X, \underset{\text{PDF}}{f(x)}, \underset{\text{CDF}}{F(x)}$: define a r.v. & its distribution

Only 2 conditions on $f(x)$

$$\boxed{\begin{array}{l} ① \ f(x) \geq 0 \\ ② \ \displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1 \end{array}}$$

- Probabilities are now computed using integrals, not sums
- To compute the prob. that X is in the interval (a,b) we compute the area under the curve $y = f(x)$ over (a,b)
- $P(X \underset{\text{"is in"}}{\overset{}{\in}} (a,b)) = P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

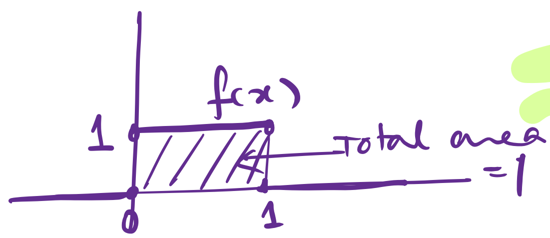
$= P(a \leq X < b)$
$= P(a \leq X \leq b) = P(a < X \leq b)$

$P(X=a)=0$ for a continuous r.v. & any a since a single value has no area.

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

$$F'(x) = f(x)$$ (by the fundamental theorem of Calculus)

Ex X takes any value in $(0,1)$ "uniformly"

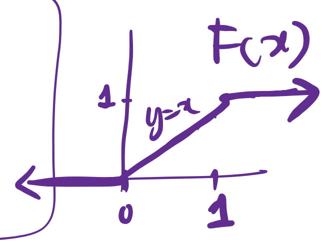


$$f(x) = \begin{cases} 0, & x \le 0 \\ 1 & 0 < x < 1 \\ 0, & x \ge 1 \end{cases}$$

Such a r.v. is called the uniform r.v.

$X \sim U[0,1]$

$$\boxed{F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0, & x \le 0 \\ x, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}}$$



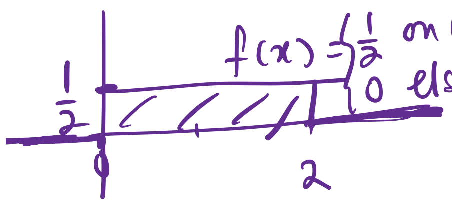
$$F\left(\tfrac{1}{2}\right) = \int_{-\infty}^{\frac{1}{2}} f(t)\,dt = \int_{0}^{\frac{1}{2}} f(t)\,dt = \int_{0}^{\frac{1}{2}} 1\,dt = \frac{1}{2}$$

$$F(x) \quad \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} 1\,dt = x, \quad 0 < x < 1$$

Ex 2 X is uniform on $(0,2)$

(a) Write down the density function $f(x)$

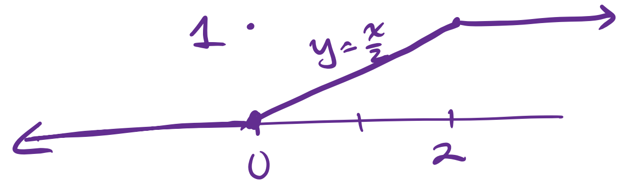
(b) " " dsn function $F(x)$



$f(x) = \begin{cases} \frac{1}{2} & \text{on } (0,2) \\ 0 & \text{elsewhere} \end{cases}$

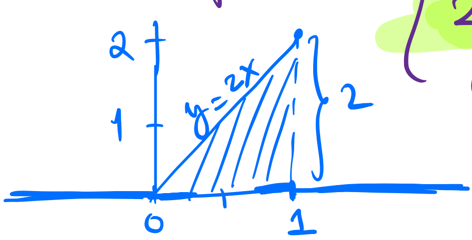
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} \frac{1}{2}\,dt = \frac{x}{2}, \quad x \in (0,2)$$

$$F(x) = \begin{cases} 0 & , x \le 0 \\ \frac{x}{2} & , 0 < x < 2 \\ 1 & , x > 2 \end{cases}$$



Ex3 Verify that following functions define probability densities. Also find expressions for the associated cdf.

(a) $f(x) = \begin{cases} 0, & x \le 0 \\ 2x, & 0 < x < 1 \\ 0, & x \ge 1 \end{cases}$



$f(1) = 2 \times 1 = 2 > 1$

Density $\neq$ probability

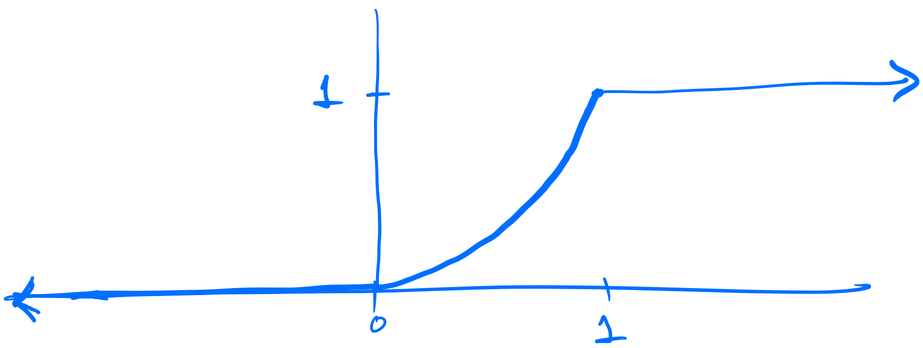
Total area $= \frac{1}{2} \cdot 1 \cdot 2 = 1$ so $\int_{-\infty}^{\infty} f(t)\,dt = 1$, so f is a density function

Exercise: Write down $F(x)$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} 2t\,dt \quad , \quad 0 < x < 1$$

$$= t^2 \Big|_0^x = x^2 \quad , \quad 0 < x < 1$$

$$F(x) = \begin{cases} 0 \, , & x < 0 \\ x^2 \, , & 0 \leq x < 1 \\ 1 \, , & x > 1 \end{cases}$$



$$0 \leq F(x) \leq 1$$

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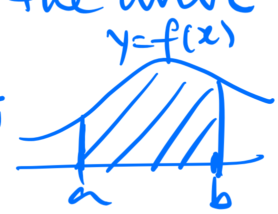
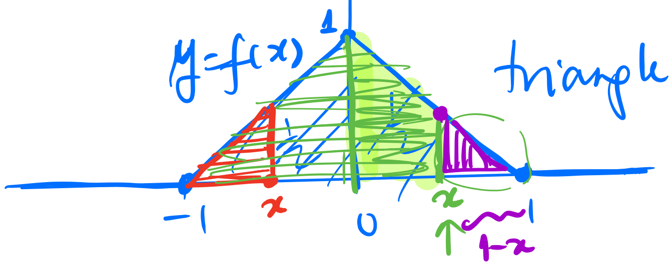
cdf $= F(x)$ is area under the curve $y = f(x)$ upto the value x

So $P(a < X < b)$ = area under the curve $y = f(x)$ between a & b $\int_a^b f(t)\,dt$



or $\boxed{\int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt = F(b) - F(a) = P(a < X < b)}$

(b) $f(x) = \begin{cases} 0, & x < -1 \\ x+1, & -1 \leq x < 0 \\ 1-x, & 0 \leq x < 1 \\ 0, & x \geq 1 \end{cases}$



triangle function

total area $= \frac{1}{2} + \frac{1}{2}$
$= 1$

$F(x)$ = green area
$= \frac{1}{2} + \underbrace{\square}_{\frac{1}{2} - \blacktriangle} = 1 - \blacktriangle$

<u>Moral</u> : Can use basic geometry.

<u>Ex 4</u> : A very special density function

$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

① Verify that $f(x)$ is a density fn.

② find $F(x)$

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_0^{\infty} e^{-x}\,dx = -e^{-x}\Big|_0^{\infty} = 0-(-1)$$
$$= 1$$

$$F(x) = \int_0^x f(t)\,dt = \int_0^x e^{-t}\,dt = \begin{cases} 1-e^{-x}, & x>0 \\ 0, & x\le 0 \end{cases}$$

X with this pdf is called an exponential r.v.

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① A prob. density function for a continuous r.v X is a function $f(x)$ s.t. ① $f(x)\ge 0$, ② $\int_{-\infty}^{\infty} f(x)\,dx = 1$

② cdf $F(x) = P(X\le x) = \int_{-\infty}^{x} f(t)\,dt$

③ $P(X=x)=0$

④ $P(a<X<b) = P(a\le X< b)$

$= P(a\le X\le b) = P(a<X\le b)$

$= \int_a^b f(x)\,dx$ = area under the curve $y=f(x)$ over (a,b)

$= F(b)-F(a)$

Expected Value of a continuous r.v. X

Let X be a continuous r.v. with p.d.f $f(x)$

then $\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

For ex. Let $X \sim U(0,1)$

$$\mu = \mathbb{E}(X) = \int_0^1 x \cdot 1 \cdot dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

$$= \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx - \left(\int_{-\infty}^{\infty} x f(x)\,dx\right)^2$$

$$= \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx - \mu^2$$

If $X \sim U[0,1]$, $\mathbb{E}(X) = \mu = \frac{1}{2}$

$\mathrm{Var}(X) = ?$ $\quad \mathbb{E}(X^2) = \int_0^1 x^2 \cdot f(x)\,dx = \int_0^1 x^2 \cdot 1\,dx$

$$= \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Let $g(X)$ be a function of X

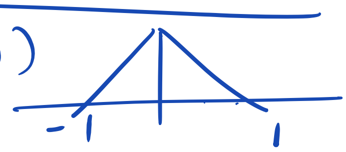
What will be $\mathbb{E}(g(X))$?

Say: $g(x) = x^2$, $X \sim f(x)$, where $f(x) = 2x$, $x \in (0,1)$ & 0 otherwise

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx = \int_0^1 x^2 [2x\,dx]$$

$$= \int_0^1 2x^3\,dx = \frac{\cancel{2}x^4}{\cancel{4}2}\Bigg|_0^1 = \frac{1}{2}$$

Exercise: Let $f(x)$ be as in 3(b)



Find ① $P(\frac{1}{2} < X < 1)$

② $P(0 < X < \frac{1}{2})$