Last time

Sec 9.4 Interpretation of CI

Sec 10.1 Density

Sec 10.2 Expectation & Var. $\Big\}$ Cts. R.V.'s

Today.

2 Named Cts. R.V.'s

## Sec. 10.3 Exponential

$\lambda$ positive constant

Say $X \sim Exp(\lambda)$ ← Exponential lambda

$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$ $\qquad \int_0^\infty e^{-x} dx = 1$

In particular, if $\lambda = 1$, $f(x) = e^{-x}$, $x \geq 0$

CDF. $\qquad -\frac{d}{dt} e^{-\lambda t} = \lambda e^{-\lambda t}$

$F(x) = \int_0^x \lambda e^{-\lambda t} dt$

$(s = \lambda t, \; ds = \lambda dt) \quad = \int_0^{\lambda x} e^{-s} ds$

$= -e^{-s} \Big|_0^{\lambda x} = -\left(e^{-\lambda x} - 1\right) = 1 - e^{-\lambda x}$

We often use exponential distr. to model random lifetimes that has the following "interesting" property.

e.g. $X$ represent the lifetime of a lightbulb / radioactive atom

From CDF, $F(x) = \mathbb{P}(X \leq x)$

$= \mathbb{P}(\text{Die before } x)$

Define the survival function $S(x) = 1 - F(x) = \mathbb{P}(\text{Survive at time } x)$

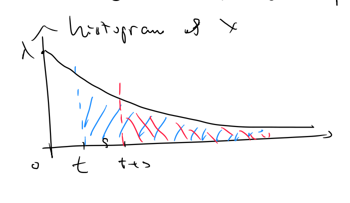
$= e^{-\lambda x}$

From this, consider the cond. prob of $(X > t+s \mid X > t)$

live another $s$ units of time $\qquad$ alive at $t$

has $t$ on the LHS

histogram of $X$



$\mathbb{P}(X > t+s \mid X > t)$

$= \frac{\mathbb{P}(X > t+s)}{\mathbb{P}(X > t)}$

$= \frac{S(t+s)}{S(t)} = \frac{e^{-\lambda(\cancel{t}+s)}}{\cancel{e^{-\lambda t}}} = e^{-\lambda s}$ $\quad$ no $t$.

This cond. prob. has nothing to do with $t$, as if it forgets how long it has survived

"memoryless property"

Example We have a lightbulb, whose lifetime $T \sim Exp(\lambda)$ with mean $\mathbb{E}T = 10$ years.

Suppose that we know this lightbulb has worked 20 years.

How long do you expect it to keep working?

10 years! b/c. memoryless property.

Remark. Use $Exp(\lambda)$ as the lifetime of objects that do not die of ages.

Expectation.

$\mathbb{E}X = \int_0^\infty x \underbrace{\lambda e^{-\lambda x}}_{\text{pdf / density}} dx$

$(s := \lambda x) \quad = \frac{1}{\lambda} \boxed{\int_0^\infty s e^{-s} ds}$ $=1$ (Integration by parts)

$= \frac{1}{\lambda} \int_0^\infty s \, d(-e^{-s})$

$\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du$

$= \frac{1}{\lambda}\left(\cancel{-se^{-s}\Big|_0^\infty} \to 0 + \int_0^\infty e^{-s} ds\right)$ density of $Exp(1)$

$= \frac{1}{\lambda}$ → $\lambda$ is the "rate" of Exp. R.V. since the larger $\lambda$ is, the faster the item is expected to die.

$Var(X)$

$\mathbb{E}X^2 = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$ — Integration by parts

$(s = \lambda x) \quad = \frac{1}{\lambda^2} \int_0^\infty s^2 e^{-s} ds$

$= \frac{1}{\lambda^2} \int_0^\infty s^2 d(-e^{-s})$

$\int u \, dv$

$= \frac{1}{\lambda^2}\left(\cancel{-s^2 e^{-s}\Big|_0^\infty} \to 0 + \int_0^\infty e^{-s} ds^2\right)$

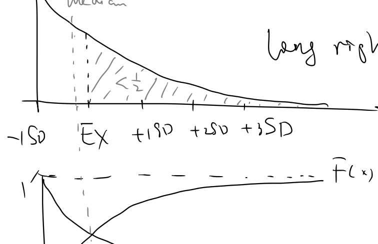
$= \frac{1}{\lambda^2} \int_0^\infty 2 s e^{-s} ds$ $=1$

$= \frac{2}{\lambda^2}$

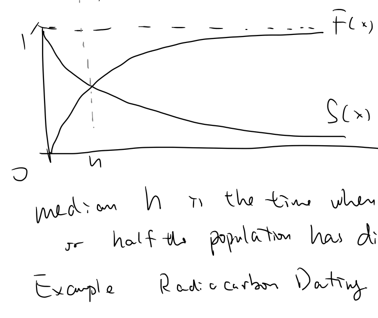
$Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

$SD(X) = \frac{1}{\lambda} = \mathbb{E}X.$

histogram of $x$




long right tail ⟹ mean > median




$F(x) + S(x) = 1$

The intersection: $F(x) = S(x) = \frac{1}{2}$

$e^{-\lambda x} = \frac{1}{2}$

$\Rightarrow e^{\lambda x} = 2 \Rightarrow$ median $x = \frac{1}{\lambda} \cdot \ln 2$

$= \underbrace{\ln 2}_{<1} \cdot \mathbb{E}X$

median $h$ is the time when it die with prob. $1/2$ or half the population has died. → "half-life"

Example Radiocarbon Dating

proportion of $^{14}C$ remaining.

half-life of $^{14}C$ is 5730 yrs.

Suppose we know there is 50% $^{14}C$ remaining. ⟹ 5730 yrs old

25% ⟹ $2 \times 5730$ yrs old

$\left(\frac{1}{2}\right)^2 = 25\%$

35% $\qquad$ $k \cdot 5730$ yrs old

$\left(\frac{1}{2}\right)^k = 35\% \Rightarrow k = \log_{\frac{1}{2}}(35\%)$

## Sec. 10.4 Normal distr.

Formalize it as a distr.

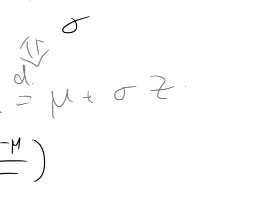
Say $X \sim N(\mu, \sigma^2)$ if the density is

↑ normal $\quad$ ↑ mean $\mu$ $\quad$ ↖ var. $\sigma^2$, sd $\sigma$

$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2} \cdot \frac{(x-\mu)^2}{\sigma^2}}$

In particular, if $Z \sim N(0,1)$, the density is standard normal curve

$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} = \phi(z)$




CDF.

$F_Z(z) = \Phi(z)$ $\qquad Z \overset{d}{=} \frac{X-\mu}{\sigma}$

$F_X(x) = \mathbb{P}(X \leq x)$ $\qquad \Updownarrow$

$= \mathbb{P}\left(\underbrace{\frac{X-\mu}{\sigma}}_{\overset{d}{=} Z} \leq \frac{x-\mu}{\sigma}\right)$ $\qquad X \overset{d}{=} \mu + \sigma Z$

$= F_Z\left(\frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

Check expectation and var.

$\mathbb{E}X = \mathbb{E}(\mu + \underbrace{\sigma Z}_{0}) = \mu$

$Var(X) = Var(\underbrace{\mu}_{0} + \sigma Z) = \sigma^2 Var(Z) = \sigma^2$

Sum of indpt. Normal

$X, Y$ indpt. Normals

mean $\mu_X \; \mu_Y$

var $\sigma_X^2 \; \sigma_Y^2$

Consider $X+Y$ $\qquad \mathbb{E}(X+Y) = \mu_X + \mu_Y$

$Var(X+Y) = \sigma_X^2 + \sigma_Y^2$

$SD(X+Y) = \sqrt{\sigma_X^2 + \sigma_Y^2}$.

State (and use) without proof:

$X+Y$ is also a normal R.V.

$X+Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

In particular, $X - Y \sim N(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2)$

since $var(X-Y) = var(X) + var(Y)$.

generally, for indpt Normals $X_1, X_2, \cdots, X_n$,

$a_1 X_1 + a_2 X_2 + a_3 X_3 + \cdots + a_n X_n + c$

$\sim N(a_1\mu_{X_1} + a_2\mu_{X_2} + \cdots + a_n\mu_{X_n} + c,$

$a_1^2 \sigma_{X_1}^2 + \cdots + a_n^2 \sigma_{X_n}^2)$

Applications.

① CI for diff. between means.

Setup. $\begin{cases} X_1, X_2, \cdots, X_n & \text{i.i.d. with mean } \mu_X, \text{ sd } \sigma_X \\ Y_1, Y_2, \cdots, Y_m & \text{i.i.d. w/. mean } \mu_Y, \text{ sd } \sigma_Y \end{cases}$ $\Big\}$ indpt

want to estimate the difference $\mu_X - \mu_Y$.

use $\bar{X} - \bar{Y}$ to give esitimation, thanks to law of larger number.

by CLT. $\bar{X} \sim N(\mu_X, \frac{\sigma_X^2}{n})$

$\bar{Y} \sim N(\mu_Y, \frac{\sigma_Y^2}{m})$

$\bar{X} - \bar{Y} \sim N(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m})$ $\Rightarrow \mathbb{P}\left(-2 \leq \frac{(\bar{X}-\bar{Y}) - \boxed{(\mu_X - \mu_Y)}}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \leq +2\right) \approx 95\%$

95% CI is given by

$\bar{X} - \bar{Y} \pm 2 \cdot \sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}$.

| example | Sample size | Sample mean | sd |
|---|---|---|---|
| City X | $n_X = 400$ | $\bar{X} = \$70k$ | $\sigma_X = \$40k$ |
| City Y | $n_Y = 600$ | $\bar{Y} = \$80k$ | $\sigma_Y = \$50k$. |

give 95% CI for $\mu_X - \mu_Y$ (diff. in population mean).

unit = \$1000

$(70 - 80) \pm 2 \cdot \underbrace{\sqrt{\frac{40^2}{400} + \frac{50^2}{600}}}_{\approx 2.86}$

$\sim 10 \pm 2 \times 2.86$ $\quad (\times \$1000)$

It makes sense to have negative number here

Test for the equality of two means.

(also an A/B test) $\qquad H_0: \mu_X - \mu_Y = 0$

$\qquad H_a: \mu_X - \mu_Y < 0$

$H_0$: the means are the same

$H_a$: - - - - - - different (two sided)

$\mu_X < \mu_Y$ (one sided) $\Big\}$ choose one only.

Note: we only assumed the means are the same, but under $H_0$ the variance or the distr. could be different.

test statistic: $\bar{X} - \bar{Y}$

distr. under $H_0$: $\bar{X} - \bar{Y} \sim N(\underbrace{0}_{\mu_X - \mu_Y}, \overbrace{\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}}^{2.86^2})$

observed value: $\bar{X} - \bar{Y} = -10 < -3SD$

— very significant —

evidence towards $H_a$.

p-value is very small)

Conclusion: reject $H_0$