Linear Algebra

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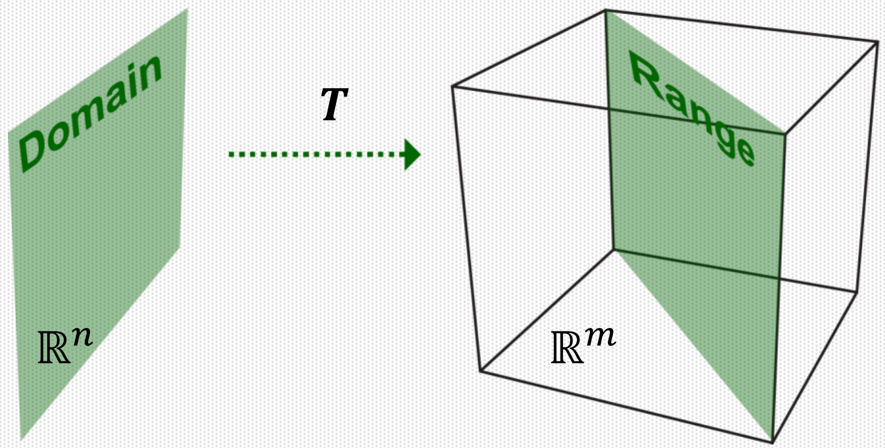
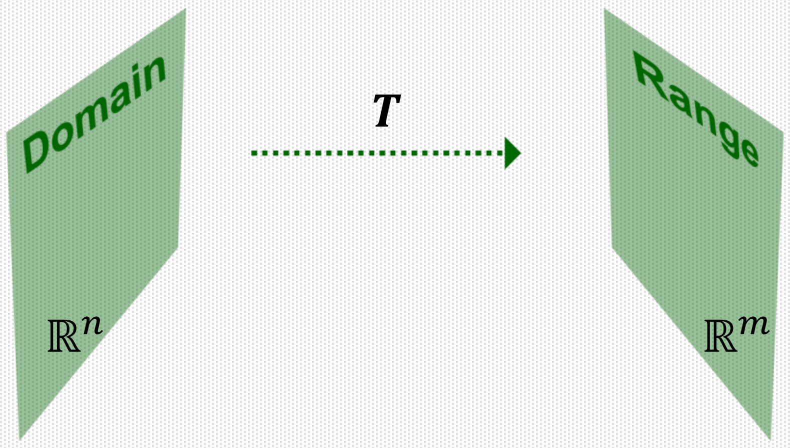
ONTO and ONE-TO-ONE

- **Definition:** A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $\mathbf{b} \in \mathbb{R}^m$ is the image of at least one $\mathbf{x} \in \mathbb{R}^n$.
  That is, the range is equal to the co-domain.



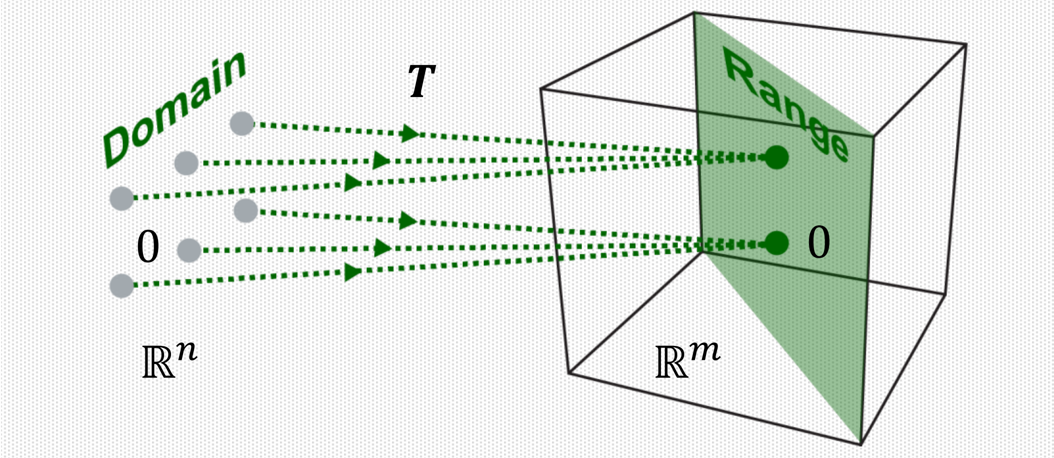
T is NOT onto $\mathbb{R}^m$



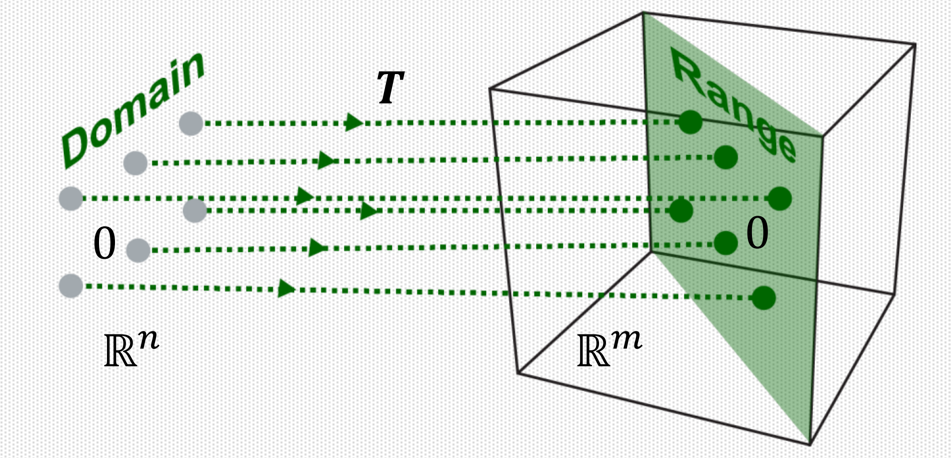
T is onto $\mathbb{R}^m$

ONTO and ONE-TO-ONE

- **Definition:** A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be one-to-one if each $\mathbf{b} \in \mathbb{R}^m$ is the image of at most one $\mathbf{x} \in \mathbb{R}^n$.
  That is, each output vector in the range is mapped by only one input vector, no more than that.



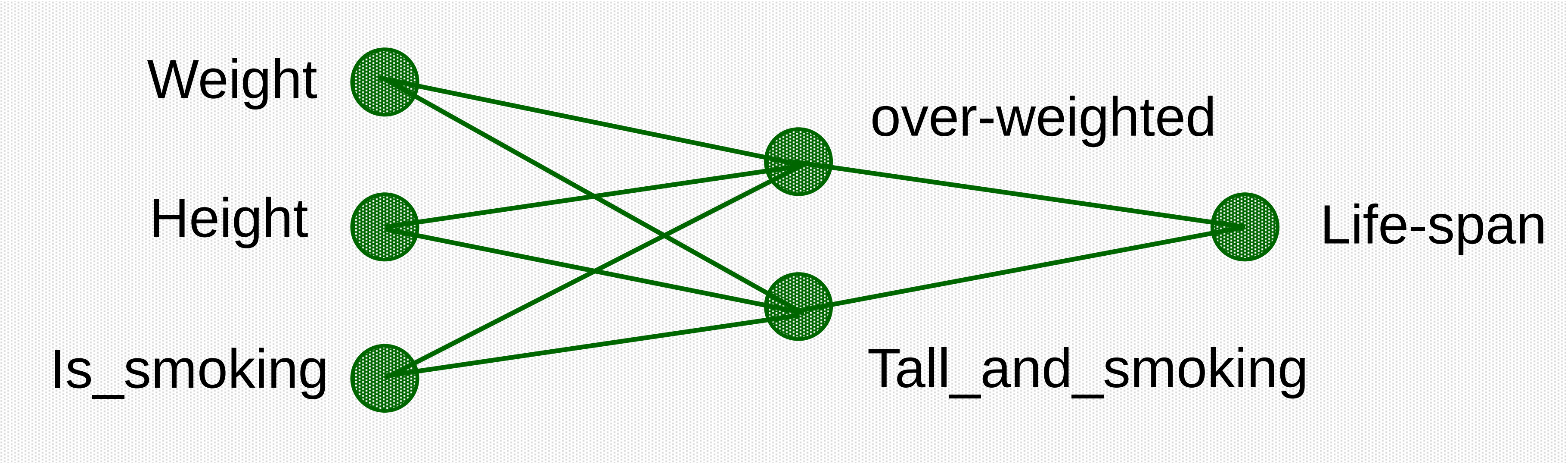
T is NOT one-to-one



T is one-to-one

Neural Network Example

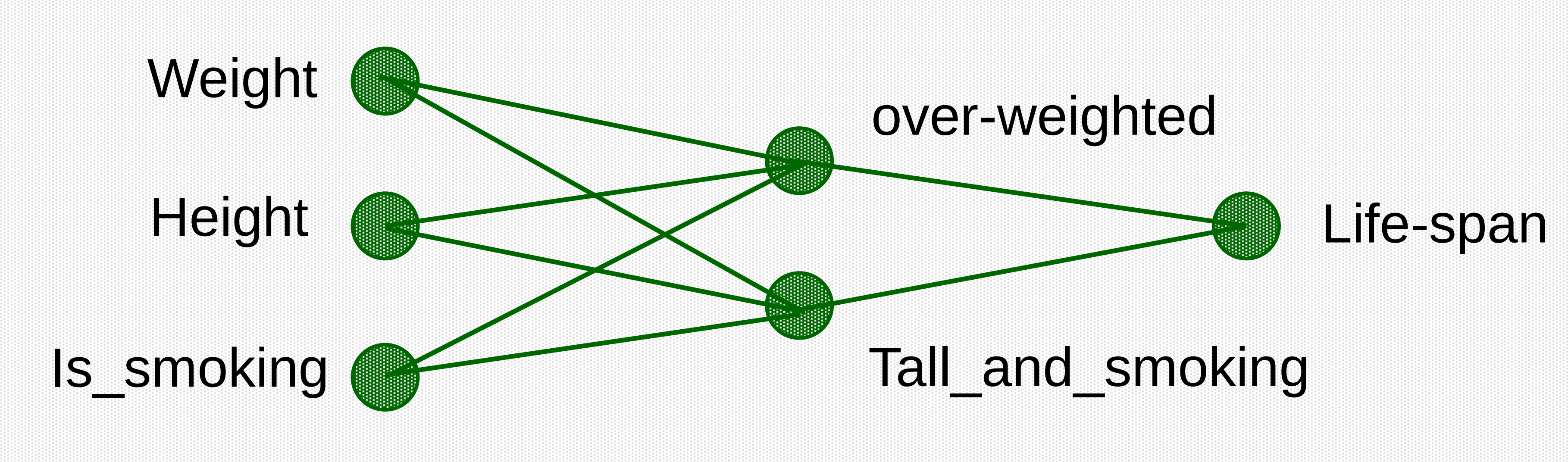
- Fully-connected layers



$$\mathbf{x} \xrightarrow[T_1]{} \mathbf{y} \xrightarrow[T_2]{} \mathbf{z}$$

Neural Network Example: ONE-TO-ONE

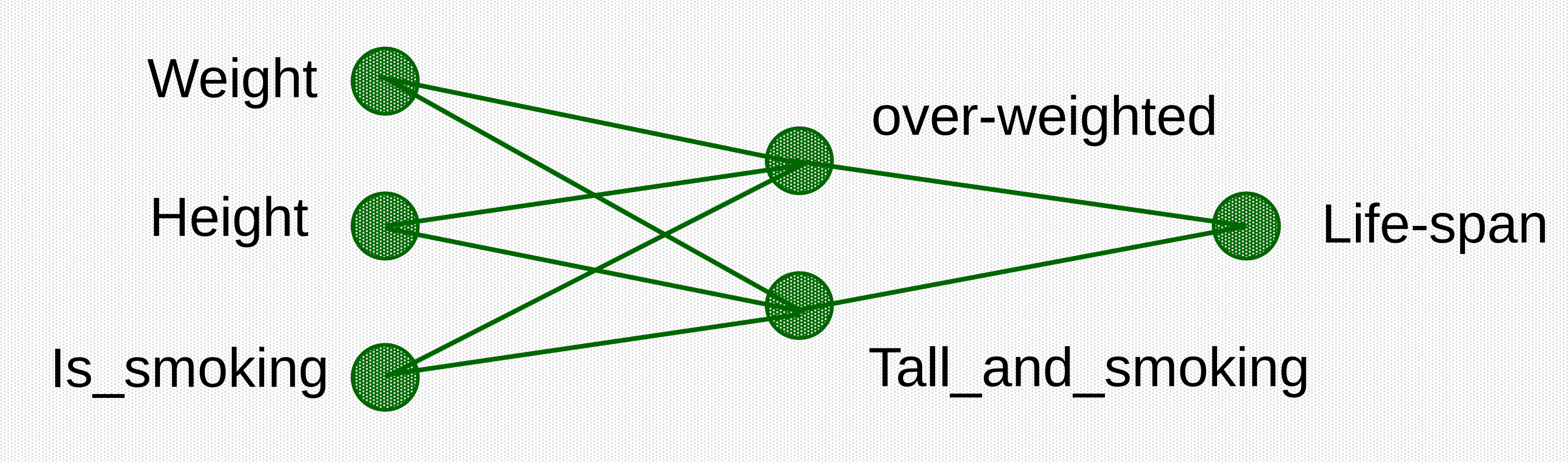
- Will there be many (or unique) people mapped to the same (over_weighted, tall_and_smoking)?



$$\mathbf{x} \xrightarrow[T_1]{} \mathbf{y} \xrightarrow[T_2]{} \mathbf{z}$$

Neural Network Example: ONTO

- Is there any (over_weighted, tall_and_smoking) that does not exist at all?



$$\mathbf{x} \xrightarrow[T_1]{} \mathbf{y} \xrightarrow[T_2]{} \mathbf{z}$$

ONTO and ONE-TO-ONE

- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, i.e.,

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

- T is one-to-one if and only if the columns of A are linearly independent.

- T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.

ONTO and ONE-TO-ONE

- **Example:**

  Let $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- Is T one-to-one?
- Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?

ONTO and ONE-TO-ONE

- **Example:**

  Let $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

- Is T one-to-one?
- Does T map $\mathbb{R}^3$ onto $\mathbb{R}^2$?

Further Study

- Gaussian elimination, row reduction, echelon form
  - Lay Ch1.2,

- LU factorization: efficiently solving linear systems
  - Lay Ch2.5

- Computing invertible matrices
  - Lay Ch2.2

- Invertible matrix theorem for square matrices
  - Lay Ch2.3, Ch2.9