Stat 88 lec 10

warm-up 2:00-2:10

exercise 4.5.10

Suppose you are running independent success / failure trials with probability 0.7 of success on each trial,

a) what is the chance your first success is on the 3rd trial? qqP $(.3)^2(.7)$

b) What is the chance that you get 10 failures before the 15th Success?

15th success is on the 25th trial

14 successes in 24 trials and a 15th success on the 25th trial.

$$\boxed{\binom{24}{14}(.7)^{14}(.3)^{10}(.7)}$$

last time

**Sec 4.2**

## Waiting time for first success

If we have independent and identically distributed (i.i.d.) trials with probability p for Success and q for failure

Let T1=# trials until the first success.

$T1 \sim \text{Geom}(p)$

$P(T_1 = k) = q^{k-1} p$

$P(T1 > k) = q^k$

## Waiting time for r th success

Let $T_r$ = # trials until the $r^{th}$ success

$$
\begin{aligned}
&P(T_r = k) \\
&= P(r-1 \text{ of the first } k-1 \text{ trials are successes and trial } k \text{ is a success}) \\
&= \binom{k-1}{r-1} p^{r-1} q^{k-1-(r-1)} \cdot p \\
&= \binom{k-1}{r-1} p^{r-1} q^{k-r} \cdot p
\end{aligned}
$$

Picture

$$
\underbrace{\overset{1}{\_}\ \overset{2}{\_}\ \overset{3}{\_}\ \cdots\ \overset{k-1}{\_}}_{r-1 \text{ successes}}\ \overset{k}{\underline{S}}
$$

$r^{th}$ success.

$$
\begin{aligned}
&P(T_r > k) \\
&= P(\text{at most } r-1 \text{ successes in the first } k \text{ trials}) \\
&= \sum_{j=0}^{r-1} \binom{k}{j} p^j q^{k-j}
\end{aligned}
$$

Sec 4.3

Exponential approximation

$\log(1+\delta) \approx \delta$ when $\delta$ is a small number,

↖ base e

ex $\log\left(1 - \frac{1}{1000}\right) \approx -\frac{1}{1000}$

$\underbrace{1 - \frac{1}{1000}}_{1 + \left(-\frac{1}{1000}\right)}$

Today

(0) Go over concept test from last time.

(1) Sec 4.3 exponential approximation

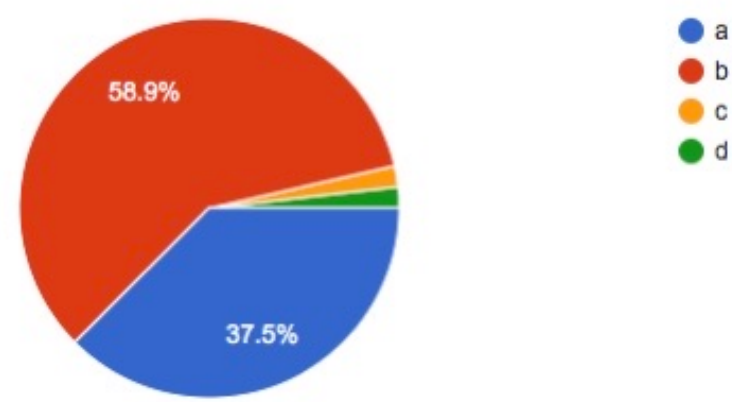
(2) Sec 4.4 Poisson distribution

⊙ Concept test last time

In a population, 30% of the individuals are green and the rest are blue. Suppose you draw individuals **with replacement** until you draw a blue. Is the binomial formula applicable to find the chance that you draw 10 times?

**a** yes

**b** no




| | |
|---|---|
| a | "With replacement" indicated binomial because the probability of success p does not change! |

| | |
|---|---|
| b | Even though there is replacement, there is no fixed # of trials |

## (1) Exponential approximation.

We know $\log(1+\delta) \approx \delta$ for small $\delta$.

ex Approximate $x = \left(1 - \frac{3}{100}\right)^{100}$

$$\log x = \log\left(1 - \frac{3}{100}\right)^{100} = 100 \log\left(1 - \frac{3}{100}\right)$$

$$\approx 100\left(-\frac{3}{100}\right) = -3$$

$\Rightarrow \boxed{x = e^{-3}}$ Exponential approximation

ex Give exponential approx for

a) $x = \left(1 - \frac{2}{1000}\right)^{5000}$

$$\log x = 5000 \log\left(1 - \frac{2}{1000}\right)$$

$$\approx 5000\left(-\frac{2}{1000}\right) = -10 \Rightarrow \boxed{x = e^{-10}}$$

b) $x = (1-p)^n$ for small $p$

$$\log x = n \log(1-p) \approx -np$$

$$\Rightarrow \boxed{x = e^{-np}}$$

Suppose we have $n$ iid success/failure trials where $n$ is large and $p$ is small and average number of successes $\mu = np$

$1-p$ chance of failure

$(1-p)^n$ chance no success in $n$ trials

$(1-p)^n \approx e^{-np} = e^{-\mu}$ approximation of chance of no success in $n$ trials.

Note Fixed mistake from class.

ex A chapter of a book has $n=100{,}000$ words and the chance a word of the chapter has a misprint is very small $p=\frac{1}{1{,}000{,}000}$. Give an approximation of the chance the chapter doesn't have a misprint.

Here a misprint is a success.

Chance no misprints out of n words $= (1-p)^n = \left(1-\frac{1}{1{,}000{,}000}\right)^{100{,}000}$.

Let $\mu = np = .1$ Exponential approx $= e^{-\mu} = \boxed{e^{-.1}}$

## (2) Poisson distribution

$X$ = # of successes of an event with small chance of success out of many independent trials

a count

$X \sim \text{Pois}(\mu)$

parameter

$$P(X=k) = e^{-\mu}\frac{\mu^k}{k!} \quad k=0,1,2,\ldots$$

Poisson Formula

Here $\mu$ is the average number of successes

This sums to 1

$$\sum_{k=0}^{\infty} e^{-\mu} \frac{\mu^k}{k!} = e^{-\mu} \underbrace{\sum_{k=0}^{\infty} \frac{\mu^k}{k!}}_{= e^{\mu}} = e^{-\mu} e^{\mu} = 1$$

Note Fixed mistake from class

It is rare that a word in a chapter of a book has a misprint. There are many words in a chapter and each is an independent trial for a misprint. The number of words in a Chapter that are misprinted can be modeled by a Poisson distribution.

$X \sim \text{Pois}(\mu)$ ← parameter

$$P(X=k) = e^{-\mu} \frac{\mu^k}{k!} \quad k=0,1,2,\ldots$$

Poisson Formula

ex 4.5.7

**7.** A book has 20 chapters. In each chapter the number of misprints has the Poisson distribution with parameter 2, independently of the misprints in other chapters.

**a)** Find the chance that Chapter 1 has more than two misprints.

$X$ = # misprints in chapter 1

$X \sim \text{Pois}(2)$

$P(X>2) = 1 - P(X \leq 2) = 1 - P(X=0) - P(X=1) - P(X=2)$

$P(X=0) = \frac{e^{-\mu} \mu^0}{0!} = e^{-\mu} = e^{-2}$

$$\boxed{P(X>2) = 1 - e^{-2} - \frac{e^{-2} 2^1}{1} - \frac{e^{-2} 2^2}{2!}}$$

**b)** Find the chance that two of the chapters have three misprints each.

$X$ = # misprints in the chapter of a book

$X \sim \text{Pois}(2)$

$$P = P(X=3) = \frac{e^{-2} 2^3}{3!}$$

There are 20 independent chapters each with chance $p = \frac{e^{-2} 2^3}{3!}$ to have exactly 3 misprints,

Hence the chance 2 chapters have 3 misprints and 18 chapters dont have 3 misprints is

$$\boxed{\binom{20}{2} p^2 (1-p)^{18} \quad \text{where } p = \frac{e^{-2} 2^3}{3!}}$$

## Poisson(np) as an approximation of Binomial(n, p) when n is large and p is small

Let $X = \text{Binomial}(n, p)$ for n large, p small.
Let $Y = \text{Poisson}(\mu = np)$.
We show that $P(X=0) = P(Y=0)$:

$$P(X=0) = \binom{n}{0} p^0 (1-p)^n = (1-p)^n \underbrace{\approx}_{\text{exponential approx.}} e^{-np} = \underbrace{e^{-\mu}}_{P(Y=0) \text{ for } Y \sim \text{Pois}(\mu)}$$

similarly,

$$P(X=1) = \underbrace{\binom{n}{1}}_{= n} p^1 (1-p)^{n-1} = np \frac{(1-p)^n}{\underbrace{1-p}_{\text{close to } 1}} \approx \underbrace{\frac{\mu e^{-\mu}}{1}}_{P(Y=1) \text{ for } Y \sim \text{Pois}(\mu)}$$

similarly,

$$P(X=2) = \underbrace{\binom{n}{2}}_{= \frac{n(n-1)}{2}} p^2 (1-p)^{n-2} = \frac{\overbrace{n(n-1)}^{\text{close to } n^2 \text{ for } n \text{ large}}}{2} p^2 \frac{(1-p)^n}{\underbrace{(1-p)^2}_{\text{close to } 1 \text{ for } p \text{ small}}} \approx \frac{(np)^2 (1-p)^n}{2 \cdot 1} = \underbrace{\frac{\mu^2 e^{-\mu}}{2!}}_{P(Y=2)}$$

So Poisson is approximately Binomial for small p, large n.