Last time:

Sec 9.1 Hypotheses test.

Sec 9.2 A/B testing

Sec 9.3 CI (for population mean)

Today.

$\begin{cases} \text{finish CI} \\ \underline{\text{cts.}}\ \text{R.V.} \end{cases}$

continuous

Sec. 9.4 Interpretation of CI.

e.g. $X_1, X_2, X_3, \dots, X_n$ sample size $= n$

from population with $\underline{\text{mean } \mu}$, $\underline{\text{sd } \sigma}$.

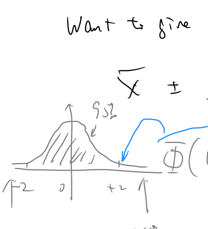
unknown

known

unknown, but can be estimated by sample sd

Want to give 95% CI of μ:

$\bar{X} \pm 2 \frac{\sigma}{\sqrt{n}}$



$2 = \Phi(1 - \frac{5\%}{2})$

$\frac{\sigma}{\sqrt{n}}$: sd of sample mean $\bar{X}$

$sd\left(\frac{1}{n}(X_1 + \dots + X_n)\right)$

$P\left(\mu \in \left(\bar{X} - 2\frac{\sigma}{\sqrt{n}}, \bar{X} + 2\frac{\sigma}{\sqrt{n}}\right)\right) \approx 0.95$

This means:

- The CI, as a Random Interval, has 0.95 chance to cover the true population mean
- Imagine you repeat this 100 times, and generate 100 CI's from them, then you expect 95 of these CI's will cover μ

This does not mean: (All wrong)

- μ has 0.95 chance to fall in CI.
- X has 0.95 chance to be in CI

There is no 'chance' about the constant parameter μ. CI only covers population mean, it cannot be used to describe the population distr.

- $\bar{X}$ has 0.95 chance to be in CI.

$\begin{cases} \text{same data} \to \text{chance} = 1 \\ \text{two sets of data} \to \mathbb{P}(\bar{X}' \in CI) \end{cases}$

random random

How to decrease the width of a CI? (by half)

$\bar{X} \pm \underbrace{\Phi\left(1 - \frac{1-95\%}{2}\right) \frac{\sigma}{\sqrt{n}}}_{\text{half the width}}$ ① ②

① decrease the significance level, to α such that

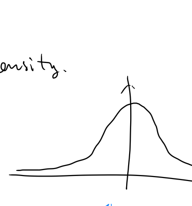
$\Phi\left(1 - \frac{1-\alpha}{2}\right) = \frac{1}{2}\Phi\left(1 - \frac{1-95\%}{2}\right)$ (Usually not wise)

② increase sample size from n to $4n$.

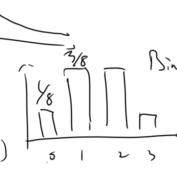
$\frac{1}{\sqrt{n}} \to \frac{1}{2} \cdot \frac{1}{\sqrt{n}}$.

Continuous R.V. and Density.

Standard Normal Curve



Discrete R.V. has its distr attached to each of the possible values



Cts. R.V. has an interval (or many intervals) of possible values such as $\mathbb{R}$, $(0,1)$, $(0,\infty)$ ...

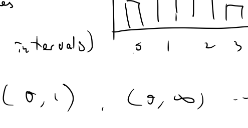
Sec 6.1 Density. (Probability Density Function, pdf. PDF)

Density is a function describe a distribution, like the normal curve.

Defn We say a function f is a density, if

$\mathbb{R}$-valued on $\mathbb{R}$

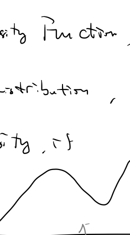
- $f \geq 0$
- $\int_{-\infty}^{\infty} f(x)\,dx = 1$



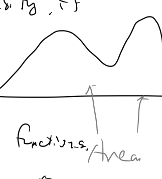
Usually, we consider only piecewise smooth functions.

e.g. 1. Normal curve

2. $f(x) = \begin{cases} 2x & x \in (0,1) \\ 0 & o/w. \end{cases}$



3. $f(x) = \begin{cases} x & x \in (0,1] \\ 2-x & x \in (1,2) \\ 0 & o/w. \end{cases}$

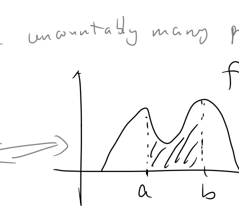


- Density is not prob. !

Another reason: there are uncountably many possible values.

- Areas are probs!

$\mathbb{P}(X \in (a,b])$



$= \int_a^b f(x)\,dx$

$= \mathbb{P}(X \in [a,b])$

$= \mathbb{P}(X \in (a,b))$

$= \mathbb{P}(X \in [a,b))$

Since the Area of a single line $= 0$

$\Rightarrow \mathbb{P}(X = a) = 0$

"<" is the same as "≤" in cts. case

CDF $F(x) = \mathbb{P}(X \leq x)$ $\left(= \sum_{j \leq x} \mathbb{P}(X = j)\right)$

$= \mathbb{P}(X \in (-\infty, x])$

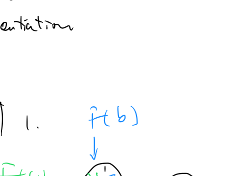
$= \int_{-\infty}^{x} f(x)\,dx$

- has a clear prob.
- pdf & cdf can be converted to one another easily through integration and differentiation
- $F(x) \in [0,1]$

$\lim_{x \to -\infty} F(x) \downarrow 0$ $\lim_{x \to +\infty} F(x) \uparrow 1$.

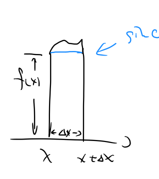
non-decreasing.

- $\mathbb{P}(X \in (a,b]) = F(b) - F(a)$



- Another interpretation using Δx (a short interval $\Delta x \to 0$)

$\mathbb{P}(X \in (x, x + \Delta x))$



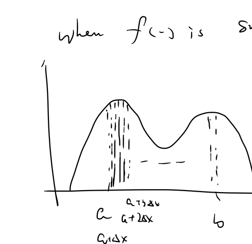
$\approx \Delta x\, f(x)$

$f(x) \approx \frac{\mathbb{P}(X \in (x, x+\Delta x))}{\Delta x}$

$f(x) = \lim_{\Delta x \downarrow 0} \frac{\mathbb{P}(X \in (x, x + \Delta x))}{\Delta x}$ when $f(\cdot)$ is smooth at x

$\mathbb{P}(X \in [a,b])$

$= \sum_{x \in [a,b]} p(x)$ (for discrete)



$= \int_a^b f(x)\,dx$ (for cts.)

Sec 9.2 Expectation and Variance.

$\mathbb{E}X = \sum_x x\,p(x)$

$\int_{\mathbb{R}} x f(x)\,dx$

$Var(X) = \mathbb{E}X^2 - \boxed{(\mathbb{E}X)^2}$

$\mathbb{E}X^2 = \sum_x x^2 p(x)$

$\int_{\mathbb{R}} x^2 f(x)\,dx$

cts.

discrete

Note: these integrals are allowed to be infinite or undefined. but in this course all that you might encounter are finite.

Properties involving $\mathbb{E}X$, $Var(X)$ & $SD(X)$ are the same as discrete ones.

- linearity of expectation.

$\mathbb{E}(aX + bY + c) = a\mathbb{E}X + b\mathbb{E}Y + c$

- linear transform of Var / SD.

$Var(aX + c) = a^2 Var(X)$

$SD(aX + c) = |a|\,SD(X)$

- Two ways of Var calculation

$Var(X) = \mathbb{E}(X - \mathbb{E}X)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2$

- Additivity of Var of indpt. sums.

$X_1, X_2, \dots, X_n$ indpt

$Var(X_1 + \dots + X_n) = Var(X_1) + Var(X_2) + \dots + Var(X_n)$

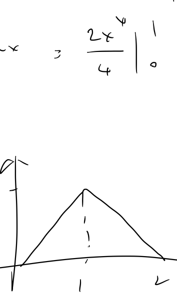
- (Weak) Law of large numbers / CLT.

$\bar{X} \xrightarrow{d} \mu$

$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d}$ standard normal $N(0,1)$

Example

1. $f(x) = \begin{cases} 2x & x \in (0,1) \\ 0 & o/w. \end{cases}$



$\mathbb{P}(X \in (0, \tfrac{1}{2})) = \int_0^{\frac{1}{2}} 2x\,dx$

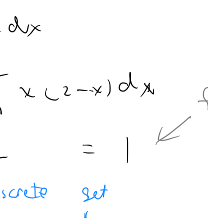
$= x^2 \Big|_0^{\frac{1}{2}} = \frac{1}{4}$

$\mathbb{E}X = \int x f(x)\,dx = \int_0^1 x \cdot 2x\,dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$.

$\mathbb{E}X^2 = \int x^2 f(x)\,dx = \int_0^1 x^2 \cdot 2x\,dx = \frac{2x^4}{4}\Big|_0^1 = \frac{1}{2}$.

$Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2$

2. $f(x) = \begin{cases} x & x \in (0,1] \\ 2-x & x \in (1,2) \\ 0 & o/w \end{cases}$



$\mathbb{E}X = \int x f(x)\,dx = \left(\int_0^1 + \int_1^2\right) x f(x)\,dx$

$= \int_0^1 x \cdot x\,dx + \int_1^2 x(2-x)\,dx$

$= \frac{1}{2}x^2\Big|_0^1 + \dots = 1$ from symmetry

3. Uniform on $(0,1)$ two numbers.

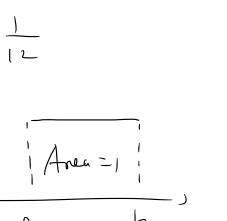
discrete set

$Unif(\{1,2,3,\dots,6\})$

$Unif(0,1)$.

$Unif((0,1))$ $\begin{cases} 1 & x \in (0,1) \\ 0 & o/w. \end{cases}$

$Unif([0,1])$



$\mathbb{E}X = \int_0^1 x \cdot 1\,dx = \frac{1}{2}$

also from symmetry.

$\mathbb{E}X^2 = \int_0^1 x^2 \cdot 1\,dx = \frac{1}{3}$

$Var(X) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$Unif(a,b)$ $f(x) = \begin{cases} \frac{1}{b-a} & (a,b) \\ 0 & o/w. \end{cases}$

$\mathbb{E}X = \frac{a+b}{2}$

$\mathbb{E}X^2 = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx$

$X - a \sim Unif(0, b-a)$

$\frac{X-a}{b-a} \sim Unif(0,1)$

$\frac{1}{12} = Var\left(\frac{X-a}{b-a}\right) = \left(\frac{1}{b-a}\right)^2 Var(X)$

$\Rightarrow Var(X) = \frac{(b-a)^2}{12}$