Stat 88 lec 12

Warmup 2:00–2:10

Exercise 5.7.3

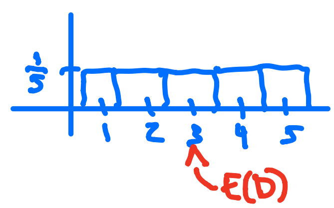
3. A box contains four blank index cards and one that has a gold star on it. Cards are drawn one by one at random without replacement until the gold star appears. Let D be the number of cards drawn.

a) Find the distribution of D.

D	1	2	3	4	5
	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

b) Find $E(D)$.

$$E(D) = \frac{1}{5}(1+2+3+4+5) = 3$$



c) Suppose all five cards were dealt one by one at random without replacement. If you saw the sequence of cards, would you be able to tell whether you were looking at the sequence forwards (that is, in the order in which the cards were drawn) or backwards? If the answer is "no", can you use it to give another justification for the answer to Part **b**?

$D \sim \text{Uniform}\{1, 2, 3, 4, 5\}$

Uniform on $\{1, 2, 3, \dots, n\}$

Let n be a fixed positive integer. A random variable X has the *uniform* distribution on the integers 1 through n if X is equally likely to have any of the values 1 through n.

Last time

Sec 5.1 Expectation

The *expectation* of a random variable X, denoted $E(X)$, is the average of the possible values of X weighted by their probabilities:

$$E(X) = \sum_{\text{all } x} x P(X = x)$$

ex (Indicator RV)

$$X = \begin{cases} 1 \text{ (success) with prob } p \\ 0 \text{ (failure) with prob } 1-p \end{cases}$$

$$E(X) = 1 \cdot p + 0 \cdot (1-p) = p$$

Poisson (μ)

Let X have the Poisson (μ) distribution. Then

ex X = # raindrops landing on a tile in 10 sec.

$$E(X) = \sum_{k=0}^{\infty} \cancel{k} e^{-\mu} \frac{\mu^k}{k!} \qquad = \cancel{k}(k-1)!$$

$$P(X=k) = \frac{e^{-\mu}\mu^k}{k!}$$

$$= e^{-\mu}\mu \sum_{k=1}^{\infty} \frac{\mu^{k-1}}{(k-1)!}$$

$$= e^{-\mu}\mu \sum_{j=0}^{\infty} \frac{\mu^j}{j!}$$

$$= e^{-\mu}\mu e^{\mu}$$

$$= \mu$$

parameter μ is the expected number of raindrops.

We have seen a RV, X, belong to different distributions depending on the application.
Uniform
Binomial
Hypergeometric
Geometric
Poisson
E(X) is the center of the distribution of X and an important summary statistic.

Today

Sec 5.2 Functions of a RV
Sec 5.3 Method of Indicators

1 <u>Sec 5.2</u> Functions of Random Variables.

When we work with RVs we often want to consider functions of them,

<u>ex</u> let $X \sim \text{Unif}\{-1, 0, 1\}$ and $Y = X^2$

Here is a distribution table for X, with values of Y as well.

$y = x^2$	1	0	1
x	-1	0	1
$P(X = x)$	1/3	1/3	1/3

Then $\underset{\underset{Y}{\parallel}}{E(X^2)} = (-1)^2\left(\frac{1}{3}\right) + 0^2\left(\frac{1}{3}\right) + (1)^2\left(\frac{1}{3}\right) = 2/3$

more generally,

Let $Y = g(X)$ be a function of the random variable X. Then

$$E(g(X)) = \sum_{\text{all } x} g(x)P(X = x)$$

<u>ex</u> let $W = \min(X, .5)$

Fill out table and find

Find $E(W)$

$w = \min(x, .5)$	-1	0	$.5$
x	-1	0	1
$P(X = x)$	1/3	1/3	1/3

$E(W) = \frac{1}{3}\left(-1 + 0 + \frac{1}{2}\right)$

$= \boxed{-\frac{1}{6}}$.

Remember that a RV is a function on the outcome space Ω.

$$X: \Omega \longrightarrow \mathbb{R}$$

ex Flip a fair coin twice

$\Omega = \{HH, HT, TH, TT\}$

X = # heads in 2 coin tosses.

$X(HH) = 2$

If $Y = g(X) = X^2$, this is also a function of the outcome space

$Y(HH) = (X(HH))^2 = 4$

So a function of a RV is itself a RV.

Joint Distribution — distribution of (X_1, X_2)

Suppose two draws are made at random without replacement from a population that has five elements labeled $1, 2, 2, 3, 3$. Define the following random variables:

- X_1 is the number on the first draw
- X_2 is the number on the second draw

The pair (X_1, X_2) is a RV and we describe its distribution in a table

$P(X_1=1, X_2=1) = P(X_1=1)\cdot P(X_2=1 \mid X_1=1)$ by mult. rule.

$= \frac{1}{5}\cdot 0 = \boxed{0}$

$P(X_1=1, X_2=2) = ?$ $P(X_1=1)P(X_2=2 \mid X_1=1) = \frac{1}{5} \cdot \frac{3}{4} = \boxed{\frac{3}{20}}$

This matrix is symmetric

$P(X_1=3, X_2=1) = P(X_1=1, X_2=2)$

This fills out the table!

	$X_2 = 1$	$X_2 = 2$	$X_2 = 3$
$X_1 = 1$	0	$\frac{2}{20}$	$\frac{2}{20}$
$X_1 = 2$	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{4}{20}$
$X_1 = 3$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{2}{20}$

Find $P(X_1 + X_2 = 4)$? $= \frac{2}{20} + \frac{2}{20} + \frac{2}{20}$

Marginal Distribution

By the addition rule,

$$P(X_1=1) = P(X_1=1, X_2=1) + P(X_1=1, X_2=2) + P(X_1=1, X_2=3).$$

So we can recover the distribution of X_1 from the joint distribution.

	$X_2 = 1$	$X_2 = 2$	$X_2 = 3$	Dist of X_1
$X_1 = 1$	0	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{4}{20} = \frac{1}{5}$
$X_1 = 2$	$\frac{2}{20}$	$\frac{2}{20} \neq \left(\frac{2}{5}\right)^2$	$\frac{4}{20}$	$\frac{8}{20} = \frac{2}{5}$
$X_1 = 3$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{2}{20}$	$\frac{8}{20} = \frac{2}{5}$
Dist of X_2	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	

Are X_1 and X_2 independent?

No! For example $P(X_1=2, X_2=2) \neq P(X_1=2) \cdot P(X_2=2)$

$$\underbrace{P(X_1=2, X_2=2)}_{=\frac{2}{20}} \neq \underbrace{P(X_1=2)}_{=\frac{2}{5}} \cdot \underbrace{P(X_2=2)}_{=\frac{2}{5}}$$

X_1 and X_2 are <u>independent</u> if

$$P(X_1=a, X_2=b) = P(X_1=a)P(X_2=b)$$

for all cells in the joint table.

If draw cards w/ replacement then X_1, X_2 would be independent.

Expectation of a function of RVs

- Take a cell of the joint distribution table of X and Y. This corresponds to one possible value (x, y) of the pair (X, Y).
- Apply the function g to get $g(x, y)$.
- Weight this by the probability in the cell, to get the product $g(x, y)P(X = x, Y = y)$
- Add these products over all the cells of the table.

ex Find $E(|X_1 - X_2|)$

	$X_2 = 1$	$X_2 = 2$	$X_2 = 3$
$X_1 = 1$	0	$\frac{2}{20}$	$\frac{2}{20}$
$X_1 = 2$	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{4}{20}$
$X_1 = 3$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{2}{20}$

$$= \underbrace{g(1,1)}_{=0} \cdot 0 + \underbrace{g(1,2)}_{=1} \cdot \frac{2}{20} + \underbrace{g(1,3)}_{=2} \cdot \frac{2}{20}$$

$$+ \underbrace{g(2,1)}_{=1} \cdot \frac{2}{20} + \ldots \qquad = \boxed{1}$$

9 terms

Tinyurl.com/feb19-pt1

Stats 88

Wednesday February 19 2020

1. A joint distribution for two random variables M and S is given below.

	$M=2$	$M=3$
$S=2$	0	$\frac{1}{3}$
$S=1$	$\frac{1}{3}$	$\frac{1}{3}$

dist M $1/3$ $2/3$

E(M) equals:

$E(M) = 2 \cdot 1/3 + 3 \cdot 1/3 = \boxed{2.67}$

a 1.67

b 2.33

c 2.67

d none of the above

Alternatively

$E(M) = 2 \cdot 0 + 3 \cdot 1/3 + 2 \cdot 1/3 + 3 \cdot 1/3 = 2.67$

as we go through each cell of the table.

Additivity of Expectation

$$E(X+Y) = E(X) + E(Y)$$

This doesnt require independance of X and Y.

We will skip the proof.

② Sec 5.3 Method of Indicators to find $E(X)$

Key idea

Counting the number of successful trials is the same as adding zeros and ones.

ex A success is blue, and failure non blue

B R R G B R B B

1 0 0 0 1 0 1 1

blue = 1+0+0+0+1+0+1+1 = 4

Recall (Indicator RV)

$$X = \begin{cases} 1 \xleftarrow{\text{success}} & \text{with prob } p \\ 0 \xleftarrow{\text{failure}} & \text{with prob } 1-p \end{cases}$$

$$E(X) = 1 \cdot p + 0 \cdot (1-p) = p$$

method of indicators

If X is a count of n
successful trials all with the same
probability p of success, to find
E(X), write X as a sum of
n indicators.

$$X = I_1 + \cdots + I_n$$

prob p

where $I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ trial is success} \\ 0 & \text{else} \end{cases}$

$$\begin{aligned} E(X) &= E(I_1 + \cdots + I_n) \\ &= E(I_1) + \cdots + E(I_n) \\ &= \boxed{np} \end{aligned}$$

exercise 5.7.6

6. A die is rolled 12 times. Find the expectation of

a) the number of times the face with five spots appears

X = # of 5 out of 12

$X = I_1 + \cdots + I_{12}$ $\quad I_2 = \begin{cases} 1 \text{ if } 2^{nd} \text{ toss is } 5 \end{cases}$ $p = 1/6$

$E(X) = np = 12\left(\frac{1}{6}\right) = \boxed{2}$