Linear Algebra

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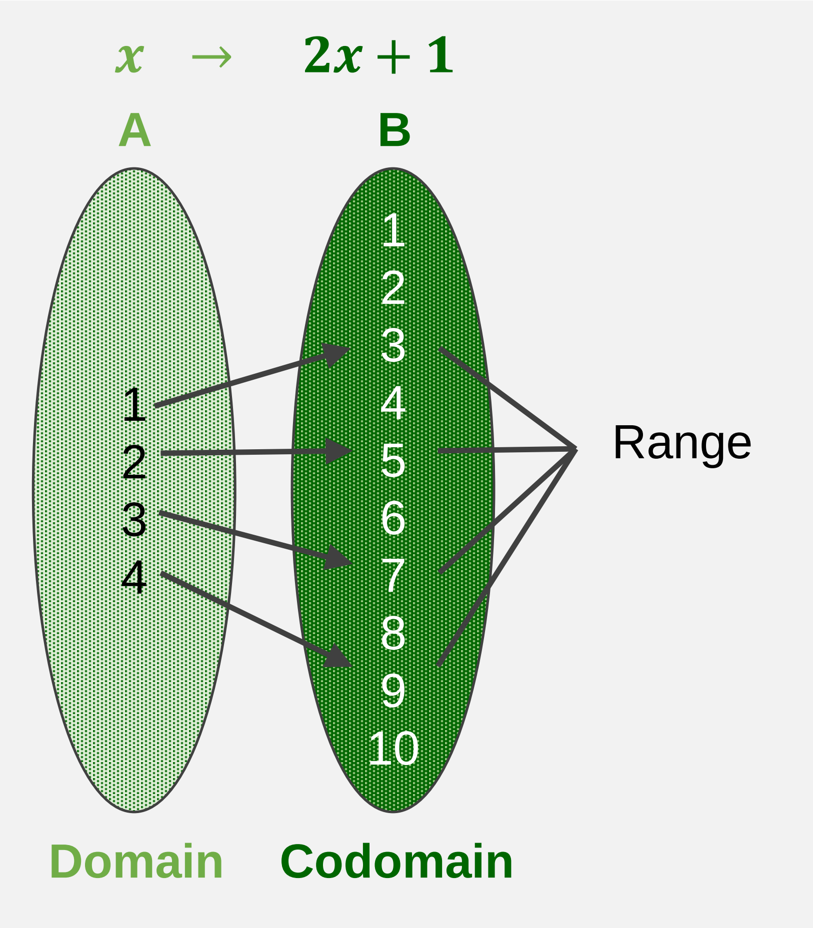
Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
  Four views of matrix multiplication
- Linear independence, span, and subspace
- <u>Linear transformation</u>
- Least squares
- Eigendecomposition
- Singular value decomposition

Transformation

- A **transformation**, **function**, or **mapping**, T maps an input x to an output y
  - Mathematical notation: T: $x \mapsto y$
- **Domain**: Set of all the possible values of x
- **Co-domain**: Set of all the possible values of y
- **Image**: a mapped output y, given x
- **Range**: Set of all the output values mapped by each x in the domain

- **Note**: the output mapped by a particular x is uniquely determined.

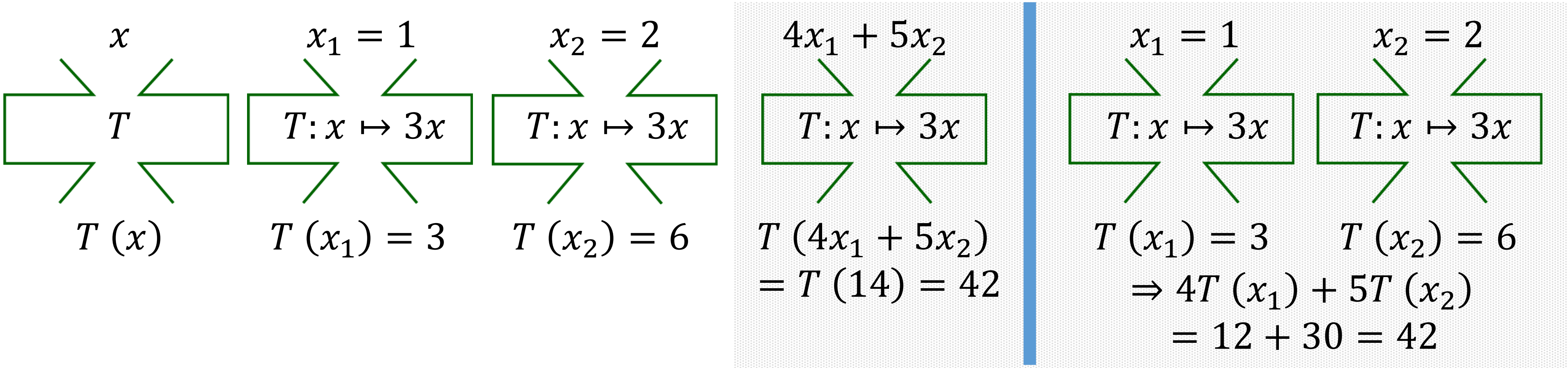


Linear Transformation

- **Definition**: A transformation (or mapping) T is **linear** if:

  I. $T\,(c\mathbf{u} + d\mathbf{v}) = cT\,(\mathbf{u}) + dT\,(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T and for all scalars c and d

- Simple example: T: $x \mapsto y$, $T(x) = y\ = 3x$



Transformations between Vectors

- $T: \mathbf{x} \in \mathbb{R}^n \mapsto \mathbf{y} \in \mathbb{R}^m$: Mapping n-dim vector to m-dim vector
- Example:

$$T: \mathbf{x} \in \mathbb{R}^3 \mapsto \mathbf{y} \in \mathbb{R}^2 \quad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3 \quad \mapsto \quad \mathbf{y} = T(\mathbf{x}) = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \in \mathbb{R}^2$$

Matrix of Linear Transformation

- Example: Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\-1\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\\2\end{bmatrix}$. With no additional information, find a formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\end{bmatrix}$$

$$\Rightarrow T(\mathbf{x})=T\left(x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\end{bmatrix}\right)=x_1T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)+x_2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$=x_1\begin{bmatrix}2\\-1\\1\end{bmatrix}+x_2\begin{bmatrix}0\\1\\2\end{bmatrix}=\begin{bmatrix}2&0\\-1&1\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Matrix of Linear Transformation

- In general, let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then T is always written as a matrix-vector multiplication, i.e.,

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

- In fact, the j-th column of $A \in \mathbb{R}^{m\times n}$ is equal to the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j-th column of the identity matrix in $\mathbb{R}^{n\times n}$:

$$A = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)]$$

- Here, the matrix A is called the **standard matrix** of the linear transformation T

Matrix of Linear Transformation

- Example: Find the standard matrix A of a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ such that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\end{bmatrix}, T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}4\\3\end{bmatrix}, \text{ and } T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}5\\6\end{bmatrix}.$$

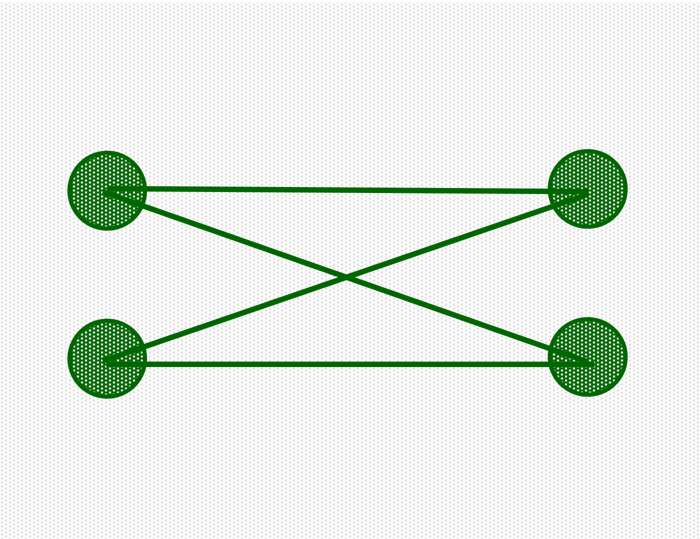
$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\Rightarrow T(\mathbf{x})=T\left(x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=x_1T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)+x_2T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)+x_3T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

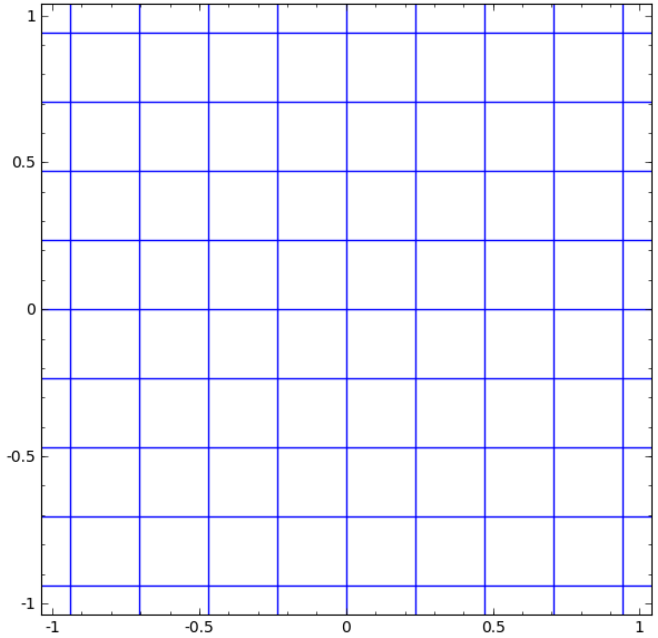
$$=x_1\begin{bmatrix}1\\2\end{bmatrix}+x_2\begin{bmatrix}4\\3\end{bmatrix}+x_2\begin{bmatrix}5\\6\end{bmatrix}=\begin{bmatrix}1&4&5\\2&3&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=A\mathbf{x}$$

Linear Transformation in Neural Networks

- Fully-connected layers (linear layer)



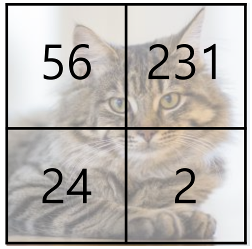
$$\mathbf{x} \underset{T_1}{\longrightarrow} \mathbf{y}$$

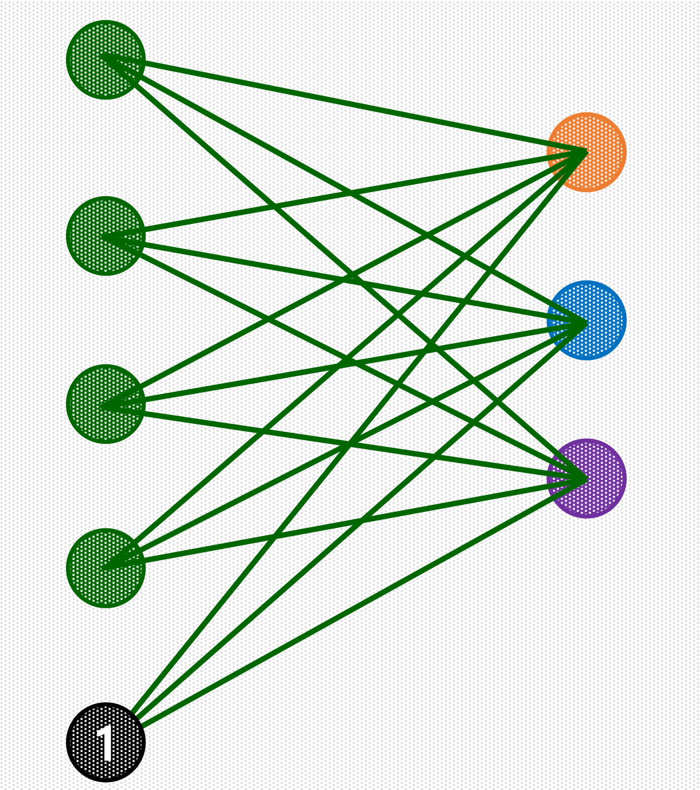


https://colah.github.io/posts/2014-03-NN-Manifolds-Topology/

Affine Layer in Neural Networks

- Fully-connected layers usually involve a bias term. That's why we call it an affine layer, but not a linear layer.
- Example: Image with 4 pixels and 3 classes (cat/dog/ship)





$$\begin{bmatrix}0.2&-0.5&0.1&2\\1.5&1.3&2.1&1\\-.2&0.3&0.7&-1.3\end{bmatrix}\begin{bmatrix}56\\231\\24\\2\end{bmatrix}+\begin{bmatrix}1.1\\3.2\\-1.2\end{bmatrix}=\begin{bmatrix}-96.8\\439.9\\71.1\end{bmatrix}$$

$$=56\begin{bmatrix}0.2\\1.5\\-.2\end{bmatrix}+231\begin{bmatrix}-0.5\\1.3\\0.3\end{bmatrix}+24\begin{bmatrix}0.1\\2.1\\0.7\end{bmatrix}+2\begin{bmatrix}2\\1\\-1.3\end{bmatrix}+1\begin{bmatrix}1.1\\3.2\\-1.2\end{bmatrix}$$

$$=\begin{bmatrix}0.2&-0.5&0.1&2&1.1\\1.5&1.3&2.1&1&3.2\\-.2&0.3&0.7&-1.3&-1.2\end{bmatrix}\begin{bmatrix}56\\231\\24\\2\\1\end{bmatrix}$$