Stat 88 lec 16

warmup 2:00-2:10

Let S and M have joint distribution

| | $M=2$ ($\frac{1}{3}$) | $M=3$ ($\frac{2}{3}$) |
|---|---|---|
| $S=2$ | $0$ | $\frac{1}{3}$ |
| $S=1$ | $\frac{1}{3}$ | $\frac{1}{3}$ |

a) Find $E(S|M=3)$

b) Find $E(S)$

a) $2 \cdot \underbrace{P(S=2|M=3)}_{=\frac{1/3}{2/3}=1/2} + 1 \cdot \underbrace{P(S=1|M=3)}_{=\frac{1/3}{2/3}=1/2} = \boxed{1.5}$

$E(S|M=2) = \boxed{1}$

b) $E(S) = \frac{1}{3}(1) + \frac{2}{3}(1.5) = \boxed{\frac{4}{3}}$

Midterm review materials today

<u>Last time</u> sec 5.5, 5.6 <u>Conditional Expectation.</u>

Let M, S be two RVs with joint distribution

| | $M=2$ | $M=3$ |
|---|---|---|
| $S=2$ | $0$ | $\frac{1}{3}$ |
| $S=1$ | $\frac{1}{3}$ | $\frac{1}{3}$ |

$$\boxed{E(S \mid M=m) = \sum_{\text{all } s} s \cdot P(S=s \mid M=m)}$$

and

$$\boxed{E(S) = \sum_{\text{all } m} E(S \mid M=m) P(M=m)}$$

<u>Note</u> $E(S \mid M=m)$ is a RV
(a function of M).

$$E(f(m)) = \sum_{\text{all } m} f(m) P(M=m)$$

Let $f(m) = E(S|M)$

then

$$E(E(S|M)) = \sum_{\text{all } m} E(S|M=m)P(M=m)$$

$$= E(S)$$

This proves the law of iterated expectation

$$\boxed{E(S) = E(E(S|M))}$$

Today

1. Concept test from last time
2. Expectation of Geometric(p)
3. Sec 5.6 Practice with conditional expectation

① Concept test

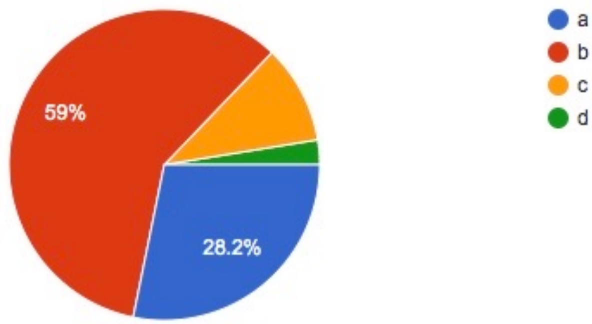
You flip a fair coin N times where N is a random variable $N \sim$Poisson(5). What is the expected number of heads you will get?

**a** 5

**b** $\frac{5}{2}$

**c** $\frac{N}{2}$

**d** none of the above




| | |
|---|---|
| a | Poisson tells you how many times you expect to get heads in a Certain amount of flips. This is N flips, so you would expect 5 heads in N flips |

| | |
|---|---|
| c | Half of the flips should be heads |

N = # coin tosses in 1 minute

H = # coin tosses landing heads in 1 minute

$H \sim \text{Poisson}(\frac{5}{2})$ since half of flips are head

$E(H) = 5/2.$

| | |
|---|---|
| b | The expected number of coin flips is 5, and we expect half of those to be heads so 5/2. |

$N \sim \text{Poisson}(5)$ is number of coin tosses in 1 min

$H = \#$ heads in 1 min

$H \mid N \sim \text{Binomial}(N, \tfrac{1}{2})$

$E(H \mid N) = \frac{N}{2}$

$E(H) = E(E(H \mid N)) = E\left(\frac{N}{2}\right) = \frac{1}{2} E(N)$

$= \boxed{\frac{5}{2}}$

$\approx 5$

## ② Expectation of Geometric $(p)$.

Let $X = \#$ $p$-coin tosses till first heads

$X \sim \text{Geometric}(p)$

$X$ has values $1, 2, 3, \ldots$

recall
$$P(X>1) = qp + q^2p + q^3p + \cdots$$
$$= qp\left(1+q+q^2+\cdots\right)$$
$$= qp\left(\frac{1}{1-q}\right) = q\frac{p}{p} = \boxed{q}$$

Dont use method of indicators to find expected waiting time since you dont know how many indicators you need.

Use conditional expectation

$$E(X) = \underbrace{E(X \mid X=1)}_{=1}\underbrace{P(X=1)}_{=p} + \underbrace{E(X \mid X>1)}_{=1+E(X)}\underbrace{P(X>1)}_{=q}$$

$E(X \mid X > 1) = \ ?$ — $1 + E(X)$

— knowing that 1st coin is tail increases the expected number of trials to get a head by 1. Just think of starting over after the first tails. Future coin tosses are independent of the 1st coin toss.

$$E(X) = p + (1 + E(X))q$$

$$E(X) = \underbrace{p + q}_{1} + q(E(X))$$

$$E(X) - qE(X) = 1$$

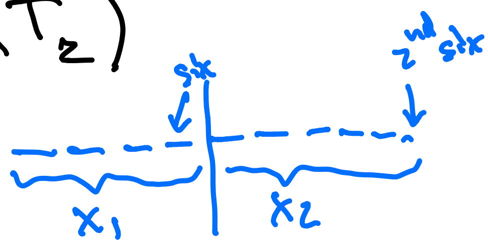
$$E(X)(1 - q) = 1$$

$$\boxed{E(X) = \frac{1}{p}}$$

## ex Waiting till 2 sixes

Let $T_2$ be the number of rolls of a die till a total of 2 sixes have appeared.

Find $E(T_2)$




$X_1 \sim \text{Geometric}(\frac{1}{6})$

$X_2 \sim \text{Geometric}(1/6)$

$T_2 = X_1 + X_2$

$$E(T_2) = \underset{6}{\underset{\shortparallel}{E(X_1)}} + \underset{6}{\underset{\shortparallel}{E(X_2)}} = 12$$

③ Sec 5.5, 5.6 Practice

a) A die is rolled repeatedly. Find the expected number of rolls till a total of 5 sixes appear.

$T_5 = X_1 + \cdots + X_5$

$E(T_5) = 5\left(\frac{1}{1/6}\right) = \boxed{30}$

a) A die is rolled repeatedly. Find the expected number of rolls till two diffent faces appear.

anything

__ __ __ __ __

$X$ = # rolls till any face $\sim$ Geom(1)

$E(X) = 1$

$Y$ = # rolls till diffent face $\sim$ Geom$\left(\frac{5}{6}\right)$

$E(X+Y) = \underbrace{E(X)}_{=1} + \underbrace{E(Y)}_{=6/5} = \boxed{11/5}$

ex

A fair coin is tossed 3 times

X = # heads first two tosses
Y = # heads last two tosses.

Find $E(Y|X=2)$ $P(X=2)=\frac{1}{4}$

$$= \sum_{\text{all } y} y \cdot P(Y=y|X=2)$$

$$= 1 \cdot \frac{\frac{1}{8}}{\frac{1}{4}} + 2 \cdot \frac{\frac{1}{8}}{\frac{1}{4}} = \boxed{3/2}$$

| outcomes |
| --- |
| HHH |
| HHT |
| → HTH |
| → THH |
| TTH |
| → THT |
| → HTT |
| TTT |

$E(Y|X=1)$ $P(X=1)=1/2$

$$0 \cdot \frac{\frac{1}{8}}{1/2} + 1 \cdot \frac{\frac{1}{4}}{1/2} + 2 \cdot \frac{\frac{1}{8}}{1/2}$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = \boxed{1}$$

$E(Y|X=0) = \boxed{\frac{1}{2}}$

Find $E(Y)$

$$\frac{3}{2}\left(\frac{1}{4}\right) + 1\left(\frac{1}{2}\right) + \frac{1}{2}\left(1/4\right) = \boxed{1}$$ ✓

makes sense!! avg # heads in 2 coin tosses is 1.