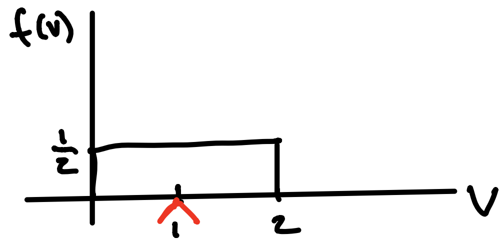
Stat 88 lec 31

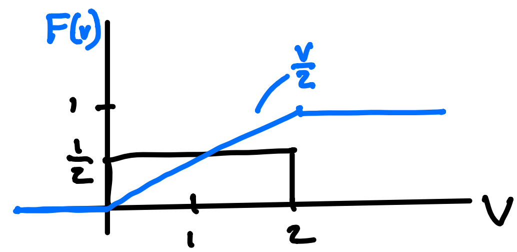
Warmup 2:00-2:10

Let V have density $f(v) = \begin{cases} \frac{1}{2} & \text{for } 0 \le v \le 2 \\ 0 & \text{else.} \end{cases}$



a) Find the cdf of V

b) Find Var(V)



$F(v) = \begin{cases} \frac{v}{2} & 0 \le v \le 2 \\ 0 & \text{else} \end{cases}$

$Var(V) = E(V^2) - (E(V))^2$

$E(V) = 1$

$E(V^2) = \int_0^2 v^2 \cdot \frac{1}{2}\, dv = \frac{v^3}{6}\Big|_0^2 = 4/3$

$Var(V) = 4/3 - 1^2 = \boxed{1/3}$

Last time

sec 10.1 density.

For discrete RVs, such as $X \sim \text{Binomial}(n,p)$, the probability mass function, $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$, or the cdf $P(X \le k)$ describes the distribution.

For continuous RVs, such as $Z \sim N(0,1)$, the density, $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$, or the cdf $P(Z \le z) = \Phi(z)$, describes the distribution.

A density is always nonnegative and integrates to 1.

If X is a continuous RV, f is a density of X if $P(X \in (x, x+dx)) = f(x)\,dx$ and

$$P(X \in (a,b)) = \int_a^b f(x)\,dx.$$

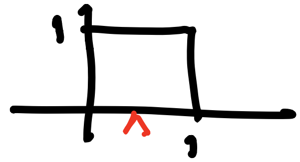
sec 10.2 expectation and variance

If a continuous RV, X, has density f,

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \quad \text{and} \quad E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

and $Var(X) = E(X^2) - (E(X))^2$

ex Let $U \sim \text{Unif}(0,1)$



Find $\text{Var}(U)$

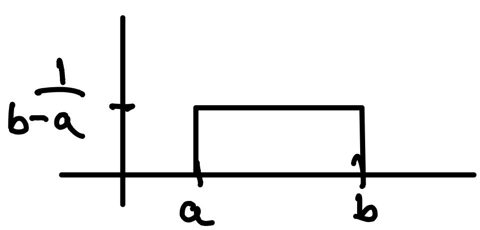
$$E(U^2) = \int_0^1 u^2 \, 1 \, du = \frac{u^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\text{Var}(U) = E(U^2) - (E(U))^2$$
$$= \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}}$$

Today

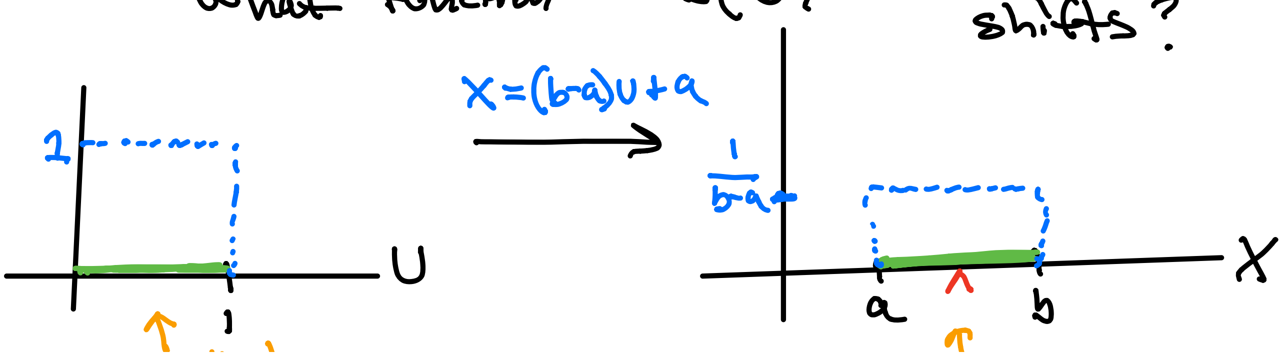
1. Sec 10.2 $U(a,b)$
2. Sec 10.3 The Exponential Distribution

1. Uniform (a,b)



$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{else} \end{cases}$$

What function $X = g(U)$ stretches and shifts?



length 1

length $b-a$

Find

$$E(X) = E\big((b-a)U + a\big) = (b-a)\underbrace{E(U)}_{=1/2} + a = \boxed{\frac{b+a}{2}}$$

$$\begin{aligned} Var(X) &= Var\big((b-a)U + a\big) \\ &= (b-a)^2 \underbrace{Var(U)}_{=1/12} \\ &= \boxed{\frac{(b-a)^2}{12}} \end{aligned}$$

exercise 10.5.2

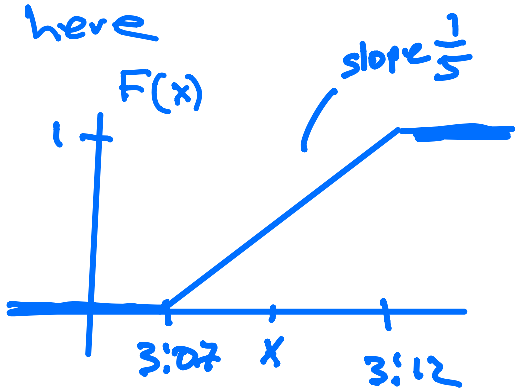
2. A class starts at 3:10 p.m. Seven students in the class arrive at random times $T_1, T_2, \ldots, T_7$ that are i.i.d. with the uniform distribution on the interval 3:07 to 3:12.

c) Let $X = \max(T_1, T_2, \ldots, T_7)$ be the time when the last of the seven students arrives. Find the cdf of X.

$$P(\overset{\text{max}}{X} \le x) = P(T_1 \le x, T_2 \le x, \ldots, T_7 \le x)$$

$$= \big(P(T_1 \le x)\big)^7$$

So we need to find the cdf of T_1, $F(x)$ which I graph here



The equation of the line between 3:07 and 3:12

$$F(x) = \frac{x - 3{:}07}{5} \quad \text{for } 3{:}07 \leq x \leq 3{:}12$$

For $x > 3{:}12$, $F(x) = 1$

and for $x < 3{:}07$, $F(x) = 1$

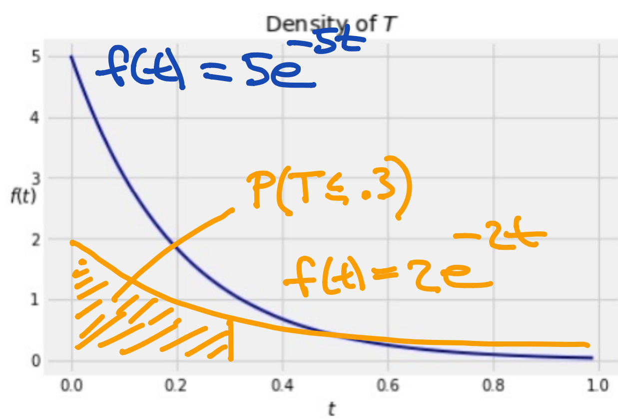
Hence,

$$\boxed{P(X \leq x) = \begin{cases} 1 & \text{if } x > 3{:}12 \\ \left(\dfrac{x - 3{:}07}{5}\right)^7 & \text{if } 3{:}07 \leq x \leq 3{:}12 \\ 0 & \text{if } x < 3{:}07 \end{cases}}$$

② The Exponential Distribution

Let $\lambda > 0$,

A RV T has the exponential distribution with rate λ (written $T \sim Exp(\lambda)$), if the density of T is $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$.



ex T = lifetime of a lightbulb.
λ = rate a lightbulb burns out = 2 bulbs/decade

$$P(T \leq .3) = \int_0^{.3} 2e^{-2t}\,dt = -e^{-2t}\Big|_0^{.3}$$

3 years

$$= -e^{-.6} + 1$$

$$= 1 - e^{-.6} = .45$$

(the chance a bulb dies in 3 years is .45)

CDF and survival function

The cdf, $P(T \le t)$ = chance the lightbulb dies before time t

$$F(t) = P(T \le t) = \int_0^t \lambda e^{-\lambda s} ds$$

$$= -e^{-\lambda s} \Big|_0^t = \boxed{1 - e^{-\lambda t}}$$

The survival function, $S(t)$, is the chance the lightbulb last longer than time t.

$$S(t) = P(T > t) = 1 - F(t) = \boxed{e^{-\lambda t}}$$

Memoryless Property

Lets find the chance T survives time $t+s$, given that it survives time t:

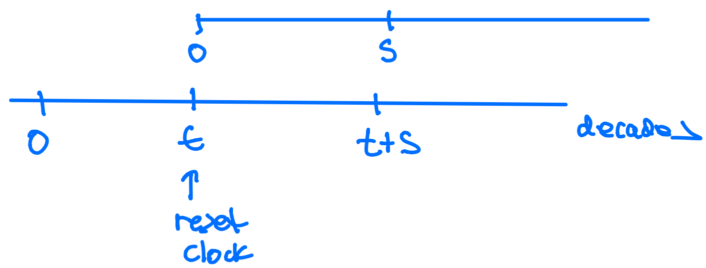
$S(t) = e^{-\lambda t}$

$$P(T > t+s \mid T > t) \overset{\text{Baye's rule}}{=} \frac{P(T > t+s, T > t)}{P(T > t)}$$

$$= \frac{P(T > t+s)}{P(T > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s}$$

$= P(T > s)$

Picture



So if the bulb lasts a decade the chance it will last another decade is the same as that it lasts 1 decade (it forgets that it lived for 1 decade already).

The exponential distribution is the continuous analog to the geometric distribution. Both have the memoryless property.

Mean and SD

Let $T \sim Exp(\lambda)$

$$E(T) = \int_0^\infty t \underbrace{\lambda e^{-\lambda t}}_{f(t)} dt = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$u = t \qquad v = -e^{-\lambda t}$

$du = dt \qquad dv = \lambda e^{-\lambda t}$

$$= \underbrace{-te^{-\lambda t}\Big|_0^\infty}_{=0} - \int_0^\infty -e^{-\lambda t} dt$$

$$= \int_0^\infty e^{-\lambda t} dt = -\frac{1}{\lambda} e^{-\lambda t}\Big|_0^\infty$$

$$= 0 - \left(-\frac{1}{\lambda}\right) = \boxed{1/\lambda}$$

The bigger λ is the sooner the bulb dies,

$E(T^2) = 2/\lambda^2$ (integration by parts)

$Var(T) = E(T^2) - (E(T))^2$

$= 2/\lambda^2 - \frac{1}{\lambda^2} = 1/\lambda^2$

$\boxed{SD(T) = 1/\lambda}$

exercise 10.5.3

3. Let X have the exponential distribution with mean 24 hours. Assume that X is measured in hours.

$E(X) = \frac{1}{\lambda} = 24 \Rightarrow \lambda = \frac{1}{24}$

b) Find $P(X > \underbrace{72}_{s+t} \mid X > \underbrace{24}_{t})$.

$= P(x > 48) = e^{-\frac{1}{24}(48)} = \boxed{e^{-2}}$