Theoretical Statistics: Topics for a Core Course
Problem Solutions

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Update: November 25, 2021

Contents

Chapter 1

Probability and Measure

Problem 1.1

Prove (1.1). If measurable sets $B_n,\ n \geq 1$, are increasing, with $B = \bigcup_{n=1}^{\infty} B_n$, called the limit of the sequence, then

$$\mu(B) = \lim_{n\to\infty} \mu(B_n).$$

Solution.
First, we showed that A_n's are disjoint. If $j < k$, then $B_j \subseteq B_{k-1}$.
Since $A_j \subset B_j \subseteq B_{k-1}$ and $A_k \subset B_{k-1}^c$, A_j and A_k are disjoint.
Also $B_n = \bigcup_{j=1}^{n} A_j$ and $\bigcup_{n=1}^{\infty} A_n = B$,

$$\mu(B) = \sum_{i=1}^{\infty} \mu(A_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} A_i\right) = \lim_{n\to\infty} \mu(B_n).$$

□

Problem 1.8

Prove *Boole's inequality*: For any events $B_1, B_2, \ldots,$

$$P\left(\bigcup_{i\geq 1} B_i\right) \leq \sum_{i\geq 1} P(B_i).$$

Solution.
Let $B = \bigcup_{i=1}^{\infty} B_i$, then $1_B \leq \sum 1_{B_i}$. Then by Fubini's theorem,

$$P(B) = \int 1_B dP \leq \int \sum 1_{B_i} dP = \sum \int 1_{B_i} dP = \sum P(B_i).$$

□

Problem 1.10

Let μ and ν be measures on $(\mathcal{E}, \mathcal{B})$.

a) Show that the sum η defined by $\eta(B) = \mu(B) + \nu(B)$ is also a measure.

Solution.
We should check following 2 conditions.

(i) For arbitrary set $A \in \mathcal{B}$, $\mu(A) \geq 0$ and $\nu(A) \geq 0$. $\Rightarrow \eta(A) = \mu(A) + \nu(A) \geq 0$.
$\therefore \eta : \mathcal{B} \to [0, \infty]$.

(ii) For disjoint set $B_1, B_2, \dots \subset \mathcal{B}$, then

$$\begin{aligned}\eta\left(\bigcup_{i=1}^{\infty} B_i\right) &= \mu\left(\bigcup B_i\right) + \nu\left(\bigcup B_i\right)\\ &= \sum \mu(B_i) + \sum \nu(B_i)\\ &= \sum \{\mu(B_i) + \nu(B_i)\}\\ &= \sum \eta(B_i)\end{aligned}$$

$\therefore \eta$ is a measure. □

b) If f is a non-negative measurable function, show that

$$\int f d\eta = \int f d\mu + \int f d\nu.$$

Solution.
We show it by 2 stage.

(i) Let $f = \sum_{i=1}^{n} a_i 1_{A_i}$, the non-negative simple function. Then,

$$\begin{aligned}\int f d\eta &= \int \sum_{i=1}^{n} a_i 1_{A_i} d\eta = \sum a_i \eta(A_i)\\ &= \sum a_i \{\mu(A_i) + \nu(A_i)\} = \int f d\mu + \int f d\nu.\end{aligned}$$

(ii) For general case, let f_n is the sequence of non-negative simple functions increasing to f. (i.e. $f_1 \leq f_2 \leq \dots \leq f$)

$$\begin{aligned}\int f d\eta &= \lim_{n\to\infty} \int f_n d\eta = \lim_{n\to\infty} \left(\int f_n d\mu + \int f_n d\nu\right)\\ &= \lim_{n\to\infty} \int f_n d\mu + \lim_{n\to\infty} \int f_n d\nu = \int f d\mu + \int f d\nu.\end{aligned}$$

□

Problem 1.11

Suppose f is the simple function $1_{(1/2,\pi]} + 21_{(1,2]}$, and let μ be a measure on $\mathbb{R}$ with $\mu\{(0, a^2]\} = a,\ a > 0$. Evaluate $\int f d\mu$.

Solution.
By the integral of simple function and the finite additivity, it can be simply computed as

$$\begin{aligned}\int f d\mu &= \mu\{(1/2,\pi]\} + 2\mu\{(1,2]\} \\ &= [\mu\{(0,\pi]\} - \mu\{(1/2,\pi]\}] + 2\left[\mu\{(0,2]\} - \mu\{(1,2]\}\right] \quad \text{(finite additivity)} \\ &= \left(\sqrt{\pi} - 1/\sqrt{2}\right) + 2\left(\sqrt{2}-1\right)\end{aligned}$$

□

Problem 1.12

Suppose that $\mu\{(0,a)\} = a^2$ for $a > 0$ and that f is defined by

$$f(x) = \begin{cases} 0, & x \le 0, \\ 1, & 0 < x < 2, \\ \pi, & 2 \ge x < 5, \\ 0, & x \ge 5. \end{cases}$$

Compute $\int f d\mu$.

Solution.
Since $f = 1_{(0,2)} + 1_{[2,5)}$ is simple, the integral can be computed as

$$\begin{aligned}\int f d\mu &= \mu\{(0,2)\} + \pi\mu\{[2,5)\} \\ &= \mu\{(0,2)\} + \pi\left[\mu\{(0,5)\} - \mu\{(0,2)\}\right] \\ &= 4 - \pi(25-4)\end{aligned}$$

□

Problem 1.13

Define the function f by

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find simple functions $f_1 \le f_2 \le \cdots$ increasing to f (i.e. $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in \mathbb{R}$). Let μ be Lebesgue measure on $\mathbb{R}$. Using our formal definition of an integral and the fact that $\mu\big((a,b]\big) = b-a$ whenever $b > a$ (this might be used to formally define Lebesgue measure), show that $\int f d\mu = 1/2$.

Solution.
Let $f_n = \lfloor 2^n x \rfloor / 2^n$ for $0 < x \le 1$ and 0 otherwise. ($\lfloor y \rfloor$ is a floor function.) Then,

$$f_1(x) = \lfloor 2x \rfloor / 2 = \begin{cases} 0, & x < 1/2, \\ 1/2, & 1/2 \le x < 1, \\ 1, & x = 1 \end{cases}$$

$$f_2(x) = \lfloor 2^2 x \rfloor / 2^2 = \begin{cases} 0, & x < 1/2^2, \\ 1/2^2, & 1/2^2 \le x < 2/2^2, \\ 2/2^2, & 2/2^2 \le x < 3/2^2, \\ 3/2^2, & 3/2^2 \le x < 1, \\ 1, & x = 1 \end{cases}$$

$$\vdots$$

$$f_n(x) = \lfloor 2^n x \rfloor / 2^n = \begin{cases} 0, & x < 1/2^n, \\ 1/2^n, & 1/2^n \le x < 2/2^n, \\ \vdots \\ (2^n-1)/2^n, & (2^n-1)/2^n \le x < 1, \\ 1, & x = 1 \end{cases}$$

$$\begin{aligned} \therefore \int f_n d\mu &= \frac{1}{2^n}\left(\frac{1}{2^n} + \frac{2}{2^n} + \cdots + \frac{2^n-1}{2^n}\right) \\ &= \frac{1+2+\cdots+(2^n-1)}{4^n} \\ &= \frac{2^n(2^n-1)}{2 \cdot 4^n} \\ &\longrightarrow \frac{1}{2}. \end{aligned}$$

My solution

Let the simple function $f_n = \sum_{i=1}^n \frac{i}{n} 1_{(\frac{i-1}{n}, \frac{i}{n}]}$. Then,

$$\int f_n d\mu = \sum_{i=1}^n \frac{i}{n}\frac{1}{n} = \frac{1}{n^2}\frac{n(n+1)}{2} \longrightarrow \frac{1}{2}.$$

□

Problem 1.16

Define $F(a-) = \lim_{x \uparrow a} F(x)$. Then, if F is non-decreasing, $F(a-) = \lim_{n\to\infty} F(a-1/n)$. Use (1.1)[Continuity of measure] to show that if a random variable X has cumulative distribution function F_X,

$$P(X < a) = F_X(a-).$$

Also, show that

$$P(X = a) = F_X(a) - F_X(a-).$$

Solution.

(i) Let $B_n = \{X \le a - 1/n\}$ and $\bigcup_{n=1}^\infty B_n = \{X < a\}$. By continuity of measure,

$$P(B) = P(X < a) = \lim_{n\to\infty} P(X \le a - 1/n) = \lim_{n\to\infty} F_X(a - 1/n) = F_X(a-).$$

(ii) Since $\{X < a\}$ and $\{X = a\}$ are disjoint with union $\{X \le a\}$,

$$P(X < a) + P(X = a) = P(X \le a).$$

$$\therefore P(X = a) = F_X(a) - F_X(a-).$$

□

Problem 1.17

Suppose X is a geometric random variable with mass function

$$p(x) = P(X = x) = \theta(1-\theta)^x, \quad x = 0, 1, \ldots,$$

where $\theta \in (0, 1)$ is a constant. Find the probability that X is even.

Solution.

$$\begin{aligned} P(X \text{ is even}) &= P(X = 0) + P(X = 2) + P(X = 4) \cdots \\ &= \theta + \theta(1-\theta)^2 + \theta(1-\theta)^4 + \cdots \\ &= \frac{\theta}{1-(1-\theta)^2} \\ &= \frac{1}{2-\theta} \end{aligned}$$

□

Problem 1.18

Let X be a function mapping $\mathcal{E}$ into $\mathbb{R}$. Recall that if B is a subset of $\mathbb{R}$, then $X^{-1}(B) = \{e \in \mathcal{E} : X(e) \in B\}$. Use this definition to prove that

$$X^{-1}(A \cap B) = X^{-1}(A) \cap X^{-1}(B),$$

$$X^{-1}(A \cup B) = X^{-1}(A) \cup X^{-1}(B),$$

and

$$X^{-1}\left(\bigcup_{i=0}^{\infty} A_i\right) = \bigcup_{i=0}^{\infty} X^{-1}(A_i).$$

Solution.

(i)

$$\begin{aligned} e \in X^{-1}(A \cap B) &\Leftrightarrow X(e) \in A \cap B \\ &\Leftrightarrow X(e) \in A \text{ and } X(e) \in B \\ &\Leftrightarrow e \in X^{-1}(A) \text{ and } e \in X^{-1}(B) \\ &\Leftrightarrow e \in X^{-1}(A) \cap X^{-1}(B). \end{aligned}$$

(ii) Similarly,

$$\begin{aligned} e \in X^{-1}(A \cup B) &\Leftrightarrow X(e) \in A \cup B \\ &\Leftrightarrow X(e) \in A \text{ or } X(e) \in B \\ &\Leftrightarrow e \in X^{-1}(A) \text{ or } e \in X^{-1}(B) \\ &\Leftrightarrow e \in X^{-1}(A) \cup X^{-1}(B). \end{aligned}$$

(iii)

$$\begin{aligned} e \in X^{-1}(\bigcup_{i=0}^{\infty} A_i) &\Leftrightarrow X(e) \in \bigcup_{i=0}^{\infty} A_i \\ &\Leftrightarrow X(e) \in A_i \text{ for some } i \\ &\Leftrightarrow e \in X^{-1}(A_i) \text{ for some } i \\ &\Leftrightarrow e \in \bigcup_{i=0}^{\infty} X^{-1}(A_i). \end{aligned}$$

□

Problem 1.19

Let P be a probability measure on $(\mathcal{E}, \mathcal{B})$, and let X be a random variable. Show that the distribution P_X of X defined by $P_X(B) = P(X \in B) = P(X^{-1}(B))$ is a measure (on the Borel sets of $\mathbb{R}$).

***Solution*.**
To show that P_X is a measure, we should check 2 conditions.

(i) $P_X(B) = P\big(X^{-1}(B)\big) \geq 0$.

(ii) From Problem 1.18, $X^{-1}\left(\bigcup_i A_i\right) = \bigcup_i X^{-1}(A_i)$ is hold.
Also, if B_i and B_j are disjoint, then $X^{-1}(B_i)$ and $X^{-1}(B_j)$ are disjoint. ($\because$ If $e \in \mathcal{E}$ lie in both $X^{-1}(B_1)$ and $X^{-1}(B_2)$, $X(e)$ lies in B_1 and B_2.)
Therefore, for arbitrary disjoint set $B_1, B_2, \ldots$ with union $B = \bigcup_i B_i$, $X^{-1}(B_1), X^{-1}(B_2), \ldots$ are also disjoint with union $X^{-1}(B)$.

$$\therefore \sum_i P_X(B_i) = \sum_i P\big(X^{-1}(B_i)\big) = P\big(X^{-1}(B)\big) = P_X(B).$$

The above 2 conditions are hold, P_X is a measure. □

Problem 1.21

Let X have a uniform distribution on $(0,1)$; that is, X is absolutely continuous with density p defined by

$$p(x) = \begin{cases} 1, & x \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Let Y_1 and Y_2 denote the first two digits of X when X is written as a binary decimal (so $Y_1 = 0$ if $X \in (0, 1/2)$ for instance). Find $P(Y_1 = i, Y_2 = j)$, $i = 0$ or $1, j = 0$ or 1.

Solution.
For all cases, the probability are same. In other words,

$$P(Y_1 = 0, Y_2 = 0) = P(Y_1 = 1, Y_2 = 0) = P(Y_1 = 0, Y_2 = 1) = P(Y_1 = 1, Y_2 = 1) = 1/4.$$

□

Problem 1.22

Let $\mathcal{E} = (0,1)$, let $\mathcal{B}$ be the Borel subsets of $\mathcal{E}$, and let $P(A)$ be the length of A for $A \in \mathcal{B}$. (P would be called the *uniform probability measure* on $(0,1)$.) Define the random variable X by

$$X(e) = \min\{e, 1/2\}.$$

Let μ be the sum of Lebesgue measure on $\mathbb{R}$ and counting measure on $\mathcal{X}_0 = \{1/2\}$. Show that the distribution P_X of X is absolutely continuous with respect to μ and find the density of P_X.

Solution.

(i) Claim: For any $B \in \mathbb{R}$, $\mu(B) = 0 \Longrightarrow P_X(B) = 0$.

$$\begin{aligned} \{X \in B\} &= \{y \in (0,1) : X(y) \in B\} \\ &= \begin{cases} B \cap (0, 1/2), & 1/2 \notin B, \\ \big(B \cap (0,1/2)\big) \cup [1/2, 1), & 1/2 \in B. \end{cases} \end{aligned}$$

Let λ be a Lebesgue measure and ν be a counting measure on $\{1/2\}$. Then,

$$\mu(B) = 0 \Longleftrightarrow \lambda(B) = 0 \text{ and } 1/2 \notin B.\ (\mu(B) = \nu(B) = 0)$$

$$\Longrightarrow P_X(B) = P(X \in B) = P\big(B \cap (0,1/2)\big) = \lambda\big(B \cap (0,1/2)\big) = 0.$$

$$\therefore P_X \text{ is absolutely continuous w.r.t. } \lambda + \nu \overset{\text{def}}{=} \mu.$$

(ii) Find the density of P_X.
From the above equation,

$$P(X \in B) = \lambda\big(B \cap (0,1/2)\big) + \frac{1}{2} 1_B(1/2). \tag{1.1}$$

If f is the density, then (1.1) should be equal to $\int_B f d(\lambda + \nu)$.

$$\begin{aligned} \int_B f d(\lambda + \nu) &= \int f 1_B d(\lambda + \nu) \\ &= \int \left(1_{B \cap (0,1/2)} + \frac{1}{2} 1_{\{1/2\} \cap B} \right) d(\lambda + \nu) \\ &= \lambda\big(B \cap (0,1/2)\big) + \nu\big(B \cap (0,1/2)\big) + \frac{1}{2}\lambda\big(\{1/2\} \cap B\big) + \frac{1}{2}\nu\big(\{1/2\} \cap B\big) \\ &= \lambda\big(B \cap (0,1/2)\big) + \frac{1}{2} 1_B(1/2). \end{aligned}$$

$$\therefore f = 1_{(0,1/2} + \frac{1}{2} 1_{\{1/2\}}.$$

□

Problem 1.23

The standard normal distribution $N(0,1)$ has density ϕ given by

$$\phi(x) = \frac{e^{-x^2/2}}{\sqrt{w\pi}}, \qquad x \in \mathbb{R},$$

with respect to Lebesgue measure λ on $\mathbb{R}$. The corresponding cumulative distribution function is Φ, so

$$\Phi(x) = \int_{-\infty}^{x} \phi(z)dz$$

for $x \in \mathbb{R}$. Suppose that $X \sim N(0,1)$ and that the random variable Y equals X when $|X| < 1$ and is 0 otherwise. Let P_Y denote the distribution of Y and let μ be counting measure on $\{0\}$. Find the density of P_Y with respect to $\lambda + \mu$.

Solution.
Let $g(x) = x1_{|x|<1}$, then $Y = g(X)$.
For any integrable function f, the expectation of $f(Y)$ against the density of X is

$$Ef(Y) = Ef\big(g(X)\big) = \int f\big(g(x)\big)\phi(x)dx = \int f\big(g(x)\big)\phi(x)1_{(-1,1)}(x)dx + cf(0), \tag{1.2}$$

where $c = \int \phi(x)1_{|x|>1}(x)dx = 2\Phi(-1)$.
Similarly, the expectation of $f(Y)$ against the density p of Y is

$$Ef(Y) = \int fpd(\lambda+\mu) = \int fpd\lambda + \int fpd\mu = \int f(x)p(x)dx + f(0)p(0). \tag{1.3}$$

Then, (1.2) and (1.3) should be same. Therefore,

$$\begin{aligned} f = 1_{\{0\}} &\Longrightarrow p(0) = 2\Phi(-1), \\ f(0) = 0 &\Longrightarrow \int f(x)\phi(x)1_{(-1,1)}(x)dx = \int f(x)p(x)dx \\ &\Longrightarrow p(x) = \phi(x) \text{ for } 0 < |x| < 1. \end{aligned}$$

$$\therefore p(x) = 2\Phi(-1)1_{\{0\}}(x) + \phi(x)1_{(0,1)}(|x|) \text{ is the density of } P_Y.$$

□

Problem 1.24

Let μ be a σ-finite measure on a measurable space $(X, \mathcal{B})$. Show that μ is absolutely continuous with respect to some probability measure P.
Hint: You can use the fact that if $\mu_1, \mu_2, \ldots$ are probability measures and $c_1, c_2, \ldots$ are non-negative constants, then $\sum c_i\mu_i$ is a measure. (The proof for Problem 1.10 extents easily to this case.) The measure μ_i, you will want to consider, are truncations of μ to sets A_i covering X with $\mu(A_i) < \infty$, given by $\mu_i(B) = \mu(B \cap A_i)$. With the constants c_i chosen properly, $\sum c_i\mu_i$ will be a probability measure.

Solution.

(i) μ is finite $\Longrightarrow$ Trivial.
$\because$ Since $\mu(X) < \infty$, $P = \mu/\mu(X)$ be a probablity measure. Therefore, $P(N) = 0$ implies $\mu(N) = 0$.

(ii) μ is infinite but σ-finite.
$\implies \exists A_1, A_2, \cdots \in \mathcal{B}$ s.t. $\bigcup A_i = X$ and $0 < \mu(A_i) < \infty\ \forall i$.
From Hint, $\mu_i(B) = \mu(B \cap A_i)$ and also $\mu_i(X) = \mu(A_i) \implies \mu_i$ is a finite measure.
Let $b_i = 1/2^i$ (or any other seq of positive constants s.t. $\sum b_i = 1$) and define $c_i = b_i/\mu(A_i)$.
Then $P = \sum_i c_i\mu_i$ is a probability measure, since $P(X) = \sum_i c_i\mu_i(X) = \sum_i \left(\frac{b_i}{\mu(A_i)}\right)\mu(A_i) = 1$.
Suppose $P(N) = \sum_i c_i\mu_i(N) = 0$. Then $\mu_i(N) = \mu(N \cap A_i) = 0\ \forall i$.
By Boole's inequality,

$$\mu(N) = \mu\big(\bigcup_i (N \cap A_i)\big) \le \sum_i \mu(N \cap A_i) = 0.$$

For any null set for P is a null set for μ.
$\therefore \mu \ll P$.

$\square$

Problem 1.25

The monotone convergence theorem states that if $0 \le f_1 \le f_2 \le \cdots$ are measurable functions and $f = \lim_{n\to\infty} f_n$, then $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$. Use this result to prove the following assertions.

a) Show that if $X \sim P_X$ is a random variable on $(\mathcal{E}, \mathcal{B}, P)$ and f is a non-negative measurable function, then

$$\int f\big(X(e)\big)dP(e) = \int f(x)dP_X(x).$$

Hint: Try it first with f an indicator function. For the general case, let f_n be a sequence of simple functions increasing to f.

Solution.
We will show the equation by 3 steps.

(i) Suppose $f = 1_A$ (indicator function).
Then, $f\big(X(e)\big) = 1$ for $X(e) \in A$.
$\Rightarrow f \circ X = 1_B$ where $B = \{e : X(e) \in A\}$.
By definition, $P_X(A) = P(B)$.

$$\int f\big(X(e)\big)dP(e) = \int 1_B(e)dP(e) = P(B),$$
$$\int f(x)dP_X(x) = \int 1_A(x)dP_X(x) = P_X(A).$$

$\therefore$ The statement is holds.

(ii) Suppose $f = \sum_{i=1}^n c_i 1_{A_i}$ (simple function).

$$\begin{aligned}\therefore \int f\big(X(e)\big)dP(e) &= \int \sum_i c_i 1_{A_i}\big(X(e)\big)dP(e)\\ &= \sum_i c_i \int 1_{A_i}\big(X(e)\big)dP(e)\\ &= \sum_i c_i \int 1_{A_i}(x)dP_X(x) \quad \big(\text{by (i)}\big)\end{aligned}$$

$$= \int \sum_i c_i 1_{A_i}(x) dP_X(x)$$
$$= \int f(x) dP_X(x).$$

(iii) Suppose f is a non-negative measurable function (general case), and f_n is non-negative simple functions increasing to f.
$\Rightarrow f_n \circ X \nearrow f \circ X$.

$$\begin{aligned}\therefore \int f\big(X(e)\big) dP(e) &= \lim_{n\to\infty} \int f_n\big(X(e)\big) dP(e) \qquad \big(\text{by M.C.T.}\big) \\ &= \lim_{n\to\infty} \int f_n(x) dP_X(x) \qquad \big(\text{by (ii)}\big) \\ &= \int f(x) dP_X(x) \qquad \big(\text{by M.C.T.}\big)\end{aligned}$$

□

b) Suppose that P_X has density p with respect to μ, and let f be a non-negative measurable function. Show that
$$\int f dP_X = \int f p d\mu.$$

Solution.
It can be also showed by 3 steps.

(i) Let $f = 1_A$ (indicator function).
$$\int f dP_X = \int 1_A dP_X = P_X(A) = \int_A p d\mu = \int 1_A p d\mu = \int f p d\mu.$$

(ii) Suppose $f = \sum_{i=1}^n c_i 1_{A_i}$ (simple function).
$$\int f dP_X = \int \sum_i c_i 1_{A_i} dP_X = \sum_i c_i \int 1_{A_i} dP_X \overset{\text{(i)}}{=} \sum_i c_i \int 1_{A_i} p d\mu = \int \sum_i c_i 1_{A_i} p d\mu = \int f p d\mu.$$

(iii) For general case, let a non-negative measurable function f and non-negative simple functions f_n s.t. $f_n \nearrow f$. Then by M.C.T.,
$$\int f dP_X \overset{\text{M.C.T.}}{=} \lim_{n\to\infty} \int f_n dP_X \overset{\text{(ii)}}{=} \lim_{n\to\infty} \int f_n p d\mu \overset{\text{M.C.T.}}{=} \int f p d\mu.$$

□

Problem 1.26

The gamma distribution.

a) The gamma function is defined for $\alpha > 0$ by
$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$. Show that $\Gamma(X+1) = x!$ for $x = 0, 1, \ldots$.

Solution.
By integration by parts,

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = -t^x e^{-t}\Big|_0^\infty + (\alpha - 1)\int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x).$$

Since $\Gamma(1) = \int_0^\infty e^{-x} dx = 1$,

$$\Gamma(x+1) = x\Gamma(x) = x(x-1)\Gamma(x-1) = \cdots = x(x-1)\cdots 2 \cdot 1 \cdot \Gamma(1) = x!.$$

$\square$

b) Show that the function

$$p(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & \text{otherwise}, \end{cases}$$

is a (Lebesgue) probability density when $\alpha > 0$ and $\beta > 0$. This density is called the gamma density with parameters α and β. The corresponding probability distribution is denoted $\Gamma(\alpha, \beta)$.

Solution.
Using change of variable, $y = x/\beta \Rightarrow dx = \beta dy$. Therefore,

$$\begin{aligned} \int p(x)dx &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx \\ &= \int \frac{1}{\Gamma(\alpha)\beta^\alpha} (\beta y)^{\alpha-1} e^{-y} \beta dy \\ &= \frac{1}{\Gamma(\alpha)} \int y^{\alpha-1} e^{-y} dy \\ &= 1. \end{aligned}$$

$\square$

c) Show that if $X \sim \Gamma(\alpha, \beta)$, then $EX^r = \beta^r \Gamma(\alpha + r)/\Gamma(\alpha)$. Use this formula to find the mean and variance of X.

Solution.

$$\begin{aligned} EX^r &= \int_0^\infty x^r p(x) dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha+r-1} e^{-x/\beta} dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha+r)\beta^{\alpha+r}} x^{\alpha+r-1} e^{-x/\beta} dx \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \beta^r \\ &= \beta^r \Gamma(\alpha+r)/\Gamma(\alpha). \end{aligned}$$

By using this,

$$EX = \beta\Gamma(\alpha+1)/\Gamma(\alpha) = \alpha\beta,$$

$$EX^2 = \beta^2\Gamma(\alpha+2)/\Gamma(\alpha) = (\alpha+1)\alpha\beta^2,$$

$$Var(X) = EX^2 - \{EX\}^2 = \alpha\beta^2.$$

$\square$

Problem 1.27

Suppose X has a uniform distribution on $(0,1)$. Find the mean and covariance matrix of the random vector $\binom{X}{X^2}$.

Solution.
Since $X \sim \text{Uniform}(0,1)$, the density of X is $p(x) = 1$ for $x \in (0,1)$. Thus,

$$EX^r = \int x^r dx = \frac{1}{r+1}.$$

Therefore,

$$E\left(\begin{pmatrix} X \\ X^2 \end{pmatrix}\right) = \begin{pmatrix} EX \\ EX^2 \end{pmatrix},$$

$$Cov\left(\begin{pmatrix} X \\ X^2 \end{pmatrix}\right) = \begin{pmatrix} Var(X) & Cov(X, X^2) \\ Cov(X^2, X) & Var(X^2) \end{pmatrix} = \begin{pmatrix} EX^2 - \{EX\}^2 & EX^3 - EXEX^2 \\ EX^3 - EXEX^2 & EX^4 - \{EX^2\}^2 \end{pmatrix}.$$

□

Problem 1.28

If $X \sim N(0,1)$, find the mean and covariance matrix of the random vector $\binom{X}{I\{X>c\}}$.

Solution.
Let $\Phi(x)$ is a cumulative distribution function of X, and ϕ is a density of X. Then,

$$E\left(\begin{pmatrix} X \\ I\{X > c\} \end{pmatrix}\right) = \begin{pmatrix} EX \\ EI\{X > c\} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 - \Phi(c) \end{pmatrix},$$

$$Cov\left(\begin{pmatrix} X \\ I\{X > c\} \end{pmatrix}\right) = \begin{pmatrix} Var(X) & EX1_{X>c} - EXE1_{X>c} \\ EX1_{X>c} - EXE1_{X>c} & E1_{X>c}^2 - \{E1_{X>c}\}^2 \end{pmatrix} = \begin{pmatrix} 1 & \phi(c) \\ \phi(c) & \Phi(c)(1 - \Phi(c)) \end{pmatrix}.$$

$$\because EX1_{X>c} = \int_c^\infty \frac{1}{\sqrt{2\pi}} x e^{-x^2/2} dx = \phi(c) \text{ and } 1_{X>c}^2 = 1_{X>c}.$$

□

Problem 1.32

Suppose $E|X| < \infty$ and let

$$h(t) = \frac{1 - E\cos(tX)}{t^2}.$$

Use Fubini's theorem to find $\int_0^\infty h(t)dt$. <u>Hint</u>:

$$\int_0^\infty \big(1 - \cos(u)\big)u^{-2} du = \frac{\pi}{2}.$$

Solution.
By Funini's theorem,

$$\int_0^\infty h(t)dt = \int_0^\infty \left(\frac{1 - E\cos(tX)}{t^2}\right) dt$$

$$
\begin{aligned}
&= \int_0^\infty E\left(\frac{1-\cos(tX)}{t^2}\right) dt \\
&= \int_0^\infty \int \frac{1-\cos(tx)}{t^2} dP_X(x)dt \\
&= \int \int_0^\infty \frac{1-\cos(tx)}{t^2} dtdP_X(x) \quad \text{(by Funini's theorem)} \\
&\quad u = tx \Rightarrow du = xdt, t = x/u \\
&= \int \int_0^\infty |x| \frac{1-\cos(u)}{u^2} dudP_X(x) \\
&= \int \frac{\pi}{2} |x| dP_X(x) \\
&= \frac{\pi}{2} E|X|.
\end{aligned}
$$

□

Problem 1.33

Suppose X is absolutely continuous with density $p_X(x) = xe^{-x},\ x > 0$ and $p_X(x) = 0,\ x \le 0$. Define $c_n = E(1+X)^{-n}$. Use Funini's theorem to evaluate $\sum_{n=1}^\infty c_n$.

Solution.
By Funini's theorem,

$$
\begin{aligned}
\sum_{n=1}^\infty c_n &= \sum_n E(1+X)^{-n} \\
&= \sum_n \int_0^\infty (1+x)^{-n} xe^{-x} dx \\
&= \int_0^\infty \sum_n (1+x)^{-n} xe^{-x} dx \\
&\text{Since } \left|\frac{1}{1+x}\right| < 1,\ \sum_{n=1}^\infty \left(\frac{1}{1+x}\right)^n = \frac{1/(1+x)}{1-1/(1+x)} = \frac{1}{x}. \\
&= \int_0^\infty \frac{1}{x} xe^{-x} dx \\
&= \int_0^\infty e^{-x} dx \\
&= 1.
\end{aligned}
$$

□

Problem 1.36

Suppose X and Y are independent random variables, and let F_X and F_Y denote their cumulative distribution functions.

a) Use smoothing to show that the cumulative distribution function of $S = X + Y$ is

$$F_S(s) = P(X+Y \le s) = EF_X(s-Y). \tag{1.4}$$

Solution.
Since X and Y are independent, $P(X+Y \le s|Y=y) = P(X \le s-y) = F_X(s-y)$.
Thus, $P(X+Y \le s|Y) = F_X(s-Y)$.

$$\therefore F_S(s) = P(X+Y \le s) = EP(X+Y \le s|Y) = EF_X(s-Y).$$

□

b) If X and Y are independent and Y is almost surely positive, use smoothing to show that the cumulative distribution function of $W = XY$ is $F_W(w) = EF_X(w/Y)$ for $w > 0$.

Solution.
Since X and Y are independent, $P(XY \le w|Y=y) = P(X \le w/y|Y=y) = P(X \le w/y) = F_X(w/y)$, for $y > 0$.
Thus, $P(XY \le w|Y) = F_X(w/Y)$ a.s.

$$\therefore F_W(w) = P(XY \le w) = EP(XY \le w|Y) = EF_X(w/Y).$$

□

Problem 1.37

Differentiating (1.4) with respect to s one can show that if X is absolutely continuous with density p_X, then $S = X+Y$ is absolutely continuous with density

$$p_S(s) = Ep_X(s-Y)$$

for $s \in \mathbb{R}$. Use this formula to show that if X and Y are independent with $X \sim \Gamma(\alpha,1)$ and $Y \sim \Gamma(\beta,1)$, then $X+Y \sim \Gamma(\alpha+\beta,1)$.

Solution.
Since $X \sim \Gamma(\alpha,1)$, $P_X(x-Y) = 0$ for $Y \ge x$ ($\because p_X(x) = 0,\ \forall x < 0$). Then,

$$\begin{aligned}
P_S(x) &= EP_X(x-Y) \\
&= E\left[\frac{1}{\Gamma(\alpha)}(x-Y)^{\alpha-1}e^{-(x-Y)}1_{(0,x)}(Y)\right] \\
&= \int_0^x \frac{1}{\Gamma(\alpha)}(x-y)^{\alpha-1}e^{-(x-y)}\frac{1}{\Gamma(\beta)}y^{\beta-1}e^{-y}dy \\
&= \int_0^x \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(x-y)^{\alpha-1}y^{\beta-1}e^{-x}dy \\
&\text{Change the variable by } u = y/x \Rightarrow du = 1/xdy. \\
&= \int_0^1 \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(x-xu)^{\alpha-1}(xu)^{\beta-1}e^{-x}xdu \\
&= \int_0^1 \frac{1}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha+\beta-1}(1-u)^{\alpha-1}u^{\beta-1}e^{-x}du \\
&= \frac{1}{\Gamma(\alpha+\beta)}x^{\alpha+\beta-1}e^{-x}, \quad \text{for } x > 0.
\end{aligned}$$

$\therefore S = X+Y \sim \Gamma(\alpha+\beta,1)$. □

Problem 1.38

Let Q_λ denote the exponential distribution with failure rate λ, given in Problem 1.30. Let X be a discrete random variable taking values in $\{1, \dots, n\}$ with mass function

$$P(X = k) = \frac{2k}{n(n+1)}, \qquad k = 1, \dots, n,$$

and assume that the conditional distribution of Y given $X = x$ is exponential with failure rate x,

$$Y|X = x \sim Q_x.$$

a) Find $E[Y|X]$.

Solution.
From Problem 1.30, $q_\lambda(x) = \lambda 3^{-\lambda x},\ \lambda > 0$. Then, the mean of Q_λ is

$$\int x dQ_\lambda(x) = \int x q_\lambda(x) dx = \int_0^\infty x\lambda e^{-\lambda x} dx = \frac{1}{\lambda}.$$

$$\therefore E(Y|X = x) = \frac{1}{x} \Rightarrow EY|X = \frac{1}{X}.$$

□

b) Use smoothing to compute EY.

Solution.

$$EY = E[E(Y|X)] = E\frac{1}{X} = \sum_{x=1}^{n} \frac{1}{x}\frac{2x}{n(n+1)} = \sum_{x=1}^{n} \frac{2}{n(n+1)} = \frac{2}{n+1}.$$

□

Problem 1.39

Let X be a discrete random variable uniformly distributed on $\{1, \dots, n\}$, so $P(X = k) = 1/n,\ k = 1, \dots, n$, and assume that the conditional distribution of Y given $X = x$ is exponential with failure rate x.

a) For $y > 0$ find $P[Y > y|X]$.

Solution.

$$P(Y > y|X = x) = \int_y^\infty x e^{-xt} dt = -e^{-xt}\Big|_y^\infty = e^{-xy}.$$

$$\therefore P(X > y|X) = e^{-Xy}.$$

□

b) Use smoothing to compute $P(Y > y)$.

Solution.
Using the sum of the geometric sequence (등비급수의 합),

$$P(Y > y) = EP(Y > y|X) = Ee^{-Xy} = \sum_{x=1}^{n} e^{-xy}\frac{1}{n} = \frac{1}{n}\frac{e^{-y}(1 - e^{-ny})}{1 - e^{-y}} = \frac{1 - e^{-ny}}{n(e^y - 1)}.$$

□

c) Determine the density of Y.

Solution.
Let the density of Y is p_Y. Then,

$$\begin{aligned}
p_Y(y) &= \frac{d}{dy}\big(1 - P(Y > y)\big) \\
&= \frac{d}{dy}\left(\frac{1 - e^{-ny}}{n(e^y - 1)}\right) \\
&= \frac{-ne^{-ny}(ne^y - n) + ne^y(1 - e^{-ny})}{n^2(e^y - 1)^2} \\
&= \frac{-ne^y e^{-ny} + ne^{-ny} + e^y - e^y e^{-ny}}{n(e^y - 1)^2} \\
&= \frac{e^y(1 - e^{-ny})}{n(e^y - 1)^2} - \frac{e^{-ny}}{e^y - 1}.
\end{aligned}$$

□

Chapter 2

Exponential Families

Problem 2.1

Consider independent Bernoulli trials with success probability p and let X be the number of failures before the first success. Then $P(X = x) = p(1-p)^x$, for $x = 0, 1, \dots$, and X has the geometric distribution with parameter p, introduced in Problem 1.17.

a) Show that the geometric distributions form an exponential family.

Solution.

$$P(X = x) = p(1-p)^x = \exp\big\{x \log(1-p) - (-\log p)\big\}.$$

$\therefore$ the geometric distribution is an exponential family. □

b) Write the densities for the family in canonical form, identifying the canonical parameter η, and the function $A(\eta)$.

Solution.
Let $\eta(p) = \log(1-p)$. Then, $P(X = x) = \exp\big\{\eta x - \big(-\log(1-e^\eta)\big)\big\}$.
$\therefore A(\eta) = -\log(1-e^\eta)$ with $T(x) = x$. □

c) Find the mean of the geometric distribution using a differential identity.

Solution.

$$E_\eta(T) = A'(\eta) = \frac{e^\eta}{1-e^\eta} = \frac{1-p}{p}.$$

□

d) Suppose $X_1, \dots, X_n$ are i.i.d. from a geometric distribution. Show that the joint distributions form an exponential family, and find the mean and variance of T.

Solution.

$$P(X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^n P(X_i = x_i) = p^n (1-p)^{\sum_{i=1}^n x_i} = \exp\Big\{\sum_i x_i \log(1-p) - (-n \log p)\Big\}.$$

Therefore, the joint distribution is also exponential family.
Now, let $\eta = \log(1-p)$. Then, the canonical exponential family is obtained with $A(\eta) = -n\log(1-e^\eta)$ and $T = \sum_i x_i$.

$$ET = \kappa_1 = A'(\eta) = \frac{ne^\eta}{1-e^\eta} = \frac{n(1-p)}{p},$$

$$Var(T) = \kappa_2 = A''(\eta) = \frac{ne^\eta(1-e^\eta) + ne^{2\eta}}{(1-e^\eta)^2} = \frac{np(1-p) + n(1-p)^2}{p^2} = \frac{n(1-p)}{p^2}.$$

□

Problem 2.2

Determine the canonical parameter space Ξ, and find densities for the one-parameter exponential family with μ Lebesgue measure on $\mathbb{R}^2$, $h(x,y) = \exp\big[-(x^2+y^2)/2\big]/(2\pi)$, and $T(x,y) = xy$.

Solution.
By definition of the canonical exponential family, the pdf be represented as

$$p(x,y) = \exp\big(\eta T(x,y) - A(\eta)\big)h(x,y) = \exp\big\{\eta xy - A(\eta) - (x^2+y^2)/2\big\}/(2\pi).$$

Thus,

$$\begin{aligned}\int p(x,y)d\mu(x,y) = 1 &\iff \int\int \exp\big\{\eta xy - A(\eta) - (x^2+y^2)/2\big\}/(2\pi)dxdy = 1\\ &\iff e^{A(\eta)} = \int\int \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(x-\eta y)^2\right\}dx \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y^2-\eta^2y^2)}dy\\ &\iff e^{A(\eta)} = \frac{1}{\sqrt{2\pi}}\int e^{-\frac{1}{2}y^2(1-\eta^2)}dy \underset{\substack{\uparrow\\ \text{the integral is finite } \Leftrightarrow |\eta|<1}}{=} (1-\eta^2)^{-1/2}.\end{aligned}$$

$$\therefore A(\eta) = -\frac{1}{2}\log(1-\eta^2) \text{ for } \eta \in \Xi = (-1,1).$$

$\therefore$ The canonical exponential density is

$$\exp\big\{\eta xy + \frac{1}{2}\log(1-\eta^2) - (x^2+y^2)/2\big\}/(2\pi).$$

□

Problem 2.4

Find the natural parameter space Ξ and densities p_η for a canonical one-parameter exponential family with μ Lebesgue measure on $\mathbb{R}$, $T_1(x) = \log x$, and $h(x) = (1-x)^2,\ x \in (0,1)$, and $h(x) = 0,\ x \notin (0,1)$.

Solution.

$$\begin{aligned}e^{A(\eta)} &= \int_0^1 e^{\eta\log x}(1-x)^2dx\\ &\text{Let } t = \log x \Rightarrow dt = 1/xdx = e^{-t}dx,\ -\infty < t < 0\\ &= \int_{-\infty}^0 e^{\eta t}(1-e^t)^2e^tdt\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{0} \left(e^{t(\eta+1)} - 2e^{t(\eta+2)} + e^{t(\eta+3)} \right) dt \\
&= \left(\frac{1}{\eta+1} e^{t(\eta+1)} - \frac{2}{\eta+2} e^{t(\eta+2)} + \frac{1}{\eta+3} e^{t(\eta+3)} \right) \Big|_{-\infty}^{0} \\
&= \frac{1}{\eta+1} - \frac{2}{\eta+2} + \frac{1}{\eta+3} \\
&= \frac{2}{(\eta+1)(\eta+2)(\eta+3)}, \quad \eta \in \Xi = (-1, \infty).
\end{aligned}
$$

($\because$ For a finite integral, $\eta + 1 > 0,\ \eta + 2 > 0$, and $\eta + 3 > 0 \Rightarrow \eta > -1$.)

$$
\therefore p_\eta(x) = \exp\big(\eta T(x) - A(\eta)\big) h(x) = \frac{1}{2}(\eta+1)(\eta+2)(\eta+3)(1-x)^2 x^\eta, \quad x \in (0,1).
$$

$\square$

Problem 2.5

Find the natural parameter space Ξ and densities p_η for a canonical one-parameter exponential family with μ Lebesgue measure on $\mathbb{R}$, $T_1(x) = -x$, and $h(x) = e^{-2\sqrt{x}}/\sqrt{x},\ x > 0$, and $h(x) = 0,\ x \le 0$. (Hint: After a change of variables, relevant integrals will look like integrals against a normal density. You should be able to express the answer using Φ, the standard normal cumulative distribution function.) Also, determine the mean and variance for a variable X with this density.

Solution.

$$
e^{A(\eta)} = \int_0^\infty e^{-\eta x - 2\sqrt{x}} \frac{1}{\sqrt{x}} dx
$$

To the integral is finite, $\eta \in \Xi = [0, \infty)$,

Let $t = \sqrt{x} \Rightarrow t^2 = x,\ 2tdt = dx$

$$
\begin{aligned}
&= \int_0^\infty \frac{1}{t} e^{-\eta t^2 - 2t} 2tdt \\
&= 2 \int_0^\infty e^{-\eta\left(t^2 + \frac{2}{\eta} t\right)} dt \\
&= 2 \int_0^\infty e^{-\frac{2\eta}{2}\left(t + \frac{1}{\eta}\right)^2} dt \cdot e^{-\eta(-\frac{1}{\eta^2})}
\end{aligned}
$$

Since inner term of integration $\sim N(-1/\eta, 1/2\eta)$,

$$
\begin{aligned}
&= 2\sqrt{2\pi}/\sqrt{2\eta} \cdot e^{\frac{1}{\eta}} \Phi\left(\frac{-1/\eta}{\sqrt{1/(2\eta)}} \right) \\
&= \frac{2\sqrt{\pi}}{\sqrt{\eta}} e^{1/\eta} \Phi(-\sqrt{2/\eta})
\end{aligned}
$$

$$
\Rightarrow A(\eta) = \log 2\sqrt{\pi} - \frac{1}{2}\log \eta + \frac{1}{\eta} + \log \Phi\left(-\sqrt{\frac{2}{\eta}} \right).
$$

$$
\therefore p_\eta(x) = \exp\big\{ -\eta x - 2\sqrt{x} - A(\eta) \big\} \frac{1}{\sqrt{x}}, \quad x > 0.
$$

The mean of X is

$$
E_\eta(X) = -E_\eta T = -A'(\eta) = \frac{1}{2\eta} + \frac{1}{\eta^2} - \frac{\frac{\sqrt{2}}{2}\eta^{-3/2}\phi\left(-\sqrt{\frac{2}{\eta}}\right)}{\Phi\left(-\sqrt{\frac{2}{\eta}}\right)}.
$$

□

Problem 2.6

Find the natural parameter space Ξ and densities p_η for a canonical two-parameter exponential family with μ counting measure on $\{0, 1, 2\}$, $T_1(x) = x,\ T_2(x) = x^2$, and $h(x) = 1$ for $x \in \{0, 1, 2\}$.

Solution.

$$\begin{aligned} e^{A(\eta)} &= \sum_{x=0}^{2} e^{\eta_1 x + \eta_2 x^2} \\ &= 1 + e^{\eta_1+\eta_2} + e^{2\eta_1+4\eta_2}. \end{aligned}$$

To the sum is finite, $\eta_1 + \eta_2 < \infty$ & $2\eta_1 + 4\eta_2 < \infty \Rightarrow \eta \in \Xi = \mathbb{R}^2$.

$$\therefore p_\eta(x) = \exp\left[\eta_1 x + \eta_2 x^2 - \left(1 + e^{\eta_1+\eta_2} + e^{2\eta_1+4\eta_2}\right)\right].$$

□

Problem 2.7

Suppose $X_1, \ldots, X_n$ are independent geometric variables with p_i the success probability for X_i. Suppose these success probabilities are related to a sequence of "independent" variables $t_1, \ldots, t_n$, viewed as known constants, through

$$p_i = 1 - \exp(\alpha + \beta t_i), \quad i = 1, \ldots, n.$$

Show that the joint densities for $X_1, \ldots, X_n$ form a two-parameter exponential family, and identify the statistics T_1 and T_2.

Solution.
Since $X_i \overset{\text{ind}}{\sim} Geo(p_i)$, then the joint density is represented by

$$\begin{aligned} \prod_{i=1}^{n} p(x_i) = \prod_{i=1}^{n} (1-p_i)^{x_i} p_i &= \exp[\sum_{i=1}^{n} x_i \log(1-p_i) + \sum_{i=1}^{n} \log p_i] \\ &= \exp[\sum_{i=1}^{n} x_i(\alpha + \beta t_i) + \sum_{i=1}^{n} \log(1 - \exp(\alpha + \beta t_i))]. \end{aligned}$$

$\therefore T_1 = \sum_{i=1}^{n} X_i,\ T_2 = \sum_{i=1}^{n} t_i X_i.$ □

Problem 2.8

Assume that $X_1, \ldots, X_n$ are independent random variables with $X_i \sim N(\alpha + \beta t_i, 1)$, where $t_1, \ldots, t_n$ are observed constants and α and β are unknown parameters. Show that the joint densities for $X_1, \ldots, X_n$ form a two-parameter exponential family, and identify the statistics T_1 and T_2.

Solution.
The joint density of $X_1, \ldots, X_n$ is

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(x_i - \alpha - \beta t_i)^2\right] = (2\pi)^{-\frac{n}{2}} \exp\left[\sum_i x_i(\alpha + \beta t_i) - \frac{1}{2}\sum_i (\alpha + \beta t_i)\right] \exp\left(-\sum_i \frac{x_i^2}{2}\right).$$

$\therefore T_1 = \sum_{i=1}^{n} X_i,\ T_2 = \sum_{i=1}^{n} t_i X_i.$ □

Problem 2.9

Suppose that $X_1, \ldots, X_n$ are independent Bernoulli variables (a random variable is Bernoulli if it only takes on values 0 and 1) with

$$P(X_i = 1) = \frac{\exp(\alpha + \beta t_i)}{1 + \exp(\alpha + \beta t_i)}.$$

Show that the joint distribution for $X_1, \ldots, X_n$ form a two-parameter exponential family, and identify the statistics T_1 and T_2.

Solution.
The joint density of $X_1, \ldots, X_n$ is

$$\begin{aligned}
\prod_{i=1}^{n} p_i^{x_i}(1-p_i)^{1-x_i} &= \exp\left[\sum_i x_i \log p_i + \sum_i (1-x_i)\log(1-p_i)\right] \\
&= \exp\left[\sum_i x_i \left(\alpha + \beta t_i - \log\left(1 + e^{\alpha+\beta t_i}\right)\right) + \sum_i (1-x_i)\left(1 - \log\left(1 + e^{\alpha+\beta t_i}\right)\right)\right] \\
&= \exp\left[\sum_i x_i(\alpha + \beta t_i) - \sum_i \log\left(1 + e^{\alpha+\beta t_i}\right)\right].
\end{aligned}$$

$\therefore T_1 = \sum_{i=1}^n X_i,\ T_2 = \sum_{i=1}^n t_i X_i.$ □

Problem 2.15

For an exponential family in canonical form, $ET_j = \partial A(\eta)/\partial \eta_j$. This can be written in vector form as $ET = \nabla A(\eta)$. Derive an analogous differential formula for $E_\theta T$ for an s-parameter exponential family that is not in canonical form. Assume that Ω has dimension s.
Hint: Differentiation under the integral sign should give a system of linear equations. Write these equations in matrix form.

Solution.
Note that the exponential family form is

$$\begin{aligned}
p_\eta(x) &= \exp\left[\sum_{i=1}^{s} \eta_i T_i(x) - A(\eta)\right] h(x) \\
&= \exp\left[\sum_{i=1}^{s} \eta_i(\theta) T_i(x) - B(\theta)\right] h(x) \\
&= p_\theta(x).
\end{aligned}$$

Thus,

$$e^{B(\theta)} = \int \exp\left[\sum_{i=1}^{s} \eta_i(\theta) T_i(x)\right] h(x) d\mu(x).$$

By differentiating with respect to θ_i,

$$e^{B(\theta)} \frac{\partial B(\theta)}{\partial \theta_i} = \int \left(\sum_{j=1}^{s} \frac{\partial \eta_j(\theta)}{\partial \theta_i} T_j(x)\right) \exp\left[\sum_{j=1}^{s} \eta_j(\theta) T_j(x)\right] h(x) d\mu(x).$$

$$\begin{aligned}\frac{\partial B(\theta)}{\partial \theta_i} &= \int \left(\sum_{j=1}^{s} \frac{\partial \eta_j(\theta)}{\partial \theta_i} T_j(x) \right) p_\theta(x) d\mu(x) \\ &= \sum_{j=1}^{s} \frac{\partial \eta_j(\theta)}{\partial \theta_i} E_\theta T_j, \qquad i = 1, \dots, s.\end{aligned}$$

□

Problem 2.17

Let μ denote counting measure on $\{1, 2, \dots\}$. One common definition for $\sum_{k=1}^{\infty} f(k)$ is $\lim_{n\to\infty} \sum_{k=1}^{n} f(k)$, and another definition is $\int f d\mu$.

a) Use the dominated convergence theorem to show that the two definitions give the same answer when $\int |f| d\mu < \infty$.
Hint: Find functions f_n, $n = 1, 2, \dots$, so that $\sum_{k=1}^{n} f(k) = \int f_n d\mu$.

Solution.
Let $f_n(k) = f(k)$, $\forall k \le n$, $f_n(k) = 0$, $\forall k > n$. Then, $f_n \to f$ pointwise and $|f_n| \le |f|$. Also f_n is simple and $\int f_n d\mu = \sum_{k=1}^{n} f(k)$.
By D.C.T., $\int f_n d\mu \to \int f d\mu$. □

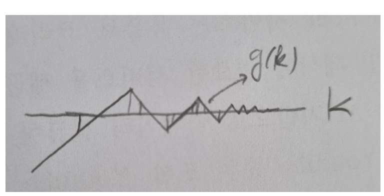
b) Use the monotone convergence theorem, give in Problem 1.25, to show the definitions agree if $f(k) \ge 0$ for all $1, 2, \dots$.

Solution.
Define f_n and f like part (a). Then, $0 \le f_1 \le f_2 \le \cdots$.
By M.C.T., $\sum_{k=1}^{n} f(k) = \int f_n d\mu \to \int f d\mu$. □

c) Suppose $\lim_{n\to\infty} f(n) = 0$ and that $\int f^+ d\mu = \int f^- d\mu = \infty$ (so that $\int f d\mu$ is undefined.) Let K be an arbitrary constant. Show that the list $f(1), f(2), \dots$ can be rearranged to form a new list $g(1), g(2), \dots$ so that

$$\lim_{n\to\infty} \sum_{k=1}^{n} g(k) = K.$$

Solution.
Take positive parts of the sequence f until the sum is exceed K. And then, take negative parts of the sequence f until the sum is below K. Repeat this procedure like the figure below. (In the figure, $g(k)$ is typo. $\sum g(k)$ is correct.) Then, the sum of the sequence goes to 0 for $k > k'$ where k' is



the number when the sum of the rearranged sequence exceed K, first. Let this sequence as g. Then, $\lim_{n\to\infty} \sum_{k=1}^{n} g(k) \to K$. □

Problem 2.19

Let $p_n, n = 1, 2, \ldots,$ and p be probability densities with respect to a measure μ, and let $P_n, n = 1, 2, \ldots,$ and P be the corresponding probability measures.

a) Show that if $p_n(x) \to p(x)$ as $n \to \infty$, then $\int |p_n - p| d\mu \to 0$.
Hint: First use the fact that $\int (p_n - p) d\mu = 0$ to argue that $\int |p_n - p| d\mu = 2 \int (p - p_n)^+ d\mu$. Then use dominated convergence.

Solution.
Since p_n and p be probability densities, $\int (p_n - p) d\mu = \int p_n d\mu - \int p d\mu = 1 - 1 = 0$. Since $p_n - p = (p - p_n)^+ - (p - p_n)^-$,

$$\int (p_n - p) d\mu = \int (p - p_n)^+ d\mu - \int (p - p_n)^- d\mu = 0,$$

$$\therefore \int (p - p_n)^+ d\mu = \int (p - p_n)^- d\mu.$$

Also $|p_n - p| = (p - p_n)^+ + (p - p_n)^-$,

$$\int |p_n - p| d\mu = \int (p - p_n)^+ d\mu + \int (p - p_n)^- d\mu = 2 \int (p - p_n)^+ d\mu.$$

Also note that $|(p - p_n)^+| \leq p \Rightarrow |(p - p_n)^+|$ is integrable. ($\because$ p is a probability density $\Rightarrow$ p is integrable.) From the condition, $(p_n \to p) \Rightarrow \big(p(x) - p_n(x)\big)^+ \to 0$.
By D.C.T.,

$$\int |p_n - p| d\mu = 2 \int (p - p_n)^+ d\mu \to 0.$$

□

b) Show that $|P_n(A) - P(A)| \leq \int |p_n - p| d\mu$.
Hint: Use indicators and the bound $|\int f d\mu| \leq \int |f| d\mu$.

Solution.

$$\begin{aligned} LHS &= \left| \int 1_A dP_n - \int 1_A dP \right| \\ &= \left| \int 1_A (p_n - p) d\mu \right| \\ &\leq \int \left| 1_A (p_n - p) \right| d\mu \\ &\leq \int \left| p_n - p \right| d\mu. \end{aligned}$$

□

Remark: Distributions $P_n,\ n \geq 1$, are said to *converge strongly* to P if $\sup_A |P_n(A) - P(A)| \to 0$. The two parts above show that pointwise convergence of p_n to p implies strong convergence. This was discovered by Scheffé.

Problem 2.22

Suppose X is absolutely continuous with density

$$p_\theta(x) = \begin{cases} \frac{e^{-(x-\theta)^2/2}}{\sqrt{2\pi}\Phi(\theta)}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the moment generating function of X. Compute the mean and variance of X.

Solution.

$$p_\theta(x) = \exp\left[-\frac{1}{2}(x^2 - 2x\theta + \theta^2) - \log\Phi(\theta)\right]\frac{1}{\sqrt{2\pi}}, \quad x > 0$$

is the exponential family with $T = X,\ A(\theta) = \frac{\theta^2}{2} + \log\Phi(\theta)$. Therefore, the moment generating function of $T = X$ is

$$\begin{aligned} M_X(u) &= \exp\left[A(\theta+u) - A(\theta)\right] \\ &= \exp\left[\frac{1}{2}(\theta+u)^2 - \frac{1}{2}\theta^2\right]\Phi(\theta+u)/\Phi(\theta) \\ &= \exp\left(\theta u + \frac{1}{2}u^2\right)\Phi(\theta+u)/\Phi(\theta). \end{aligned}$$

Meanwhile, the *c.g.f.* of $T = X$ for exponential family is $K_X(u) = A(\theta+u) - A(\theta)$. Thus,

$$EX = K_X'(u) = A'(\theta) = \theta + \frac{\phi(\theta)}{\Phi(\theta)},$$

$$\mathrm{Var}(X) = K_X''(u) = A''(\theta) = 1 + \frac{\phi'(\theta)\Phi(\theta) - \phi(\theta)^2}{\Phi(\theta)^2}.$$

$$\left(\because \phi(x) = e^{-x^2/2}/\sqrt{2\pi} \Rightarrow \phi'(x) = -x\phi(x).\right)$$

$\square$

Problem 2.23

Suppose $Z \sim N(0,1)$. Find the first four cumulants of Z^2.
<u>Hint</u>: Consider the exponential family $N(0,\sigma^2)$.

Solution.
Let $X \sim N(0,\sigma^2)$. Then, its density is

$$p_\sigma(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{s\sigma^2}} = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma^2}x^2 - \log\sigma\right],$$

and also be the exponential family.
Reparameterize $\eta = -\frac{1}{2\sigma^2}$, then $T = X^2$ and $A(\eta) = \frac{1}{2}\log -\frac{1}{2\eta} = -\frac{1}{2}\log(-2\eta)$. Thus, the cumulative generating function of $T = X^2$ is

$$\begin{aligned} K_{X^2}(u) &= A(\eta+u) - A(\eta) \\ &= -\frac{1}{2}\left[\log\big(-2(\eta+u)\big) + \log(-2\eta)\right]. \end{aligned}$$

If $\sigma = 1$, then $\eta = -\frac{1}{2}$ and $X^2 = Z^2$. Therefore,

$$K_{Z^2}(u) = -\frac{1}{2}\left[\log(-2u+1)\right].$$

$$K'_{Z^2}(u) = \frac{1}{1-2u}, \quad K''_{Z^2}(u) = \frac{2}{(1-2u)^2},$$

$$K^{(3)}_{Z^2}(u) = 8(1-2u)^{-3}, \quad K^{(4)}_{Z^2}(u) = 48(1-2u)^{-4}.$$

$\therefore$ the cumulants are 1, 2, 8, and 48, respectively. □

Problem 2.24

Find the first four cumulants of $T = XY$ when X and Y are independent standard normal variates.

Solution.
Since X and Y are independent, the function of these r.v.s are also independent. (i.e. $g(X)$ and $g(Y)$ are independent for any function g.) Then, first 2 cumulants are

$$\kappa_1 = ET = EXY = EXEY = 0,$$

$$\kappa_2 = E(T-ET)^2 = EX^2Y^2 = EX^2EY^2 = 1.$$

For third cumulants, we need to obtain EX^3.

$$\begin{aligned} EX^3 &= \int x^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &\text{Let } t = x^2/2 \Rightarrow xdx = dt \\ &= \frac{2}{\sqrt{2\pi}} \int_0^\infty te^{-t} dt \\ &= \frac{2}{\sqrt{2\pi}} \left[-te^{-t}\Big|_0^\infty + \int_0^\infty e^{-t} dt \right] \\ &= 0. \end{aligned}$$

Thus, the third cumulants is

$$\kappa_3 = E(T-ET)^3 = EX^3Y^3 = EX^3EY^3 = 0.$$

Similarly, we need to obtain EX^4. 자꾸 적분이 틀림... 다시 해보기 Gamma dist 형태 사용

$$\begin{aligned} EX^4 &= \int x^4 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &\text{Let } t = x^2/2 \Rightarrow xdx = dt \end{aligned}$$

$$\kappa_4 = E(T-ET)^4 - 3\mathrm{Var}(T)^2 = EX^4EY^4 - 3 \cdot 1^2 = 4.$$

□

Problem 2.25

Find the third and fourth cumulants of the geometric distribution.

Solution.

$$f_p(x) = (1-p)^x p = \exp\left[x\log(1-p) + \log p\right]$$

is the exponential family. Reparameterize $\eta = -\log(1-p)$ ($\Rightarrow p = 1 - e^{\eta}$), then the above form be the canonical exponential family with $T = X$ and $A(\eta) = -\log(1-e^{\eta})$.
Thus, the cumulative generating function of $T = X$ is

$$K_X(u) = A(\eta + u) - A(\eta) = -\log(1-e^{\eta+u}) + \log(1-e^{\eta}).$$

Then, the first cumulants is

$$K'_X(0) = A'(\eta) = \frac{e^{\eta}}{1-e^{\eta}} = \frac{1-p}{p}.$$

The higher order cumulants are easily obtained by using the chain rule. By using the $p' = \frac{dp}{d\eta} = -e^{\eta} = p-1$,

$$K''_X(0) = A''(\eta) = \frac{d}{dp}A'(\eta)\frac{dp}{d\eta} = \frac{-p-(1-p)}{p^2}(p-1) = \frac{1}{p^2} - \frac{1}{p},$$

$$K^{(3)}_X(0) = A^{(3)}(\eta) = \frac{d}{dp}A''(\eta)p' = -\frac{2p'}{p^3} + \frac{p'}{p^2} = -\frac{2-2p}{p^3} + \frac{p-1}{p^2} = \frac{2}{p^3} - \frac{3}{p^2} + \frac{1}{p},$$

$$K^{(4)}_X(0) = A^{(4)}(\eta) = \frac{d}{dp}A^{(3)}(\eta)p' = -\frac{6p'}{p^4} + \frac{6p'}{p^3} - \frac{p'}{p^2} = \frac{6}{p^4} - \frac{12}{p^3} + \frac{7}{p^2} - \frac{1}{p}.$$

□

Problem 2.26

Find the third cumulant and third moment of the binomial distribution with n trials and success probability p.

Solution.

$$p(x) = \binom{n}{x}p^x(1-p)^{n-x} = \binom{n}{x}\left(\frac{p}{1-p}\right)^x(1-p)^n = \exp\left[x\log\frac{p}{1-p} + n\log(1-p)\right]\binom{n}{x}$$

is the exponential family for $\eta = \log\frac{p}{1-p}$ with $T = X$ and $A(\eta) = -n\log(1-p)$. To use the chain rule,

$$p = \frac{e^{\eta}}{1+e^{\eta}} \Rightarrow \frac{dp}{d\eta} = \frac{e^{\eta}(1+e^{\eta}) - (e^{\eta})^2}{(1+e^{\eta})^2} = p(1-p).$$

By using the above fact,

$$\kappa_1 = A'(\eta) = \frac{d}{dp}A(\eta)\frac{dp}{d\eta} = \frac{n}{1-p}p(1-p) = np,$$

$$\kappa_2 = A''(\eta) = \frac{d}{dp}A'(\eta)\frac{dp}{d\eta} = np(1-p),$$

$$\kappa_3 = A'''(\eta) = \frac{d}{dp}A''(\eta)\frac{dp}{d\eta} = (n-2np)p(1-p) = np(1-p)(1-2p).$$

The third moment for $T = X$ can be obtained by

$$EX^3 = \kappa_3 + 3\kappa_1\kappa_2 + \kappa_1^3.$$

□

Problem 2.27

Let T be a random vector in $\mathbb{R}^2$.

a) Express $\kappa_{2,1}$ as a function of the moments of T.

***Solution*.**
To make simple derivation, we denote $f_{ij}(u) = \frac{\partial^{i+j} f(u)}{\partial u_1^i \partial u_2^j}$. Note that $K(u) = \log M(u)$, by taking derivative,

$$K_{10} = \frac{M_{10}}{M}, \quad K_{20} = \frac{M_{20}M - M_{10}^2}{M^2},$$

$$\begin{aligned}
K_{21} &= \frac{1}{M^4}\left[(M_{21}M + M_{20}M_{01} - 2M_{10}M_{11})M^2 - 2MM_{01}(M_{20}M - M_{10}^2)\right] \\
&= \frac{1}{M^3}[M_{21}M^2 + M_{20}M_{01}M - 2M_{10}M_{11}M - 2M_{01}M_{20}M - 2M_{01}M_{10}^2] \\
&= \frac{1}{M^3}[M_{21}M^2 - M_{20}M_{01}M - 2M_{10}M_{11}M - 2M_{01}M_{10}^2].
\end{aligned}$$

Taking $u = 0$, then

$$\kappa_{2,1} = K_{2,1}|_{u=0} = ET_1^2T_2 - (ET_2)(ET_1^2) - 2(ET_1)(ET_1T_2) - 2(ET_2)(ET_1)^2.$$

□

b) Assume $ET_1 = ET_2 = 0$ and give an expression for $\kappa_{2,2}$ in terms of moments of T.

***Solution*.**
$\kappa_{2,2}$ can be obtain by one more derivative for K_{21}. (Too complicated. See Keener page 462.) □

Problem 2.28

Suppose $X \sim \Gamma(\alpha, 1/\lambda)$, with density

$$\frac{\lambda^\alpha x^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)}, \qquad x > 0.$$

Find the cumulants of $T = (X, \log X)$ of order 3 or less. The answer will involve $\psi(\alpha) = d\log\Gamma(\alpha)/d\alpha = \Gamma'(\alpha)/\Gamma(\alpha)$.

***Solution*.**
Since Gamma distribution is the exponential family,

$$\begin{aligned}
p(x) &= \exp\left[\alpha\log\lambda + (\alpha-1)\log x - \lambda x - \log\Gamma(\alpha)\right] \\
&= \exp\left[-\lambda x + \alpha\log x - \big(\log\Gamma(\alpha) - \alpha\log\lambda\big) - \log x\right].
\end{aligned}$$

Let $\eta = (-\lambda, \alpha)$, and $A(\eta) = \log\Gamma(\eta_2) - \eta_2\log(-\eta_1)$.
Then, the cumulants are obtained by taking derivatives:

$$\kappa_{1,0} = \frac{\partial A(\eta)}{\partial\eta_1} = -\frac{\eta_2}{\eta_1} = \frac{\alpha}{\lambda},$$

$$\kappa_{0,1} = \frac{\partial A(\eta)}{\partial\eta_2} = \psi(\eta_2) - \log(-\eta_1) = \psi(\alpha) - \log\lambda,$$

$$\kappa_{2,0} = \frac{\partial^2 A(\eta)}{\partial \eta_1^2} = \frac{\eta_2}{\eta_1^2} = \frac{\alpha}{\lambda^2},$$
$$\kappa_{1,1} = \frac{\partial^2 A(\eta)}{\partial \eta_1 \partial \eta_2} = -\frac{1}{\eta_1} = \frac{1}{\lambda},$$
$$\kappa_{0,2} = \frac{\partial^2 A(\eta)}{\partial \eta_2^2} = \psi'(\eta_2) = \psi'(\alpha),$$

$$\kappa_{3,0} = \frac{\partial^3 A(\eta)}{\partial \eta_1^3} = -\frac{2\eta_2}{\eta_1^3} = \frac{2\alpha}{\lambda^3},$$
$$\kappa_{2,1} = \frac{\partial^3 A(\eta)}{\partial \eta_1^2 \partial \eta_2} = \frac{1}{\eta_1^2} = frac1\lambda^2,$$
$$\kappa_{1,2} = \frac{\partial^3 A(\eta)}{\partial \eta_1 \partial \eta_2^2} = 0,$$
$$\kappa_{0,3} = \frac{\partial^3 A(\eta)}{\partial \eta_2^3} = \psi''(\eta_2) = \psi''(\alpha).$$

□

Chapter 3

Risk, Sufficiency, Completeness, and Ancillarity

Problem 3.2

Suppose data $X_1, \dots, X_n$ are independent with

$$P_\theta(X_i \le x) = x^{t_i\theta}, \qquad x \in (0,1).$$

where $\theta > 0$ is the unknown parameter, and $t_1, \dots, t_n$ are known positive constants. Find a one-dimensional sufficient statistic T.

Solution.
Take a first derivative for the above cdf, then the pdf of X be $p_\theta(x) = t_i\theta x_i^{t_i\theta-1}$. Then, the joint density is

$$p_\theta(x_1, \dots, x_n) = \prod_{i=1}^n t_i\theta x_i^{t_i\theta-1} = \theta^n \left(\prod_{i=1}^n x_i^{t_i}\right)^\theta \frac{\prod_{i=1}^n t_i}{\prod_{i=1}^n x_i}.$$

By factorization theorem, the sufficient statistic for θ is

$$T = \prod_{i=1}^n x_i^{t_i}.$$

□

Problem 3.3

An object with weight θ is weighted on scales with different precision. The data $X_1, \dots, X_n$ are independent, with $X_i \sim N(\theta, \sigma_i^2),\ i = 1, \dots, n$, with the standard deviations $\sigma_1, \dots, \sigma_n$ known constants. Use sufficiency to suggest a weighted average of $X_1, \dots, X_n$ to estimate θ. (A weighted average would have form $\sum_{i=1}^n w_iX_i$, where the w_i are positive and sum to one.)

Solution.
The pdf of X_i is

$$p_\mu(x_i) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2\sigma_i^2}(x_i-\theta)^2} = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left[\frac{\theta x_i}{\sigma_i^2} - \frac{x_i^2}{2\sigma_i^2} - \frac{\theta^2}{2\sigma_i^2}\right].$$

Then, the joint density is

$$p_\mu(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2} \prod_i \sigma_i} \exp\left[\sum_i \frac{\theta x_i}{\sigma_i^2} - \sum_i \frac{x_i^2}{2\sigma_i^2} - \sum_i \frac{\theta^2}{2\sigma_i^2}\right].$$

By factorization theorem,

$$T = \sum_{i=1}^n \frac{x_i}{\sigma_i^2}.$$

To estimate θ using a weighted average form, let

$$w_i = \frac{T}{\sum_{i=1}^n \sigma_i^{-2}},$$

then the weighted average of $X_1, \ldots, X_n$ be an estimator for θ satisfied the sufficiency. □

Problem 3.4

Let $X_1, \ldots, X_n$ be a random sample from an arbitrary discrete distribution P on $\{1, 2, 3\}$. Find a two-dimensional sufficient statistic.

***Solution*.**
Let $p_i = P(\{i\})$ and $n_i(x) = \sum_{j=1}^n 1_{\{x_j = i\}}(x)$ for $i = 1, 2, 3$. Then, $n_1(x) + n_2(x) + n_3(x) = n$. Since X_is are i.i.d., the joint pdf is

$$\begin{aligned} P(X_1 = x_1, \ldots, X_n = x_n) &= p_1^{n_1(x)} p_2^{n_2(x)} p_3^{n_3(x)} \\ &= p_1^{n_1(x)} p_2^{n_2(x)} p_3^{n - n_1(x) - n_2(x)}. \end{aligned}$$

By factorization theorem, the sufficient statistic $T = (n_1, n_2)$. □

Problem 3.6

The beta distribution with parameters $\alpha > 0$ and $\beta > 0$ has density

$$f_{\alpha,\beta}(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & x \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $X_1, \ldots, X_n$ are i.i.d. from a beta distribution.

a) Determine a minimal sufficient statistic (for the family of joint distributions) if α and β vary freely.

***Solution*.**
The joint density

$$\begin{aligned} f_{\alpha,\beta}(x_1, \ldots, x_n) &= \exp\left[\sum_i (\alpha - 1)\log x_i + \sum_i (\beta - 1)\log(1 - x_i) + n \log \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \\ &= \exp\left[\sum_i \alpha \log x_i + \sum_i \beta \log(1 - x_i) + n \log \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} - \sum_i \log x_i(1 - x_i)\right], \end{aligned}$$

is the exponential family and full-rank. Then,

$$T = \left(\sum_{i=1}^{n} \log X_i, \sum_{i=1}^{n} \log(1 - X_i) \right)$$

is complete.
By factorization theorem, T is also sufficient.
Therefore, T is the minimal sufficient statistic for (α, β). (See Theorem 3.19) □

b) Determine a minimal sufficient statistic if $\alpha = 2\beta$.

Solution.
Plug into the joint density,

$$\begin{aligned} f_\beta(x_1, \ldots, x_n) &= \exp\left[\sum_i 2\beta \log x_i + \sum_i \beta \log(1 - x_i) + n \log \frac{\Gamma(2\beta + \beta)}{\Gamma(2\beta)\Gamma(\beta)} - \sum_i \log x_i(1 - x_i) \right] \\ &= \exp\left[\beta \sum_i \left(\log x_i^2 + \log(1 - x_i) \right) + n \log \frac{\Gamma(3\beta)}{\Gamma(2\beta)\Gamma(\beta)} - \sum_i \log x_i(1 - x_i) \right] \end{aligned}$$

is the one-parameter exponential family and full-rank. Thus,

$$T = \sum_i \left(\log x_i^2 + \log(1 - x_i) \right) = 2T_1 + T_2$$

is complete, and also sufficient by factorization theorem.
Therefore, T is the minimal sufficient statistic for β. □

c) Determine a minimal sufficient statistic if $\alpha = \beta^2$.

Solution.
Now, the joint density is

$$\begin{aligned} p_\beta(x_1, \ldots, x_n) &= \exp\left[\sum_i (\beta^2 - 1) \log x_i + \sum_i (\beta - 1) \log(1 - x_i) + n \log \frac{\Gamma(2\beta + \beta)}{\Gamma(2\beta)\Gamma(\beta)} \right] \\ &= \exp\left[(\beta^2 - 1) T_1(x) + (\beta - 1) T_2(x) + n \log \frac{\Gamma(\beta^2 + \beta)}{\Gamma(\beta^2)\Gamma(\beta)} \right]. \end{aligned}$$

Suppose $p_\theta(x) \propto_\theta p_\theta(y)$. Then W.L.O.G. we consider 2 cases as follows:

$$\begin{aligned} \frac{p_1(y)}{p_1(x)} = \frac{p_2(y)}{p_2(x)} &\Longrightarrow \frac{p_2(x)}{p_1(x)} = \frac{p_2(y)}{p_1(y)}, \\ \frac{p_1(y)}{p_1(x)} = \frac{p_3(y)}{p_3(x)} &\Longrightarrow \frac{p_3(x)}{p_1(x)} = \frac{p_3(y)}{p_1(y)}. \end{aligned}$$

Since $p_1(x) = p_1(y) = 1$, from the above equation,

$$\begin{aligned} 3T_1(x) + T_2(x) + n \log \frac{\Gamma(2^2 + 2)}{\Gamma(2^2)\Gamma(2)} &= 3T_1(y) + T_2(y) + n \log \frac{\Gamma(2^2 + 2)}{\Gamma(2^2)\Gamma(2)}, \\ 8T_1(x) + 2T_2(x) + n \log \frac{\Gamma(3^2 + 3)}{\Gamma(3^2)\Gamma(3)} &= 8T_1(y) + 2T_2(y) + n \log \frac{\Gamma(3^2 + 3)}{\Gamma(3^2)\Gamma(3)}. \end{aligned}$$

These 2 equations imply $T(x) = T(y)$.
Therefore, T is the minimal sufficient statistic by Theorem 3.11. □

Problem 3.7

Logistic regression. Let $X_1, \ldots, X_n$ be independent Bernoulli variables, with $p_i = P(X_i = 1),\ i = 1, \ldots, n$. Let $t_1, \ldots, t_n$ be a sequence of known constants that are related to the p_i via

$$\log \frac{p_i}{1-p_i} = \alpha + \beta t_i,$$

where α and β are unknown parameters. Determine a minimal sufficient statistic for the family of joint distributions.

Solution.
From the above equation,

$$p_i = \frac{e^{\alpha+\beta t_i}}{1+e^{\alpha+\beta t_i}}.$$

Then, the density of X_i is

$$\begin{aligned}
p_{\alpha,\beta}(x_i) &= p_i^{x_i}(1-p_i)^{1-x_i} \\
&= \left(\frac{e^{\alpha+\beta t_i}}{1+e^{\alpha+\beta t_i}}\right)^{x_i} \left(\frac{1}{1+e^{\alpha+\beta t_i}}\right)^{1-x_i} \\
&= \exp\left[x_i(\alpha+\beta t_i) - x_i \log(1+e^{\alpha+\beta t_i}) - (1-x_i)\log(1+e^{\alpha+\beta t_i})\right] \\
&= \exp\left[\alpha x_i + \beta x_i t_i - \log(1+e^{\alpha+\beta t_i})\right].
\end{aligned}$$

Therefore, we can easily obtain the following sufficient statistic by factorization theorem for the joint density,

$$T = \left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i t_i\right).$$

And also, the joint density is exponential family and full-rank. Thus T is complete.
$\therefore$ T is minimal sufficient. □

Problem 3.8

The multinomial distribution, derived later in Section 5.3, is a discrete distribution with mass function

$$\frac{n!}{x_1! \times \cdots \times x_s!} p_1^{x_1} \times \cdots \times p_s^{x_s},$$

where $x_1, \ldots, x_s$ are non-negative integers summing to n, where $p_1, \ldots, p_s$ are non-negative probabilities summing to one, and n is the sample size. Let N_{11}, N_{21}, N_{22} have a multinomial distribution with n trials and success probabilities $p_{11}, p_{12}, p_{21}, p_{22}$. (A common model for a two-by-two contingency table.)

a) Give a minimal sufficient statistic if the success probabilities vary freely over the unit simplex in $\mathbb{R}^4$. (The unit simplex in $\mathbb{R}^p$ is the set of all vectors with non-negative entries summing to one.)

   Solution.
   The density be the exponential family as follows:

$$\begin{aligned}
p(\mathbf{N}) &= \frac{n!}{N_{11}!N_{12}!N_{21}!N_{22}!} p_{11}^{N_{11}} p_{12}^{N_{12}} p_{21}^{N_{21}} p_{22}^{N_{22}} \\
&= \frac{n!}{N_{11}!N_{12}!N_{21}!N_{22}!} \exp\left[N_{11}\log p_{11} + N_{12}\log p_{12} + N_{21}\log p_{21} + N_{22}\log p_{22}\right] \qquad (*)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n!}{N_{11}!N_{12}!N_{21}!N_{22}!} \exp\left[N_{11}\log p_{11} + N_{12}\log p_{12} + N_{21}\log p_{21}\right. \\
&\qquad\qquad \left. +(n - N_{11} - N_{12} - N_{21})\log(1 - p_{11} - p_{12} - p_{21})\right] \\
&= \frac{n!}{N_{11}!N_{12}!N_{21}!N_{22}!} \exp\left[N_{11}\log\left(\frac{p_{11}}{1 - p_{11} - p_{12} - p_{21}}\right) + N_{12}\log\left(\frac{p_{12}}{1 - p_{11} - p_{12} - p_{21}}\right)\right. \\
&\qquad\qquad \left. +N_{21}\log\left(\frac{p_{21}}{1 - p_{11} - p_{12} - p_{21}}\right) + n\log(1 - p_{11} - p_{12} - p_{21})\right].
\end{aligned}
$$

Since it is the exponential family of full-rank, $T = (N_{11}, N_{12}, N_{21})$ is complete, and also sufficient by factorization theorem. Thus $T = (N_{11}, N_{12}, N_{21})$ is minimal sufficient.
If we use the equation $(*)$, the minimal sufficient statistic is $T = (N_{11}, N_{12}, N_{21}, N_{22})$. □

b) Give a minimal sufficient statistic if the success probabilities are constrained so that $p_{11}p_{22} = p_{12}p_{21}$.

Solution.
확실하지 않음...
From the constraint, plug in $p_{11} = (p_{12}p_{21})/p_{22}$ and denote C is the term which is not depend on parameters,

$$
\begin{aligned}
p(\mathbf{N}) &= C p_{11}^{N_{11}} p_{12}^{N_{12}} p_{21}^{N_{21}} p_{22}^{N_{22}} \\
&= C\left(\frac{p_{12}p_{21}}{p_{22}}\right)^{N_{11}} p_{12}^{N_{12}} p_{21}^{N_{21}} p_{22}^{N_{22}} \\
&= C p_{12}^{N_{11}+N_{12}} p_{21}^{N_{11}+N_{21}} p_{22}^{N_{22}-N_{11}} \\
&= C p_{12}^{N_{11}+N_{12}} p_{21}^{N_{11}+N_{21}} (1 - p_{11} - p_{12} - p_{21})^{n-N_{12}-N_{21}-2N_{11}} \\
&= C\exp\left[(N_{11} + N_{12})\log\left(\frac{p_{12}}{1 - p_{11} - p_{12} - p_{21}}\right)\right. \\
&\qquad \left. +(N_{11} + N_{21})\log\left(\frac{p_{21}}{1 - p_{11} - p_{12} - p_{21}}\right) + n\log(1 - p_{11} - p_{12} - p_{21})\right].
\end{aligned}
$$

Since it is the exponential family of full-rank, $T = (N_{11}+N_{12}, N_{11}+N_{21})$ is complete, and also sufficient by factorization theorem. Thus $T = (N_{11} + N_{12}, N_{11} + N_{21})$ is minimal sufficient. □

Problem 3.9

Let f be a positive integrable function on $(0, \infty)$. Define

$$c(\theta) = 1/\int_{\theta}^{\infty} f(x)dx,$$

and take $p_\theta(x) = c(\theta)f(x)$ for $x > \theta$, and $p_\theta(x) = 0$ for $x \le \theta$. Let $X_1, \dots, X_n$ be i.i.d. with common density p_θ.

a) Show that $M = \min\{X_1, \dots, X_n\}$ is sufficient.

Solution.
The joint density is

$$p_\theta(x_1, \dots, x_n) = c(\theta)^n \prod_i f(x_i)$$

$$= c(\theta)^n \prod_i f(x_i) \cdot 1_{\min\{x_1,\dots,x_n\}>\theta}.$$

By factorization theorem, $M = \min\{X_1, \dots, X_n\}$ is sufficient. □

b) Show that M is minimal sufficient.

Solution.
Suppose $p_\theta(x) \propto_\theta p_\theta(y)$. Then, the region of 0 value should be equal. Thus $T(x) = T(y)$ should be satisfied, therefore $T = M$ is minimal sufficient. □

Problem 3.10

Suppose $X_1, \dots, X_n$ are i.i.d. with common density $f_\theta(x) = (1 + \theta x)/2,\ |x| < 1;\ f_\theta(x) = 0$, otherwise, where $\theta \in [-1, 1]$ is an unknown parameter. Show that the order statistics are minimal sufficient. (Hint: A polynomial of degree n is uniquely determined by its value on a grid of $n + 1$ points.)

Solution.
The joint density is

$$\begin{aligned} f_\theta(x_1, \dots, x_n) &= 2^{-n}(1 + \theta x_1) \cdots (1 + \theta x_n) \\ &= 2^{-n}(1 + \theta x_{(1)}) \cdots (1 + \theta x_{(n)}) 1_{\{-1<x_{(1)}<\cdots<x_{(n)}<1\}}. \end{aligned}$$

By factorization theorem, the order statistic $T = (X_{(1)}, \dots, X_{(n)})$ is sufficient.
Now, suppose $p_\theta(x) \propto_\theta p_\theta(y)$. Since the joint density is the n-th order polynomials(다항식) with root(근) $-1/x_{(i)}$, the roots of $p_\theta(x)$ and $p_\theta(x)$ should be equal which implies $T(x) = T(y)$. Therefore, $T = (X_{(1)}, \dots, X_{(n)})$ is minimal sufficient. □

Problem 3.16

Let $X_1, \dots, X_n$ be a random sample from an absolutely continuous distribution with density

$$f_\theta(x) = \begin{cases} 2x/\theta^2, & x \in (0, \theta), \\ 0, & \text{otherwise.} \end{cases}$$

a) Find a one-dimensional sufficient statistic T.

Solution.
The joint density is factorized as follows:

$$f_\theta(x_1, \dots, x_n) = 2^n \frac{x_1 \cdots x_n}{\theta^{2n}} = \frac{2^n}{\theta^{2n}} x_{(1)} \cdots x_{(n)} 1_{x_{(1)}>0} 1_{x_{(n)}<\theta}.$$

By factorization theorem, $T = \max\{X_1, \dots, X_n\}$ is the sufficient statistic for θ. □

b) Determine the density of T.

Solution.
Since $T = \max\{X_1, \dots, X_n\}$,

$$P(T \le t) = \prod_{i=1}^{n} P(X_i \le t)$$

Note that $P(X_i \le t) = \int_0^t 2x/\theta^2 dx = t^2/\theta^2$.

$$= \frac{t^{2n}}{\theta^{2n}}$$

Taking a derivative with respect to t, the density of T is

$$p_\theta(t) = \frac{2n}{\theta^{2n}} t^{2n-1}, \quad t \in (0, \theta).$$

□

c) Show directly that T is complete.

Solution.
Suppose $E_\theta(T) = c, \ \forall\theta > 0$. Then,

$$E_\theta\left[f(T) - c\right] = \frac{2n}{\theta^{2n}} \int_0^\theta [f(t) - c] t^{2n-1} dt = 0$$
$$\Rightarrow \quad [f(t) - c] t^{2n-1} = 0 \quad \text{a.e. } 0 < t < \theta.$$
$$\Rightarrow \quad f(T) = c.$$

Therefore, T is complete.

Other solution

$$E_\theta f(T) = \frac{2n}{\theta^{2n}} \int_0^\theta f(t) t^{2n-1} dt = c$$
$$\Rightarrow \quad \int_0^\theta f(t) t^{2n-1} dt = c\theta^{2n}/2n, \quad \forall\theta > 0$$
$$\Rightarrow \quad f(\theta)\theta^{2n-1} = c\theta^{2n-1} \quad \text{a.e. } \theta$$
$$\Rightarrow \quad f(t) = c \quad \text{a.e. } t.$$

□

Problem 3.17

Let $X, X_1, X_2, \dots$ be i.i.d. from an exponential distribution with failure rate λ (introduced in Problem 1.30).

a) Find the density of $Y = \lambda X$.

Solution.
Note that the density of X is $p_\lambda(x) = \lambda e^{-\lambda x}, \ \lambda > 0$, and its cdf is easily obtained by integration, $P(X \le x) = 1 - e^{-\lambda x}$. Thus the cdf of Y is

$$P(Y \le y) = P(X \le y/\lambda) = 1 - e^{-y}, \ y > 0,$$

and the density is

$$p(y) = e^{-y},\ y > 0.$$

□

b) Let $\overline{X} = (X_1 + \cdots + X_n)/n$. Show that $\overline{X}$ and $(X_1^2 + \cdots + X_n^2)/\overline{X}^2$ are independent.

Solution.
The joint density of $X_1, \ldots, X_n$ is

$$p_\lambda(x_1, \ldots, x_n) = \lambda^n e^{-\lambda \sum_i x_i} = \lambda^n e^{-n\lambda\bar{x}},$$

is the full rank exponential family, and by factorization theorem, $T = \overline{X}$ is the complete sufficient statistic.
Now, let $Y = \lambda X$, then

$$V = \frac{(Y_1^2 + \cdots + Y_n^2)}{\overline{Y}^2} = \frac{(X_1^2 + \cdots + X_n^2)}{\overline{X}^2}.$$

Since Y does not depend on parameter λ, V is ancillary for P_λ.
$\therefore$ By Basu's theorem, T and V are independent. □

Problem 3.29

Find a function on $(0, \infty)$ that is bounded and strictly convex.

Solution.
$f(x) = e^{-x}$ is bounded on range $(0, 1)$ and strictly convex.
Or $f(x) = 1/(x + 1)$ is bounded on range $(0, 1)$ and strictly convex. □

Problem 3.30

Use convexity to show that the canonical parameter space Ξ of a one-parameter exponential family must be an interval. Specifically, show that if $\eta_0 < \eta < \eta_1$, and if η_0 and η_1 both lie in Ξ, then η must lie in Ξ.

Solution.
Note that if a random variable X is a canonical exponential family, then it has the form,

$$p_\eta(x) = \exp[\eta T(x) - A(\eta)]h(x), \quad \eta \in \Xi.$$

Since $\eta_0 < \eta < \eta_1$, we define $\eta = \gamma\eta_0 + (1-\gamma)\eta_1$ for some $\gamma \in (0, 1)$. Since the exponential function is convex,

$$\exp\left[(\gamma\eta_0 + (1-\gamma)\eta_1)T(x)\right] = e^{\eta T(x)} < \gamma e^{\eta_0 T(x)} + (1-\gamma)e^{\eta_1 T(x)}.$$

From the definition of exponential family, $h(x) \geq 0$, and for the positive function, the inequality sign does not change by integration against μ. Thus, following inequality is satisfied:

$$\int e^{\eta T(x)} h(x) d\mu(x) < \gamma \int e^{\eta_0 T(x)} h(x) d\mu(x) + (1-\gamma) \int e^{\eta_1 T(x)} h(x) d\mu(x).$$

Because it should be satisfied the exponential family, RHS is finite. ($\because$ RHS $= \gamma e^{A(\eta_0)} + (1-\gamma)e^{A(\eta_1)} < \infty$.)
Therefore, η must lie in Ξ. □

Problem 3.31

Let f and g be positive probability densities on $\mathbb{R}$. Use Jensen's inequality to show that

$$\int \log\left(\frac{f(x)}{g(x)}\right) f(x)dx > 0,$$

unless $f = g$ a.e. (If $f = g$, the integral equals zero.) This integral is called the *Kullback-Liebler information*.

Solution.
Suppose X be a absolutely continuous random variable with density f. Now, define $Y = \frac{g(X)}{f(X)}$, then

$$EY = \int \frac{g(x)}{f(x)} f(x)dx = \int g(x)dx = 1.$$

Next, let $h(y) = \log\frac{1}{y} = -\log y$, then h is strictly convex on $(0, \infty)$. Then, by Jensen's inequality,

$$Eh(Y) = \int \log\left(\frac{f(x)}{g(x)}\right) f(x)dx \geq h\left(EY\right) = \log 1 = 0.$$

The equality holds when Y is a constant. (then $Y = EY = 1 \Rightarrow f(x) = g(x)$ a.e.) $\square$

Chapter 4

Unbiased Estimation

Problem 4.1

Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_n$ be independent variables with the X_i a random sample from an exponential distribution with failure rate λ_x, and the Y_j a random sample from an exponential distribution with failure rate λ_y.

a) Determine the UMVU estimator of λ_x/λ_y.

Solution.
Note that if $X \sim \text{Exp}(\lambda)$ with rate parameter λ, the denstiy $p_\lambda(x) = \lambda e^{-\lambda x}$.
Since $X_i, \forall i$ and $Y_j, \forall j$ are independent, the joint density is

$$p_{\lambda_x,\lambda_y}(x,y) = \lambda_x^n \lambda_y^n e^{-\lambda_x \sum_i x_i - \lambda_y \sum_i y_i},$$

which is the 2-parameter full-rank exponential family. Thus, by factorization theorem, $T = (\sum_{i=1}^m X_i, \sum_{i=1}^n Y_i)$ is the complete sufficient.
Since $X_1, \dots, X_m$ are i.i.d. $\text{Exp}(\lambda_x) = \Gamma(1, \lambda_x)$, $T_1 = \sum_{i=1}^m X_i \sim \Gamma(m, \lambda_x)$. Then,

$$E\frac{1}{T_1} = \int \frac{1}{t} \frac{\lambda_x^m}{\Gamma(m)} t^{m-1} e^{-\lambda_x t} dt = \frac{\Gamma(m-1)\lambda_x^m}{\Gamma(m)\lambda_x^{m-1}} = \frac{\lambda_x}{m-1}.$$

Similarly, we can obtain $ET_2 = n/\lambda_y$. ($\because EY_i = 1/\lambda_y$)
From the fact of the expectation of independent random variables,

$$E\frac{T_2}{T_1} = E\frac{1}{T_1} \cdot ET_2 = \frac{n\lambda_x}{(m-1)\lambda_y}.$$

Therefore,

$$\delta(x,y) = \frac{(m-1)T_2}{nT_1}$$

is the unbiased estimator of λ_x/λ_y, and also UMVU since it is the function of complete sufficient statistic T. (By Theorem 4.4 with Rao-Blackwellization) □

b) Under squared error loss, find the best estimator of λ_x/λ_y of form $\delta = c\overline{Y}/\overline{X}$.

Solution.
Let $\delta = c\overline{Y}/\overline{X} = dT_2/T_1$. Then, the risk of $\delta = c\overline{Y}/\overline{X}$ and λ_x/λ_y is

$$R\left(\frac{\lambda_y}{\lambda_x}, \delta\right) = E\left[d\frac{T_2}{T_1} - \frac{\lambda_x}{\lambda_y}\right]^2 = d^2 E\frac{T_2^2}{T_1^2} - 2\frac{\lambda_x}{\lambda_y} dE\frac{T_2}{T_1} + \frac{\lambda_x^2}{\lambda_y^2}. \tag{4.1}$$

Since $T_1 \sim \Gamma(m, \lambda_x)$ and $T_2 \sim \Gamma(n, \lambda_y)$,

$$ET_2^2 = \mathrm{Var}(T_2) + \{ET_2\}^2 = n/\lambda_y^2 + n^2/\lambda_y^2,$$

$$E\frac{1}{T_1^2} = \int \frac{1}{t^2}\frac{\lambda_x^m}{\Gamma(m)} t^{m-1} e^{-\lambda_x t} dt = \frac{\Gamma(m-2)\lambda_x^2}{\Gamma(m)} = \frac{\lambda_x^2}{(m-1)(m-2)}.$$

Thus, using the fact that T_1 and T_2 are independent,

$$E\frac{T_2^2}{T_1^2} = E\frac{1}{T_1^2} \cdot ET_2^2 = \frac{n(n+1)\lambda_x^2}{(m-1)(m-2)\lambda_y^2}.$$

By using the result from a), we know $ET_2/T_1 = \big(n\lambda_x\big)/\big((m-1)\lambda_y\big)$, then the risk (4.1) be

$$\begin{aligned}
R\left(\frac{\lambda_y}{\lambda_x}, \delta\right) &= d^2 E\frac{T_2^2}{T_1^2} - 2\frac{\lambda_x}{\lambda_y} dE\frac{T_2}{T_1} + \frac{\lambda_x^2}{\lambda_y^2} \\
&= d^2 \frac{n(n+1)\lambda_x^2}{(m-1)(m-2)\lambda_y^2} - 2\frac{\lambda_x}{\lambda_y} d\frac{n\lambda_x}{(m-1)\lambda_y} + \frac{\lambda_x^2}{\lambda_y^2} \\
&= \frac{\lambda_x^2}{\lambda_y^2}\left(\frac{n(n+1)}{(m-1)(m-2)} d^2 - 2\frac{n}{(m-1)} d + 1\right) \\
&= \frac{\lambda_x^2}{\lambda_y^2}\frac{n(n+1)}{(m-1)(m-2)}\left[\left(d - \frac{m-2}{n+1}\right)^2 - \left(\frac{m-2}{n+1}\right)^2 + 1\right].
\end{aligned}$$

Thus, it has the minimum when $d = \frac{m-1}{n+1}$.
Therefore, the best estimator of λ_x/λ_y is $c\overline{Y}/\overline{X} = dT_2/T_1 = \big[(m-2)n\overline{Y}\big] / \big[(n+1)m\overline{X}\big]$.

$$\therefore c = \frac{(m-2)n}{(n+1)m}.$$

□

c) Find the UMVU estimator of $e^{-\lambda_x} = P(X_1 > 1)$.

Solution.
Let δ is the unbiased estimator of $P(X_1 > 1)$. Then,

$$E\delta = P(X_1 > 1) \Rightarrow \delta = 1_{\{X_1 > 1\}}.$$

From the previous steps, we know T_1 is C.S.S., thus $\eta(T_1) = E(\delta|T_1) = P(X_1 > 1|T_1)$ is UMVU by Theorem 4.4. We follow by 2 steps.

(i) Find the joint density of X and T_1, where $X = 1_{\{X_1>1\}}$ and $T_1 = \sum_{i=1}^m X_i$.
First, we should find the joint density of X and S where $S = \sum_{i=2}^m X_i$. Since X and S are independent, the joint density is easily obtained as

$$f(x, s) = f(x)f(s) = \lambda_x e^{-\lambda_x x}\frac{\lambda_x^{m-1}}{\Gamma(m-1)} s^{m-2} e^{-\lambda_x s} = \frac{\lambda_x^m}{\Gamma(m-1)} s^{m-2} e^{-\lambda_x (x+2)}, \quad x > 0, s > 0.$$

By transformation of variables $X = X, T_1 = S + X$, the jacobian $|J| = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$. Then, the joint density of X and T_1 is

$$f(x,t) = \frac{\lambda_x^m}{\Gamma(m-2)}(t-x)^{m-2}e^{-\lambda_x t}, \quad 0 < x < t.$$

(ii) Derive $E(\delta|T_1)$.
To obtain the expectation, we derive the conditional density of X given T_1. Since $T_1 \sim \Gamma(m, \lambda_x)$, the conditional density be

$$f(x|t) = \frac{f(x,t)}{f(t)} = \frac{\Gamma(m)}{\Gamma(m-2)}(t-x)^{m-2}t^{-(m-1)} = (m-1)\frac{1}{t}\left(1-\frac{x}{t}\right)^{m-2}, \quad 0 < x < t.$$

Then,

$$\begin{aligned} E(\delta|T_1) &= P(X_1 > 1|T_1) = P(X|T_1) \\ &= \int_1^t f(x|t)dx \\ &= \int_1^t (m-1)\frac{1}{t}\left(1-\frac{x}{t}\right)^{m-2} dx \\ &= -\left(1-\frac{x}{t}\right)^{m-1}\Bigg|_1^t \\ &= \left(1-\frac{1}{t}\right)^{m-1} 1_{\{T_1>1\}}. \end{aligned}$$

□

Problem 4.2

Let $X_1, \ldots, X_n$ be a random sample from $N(\mu_x, \sigma^2)$, and let $Y_1, \ldots, Y_m$ be an independent random sample from $N(\mu_y, 2\sigma^2)$, with μ_x, μ_y, and σ^2 all unknown parameters..

a) Find a complete sufficient statisic.

<u>Solution</u>.
Since $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ are mutually independent, the joint density,

$$\begin{aligned} f(x,y) &= f(x)f(y) \\ &= (2\pi\sigma^2)^{-n/2}(4\pi\sigma^2)^{-m/2}\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_x)^2 - \frac{1}{4\sigma^2}\sum_{j=1}^m (y_j-\mu_y)^2\right] \\ &= \exp\left[-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2} - \frac{\sum_{j=1}^m y_j^2}{4\sigma^2} + \frac{\mu_x\sum_i x_i}{\sigma^2} + \frac{\mu_y\sum_j y_j}{2\sigma^2} - \frac{n\mu_x^2}{2\sigma^2} - \frac{m\mu_y^2}{4\sigma^2}\right](2\pi\sigma^2)^{-n/2}(4\pi\sigma^2)^{-m/2} \\ &= \exp\left[\frac{\mu_x\sum_i x_i}{\sigma^2} + \frac{\mu_y\sum_j y_j}{2\sigma^2} - \frac{2\sum_{i=1}^n x_i^2 + \sum_{j=1}^m y_j^2}{4\sigma^2} - \frac{n\mu_x^2}{2\sigma^2} - \frac{m\mu_y^2}{4\sigma^2}\right](2\pi\sigma^2)^{-n/2}(4\pi\sigma^2)^{-m/2}, \end{aligned}$$

is the full-rank exponential family. Therefore,

$$T = \left(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j, 2\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2\right)$$

is the complete sufficient statistic for (μ_x, μ_y, σ^2). □

b) Determine the UMVU estimator of σ^2.
Hint: Find a linear combination L of $S_x^2 = \sum_{i=1}^n (X_i - \overline{X})^2/(n-1)$ and $S_y^2 = \sum_{j=1}^m (Y_j - \overline{Y})^2/(m-1)$ so that $(\overline{X}, \overline{Y}, L)$ is complete sufficient.

Solution.
To use the C.S.S. obtained from a), let

$$2(n-1)S_x^2 + (m-1)S_y^2 = 2\left(\sum_i X_i^2 - n\overline{X}^2\right) + \left(\sum_j Y_j^2 - m\overline{Y}^2\right)$$

is the linear combination of T. Since the sample variance is unbiased, then its expectation is

$$E\left[2(n-1)S_x^2 + (m-1)S_y^2\right] = (2n + 2m - 4)\sigma^2.$$

Now, we let

$$S_p^2 = \frac{1}{2n+2m-4}\left[2(n-1)S_x^2 + (m-1)S_y^2\right],$$

then it is unbiased for σ^2 based on complete sufficient T.
By theorem 4.4, it is UMVU of σ^2. □

c) Find a UMVU estimator of $(\mu_x - \mu_y)^2$.

Solution.

$$\begin{aligned} E(\overline{X} - \overline{Y})^2 &= \mathrm{Var}(\overline{X} - \overline{Y}) + \left[E(\overline{X} - \overline{Y})\right]^2 \\ &= \mathrm{Var}(\overline{X}) + \mathrm{Var}(\overline{Y}) + \left[E\overline{X} - E\overline{Y}\right]^2 \\ &= \frac{\sigma^2}{n} + \frac{2\sigma^2}{m} + (\mu_x - \mu_y)^2. \end{aligned}$$

We obtained the unbiased estimator of σ^2 in b). Thus, we let δ as

$$\delta = (\overline{X} - \overline{Y})^2 - S_p^2\left(\frac{1}{n} - \frac{2}{m}\right),$$

then it is the unbiased estimator of $(\mu_x - \mu_y)^2$ based on complete sufficient T.
By Theorem 4.4, δ is UMVU of $(\mu_x - \mu_y)^2$. □

d) Suppose we know the $\mu_y = 3\mu_x$. What is the UMVU estimator of μ_x?

Solution.
With restriction $\mu_y = 3\mu_x$, the joint density of exponential family is changed as

$$f(x, y) = \cdots = \exp\left[\frac{\mu_x(2\sum_i x_i + 3\sum_j y_j)}{2\sigma^2} - \cdots\right]\cdots.$$

Thus, $T = \left(2\sum_i X_i + 3\sum_j Y_j,\ 2\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2\right)$ is complete sufficient.
Take expectation for T_1, then

$$ET_1 = 2n\mu_x + 3m\mu_y = (2n + 9m)\mu_x,$$

Now we let δ as

$$\delta = \frac{2\sum_i x_i + 3\sum_j y_j}{2n + 9m},$$

then, it is unbiased estimator based on T.
By Theorem 4.4, δ is UMVU of μ_x. □

Problem 4.3

Let $X_1, \ldots, X_n$ be a random sample from the Poisson distribution with mean λ. Find the UMVU for $\cos\lambda$. (Hint: For Taylor expansion, the identity $\cos\lambda = (e^{i\lambda} + e^{-i\lambda})/2$ may be useful.)

Solution.
The joint density is

$$p_\lambda(x) = \frac{\lambda^{\sum_i x_i} e^{-n\lambda}}{x_1! \cdots x_n!},$$

with $T = \sum_i X_i$, that is complete sufficient. ($\because$ full-rank exponential family, and by factorization theorem)
Note that $T \sim Poisson(n\lambda)$. Let $\delta(t)$ is an unbiased estimator of $\cos\lambda$ based on T. Then, its expectation should satisfy follows:

$$E\delta(T) = \sum_{t=0}^{\infty} \delta(t) \frac{(n\lambda)^t e^{-n\lambda}}{t!} = \cos\lambda$$

$$\Leftrightarrow \sum_{t=0}^{\infty} \delta(t) \frac{(n\lambda)^t}{t!} = e^{n\lambda}\cos\lambda$$

Apply Talyor expansion for RHS,

$$\begin{aligned}
e^{n\lambda}\cos\lambda &= e^{n\lambda}\frac{1}{2}\left(e^{i\lambda} + e^{-i\lambda}\right) \\
&= \frac{1}{2}\left(e^{(n+i)\lambda} + e^{(n-i)\lambda}\right) \\
&= \frac{1}{2}\sum_{t=0}^{\infty} \frac{\left((n+i)\lambda\right)^t + \left((n-i)\lambda\right)^t}{t!} \\
&= \sum_{t=0}^{\infty} \frac{(n\lambda)^t}{t!}\left[\frac{\left(1+\frac{i}{n}\right)^t + \left(1-\frac{i}{n}\right)^t}{2}\right].
\end{aligned}$$

Therefore,

$$\delta(t) = \frac{1}{2}\left\{\left(1+\frac{i}{n}\right)^t + \left(1-\frac{i}{n}\right)^t\right\}$$

is unbiased estimator of $\cos\lambda$ based on T.
By Theorem 4.4, $\delta(t)$ is UMVU of $\cos\lambda$. □

Problem 4.4

Let $X_1, \ldots, X_n$ be independent normal variables, each with unit variance, and with $EX_i = \alpha t_i + \beta t_i^2,\ i = 1, \ldots, n$, where α and β are unknown parameters and $t_1, \ldots, t_n$ are known constants. Find UMVU estimators of α and β.

***Solution*.**
Since $X_i \sim N(\alpha t_i + \beta t_i^2, 1)$, the joint density is

$$\begin{aligned} f(x) &= (2\pi)^{-\frac{n}{2}} \exp\left[-\frac{1}{2}\sum_i (x_i - \alpha t_i - \beta t_i^2)^2\right] \\ &= (2\pi)^{-\frac{n}{2}} \exp\left[-\frac{1}{2}\sum_i \left(x_i^2 + \alpha^2 t_i^2 + \beta^2 t_i^4 - 2\alpha x_i t_i - 2\beta x_i t_i^2 + \alpha\beta t_i^3\right)\right]. \end{aligned}$$

Since these density is full-rank exponential family,

$$T = \left(\sum_i X_i t_i,\ \sum_i X_i t_i^2\right)$$

is complete sufficient and its expected value is

$$ET_1 = \alpha \sum t_i^2 + \beta \sum t_i^3, \quad ET_2 = \alpha \sum t_i^3 + \beta \sum t_i^4.$$

By using this,

$$\widehat{\alpha} = \frac{T_1 \sum t_i^4 - T_2 \sum t_i^3}{\sum t_i^2 \sum t_i^4 - \left(\sum t_i^3\right)^2}, \quad \widehat{\beta} = \frac{T_1 \sum t_i^3 - T_2 \sum t_i^2}{\left(\sum t_i^3\right)^2 - \sum t_i^4 \sum t_i^2}$$

are unbiased estimator of α and β. Since these are the function of T, also UMVU. □

Problem 4.5

Let $X_1, \dots, X_n$ be i.i.d. from some distribution Q_θ, and let $\overline{X} = (X_1 + \cdots + X_n)/n$ be the sample average.

a) Show that $S^2 = \sum(X_i - \overline{X})^2/(n-1)$ is unbiased for $\sigma^2 = \sigma^2(\theta) = \mathrm{Var}_\theta(X_i)$.

***Solution*.**
Denote $\mu = EX_i$. Also note that $(n-1)S^2 = \sum_i X_i^2 - n\overline{X}^2$. Then,

$$\begin{aligned} E\left((n-1)S^2\right) &= \sum_i EX_i^2 - nE\overline{X}^2 \\ &= \sum_i \left[\mathrm{Var}X_i + (EX_i)^2\right] - n\left[\mathrm{Var}\overline{X} + (E\overline{X})^2\right] \\ &= n(\sigma^2 + \mu^2) - n(\sigma^2/n + \mu^2) \\ &= (n-1)\sigma^2. \end{aligned}$$

Therefore S^2 is unbiased for σ^2. □

b) If Q_θ is the Bernoulli distribution with success probability θ, show that S^2 from (a) is UMVU.

***Solution*.**
Note that the density of X_i is $\theta^{x_i}(1-\theta)^{1-x_i} = \left(\frac{\theta}{1-\theta}\right)^x (1-\theta)$.
The joint density,

$$p(x) = \exp\left[\sum_i x_i \log\frac{\theta}{1-\theta} + n\log(1-\theta)\right],$$

is the full-rank exponential family. Thus, $T = \overline{X}$ (or $\sum_i X_i$) is complete sufficient.
Meanwhile,

$$\begin{aligned}
S^2 &= \frac{1}{n-1}\left(\sum X_i^2 - n\overline{X}^2\right) \\
&= \frac{1}{n-1}\left(\sum X_i - n\overline{X}^2\right) \quad (\because X_i^2 = X_i) \\
&= \frac{n\overline{X}(1-\overline{X})}{n-1}
\end{aligned}$$

is the function of $T = \overline{X}$. Therefore, S^2 is UMVU of σ^2 by Theorem 4.4. □

c) If Q_θ is the exponential distribution with failure rate θ, find the UMVU estimator of $\sigma^2 = 1/\theta^2$. Give a formula for $E_\theta[X_i^2|\overline{X} = c]$ in this case.

Solution.
Note that the density of X_i is $\theta e^{-\theta x}$ with mean $1/\theta$ and variance $1/\theta^2$.
The joint density,

$$p(x) = \exp\left[-\theta \sum x_i + n \log \theta\right],$$

is the full-rank exponential family. Thus, $T = \overline{X}$ (or $\sum_i X_i$) is complete sufficient with $ET = 1/\theta$ and $\mathrm{Var}T = 1/(n\theta^2)$.
Meanwhile,

$$E\overline{X}^2 = \mathrm{Var}\overline{X} + (E\overline{X})^2 = \frac{1}{n\theta^2} + \frac{1}{\theta^2} = \frac{n+1}{n}\frac{1}{\theta^2} = \frac{n+1}{n}\sigma^2.$$

Thus, $\delta = \frac{n}{n+1}\overline{X}^2$ is the unbiased estimator of σ^2, and also function of $T = \overline{X}$.
By Theorem 4.4, δ is UMVU for σ^2.

Next, since X_i's are i.i.d., $E_\theta[X_1^2|\overline{X} = c] = \cdots = E_\theta[X_n^2|\overline{X} = c]$. Then,

$$\begin{aligned}
E_\theta[S^2|\overline{X} = c] &= \frac{nc^2}{n+1} \\
&= E_\theta\left[\frac{\sum_i X_i^2 - n\overline{X}^2}{n-1}\middle|\overline{X} = c\right] \\
&= E_\theta\left[\frac{\sum_i X_i^2}{n-1} - \frac{nc^2}{n-1}\middle|\overline{X} = c\right] \\
&= \frac{n}{n-1}E_\theta[X_1^2|\overline{X} = c] - \frac{nc^2}{n-1}.
\end{aligned}$$

$$\therefore E_\theta[X_i^2|\overline{X} = c] = nc^2\left(\frac{1}{n+1} + \frac{1}{n-1}\right)\frac{n-1}{n} = c^2\left(\frac{n-1}{n+1} + 1\right) = \frac{2nc^2}{n+1}.$$

□

Problem 4.6

Suppose δ is a UMVU estimator of $g(\theta)$; U is an unbiased estimator of zero, $E_\theta U = 0,\ \theta \in \Omega$; and that δ and U both have finite variances for all $\theta \in \Omega$. Show that U and δ are uncorrelated, $E_\theta U\delta = 0,\ \theta \in \Omega$.

Solution.
For the convenience, we drop θ in the notation of E, Var and Cov.
Let $\delta^* = \delta + cU$ where c is a constant. Then, δ^* is an unbiased estimator for $g(\theta)$ ($\because$ $E\delta^* = E(\delta + cU) = E\delta + cEU = g(\theta)$).
Since δ is UMVU,

$$\text{Var}(\delta + cU) = \text{Var}(\delta) + c^2\text{Var}(U) + 2c\text{Cov}(\delta, U) \geq \text{Var}(\delta),$$

$$\Rightarrow c^2\text{Var}(U) + 2c\text{Cov}(\delta, U) \geq 0.$$

Denote LHS as $h(c)$. Since h is convex and has minimum $h(0) = 0$, h should have a local minimum at $c = 0$. Since h is differentiable, we can obtain the following result by taking a first derivative at $c = 0$:

$$h'(0) = 2\text{Cov}(\delta, U) = 0.$$

Since $EU = 0$, we finally obtain $E_\theta U\delta = 0$ for all $\theta \in \Omega$. □

Problem 4.7

Suppose δ_1 is a UMVU estimator of $g_1(\theta)$, δ_2 is UMVU estimator of $g_2(\theta)$, and that δ_1 and δ_2 both have finite variance for all θ. Show that $\delta_1 + \delta_2$ is UMVU for $g_1(\theta) + g_2(\theta)$.
Hint: Use the result in the previous problem.

Solution.
We can easily show that $\delta_1 + \delta_2$ is unbiased for $g_1(\theta) + g_2(\theta)$.
Now, suppose δ is any unbiased estimator of $g_1(\theta) + g_2(\theta)$ with $\text{Var}(\delta) < \infty$. Let $U = \delta - \delta_1 - \delta_2$, then U is an unbiased estimator of 0 ($\because$ $EU = E\delta - E\delta_1 - E\delta_2 = 0$).
By Problem 4.6,

$$\text{Cov}(U, \delta_1 + \delta_2) = \text{Cov}(U, \delta_1) + \text{Cov}(U, \delta_2) = 0.$$

Since $\delta = U + \delta_1 + \delta_2$, we obtain the following result:

$$\begin{aligned}\text{Var}(\delta) &= \text{Var}(U) + \text{Var}(\delta_1 + \delta_2) \\ &\geq \text{Var}(\delta_1 + \delta_2).\end{aligned}$$

Because δ is arbitrary, $\delta_1 + \delta_2$ is UMVU. □

Problem 4.8

Let $X_1, \ldots, X_n$ be i.i.d. absolutely continuous variables with common density f_θ, $\theta > 0$, given by

$$f_\theta(x) = \begin{cases} \theta/x^2, & x > \theta, \\ 0, & x \leq \theta. \end{cases}$$

Find the UMVU estimator for $g(\theta)$ if $g(\theta)/\theta^n \to 0$ as $\theta \to \infty$ and g is differentiable.

Solution.
The joint density is

$$\theta^n/(x_1 \cdots x_n)^2 1_{\{\min x_i\}}.$$

By factorization theorem, $T = \min X_i$ is sufficient.
To obtain the density of T, the cumulative distribution function is

$$P(T \le t) = 1 - P(\min X_i > t) = 1 - \prod_i P(X_i > t) = 1 - \left(\frac{\theta}{t}\right)^n, \quad t > \theta.$$

Take a derivative, we can obtain the density,

$$f(t) = \frac{n\theta^n}{t^{n+1}}, \quad t > \theta.$$

Now, suppose $\delta(T)$ is the unbiased estimator of $g(\theta)$, then

$$\begin{aligned}
\int_\theta^\infty \delta(t)\frac{n\theta^n}{t^{n+1}}dt = g(\theta) &\iff n\int_\theta^\infty \delta(t)t^{-n-1}dt = \frac{g(\theta)}{\theta^n} \\
&\quad \text{Taking a derivative w.r.t. } \theta, \\
&\iff -n\delta(t)t^{-n-1} = g'(t)t^{-n} - ng(t)t^{-n-1}. \\
&\therefore\ \delta(t) = g(t) - \frac{g'(t)}{nt}.
\end{aligned}$$

From the above result, $g(t) = c$ for any constant c implies $\delta(t) = c$. Therefore T is complete.
In summary, T is complete sufficient, and $\delta(T)$ is the function of T. Therefore, by Theorem 4.4, $\delta(T)$ is UMVU. □

Problem 4.10

Suppose X is an exponential variable with density $p_\theta(x) = \theta e^{-\theta x},\ x > 0;\ p_\theta(x) = 0$, otherwise. Find the UMVU estimator for $1/(1+\theta)$.

Solution.
Since X is a full-rank exponential family, $T = X$ is complete sufficient.
Let $\delta(X)$ be an unbiased estimator of $1/(1+\theta)$, and represent as $\delta(x) = \sum_{n=0}^\infty c_n x^n$ by power series. Then,

$$\begin{aligned}
E\delta(X) &= \int_0^\infty \left(\sum_{n=0}^\infty c_n x^n\right)\theta e^{-\theta x}dx \\
&\quad \text{By Fubini theorem,} \\
&= \sum_{n=0}^\infty \int_0^\infty c_n\theta x^n e^{-\theta x}dx \\
&\quad \text{The inner integration follows } \Gamma(n+1, 1/\theta) \text{ with rate parameter.} \\
&= \sum_{n=0}^\infty \frac{c_n\Gamma(n+1)}{\theta^n} \\
&= \sum_{n=0}^\infty \frac{c_n n!}{\theta^n}. \qquad (*)
\end{aligned}$$

Meanwhile, $1/(1+\theta)$ can be written as the geometric series,

$$\frac{1}{1+\theta} = \frac{1/\theta}{1+1/\theta} = -\sum_{n=1}^\infty \left(-\frac{1}{\theta}\right)^n = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{\theta^n} \quad \text{for } \theta > 1. \qquad (**)$$

Since δ is unbiased, $(*)$ and $(**)$ should be identical. Thus,

$$c_0 = 0,\ c_n = \frac{(-1)^{n+1}}{n!},\ n = 1, 2, \ldots.$$

Therefore,

$$\delta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} x^n = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n!} = 1 - e^{-x} \quad \text{for } \theta > 1.$$

Since δ is the function of the complete sufficient statistic X, $\delta(X)$ is UMVU by Theorem 4.4.
If $\theta \leq 1$, direct하게 unbiased임을 보일 수 있다?? 어떻게?? □

Problem 4.11

Let $X_1, \ldots, X_3$ be i.i.d. geometric variables with common mass function $f_\theta(x) = P_\theta(X_i = x) = \theta(1-\theta)^x,\ x = 0, 1, \ldots$. Find the UMVU estimator of θ^2.

Solution.
Since the joint mass, $\theta^3(1-\theta)^{x_1+x_2+x_3}$, is the full-rank exponential family, $T = X_1 + X_2 + X_3$ is complete sufficient.
Denote $g(\theta) = \theta^2$, and let $\delta(X) = 1_{\{X_1=0, X_2=0\}}$, then $\delta(X)$ be an unbiased estimator of $g(\theta)$ ($\because\ E\delta(X) = P(X_1 = 0, X_2 = 0) = \theta^2$).
Define $\eta(T) = E[\delta(X)|T] = P(X_1 = 0, X_2 = 0|T)$. Then it is UMVU by Theorem 4.4. To obtain $\eta(t)$, we first find the density of T by using the transformation $T = X_1 + X_2 + X_3$ from the joint density,

$$P(T = t) = \binom{t+2}{2} \theta^3 (1-\theta)^t = \binom{t+2}{2} \theta^2 \left[\theta(1-\theta)^t\right].$$

Because we already have 2 successes in first $t+2$ trials of total $t+3$ trials, we should add the combination term. (See RHS!!)
Next, the joint mass of X_1, X_2, T is

$$P(X_1 = 0, X_2 = 0, T = t) = P(X_1 = 0, X_2 = 0, X_3 = t) = \theta^3(1-\theta)^t,$$

from the joint mass of X_1, X_2, X_3.
Therefore,

$$\eta(t) = \frac{P(X_1 = 0, X_2 = 0, T = t)}{P(T = t)} = 1 \Big/ \binom{t+2}{2} = \frac{2}{(t+2)(t+1)},$$

$$\therefore\ \eta(T) = \frac{2}{(T+2)(T+1)}$$

is the UMVU estimator of θ^2. □

Problem 4.12

Let X be a single observation, absolutely continuous with density

$$p_\theta(x) = \begin{cases} \frac{1}{2}(1 + \theta x), & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

Here $\theta \in [-1, 1]$ is an unknown parameter.

a) Find a constant a so that aX is unbiased for θ. The expected value is

***Solution*.**

$$EX = \int_{-1}^{1} \frac{x}{2}(1+\theta x)dx = \frac{1}{4}x^2 + \frac{\theta}{6}x^3 \Big|_{-1}^{1} = \frac{\theta}{3}.$$

Therefore, when $a = 3$, aX be unbiased. □

b) Show that $b = E_\theta|X|$ is independent of θ.

***Solution*.**
By using the procedure in a),

$$E|X| = \int_{-1}^{1} \frac{|x|}{2}(1+\theta x)dx = \frac{1}{2}.$$

Thus, $b = E_\theta|X|$ does not depend on θ. □

c) Let θ_0 be a fixed parameter value in $[-1,1]$. Determine the constant $c = c_{\theta_0}$ that minimizes the variance of the unbiased estimator $aX + c\big(|X| - b\big)$ when $\theta = \theta_0$. Is aX uniformly minimum variance unbiased?

***Solution*.**
To obtain the variance of $aX + c\big(|X| - b\big)$, we first derive the second moments.

$$EX^2 = E|X|^2 = \int_{-1}^{1} \frac{x^2}{2}(1+\theta x)dx = \frac{x^3}{6} + \frac{\theta x^4}{8}\Big|_{-1}^{1} = \frac{1}{3},$$

$$EX|X| = \int_{-1}^{1} \frac{x|x|}{2}(1+\theta x)dx = \frac{\theta}{4}.$$

From the previous steps, we obtain $a = 3$ and $b = E|X|$. Under $\theta = \theta_0$, the variance be

$$\begin{aligned}
\mathrm{Var}_{\theta_0}\Big(3X + c\big(|X| - E|X|\big)\Big) &= 9\mathrm{Var}_{\theta_0}(X) + 6c\mathrm{Cov}_{\theta_0}\big(X, |X| - E|X|\big) + c^2\mathrm{Var}_{\theta_0}\big(|X| - E|X|\big) \\
&= 9\mathrm{Var}_{\theta_0}(X) + 6c\mathrm{Cov}_{\theta_0}\big(X, |X| - E|X|\big) + c^2\mathrm{Var}_{\theta_0}\big(|X|\big) \\
&= 9\left(\frac{1}{3} - \frac{\theta_0^2}{9}\right) + 6c\left(\frac{\theta_0}{4} - \frac{1}{2}\cdot\frac{\theta_0}{3}\right) + c^2\left(\frac{1}{3} - \frac{1}{4}\right) \\
&= 3 - \theta_0^2 + \frac{\theta_0}{2}c + \frac{1}{12}c^2. \qquad (*)
\end{aligned}$$

Since $(*)$ is convex with respect to c, take a derivative,

$$\frac{\theta_0}{2} + \frac{c}{6} = 0$$

Therefore, $(*)$ has minimum when $c = -3\theta_0$.
Meanwhile, $\mathrm{Var}_{\theta_0}(3X) = 3 - \theta_0^2$ which is greater than $(*)$. Therefore, $3X$ is not UMVU. □

Problem 4.24

In the normal one-sample problem, the statistic $t = \sqrt{n}\overline{X}/S$ has the non-central t-distribution on $n-1$ degrees of freedom and non-centrality parameter $\delta = \sqrt{n}\mu/\sigma$. Use our results on distribution theory for the one-sample problem to find the mean and variance of t.

Solution.
Note that $E\overline{X} = \mu = \sigma\delta/\sqrt{n}$, $\operatorname{Var}(\overline{X}) = \sigma^2/n$, and the formula ES^r in (4.10) page 70.
Since $\overline{X}$ and S^2 are independent,

$$Et = E\left[\sqrt{n}\frac{\overline{X}}{S}\right] = \sqrt{n}(E\overline{X})(ES^{-1}) = \sqrt{n}\frac{\sigma\delta}{\sqrt{n}}\frac{\sigma^{-1}2^{-\frac{1}{2}}\Gamma\big[(-1+n-1)/2\big]}{(n-1)^{-\frac{1}{2}}\Gamma\big[(n-1)/2\big]} = \delta\frac{\sqrt{n-1}\,\Gamma\big[(n-2)/2\big]}{\sqrt{2}\,\Gamma\big[(n-1)/2\big]},$$

$$\begin{aligned}Et^2 &= n(E\overline{X}^2)(ES^2) = n\left(\frac{\sigma^2}{n} + \frac{\sigma^2\delta^2}{n}\right)\frac{\sigma^{-2}2^{-1}\,\Gamma\big[(-2+n-1)/2\big]}{(n-1)^{-1}\,\Gamma\big[(n-1)/2\big]}\\ &= (1+\delta^2)\frac{n-1}{2}\frac{\Gamma\big[(n-3)/2\big]}{\Gamma\big[(n-3)/2+1\big]} = (1+\delta^2)\frac{n-1}{n-3}.\end{aligned}$$

$$\operatorname{Var}(t) = Et^2 - (Et)^2 = \frac{n-1}{n-3} + \delta^2\left[\frac{n-1}{n-3} - \frac{(n-1)\left(\Gamma\big[(n-2)/2\big]\right)^2}{2\left(\Gamma\big[(n-1)/2\big]\right)^2}\right].$$

□

Problem 4.28

Let $X_1, \ldots, X_n$ be i.i.d. from the uniform distribution on $(0, \theta)$.

a) Use the Hammersley-Chapman-Robbins inequality to find a lower bound for the variance of an unbiased estimator of θ. This bound will depend on Δ. Note that Δ cannot vary freely but must lie in a suitable set.

   Solution.
   Since δ is the unbiased estimator of θ, $g(\theta) = \theta$ and $g(\theta+\Delta) - g(\theta) = \Delta$. By Hammersley-Chapman-Robbins inequality, the lower bound for the variance of δ be

   $$LB = \frac{\Delta^2}{E_\theta\left[\frac{p_{\theta+\Delta}(X)}{p_\theta(X)} - 1\right]^2}.$$

   Since $X_i \sim Uniform(0,\theta)$, $p_\theta(X) = 0 \Rightarrow p_{\theta+\Delta}(X) = 0$. Thus Δ should be negative and $\theta + \Delta > 0$, therefore $\Delta \in (-\theta, 0)$.

   $$\frac{p_{\theta+\Delta}(X)}{p_\theta(X)} = \begin{cases} \frac{\theta^n}{(\theta+\Delta)^n}, & \max\{X_1,\ldots,X_n\} < \theta+\Delta \text{ with probability } \frac{(\theta+\Delta)^n}{\theta^n} \text{ under } P_\theta,\\ 0, & \text{otherwise.}\end{cases}$$

   For simplicity, denote $Y = p_{\theta+\Delta}(X)/p_\theta(X)$, then $EY = 1$ and $EY^2 = \frac{\theta^n}{(\theta+\Delta)^n}$. Thus, the expectation in the denominator is

   $$E(Y-1)^2 = Var(Y) = EY^2 - [EY]^2 = \frac{\theta^n}{(\theta+\Delta)^n} - 1.$$

Therefore, the lower bound be

$$LB = \frac{\Delta^2}{\frac{\theta^n}{(\theta+\Delta)^n} - 1}. \tag{$*$}$$

□

b) In principle, the best lower bound can be found taking the supremum over Δ. This calculation cannot be done explicitly, but an approximation is possible. Suppose $\Delta = -c\theta/n$. Show that the lower bound for the variance can be written as $\theta^2 g_n(c)/n^2$. Determine $g(c) = \lim_{n\to\infty} g_n(c)$.

Solution.
Plug in $\Delta = -c\theta/n$ to $(*)$, then

$$LB = \frac{c^2\theta^2/n^2}{\left(1-\frac{c}{n}\right)^{-n} - 1}.$$

Therefore,

$$g_n(c) = \frac{c^2}{\left(1-\frac{c}{n}\right)^{-n} - 1},$$

$$g(c) = \lim_{n\to\infty} g_n(c) = \frac{c^2}{e^c - 1}.$$

□

c) Find the value c_0 that maximizes $g(c)$ over $c \in (0,1)$ and give an approximate lower bound for the variance of δ. (The value c_0 cannot be found explicitly, but you should be able to come up with a numerical value.)

Solution.
Taking a derivative,

$$\frac{\partial g(c)}{\partial c} = \frac{\partial}{\partial c}\left(\frac{c^2}{e^c-1}\right) = \frac{2c(e^c-1) - c^2e^c}{(e^c-1)^2} = 0$$

$$\begin{aligned}
&\iff 2c(e^c-1) = c^2e^c \\
&\iff 2e^c - 2 = ce^c \\
&\iff 1 - e^c = \frac{c}{2} \\
&\iff e^{-c} = 1 - \frac{c}{2}.
\end{aligned}$$

By numerical method, $c_0 = 1.59362$. Therefore, an approximated lower bound of $g(c_0)\theta^2/n^2 = 0.64761\theta^2/n^2$.

□

Problem 4.30

Suppose $X_1, \ldots, X_n$ are independent with $X_i \sim N(\alpha + \beta t_i, 1),\ i = 1, \ldots, n$, where $t_1, \ldots, t_n$ are known constants and α, β are unknown parameters.

a) Find the Fisher information matrix $I(\alpha, \beta)$.

Solution.
The log likelihood of $\theta = (\alpha, \beta)$ is

$$l(\beta) = \frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i - \alpha - \beta t_i)^2.$$

Take derivatives,

$$\frac{\partial l}{\partial \alpha} = \sum_i (x_i - \alpha - \beta t_i), \qquad \frac{\partial l}{\partial \beta} = \sum_i (x_i - \alpha - \beta t_i) t_i,$$

$$\frac{\partial^2 l}{\partial \alpha^2} = -n, \qquad \frac{\partial^2 l}{\partial \beta^2} = -\sum_i t_i^2,$$

$$\frac{\partial^2 l}{\partial \alpha \beta} = -\sum_i t_i$$

Therefore, the Fisher information matrix be

$$I(\alpha, \beta) = -E\left[\nabla l(\alpha, \beta)\right]^2 = \begin{pmatrix} n & \sum_i t_i \\ \sum_i t_i & \sum_i t_i^2 \end{pmatrix}.$$

□

b) Give a lower bound for the variance of an unbiased estimator of α.

Solution.
Since $g(\theta) = \alpha$, by Cramer-Rao lower bound,

$$\begin{aligned} LB &= \nabla g(\theta)^T I^{-1}(\alpha, \beta) \nabla g(\theta) \\ &= (1, 0)\frac{\begin{pmatrix} \sum_i t_i^2 & -\sum_i t_i \\ -\sum_i t_i & n \end{pmatrix}}{n\sum_i t_i^2 - (\sum_i t_i)^2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{n\sum_i t_i^2 - (\sum_i t_i)^2}\left(\sum_i t_i^2, \ -\sum_i t_i\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{\sum_i t_i^2}{n\sum_i t_i^2 - (\sum_i t_i)^2} \\ &= \frac{1/n}{1 - (\sum_i t_i)^2 / \sum_i t_i^2}. \end{aligned}$$

□

c) Suppose we know the value of β. Give a lower bound for the variance of an unbiased estimator of α in this case.

Solution.
Since β is known, $\theta = \alpha$. Then the lower bound be

$$LB = \frac{g'(\theta)}{I(\theta)} = \frac{1}{n}.$$

□

d) Compare the estimators in parts (b) and (c). When are the bounds the same? If the bounds are different, which is larger?

Solution.
If $\sum_i t_i = 0$, two lower bounds be same. And by comparing the lower bounds,

$$LB_{b)} = \frac{1/n}{1 - (\sum_i t_i)^2 / \sum_i t_i^2} > \frac{1}{n} = LB_{c)}.$$

□

e) Give a lower bound for the variance of an unbiased estimator of the product $\alpha\beta$.

Solution.
Since $g(\theta) = \alpha\beta$, $\nabla g(\theta) = \binom{\beta}{\alpha}$. Then the lower bound be

$$\begin{aligned}
LB &= \nabla g(\theta)^T I^{-1}(\alpha, \beta) \nabla g(\theta) \\
&= (\beta,\ \alpha) \frac{\begin{pmatrix} \sum_i t_i^2 & -\sum_i t_i \\ -\sum_i t_i & n \end{pmatrix}}{n \sum_i t_i^2 - (\sum_i t_i)^2} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \\
&= \frac{1}{n \sum_i t_i^2 - (\sum_i t_i)^2} \left(\beta \sum_i t_i^2 - \alpha \sum_i t_i,\ \ - \beta \sum_i t_i + n\alpha \right) \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \\
&= \frac{\beta^2 \sum_i t_i^2 - 2\alpha\beta \sum_i t_i + n\alpha^2}{n \sum_i t_i^2 - (\sum_i t_i)^2}.
\end{aligned}$$

□

Problem 4.31

Find the Fisher information for the Cauchy location family with densities p_θ given by

$$p_\theta(x) = \frac{1}{\pi\big[(x-\theta)^2 + 1\big]}.$$

Also, what is the Fisher information for θ^3?

Solution.
Since p_θ is the location family, by using Example 4.11 in page 75,

$$\begin{aligned}
I(\theta) = \int \frac{[f'(x)]^2}{f(x)} dx &= \int \frac{\left[-\frac{2\pi x}{\pi(x^2+1)} \right]^2}{1/[\pi(x^2+1)]} = \int \frac{4x^2}{\pi(x^2+1)^3} dx \\
&\text{Change of variable } x = \tan t \Rightarrow dx = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} dt \\
&= \frac{4}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\tan^2 t}{(\tan^2 + 1)^3} \frac{1}{\cos^2 t} dt \\
&= \frac{4}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 t}{\cos^2 t} \cos^6 t \frac{1}{\cos^2 t} dt \quad (\because \tan^2 t + 1 = 1/\cos^2 t) \\
&= \frac{4}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t \cos^2 t dt \\
&= \frac{4}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - \cos 2x}{2} \frac{1 + \cos 2x}{2} dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos^2 2x}{4} dt \\
&= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos 4x}{2} dt \\
&= \frac{1}{2\pi}\left[t - \frac{1}{4}\sin 4t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{1}{2}.
\end{aligned}
$$

Let $\xi = \theta^3$, then $\theta = \xi^{1/3} := h(\xi)$.

$$\therefore I(\xi) = [h'(\xi)]^2 I\big(h(\xi)\big) = \left(\frac{1}{3}\xi^{-2/3}\right)^2 \frac{1}{2} = \frac{1}{18\xi^{4/3}}.$$

□

Problem 4.32

Suppose X has a Poisson distribution with mean θ^2, so the parameter θ is the square root of the usual parameter $\lambda = EX$. Show that the Fisher information $I(\theta)$ is constant.

<u>Solution</u>.
$p_\theta(X) = (\theta^2)^x e^{-\theta^2}/x!$ and $l(\theta) = \log 1/x! + 2x\log\theta - \theta^2$.

$$\frac{\partial l(\theta)}{\partial \theta} = \frac{2x}{\theta} - 2\theta, \quad \frac{\partial^2 l(\theta)}{\partial \theta^2} = -\frac{2x}{\theta^2} - 2.$$

$$\therefore I(\theta) = E\left(-\frac{\partial^2 l(\theta)}{\partial \theta^2}\right) = \frac{2}{\theta^2}EX + 2 = 4.$$

□

Problem 4.33

Consider the exponential distribution with failure rate λ. Find a function h defining a new parameter $\theta = h(\lambda)$ so that Fisher information $I(\theta)$ is constant.

<u>Solution</u>.
Let the original parameter is θ, and reparameterized parameter is λ.
$p_\lambda(x) = \lambda e^{-\lambda x}$, $l(\lambda) = \log\lambda - \lambda x$.

$$\frac{\partial l(\lambda)}{\partial \lambda} = \frac{1}{\lambda} - x, \quad \frac{\partial^2 l(\lambda)}{\partial \lambda^2} = -\frac{1}{\lambda^2}.$$

$$\tilde{I}(\lambda) = \frac{1}{\lambda^2}.$$

From (4.18) in page 74,

$$I(\theta) = \frac{\tilde{I}(\lambda)}{[h'(\lambda)]^2} = \frac{1}{[\lambda h'(\lambda)]^2}.$$

Thus, if $h(\lambda) = \log\lambda$, $I(\theta)$ be constant. □

Problem 4.34

Consider an autoregressive model in which $X_1 \sim N\big(\theta, \sigma^2/(1-\rho^2)\big)$ and the conditional distribution of X_{j+1} given $X_1 = x_1, \ldots X_j = x_j$, is $N\big(\theta + \rho(x_j - \theta), \sigma^2\big)$, $j = 1, \ldots, n-1$.

a) Find the Fisher information matrix, $I(\theta, \sigma)$.

Solution.
The joint density is

$$p(x_1, \ldots, x_n) = p(x_1)p(x_2|x_1)\cdots(x_n|x_1, \ldots, x_n),$$

thus the likelihood function be

$$\begin{aligned}
l(\theta,\sigma) &= l(x_1) + \sum_{j=1}^{n-1} l(x_{j+1}|x_1, \ldots, x_j) \\
&= \log\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} - \log\sigma - \frac{1-\rho^2}{2\sigma^2}(x_1-\theta)^2 \\
&\quad - \frac{1}{\sigma^2}\sum_{j=1}^{n-1}[x_{j+1} - \theta - \rho(x_j - \theta)]^2 + \log\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{n-1} \\
&= -\frac{1-\rho^2}{2\sigma^2}(x_1-\theta)^2 - \frac{1}{2\sigma^2}\sum_{j=1}^{n-1}[x_{j+1} - \rho x_j - \theta(1-\rho)]^2 \\
&\quad - \frac{n}{2}\log(2\pi) - n\log\sigma + \log\sqrt{1-\rho^2}.
\end{aligned}$$

Take derivatives,

$$\frac{\partial l(\theta,\sigma)}{\partial\theta} = \frac{1-\rho^2}{\sigma^2}(x_1-\theta) + \frac{1-\rho}{\sigma^2}\sum_{j=1}^{n-1}[x_{j+1} - \rho x_j - \theta(1-\rho)],$$

$$\frac{\partial^2 l(\theta,\sigma)}{\partial\theta^2} = -\frac{1-\rho^2}{\sigma^2} - \frac{(n-1)(1-\rho)^2}{\sigma^2},$$

$$\frac{\partial^2 l(\theta,\sigma)}{\partial\theta\partial\sigma} = -\frac{2(1-\rho^2)\epsilon_1}{\sigma^3} + \frac{2(1-\rho)}{\sigma^3}\sum_{j=1}^{n-1}\eta_{j+1},$$

$$\frac{\partial l(\theta,\sigma)}{\partial\sigma} = \frac{(1-\rho^2)(x_1-\theta)^2}{\sigma^3} + \frac{1}{\sigma^3}\sum_{j=1}^{n-1}\eta_{j+1}^2 - \frac{n}{\sigma},$$

$$\frac{\partial^2 l(\theta,\sigma)}{\partial\sigma^2} = -\frac{3(1-\rho^2)(x_1-\theta)^2}{\sigma^4} - \frac{3}{\sigma^4}\sum_{j=1}^{n-1}\eta_{j+1}^2 + \frac{n}{\sigma^2},$$

where $\epsilon_j = x_j - \theta$, and $\eta_{j+1} = \epsilon_{j+1} - \rho\epsilon_j$. (See Hint in part c))
We can easily infer that $\epsilon_1 \sim N\big(0, \sigma^2/(1-\rho^2)\big)$ and $\eta_{j+1}|_{X_1=x_1,\ldots,X_j=x_j} = \eta_{j+1}|_{\epsilon_1,\eta_2,\ldots,\eta_j} \sim N(0,\sigma^2)$.
From this, $\eta_2, \ldots, \eta_n \sim$ i.i.d. $N(0,\sigma^2)$ and these are independent of ϵ_1. (왜???)
Therefore, the information matrix is given by taking expectation,

$$I(\theta,\sigma) = \begin{pmatrix} \frac{1-\rho^2+(n-1)(1-\rho)^2}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{pmatrix}.$$

□

b) Give a lower bound for the variance of an unbiased estimator of θ.

Solution.
Since $g(\theta, \sigma) = \theta$,

$$LB = \nabla g(\theta,\sigma)^T I^{-1}(\theta,\sigma)\nabla g(\theta,\sigma) = (1,0)I^{-1}(\theta,\sigma)\begin{pmatrix}1\\0\end{pmatrix} = \frac{\sigma^2}{1-\rho^2+(n-1)(1-\rho)^2}.$$

□

c) Show that the sample average $\overline{X} = (X_1 + \cdots + X_n)/n$ is an unbiased estimator of θ, compute its variance, and compare its variance with the lower bound.
Hint: Define $\epsilon_j = X_j - \theta$ and $\eta_{j+1} = \epsilon_{j+1} - \epsilon_j$. Use smoothing to argue that $\eta_2, \ldots, \eta_n$ are i.i.d. $N(0, \sigma^2)$ and are independent of ϵ_1. Similarly, X_i is independent of $\eta_{i+1}, \eta_{i+2}, \ldots$. Use these facts to find first $\mathrm{Var}(X_2) = \mathrm{Var}(\epsilon_2)$, then $\mathrm{Var}(X_3), \mathrm{Var}(X_4), \ldots$. Finally find $\mathrm{Cov}(X_{i+1}, X_i), n\mathrm{Cov}(X_{i+2}, X_i)$, and so on.

Solution.
Let $\bar{\epsilon} = \frac{1}{n}\sum_j \epsilon_j = \overline{X} - \theta$. Since $E\epsilon_j = 0$, $E(\overline{X} - \theta) = E(\bar{\epsilon}) = 0$. Thus $\overline{X}$ is unbiased estimator of θ.
Since $\epsilon_2 = \rho\epsilon_1 + \eta_2$, $\mathrm{Var}(\epsilon_2) = \rho^2\mathrm{Var}(\epsilon_1) + \mathrm{Var}(\eta_2) = \sigma^2 + \frac{\rho^2\sigma^2}{1-\rho^2} = \frac{\sigma^2}{1-\rho^2}$. In general, $\epsilon_i = \rho\epsilon_{i-1} + \eta_i$ and we can obtain $\mathrm{Var}(X_i) = \mathrm{Var}(\epsilon_i) = \frac{\sigma^2}{1-\rho^2}$.
If $i > j$,

$$\begin{aligned}
\epsilon_i &= \rho\epsilon_{i-1} + \eta_i \\
&= \rho^2(\epsilon_{i-2} + \eta_{i-1}) + \eta_i \\
&= \cdots \\
&= \rho^{i-j}\epsilon_j + \rho^{i-j+1}\eta_{j+1} + \cdots + \rho\eta_{i-1} + \eta_i.
\end{aligned}$$

From this,

$$\mathrm{Cov}(X_i, X_j) = \mathrm{Cov}(\epsilon_i, \epsilon_j) = \rho^{|i-j|}\frac{\sigma^2}{1-\rho^2}.$$

Then, the variance of $\overline{X}$ is

$$\begin{aligned}
\mathrm{Var}(\overline{X}) &= \mathrm{Var}(\bar{\epsilon}) \\
&= \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^n \epsilon_i\right) = \frac{1}{n^2}\mathrm{Cov}\left(\sum_{i=1}^n \epsilon_i, \sum_{i=1}^n \epsilon_i\right) \\
&= \frac{1}{n^2}\left[\sum_{i=1}^n \mathrm{Cov}(\epsilon_i, \epsilon_i) + 2\sum_{i>j}\mathrm{Cov}(\epsilon_i, \epsilon_j)\right] \\
&= \frac{\sigma^2}{n^2(1-\rho^2)}\left[n + 2\left\{(n-1)\rho + (n-2)\rho^2 + \cdots + \rho^{n-1}\right\}\right] \\
&= \frac{\sigma^2}{n^2(1-\rho^2)}\left[n + 2\sum_{j=1}^{n-1}(n-j)\rho^j\right] \\
&= \frac{\sigma^2}{n^2(1-\rho^2)}\left[n + 2\left(n\frac{\rho-\rho^n}{1-\rho} - \frac{\rho-\rho^{n+1}}{(1-\rho)^2} + \frac{n\rho^n}{1-\rho}\right)\right] \qquad (*) \\
&= \frac{\sigma^2}{n^2(1-\rho^2)}\left[n + 2\frac{n\rho - n\rho^2 - \rho + \rho^{n+1}}{(1-\rho)^2}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sigma^2}{n^2(1-\rho^2)} \left[\frac{n - 2n\rho + n\rho^2 + 2n\rho - 2n\rho^2 - 2\rho + 2\rho^{n+1}}{(1-\rho)^2} \right] \\
&= \frac{\sigma^2}{n^2(1-\rho^2)} \left[\frac{-n\rho^2 - 2\rho + n + 2\rho^{n+1}}{(1-\rho)^2} \right] \\
&= \frac{\sigma^2}{n^2(1-\rho^2)} \left[\frac{n(1-\rho^2) - 2(\rho - \rho^{n+1})}{(1-\rho)^2} \right] \\
&= \frac{\sigma^2}{n(1-\rho)^2} - \frac{2\sigma^2\rho(1-\rho^n)}{n^2(1-\rho^2)(1-\rho)^2}.
\end{aligned}
$$

Note. $(*)$ is obtained as follows:

$$
\begin{aligned}
\sum_{j=1}^{n-1} \rho^j &= \frac{\rho(1-\rho^{n-1})}{1-\rho}, \\
\sum_{j=1}^{n-1} j\rho^j &= \sum_j \rho \frac{d}{d\rho}\rho^j = \rho \frac{d}{d\rho} \sum_{j=1}^{n-1} \rho^j \\
&= \rho \frac{d}{d\rho} \left(\frac{\rho - \rho^n}{1-\rho} \right) \\
&= \rho \frac{(1 - n\rho^{n-1})(1-\rho) + (\rho - \rho^n)}{(1-\rho)^2} \\
&= \rho \frac{1 - \rho - n\rho^{n-1} + n\rho^n + \rho - \rho^n}{(1-\rho)^2} \\
&= \rho \frac{1 - \rho^n - n\rho^{n-1}(1-\rho)}{(1-\rho)^2} \\
&= \frac{\rho - \rho^{n+1}}{(1-\rho)^2} - \frac{n\rho^n}{1-\rho}.
\end{aligned}
$$

□

Chapter 5

Curved Exponential Families

Problem 5.1

Suppose X has a binomial distribution with m trials and success probability θ, Y has a binomial distribution with n trials and success probability θ^2, and X and Y are independent.

a) Find a minimal sufficient statistic T.

Solution.
Note that $f(x) = \binom{m}{x}\theta^x(1-\theta)^{m-x}$ and $f(y) = \binom{n}{y}(\theta^2)^y(1-\theta^2)^{n-y}$. Then the joint density is

$$f(x,y) = \binom{m}{x}\binom{n}{y}\exp\left[x\log\frac{\theta}{1-\theta} + y\log\frac{\theta^2}{1-\theta^2} + m\log(1-\theta) + n\log(1-\theta^2)\right].$$

Since the canonical parameter $\eta = \left(\log\frac{\theta}{1-\theta}, \log\frac{\theta^2}{1-\theta^2}\right)$ does not satisfy a linear constraint, the joint density is the curved exponential family. Therefore, $T = (X, Y)$ is minimal sufficient. $\square$

b) Show that T is not complete, providing a non-trivial function f with $E_\theta f(T) = 0$.

Solution.
Note that $ET_1 = EX = m\theta,\ ET_1^2 = EX^2 = m\theta(1-\theta) + m^2\theta^2$ and $ET_2 = EY = n\theta^2$.
Now we let

$$f(T) = T_1^2 - T_1 - \frac{m(m-1)}{n}T_2,$$

then $E_\theta f(T) = 0$ for all $\theta \in (0,1)$. But $f(T)$ is not always 0. Thus T is not complete. $\square$

Problem 5.2

Let X and Y be independent Bernoulli variables with $P(X=1) = p$ and $P(Y=1) = h(p)$ for some known function h.

a) Show that the family of joint distributions is a curved exponential family unless

$$h(p) = \frac{1}{1+\exp\left\{a + b\log\frac{p}{1-p}\right\}}$$

for some constants a and b.

Solution.
The joint density is

$$f(x,y) = p^x(1-p)^{1-x}h(p)^y\big(1-h(p)\big)^{1-y} = \exp\left[x\log\left(\frac{p}{1-p}\right) + y\log\left(\frac{h(p)}{1-h(p)}\right)\right](1-p)\big(1-h(p)\big).$$

If $h(p)$ defined as the above for some constants a and b, then

$$\log\left(\frac{h(p)}{1-h(p)}\right) = -\left(a + b\log\left(\frac{p}{1-p}\right)\right)$$

which is the linear constraint. Therefore, unless the above case, the family of joint densities is a curved exponential family. □

b) Give two functions h, one where (X,Y) is minimal but not complete, and one where (X,Y) is minimal and complete.

Solution.

(i) Minimal but not complete
From a), the joint density is a curved exponential family except on the above h form.
If we let a function $h(p) = \frac{p}{2}$, $E_p(X-2Y) = 0$ for all $p \in (0,1)$. But $X - 2Y$ is not always 0. Therefore T is not complete.

(ii) Minimal and complete
For some function g,

$$E_p g(X,Y) = g(1,1)ph(p) + g(1,0)p\big(1-h(p)\big) + g(0,1)(1-p)h(p) + g(0,0)(1-p)\big(1-h(p)\big).$$

To show T is complete, we need to find a function h that $ph(p), p\big(1-h(p)\big), (1-p)h(p)$ are linearly independent.
If we let $h(p) = p^2$, then

$$\begin{aligned} E_p g(X,Y) = p^3 &[g(1,1) - g(1,0) - g(0,1) + g(0,0)] \\ &+ p^2[g(0,1) - g(0,0)] + p[g(1,0) - g(0,0)] + g(0,0). \end{aligned}$$

If this is 0 for all $p \in (0,1)$, it must be $g(0,0) = g(1,0) = g(0,1) = g(1,1) = 0$. Thus $g(X,Y) = 0$ and T is complete.

□

Problem 5.6

Two teams A and B play a series of games, stopping as soon as one of the team has 4 wins. Assume that game outcomes are independent and that on any given game team A has a fixed chance θ of winning. Let X and Y denote the number of games won by the first and second team, respectively.

a) Find the joint mass function for X and Y. Show that as θ varies these mass functions form a curved exponential family.

Solution.
First, suppose team A has 4 wins which is the event $X = 4$ and $Y = y$. Then, the first $(3+y)$ trials are random and last trial should be the winning of team A, thus the probability be

$$P(X = 4, Y = y) = \binom{3+y}{3}\theta^4(1-\theta)^y, \qquad y = 0, 1, 2, 3.$$

Similarly,

$$P(X = x, Y = 4) = \binom{3+x}{3}\theta^x(1-\theta)^4, \qquad x = 0, 1, 2, 3.$$

Therefore, the joint mass have the form $h(x, y)\exp\big[x\log\theta + y\log(1-\theta)\big]$. Since the canonical parameters $\log\theta$ and $\log(1-\theta)$ have non-linear relationship, the family of the joint mass is a curved exponential family. $\square$

b) Show that $T = (X, Y)$ is complete.

Solution.
Let g be an arbitrary function of X and Y, and suppose

$$E_\theta g(X, Y) = \sum_{x=0}^{3} g(x, 4)\binom{3+x}{3}\theta^x(1-\theta)^4 + \sum_{y=0}^{3} g(4, y)\binom{3+y}{3}\theta^4(1-\theta)^y = 0, \quad \theta \in (0, 1).$$

Since it is the polynomial in θ, if $\theta \to 0$, then the only constant term $g(0, 4)$ is remain (other terms are tend to 0). Thus $g(0, 4)$ should be 0.
If $g(0, 4) = 0$, then the linear term $g(1, 4)$ also should be 0 (dividing by θ and letting $\theta \to 0$). Similarly $g(2, 4) = g(3, 4) = 0$.
Then, now remain only

$$\frac{E_\theta g(X, Y)}{\theta^4} = \sum_{y=0}^{3} g(4, y)\binom{3+y}{3}(1-\theta)^y = 0,$$

and this is the polynomial in $1-\theta$. Similarly, we can obtain $g(4, 0) = \cdots = g(4, 3) = 0$. Therefore, $g(X, Y) = 0$ almost surely.
By definition of completeness, T is complete. $\square$

c) Find a UMVU estimator of θ.

Solution.
Let $\delta = 1_{\{\text{team A wins the first game}\}}$. Then δ is unbiased because $E\delta = P(\text{team A wins the first game}) = \theta$. Thus the UMVU estimator of θ be $E[\delta|T] = P[\delta = 1|X, Y]$. By using the joint mass function,

$$\begin{aligned}
E[\delta = 1|X = 4, Y = y] &= P(\delta = 1|X = 4, Y = y) \\
&= \frac{P(\delta = 1, X = 4, Y = y)}{P(X = 4, Y = y)} \\
&= \frac{\theta^2\binom{2+y}{2}\theta^2(1-\theta)^y}{\binom{3+y}{3}\theta^4(1-\theta)^y} \\
&= \frac{3}{3+y},
\end{aligned}$$

and

$$P(\delta = 1 | X = x, Y = 4) = \frac{\theta(1-\theta)\binom{2+x}{x-1}\theta^{x-1}(1-\theta)^3}{\binom{3+x}{3}\theta^x(1-\theta)^4} = \frac{x}{3+x}.$$

By combining 2 results, the UMVU estimator of θ is

$$\frac{X - 1_{\{X=4\}}}{X+Y-1}.$$

□

Problem 5.7

Consider a sequential experiment in which observations are i.i.d. from a Poisson distribution with mean λ. If the first observation X is zero, the experiment stops, and if $X > 0$, a second observation Y is observed. Let $T = 0$ if $X = 0$, and let $T = 1 + X + Y$ if $X > 0$.

a) Find the mass function for T.

Solution.
$P(T = 0) = P(X = 0) = e^{-\lambda}$.
For $k = 0, 1, \ldots,$,

$$\begin{aligned} P(T = k+1) &= P(X+Y = k, X > 0) \\ &= P(X+Y = k) - P(X+Y = k, X = 0) \\ &= \frac{(2\lambda)^k e^{-2\lambda}}{k!} - \frac{\lambda^k e^{-2\lambda}}{k!}. \end{aligned}$$

이게 왜 아니지??

$$\begin{aligned} P(T = k+1) &= P(X+Y = k, X > 0) \\ &= P(X+Y = k) - P(X+Y = k, X = 0) \\ &= P(X+Y = k) - P(Y = k) \\ &= \frac{(2\lambda)^k e^{-2\lambda}}{k!} - \frac{\lambda^k e^{-\lambda}}{k!}. \end{aligned}$$

□

b) Show that T is minimal sufficient.

Solution.
Part c) first!!
From c), (W, N) is minimal sufficient. Therefore T that is one-to-one function of (W, N) is also minimal sufficient. □

c) Does this experiment give a curved two-parameter exponential family or full rank one-parameter exponential family?

Solution.
Let N is the sample size(or # of experiments), and W as follows:

$$N = \begin{cases} 1, & \text{if } X = 0, \\ 2, & \text{if } X > 0, \end{cases} \qquad W = \begin{cases} 0, & \text{if } X = 0, \\ X + Y, & \text{if } X > 0. \end{cases}$$

Note that X and Y are from i.i.d. Poisson(λ) and it satisfies the condition in Theorem 5.4.
By Theorem 5.4, the joint density is an exponential family with canonical parameter $\eta = (\log \lambda, -\lambda)$ and sufficient statistic $(X + Y, N) = (W, N)$.
Since η does not satisfy a linear constraint, the density of W and N is a curved exponential family.
Thus, (W, N) is minimal sufficient. □

d) Is T a complete sufficient statistic?
Hint: Write $e^{\lambda} E_{\lambda} g(T)$ as a power series in λ and derive equations for g setting coefficients for λ^x to zero.

Solution.
Suppose $E_{\lambda} g(T) = 0$ for all $\lambda > 0$ and any function g. Then

$$E_{\lambda} g(T) = g(0) e^{-\lambda} + \sum_{k=0}^{\infty} g(k+1) \frac{(2\lambda)^k e^{-2\lambda} - \lambda^k e^{-2\lambda}}{k!} = 0.$$

Multiplying $e^{2\lambda}$,

$$e^{2\lambda} E_{\lambda} g(T) = g(0) e^{\lambda} + \sum_{k=0}^{\infty} \frac{g(k+1)\left(2^k - 1\right)}{k!} \lambda^k = 0.$$

This is the power series in λ, so the coefficients of the power should be 0. ($g(0) = 0$ and $g(k+1) = 0$ for $k = 1, 2, \ldots$.)
Note that $T \neq 1$($\because$ If $X > 0$, then $Y \geq 0$ and $T > 1$). Therefore, $g(T) = 0$ thus T is complete. □

Problem 5.13

Consider a single two-way contingency table and define $R = N_{11} + N_{12}$ (the first row sum), $C = N_{11} + N_{21}$ (the first column sum), and $D = N_{11} + N_{22}$ (the sum of the diagonal entries).

a) Show that the joint mass function can be written as a full rank three-parameter exponential family with $T = (R, C, D)$ as the canonical sufficient statistic.

Solution.
The contingency table be as follows:

N_{11}	N_{12}	R
N_{21}	N_{22}	
C		n

By representing the elements with R, C, D, and n,

$$N_{11} = (R + C + D - n)/2, \quad N_{12} = (n - D - C + R)/2,$$

$$N_{21} = (n - D - R + C)/2, \quad N_{22} = (n + D - R - C)/2.$$

Thus the joint mass function is

$$\begin{aligned}\binom{n}{n_{11}, \ldots, n_{22}} p_{11}^{n_{11}} \cdots p_{22}^{n_{22}} &= \binom{n}{n_{11}, \ldots, n_{22}} (\sqrt{p_{11}})^{r+c+d-n} (\sqrt{p_{12}})^{n-d-c+r} (\sqrt{p_{21}})^{n-d-r+c} (\sqrt{p_{22}})^{n+d-r-c} \\ &= \binom{n}{n_{11}, \ldots, n_{22}} \sqrt{\frac{p_{11}p_{12}}{p_{21}p_{22}}}^{r} \sqrt{\frac{p_{11}p_{21}}{p_{12}p_{22}}}^{c} \sqrt{\frac{p_{11}p_{22}}{p_{12}p_{21}}}^{d} \sqrt{\frac{p_{12}p_{21}p_{22}}{p_{11}}}^{n} \\ &= \binom{n}{n_{11}, \ldots, n_{22}} \exp\left[r \log \sqrt{\frac{p_{11}p_{12}}{p_{21}p_{22}}} + c \log \sqrt{\frac{p_{11}p_{21}}{p_{12}p_{22}}}\right. \\ &\qquad \left. + d \log \sqrt{\frac{p_{11}p_{22}}{p_{12}p_{21}}} + n \log \sqrt{\frac{p_{12}p_{21}p_{22}}{p_{11}}}\right],\end{aligned}$$

where r, c, and d are defined as before.
The density is a full-rank(3-parameter) exponential family, and the sufficient statistic (R, C, D) do not satisfy a linear constraint because there is a one-to-one linear association between it and (N_{11}, N_{12}, N_{21}), and the 3 canonical parameters η_r, η_c, and η_d can vary freely over $\mathbb{R}^3$. 마지막 부분 잘 이해 안됨,,, □

b) Relate the canonical parameter associated with D to the "cross-product ratio" α defined as $\alpha = p_{11}p_{22}/(p_{12}p_{21})$.

Solution.
$\eta_d = \log \sqrt{\alpha}$ □

c) Suppose we observe m independent two-by-two contingency tables. Let $n_i,\ i = 1, \ldots, m$, denote the trials for table i. Assume that cell probabilities for the tables may differ, but that the cross-product ratios for all m tables are all the same. Show that the joint mass functions form a full rank exponential family. Express the sufficient statistic as a function of the variables $R_1, \ldots, R_m, C_1, \ldots, C_m$, and $D_1, \ldots, D_m$.

Solution.
The joint mass can be obtained as

$$\exp\left[\sum_{i=1}^{m} r_i \eta_{r,i} + \sum_{i=1}^{m} c_i \eta_{c,i} + \eta_d \sum_{i=1}^{m} d_i - \sum_{i=1}^{m} A_i(\eta_i)\right] h(x),$$

which is a full-rank($2m + 1$ parameter) exponential family. Therefore, the complete sufficient statistic is

$$T = \left(R_1, \ldots, R_m, C_1, \ldots, C_m, \sum_{i=1}^{m} D_i\right),$$

which is a function of $R_1, \ldots, R_m, C_1, \ldots, C_m$. □

Problem 5.16

For an $I \times J$ contingency table with independence, the UMVU estimator of p_{ij} is $\hat{p}_{i+}\hat{p}_{+j} = N_{i+}N_{+j}/n^2$.

a) Determine the variance of this estimator, $\text{Var}(\hat{p}_{i+}\hat{p}_{+j})$.

Solution.
Since N_{i+} and N_{+j} are independent with

$$N_{i+} \sim \text{Binomial}(n, p_{i+}), \qquad N_{+j} \sim \text{Binomial}(n, p_{+j}).$$

$E\hat{p}_{i+} = \frac{1}{n} EN_{i+} = p_{i+}$ and similarly, $E\hat{p}_{+j} = p_{+j}$.

$$\begin{aligned}
E\hat{p}_{i+}^2 \hat{p}_{+j}^2 &= \frac{1}{n^4} EN_{i+}^2 N_{+j}^2 \\
&= \frac{1}{n^4} \left[np_{i+}(1-p_{i+}) + n^2 p_{i+}^2\right] \left[np_{+j}(1-p_{+j}) + n^2 p_{+j}^2\right] \\
&= \frac{1}{n^2} \left[p_{i+}(1-p_{i+}) + np_{i+}^2\right] \left[p_{+j}(1-p_{+j}) + np_{+j}^2\right].
\end{aligned}$$

$$\begin{aligned}
\therefore \operatorname{Var}(\hat{p}_{i+}\hat{p}_{+j}) &= E\hat{p}_{i+}^2 \hat{p}_{+j}^2 - \left[E\hat{p}_{i+}\hat{p}_{+j}\right]^2 \\
&= \frac{1}{n^2} \left[p_{i+}(1-p_{i+}) + np_{i+}^2\right] \left[p_{+j}(1-p_{+j}) + np_{+j}^2\right] - p_{i+}^2 p_{+j}^2 \\
&= \frac{1}{n^2} \left[p_{i+}(1-p_{i+})p_{+j}(1-p_{+j}) + np_{i+}(1-p_{i+})p_{+j}^2 + np_{i+}^2 p_{+j}(1-p_{+j})\right] \\
&= \frac{p_{i+}(1-p_{i+})p_{+j}^2 + p_{+j}(1-p_{+j})p_{i+}^2}{n} + \frac{p_{i+}(1-p_{i+})p_{+j}(1-p_{+j})}{n^2}.
\end{aligned}$$

□

b) Find the UMVU estimator of the variance in (a).

Solution.
First, we should find the unbiased estimators in the numerator term.

$$\begin{aligned}
EN_{i+}^2 &= np_{i+}(1-p_{i+}) + n^2 p_{i+}^2 \\
&= np_{i+} - np_{i+}^2 + n^2 p_{i+}^2 \\
&= EN_{i+} + (n^2 - n)p_{i+}^2
\end{aligned}$$

$$\Rightarrow E\left(\frac{N_{i+}(N_{i+}-1)}{n^2-n}\right) = p_{i+}^2,$$

$$\begin{aligned}
p_{i+}(1-p_{i+}) &= p_{i+} - p_{i+}^2 \\
&= E\frac{N_{i+}}{n} - E\left(\frac{N_{i+}(N_{i+}-1)}{n^2-n}\right) \\
&= E\left(\frac{N_{i+}(n-N_{i+})}{n^2-n}\right),
\end{aligned}$$

From the above, we obtain the unbiased estimators as follows:

$$\begin{aligned}
E\left(\frac{N_{i+}(N_{i+}-1)}{n^2-n}\right) &= p_{i+}^2, \\
E\left(\frac{N_{i+}(n-N_{i+})}{n^2-n}\right) &= p_{i+}(1-p_{i+}), \\
E\left(\frac{N_{+j}(N_{+j}-1)}{n^2-n}\right) &= p_{+j}^2,
\end{aligned}$$

$$E\left(\frac{N_{+j}(n-N_{+j})}{n^2-n}\right)=p_{+j}(1-p_{+j}).$$

By using these, the expectation of $\mathrm{Var}(\hat{p}_{i+}\hat{p}_{+j})$ is

$$\begin{aligned}E\left[\mathrm{Var}(\hat{p}_{i+}\hat{p}_{+j})\right]&=E\left[\frac{p_{i+}(1-p_{i+})p_{+j}^2+p_{+j}(1-p_{+j})p_{i+}^2}{n}+\frac{p_{i+}(1-p_{i+})p_{+j}(1-p_{+j})}{n^2}\right]\\&=\frac{N_{i+}(n-N_{i+}N_{+j}(N_{+j}-1)+N_{+j}(n-N_{+j})N_{i+}(N_{i+}-1)}{n(n^2-n)^2}\\&\quad+\frac{N_{i+}(n-N_{i+})N_{+j}(n-N_{+j})}{n^2(n^2-n)^2}.\end{aligned}\tag{$*$}$$

Therefore, $(*)$ is UMVU estimator of $\mathrm{Var}(\hat{p}_{i+}\hat{p}_{+j})$. □

Chapter 6

Conditional Distributions

Problem 6.3

Suppose that X and Y are independent and positive. Use a smoothing argument to show that if $x \in (0,1)$, then

$$P\left(\frac{X}{X+Y} \le x\right) = EF_X\left(\frac{xY}{1-x}\right), \tag{6.8}$$

where F_X is the cumulative distribution function of X.

Solution.
Since X and Y are positive and $x \in (0,1)$,

$$\frac{X}{X+Y} \le x \Leftrightarrow X \le \frac{xY}{1-x}.$$

Then

$$\begin{aligned} P\left(\frac{X}{X+Y} \le x \middle| Y=y\right) &= P\left(X \le \frac{xY}{1-x} \middle| Y=y\right) \\ &= E\left[I\left(X \le \frac{xY}{1-x}\right) \middle| Y=y\right] \\ &= F_X\left(X \le \frac{xy}{1-x}\right). \end{aligned}$$

Thus we obtain $P\left(\frac{X}{X+Y} \le x \middle| Y\right) = F_X\left(X \le \frac{xY}{1-x}\right)$, and taking the expectation for the both sides,

$$EP\left(\frac{X}{X+Y} \le x \middle| Y\right) = P\left(\frac{X}{X+Y} \le x\right) = EF_X\left(\frac{xY}{1-x}\right).$$

$\square$

Problem 6.11

Suppose X and Y are independent, both absolutely continuous with common density f. Let $M = \max\{X,Y\}$ and $Z = \min\{X,Y\}$. Determine the conditional distribution for the pair (X,Y) given (M,Z).

Solution.
Given $M = m, Z = z$, the natural solution can be

$$P(X = m, Y = z|M = m, Z = z) = P(X = z, Y = m|M = m, Z = z) = \frac{1}{2}.$$

To see that the above is true, we check by following procedure.
Let $g(x, y) = h(x, y, x \vee y, x \wedge y)$ so that $h(X, Y, M, Z) = g(X, Y)$. Then

$$\begin{aligned} Eh(X, Y, M, Z|M, Z) &= E\left[g(X, Y)|M, Z\right] \\ &= \frac{1}{2}g(M, Z) + \frac{1}{2}g(Z, M). \end{aligned}$$

To show the last equality holds, we should check the smoothing works.
Taking the expectations on the both sides,

$$\begin{aligned} \int\int g(x, y)f(x)f(y)dxdy &= \int\int \frac{1}{2}\big[g(x \vee y, x \wedge y) + g(x \wedge y, x \vee y)\big]f(x)f(y)dxdy \\ &= \int\int \frac{1}{2}\big[g(x, y) + g(y, x)\big]f(x)f(y)dxdy. \end{aligned}$$

Since $\int\int g(x, y)dxdy = \int\int g(y, x)dxdy$, the above equality holds and the smoothing works. Therefore, the conditional distribution is correct. □

Problem 6.14

Let X and Y be absolutely continuous with density $p(x, y) = e^{-x}$, if $0 < y < x$; $p(x, y) = 0$, otherwise.

a) Find the marginal densities of X and Y.

Solution.
Take the integration with respect to x and y, respectively,

$$\begin{aligned} f(x) &= \int_0^x p(x, y)dy = e^{-x}y\Big|_0^x = xe^{-x}, \quad 0 < x < \infty, \\ f(y) &= \int_0^x p(x, y)dx = -e^{-x}\Big|_0^y = e^{-y}, \quad 0 < y < \infty. \end{aligned}$$

□

b) Compute EY and EY^2 integrating against the marginal density of Y.

Solution.

$$\begin{aligned} EY &= \int_0^\infty ye^{-y}dy = 1 \quad (\because \text{the density of } \Gamma(2, 1)) \\ EY^2 &= \int_0^\infty y \cdot ye^{-y}dy = 2 \cdot 1 = 2 \end{aligned}$$

□

c) Find the conditional density of Y given $X = x$, and use it to compute $E[Y|X]$ and $E[Y^2|X]$.

***Solution*.**
The conditional density is $p(y|x) = p(y,x)/p(x) = 1/x, \quad 0 < y < x$. Then

$$E[Y|X = x] = \int_0^x \frac{1}{x} y dy = \frac{1}{x} \cdot \frac{1}{2} y^2 \Big|_0^x = \frac{x}{2},$$
$$E[Y^2|X = x] = \int_0^x \frac{1}{x} y^2 dy = \frac{x^2}{3}.$$

Therefore, $E[Y|X] = \frac{X}{2}$ and $E[Y^2|X] = \frac{X^2}{3}$. □

d) Find the expectations of $E[Y|X]$ and $E[Y^2|X]$ integrating against the marginal density of X.

***Solution*.**
By using the fact of $\Gamma(2,1)$,

$$EE[Y|X] = E\left[\frac{X}{2}\right] = \frac{1}{2}\int_0^\infty x \cdot xe^{-x} dx = \frac{1}{2} \cdot 2 = 1,$$
$$EE[Y^2|X] = E\left[\frac{X^2}{3}\right] = \frac{1}{3}\int_0^\infty x^2 \cdot xe^{-x} dx = \frac{1}{3}(2 + 2^2) = 2.$$

□

Chapter 7

Bayesian Estimation