Last time:

Sec. 4.3 { Exp. Approx. $(1-p)^n$

4.4 { Poisson distr. - Law of small numbers. (Binom (n,p) for n large, p small)

Sec. 5.1 Expectation of { - a single R.V.

5.2 { - a function (of a single R.V.)

{ - a function of more than one R.V.'s

esp. sum of R.V.s

'Additivity' (Linearity)

$$\mathbb{E}(X+Y) = \mathbb{E}X + \mathbb{E}Y.$$

Proof of additivity.

$Y \backslash X$	x_1	x_2	x_3	...
y_1				
y_2				
y_3				
$\vdots$				

marginal prob.

$$\mathbb{P}(X = x_i) = \sum_j \mathbb{P}(X = x_i, Y = y_j) \qquad \mathbb{P}(X = x_i) = \sum_j p_{ij}$$

$$\mathbb{P}(Y = y_j) = \sum_i \mathbb{P}(X = x_i, Y = y_j). \qquad \mathbb{P}(Y = y_j) = \sum_i p_{ij}$$

$$\mathbb{E}(X+Y) = \sum_{i,j} (x_i + y_j)\,\mathbb{P}(X = x_i, Y = y_j).$$

$$= \sum_{i,j} x_i p_{ij} + \sum_{i,j} y_j p_{ij}$$

$$= \sum_i x_i \sum_j p_{ij} + \sum_j y_j \sum_i p_{ij}$$

$$= \sum_i x_i \mathbb{P}(X = x_i) + \sum_j y_j \mathbb{P}(Y = y_j)$$

$$= \mathbb{E}X + \mathbb{E}Y.$$

$$\mathbb{E}((X+Y)+Z) = \mathbb{E}(X+Y) + \mathbb{E}Z$$

$$= \mathbb{E}X + \mathbb{E}Y + \mathbb{E}Z.$$

Today:

Sec. 5.3 Method of indicators — A counting technique

1st 2nd 3rd

Counting.

$= 3 \leftarrow$ Count of H's

Recall indicator of an event A $\underbrace{(1_A \text{ or } I_A)}_{\text{Indicators}}$ is a R.V.

$$I_A = \begin{cases} 1 & \text{on } A \iff A \text{ happens.} \\ 0 & \text{on } A^c \end{cases}$$

$$\mathbb{E} I_A = \mathbb{P}(A)$$

Example. X = # of H's in 6 coin tosses (fair).

Find $\mathbb{E}X$

Step 1: $I_j = \begin{cases} 1 & \text{if the } j^{th} \text{ toss is H} \\ 0 & \text{-- -- ---- -- T} \end{cases}$

H T T H H T

Sum (1 0 0 1 1 0) = 3

Hence, $I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = X$

Step 2: $\mathbb{E}X = \mathbb{E}(I_1 + I_2 + \cdots + I_6)$

Apply additivity $= \mathbb{E}I_1 + \mathbb{E}I_2 + \cdots + \mathbb{E}I_6.$

"Generalized additivity" $\mathbb{E}\sum_{i=1}^n X_i = \sum_{i=1}^n \mathbb{E}X_i.$

Step 3: $\mathbb{E}I_j = \mathbb{P}(j^{th} \text{ toss lands H}) = \frac{1}{2}$, $\forall j = 1, 2, \ldots, 6$

$\mathbb{E}X = 6\,\mathbb{E}I_1 = 6 \times \frac{1}{2} = 3.$

Generally, for any $X \sim \text{Binom}(n,p)$, $\mathbb{E}X = np.$

since X stands for the # of H's in n tosses of a p-coin

- Expect 50 H's in 100 coin tosses
- 10 sixes 60 die rolls
- Recall Poisson (μ) has expectation μ, and in law of small numbers, $\text{Binom}(n,p) \approx \text{Poisson}(\mu = np)$. has the same expectation.

Example 2. $Y \sim \text{Hypergeom}(N = 52, G = 4, n = 5)$

5 cards from a standard deck. Y = # of Aces.

Step 1: $I_j = \begin{cases} 1 & \text{the } j^{th} \text{ card is an Ace} \\ 0 & \text{o/w.} \end{cases}$ $j = 1, 2, \ldots, 5$

Try this. Step 1*: $I_k^* = \begin{cases} 1 & \text{if the } k^{th} \text{ Ace is in hand} \\ 0 & \text{o/w.} \end{cases}$ $k = 1, 2, 3, 4$

$Y = I_1^* + \cdots + I_4^*$

Hence $Y = I_1 + \cdots + I_5.$

Step 2: $\mathbb{E}Y = \mathbb{E}I_1 + \mathbb{E}I_2 + \cdots + \mathbb{E}I_5$ — Since additivity requires no independence.

Step 3. $\mathbb{E}I_j = \mathbb{P}(j^{th} \text{ card is an Ace})$

$= \frac{4}{52}$ (by symmetry)

$\mathbb{E}Y = 5\,\mathbb{E}I_1 = 5 \times \frac{4}{52}.$

Generally, the expectation of a Hypergeom(N, G, n) R.V. is $\frac{nG}{N}$.

- Binom $(n, \frac{G}{N})$ is sampling w/. repl. from a population of size N with G target items, whose expectation is $\frac{nG}{N}$, the same as Hypergeom(N, G, n)

In words. sampling w/. or w/o. repl. shares the same expectation.

Example 3. "missing classes"

Consider a population in which each element belongs to exactly one class: A 40%, B 35%, C 25%.

p_A p_B p_C

Suppose we sample w/. repl., size $= n$, what is the expected number of classes not appearing?

Define $I_A = \begin{cases} 1 & \text{if class A is missing} \\ 0 & \text{o/w.} \end{cases}$, I_B, I_C similarly

X = # of missing classes $= I_A + I_B + I_C$

$$\mathbb{E}X = \mathbb{E}I_A + \mathbb{E}I_B + \mathbb{E}I_C$$

$$= \mathbb{P}(A \text{ missing}) + \mathbb{P}(B \text{ missing}) + \mathbb{P}(C \text{ missing})$$

$$= (1-p_A)^n + (1-p_B)^n + (1-p_C)^n.$$

Y = # of classes appearing

$\Rightarrow X + Y = 3 \quad \Rightarrow \mathbb{E}X + \mathbb{E}Y = \mathbb{E}3 = 3$

$\Rightarrow \mathbb{E}Y = 3 - \mathbb{E}X.$

To summarize method of indicators may help us calculate the expectation of something countable.

(Skip Sec. 5.4)

Sec. 5.5. Conditional expectation.

Example. toss a fair coin 3 times. Let X = # of H in the first toss, Y = # of H's in the last 2 tosses

$S = X + Y$:= # of H's in all three tosses.

What is the distribution of X, given $S = 2$?

joint distr. table

$X \sim \text{Bern}(\frac{1}{2})$, $Y = \text{Binom}(2, \frac{1}{2})$

$S \backslash X$	0	1
0	$\frac{1}{8}$	0
1	$\frac{1}{4}$	$\frac{1}{8}$
2	$\frac{1}{8}$	$\frac{1}{4}$
3	0	$\frac{1}{8}$

$\mathbb{P}(X = 0, S = 0) = \mathbb{P}(X = 0, Y = 0)$

$= \mathbb{P}(X = 0)\,\mathbb{P}(Y = 0)$

$= \frac{1}{2} \times \frac{1}{4}$

In this row. it contains all information about $S = 2$

We may turn this into a distr. by normalizing, i.e. divide each entry by their sum. (marginal prob.)

$\frac{\frac{1}{8}}{\frac{1}{8} + \frac{1}{4}} = \frac{1}{3}$, $\frac{\frac{1}{4}}{\frac{1}{8} + \frac{1}{4}} = \frac{2}{3}$

Divide each row by the marginal of S

Conditional prob of $X = x$ given $S = s$

$S \backslash X$	0	1
0	1	0
1	$\frac{2}{3}$	$\frac{1}{3}$
2	$\frac{1}{3}$	$\frac{2}{3}$
3	0	1

Each row is a conditional distr.

$\mathbb{E}(X \mid S = 2) = 0 \cdot \mathbb{P}(X = 0 \mid S = 2) + 1 \cdot \mathbb{P}(X = 1 \mid S = 2) = \frac{2}{3}$

$\mathbb{E}X = 0 \cdot \mathbb{P}(X = 0) + 1 \cdot \mathbb{P}(X = 1) = \frac{1}{2}$

Generally, for R.V. X and event A, the conditional expectation of X given A is

$$\mathbb{E}(X \mid A) = \sum_x x\,\mathbb{P}(X = x \mid A)$$

In particular, $A = \{Y = y\}$

Sec. 5.6. Expectation by conditioning.

"weighted average"

Example class with 30 students. { Sec 1 12 students 75 / Sec 2 18 80

What is the avg. of the whole class?

$(\mathbb{E}X =)\ 75 \times \frac{12}{30} + 80 \times \frac{18}{30} = 78$

Randomize the question: Select a student at random, his score is X. What is $\mathbb{E}X$?

$\frac{12}{30} = \mathbb{P}(S = 1)$ $\quad \frac{18}{30} = \mathbb{P}(S = 2)$ $\quad S$ = Sec the student is from

$75 = \mathbb{E}(X \mid S = 1)$, $80 = \mathbb{E}(X \mid S = 2)$

$$\mathbb{E}X = \mathbb{E}(X \mid S = 1) \cdot \mathbb{P}(S = 1) + \mathbb{E}(X \mid S = 2)\,\mathbb{P}(S = 2)$$

Generally, for R.V.'s X, Y,

$$\mathbb{E}X = \sum_y \mathbb{E}(X \mid Y = y)\,\mathbb{P}(Y = y).$$

where $\sum_y$ takes all possible values of Y.

Example 2. catching misprints.

# of misprints in a document is $N \sim \text{Poisson}(5)$, each is caught with prob. $p = 0.95$, indpt. of the others.

What is the expected # of misprints we catch?

X = # of misprints caught.

Step 1: Conditional on $N = n$, $(X \mid N = n) \sim \text{Binom}(n, p)$

$X \sim \text{Binom}(N, p)$ ✗

Step 2: $\mathbb{E}(X \mid N = n) = np.$

Step 3: $\mathbb{E}X = \sum_{n=0}^{\infty} \mathbb{E}(X \mid N = n)\,\mathbb{P}(N = n)$

$= \sum_{n=0}^{\infty} np \cdot \mathbb{P}(N = n)$

$= p\left(\sum_{n=0}^{\infty} n\,\mathbb{P}(N = n)\right) = p\,\mathbb{E}N = 0.95 \times 5.$

Example 3. $X \sim \text{Geom}(p)$ on $1, 2, 3, \ldots$

Conditional on the result of the first trial $I_1 = \begin{cases} 1 & \text{success} \\ 0 & \text{failure} \end{cases}$

$(X \mid I_1 = 1) = 1$

$(X \mid I_1 = 0) \overset{d}{=} 1 + X$ or $(X \mid I_1 = 0) = 1 + X_1$ when $X_1 \sim \text{Geom}(p)$.

$\mathbb{E}(X \mid I_1 = 1) = \mathbb{E}1 = 1$

$\mathbb{E}(X \mid I_1 = 0) = \mathbb{E}1 + \mathbb{E}X = 1 + \mathbb{E}X.$

$$\mathbb{E}X = \mathbb{E}(X \mid I_1 = 1)\,\mathbb{P}(I_1 = 1) + \mathbb{E}(X \mid I_1 = 0)\,\mathbb{P}(I_1 = 0)$$

$$= 1 \times p + (1 + \mathbb{E}X) \times (1-p).$$

$$= 1 + (1-p)\,\mathbb{E}X. \quad \Rightarrow \mathbb{E}X = \frac{1}{p}.$$

$x := \mathbb{E}X.$ $\quad x = 1 + (1-p)x \Rightarrow x = \frac{1}{p}.$

This means: - Expect 2 tosses for the first H

- expect 6 rolls for the first six.