

Theoretical Statistics: Topics for a Core Course

Problem Solutions

Robert W. Keener

HYUNSUNG KIM
Department of Statistics
Chung-Ang University

Update: September 7, 2021

Contents

Preface	2
1 Probability and Measure	4
2 Exponential Families	20
3 Risk, Sufficiency, Completeness, and Ancillarity	32

Preface

This note contains the solution of the problems in the textbook, *Theoretical Statistics: Topics for a Core Course*, and it was created by Hyunsung Kim, who is a Ph.D. student. I wrote it when I study a theoretical statistics based on this textbook on my own by solving some problems and also referred to the solution manual in the textbook.

It contains a few selected problems in the textbook what I studied, and also note that it may not be the exact solutions. If you want to refer to this note, you should study with doubt about the answer.

Textbook

- Keener, *Theoretical Statistics: Topics for a Core Course*.

Reference

- Durrett, *Probability: Theory and Examples, 5th edition*.
- Royden, *Real Analysis, 4th edition*.

Chapter 1

Probability and Measure

Problem 1.1

Prove (1.1). If measurable sets B_n , $n \geq 1$, are increasing, with $B = \bigcup_{n=1}^{\infty} B_n$, called the limit of the sequence, then

$$\mu(B) = \lim_{n \rightarrow \infty} \mu(B_n).$$

Solution.

First, we showed that A_n 's are disjoint. If $j < k$, then $B_j \subseteq B_{k-1}$.

Since $A_j \subset B_j \subseteq B_{k-1}$ and $A_k \subset B_{k-1}^c$, A_j and A_k are disjoint.

Also $B_n = \bigcup_{j=1}^n A_j$ and $\bigcup_{n=1}^{\infty} A_n = B$,

$$\mu(B) = \sum_{i=1}^{\infty} \mu(A_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(A_i) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{i=1}^n A_i\right) = \lim_{n \rightarrow \infty} \mu(B_n).$$

□

Problem 1.8

Prove *Boole's inequality*: For any events B_1, B_2, \dots ,

$$P\left(\bigcup_{i \geq 1} B_i\right) \leq \sum_{i \geq 1} P(B_i).$$

Solution.

Let $B = \bigcup_{i=1}^{\infty} B_i$, then $1_B \leq \sum 1_{B_i}$. Then by Fubini's theorem,

$$P(B) = \int 1_B dP \leq \int \sum 1_{B_i} dP = \sum \int 1_{B_i} dP = \sum P(B_i).$$

□

Problem 1.10

Let μ and ν be measures on $(\mathcal{E}, \mathcal{B})$.

- a) Show that the sum η defined by $\eta(B) = \mu(B) + \nu(B)$ is also a measure.

Solution.

We should check following 2 conditions.

- (i) For arbitrary set $A \in \mathcal{B}$, $\mu(A) \geq 0$ and $\nu(A) \geq 0 \Rightarrow \eta(A) = \mu(A) + \nu(A) \geq 0$.
 $\therefore \eta : \mathcal{B} \rightarrow [0, \infty]$.

- (ii) For disjoint set $B_1, B_2, \dots \subset \mathcal{B}$, then

$$\begin{aligned}\eta\left(\bigcup_{i=1}^{\infty} B_i\right) &= \mu\left(\bigcup B_i\right) + \nu\left(\bigcup B_i\right) \\ &= \sum \mu(B_i) + \sum \nu(B_i) \\ &= \sum \{\mu(B_i) + \nu(B_i)\} \\ &= \sum \eta(B_i)\end{aligned}$$

$\therefore \eta$ is a measure. □

- b) If f is a non-negative measurable function, show that

$$\int f d\eta = \int f d\mu + \int f d\nu.$$

Solution.

We show it by 2 stage.

- (i) Let $f = \sum_{i=1}^n a_i 1_{A_i}$, the non-negative simple function. Then,

$$\begin{aligned}\int f d\eta &= \int \sum_{i=1}^n a_i 1_{A_i} d\eta = \sum a_i \eta(A_i) \\ &= \sum a_i \{\mu(A_i) + \nu(A_i)\} = \int f d\mu + \int f d\nu.\end{aligned}$$

- (ii) For general case, let f_n is the sequence of non-negative simple functions increasing to f . (i.e. $f_1 \leq f_2 \leq \dots \leq f$)

$$\begin{aligned}\int f d\eta &= \lim_{n \rightarrow \infty} \int f_n d\eta = \lim_{n \rightarrow \infty} \left(\int f_n d\mu + \int f_n d\nu \right) \\ &= \lim_{n \rightarrow \infty} \int f_n d\mu + \lim_{n \rightarrow \infty} \int f_n d\nu = \int f d\mu + \int f d\nu.\end{aligned}$$

□

Problem 1.11

Suppose f is the simple function $1_{(1/2, \pi]} + 21_{(1, 2]}$, and let μ be a measure on \mathbb{R} with $\mu\{(0, a^2]\} = a$, $a > 0$. Evaluate $\int f d\mu$.

Solution.

By the integral of simple function and the finite additivity, it can be simply computed as

$$\begin{aligned}\int f d\mu &= \mu\{(1/2, \pi]\} + 2\mu\{(1, 2]\} \\ &= [\mu\{(0, \pi]\} - \mu\{(1/2, \pi]\}] + 2[\mu\{(0, 2]\} - \mu\{(1, 2]\}] \quad (\text{finite additivity}) \\ &= (\sqrt{\pi} - 1/\sqrt{2}) + 2(\sqrt{2} - 1)\end{aligned}$$

□

Problem 1.12

Suppose that $\mu\{(0, a)\} = a^2$ for $a > 0$ and that f is defined by

$$f(x) = \begin{cases} 0, & x \leq 0, \\ 1, & 0 < x < 2, \\ \pi, & 2 \geq x < 5, \\ 0, & x \geq 5. \end{cases}$$

Compute $\int f d\mu$.

Solution.

Since $f = 1_{(0,2)} + 1_{[2,5)}$ is simple, the integral can be computed as

$$\begin{aligned}\int f d\mu &= \mu\{(0, 2)\} + \pi\mu\{[2, 5)\} \\ &= \mu\{(0, 2)\} + \pi[\mu\{(0, 5)\} - \mu\{(0, 2)\}] \\ &= 4 - \pi(25 - 4)\end{aligned}$$

□

Problem 1.13

Define the function f by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find simple functions $f_1 \leq f_2 \leq \dots$ increasing to f (i.e. $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for all $x \in \mathbb{R}$). Let μ be Lebesgue measure on \mathbb{R} . Using our formal definition of an integral and the fact that $\mu((a, b]) = b - a$ whenever $b > a$ (this might be used to formally define Lebesgue measure), show that $\int f d\mu = 1/2$.

Solution.

Let $f_n = \lfloor 2^n x \rfloor / 2^n$ for $0 < x \leq 1$ and 0 otherwise. ($\lfloor y \rfloor$ is a floor function.) Then,

$$f_1(x) = \lfloor 2x \rfloor / 2 = \begin{cases} 0, & x < 1/2, \\ 1/2, & 1/2 \leq x < 1, \\ 1, & x = 1 \end{cases}$$

$$f_2(x) = \lfloor 2^2 x \rfloor / 2^2 = \begin{cases} 0, & x < 1/2^2, \\ 1/2^2, & 1/2^2 \leq x < 2/2^2, \\ 2/2^2, & 2/2^2 \leq x < 3/2^2, \\ 3/2^2, & 3/2^2 \leq x < 1, \\ 1, & x = 1 \end{cases}$$

⋮

$$f_n(x) = \lfloor 2^n x \rfloor / 2^n = \begin{cases} 0, & x < 1/2^n, \\ 1/2^n, & 1/2^n \leq x < 2/2^n, \\ \vdots & \\ (2^n - 1)/2^n, & (2^n - 1)/2^n \leq x < 1, \\ 1, & x = 1 \end{cases}$$

$$\begin{aligned} \therefore \int f_n d\mu &= \frac{1}{2^n} \left(\frac{1}{2^n} + \frac{2}{2^n} + \cdots + \frac{2^n - 1}{2^n} \right) \\ &= \frac{1 + 2 + \cdots + (2^n - 1)}{4^n} \\ &= \frac{2^n(2^n - 1)}{2 \cdot 4^n} \\ &\longrightarrow \frac{1}{2}. \end{aligned}$$

My solution

Let the simple function $f_n = \sum_{i=1}^n \frac{i}{n} 1_{(\frac{i-1}{n}, \frac{i}{n}]}$. Then,

$$\int f_n d\mu = \sum_{i=1}^n \frac{i}{n} \frac{1}{n} = \frac{1}{n^2} \frac{n(n+1)}{2} \longrightarrow \frac{1}{2}.$$

□

Problem 1.16

Define $F(a-) = \lim_{x \uparrow a} F(x)$. Then, if F is non-decreasing, $F(a-) = \lim_{n \rightarrow \infty} F(a - 1/n)$. Use (1.1)[Continuity of measure] to show that if a random variable X has cumulative distribution function F_X ,

$$P(X < a) = F_X(a-).$$

Also, show that

$$P(X = a) = F_X(a) - F_X(a-).$$

Solution.

(i) Let $B_n = \{X \leq a - 1/n\}$ and $\bigcup_{n=1}^{\infty} B_n = \{X < a\}$. By continuity of measure,

$$P(B) = P(X < a) = \lim_{n \rightarrow \infty} P(X \leq a - 1/n) = \lim_{n \rightarrow \infty} F_X(a - 1/n) = F_X(a-).$$

(ii) Since $\{X < a\}$ and $\{X = a\}$ are disjoint with union $\{X \leq a\}$,

$$P(X < a) + P(X = a) = P(X \leq a).$$

$$\therefore P(X = a) = F_X(a) - F_X(a-).$$

□

Problem 1.17

Suppose X is a geometric random variable with mass function

$$p(x) = P(X = x) = \theta(1 - \theta)^x, \quad x = 0, 1, \dots,$$

where $\theta \in (0, 1)$ is a constant. Find the probability that X is even.

Solution.

$$\begin{aligned} P(X \text{ is even}) &= P(X = 0) + P(X = 2) + P(X = 4) \dots \\ &= \theta + \theta(1 - \theta)^2 + \theta(1 - \theta)^4 + \dots \\ &= \frac{\theta}{1 - (1 - \theta)^2} \\ &= \frac{1}{2 - \theta} \end{aligned}$$

□

Problem 1.18

Let X be a function mapping \mathcal{E} into \mathbb{R} . Recall that if B is a subset of \mathbb{R} , then $X^{-1}(B) = \{e \in \mathcal{E} : X(e) \in B\}$. Use this definition to prove that

$$X^{-1}(A \cap B) = X^{-1}(A) \cap X^{-1}(B),$$

$$X^{-1}(A \cup B) = X^{-1}(A) \cup X^{-1}(B),$$

and

$$X^{-1}\left(\bigcup_{i=0}^{\infty} A_i\right) = \bigcup_{i=0}^{\infty} X^{-1}(A_i).$$

Solution.

(i)

$$\begin{aligned} e \in X^{-1}(A \cap B) &\Leftrightarrow X(e) \in A \cap B \\ &\Leftrightarrow X(e) \in A \text{ and } X(e) \in B \\ &\Leftrightarrow e \in X^{-1}(A) \text{ and } e \in X^{-1}(B) \\ &\Leftrightarrow e \in X^{-1}(A) \cap X^{-1}(B). \end{aligned}$$

(ii) Similarly,

$$\begin{aligned} e \in X^{-1}(A \cup B) &\Leftrightarrow X(e) \in A \cup B \\ &\Leftrightarrow X(e) \in A \text{ or } X(e) \in B \\ &\Leftrightarrow e \in X^{-1}(A) \text{ or } e \in X^{-1}(B) \\ &\Leftrightarrow e \in X^{-1}(A) \cup X^{-1}(B). \end{aligned}$$

(iii)

$$\begin{aligned} e \in X^{-1}\left(\bigcup_{i=0}^{\infty} A_i\right) &\Leftrightarrow X(e) \in \bigcup_{i=0}^{\infty} A_i \\ &\Leftrightarrow X(e) \in A_i \text{ for some } i \\ &\Leftrightarrow e \in X^{-1}(A_i) \text{ for some } i \\ &\Leftrightarrow e \in \bigcup_{i=0}^{\infty} X^{-1}(A_i). \end{aligned}$$

□

Problem 1.19

Let P be a probability measure on $(\mathcal{E}, \mathcal{B})$, and let X be a random variable. Show that the distribution P_X of X defined by $P_X(B) = P(X \in B) = P(X^{-1}(B))$ is a measure (on the Borel sets of \mathbb{R}).

Solution.

To show that P_X is a measure, we should check 2 conditions.

(i) $P_X(B) = P(X^{-1}(B)) \geq 0$.

(ii) From Problem 1.18, $X^{-1}(\bigcup_i A_i) = \bigcup_i X^{-1}(A_i)$ is hold.

Also, if B_i and B_j are disjoint, then $X^{-1}(B_i)$ and $X^{-1}(B_j)$ are disjoint. (\because If $e \in \mathcal{E}$ lie in both $X^{-1}(B_1)$ and $X^{-1}(B_2)$, $X(e)$ lies in B_1 and B_2 .)

Therefore, for arbitrary disjoint set B_1, B_2, \dots with union $B = \bigcup_i B_i$, $X^{-1}(B_1), X^{-1}(B_2), \dots$ are also disjoint with union $X^{-1}(B)$.

$$\therefore \sum_i P_X(B_i) = \sum_i P(X^{-1}(B_i)) = P(X^{-1}(B)) = P_X(B).$$

The above 2 conditions are hold, P_X is a measure. □

Problem 1.21

Let X have a uniform distribution on $(0, 1)$; that is, X is absolutely continuous with density p defined by

$$p(x) = \begin{cases} 1, & x \in (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Let Y_1 and Y_2 denote the first two digits of X when X is written as a binary decimal (so $Y_1 = 0$ if $X \in (0, 1/2)$ for instance). Find $P(Y_1 = i, Y_2 = j)$, $i = 0$ or 1 , $j = 0$ or 1 .

Solution.

For all cases, the probability are same. In other words,

$$P(Y_1 = 0, Y_2 = 0) = P(Y_1 = 1, Y_2 = 0) = P(Y_1 = 0, Y_2 = 1) = P(Y_1 = 1, Y_2 = 1) = 1/4.$$

□

Problem 1.22

Let $\mathcal{E} = (0, 1)$, let \mathcal{B} be the Borel subsets of \mathcal{E} , and let $P(A)$ be the length of A for $A \in \mathcal{B}$. (P would be called the *uniform probability measure* on $(0, 1)$.) Define the random variable X by

$$X(e) = \min\{e, 1/2\}.$$

Let μ be the sum of Lebesgue measure on \mathbb{R} and counting measure on $\mathcal{X}_0 = \{1/2\}$. Show that the distribution P_X of X is absolutely continuous with respect to μ and find the density of P_X .

Solution.

(i) Claim: For any $B \in \mathbb{R}$, $\mu(B) = 0 \implies P_X(B) = 0$.

$$\begin{aligned} \{X \in B\} &= \{y \in (0, 1) : X(y) \in B\} \\ &= \begin{cases} B \cap (0, 1/2), & 1/2 \notin B, \\ (B \cap (0, 1/2)) \cup [1/2, 1), & 1/2 \in B. \end{cases} \end{aligned}$$

Let λ be a Lebesgue measure and ν be a counting measure on $\{1/2\}$. Then,

$$\begin{aligned} \mu(B) = 0 &\iff \lambda(B) = 0 \text{ and } 1/2 \notin B. (\mu(B) = \nu(B) = 0) \\ \implies P_X(B) &= P(X \in B) = P(B \cap (0, 1/2)) = \lambda(B \cap (0, 1/2)) = 0. \\ \therefore P_X &\text{ is absolutely continuous w.r.t. } \lambda + \nu \stackrel{\text{def}}{=} \mu. \end{aligned}$$

(ii) Find the density of P_X .

From the above equation,

$$P(X \in B) = \lambda(B \cap (0, 1/2)) + \frac{1}{2}1_B(1/2). \quad (1.1)$$

If f is the density, then (1.1) should be equal to $\int_B f d(\lambda + \nu)$.

$$\begin{aligned} \int_B f d(\lambda + \nu) &= \int f 1_B d(\lambda + \nu) \\ &= \int \left(1_{B \cap (0, 1/2)} + \frac{1}{2}1_{\{1/2\} \cap B} \right) d(\lambda + \nu) \\ &= \lambda(B \cap (0, 1/2)) + \nu(B \cap (0, 1/2)) + \frac{1}{2}\lambda(\{1/2\} \cap B) + \frac{1}{2}\nu(\{1/2\} \cap B) \\ &= \lambda(B \cap (0, 1/2)) + \frac{1}{2}1_B(1/2). \\ \therefore f &= 1_{(0, 1/2)} + \frac{1}{2}1_{\{1/2\}}. \end{aligned}$$

□

Problem 1.23

The standard normal distribution $N(0, 1)$ has density ϕ given by

$$\phi(x) = \frac{e^{-x^2/2}}{\sqrt{w\pi}}, \quad x \in \mathbb{R},$$

with respect to Lebesgue measure λ on \mathbb{R} . The corresponding cumulative distribution function is Φ , so

$$\Phi(x) = \int_{-\infty}^x \phi(z) dz$$

for $x \in \mathbb{R}$. Suppose that $X \sim N(0, 1)$ and that the random variable Y equals X when $|X| < 1$ and is 0 otherwise. Let P_Y denote the distribution of Y and let μ be counting measure on $\{0\}$. Find the density of P_Y with respect to $\lambda + \mu$.

Solution.

Let $g(x) = x1_{|x|<1}$, then $Y = g(X)$.

For any integrable function f , the expectation of $f(Y)$ against the density of X is

$$Ef(Y) = Ef(g(X)) = \int f(g(x))\phi(x)dx = \int f(g(x))\phi(x)1_{(-1,1)}(x)dx + cf(0), \quad (1.2)$$

where $c = \int \phi(x)1_{|x|>1}(x)dx = 2\Phi(-1)$.

Similarly, the expectation of $f(Y)$ against the density p of Y is

$$Ef(Y) = \int fpd(\lambda + \mu) = \int fpd\lambda + \int fpd\mu = \int f(x)p(x)dx + f(0)p(0). \quad (1.3)$$

Then, (1.2) and (1.3) should be same. Therefore,

$$\begin{aligned} f = 1_{\{0\}} &\implies p(0) = 2\Phi(-1), \\ f(0) = 0 &\implies \int f(x)\phi(x)1_{(-1,1)}(x)dx = \int f(x)p(x)dx \\ &\implies p(x) = \phi(x) \text{ for } 0 < |x| < 1. \end{aligned}$$

$\therefore p(x) = 2\Phi(-1)1_{\{0\}}(x) + \phi(x)1_{(0,1)}(|x|)$ is the density of P_Y .

□

Problem 1.24

Let μ be a σ -finite measure on a measurable space (X, \mathcal{B}) . Show that μ is absolutely continuous with respect to some probability measure P .

Hint: You can use the fact that if μ_1, μ_2, \dots are probability measures and c_1, c_2, \dots are non-negative constants, then $\sum c_i \mu_i$ is a measure. (The proof for Problem 1.10 extends easily to this case.) The measure μ_i , you will want to consider, are truncations of μ to sets A_i covering X with $\mu(A_i) < \infty$, given by $\mu_i(B) = \mu(B \cap A_i)$. With the constants c_i chosen properly, $\sum c_i \mu_i$ will be a probability measure.

Solution.

(i) μ is finite \implies Trivial.

\therefore Since $\mu(X) < \infty$, $P = \mu/\mu(X)$ be a probability measure. Therefore, $P(N) = 0$ implies $\mu(N) = 0$.

(ii) μ is infinite but σ -finite.

$$\implies \exists A_1, A_2, \dots \in \mathcal{B} \text{ s.t. } \bigcup A_i = X \text{ and } 0 < \mu(A_i) < \infty \forall i.$$

From Hint, $\mu_i(B) = \mu(B \cap A_i)$ and also $\mu_i(X) = \mu(A_i) \implies \mu_i$ is a finite measure.

Let $b_i = 1/2^i$ (or any other seq of positive constants s.t. $\sum b_i = 1$) and define $c_i = b_i/\mu(A_i)$.

Then $P = \sum_i c_i \mu_i$ is a probability measure, since $P(X) = \sum_i c_i \mu_i(X) = \sum_i \left(\frac{b_i}{\mu(A_i)}\right) \mu(A_i) = 1$.

Suppose $P(N) = \sum_i c_i \mu_i(N) = 0$. Then $\mu_i(N) = \mu(N \cap A_i) = 0 \forall i$.

By Boole's inequality,

$$\mu(N) = \mu\left(\bigcup_i (N \cap A_i)\right) \leq \sum_i \mu(N \cap A_i) = 0.$$

For any null set for P is a null set for μ .

$$\therefore \mu \ll P.$$

□

Problem 1.25

The monotone convergence theorem states that if $0 \leq f_1 \leq f_2 \leq \dots$ are measurable functions and $f = \lim_{n \rightarrow \infty} f_n$, then $\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$. Use this result to prove the following assertions.

- a) Show that if $X \sim P_X$ is a random variable on $(\mathcal{E}, \mathcal{B}, P)$ and f is a non-negative measurable function, then

$$\int f(X(e)) dP(e) = \int f(x) dP_X(x).$$

Hint: Try it first with f an indicator function. For the general case, let f_n be a sequence of simple functions increasing to f .

Solution.

We will show the equation by 3 steps.

- (i) Suppose $f = 1_A$ (indicator function).

Then, $f(X(e)) = 1$ for $X(e) \in A$.

$\Rightarrow f \circ X = 1_B$ where $B = \{e : X(e) \in A\}$.

By definition, $P_X(A) = P(B)$.

$$\int f(X(e)) dP(e) = \int 1_B(e) dP(e) = P(B),$$

$$\int f(x) dP_X(x) = \int 1_A(x) dP_X(x) = P_X(A).$$

\therefore The statement is holds.

- (ii) Suppose $f = \sum_{i=1}^n c_i 1_{A_i}$ (simple function).

$$\begin{aligned} \therefore \int f(X(e)) dP(e) &= \int \sum_i c_i 1_{A_i}(X(e)) dP(e) \\ &= \sum_i c_i \int 1_{A_i}(X(e)) dP(e) \\ &= \sum_i c_i \int 1_{A_i}(x) dP_X(x) \quad (\text{by (i)}) \end{aligned}$$

$$\begin{aligned}
&= \int \sum_i c_i 1_{A_i}(x) dP_X(x) \\
&= \int f(x) dP_X(x).
\end{aligned}$$

- (iii) Suppose f is a non-negative measurable function (general case), and f_n is non-negative simple functions increasing to f .

$$\Rightarrow f_n \circ X \nearrow f \circ X.$$

$$\begin{aligned}
\therefore \int f(X(e)) dP(e) &= \lim_{n \rightarrow \infty} \int f_n(X(e)) dP(e) \quad (\text{by M.C.T.}) \\
&= \lim_{n \rightarrow \infty} \int f_n(x) dP_X(x) \quad (\text{by (ii)}) \\
&= \int f(x) dP_X(x) \quad (\text{by M.C.T.})
\end{aligned}$$

□

- b) Suppose that P_X has density p with respect to μ , and let f be a non-negative measurable function.

Show that

$$\int f dP_X = \int f p d\mu.$$

Solution.

It can be also showed by 3 steps.

- (i) Let $f = 1_A$ (indicator function).

$$\int f dP_X = \int 1_A dP_X = P_X(A) = \int_A p d\mu = \int 1_A p d\mu = \int f p d\mu.$$

- (ii) Suppose $f = \sum_{i=1}^n c_i 1_{A_i}$ (simple function).

$$\int f dP_X = \int \sum_i c_i 1_{A_i} dP_X = \sum_i c_i \int 1_{A_i} dP_X \stackrel{(i)}{\doteq} \sum_i c_i \int 1_{A_i} p d\mu = \int \sum_i c_i 1_{A_i} p d\mu = \int f p d\mu.$$

- (iii) For general case, let a non-negative measurable function f and non-negative simple functions f_n s.t. $f_n \nearrow f$. Then by M.C.T.,

$$\int f dP_X \stackrel{\text{M.C.T.}}{\doteq} \lim_{n \rightarrow \infty} \int f_n dP_X \stackrel{(ii)}{\doteq} \lim_{n \rightarrow \infty} \int f_n p d\mu \stackrel{\text{M.C.T.}}{\doteq} \int f p d\mu.$$

□

Problem 1.26

The gamma distribution.

- a) The gamma function is defined for $\alpha > 0$ by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$. Show that $\Gamma(X+1) = x!$ for $x = 0, 1, \dots$

Solution.

By integration by parts,

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = -t^x e^{-t} \Big|_0^\infty + (\alpha-1) \int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x).$$

Since $\Gamma(1) = \int_0^\infty e^{-x} dx = 1$,

$$\Gamma(x+1) = x\Gamma(x) = x(x-1)\Gamma(x-1) = \cdots = x(x-1)\cdots 2 \cdot 1 \cdot \Gamma(1) = x!.$$

□

b) Show that the function

$$p(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

is a (Lebesgue) probability density when $\alpha > 0$ and $\beta > 0$. This density is called the gamma density with parameters α and β . The corresponding probability distribution is denoted $\Gamma(\alpha, \beta)$.

Solution.

Using change of variable, $y = x/\beta \Rightarrow dx = \beta dy$. Therefore,

$$\begin{aligned} \int p(x)dx &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx \\ &= \int \frac{1}{\Gamma(\alpha)\beta^\alpha} (\beta y)^{\alpha-1} e^{-y} \beta dy \\ &= \frac{1}{\Gamma(\alpha)} \int y^{\alpha-1} e^{-y} dy \\ &= 1. \end{aligned}$$

□

c) Show that if $X \sim \Gamma(\alpha, \beta)$, then $EX^r = \beta^r \Gamma(\alpha+r)/\Gamma(\alpha)$. Use this formula to find the mean and variance of X .

Solution.

$$\begin{aligned} EX^r &= \int_0^\infty x^r p(x)dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha+r-1} e^{-x/\beta} dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha+r)\beta^{\alpha+r}} x^{\alpha+r-1} e^{-x/\beta} dx \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \beta^r \\ &= \beta^r \Gamma(\alpha+r)/\Gamma(\alpha). \end{aligned}$$

By using this,

$$EX = \beta \Gamma(\alpha+1)/\Gamma(\alpha) = \alpha\beta,$$

$$EX^2 = \beta^2 \Gamma(\alpha+2)/\Gamma(\alpha) = (\alpha+1)\alpha\beta^2,$$

$$Var(X) = EX^2 - \{EX\}^2 = \alpha\beta^2.$$

□

Problem 1.27

Suppose X has a uniform distribution on $(0, 1)$. Find the mean and covariance matrix of the random vector $\begin{pmatrix} X \\ X^2 \end{pmatrix}$.

Solution.

Since $X \sim \text{Uniform}(0, 1)$, the density of X is $p(x) = 1$ for $x \in (0, 1)$. Thus,

$$EX^r = \int x^r dx = \frac{1}{r+1}.$$

Therefore,

$$\begin{aligned} E\left(\begin{pmatrix} X \\ X^2 \end{pmatrix}\right) &= \begin{pmatrix} EX \\ EX^2 \end{pmatrix}, \\ Cov\left(\begin{pmatrix} X \\ X^2 \end{pmatrix}\right) &= \begin{pmatrix} Var(X) & Cov(X, X^2) \\ Cov(X^2, X) & Var(X^2) \end{pmatrix} = \begin{pmatrix} EX^2 - \{EX\}^2 & EX^3 - EXEX^2 \\ EX^3 - EXEX^2 & EX^4 - \{EX^2\}^2 \end{pmatrix}. \end{aligned}$$

□

Problem 1.28

If $X \sim N(0, 1)$, find the mean and covariance matrix of the random vector $\begin{pmatrix} X \\ I\{X>c\} \end{pmatrix}$.

Solution.

Let $\Phi(x)$ is a cumulative distribution function of X , and ϕ is a density of X . Then,

$$\begin{aligned} E\left(\begin{pmatrix} X \\ I\{X>c\} \end{pmatrix}\right) &= \begin{pmatrix} EX \\ EI\{X>c\} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 - \Phi(c) \end{pmatrix}, \\ Cov\left(\begin{pmatrix} X \\ I\{X>c\} \end{pmatrix}\right) &= \begin{pmatrix} Var(X) & EX1_{X>c} - EXE1_{X>c} \\ EX1_{X>c} - EXE1_{X>c} & E1_{X>c}^2 - \{E1_{X>c}\}^2 \end{pmatrix} = \begin{pmatrix} 1 & \phi(c) \\ \phi(c) & \Phi(c)(1 - \Phi(c)) \end{pmatrix}. \\ \because EX1_{X>c} &= \int_c^\infty \frac{1}{\sqrt{2\pi}} xe^{-x^2/2} dx = \phi(c) \text{ and } 1_{X>c}^2 = 1_{X>c}. \end{aligned}$$

□

Problem 1.32

Suppose $E|X| < \infty$ and let

$$h(t) = \frac{1 - E \cos(tX)}{t^2}.$$

Use Fubini's theorem to find $\int_0^\infty h(t)dt$. Hint:

$$\int_0^\infty (1 - \cos(u)) u^{-2} du = \frac{\pi}{2}.$$

Solution.

By Funini's theorem,

$$\int_0^\infty h(t)dt = \int_0^\infty \left(\frac{1 - E \cos(tX)}{t^2} \right) dt$$

$$\begin{aligned}
&= \int_0^\infty E \left(\frac{1 - \cos(tX)}{t^2} \right) dt \\
&= \int_0^\infty \int \frac{1 - \cos(tx)}{t^2} dP_X(x) dt \\
&= \int \int_0^\infty \frac{1 - \cos(tx)}{t^2} dt dP_X(x) \quad (\text{by Funini's theorem}) \\
&\quad u = tx \Rightarrow du = xdt, t = x/u \\
&= \int \int_0^\infty |x| \frac{1 - \cos(u)}{u^2} du dP_X(x) \\
&= \int \frac{\pi}{2} |x| dP_X(x) \\
&= \frac{\pi}{2} E|X|.
\end{aligned}$$

□

Problem 1.33

Suppose X is absolutely continuous with density $p_X(x) = xe^{-x}$, $x > 0$ and $p_X(x) = 0$, $x \leq 0$. Define $c_n = E(1 + X)^{-n}$. Use Funini's theorem to evaluate $\sum_{n=1}^\infty c_n$.

Solution.

By Funini's theorem,

$$\begin{aligned}
\sum_{n=1}^\infty c_n &= \sum_n E(1 + X)^{-n} \\
&= \sum_n \int_0^\infty (1 + x)^{-n} xe^{-x} dx \\
&= \int_0^\infty \sum_n (1 + x)^{-n} xe^{-x} dx \\
&\quad \text{Since } \left| \frac{1}{1+x} \right| < 1, \quad \sum_{n=1}^\infty \left(\frac{1}{1+x} \right)^n = \frac{1/(1+x)}{1 - 1/(1+x)} = \frac{1}{x}. \\
&= \int_0^\infty \frac{1}{x} xe^{-x} dx \\
&= \int_0^\infty e^{-x} dx \\
&= 1.
\end{aligned}$$

□

Problem 1.36

Suppose X and Y are independent random variables, and let F_X and F_Y denote their cumulative distribution functions.

- a) Use smoothing to show that the cumulative distribution function of $S = X + Y$ is

$$F_S(s) = P(X + Y \leq s) = EF_X(s - Y). \quad (1.4)$$

Solution.

Since X and Y are independent, $P(X + Y \leq s|Y = y) = P(X \leq s - y) = F_X(s - y)$.
 Thus, $P(X + Y \leq s|Y) = F_X(s - Y)$.

$$\therefore F_S(s) = P(X + Y \leq s) = EP(X + Y \leq s|Y) = EF_X(s - Y).$$

□

- b) If X and Y are independent and Y is almost surely positive, use smoothing to show that the cumulative distribution function of $W = XY$ is $F_W(w) = EF_X(w/Y)$ for $w > 0$.

Solution.

Since X and Y are independent, $P(XY \leq w|Y = y) = P(X \leq w/y|Y = y) = P(X \leq w/y) = F_X(w/y)$, for $y > 0$.

Thus, $P(XY \leq w|Y) = F_X(w/Y)$ a.s.

$$\therefore F_W(w) = P(XY \leq w) = EP(XY \leq w|Y) = EF_X(w/Y).$$

□

Problem 1.37

Differentiating (1.4) with respect to s one can show that if X is absolutely continuous with density p_X , then $S = X + Y$ is absolutely continuous with density

$$p_S(s) = Ep_X(s - Y)$$

for $s \in \mathbb{R}$. Use this formula to show that if X and Y are independent with $X \sim \Gamma(\alpha, 1)$ and $Y \sim \Gamma(\beta, 1)$, then $X + Y \sim \Gamma(\alpha + \beta, 1)$.

Solution.

Since $X \sim \Gamma(\alpha, 1)$, $P_X(x - Y) = 0$ for $Y \geq x$ ($\because p_X(x) = 0, \forall x < 0$). Then,

$$\begin{aligned} P_S(x) &= EP_X(x - Y) \\ &= E \left[\frac{1}{\Gamma(\alpha)} (x - Y)^{\alpha-1} e^{-(x-Y)} 1_{(0,x)}(Y) \right] \\ &= \int_0^x \frac{1}{\Gamma(\alpha)} (x - y)^{\alpha-1} e^{-(x-y)} \frac{1}{\Gamma(\beta)} y^{\beta-1} e^{-y} dy \\ &= \int_0^x \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (x - y)^{\alpha-1} y^{\beta-1} e^{-x} dy \end{aligned}$$

Change the variable by $u = y/x \Rightarrow du = 1/xdy$.

$$\begin{aligned} &= \int_0^1 \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (x - xu)^{\alpha-1} (xu)^{\beta-1} e^{-x} x du \\ &= \int_0^1 \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha+\beta-1} (1-u)^{\alpha-1} u^{\beta-1} e^{-x} du \\ &= \frac{1}{\Gamma(\alpha+\beta)} x^{\alpha+\beta-1} e^{-x}, \quad \text{for } x > 0. \end{aligned}$$

$$\therefore S = X + Y \sim \Gamma(\alpha + \beta, 1).$$

□

Problem 1.38

Let Q_λ denote the exponential distribution with failure rate λ , given in Problem 1.30. Let X be a discrete random variable taking values in $\{1, \dots, n\}$ with mass function

$$P(X = k) = \frac{2k}{n(n+1)}, \quad k = 1, \dots, n,$$

and assume that the conditional distribution of Y given $X = x$ is exponential with failure rate x ,

$$Y|X = x \sim Q_x.$$

- a) Find $E[Y|X]$.

Solution.

From Problem 1.30, $q_\lambda(x) = \lambda 3^{-\lambda x}$, $\lambda > 0$. Then, the mean of Q_λ is

$$\begin{aligned} \int x dQ_\lambda(x) &= \int x q_\lambda(x) dx = \int_0^\infty x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}. \\ \therefore E(Y|X = x) &= \frac{1}{x} \Rightarrow EY|X = \frac{1}{X}. \end{aligned}$$

□

- b) Use smoothing to compute EY .

Solution.

$$EY = E[E(Y|X)] = E \frac{1}{X} = \sum_{x=1}^n \frac{1}{x} \frac{2x}{n(n+1)} = \sum_{x=1}^n \frac{2}{n(n+1)} = \frac{2}{n+1}.$$

□

Problem 1.39

Let X be a discrete random variable uniformly distributed on $\{1, \dots, n\}$, so $P(X = k) = 1/n$, $k = 1, \dots, n$, and assume that the conditional distribution of Y given $X = x$ is exponential with failure rate x .

- a) For $y > 0$ find $P(Y > y|X)$.

Solution.

$$\begin{aligned} P(Y > y|X = x) &= \int_y^\infty xe^{-xt} dt = -e^{-xt} \Big|_y^\infty = e^{-xy}. \\ \therefore P(Y > y|X) &= e^{-Xy}. \end{aligned}$$

□

- b) Use smoothing to compute $P(Y > y)$.

Solution.

Using the sum of the geometric sequence ($\overline{\text{등비급수의 합}}$),

$$P(Y > y) = EP(Y > y|X) = Ee^{-Xy} = \sum_{x=1}^n e^{-xy} \frac{1}{n} = \frac{1}{n} \frac{e^{-y}(1 - e^{-ny})}{1 - e^{-y}} = \frac{1 - e^{-ny}}{n(e^y - 1)}.$$

□

c) Determine the density of Y .

Solution.

Let the density of Y is p_Y . Then,

$$\begin{aligned} p_Y(y) &= \frac{d}{dy}(1 - P(Y > y)) \\ &= \frac{d}{dy} \left(\frac{1 - e^{-ny}}{n(e^y - 1)} \right) \\ &= \frac{-ne^{-ny}(ne^y - n) + ne^y(1 - e^{-ny})}{n^2(e^y - 1)^2} \\ &= \frac{-ne^y e^{-ny} + ne^{-ny} + e^y - e^y e^{-ny}}{n(e^y - 1)^2} \\ &= \frac{e^y(1 - e^{-ny})}{n(e^y - 1)^2} - \frac{e^{-ny}}{e^y - 1}. \end{aligned}$$

□

Chapter 2

Exponential Families

Problem 2.1

Consider independent Bernoulli trials with success probability p and let X be the number of failures before the first success. Then $P(X = x) = p(1 - p)^x$, for $x = 0, 1, \dots$, and X has the geometric distribution with parameter p , introduced in Problem 1.17.

- a) Show that the geometric distributions form an exponential family.

Solution.

$$P(X = x) = p(1 - p)^x = \exp \{x \log(1 - p) - (-\log p)\}.$$

\therefore the geometric distribution is an exponential family. \square

- b) Write the densities for the family in canonical form, identifying the canonical parameter η , and the function $A(\eta)$.

Solution.

Let $\eta(p) = \log(1 - p)$. Then, $P(X = x) = \exp \{\eta x - (-\log(1 - e^\eta))\}$.

$\therefore A(\eta) = -\log(1 - e^\eta)$ with $T(x) = x$. \square

- c) Find the mean of the geometric distribution using a differential identity.

Solution.

$$E_\eta(T) = A'(\eta) = \frac{e^\eta}{1 - e^\eta} = \frac{1 - p}{p}.$$

\square

- d) Suppose X_1, \dots, X_n are i.i.d. from a geometric distribution. Show that the joint distributions form an exponential family, and find the mean and variance of T .

Solution.

$$P(X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^n P(X_i = x_i) = p^n (1 - p)^{\sum_{i=1}^n x_i} = \exp \left\{ \sum_i x_i \log(1 - p) - (-n \log p) \right\}.$$

Therefore, the joint distribution is also exponential family.

Now, let $\eta = \log(1-p)$. Then, the canonical exponential family is obtained with $A(\eta) = -n \log(1-e^\eta)$ and $T = \sum_i x_i$.

$$ET = \kappa_1 = A'(\eta) = \frac{ne^\eta}{1-e^\eta} = \frac{n(1-p)}{p},$$

$$Var(T) = \kappa_2 = A''(\eta) = \frac{ne^\eta(1-e^\eta) + ne^{2\eta}}{(1-e^\eta)^2} = \frac{np(1-p) + n(1-p)^2}{p^2} = \frac{n(1-p)}{p^2}.$$

□

Problem 2.2

Determine the canonical parameter space Ξ , and find densities for the one-parameter exponential family with μ Lebesgue measure on \mathbb{R}^2 , $h(x, y) = \exp[-(x^2 + y^2)/2]/(2\pi)$, and $T(x, y) = xy$.

Solution.

By definition of the canonical exponential family, the pdf be represented as

$$p(x, y) = \exp(\eta T(x, y) - A(\eta)) h(x, y) = \exp\{\eta xy - A(\eta) - (x^2 + y^2)/2\}/(2\pi).$$

Thus,

$$\begin{aligned} \int p(x, y) d\mu(x, y) = 1 &\iff \int \int \exp\{\eta xy - A(\eta) - (x^2 + y^2)/2\}/(2\pi) dx dy = 1 \\ &\iff e^{A(\eta)} = \int \int \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x - \eta y)^2\right\} dx \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y^2 - \eta^2 y^2)} dy \\ &\iff e^{A(\eta)} = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}y^2(1-\eta^2)} dy \quad \stackrel{\uparrow}{=} \quad (1-\eta^2)^{-1/2}. \\ &\quad \text{the integral is finite } \Leftrightarrow |\eta| < 1 \\ \therefore A(\eta) &= -\frac{1}{2} \log(1-\eta^2) \text{ for } \eta \in \Xi = (-1, 1). \end{aligned}$$

∴ The canonical exponential density is

$$\exp\left\{\eta xy + \frac{1}{2} \log(1-\eta^2) - (x^2 + y^2)/2\right\}/(2\pi).$$

□

Problem 2.4

Find the natural parameter space Ξ and densities p_η for a canonical one-parameter exponential family with μ Lebesgue measure on \mathbb{R} , $T_1(x) = \log x$, and $h(x) = (1-x)^2$, $x \in (0, 1)$, and $h(x) = 0$, $x \notin (0, 1)$.

Solution.

$$\begin{aligned} e^{A(\eta)} &= \int_0^1 e^{\eta \log x} (1-x)^2 dx \\ &\text{Let } t = \log x \Rightarrow dt = 1/x dx = e^{-t} dx, \quad -\infty < t < 0 \\ &= \int_{-\infty}^0 e^{\eta t} (1-e^t)^2 e^t dt \end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^0 \left(e^{t(\eta+1)} - 2e^{t(\eta+2)} + e^{t(\eta+3)} \right) dt \\
&= \left(\frac{1}{\eta+1} e^{t(\eta+1)} - \frac{2}{\eta+2} e^{t(\eta+2)} + \frac{1}{\eta+3} e^{t(\eta+3)} \right) \Big|_{-\infty}^0 \\
&= \frac{1}{\eta+1} - \frac{2}{\eta+2} + \frac{1}{\eta+3} \\
&= \frac{2}{(\eta+1)(\eta+2)(\eta+3)}, \quad \eta \in \Xi = (-1, \infty).
\end{aligned}$$

(\because For a finite integral, $\eta+1 > 0$, $\eta+2 > 0$, and $\eta+3 > 0 \Rightarrow \eta > -1$.)

$$\therefore p_\eta(x) = \exp(\eta T(x) - A(\eta)) h(x) = \frac{1}{2}(\eta+1)(\eta+2)(\eta+3)(1-x)^2 x^\eta, \quad x \in (0, 1).$$

□

Problem 2.5

Find the natural parameter space Ξ and densities p_η for a canonical one-parameter exponential family with μ Lebesgue measure on \mathbb{R} , $T_1(x) = -x$, and $h(x) = e^{-2\sqrt{x}}/\sqrt{x}$, $x > 0$, and $h(x) = 0$, $x \leq 0$. (Hint: After a change of variables, relevant integrals will look like integrals against a normal density. You should be able to express the answer using Φ , the standard normal cumulative distribution function.) Also, determine the mean and variance for a variable X with this density.

Solution.

$$e^{A(\eta)} = \int_0^\infty e^{-\eta x - 2\sqrt{x}} \frac{1}{\sqrt{x}} dx$$

To the integral is finite, $\eta \in \Xi = [0, \infty)$,

Let $t = \sqrt{x} \Rightarrow t^2 = x$, $2tdt = dx$

$$\begin{aligned}
&= \int_0^\infty \frac{1}{t} e^{-\eta t^2 - 2t} 2tdt \\
&= 2 \int_0^\infty e^{-\eta(t^2 + \frac{2}{\eta}t)} dt \\
&= 2 \int_0^\infty e^{-\frac{2\eta}{2} \left(t + \frac{1}{\eta}\right)^2} dt \cdot e^{-\eta(-\frac{1}{\eta^2})}
\end{aligned}$$

Since inner term of integration $\sim N(-1/\eta, 1/2\eta)$,

$$\begin{aligned}
&= 2\sqrt{2\pi}/\sqrt{2\eta} \cdot e^{\frac{1}{\eta}} \Phi\left(\frac{-1/\eta}{\sqrt{1/(2\eta)}}\right) \\
&= \frac{2\sqrt{\pi}}{\sqrt{\eta}} e^{1/\eta} \Phi(-\sqrt{2/\eta})
\end{aligned}$$

$$\Rightarrow A(\eta) = \log 2\sqrt{\pi} - \frac{1}{2} \log \eta + \frac{1}{\eta} + \log \Phi\left(-\sqrt{\frac{2}{\eta}}\right).$$

$$\therefore p_\eta(x) = \exp\{-\eta x - 2\sqrt{x} - A(\eta)\} \frac{1}{\sqrt{x}}, \quad x > 0.$$

The mean of X is

$$E_\eta(X) = -E_\eta T = -A'(\eta) = \frac{1}{2\eta} + \frac{1}{\eta^2} - \frac{\frac{\sqrt{2}}{2} \eta^{-3/2} \phi\left(-\sqrt{\frac{2}{\eta}}\right)}{\Phi\left(-\sqrt{\frac{2}{\eta}}\right)}.$$

□

Problem 2.6

Find the natural parameter space Ξ and densities p_η for a canonical two-parameter exponential family with μ counting measure on $\{0, 1, 2\}$, $T_1(x) = x$, $T_2(x) = x^2$, and $h(x) = 1$ for $x \in \{0, 1, 2\}$.

Solution.

$$\begin{aligned} e^{A(\eta)} &= \sum_{x=0}^2 e^{\eta_1 x + \eta_2 x^2} \\ &= 1 + e^{\eta_1 + \eta_2} + e^{2\eta_1 + 4\eta_2}. \end{aligned}$$

To the sum is finite, $\eta_1 + \eta_2 < \infty$ & $2\eta_1 + 4\eta_2 < \infty \Rightarrow \eta \in \Xi = \mathbb{R}^2$.

$$\therefore p_\eta(x) = \exp [\eta_1 x + \eta_2 x^2 - (1 + e^{\eta_1 + \eta_2} + e^{2\eta_1 + 4\eta_2})].$$

□

Problem 2.7

Suppose X_1, \dots, X_n are independent geometric variables with p_i the success probability for X_i . Suppose these success probabilities are related to a sequence of "independent" variables t_1, \dots, t_n , viewed as known constants, through

$$p_i = 1 - \exp(\alpha + \beta t_i), \quad i = 1, \dots, n.$$

Show that the joint densities for X_1, \dots, X_n form a two-parameter exponential family, and identify the statistics T_1 and T_2 .

Solution.

Since $X_i \stackrel{\text{ind}}{\sim} \text{Geo}(p_i)$, then the joint density is represented by

$$\begin{aligned} \prod_{i=1}^n p(x_i) &= \prod_{i=1}^n (1 - p_i)^{x_i} p_i = \exp \left[\sum_{i=1}^n x_i \log(1 - p_i) + \sum_{i=1}^n \log p_i \right] \\ &= \exp \left[\sum_{i=1}^n x_i (\alpha + \beta t_i) + \sum_{i=1}^n \log(1 - \exp(\alpha + \beta t_i)) \right]. \end{aligned}$$

$$\therefore T_1 = \sum_{i=1}^n X_i, \quad T_2 = \sum_{i=1}^n t_i X_i.$$

□

Problem 2.8

Assume that X_1, \dots, X_n are independent random variables with $X_i \sim N(\alpha + \beta t_i, 1)$, where t_1, \dots, t_n are observed constants and α and β are unknown parameters. Show that the joint densities for X_1, \dots, X_n form a two-parameter exponential family, and identify the statistics T_1 and T_2 .

Solution.

The joint density of X_1, \dots, X_n is

$$\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{1}{2} (x_i - \alpha - \beta t_i)^2 \right] = (2\pi)^{-\frac{n}{2}} \exp \left[\sum_i x_i (\alpha + \beta t_i) - \frac{1}{2} \sum_i (\alpha + \beta t_i)^2 \right] \exp \left(-\sum_i \frac{x_i^2}{2} \right).$$

$$\therefore T_1 = \sum_{i=1}^n X_i, \quad T_2 = \sum_{i=1}^n t_i X_i.$$

□

Problem 2.9

Suppose that X_1, \dots, X_n are independent Bernoulli variables (a random variable is Bernoulli if it only takes on values 0 and 1) with

$$P(X_i = 1) = \frac{\exp(\alpha + \beta t_i)}{1 + \exp(\alpha + \beta t_i)}.$$

Show that the joint distribution for X_1, \dots, X_n form a two-parameter exponential family, and identify the statistics T_1 and T_2 .

Solution.

The joint density of X_1, \dots, X_n is

$$\begin{aligned} \prod_{i=1}^n p_i^{x_i} (1 - p_i)^{1-x_i} &= \exp \left[\sum_i x_i \log p_i + \sum_i (1 - x_i) \log(1 - p_i) \right] \\ &= \exp \left[\sum_i x_i (\alpha + \beta t_i - \log(1 + e^{\alpha + \beta t_i})) + \sum_i (1 - x_i) (1 - \log(1 + e^{\alpha + \beta t_i})) \right] \\ &= \exp \left[\sum_i x_i (\alpha + \beta t_i) - \sum_i \log(1 + e^{\alpha + \beta t_i}) \right]. \end{aligned}$$

$$\therefore T_1 = \sum_{i=1}^n X_i, \quad T_2 = \sum_{i=1}^n t_i X_i.$$

□

Problem 2.15

For an exponential family in canonical form, $ET_j = \partial A(\eta)/\partial \eta_j$. This can be written in vector form as $ET = \nabla A(\eta)$. Derive an analogous differential formula for $E_\theta T$ for an s -parameter exponential family that is not in canonical form. Assume that Ω has dimension s .

Hint: Differentiation under the integral sign should give a system of linear equations. Write these equations in matrix form.

Solution.

Note that the exponential family form is

$$\begin{aligned} p_\eta(x) &= \exp \left[\sum_{i=1}^s \eta_i T_i(x) - A(\eta) \right] h(x) \\ &= \exp \left[\sum_{i=1}^s \eta_i(\theta) T_i(x) - B(\theta) \right] h(x) \\ &= p_\theta(x). \end{aligned}$$

Thus,

$$e^{B(\theta)} = \int \exp \left[\sum_{i=1}^s \eta_i(\theta) T_i(x) \right] h(x) d\mu(x).$$

By differentiating with respect to θ_i ,

$$e^{B(\theta)} \frac{\partial B(\theta)}{\partial \theta_i} = \int \left(\sum_{j=1}^s \frac{\partial \eta_j(\theta)}{\partial \theta_i} T_j(x) \right) \exp \left[\sum_{j=1}^s \eta_j(\theta) T_j(x) \right] h(x) d\mu(x).$$

$$\begin{aligned}\frac{\partial B(\theta)}{\partial \theta_i} &= \int \left(\sum_{j=1}^s \frac{\partial \eta_j(\theta)}{\partial \theta_i} T_j(x) \right) p_\theta(x) d\mu(x) \\ &= \sum_{j=1}^s \frac{\partial \eta_j(\theta)}{\partial \theta_i} E_\theta T_j, \quad i = 1, \dots, s.\end{aligned}$$

□

Problem 2.17

Let μ denote counting measure on $\{1, 2, \dots\}$. One common definition for $\sum_{k=1}^\infty f(k)$ is $\lim_{n \rightarrow \infty} \sum_{k=1}^n f(k)$, and another definition is $\int f d\mu$.

- a) Use the dominated convergence theorem to show that the two definitions give the same answer when $\int |f| d\mu < \infty$.

Hint: Find functions f_n , $n = 1, 2, \dots$, so that $\sum_{k=1}^n f(k) = \int f_n d\mu$.

Solution.

Let $f_n(k) = f(k)$, $\forall k \leq n$, $f_n(k) = 0$, $\forall k > n$. Then, $f_n \rightarrow f$ pointwise and $|f_n| \leq |f|$. Also f_n is simple and $\int f_n d\mu = \sum_{k=1}^n f(k)$.

By D.C.T., $\int f_n d\mu \rightarrow \int f d\mu$. □

- b) Use the monotone convergence theorem, give in Problem 1.25, to show the definitions agree if $f(k) \geq 0$ for all $1, 2, \dots$

Solution.

Define f_n and f like part (a). Then, $0 \leq f_1 \leq f_2 \leq \dots$

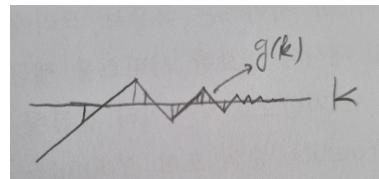
By M.C.T., $\sum_{k=1}^n f(k) = \int f_n d\mu \rightarrow \int f d\mu$. □

- c) Suppose $\lim_{n \rightarrow \infty} f(n) = 0$ and that $\int f^+ d\mu = \int f^- d\mu = \infty$ (so that $\int f d\mu$ is undefined.) Let K be an arbitrary constant. Show that the list $f(1), f(2), \dots$ can be rearranged to form a new list $g(1), g(2), \dots$ so that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n g(k) = K.$$

Solution.

Take positive parts of the sequence f until the sum is exceed K . And then, take negative parts of the sequence f until the sum is below K . Repeat this procedure like the figure below. (In the figure, $g(k)$ is typo. $\sum g(k)$ is correct.) Then, the sum of the sequence goes to 0 for $k > k'$ where k' is



the number when the sum of the rearranged sequence exceed K , first. Let this sequence as g . Then, $\lim_{n \rightarrow \infty} \sum_{k=1}^n g(k) \rightarrow K$. □

Problem 2.19

Let $p_n, n = 1, 2, \dots$, and p be probability densities with respect to a measure μ , and let $P_n, n = 1, 2, \dots$, and P be the corresponding probability measures.

- a) Show that if $p_n(x) \rightarrow p(x)$ as $n \rightarrow \infty$, then $\int |p_n - p| d\mu \rightarrow 0$.

Hint: First use the fact that $\int (p_n - p) d\mu = 0$ to argue that $\int |p_n - p| d\mu = 2 \int (p - p_n)^+ d\mu$. Then use dominated convergence.

Solution.

Since p_n and p be probability densities, $\int (p_n - p) d\mu = \int p_n d\mu - \int p d\mu = 1 - 1 = 0$. Since $p_n - p = (p - p_n)^+ - (p - p_n)^-$,

$$\int (p_n - p) d\mu = \int (p - p_n)^+ d\mu - \int (p - p_n)^- d\mu = 0,$$

$$\therefore \int (p - p_n)^+ d\mu = \int (p - p_n)^- d\mu.$$

Also $|p_n - p| = (p - p_n)^+ + (p - p_n)^-$,

$$\int |p_n - p| d\mu = \int (p - p_n)^+ d\mu + \int (p - p_n)^- d\mu = 2 \int (p - p_n)^+ d\mu.$$

Also note that $|(p - p_n)^+| \leq p \Rightarrow |(p - p_n)^+|$ is integrable. ($\because p$ is a probability density $\Rightarrow p$ is integrable.) From the condition, $(p_n \rightarrow p) \Rightarrow (p(x) - p_n(x))^+ \rightarrow 0$.

By D.C.T.,

$$\int |p_n - p| d\mu = 2 \int (p - p_n)^+ d\mu \rightarrow 0.$$

□

- b) Show that $|P_n(A) - P(A)| \leq \int |p_n - p| d\mu$.

Hint: Use indicators and the bound $|\int f d\mu| \leq \int |f| d\mu$.

Solution.

$$\begin{aligned} LHS &= \left| \int 1_A dP_n - \int 1_A dP \right| \\ &= \left| \int 1_A (p_n - p) d\mu \right| \\ &\leq \int |1_A (p_n - p)| d\mu \\ &\leq \int |p_n - p| d\mu. \end{aligned}$$

□

Remark: Distributions $P_n, n \geq 1$, are said to *converge strongly* to P if $\sup_A |P_n(A) - P(A)| \rightarrow 0$. The two parts above show that pointwise convergence of p_n to p implies strong convergence. This was discovered by Scheffé.

Problem 2.22

Suppose X is absolutely continuous with density

$$p_\theta(x) = \begin{cases} \frac{e^{-(x-\theta)^2/2}}{\sqrt{2\pi}\Phi(\theta)}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the moment generating function of X . Compute the mean and variance of X .

Solution.

$$p_\theta(x) = \exp \left[-\frac{1}{2}(x^2 - 2x\theta + \theta^2) - \log \Phi(\theta) \right] \frac{1}{\sqrt{2\pi}}, \quad x > 0$$

is the exponential family with $T = X$, $A(\theta) = \frac{\theta^2}{2} + \log \Phi(\theta)$. Therefore, the moment generating function of $T = X$ is

$$\begin{aligned} M_X(u) &= \exp [A(\theta + u) - A(\theta)] \\ &= \exp \left[\frac{1}{2}(\theta + u)^2 - \frac{1}{2}\theta^2 \right] \Phi(\theta + u)/\Phi(\theta) \\ &= \exp \left(\theta u + \frac{1}{2}u^2 \right) \Phi(\theta + u)/\Phi(\theta). \end{aligned}$$

Meanwhile, the *c.g.f.* of $T = X$ for exponential family is $K_X(u) = A(\theta + u) - A(\theta)$. Thus,

$$\begin{aligned} EX &= K'_X(u) = A'(\theta) = \theta + \frac{\phi(\theta)}{\Phi(\theta)}, \\ \text{Var}(X) &= K''_X(u) = A''(\theta) = 1 + \frac{\phi'(\theta)\Phi(\theta) - \phi(\theta)^2}{\Phi(\theta)^2}. \\ (\because \phi(x) &= e^{-x^2/2}/\sqrt{2\pi} \Rightarrow \phi'(x) = -x\phi(x).) \end{aligned}$$

□

Problem 2.23

Suppose $Z \sim N(0, 1)$. Find the first four cumulants of Z^2 .

Hint: Consider the exponential family $N(0, \sigma^2)$.

Solution.

Let $X \sim N(0, \sigma^2)$. Then, its density is

$$p_\sigma(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{\sigma^2}} = \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{1}{2\sigma^2} x^2 - \log \sigma \right],$$

and also be the exponential family.

Reparameterize $\eta = -\frac{1}{2\sigma^2}$, then $T = X^2$ and $A(\eta) = \frac{1}{2} \log -\frac{1}{2\eta} = -\frac{1}{2} \log(-2\eta)$. Thus, the cumulative generating function of $T = X^2$ is

$$\begin{aligned} K_{X^2}(u) &= A(\eta + u) - A(\eta) \\ &= -\frac{1}{2} [\log(-2(\eta + u)) + \log(-2\eta)]. \end{aligned}$$

If $\sigma = 1$, then $\eta = -\frac{1}{2}$ and $X^2 = Z^2$. Therefore,

$$K_{Z^2}(u) = -\frac{1}{2} [\log(-2u + 1)].$$

$$K'_{Z^2}(u) = \frac{1}{1-2u}, \quad K''_{Z^2}(u) = \frac{2}{(1-2u)^2},$$

$$K^{(3)}_{Z^2}(u) = 8(1-2u)^{-3}, \quad K^{(4)}_{Z^2}(u) = 48(1-2u)^{-4}.$$

\therefore the cumulants are 1, 2, 8, and 48, respectively. \square

Problem 2.24

Find the first four cumulants of $T = XY$ when X and Y are independent standard normal variates.

Solution.

Since X and Y are independent, the function of these r.v.s are also independent. (i.e. $g(X)$ and $g(Y)$ are independent for any function g .) Then, first 2 cumulants are

$$\kappa_1 = ET = EXY = EXEY = 0,$$

$$\kappa_2 = E(T - ET)^2 = EX^2Y^2 = EX^2EY^2 = 1.$$

For third cumulants, we need to obtain EX^3 .

$$\begin{aligned} EX^3 &= \int x^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &\text{Let } t = x^2/2 \Rightarrow xdx = dt \\ &= \frac{2}{\sqrt{2\pi}} \int_0^\infty te^{-t} dt \\ &= \frac{2}{\sqrt{2\pi}} \left[-te^{-t} \Big|_0^\infty + \int_0^\infty e^{-t} dt \right] \\ &= 0. \end{aligned}$$

Thus, the third cumulants is

$$\kappa_3 = E(T - ET)^3 = EX^3Y^3 = EX^3EY^3 = 0.$$

Similarly, we need to obtain EX^4 . 자꾸 적분이 틀림... 다시 해보기 Gamma dist 형태 사용

$$\begin{aligned} EX^4 &= \int x^4 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &\text{Let } t = x^2/2 \Rightarrow xdx = dt \end{aligned}$$

$$\kappa_4 = E(T - ET)^4 - 3\text{Var}(T)^2 = EX^4EY^4 - 3 \cdot 1^2 = 4.$$

\square

Problem 2.25

Find the third and fourth cumulants of the geometric distribution.

Solution.

$$f_p(x) = (1-p)^x p = \exp[x \log(1-p) + \log p]$$

is the exponential family. Reparameterize $\eta = -\log(1-p)$ ($\Rightarrow p = 1 - e^\eta$), then the above form be the canonical exponential family with $T = X$ and $A(\eta) = -\log(1 - e^\eta)$.

Thus, the cumulative generating function of $T = X$ is

$$K_X(u) = A(\eta + u) - A(\eta) = -\log(1 - e^{\eta+u}) + \log(1 - e^\eta).$$

Then, the first cumulants is

$$K'_X(0) = A'(\eta) = \frac{e^\eta}{1 - e^\eta} = \frac{1 - p}{p}.$$

The higher order cumulants are easily obtained by using the chain rule. By using the $p' = \frac{dp}{d\eta} = -e^\eta = p - 1$,

$$\begin{aligned} K''_X(0) &= A''(\eta) = \frac{d}{dp} A'(\eta) \frac{dp}{d\eta} = \frac{-p - (1 - p)}{p^2} (p - 1) = \frac{1}{p^2} - \frac{1}{p}, \\ K^{(3)}_X(0) &= A^{(3)}(\eta) = \frac{d}{dp} A''(\eta) p' = -\frac{2p'}{p^3} + \frac{p'}{p^2} = -\frac{2 - 2p}{p^3} + \frac{p - 1}{p^2} = \frac{2}{p^3} - \frac{3}{p^2} + \frac{1}{p}, \\ K^{(4)}_X(0) &= A^{(4)}(\eta) = \frac{d}{dp} A^{(3)}(\eta) p' = -\frac{6p'}{p^4} + \frac{6p'}{p^3} - \frac{p'}{p^2} = \frac{6}{p^4} - \frac{12}{p^3} + \frac{7}{p^2} - \frac{1}{p}. \end{aligned}$$

□

Problem 2.26

Find the third cumulant and third moment of the binomial distribution with n trials and success probability p .

Solution.

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} \left(\frac{p}{1-p}\right)^x (1-p)^n = \exp\left[x \log \frac{p}{1-p} + n \log(1-p)\right] \binom{n}{x}$$

is the exponential family for $\eta = \log \frac{p}{1-p}$ with $T = X$ and $A(\eta) = -n \log(1-p)$. To use the chain rule,

$$p = \frac{e^\eta}{1 + e^\eta} \Rightarrow \frac{dp}{d\eta} = \frac{e^\eta(1 + e^\eta) - (e^\eta)^2}{(1 + e^\eta)^2} = p(1 - p).$$

By using the above fact,

$$\kappa_1 = A'(\eta) = \frac{d}{dp} A(\eta) \frac{dp}{d\eta} = \frac{n}{1-p} p(1-p) = np,$$

$$\kappa_2 = A''(\eta) = \frac{d}{dp} A'(\eta) \frac{dp}{d\eta} = np(1-p),$$

$$\kappa_3 = A'''(\eta) = \frac{d}{dp} A''(\eta) \frac{dp}{d\eta} = (n - 2np)p(1-p) = np(1-p)(1-2p).$$

The third moment for $T = X$ can be obtained by

$$EX^3 = \kappa_3 + 3\kappa_1\kappa_2 + \kappa_1^3.$$

□

Problem 2.27

Let T be a random vector in \mathbb{R}^2 .

- a) Express $\kappa_{2,1}$ as a function of the moments of T .

Solution.

To make simple derivation, we denote $f_{ij}(u) = \frac{\partial^{i+j} f(u)}{\partial u_1^i \partial u_2^j}$. Note that $K(u) = \log M(u)$, by taking derivative,

$$K_{10} = \frac{M_{10}}{M}, \quad K_{20} = \frac{M_{20}M - M_{10}^2}{M^2},$$

$$\begin{aligned} K_{21} &= \frac{1}{M^4} [(M_{21}M + M_{20}M_{01} - 2M_{10}M_{11})M^2 - 2MM_{01}(M_{20}M - M_{10}^2)] \\ &= \frac{1}{M^3} [M_{21}M^2 + M_{20}M_{01}M - 2M_{10}M_{11}M - 2M_{01}M_{20}M - 2M_{01}M_{10}^2] \\ &= \frac{1}{M^3} [M_{21}M^2 - M_{20}M_{01}M - 2M_{10}M_{11}M - 2M_{01}M_{10}^2]. \end{aligned}$$

Taking $u = 0$, then

$$\kappa_{2,1} = K_{2,1}|_{u=0} = ET_1^2T_2 - (ET_2)(ET_1^2) - 2(ET_1)(ET_1T_2) - 2(ET_2)(ET_1)^2.$$

□

- b) Assume $ET_1 = ET_2 = 0$ and give an expression for $\kappa_{2,2}$ in terms of moments of T .

Solution.

$\kappa_{2,2}$ can be obtain by one more derivative for K_{21} . (Too complicated. See Keener page 462.)

□

Problem 2.28

Suppose $X \sim \Gamma(\alpha, 1/\lambda)$, with density

$$\frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0.$$

Find the cumulants of $T = (X, \log X)$ of order 3 or less. The answer will involve $\psi(\alpha) = d \log \Gamma(\alpha)/d\alpha = \Gamma'(\alpha)/\Gamma(\alpha)$.

Solution.

Since Gamma distribution is the exponential family,

$$\begin{aligned} p(x) &= \exp [\alpha \log \lambda + (\alpha - 1) \log x - \lambda x - \log \Gamma(\alpha)] \\ &= \exp [-\lambda x + \alpha \log x - (\log \Gamma(\alpha) - \alpha \log \lambda) - \log x]. \end{aligned}$$

Let $\eta = (-\lambda, \alpha)$, and $A(\eta) = \log \Gamma(\eta_2) - \eta_2 \log(-\eta_1)$.

Then, the cumulants are obtained by taking derivatives:

$$\kappa_{1,0} = \frac{\partial A(\eta)}{\partial \eta_1} = -\frac{\eta_2}{\eta_1} = \frac{\alpha}{\lambda},$$

$$\kappa_{0,1} = \frac{\partial A(\eta)}{\partial \eta_2} = \psi(\eta_2) - \log(-\eta_1) = \psi(\alpha) - \log \lambda,$$

$$\begin{aligned}
\kappa_{2,0} &= \frac{\partial^2 A(\eta)}{\partial \eta_1^2} = \frac{\eta_2}{\eta_1^2} = \frac{\alpha}{\lambda^2}, \\
\kappa_{1,1} &= \frac{\partial^2 A(\eta)}{\partial \eta_1 \partial \eta_2} = -\frac{1}{\eta_1} = \frac{1}{\lambda}, \\
\kappa_{0,2} &= \frac{\partial^2 A(\eta)}{\partial \eta_2^2} = \psi'(\eta_2) = \psi'(\alpha), \\
\kappa_{3,0} &= \frac{\partial^3 A(\eta)}{\partial \eta_1^3} = -\frac{2\eta_2}{\eta_1^3} = \frac{2\alpha}{\lambda^3}, \\
\kappa_{2,1} &= \frac{\partial^3 A(\eta)}{\partial \eta_1^2 \partial \eta_2} = \frac{1}{\eta_1^2} = \text{frac1}\lambda^2, \\
\kappa_{1,2} &= \frac{\partial^3 A(\eta)}{\partial \eta_1 \partial \eta_2^2} = 0, \\
\kappa_{0,3} &= \frac{\partial^3 A(\eta)}{\partial \eta_2^3} = \psi''(\eta_2) = \psi''(\alpha).
\end{aligned}$$

□

Chapter 3

Risk, Sufficiency, Completeness, and Ancillarity

Problem 3.2

Suppose data X_1, \dots, X_n are independent with

$$P_\theta(X_i \leq x) = x^{t_i \theta}, \quad x \in (0, 1).$$

where $\theta > 0$ is the unknown parameter, and t_1, \dots, t_n are known positive constants. Find a one-dimensional sufficient statistic T .

Solution.

□

Problem 3.3

An object with weight θ is weighted on scales with different precision. The data X_1, \dots, X_n are independent, with $X_i \sim N(\theta, \sigma_i^2)$, $i = 1, \dots, n$, with the standard deviations $\sigma_1, \dots, \sigma_n$ known constants. Use sufficiency to suggest a weighted average of X_1, \dots, X_n to estimate θ . (A weighted average would have form $\sum_{i=1}^n w_i X_i$, where the w_i are positive and sum to one.)

Solution.

□

Problem 3.4

Let X_1, \dots, X_n be a random sample from an arbitrary discrete distribution P on $\{1, 2, 3\}$. Find a two-dimensional sufficient statistic.

Solution.

□

Problem 3.6

The beta distribution with parameters $\alpha > 0$ and $\beta > 0$ has density

$$f_{\alpha,\beta}(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}, & x \in (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Suppose X_1, \dots, X_n are i.i.d. from a beta distribution.

- a) Determine a minimal sufficient statistic (for the family of joint distributions) if α and β vary freely.

Solution.

□

- b) Determine a minimal sufficient statistic if $\alpha = 2\beta$.

Solution.

□

- c) Determine a minimal sufficient statistic if $\alpha = \beta^2$.

Solution.

□

Problem 3.7

Logistic regression. Let X_1, \dots, X_n be independent Bernoulli variables, with $p_i = P(X_i = 1)$, $i = 1, \dots, n$. Let t_1, \dots, t_n be a sequence of known constants that are related to the p_i via

$$\log \frac{p_i}{1 - p_i} = \alpha + \beta t_i,$$

where α and β are unknown parameters. Determine a minimal sufficient statistic for the family of joint distributions.

Solution.

□

Problem 3.8

The multinomial distribution, derived later in Section 5.3, is a discrete distribution with mass function

$$\frac{n!}{x_1! \times \cdots \times x_s!} p_1^{x_1} \times \cdots \times p_s^{x_s},$$

where x_0, \dots, x_s are non-negative integers summing to n , where p_1, \dots, p_s are non-negative probabilities summing to one, and n is the sample size. Let N_{11}, N_{21}, N_{22} have a multinomial distribution with n trials and success probabilities $p_{11}, p_{12}, p_{21}, p_{22}$. (A common model for a two-by-two contingency table.)

- a) Give a minimal sufficient statistic if the success probabilities vary freely over the unit simplex in \mathbb{R}^4 .
 (The unit simplex in \mathbb{R}^p is the set of all vectors with non-negative entries summing to one.)

Solution.

□

- b) Give a minimal sufficient statistic if the success probabilities are constrained so that $p_{11}p_{22} = p_{12}p_{21}$.

Solution.

□

Problem 3.9

Let f be a positive integrable function on $(0, \infty)$. Define

$$c(\theta) = 1 / \int_{\theta}^{\infty} f(x) dx,$$

and take $p_{\theta}(x) = c(\theta)f(x)$ for $x > \theta$, and $p_{\theta}(x) = 0$ for $x \leq \theta$. Let X_1, \dots, X_n be i.i.d. with common density p_{θ} .

- a) Show that $M = \min\{X_1, \dots, X_n\}$ is sufficient.

Solution.

□

- b) Show that M is minimal sufficient.

Solution.

□

Problem 3.10

Suppose X_1, \dots, X_n are i.i.d. with common density $f_\theta(x) = (1 + \theta x)/2$, $|x| < 1$; $f_\theta(x) = 0$, otherwise, where $\theta \in [-1, 1]$ is an unknown parameter. Show that the order statistics are minimal sufficient. (Hint: A polynomial of degree n is uniquely determined by its value on a grid of $n + 1$ points.)

Solution.

□

Problem 3.16

Let X_1, \dots, X_n be a random sample from an absolutely continuous distribution with density

$$f_\theta(x) = \begin{cases} 2x/\theta^2, & x \in (0, \theta), \\ 0, & \text{otherwise.} \end{cases}$$

- a) Find a one-dimensional sufficient statistic T .

Solution.

□

- b) Determine the density of T .

Solution.

□

- c) Show directly that T is complete.

Solution.

□

Problem 3.17

Let X, X_1, X_2, \dots be i.i.d. from an exponential distribution with failure rate λ (introduced in Problem 1.30).

- a) Find the density of $Y = \lambda X$.

Solution.

□

- b) Let $\bar{X} = (X_1 + \dots + X_n)/n$. Show that \bar{X} and $(X_1^2 + \dots + X_n^2)/\bar{X}^2$ are independent.

Solution.

□

Problem 3.29

Find a function on $(0, \infty)$ that is bounded and strictly convex.

Solution.

□

Problem 3.30

Use convexity to show that the canonical parameter space Ξ of a one-parameter exponential family must be an interval. Specifically, show that if $\eta_0 < \eta < \eta_1$, and if η_0 and η_1 both lie in Ξ , then η must lie in Ξ .

Solution.

□

Problem 3.31

Let f and g be positive probability densities on \mathbb{R} . Use Jensen's inequality to show that

$$\int \log\left(\frac{f(x)}{g(x)}\right) f(x) dx > 0,$$

unless $f = g$ a.e. (If $f = g$, the integral equals zero.) This integral is called the *Kullback-Liebler information*.

Solution.

□