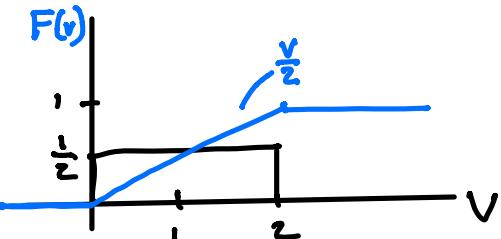
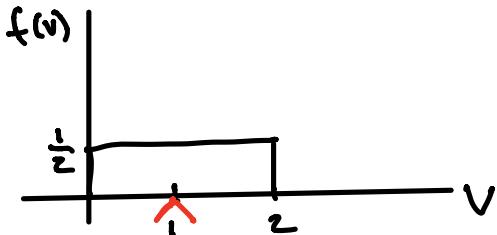


Stat 88 Lec 31

Warmup 2:00 - 2:10

Let  $V$  have density  $f(v) = \begin{cases} \frac{1}{2} & \text{for } 0 \leq v \leq 2 \\ 0 & \text{else.} \end{cases}$



a) Find the cdf of  $V$

b) Find  $\text{Var}(V)$

$$\text{Var}(v) = E(V^2) - (E(V))^2$$

$$E(V) = 1$$

$$E(V^2) = \int_0^2 v^2 \cdot \frac{1}{2} dv = \frac{v^3}{6} \Big|_0^2 = \frac{4}{3}$$

$$\text{Var}(V) = \frac{4}{3} - 1^2 = \boxed{\frac{1}{3}}$$

$$F(v) = \begin{cases} \frac{v}{2} & 0 \leq v \leq 2 \\ 0 & \text{else} \end{cases}$$

## Last time

### Sec 10.1 density.

For discrete RVs, such as  $X \sim \text{Binomial}(n, p)$ , the Probability mass function,  $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$ , or the cdf  $P(X \leq k)$  describes the distribution.

For continuous RVs, such as  $Z \sim N(0, 1)$ , the density,  $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ , or the cdf  $P(Z \leq z) = \Phi(z)$ , describes the distribution.

A density is always nonnegative and integrates to 1.

If  $X$  is a continuous RV,  $f$  is a density of  $X$  if  $P(X \in (x, x+dx)) = f(x)dx$  and

$$P(X \in (a, b)) = \int_a^b f(x)dx.$$

### Sec 10.2 expectation and variance

If a continuous RV,  $X$ , has density  $f$ ,

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx \quad \text{and} \quad E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

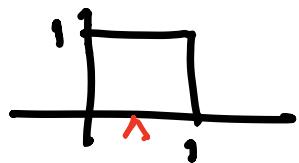
$$\text{and } \text{Var}(X) = E(X^2) - (E(X))^2$$

$\Leftarrow$  Let  $U \sim \text{Unif}(0,1)$

Find  $\text{Var}(U)$

$$E(U^2) = \int_0^1 u^2 1 du = u^3 \Big|_0^1 = \frac{1}{3}$$

$$\begin{aligned} \text{Var}(U) &= E(U^2) - (E(U))^2 \\ &= \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}} \end{aligned}$$

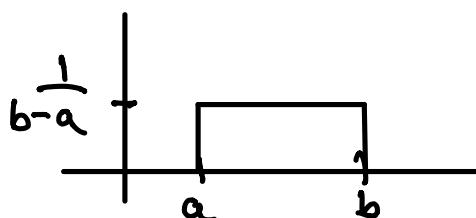


Today

① Sec 10.2  $U(a,b)$

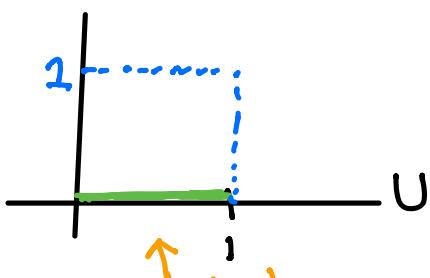
② Sec 10.3 The Exponential Distribution

① Uniform  $(a,b)$

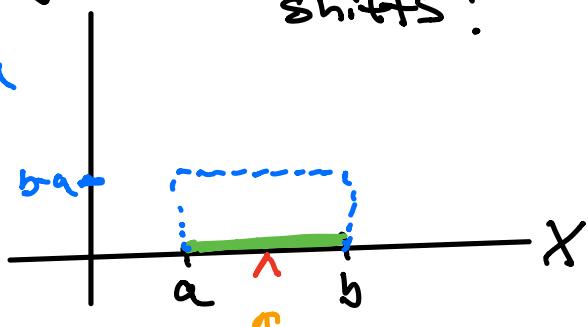


$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{else} \end{cases}$$

What function  $X = g(U)$  stretches and shifts?



$$X = (b-a)U + a$$



length 1

Find

$$E(X) = E((b-a)U + a) = (b-a)E(U) + a$$

$$\text{Var}(X) = \text{Var}((b-a)U + a) = \frac{b-a}{12} \cdot \frac{(b-a)^2}{12} \text{Var}(U)$$

$$= \frac{(b-a)^2}{12}$$

exercise 10.5.2

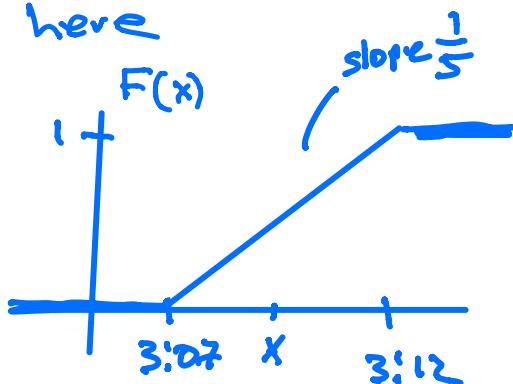
2. A class starts at 3:10 p.m. Seven students in the class arrive at random times  $T_1, T_2, \dots, T_7$  that are i.i.d. with the uniform distribution on the interval 3:07 to 3:12.

c) Let  $X = \max(T_1, T_2, \dots, T_7)$  be the time when the last of the seven students arrives. Find the cdf of  $X$ .

$$P(\overset{\leftarrow}{X} \leq x) = P(T_1 \leq x, T_2 \leq x, \dots, T_7 \leq x)$$

$$= (P(T_1 \leq x))^7$$

So we need to find the cdf of  $T_1$ ,  $F(x)$   
which I graph here



The equation of the line between  
3:07 and 3:12

$$F(x) = \frac{x - 3:07}{5} \text{ for } 3:07 \leq x \leq 3:12$$

For  $x > 3:12$ ,  $F(x) = 1$

and for  $x < 3:07$ ,  $F(x) = 0$

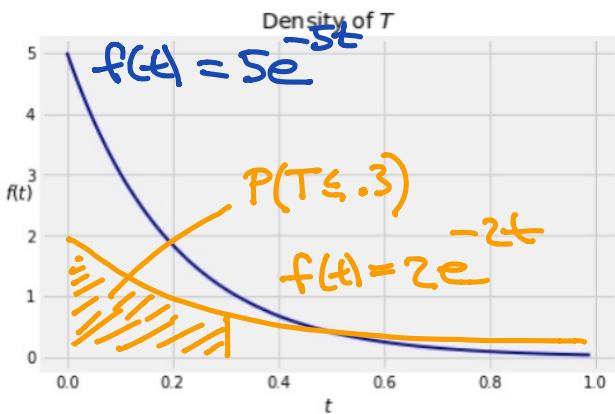
Hence,

$$P(X \leq x) = \begin{cases} 0 & \text{if } x < 3:07 \\ \left(\frac{x - 3:07}{5}\right)^7 & \text{if } 3:07 \leq x \leq 3:12 \\ 1 & \text{if } x \geq 3:12 \end{cases}$$

## ② The Exponential Distribution

Let  $\lambda > 0$ ,

A RV  $T$  has the exponential distribution with rate  $\lambda$  (written  $T \sim \text{Exp}(\lambda)$ ), if the density of  $T$  is  $f(t) = \lambda e^{-\lambda t}$  for  $t \geq 0$ .



$\Leftrightarrow T = \text{lifetime of a lightbulb}$ .

$\lambda = \text{rate a lightbulb burns out} = \frac{2 \text{ bulbs}}{\text{decade}}$

$$\begin{aligned} P(T \leq 0.3) &= \int_0^{0.3} 2e^{-2t} dt = -e^{-2t} \Big|_0^{0.3} \\ &= -e^{-0.6} + 1 \\ &= 1 - e^{-0.6} = 0.45 \end{aligned}$$

(the chance a bulb dies in 3 years is .45)

## CDF and Survival function

The cdf,  $P(T \leq t)$  = chance the lightbulb dies before time  $t$

$$F(t) = P(T \leq t) = \int_0^t \lambda e^{-\lambda s} ds$$

$$= -e^{-\lambda s} \Big|_0^t = \boxed{1 - e^{-\lambda t}}$$

The survival function,  $S(t)$ , is the chance the lightbulb last longer than time  $t$ .

$$S(t) = P(T > t) = 1 - F(t) = \boxed{e^{-\lambda t}}$$

## Memoryless Property

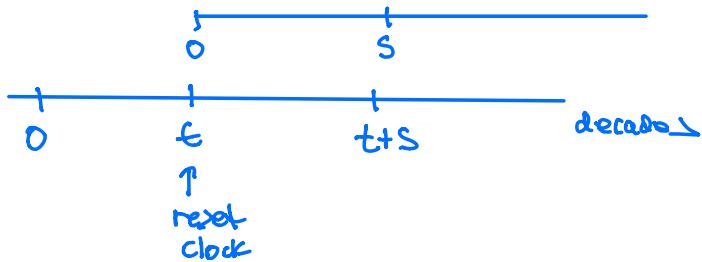
Let's find the chance  $T$  survives time  $t+s$ , given that it survives time  $t$ :

$$\begin{aligned} P(T > t+s | T > t) &= \frac{P(T > t+s, T > t)}{P(T > t)} \\ &= \frac{P(T > t+s)}{P(T > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} \end{aligned}$$

$$S(t) = \boxed{e^{-\lambda t}}$$

Picture

$$= P(T > s)$$



So if the bulb lasts a decade the chance it will last another decade is the same as that it lasts 1 decade (it forgets that it lived for 1 decade already).

The exponential distribution is the continuous analog to the geometric distribution. Both have the memoryless property.

mean and SD

$$\begin{aligned}
 \text{Let } T &\sim \text{Exp}(\lambda) & f(t) \\
 E(T) &= \int_0^\infty t \lambda e^{-\lambda t} dt & = UV \Big|_0^\infty - \int_0^\infty V dU \\
 U &= t & V &= -e^{-\lambda t} \\
 dU &= dt & dV &= \lambda e^{-\lambda t} dt \\
 &&& \boxed{\text{if}}
 \end{aligned}$$

$$\begin{aligned}
 &= -te^{-\lambda t} \Big|_0^\infty - \int_0^\infty -e^{-\lambda t} dt \\
 &= \int_0^\infty e^{-\lambda t} dt = -\frac{1}{\lambda} e^{-\lambda t} \Big|_0^\infty \\
 &= 0 - \left(-\frac{1}{\lambda}\right) = \boxed{\frac{1}{\lambda}}
 \end{aligned}$$

The bigger  $\lambda$  is the sooner the bulb dies.

$$E(T^2) = \frac{2}{\lambda^2} \quad (\text{integration by parts})$$

$$\begin{aligned} \text{Var}(T) &= E(T^2) - (E(T))^2 \\ &= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \end{aligned}$$

$$\boxed{\text{SD}(T) = \frac{1}{\lambda}}$$

### Exercise 10.5.3

3. Let  $X$  have the exponential distribution with mean 24 hours. Assume that  $X$  is measured in hours.

$$E(X) = \frac{1}{\lambda} = 24 \Rightarrow \lambda = \frac{1}{24}$$

- b) Find  $P(X > 72 | X > 24)$ .

$$P(X > 72 | X > 24) = e^{-\frac{1}{24}(48)} = \boxed{e^{-2}}$$