

# Theoretical Statistics: Topics for a Core Course

## Summary Note

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# Preface

This note contains summaries of the textbook, *Theoretical Statistics: Topics for a Core Course*, and it was created by Hyunsung Kim, who is a Ph.D. student. I made it when I study a theoretical statistics based on this textbook on my own.

## Textbook

- Keener, *Theoretical Statistics: Topics for a Core Course*.

## Reference

- Durrett, *Probability: Theory and Examples, 5th edition*.
- Royden, *Real Analysis, 4th edition*.

# Chapter 1

## Probability and Measure

### 1.1 Measures

**Definition 1.1.1** ( $\sigma$ -algebra). *A collection of  $\mathcal{A}$  of subsets of a set  $\mathcal{X}$  is a  $\sigma$ -algebra (or  $\sigma$ -field) if*

1.  $\mathcal{X} \in \mathcal{A}$  and  $\emptyset \in \mathcal{A}$ ,
2. If  $A \in \mathcal{A}$ , then  $A^c = \mathcal{X} - A \in \mathcal{A}$ ,
3. If  $A_1, A_2, \dots \in \mathcal{A}$ , then  $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ .

**Definition 1.1.2** (Measure). *A function  $\mu$  on a  $\sigma$ -algebra  $\mathcal{A}$  of  $\mathcal{X}$  is a measure if*

1. For every  $A \in \mathcal{A}$ ,  $0 \leq \mu(A) \leq \infty$ ; that is,  $\mu : \mathcal{A} \rightarrow [0, \infty]$ .
2. (Countable Additivity) If  $A_1, A_2, \dots$  are disjoint elements of  $\mathcal{A}$ , then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

**Example 1.1.3.** For a measure  $\mu$  on a set  $\mathcal{X}$  and subsets  $A \in \mathcal{X}$ ,

1.  $\mu$  is *counting measure* on  $\mathcal{X}$  if  $\mathcal{X}$  is countable, and define

$$\mu(A) = \#A = \text{number of points in } A,$$

and also its  $\sigma$ -algebra  $\mathcal{A}$  be the *power set* of  $\mathcal{X}$ , denoted  $\mathcal{A} = 2^{\mathcal{X}}$ .

2.  $\mu$  is *Lebesgue measure* on  $\mathcal{X}$  if  $\mathcal{X} = \mathbb{R}^n$ , and define

$$\mu(A) = \int_A \cdots \int dx_1 \cdots dx_n,$$

and for any set  $A$  in a  $\sigma$ -algebra  $\mathcal{A}$  called the *Borel sets* of  $\mathcal{X} = \mathbb{R}^n$ , and formally  $\mathcal{A}$  is the smallest  $\sigma$ -algebra that contains all open set in  $\mathbb{R}^n$  ("rectangles").

3.  $\mu$  is probability measure on  $\mathcal{X}$  if  $\mu(\mathcal{X}) = 1$ .

**Remark 1.1.4.** If  $\mathcal{A}$  is a  $\sigma$ -algebra of subsets of  $\mathcal{X}$ , the pair  $(\mathcal{X}, \mathcal{A})$  is called a *measurable space*, and if  $\mu$  is a measure on  $\mathcal{A}$ , the triple  $(\mathcal{X}, \mathcal{A}, \mu)$  is called a *measure space*.

**Proposition 1.1.5** (Continuity property of measures). *If measurable sets  $B_n$ ,  $n \geq 1$ , are increasing ( $B_1 \subset B_2 \subset \dots$ ), with  $B = \bigcup_{i=1}^{\infty} B_n$ , called the limit of the sequence, then*

$$\mu(B) = \lim_{n \rightarrow \infty} \mu(B_n).$$

*If a measure  $\mu$  is a probability measure, it also holds for decreasing sets ( $B_1 \supset B_2 \supset \dots$ ) and its limit  $B = \bigcap_{n=1}^{\infty} B_n$ .*

*Proof.* 생략 □

**Definition 1.1.6** ( $\sigma$ -finite). *There is some notations and definitions about the measure.*

1. A measure  $\mu$  is finite if  $\mu(\mathcal{X}) < \infty$ .
2. A measure  $\mu$  is  $\sigma$ -finite if  $\exists A_1, A_2, \dots \in \mathcal{A}$  with  $\mu(A_i) < \infty \forall i = 1, 2, \dots$  and  $\bigcup_{i=1}^{\infty} A_i = \mathcal{X}$ .

**Remark 1.1.7.** A probability measure is  $\sigma$ -finite.

## 1.2 Integration

**Example 1.2.1.** The integral of "nice" function  $f$  against a measure  $\mu$  is written as  $\int f d\mu$  or  $\int f(x) d\mu(x)$ .

1. If  $\mu$  is *counting measure* on  $\mathcal{X}$ , then the integral of  $f$  against  $\mu$  is

$$\int f d\mu = \sum_{x \in \mathcal{X}} f(x).$$

2. If  $\mu$  is *Lebesgue measure* on  $\mathbb{R}^n$ , then the integral of  $f$  against  $\mu$  is

$$\int f d\mu = \int \dots \int f(x_1, \dots, x_n) dx_1 \dots dx_n.$$

**Definition 1.2.2** (Measurable). *If  $(\mathcal{X}, \mathcal{A})$  is a measurable space and  $f$  is a real-valued function on  $\mathcal{X}$ , then  $f$  is measurable if*

$$f^{-1}(B) \stackrel{\text{def}}{=} \{x \in \mathcal{X} : f(x) \in B\} \in \mathcal{A},$$

for every Borel set  $B$ .

**Definition 1.2.3** (Indicator function). *The indicator function  $1_A$  of a set  $A$  is defined as*

$$1_A(x) = I\{x \in A\} = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

**Proposition 1.2.4.** *The integral of the indicator function  $1_A$  has following properties:*

1. *For any set  $A \in \mathcal{A}$ ,  $\int 1_A d\mu = \mu(A)$ .*
2. *If  $f$  and  $g$  are non-negative measurable functions, and if  $a$  and  $b$  are positive constants,*

$$\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu.$$

3. *If  $f_1 \leq f_2 \leq \dots$  are non-negative measurable functions, and if  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ , then*

$$\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

*Proof.* 생략 □

**Definition 1.2.5** (Simple function). *A function  $f$  is simple if*

$$f = \sum_{i=1}^m a_i 1_{A_i},$$

where  $a_1, \dots, a_m$  are positive constants, and  $A_1, \dots, A_m$  are sets in  $\mathcal{A}$

**Remark 1.2.6.** By using 1 and 2 in Proposition 1.2.4, the integral of simple function is obtained as

$$\int \left( \sum_{i=1}^m a_i 1_{A_i} \right) d\mu = \sum_{i=1}^m a_i \mu(A_i).$$

**Theorem 1.2.7.** *If  $f$  is non-negative and measurable, then there exist non-negative simple functions  $f_1 \leq f_2 \leq \dots$  with  $f = \lim_{n \rightarrow \infty} f_n$ .*

*Proof.* 생략 □

**Definition 1.2.8** (Integrable). *A measurable function  $f$  is integrable if  $\int |f| d\mu < \infty$ .*

### 1.3 Events, Probabilities, and Random Variables

**Definition 1.3.1** (Probability space). *If  $\mathcal{E}$  is a sample space,  $\mathcal{B}$  is a  $\sigma$ -algebra of subsets of  $\mathcal{X}$ , and  $P$  is a probability measure, then the triple  $(\mathcal{X}, \mathcal{A}, P)$  is called a probability space. Also  $B \in \mathcal{B}$  are called events, points  $e \in \mathcal{E}$  are called outcomes, and  $P(B)$  is called the probability of  $B$ .*

**Definition 1.3.2** (Random variable). *A measurable function  $X : \mathcal{E} \rightarrow \mathbb{R}$  is called a random variable.*

**Definition 1.3.3** (Distribution). *The probability measure  $P_X$  is defined by*

$$P_X(A) = P(\{e \in \mathcal{E} : X(e) \in A\}) \stackrel{\text{def}}{=} P(X \in A),$$

for Borel sets  $A$  is called distribution of  $X$ , and if we let  $P_X := Q$ , then it is denoted as  $X \sim Q$ , means that a random variable  $X$  has distribution  $Q$ .

**Definition 1.3.4** (Cumulative distribution function). *The cumulative distribution function of  $X$  is defined by*

$$F_X(x) = P(X \leq x) = P(\{e \in \mathcal{E} : X(e) \leq x\}) = P_X((-\infty, x]),$$

for  $x \in \mathbb{R}$ .

## 1.4 Null Sets

**Definition 1.4.1** (Null set). *For a measure  $\mu$  on the measurable space  $(\mathcal{X}, \mathcal{A})$ , a set  $N$  is called null with respect to  $\mu$  if*

$$\mu(N) = 0.$$

**Definition 1.4.2** (Almost everywhere). *If a statement holds for  $x \in \mathcal{X} - N$  with  $N$  null, the statement is said to hold almost everywhere (a.e.) or a.e.  $\mu$ .*

**Example 1.4.3.**  $f = 0$  a.e.  $\mu \iff \mu(\{x \in \mathcal{X} : f(x) \neq 0\}) = 0$ .

**Remark 1.4.4** (Almost sure). For the probability space, the statement holds a.e. if and only if  $\mu(B^c) = 0$  or  $\mu(B) = 1$ . Especially, it can be denoted that the statement holds *almost surely (a.s.)* or *with probability one*.

**Proposition 1.4.5.** *There are a few useful facts about integration:*

1. If  $f = 0$  (a.e.  $\mu$ ), then  $\int f d\mu = 0$ .
2. If  $f \geq 0$  and  $\int f d\mu = 0$ , then  $f = 0$  (a.e.  $\mu$ ).
3. If  $f = g$  (a.e.  $\mu$ ), then  $\int f d\mu = \int g d\mu$  for at least of the integral exists.
4. If  $\int 1_{(c,x)} f d\mu = 0$  for all  $x > c$ , then  $f(x) = 0$  for a.e.  $x > c$ . The constant  $c$  here can be  $-\infty$ .
5. If  $f$  and  $g$  are integrable and  $f > g$ , then  $\int f d\mu > \int g d\mu$ . (From 2)

*Proof.* 생략 □

## 1.5 Densities

**Definition 1.5.1** (Absolutely continuity). *Let  $P$  and  $\mu$  be measures on a  $\sigma$ -field  $\mathcal{A}$  of  $\mathcal{X}$ . Then  $P$  is called absolutely continuous with respect to  $\mu$ , written  $P \ll \mu$ , if  $P(A) = 0$  whenever  $\mu(A) = 0$ .*

**Theorem 1.5.2** (Radon-Nikodym). *If a finite measure  $P$  is absolutely continuous with respect to a  $\sigma$ -finite measure  $\mu$ , then there exists a non-negative measurable function  $f$  such that*

$$P(A) = \int_A f d\mu \stackrel{\text{def}}{=} \int f 1_A d\mu.$$

**Remark 1.5.3.** The function  $f$  in Theorem 1.5.2 is called Radon-Nikodym derivative of  $P$  with respect to  $\mu$ , or the *density* of  $P$  with respect to  $\mu$ , denoted

$$f = \frac{dP}{d\mu}.$$

**Remark 1.5.4.** If  $X \sim P_X$  and  $P_X$  is absolutely continuous with respect to  $\mu$  with density  $p = dP_X/d\mu$ , it is called that  $X$  has *density p with respect to  $\mu$* .

**Example 1.5.5** (Absolutely continuous random variables). If a random variable  $X$  has density  $p$  with respect to Lebesgue measure on  $\mathbb{R}$ , then  $X$  or its distribution  $P_X$  is called *absolutely continuous* with density  $p$ . Then, from the Radon-Nikodym theorem,

$$F_X(x) = P(X \leq x) = P_X((-\infty, x]) = \int_{-\infty}^x p(u)du$$

And also by fundamental theorem of calculus,  $p(x) = F'_X(x)$ .

**Example 1.5.6** (Discrete random variables). Let  $\mathcal{X}_0$  be a countable subset of  $\mathbb{R}$ . The measure  $\mu$  defined by

$$\mu(B) = \#(\mathcal{X}_0 \cap B)$$

for Borel sets  $B$  is (also) called *counting measure on  $\mathcal{X}_0$* . Then, the integration is defined as

$$\int f d\mu = \sum_{x \in \mathcal{X}_0} f(x).$$

If a random variable  $X$  satisfies

$$P(X \in \mathcal{X}_0) = P_X(\mathcal{X}_0) = 1,$$

then  $X$  is called a *discrete random variable*. And also  $P_X$  is absolutely continuous with respect to  $\mu$ . (See the textbook page 8.)

The density  $p$  of  $P_X$  with respect to  $\mu$  satisfies

$$P(X \in A) = P_X(A) = \int_A p d\mu = \sum_{x \in \mathcal{X}_0} p(x)1_A(x).$$

If  $A = \{y\}$  with  $y \in \mathcal{X}_0$ , then

$$P(X = y) = \sum_{x \in \mathcal{X}_0} p(x)1_{\{y\}}(x) = p(y).$$

This density  $p$  is called the *mass function* for  $X$ .

## 1.6 Expectation

**Definition 1.6.1** (Expectation). *If  $X$  is a random variable on a probability space  $(\mathcal{E}, \mathcal{B}, P)$ , then the expectation or expected value of  $X$  is defined as*

$$EX = \int X dP.$$

If  $X \sim P_X$ , then

$$EX = \int x dP_X(x).$$

Also, if  $Y = f(X)$ , then

$$EY = Ef(X) = \int f dP_X.$$

**Remark 1.6.2.** If  $P_X$  has density  $p$  with respect to  $\mu$ , then

$$\int f dP_X = \int f p d\mu.$$

**Example 1.6.3.** If  $X$  is an absolutely continuous random variable with density  $p$ , then

$$EX = \int x dP_X(x) = \int xp(x) dx$$

and

$$Ef(X) = \int f(x)p(x) dx.$$

**Example 1.6.4.** If  $X$  is a discrete with  $P(X \in \mathcal{X}_0) = 1$  for a countable set  $\mathcal{X}_0$ , if  $\mu$  is counting measure on  $\mathcal{X}_0$ , and if  $p$  is the mass function given by  $p(x) = P(X = x)$ , then

$$EX = \int x dP_X(x) = \int xp(x) d\mu(x) = \sum_{x \in \mathcal{X}_0} xp(x)$$

and

$$Ef(X) = \sum_{x \in \mathcal{X}_0} f(x)p(x).$$

**Proposition 1.6.5.** *If  $X$  and  $Y$  are random variables with finite expectations, the followings are satisfied.*

1. *For  $a$  and  $b$  are nonzero constants,*

$$E(aX + bY) = aEX + bEY.$$

2. *If  $X \leq y$  (a.e.  $P$ ), then  $EX \leq EY$ , and equality holds if  $X = Y$  (a.e.  $P$ ).*

*Proof.* 생략 □

**Definition 1.6.6** (Variance). *The variance of a random variable  $X$  with finite expectation is defined as*

$$\text{Var}(X) = E(X - EX)^2 = EX^2 - (EX)^2.$$

1. If  $X$  is absolutely continuous with density  $p$ ,

$$\text{Var}(X) = \int (x - EX)^2 p(x) dx.$$

2. If  $X$  is discrete with mass function  $p$ ,

$$\text{Var}(X) = \sum_{x \in \mathcal{X}_0} (x - EX)^2 p(x).$$

**Definition 1.6.7** (Covariance). *The covariance between two random variables  $X$  and  $Y$  with finite expectation is defined as*

$$\text{Cov}(X, Y) = E(X - EX)(Y - EY) = EXY - (EX)(EY).$$

**Definition 1.6.8** (Correlation). *The correlation between two random variables  $X$  and  $Y$  with finite expectation is defined as*

$$\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}},$$

and it always lie in  $[-1, 1]$ .

## 1.7 Random Vectors

**Definition 1.7.1** (Random vector). *If  $X_1, \dots, X_n$  are random variables, then the function  $X : \mathcal{E} \rightarrow \mathbb{R}^n$  defined by*

$$X(e) = \begin{pmatrix} X_1(e) \\ \vdots \\ X_n(e) \end{pmatrix}, \quad e \in \mathcal{E},$$

*is called a random vector.*

**Remark 1.7.2.** Similarly with the random variables, the notations of random vectors is directly extended by follows:

1. The distribution of  $P_X$  of  $X$  is defined by

$$P_X(B) = P(X \in B) \stackrel{\text{def}}{=} P(\{e \in \mathcal{E} : X(e) \in B\})$$

for Borel sets  $B \in \mathbb{R}^n$ , and denote  $X \sim P_X$ .

2. The random vector  $X$  or its distribution  $P_X$  is called *absolutely continuous* with density  $p$  if  $P_X$  is absolutely continuous with respect to Lebesgue measure on  $\mathbb{R}^n$ . And the probability is defined as

$$P(X \in B) = \int \cdots \int_B p(x) dx.$$

3. The random vector  $X$  is *discrete* if  $P(X \in \mathcal{X}_0) = 1$  for some countable set  $\mathcal{X}_0 \subset \mathbb{R}^n$ . If  $p(x) = P(X = x)$ , then  $P_X$  has density  $p$  with respect to counting measure on  $\mathcal{X}_0$  and

$$P(X \in B) = \sum_{x \in \mathcal{X} \cap B} p(x).$$

4. The expectation of random vector  $X$  is defined as

$$EX = \begin{pmatrix} EX_1 \\ \vdots \\ EX_n \end{pmatrix}.$$

If  $T : \mathbb{R}^n \rightarrow \mathbb{R}$  is a measurable function, then  $T(X)$  is a random variable, and with finite integral,

$$ET(X) = \int T dP_X.$$

If  $P_X$  has a density  $p$  with respect to a dominating measure  $\mu$ , this integral can be expressed as  $\int T p d\mu$ , which becomes

$$\sum_{x \in \mathcal{X}_0} T(x)p(x) \text{ or } \int \cdots \int T(x)p(x)dx,$$

where discrete and absolutely continuous cases with  $\mu$  counting or Lebesgue measure, respectively.

## 1.8 Covariance Matrices

**Definition 1.8.1** (Random matrix). *A matrix  $W$  is called a random matrix if the entires  $W_{ij}$  are random variables.*

**Definition 1.8.2** (Expectation of random matrix). *If  $W$  is a random matrix, then  $EW$  is the matrix of expectations of the entries,*

$$(EW)_{ij} = EW_{ij}.$$

**Definition 1.8.3** (Covariance matrix). *The covariance of a random vector  $X$  is the matrix of covariances of the variables in  $X$ ; that is,*

$$[\text{Cov}(X)]_{ij} = \text{Cov}(X_i, X_j).$$

*If  $\mu = EX$  and  $(X - \mu)'$  denotes the transpose of  $X - \mu$ , a (random) row vector, then*

$$\text{Cov}(X_i, X_j) = E(X_i - \mu_i)(X_j - \mu_j) = E[(X - \mu)(X - \mu)']_{ij},$$

*and so*

$$\text{Cov}(X) = E(X - \mu)(X - \mu)' = EXX' - \mu\mu'.$$

**Remark 1.8.4.** If  $v$  is a constant vector,  $A$ ,  $B$ , and  $C$  are constant matrices,  $X$  is a random vector, and  $W$  is a random matrix, then

1.  $E[v + AX] = v + AE(X)$ .
2.  $E[A + BWC] = A + B(EW)C$ .
3.  $\text{Cov}(v + AX) = AC\text{Cov}(X)A'$ .

*Proof.* By using the definitions and the simple matrix algebra, these are trivial.  $\square$

## 1.9 Product Measures and Independence

**Definition 1.9.1** (Product measure). *Let  $(\mathcal{X}, \mathcal{A}, \mu)$  and  $(\mathcal{Y}, \mathcal{B}, \nu)$  are measure spaces. Then there exists a unique measure  $\mu \times \nu$ , called the product measure, on  $(\mathcal{X} \times \mathcal{Y}, \mathcal{A} \vee \mathcal{B})$  such that*

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B),$$

for all  $A \in \mathcal{A}$  and all  $B \in \mathcal{B}$ . The  $\sigma$ -algebra  $\mathcal{A} \vee \mathcal{B}$  is defined formally as the smallest  $\sigma$ -algebra containing all sets  $A \times B$  with  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ .

**Example 1.9.2.** These are simple examples of product measure.

1. If  $\mu$  and  $\nu$  are Lebesgue measures on  $\mathbb{R}^n$  and  $\mathbb{R}^m$ , respectively, then  $\mu \times \nu$  is Lebesgue measure on  $\mathbb{R}^{n+m}$ .
2. If  $\mu$  and  $\nu$  are counting measures on sets  $\mathcal{X}_0$  and  $\mathcal{Y}_0$ , then  $\mu \times \nu$  is counting measure on  $\mathcal{X}_0 \times \mathcal{Y}_0$ .

**Theorem 1.9.3** (Fubini). *If  $f \geq 0$  or  $f$  is integrable which means  $\int |f|d(\mu \times \nu) < \infty$ , then*

$$\begin{aligned} \int f d(\mu \times \nu) &= \int \left[ \int f(x, y) d\nu(y) \right] d\mu(x) \\ &= \int \left[ \int f(x, y) d\mu(x) \right] d\nu(y). \end{aligned}$$

*Proof.* See Royden p.418(Statement) and p.421(Proof), or Durrett p.34 Theorem 1.7.2.  $\square$

**Definition 1.9.4** (Independence). *Two random vectors,  $X \in \mathbb{R}^n$  and  $Y \in \mathbb{R}^m$  are independent if*

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B),$$

for all Borel sets  $A$  and  $B$ .