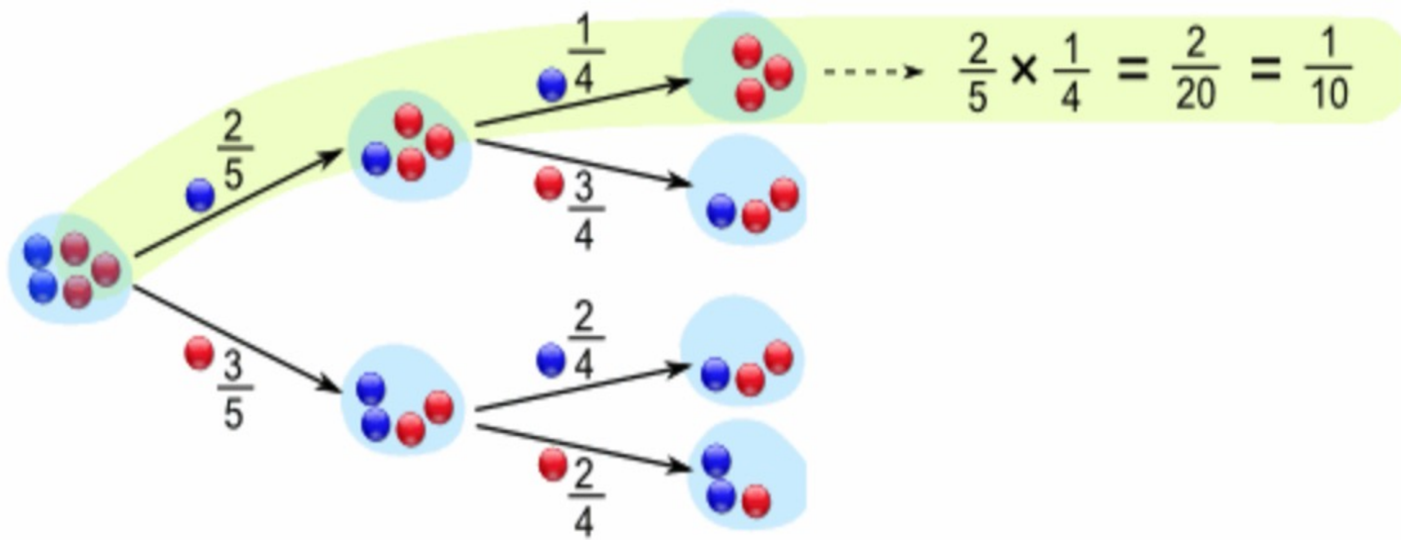
Stat 88: Probability and Mathematical Statistics in Data Science



Lecture 4: 1/27/2022

Multiplication Rule, Symmetry in Sampling, Bayes' Rule

Sections 2.1, 2.2, 2.3

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Warm-up 1: Exercise 1.4.5

- Here's a question from Quora: "If a student applies to ten colleges with a 20% chance of being accepted to each, what are the chances that he will be accepted by at least one college?" Without making any further assumptions, what can you say about this chance?

Let A_i be the event that the student is accepted to the i^{th} college $(i = 1, 2, \ldots, 10)$

$P(A_i) = 20\%$ for each i

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_{10}) \leq \underbrace{\sum_{i=1}^{10} P(A_i)}_{} = 2$$

$$\sum_{i=1}^{10} P(A_i) = P(A_1) + P(A_2) + P(A_3) + \cdots + P(A_{10})$$

$$\bigcup_{i=1}^{10} A_i = A_1 \cup A_2 \cup \cdots \cup A_{10}$$

$$\xrightarrow{\text{product}} \prod_{i=1}^{10} x_i = x_1 \cdot x_2 \cdot x_3 \cdots \cdot x_{10}$$

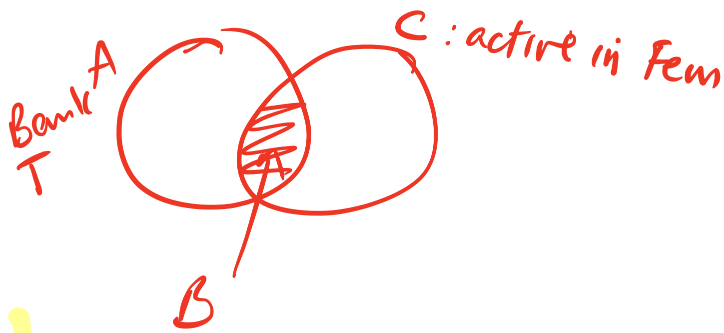
$$0.2 \leq P\left(\bigcup_{i=1}^{10} A_i\right) \leq 1$$

$\boxed{0}$

Warm up 2 $\rightarrow$ Kahneman & Tversky. (Thinking, Fast & Slow)

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations. which of the following is more likely:
(A) $\underbrace{\text{Linda is a bank teller}}_{A}$ A
(B) $\underbrace{\text{Linda is a bank teller}}_{A}$ and $\underbrace{\text{active in the feminist movement}}_{C}$

$\underline{\underline{A}}$ $\qquad$ $B \subseteq A$



$$P(A \cap B) \leq P(A), P(B)$$

(a) Somewhere in N. America, ~~[illegible]~~ there is a flood & scores of people drown

(b) In CA, an earthquake causes a flood & scores of people drown.

Agenda

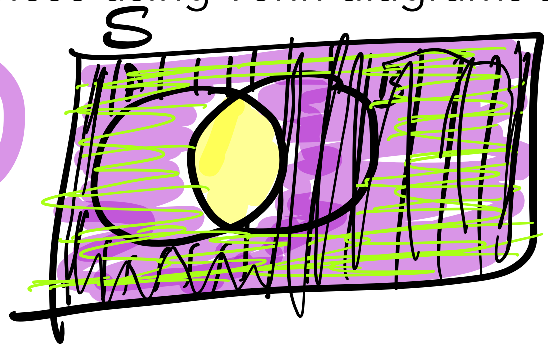
- De Morgan's laws
- Review the multiplication rule
- Addition rule
- Inclusion Exclusion
- De Méré's paradox & other exercises
- Symmetries in simple random sampling
- Bayes' rule

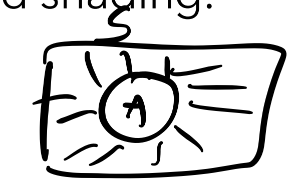
Exercise: De Morgan's Laws

A^c : A complement
not A

- Exercise: Try to show these using Venn diagrams and shading:

1. $(A \cap B)^c = A^c \cup B^c$





2. $(A \cup B)^c = A^c \cap B^c$

Exercise

Probability of an intersection

Draw 2 without repl.

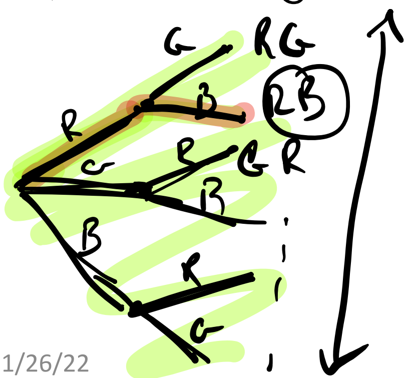
- Say we have three colored balls in an urn (red, blue, green), and we draw two balls *without* replacement.
- Find the probability that the first ball is red, and the second is blue
- Write down the outcome space and compute the probability

$\Omega = \{RB, RG, BR, BG, GR, GB\}$ $\quad P(1^{st} \text{ is } R \,\&\, 2^{nd} \text{ is } B)$

$= \frac{1}{6}$

$P(A \cap B) = P(A|B)P(B)$ suppose $A \cap B = \phi$, $P(A|B)$

- We can also write it down in sequence: P(first red, then blue) = P(first drawing a red ball)P(second ball is blue, given 1st was red)



$$P(RB) = \underbrace{P(1^{st} R)}_{\frac{1}{3}} \cdot \underbrace{P(2^{nd} B \mid 1^{st} R)}_{\frac{1}{2}}$$

$$P(1^{st} R \cap 2^{nd} B) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

Let A be event that 1^{st} ball is red.
Let B be " " 2^{nd} ball is red

$A \cap B = \phi$

$$\underbrace{P(A|B)}_{\text{conditional prob. of } A \text{ given } B} = 0$$

$$\underbrace{P(A)}_{\text{unconditional prob of } A} = \frac{1}{3}$$

Multiplication rule

- Conditional probability written as P(B|A), read as "the probability of the event B, given that the event A has occurred"
- Chance that two things will both happen is the chance that the first happens, *multiplied* by the chance that the second will happen *given* that the first has happened.

$P(A|B)$ & $P(B|A)$ can be very different

- Let $A, B \subseteq \Omega, P(A) > 0, P(B) > 0$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$P(B) \neq 0$

- Multiplication rule:

$$P(A \cap B) = P(A|B) \times P(B)$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad P(A \cap B) = P(B \cap A) = P(B) \times P(A|B)$$

If A & B are independent

$$P(A|B) = P(A) \Rightarrow \boxed{P(A \cap B) = P(A) \cdot P(B)}$$

Exercise: Think of events A, B s.t. $P(A|B) \neq P(B|A)$.

Multiplication rule

$$\boxed{P(A \cap B) = P(A|B) \times P(B)}$$

- Ex.: Draw a card at random, from a standard deck of 52
  - P(King of hearts) =? $\frac{1}{52}$

- Draw **2** cards one by one, **without** replacement.
  - P(1^{st} card is K of hearts)= $1/52$
  - P(2^{nd} card is Q of hearts | 1^{st} is K of hearts) = $\frac{1}{51}$
  - P(1^{st} card is K of hearts AND 2^{nd} is Q of hearts)= $\frac{1}{52} \cdot \frac{1}{51}$

$P(Q♡ \text{ is } 2^{nd} | K♡ \text{ is } 1^{st})$

$P(K♡)$

$P(Q♡_{2^{nd}} | K♡_{1^{st}})$

$\cancel{P(Q♡)}$

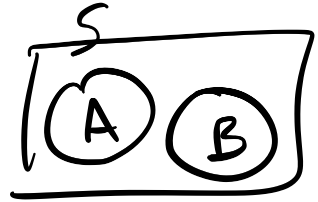
- We can also write the "Division Rule" for conditional probability:

$A \cap B = AB$

$$\boxed{P(A|B) = \frac{P(AB)}{P(B)}, P(B) \neq 0}$$

$$\frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)} = \frac{P(A|B)\,P(B)}{P(B)}$$

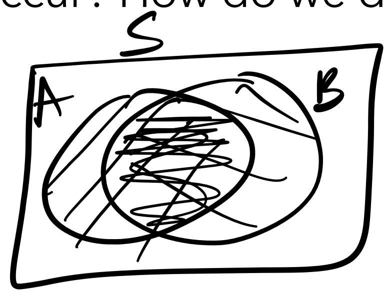
Addition rule:



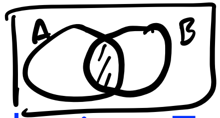
- Addition rule: If A and B are ***mutually exclusive*** events, then the probability that **at least one** of the events will occur is the sum of their probabilities:

$$P(A \cup B) = P(A) + P(B)$$

- If they are not mutually exclusive, does this still hold?
- How do we write the event that "at least one of the events A or B will occur? How do we draw it?

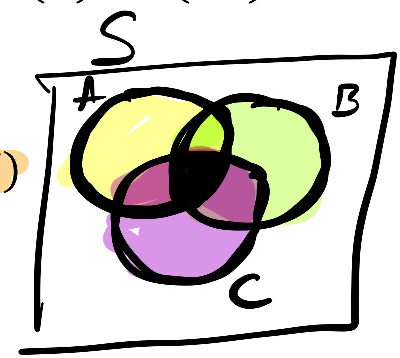


$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$



Inclusion-Exclusion Formula (general addition rule)

- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(AB)$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$

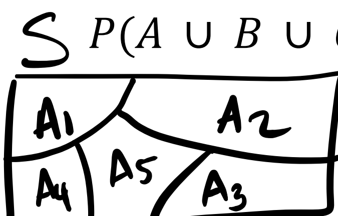


- (Draw a Venn diagram) $\boxed{P(A \cap B) = 0}$
- Of course, if A and B (or A and B and C) *don't* intersect, then the general addition rule becomes the **simple** addition rule of

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = P(A) + P(B)$, *or*

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$ } when A, B, C are mutually exclusive



$A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 = S$

$A_1 \cap A_2 = \emptyset$ $A_1 \cap A_3 = \emptyset$ etc

$A_i \cap A_j = \emptyset$ for $i \neq j$

Then we call $A_1, \ldots, A_5$ a PARTITION of S.

Examples

Exercise for Tuesday

- Deal 5 cards from the top of a well shuffled deck. What is the probability that all are hearts? (Extend the multiplication rule)

H_k : k^{th} card is ♡

$P(H_1) = \frac{13}{52}$

$$P(H_1 \,\&\, H_2) = \frac{13}{52} \cdot \underbrace{P(H_2 | H_1)} = \frac{13}{52} \cdot \frac{12}{51}$$

$$P(H_1 \,\&\, H_2 \,\&\, H_3) = P(H_3 | H_1 \,\&\, H_2)\, P(H_1 \,\&\, H_2) \frac{12}{51}$$

- Deal 5 cards, what is the chance that they are all the same suit? (flush)

$$P(\text{all same suit}) = P(\text{all ♡ or all ◇ or all ♣ or all ♠})$$
$$= P(\text{all ♡}) + P(\text{all ◇}) + \cdots$$

De Méré's paradox:

Find the probability of $\overbrace{\text{at least 1 six in 4 throws}}^{A}$ of a fair die, and $\underbrace{\text{at least a double six in 24 throws of a pair of dice}}_{B}$.

$A_k = 6$ on k^{th} roll)

De Méré's computation

$P(A) = \underbrace{\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}}$ = P(6 in 1st or 6 in 2nd or 6 in 3rd or 6 in 4th)

$P(A) = P(A_1 \cup A_2 \cup A_3 \cup A_4)$

$= \sum_{k=1}^{4} P(A_k) = \cancel{\frac{4}{6}}$

since A_1, A_2, A_3, A_4 are not mutually exclusive

$P(A) = 1 - P(A^c)$

$= 1 - \left(\frac{5}{6}\right)^4$

$P(B) = 1 - \left(\frac{35}{36}\right)^{24}$

$P(B) = \overbrace{\frac{1}{36} + \frac{1}{36} + \dots + \frac{1}{36}}^{24 \text{ times}} = \frac{24}{36} = \cancel{\frac{4}{6}}$

A_k = six on k^{th} roll

$$P(\overbrace{\text{at least 1 six in 4 rolls}}^{A})$$

$$= P(A_1 \cup A_2 \cup A_3 \cup A_4)$$

$$= P(A_1)+P(A_2)+P(A_3)+P(A_4)$$
$$-P(A_1A_2)-P(A_1A_3) - \cdots$$
$$+P(A_1A_2A_3)+\cdots$$
$$-P(A_1A_2A_3A_4)$$

$$P(A_1A_2A_3) = \left(\tfrac{1}{6}\right)^3$$

$$= 1 - P(A^c) = 1 - P(\text{no sixes in any rolls})$$
$$= 1 - \underbrace{\left(\tfrac{5}{6}\right)^4}_{\text{b/c rolls are indep}} = 0.5177 \approx 0.52$$

B : at least 1 double six in 24 rolls

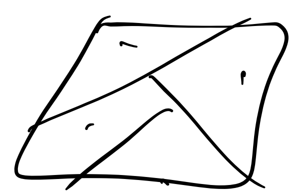
$$P(B) = 1 - P(B^c) = 1 - \underbrace{\left(\tfrac{35}{36}\right)^{24}}_{\text{mult. rule b/c indep}}$$

$$\approx 0.4914$$

$$P(\overbrace{\text{exactly 1 six in 4 rolls}}^{A})$$

A_1 : six on 1^{st} roll & no six on other 3.

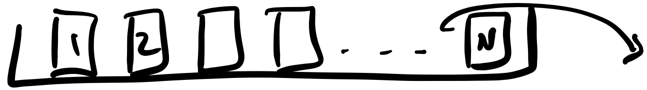
$$= P(A_1 \cup A_2 \cup A_3 \cup A_4)$$ but now $A_1, \ldots, A_4$ do form a partition



$$P(A_1) = P(\text{six on } 1^{st} \text{ \& no six on others})$$
$$= \tfrac{1}{6}\cdot\tfrac{5}{6}\cdot\tfrac{5}{6}\cdot\tfrac{5}{6} = \tfrac{1}{6}\left(\tfrac{5}{6}\right)^3$$

$$P(A_2) = \tfrac{5}{6}\cdot\tfrac{1}{6}\cdot\tfrac{5}{6}\cdot\tfrac{5}{6}$$

$$P(A) \text{ in this case} = 4\cdot\tfrac{1}{6}\cdot\left(\tfrac{5}{6}\right)^3$$



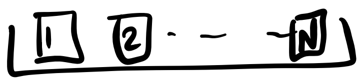
n : drawn tickets

Sec. 2.2: Symmetry in Simple Random Sampling

sampling without replacement

- One of the topics we will revisit many times is ***simple random sampling***.
- Sampling **without** replacement, each time with equally likely probabilities, that is, all the remaining tickets are equally likely
- Example to keep in mind: dealing cards from a deck

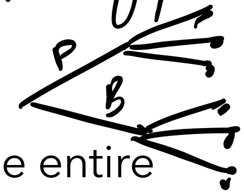
- Sampling with replacement: We keep putting the sampled outcomes back before sampling again. (Can do this to simulate die rolls.)



- Need to count number of possible outcomes from repeating an action such as sampling, will use the product rule of counting.

Product rule of counting

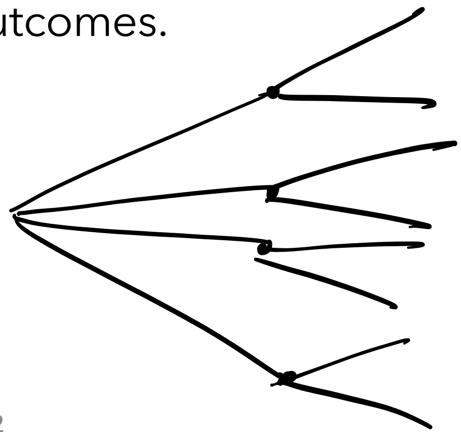
3 Shirts, 2 pairs of pants



- If a set of actions (call them $A_1, A_2, \ldots, A_n$) can result, respectively, in $k_1, k_2, \ldots, k_n$ possible outcomes, then the entire set of actions can result in:

$$k_1 \times k_2 \times k_3 \times \cdots \times k_n \text{ possible outcomes}$$

- For example: toss a coin twice. Each toss can have 2 possible outcomes, therefore 2 tosses can have 4 possible outcomes.
- So we can count the outcomes for *each* action and multiply these counts to get the number of possible sequences of outcomes.



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How many ways to arrange…

- Consider the box that contains $\boxed{O}\boxed{R}\boxed{A}\boxed{N}\boxed{G}\boxed{E}$:
- How many ways can we rearrange these letters?

$$\underline{6}\cdot\underline{5}\cdot\underline{4}\cdot\underline{3}\cdot\underline{2}\cdot\underline{1} = 6!$$

- Now say we only want to choose **2 letters** out of the six: $\underline{6}\cdot\underline{5}$

$$= \frac{6!}{4!}$$

Symmetries in cards

- Deal 2 cards from top of the deck.
  - How many possible sequences of 2 cards?
  - What is the chance that the second card is red?

- P(5th card is red)

- $P(R_{21} \cap R_{35})$ = (write it using conditional prob)

- P(7th card is a queen)

- $P(B_{52} \mid R_{21}R_{35})$

Section 2.3: Bayes' Rule:

- I have two containers: a jar and a box. Each container has five balls: The jar has three red balls and two green balls, and the box has one red and four green balls.
- Say I pick one of the containers at random, and then pick a ball at random. What is the chance that I picked the box, if I ended with a red ball?

Prior and Posterior probabilities

- The ***prior*** probability of drawing the box = ___ (before we knew anything about the balls drawn)

- The ***posterior*** probability of drawing the box = ____ (this is after we *updated* our probability, *given* the information about which ball was drawn)

Computing Posterior Probabilities: Bayes' Rule

- We want the *posterior* probability. That is, the conditional prob for the first stage, given the second.

- Division rule (for conditional probability) =

- Using the multiplication rule on P(AB), we get:

- Rule first written down by Rev. Thomas Bayes in the 18^{th} century. Helps us compute posterior probability, given prior prob. And likelihoods (which are conditional probabilities for the *second* stage given the first)

Exercise 2.6.9

A factory has two widget-producing machines. Machine I produces 80% of the factory's widgets and Machine II produces the rest. Of the widgets produced by Machine I, 95% are of acceptable quality. Machine II is less reliable – only 85% of its widgets are acceptable.

Suppose you pick a widget at random from those produced at the factory.

a) Find the chance that the widget is acceptable, given that it is produced by Machine I.

b) Find the chance that the widget is produced by Machine I, given that it is acceptable.

XS

(This example is from the text *Intro Stats* by De Veaux, Velleman, and Bock)

For men, binge drinking is defined as having 5 or more drinks in a row and for women as having 4 or more drinks in a row.
(The difference is because of the average difference in weight.)

According to a study by the Harvard School of Public Health (H.Wechsler, G. W. Dowdall, A. Davenport, and W. Dejong, "*Binge Drinking on Campus: Results of a National Study*"):

- 44% of college students engage in binge drinking,
  37% drink moderately, and 19% abstain entirely.

- Another study, published in American journal of Health Behavior, finds that among binge drinkers aged 21 to 34, 17% have been involved in an alcohol related automobile accident, while among nonbingers of the same age, only 9% have been involved in such accidents.