Theoretical Statistics: Topics for a Core Course
Problem Solutions

Robert W. Keener

Hyunsung Kim
Department of Statistics
Chung-Ang University

Update: September 21, 2021

Contents

Chapter 1

Probability and Measure

Problem 1.1

Prove (1.1). If measurable sets $B_n,\ n \geq 1$, are increasing, with $B = \bigcup_{n=1}^{\infty} B_n$, called the limit of the sequence, then

$$\mu(B) = \lim_{n\to\infty} \mu(B_n).$$

Solution.
First, we showed that A_n's are disjoint. If $j < k$, then $B_j \subseteq B_{k-1}$.
Since $A_j \subset B_j \subseteq B_{k-1}$ and $A_k \subset B_{k-1}^c$, A_j and A_k are disjoint.
Also $B_n = \bigcup_{j=1}^{n} A_j$ and $\bigcup_{n=1}^{\infty} A_n = B$,

$$\mu(B) = \sum_{i=1}^{\infty} \mu(A_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} A_i\right) = \lim_{n\to\infty} \mu(B_n).$$

□

Problem 1.8

Prove *Boole's inequality*: For any events $B_1, B_2, \ldots$,

$$P\left(\bigcup_{i\geq 1} B_i\right) \leq \sum_{i\geq 1} P(B_i).$$

Solution.
Let $B = \bigcup_{i=1}^{\infty} B_i$, then $1_B \leq \sum 1_{B_i}$. Then by Fubini's theorem,

$$P(B) = \int 1_B dP \leq \int \sum 1_{B_i} dP = \sum \int 1_{B_i} dP = \sum P(B_i).$$

□

Problem 1.10

Let μ and ν be measures on $(\mathcal{E}, \mathcal{B})$.

a) Show that the sum η defined by $\eta(B) = \mu(B) + \nu(B)$ is also a measure.

***Solution*.**
We should check following 2 conditions.

(i) For arbitrary set $A \in \mathcal{B}$, $\mu(A) \geq 0$ and $\nu(A) \geq 0$. $\Rightarrow \eta(A) = \mu(A) + \nu(A) \geq 0$.
$\therefore \eta : \mathcal{B} \to [0, \infty]$.

(ii) For disjoint set $B_1, B_2, \cdots \subset \mathcal{B}$, then

$$\begin{aligned}\eta\left(\bigcup_{i=1}^{\infty} B_i\right) &= \mu\left(\bigcup B_i\right) + \nu\left(\bigcup B_i\right) \\ &= \sum \mu(B_i) + \sum \nu(B_i) \\ &= \sum \{\mu(B_i) + \nu(B_i)\} \\ &= \sum \eta(B_i)\end{aligned}$$

$\therefore \eta$ is a measure. □

b) If f is a non-negative measurable function, show that

$$\int f d\eta = \int f d\mu + \int f d\nu.$$

***Solution*.**
We show it by 2 stage.

(i) Let $f = \sum_{i=1}^{n} a_i 1_{A_i}$, the non-negative simple function. Then,

$$\begin{aligned}\int f d\eta &= \int \sum_{i=1}^{n} a_i 1_{A_i} d\eta = \sum a_i \eta(A_i) \\ &= \sum a_i \{\mu(A_i) + \nu(A_i)\} = \int f d\mu + \int f d\nu.\end{aligned}$$

(ii) For general case, let f_n is the sequence of non-negative simple functions increasing to f. (i.e. $f_1 \leq f_2 \leq \cdots \leq f$)

$$\begin{aligned}\int f d\eta &= \lim_{n\to\infty} \int f_n d\eta = \lim_{n\to\infty} \left(\int f_n d\mu + \int f_n d\nu\right) \\ &= \lim_{n\to\infty} \int f_n d\mu + \lim_{n\to\infty} \int f_n d\nu = \int f d\mu + \int f d\nu.\end{aligned}$$

□

Problem 1.11

Suppose f is the simple function $1_{(1/2,\pi]} + 21_{(1,2]}$, and let μ be a measure on $\mathbb{R}$ with $\mu\{(0, a^2]\} = a,\ a > 0$. Evaluate $\int f d\mu$.

Solution.
By the integral of simple function and the finite additivity, it can be simply computed as

$$\begin{aligned}\int f d\mu &= \mu\{(1/2,\pi]\} + 2\mu\{(1,2]\} \\ &= [\mu\{(0,\pi]\} - \mu\{(1/2,\pi]\}] + 2\left[\mu\{(0,2]\} - \mu\{(1,2]\}\right] \quad \text{(finite additivity)} \\ &= \left(\sqrt{\pi} - 1/\sqrt{2}\right) + 2\left(\sqrt{2} - 1\right)\end{aligned}$$

□

Problem 1.12

Suppose that $\mu\{(0,a)\} = a^2$ for $a > 0$ and that f is defined by

$$f(x) = \begin{cases} 0, & x \leq 0, \\ 1, & 0 < x < 2, \\ \pi, & 2 \geq x < 5, \\ 0, & x \geq 5. \end{cases}$$

Compute $\int f d\mu$.

Solution.
Since $f = 1_{(0,2)} + 1_{[2,5)}$ is simple, the integral can be computed as

$$\begin{aligned}\int f d\mu &= \mu\{(0,2)\} + \pi\mu\{[2,5)\} \\ &= \mu\{(0,2)\} + \pi\left[\mu\{(0,5)\} - \mu\{(0,2)\}\right] \\ &= 4 - \pi(25 - 4)\end{aligned}$$

□

Problem 1.13

Define the function f by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find simple functions $f_1 \leq f_2 \leq \cdots$ increasing to f (i.e. $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in \mathbb{R}$). Let μ be Lebesgue measure on $\mathbb{R}$. Using our formal definition of an integral and the fact that $\mu\big((a,b]\big) = b-a$ whenever $b > a$ (this might be used to formally define Lebesgue measure), show that $\int f d\mu = 1/2$.

Solution.
Let $f_n = \lfloor 2^n x \rfloor / 2^n$ for $0 < x \leq 1$ and 0 otherwise. ($\lfloor y \rfloor$ is a floor function.) Then,

$$f_1(x) = \lfloor 2x \rfloor / 2 = \begin{cases} 0, & x < 1/2, \\ 1/2, & 1/2 \leq x < 1, \\ 1, & x = 1 \end{cases}$$

$$f_2(x) = \lfloor 2^2 x \rfloor / 2^2 = \begin{cases} 0, & x < 1/2^2, \\ 1/2^2, & 1/2^2 \le x < 2/2^2, \\ 2/2^2, & 2/2^2 \le x < 3/2^2, \\ 3/2^2, & 3/2^2 \le x < 1, \\ 1, & x = 1 \end{cases}$$

$$\vdots$$

$$f_n(x) = \lfloor 2^n x \rfloor / 2^n = \begin{cases} 0, & x < 1/2^n, \\ 1/2^n, & 1/2^n \le x < 2/2^n, \\ \vdots & \\ (2^n-1)/2^n, & (2^n-1)/2^n \le x < 1, \\ 1, & x = 1 \end{cases}$$

$$\begin{aligned} \therefore \int f_n d\mu &= \frac{1}{2^n}\left(\frac{1}{2^n} + \frac{2}{2^n} + \cdots + \frac{2^n-1}{2^n}\right) \\ &= \frac{1+2+\cdots+(2^n-1)}{4^n} \\ &= \frac{2^n(2^n-1)}{2\cdot 4^n} \\ &\longrightarrow \frac{1}{2}. \end{aligned}$$

My solution

Let the simple function $f_n = \sum_{i=1}^n \frac{i}{n} 1_{(\frac{i-1}{n}, \frac{i}{n}]}$. Then,

$$\int f_n d\mu = \sum_{i=1}^n \frac{i}{n}\frac{1}{n} = \frac{1}{n^2}\frac{n(n+1)}{2} \longrightarrow \frac{1}{2}.$$

□

Problem 1.16

Define $F(a-) = \lim_{x\uparrow a} F(x)$. Then, if F is non-decreasing, $F(a-) = \lim_{n\to\infty} F(a-1/n)$. Use (1.1)[Continuity of measure] to show that if a random variable X has cumulative distribution function F_X,

$$P(X < a) = F_X(a-).$$

Also, show that

$$P(X = a) = F_X(a) - F_X(a-).$$

Solution.

(i) Let $B_n = \{X \le a - 1/n\}$ and $\bigcup_{n=1}^\infty B_n = \{X < a\}$. By continuity of measure,

$$P(B) = P(X < a) = \lim_{n\to\infty} P(X \le a - 1/n) = \lim_{n\to\infty} F_X(a - 1/n) = F_X(a-).$$

(ii) Since $\{X < a\}$ and $\{X = a\}$ are disjoint with union $\{X \le a\}$,

$$P(X < a) + P(X = a) = P(X \le a).$$

$$\therefore P(X = a) = F_X(a) - F_X(a-).$$

$\square$

Problem 1.17

Suppose X is a geometric random variable with mass function

$$p(x) = P(X = x) = \theta(1-\theta)^x, \quad x = 0, 1, \ldots,$$

where $\theta \in (0, 1)$ is a constant. Find the probability that X is even.

Solution.

$$\begin{aligned} P(X \text{ is even}) &= P(X = 0) + P(X = 2) + P(X = 4)\cdots \\ &= \theta + \theta(1-\theta)^2 + \theta(1-\theta)^4 + \cdots \\ &= \frac{\theta}{1-(1-\theta)^2} \\ &= \frac{1}{2-\theta} \end{aligned}$$

$\square$

Problem 1.18

Let X be a function mapping $\mathcal{E}$ into $\mathbb{R}$. Recall that if B is a subset of $\mathbb{R}$, then $X^{-1}(B) = \{e \in \mathcal{E} : X(e) \in B\}$. Use this definition to prove that

$$X^{-1}(A \cap B) = X^{-1}(A) \cap X^{-1}(B),$$

$$X^{-1}(A \cup B) = X^{-1}(A) \cup X^{-1}(B),$$

and

$$X^{-1}\left(\bigcup_{i=0}^{\infty} A_i\right) = \bigcup_{i=0}^{\infty} X^{-1}(A_i).$$

Solution.

(i)

$$\begin{aligned} e \in X^{-1}(A \cap B) &\Leftrightarrow X(e) \in A \cap B \\ &\Leftrightarrow X(e) \in A \text{ and } X(e) \in B \\ &\Leftrightarrow e \in X^{-1}(A) \text{ and } e \in X^{-1}(B) \\ &\Leftrightarrow e \in X^{-1}(A) \cap X^{-1}(B). \end{aligned}$$

(ii) Similarly,

$$\begin{aligned} e \in X^{-1}(A \cup B) &\Leftrightarrow X(e) \in A \cup B \\ &\Leftrightarrow X(e) \in A \text{ or } X(e) \in B \\ &\Leftrightarrow e \in X^{-1}(A) \text{ or } e \in X^{-1}(B) \\ &\Leftrightarrow e \in X^{-1}(A) \cup X^{-1}(B). \end{aligned}$$

(iii)

$$\begin{aligned} e \in X^{-1}(\bigcup_{i=0}^{\infty} A_i) &\Leftrightarrow X(e) \in \bigcup_{i=0}^{\infty} A_i \\ &\Leftrightarrow X(e) \in A_i \text{ for some } i \\ &\Leftrightarrow e \in X^{-1}(A_i) \text{ for some } i \\ &\Leftrightarrow e \in \bigcup_{i=0}^{\infty} X^{-1}(A_i). \end{aligned}$$

□

Problem 1.19

Let P be a probability measure on $(\mathcal{E}, \mathcal{B})$, and let X be a random variable. Show that the distribution P_X of X defined by $P_X(B) = P(X \in B) = P(X^{-1}(B))$ is a measure (on the Borel sets of $\mathbb{R}$).

Solution.
To show that P_X is a measure, we should check 2 conditions.

(i) $P_X(B) = P\big(X^{-1}(B)\big) \geq 0$.

(ii) From Problem 1.18, $X^{-1}\left(\bigcup_i A_i\right) = \bigcup_i X^{-1}(A_i)$ is hold.
Also, if B_i and B_j are disjoint, then $X^{-1}(B_i)$ and $X^{-1}(B_j)$ are disjoint. ($\because$ If $e \in \mathcal{E}$ lie in both $X^{-1}(B_1)$ and $X^{-1}(B_2)$, $X(e)$ lies in B_1 and B_2.)
Therefore, for arbitrary disjoint set $B_1, B_2, \ldots$ with union $B = \bigcup_i B_i$, $X^{-1}(B_1), X^{-1}(B_2), \ldots$ are also disjoint with union $X^{-1}(B)$.

$$\therefore \sum_i P_X(B_i) = \sum_i P\big(X^{-1}(B_i)\big) = P\big(X^{-1}(B)\big) = P_X(B).$$

The above 2 conditions are hold, P_X is a measure. □

Problem 1.21

Let X have a uniform distribution on $(0,1)$; that is, X is absolutely continuous with density p defined by

$$p(x) = \begin{cases} 1, & x \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Let Y_1 and Y_2 denote the first two digits of X when X is written as a binary decimal (so $Y_1 = 0$ if $X \in (0, 1/2)$ for instance). Find $P(Y_1 = i, Y_2 = j)$, $i = 0$ or $1, j = 0$ or 1.

Solution.
For all cases, the probability are same. In other words,

$$P(Y_1 = 0, Y_2 = 0) = P(Y_1 = 1, Y_2 = 0) = P(Y_1 = 0, Y_2 = 1) = P(Y_1 = 1, Y_2 = 1) = 1/4.$$

□

Problem 1.22

Let $\mathcal{E} = (0,1)$, let $\mathcal{B}$ be the Borel subsets of $\mathcal{E}$, and let $P(A)$ be the length of A for $A \in \mathcal{B}$. (P would be called the *uniform probability measure* on $(0,1)$.) Define the random variable X by

$$X(e) = \min\{e, 1/2\}.$$

Let μ be the sum of Lebesgue measure on $\mathbb{R}$ and counting measure on $\mathcal{X}_0 = \{1/2\}$. Show that the distribution P_X of X is absolutely continuous with respect to μ and find the density of P_X.

Solution.

(i) Claim: For any $B \in \mathbb{R}$, $\mu(B) = 0 \Longrightarrow P_X(B) = 0$.

$$\begin{aligned}\{X \in B\} &= \{y \in (0,1) : X(y) \in B\} \\ &= \begin{cases} B \cap (0,1/2), & 1/2 \notin B, \\ \big(B \cap (0,1/2)\big) \cup [1/2, 1), & 1/2 \in B. \end{cases}\end{aligned}$$

Let λ be a Lebesgue measure and ν be a counting measure on $\{1/2\}$. Then,

$$\mu(B) = 0 \Longleftrightarrow \lambda(B) = 0 \text{ and } 1/2 \notin B. \ (\mu(B) = \nu(B) = 0)$$

$$\Longrightarrow P_X(B) = P(X \in B) = P\big(B \cap (0,1/2)\big) = \lambda\big(B \cap (0,1/2)\big) = 0.$$

$$\therefore P_X \text{ is absolutely continuous w.r.t. } \lambda + \nu \overset{\text{def}}{=} \mu.$$

(ii) Find the density of P_X.
From the above equation,

$$P(X \in B) = \lambda\big(B \cap (0,1/2)\big) + \frac{1}{2} 1_B(1/2). \tag{1.1}$$

If f is the density, then (1.1) should be equal to $\int_B f d(\lambda + \nu)$.

$$\begin{aligned}\int_B f d(\lambda + \nu) &= \int f 1_B d(\lambda + \nu) \\ &= \int \left(1_{B \cap (0,1/2)} + \frac{1}{2} 1_{\{1/2\} \cap B}\right) d(\lambda + \nu) \\ &= \lambda\big(B \cap (0,1/2)\big) + \nu\big(B \cap (0,1/2)\big) + \frac{1}{2}\lambda\big(\{1/2\} \cap B\big) + \frac{1}{2}\nu\big(\{1/2\} \cap B\big) \\ &= \lambda\big(B \cap (0,1/2)\big) + \frac{1}{2} 1_B(1/2).\end{aligned}$$

$$\therefore f = 1_{(0,1/2} + \frac{1}{2} 1_{\{1/2\}}.$$

□

Problem 1.23

The standard normal distribution $N(0,1)$ has density ϕ given by

$$\phi(x) = \frac{e^{-x^2/2}}{\sqrt{w\pi}}, \qquad x \in \mathbb{R},$$

with respect to Lebesgue measure λ on $\mathbb{R}$. The corresponding cumulative distribution function is Φ, so

$$\Phi(x) = \int_{-\infty}^{x} \phi(z)dz$$

for $x \in \mathbb{R}$. Suppose that $X \sim N(0,1)$ and that the random variable Y equals X when $|X| < 1$ and is 0 otherwise. Let P_Y denote the distribution of Y and let μ be counting measure on $\{0\}$. Find the density of P_Y with respect to $\lambda + \mu$.

<u>Solution</u>.
Let $g(x) = x1_{|x|<1}$, then $Y = g(X)$.
For any integrable function f, the expectation of $f(Y)$ against the density of X is

$$Ef(Y) = Ef\big(g(X)\big) = \int f\big(g(x)\big)\phi(x)dx = \int f\big(g(x)\big)\phi(x)1_{(-1,1)}(x)dx + cf(0), \tag{1.2}$$

where $c = \int \phi(x)1_{|x|>1}(x)dx = 2\Phi(-1)$.
Similarly, the expectation of $f(Y)$ against the density p of Y is

$$Ef(Y) = \int fpd(\lambda+\mu) = \int fpd\lambda + \int fpd\mu = \int f(x)p(x)dx + f(0)p(0). \tag{1.3}$$

Then, (1.2) and (1.3) should be same. Therefore,

$$\begin{aligned} f = 1_{\{0\}} &\Longrightarrow p(0) = 2\Phi(-1), \\ f(0) = 0 &\Longrightarrow \int f(x)\phi(x)1_{(-1,1)}(x)dx = \int f(x)p(x)dx \\ &\Longrightarrow p(x) = \phi(x) \text{ for } 0 < |x| < 1. \end{aligned}$$

$$\therefore p(x) = 2\Phi(-1)1_{\{0\}}(x) + \phi(x)1_{(0,1)}(|x|) \text{ is the density of } P_Y.$$

□

Problem 1.24

Let μ be a σ-finite measure on a measurable space $(X, \mathcal{B})$. Show that μ is absolutely continuous with respect to some probability measure P.
<u>Hint</u>: You can use the fact that if $\mu_1, \mu_2, \dots$ are probability measures and $c_1, c_2, \dots$ are non-negative constants, then $\sum c_i\mu_i$ is a measure. (The proof for Problem 1.10 extents easily to this case.) The measure μ_i, you will want to consider, are truncations of μ to sets A_i covering X with $\mu(A_i) < \infty$, given by $\mu_i(B) = \mu(B \cap A_i)$. With the constants c_i chosen properly, $\sum c_i\mu_i$ will be a probability measure.

<u>Solution</u>.

(i) μ is finite $\Longrightarrow$ Trivial.
$\because$ Since $\mu(X) < \infty$, $P = \mu/\mu(X)$ be a probablity measure. Therefore, $P(N) = 0$ implies $\mu(N) = 0$.

(ii) μ is infinite but σ-finite.
$\implies \exists A_1, A_2, \cdots \in \mathcal{B}$ s.t. $\bigcup A_i = X$ and $0 < \mu(A_i) < \infty \ \forall i$.
From Hint, $\mu_i(B) = \mu(B \cap A_i)$ and also $\mu_i(X) = \mu(A_i) \implies \mu_i$ is a finite measure.
Let $b_i = 1/2^i$ (or any other seq of positive constants s.t. $\sum b_i = 1$) and define $c_i = b_i/\mu(A_i)$.
Then $P = \sum_i c_i\mu_i$ is a probability measure, since $P(X) = \sum_i c_i\mu_i(X) = \sum_i \left(\frac{b_i}{\mu(A_i)}\right)\mu(A_i) = 1$.
Suppose $P(N) = \sum_i c_i\mu_i(N) = 0$. Then $\mu_i(N) = \mu(N \cap A_i) = 0 \ \forall i$.
By Boole's inequality,

$$\mu(N) = \mu\big(\bigcup_i (N \cap A_i)\big) \le \sum_i \mu(N \cap A_i) = 0.$$

For any null set for P is a null set for μ.
$\therefore \mu \ll P$.

$\square$

Problem 1.25

The monotone convergence theorem states that if $0 \le f_1 \le f_2 \le \cdots$ are measurable functions and $f = \lim_{n\to\infty} f_n$, then $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$. Use this result to prove the following assertions.

a) Show that if $X \sim P_X$ is a random variable on $(\mathcal{E}, \mathcal{B}, P)$ and f is a non-negative measurable function, then

$$\int f\big(X(e)\big)dP(e) = \int f(x)dP_X(x).$$

Hint: Try it first with f an indicator function. For the general case, let f_n be a sequence of simple functions increasing to f.

Solution.
We will show the equation by 3 steps.

(i) Suppose $f = 1_A$ (indicator function).
Then, $f\big(X(e)\big) = 1$ for $X(e) \in A$.
$\Rightarrow f \circ X = 1_B$ where $B = \{e : X(e) \in A\}$.
By definition, $P_X(A) = P(B)$.

$$\int f\big(X(e)\big)dP(e) = \int 1_B(e)dP(e) = P(B),$$
$$\int f(x)dP_X(x) = \int 1_A(x)dP_X(x) = P_X(A).$$

$\therefore$ The statement is holds.

(ii) Suppose $f = \sum_{i=1}^n c_i 1_{A_i}$ (simple function).

$$\begin{aligned}\therefore \int f\big(X(e)\big)dP(e) &= \int \sum_i c_i 1_{A_i}\big(X(e)\big)dP(e)\\ &= \sum_i c_i \int 1_{A_i}\big(X(e)\big)dP(e)\\ &= \sum_i c_i \int 1_{A_i}(x)dP_X(x) \qquad \big(\text{by (i)}\big)\end{aligned}$$

$$= \int \sum_i c_i 1_{A_i}(x) dP_X(x)$$
$$= \int f(x) dP_X(x).$$

(iii) Suppose f is a non-negative measurable function (general case), and f_n is non-negative simple functions increasing to f.
$\Rightarrow f_n \circ X \nearrow f \circ X$.

$$\begin{aligned}\therefore \int f\big(X(e)\big)dP(e) &= \lim_{n\to\infty} \int f_n\big(X(e)\big)dP(e) \qquad \text{(by M.C.T.)}\\ &= \lim_{n\to\infty} \int f_n(x)dP_X(x) \qquad \big(\text{by (ii)}\big)\\ &= \int f(x)dP_X(x) \qquad \text{(by M.C.T.)}\end{aligned}$$

□

b) Suppose that P_X has density p with respect to μ, and let f be a non-negative measurable function. Show that
$$\int f dP_X = \int f p d\mu.$$

Solution.
It can be also showed by 3 steps.

(i) Let $f = 1_A$ (indicator function).
$$\int f dP_X = \int 1_A dP_X = P_X(A) = \int_A p d\mu = \int 1_A p d\mu = \int f p d\mu.$$

(ii) Suppose $f = \sum_{i=1}^n c_i 1_{A_i}$ (simple function).
$$\int f dP_X = \int \sum_i c_i 1_{A_i} dP_X = \sum_i c_i \int 1_{A_i} dP_X \overset{\text{(i)}}{=} \sum_i c_i \int 1_{A_i} p d\mu = \int \sum_i c_i 1_{A_i} p d\mu = \int f p d\mu.$$

(iii) For general case, let a non-negative measurable function f and non-negative simple functions f_n s.t. $f_n \nearrow f$. Then by M.C.T.,
$$\int f dP_X \overset{\text{M.C.T.}}{=} \lim_{n\to\infty} \int f_n dP_X \overset{\text{(ii)}}{=} \lim_{n\to\infty} \int f_n p d\mu \overset{\text{M.C.T.}}{=} \int f p d\mu.$$

□

Problem 1.26

The gamma distribution.

a) The gamma function is defined for $\alpha > 0$ by
$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$. Show that $\Gamma(X+1) = x!$ for $x = 0, 1, \ldots$.

Solution.
By integration by parts,

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = -t^x e^{-t}\Big|_0^\infty + (\alpha - 1)\int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x).$$

Since $\Gamma(1) = \int_0^\infty e^{-x} dx = 1$,

$$\Gamma(x+1) = x\Gamma(x) = x(x-1)\Gamma(x-1) = \cdots = x(x-1)\cdots 2\cdot 1\cdot \Gamma(1) = x!.$$

□

b) Show that the function

$$p(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & \text{otherwise}, \end{cases}$$

is a (Lebesgue) probability density when $\alpha > 0$ and $\beta > 0$. This density is called the gamma density with parameters α and β. The corresponding probability distribution is denoted $\Gamma(\alpha, \beta)$.

Solution.
Using change of variable, $y = x/\beta \Rightarrow dx = \beta dy$. Therefore,

$$\begin{aligned} \int p(x)dx &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx \\ &= \int \frac{1}{\Gamma(\alpha)\beta^\alpha} (\beta y)^{\alpha-1} e^{-y} \beta dy \\ &= \frac{1}{\Gamma(\alpha)} \int y^{\alpha-1} e^{-y} dy \\ &= 1. \end{aligned}$$

□

c) Show that if $X \sim \Gamma(\alpha, \beta)$, then $EX^r = \beta^r \Gamma(\alpha + r)/\Gamma(\alpha)$. Use this formula to find the mean and variance of X.

Solution.

$$\begin{aligned} EX^r &= \int_0^\infty x^r p(x) dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha+r-1} e^{-x/\beta} dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha+r)\beta^{\alpha+r}} x^{\alpha+r-1} e^{-x/\beta} dx \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \beta^r \\ &= \beta^r \Gamma(\alpha+r)/\Gamma(\alpha). \end{aligned}$$

By using this,

$$EX = \beta\Gamma(\alpha+1)/\Gamma(\alpha) = \alpha\beta,$$

$$EX^2 = \beta^2\Gamma(\alpha+2)/\Gamma(\alpha) = (\alpha+1)\alpha\beta^2,$$

$$Var(X) = EX^2 - \{EX\}^2 = \alpha\beta^2.$$

□

Problem 1.27

Suppose X has a uniform distribution on $(0,1)$. Find the mean and covariance matrix of the random vector $\binom{X}{X^2}$.

Solution.
Since $X \sim \text{Uniform}(0,1)$, the density of X is $p(x) = 1$ for $x \in (0,1)$. Thus,

$$EX^r = \int x^r dx = \frac{1}{r+1}.$$

Therefore,

$$E\left(\begin{pmatrix} X \\ X^2 \end{pmatrix}\right) = \begin{pmatrix} EX \\ EX^2 \end{pmatrix},$$

$$Cov\left(\begin{pmatrix} X \\ X^2 \end{pmatrix}\right) = \begin{pmatrix} Var(X) & Cov(X, X^2) \\ Cov(X^2, X) & Var(X^2) \end{pmatrix} = \begin{pmatrix} EX^2 - \{EX\}^2 & EX^3 - EXEX^2 \\ EX^3 - EXEX^2 & EX^4 - \{EX^2\}^2 \end{pmatrix}.$$

□

Problem 1.28

If $X \sim N(0,1)$, find the mean and covariance matrix of the random vector $\binom{X}{I\{X>c\}}$.

Solution.
Let $\Phi(x)$ is a cumulative distribution function of X, and ϕ is a density of X. Then,

$$E\left(\begin{pmatrix} X \\ I\{X > c\} \end{pmatrix}\right) = \begin{pmatrix} EX \\ EI\{X > c\} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 - \Phi(c) \end{pmatrix},$$

$$Cov\left(\begin{pmatrix} X \\ I\{X > c\} \end{pmatrix}\right) = \begin{pmatrix} Var(X) & EX1_{X>c} - EXE1_{X>c} \\ EX1_{X>c} - EXE1_{X>c} & E1_{X>c}^2 - \{E1_{X>c}\}^2 \end{pmatrix} = \begin{pmatrix} 1 & \phi(c) \\ \phi(c) & \Phi(c)(1 - \Phi(c)) \end{pmatrix}.$$

$$\because EX1_{X>c} = \int_c^\infty \frac{1}{\sqrt{2\pi}} xe^{-x^2/2} dx = \phi(c) \text{ and } 1_{X>c}^2 = 1_{X>c}.$$

□

Problem 1.32

Suppose $E|X| < \infty$ and let

$$h(t) = \frac{1 - E\cos(tX)}{t^2}.$$

Use Fubini's theorem to find $\int_0^\infty h(t)dt$. <u>Hint</u>:

$$\int_0^\infty \big(1 - \cos(u)\big)u^{-2} du = \frac{\pi}{2}.$$

Solution.
By Funini's theorem,

$$\int_0^\infty h(t)dt = \int_0^\infty \left(\frac{1 - E\cos(tX)}{t^2}\right) dt$$

$$
\begin{aligned}
&= \int_0^\infty E\left(\frac{1-\cos(tX)}{t^2}\right) dt \\
&= \int_0^\infty \int \frac{1-\cos(tx)}{t^2} dP_X(x) dt \\
&= \int \int_0^\infty \frac{1-\cos(tx)}{t^2} dt dP_X(x) \quad \text{(by Funini's theorem)} \\
&\quad u = tx \Rightarrow du = xdt, t = x/u \\
&= \int \int_0^\infty |x| \frac{1-\cos(u)}{u^2} du dP_X(x) \\
&= \int \frac{\pi}{2} |x| dP_X(x) \\
&= \frac{\pi}{2} E|X|.
\end{aligned}
$$

□

Problem 1.33

Suppose X is absolutely continuous with density $p_X(x) = xe^{-x},\ x > 0$ and $p_X(x) = 0,\ x \le 0$. Define $c_n = E(1+X)^{-n}$. Use Funini's theorem to evaluate $\sum_{n=1}^\infty c_n$.

***Solution*.**
By Funini's theorem,

$$
\begin{aligned}
\sum_{n=1}^\infty c_n &= \sum_n E(1+X)^{-n} \\
&= \sum_n \int_0^\infty (1+x)^{-n} xe^{-x} dx \\
&= \int_0^\infty \sum_n (1+x)^{-n} xe^{-x} dx \\
&\text{Since } \left|\frac{1}{1+x}\right| < 1,\ \sum_{n=1}^\infty \left(\frac{1}{1+x}\right)^n = \frac{1/(1+x)}{1-1/(1+x)} = \frac{1}{x}. \\
&= \int_0^\infty \frac{1}{x} xe^{-x} dx \\
&= \int_0^\infty e^{-x} dx \\
&= 1.
\end{aligned}
$$

□

Problem 1.36

Suppose X and Y are independent random variables, and let F_X and F_Y denote their cumulative distribution functions.

a) Use smoothing to show that the cumulative distribution function of $S = X + Y$ is

$$F_S(s) = P(X + Y \le s) = EF_X(s - Y). \tag{1.4}$$

Solution.
Since X and Y are independent, $P(X+Y\le s|Y=y)=P(X\le s-y)=F_X(s-y)$.
Thus, $P(X+Y\le s|Y)=F_X(s-Y)$.

$$\therefore F_S(s)=P(X+Y\le s)=EP(X+Y\le s|Y)=EF_X(s-Y).$$

$\square$

b) If X and Y are independent and Y is almost surely positive, use smoothing to show that the cumulative distribution function of $W=XY$ is $F_W(w)=EF_X(w/Y)$ for $w>0$.

Solution.
Since X and Y are independent, $P(XY\le w|Y=y)=P(X\le w/y|Y=y)=P(X\le w/y)=F_X(w/y)$, for $y>0$.
Thus, $P(XY\le w|Y)=F_X(w/Y)$ a.s.

$$\therefore F_W(w)=P(XY\le w)=EP(XY\le w|Y)=EF_X(w/Y).$$

$\square$

Problem 1.37

Differentiating (1.4) with respect to s one can show that if X is absolutely continuous with density p_X, then $S=X+Y$ is absolutely continuous with density

$$p_S(s)=Ep_X(s-Y)$$

for $s\in\mathbb{R}$. Use this formula to show that if X and Y are independent with $X\sim\Gamma(\alpha,1)$ and $Y\sim\Gamma(\beta,1)$, then $X+Y\sim\Gamma(\alpha+\beta,1)$.

Solution.
Since $X\sim\Gamma(\alpha,1)$, $P_X(x-Y)=0$ for $Y\ge x$ ($\because p_X(x)=0,\ \forall x<0$). Then,

$$\begin{aligned}
P_S(x)&=EP_X(x-Y)\\
&=E\left[\frac{1}{\Gamma(\alpha)}(x-Y)^{\alpha-1}e^{-(x-Y)}1_{(0,x)}(Y)\right]\\
&=\int_0^x\frac{1}{\Gamma(\alpha)}(x-y)^{\alpha-1}e^{-(x-y)}\frac{1}{\Gamma(\beta)}y^{\beta-1}e^{-y}dy\\
&=\int_0^x\frac{1}{\Gamma(\alpha)\Gamma(\beta)}(x-y)^{\alpha-1}y^{\beta-1}e^{-x}dy\\
&\text{Change the variable by } u=y/x\Rightarrow du=1/xdy.\\
&=\int_0^1\frac{1}{\Gamma(\alpha)\Gamma(\beta)}(x-xu)^{\alpha-1}(xu)^{\beta-1}e^{-x}xdu\\
&=\int_0^1\frac{1}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha+\beta-1}(1-u)^{\alpha-1}u^{\beta-1}e^{-x}du\\
&=\frac{1}{\Gamma(\alpha+\beta)}x^{\alpha+\beta-1}e^{-x},\quad\text{for } x>0.
\end{aligned}$$

$\therefore S=X+Y\sim\Gamma(\alpha+\beta,1)$. $\square$

Problem 1.38

Let Q_λ denote the exponential distribution with failure rate λ, given in Problem 1.30. Let X be a discrete random variable taking values in $\{1, \dots, n\}$ with mass function

$$P(X = k) = \frac{2k}{n(n+1)}, \quad k = 1, \dots, n,$$

and assume that the conditional distribution of Y given $X = x$ is exponential with failure rate x,

$$Y|X = x \sim Q_x.$$

a) Find $E[Y|X]$.

Solution.
From Problem 1.30, $q_\lambda(x) = \lambda 3^{-\lambda x},\ \lambda > 0$. Then, the mean of Q_λ is

$$\int x dQ_\lambda(x) = \int x q_\lambda(x) dx = \int_0^\infty x\lambda e^{-\lambda x} dx = \frac{1}{\lambda}.$$

$$\therefore E(Y|X = x) = \frac{1}{x} \Rightarrow EY|X = \frac{1}{X}.$$

□

b) Use smoothing to compute EY.

Solution.

$$EY = E[E(Y|X)] = E\frac{1}{X} = \sum_{x=1}^{n} \frac{1}{x}\frac{2x}{n(n+1)} = \sum_{x=1}^{n} \frac{2}{n(n+1)} = \frac{2}{n+1}.$$

□

Problem 1.39

Let X be a discrete random variable uniformly distributed on $\{1, \dots, n\}$, so $P(X = k) = 1/n,\ k = 1, \dots, n$, and assume that the conditional distribution of Y given $X = x$ is exponential with failure rate x.

a) For $y > 0$ find $P[Y > y|X]$.

Solution.

$$P(Y > y|X = x) = \int_y^\infty x e^{-xt} dt = -e^{-xt}\Big|_y^\infty = e^{-xy}.$$

$$\therefore P(X > y|X) = e^{-Xy}.$$

□

b) Use smoothing to compute $P(Y > y)$.

Solution.
Using the sum of the geometric sequence (등비급수의 합),

$$P(Y > y) = EP(Y > y|X) = Ee^{-Xy} = \sum_{x=1}^{n} e^{-xy}\frac{1}{n} = \frac{1}{n}\frac{e^{-y}(1 - e^{-ny})}{1 - e^{-y}} = \frac{1 - e^{-ny}}{n(e^y - 1)}.$$

□

c) Determine the density of Y.

Solution.
Let the density of Y is p_Y. Then,

$$\begin{aligned} p_Y(y) &= \frac{d}{dy}\big(1 - P(Y > y)\big) \\ &= \frac{d}{dy}\left(\frac{1 - e^{-ny}}{n(e^y - 1)}\right) \\ &= \frac{-ne^{-ny}(ne^y - n) + ne^y(1 - e^{-ny})}{n^2(e^y - 1)^2} \\ &= \frac{-ne^y e^{-ny} + ne^{-ny} + e^y - e^y e^{-ny}}{n(e^y - 1)^2} \\ &= \frac{e^y(1 - e^{-ny})}{n(e^y - 1)^2} - \frac{e^{-ny}}{e^y - 1}. \end{aligned}$$

□

Chapter 2

Exponential Families

Problem 2.1

Consider independent Bernoulli trials with success probability p and let X be the number of failures before the first success. Then $P(X = x) = p(1-p)^x$, for $x = 0, 1, \ldots$, and X has the geometric distribution with parameter p, introduced in Problem 1.17.

a) Show that the geometric distributions form an exponential family.

Solution.

$$P(X = x) = p(1-p)^x = \exp\big\{x \log(1-p) - (-\log p)\big\}.$$

$\therefore$ the geometric distribution is an exponential family. $\square$

b) Write the densities for the family in canonical form, identifying the canonical parameter η, and the function $A(\eta)$.

Solution.
Let $\eta(p) = \log(1-p)$. Then, $P(X = x) = \exp\big\{\eta x - \big(-\log(1-e^\eta)\big)\big\}$.
$\therefore A(\eta) = -\log(1-e^\eta)$ with $T(x) = x$. $\square$

c) Find the mean of the geometric distribution using a differential identity.

Solution.

$$E_\eta(T) = A'(\eta) = \frac{e^\eta}{1-e^\eta} = \frac{1-p}{p}.$$

$\square$

d) Suppose $X_1, \ldots, X_n$ are i.i.d. from a geometric distribution. Show that the joint distributions form an exponential family, and find the mean and variance of T.

Solution.

$$P(X_1 = x_1, \ldots, X_n = x_n) = \prod_{i=1}^n P(X_i = x_i) = p^n (1-p)^{\sum_{i=1}^n x_i} = \exp\Big\{\sum_i x_i \log(1-p) - (-n \log p)\Big\}.$$

Therefore, the joint distribution is also exponential family.
Now, let $\eta = \log(1-p)$. Then, the canonical exponential family is obtained with $A(\eta) = -n\log(1-e^\eta)$ and $T = \sum_i x_i$.

$$ET = \kappa_1 = A'(\eta) = \frac{ne^\eta}{1-e^\eta} = \frac{n(1-p)}{p},$$

$$Var(T) = \kappa_2 = A''(\eta) = \frac{ne^\eta(1-e^\eta)+ne^{2\eta}}{(1-e^\eta)^2} = \frac{np(1-p)+n(1-p)^2}{p^2} = \frac{n(1-p)}{p^2}.$$

□

Problem 2.2

Determine the canonical parameter space Ξ, and find densities for the one-parameter exponential family with μ Lebesgue measure on $\mathbb{R}^2$, $h(x,y) = \exp\big[-(x^2+y^2)/2\big]/(2\pi)$, and $T(x,y) = xy$.

Solution.
By definition of the canonical exponential family, the pdf be represented as

$$p(x,y) = \exp\big(\eta T(x,y) - A(\eta)\big)h(x,y) = \exp\big\{\eta xy - A(\eta) - (x^2+y^2)/2\big\}/(2\pi).$$

Thus,

$$\begin{aligned}\int p(x,y)d\mu(x,y) = 1 &\iff \int\int \exp\big\{\eta xy - A(\eta) - (x^2+y^2)/2\big\}/(2\pi)dxdy = 1\\ &\iff e^{A(\eta)} = \int\int \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(x-\eta y)^2\right\}dx\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y^2-\eta^2y^2)}dy\\ &\iff e^{A(\eta)} = \frac{1}{\sqrt{2\pi}}\int e^{-\frac{1}{2}y^2(1-\eta^2)}dy \underset{\substack{\uparrow\\ \text{the integral is finite}\Leftrightarrow|\eta|<1}}{=} (1-\eta^2)^{-1/2}.\end{aligned}$$

$$\therefore A(\eta) = -\frac{1}{2}\log(1-\eta^2) \text{ for } \eta\in\Xi = (-1,1).$$

$\therefore$ The canonical exponential density is

$$\exp\big\{\eta xy + \frac{1}{2}\log(1-\eta^2) - (x^2+y^2)/2\big\}/(2\pi).$$

□

Problem 2.4

Find the natural parameter space Ξ and densities p_η for a canonical one-parameter exponential family with μ Lebesgue measure on $\mathbb{R}$, $T_1(x) = \log x$, and $h(x) = (1-x)^2,\ x\in(0,1)$, and $h(x) = 0,\ x\notin(0,1)$.

Solution.

$$\begin{aligned}e^{A(\eta)} &= \int_0^1 e^{\eta\log x}(1-x)^2dx\\ &\text{Let } t = \log x \Rightarrow dt = 1/xdx = e^{-t}dx,\ -\infty<t<0\\ &= \int_{-\infty}^0 e^{\eta t}(1-e^t)^2e^tdt\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{0} \left(e^{t(\eta+1)} - 2e^{t(\eta+2)} + e^{t(\eta+3)}\right) dt \\
&= \left(\frac{1}{\eta+1}e^{t(\eta+1)} - \frac{2}{\eta+2}e^{t(\eta+2)} + \frac{1}{\eta+3}e^{t(\eta+3)}\right)\Big|_{-\infty}^{0} \\
&= \frac{1}{\eta+1} - \frac{2}{\eta+2} + \frac{1}{\eta+3} \\
&= \frac{2}{(\eta+1)(\eta+2)(\eta+3)}, \quad \eta \in \Xi = (-1, \infty).
\end{aligned}
$$

($\because$ For a finite integral, $\eta+1>0,\ \eta+2>0$, and $\eta+3>0 \Rightarrow \eta > -1$.)

$$\therefore p_\eta(x) = \exp\big(\eta T(x) - A(\eta)\big)h(x) = \frac{1}{2}(\eta+1)(\eta+2)(\eta+3)(1-x)^2x^\eta, \quad x \in (0,1).$$

$\square$

Problem 2.5

Find the natural parameter space Ξ and densities p_η for a canonical one-parameter exponential family with μ Lebesgue measure on $\mathbb{R}$, $T_1(x) = -x$, and $h(x) = e^{-2\sqrt{x}}/\sqrt{x},\ x>0$, and $h(x) = 0,\ x \le 0$. (Hint: After a change of variables, relevant integrals will look like integrals against a normal density. You should be able to express the answer using Φ, the standard normal cumulative distribution function.) Also, determine the mean and variance for a variable X with this density.

Solution.

$$
\begin{aligned}
e^{A(\eta)} &= \int_0^\infty e^{-\eta x - 2\sqrt{x}}\frac{1}{\sqrt{x}}dx \\
&\text{To the integral is finite, } \eta \in \Xi = [0,\infty), \\
&\text{Let } t = \sqrt{x} \Rightarrow t^2 = x,\ 2tdt = dx \\
&= \int_0^\infty \frac{1}{t}e^{-\eta t^2 - 2t}2tdt \\
&= 2\int_0^\infty e^{-\eta\left(t^2 + \frac{2}{\eta}t\right)}dt \\
&= 2\int_0^\infty e^{-\frac{2\eta}{2}\left(t+\frac{1}{\eta}\right)^2}dt \cdot e^{-\eta(-\frac{1}{\eta^2})} \\
&\text{Since inner term of integration } \sim N(-1/\eta, 1/2\eta), \\
&= 2\sqrt{2\pi}/\sqrt{2\eta}\cdot e^{\frac{1}{\eta}}\Phi\left(\frac{-1/\eta}{\sqrt{1/(2\eta)}}\right) \\
&= \frac{2\sqrt{\pi}}{\sqrt{\eta}}e^{1/\eta}\Phi(-\sqrt{2/\eta})
\end{aligned}
$$

$$\Rightarrow A(\eta) = \log 2\sqrt{\pi} - \frac{1}{2}\log\eta + \frac{1}{\eta} + \log\Phi\left(-\sqrt{\frac{2}{\eta}}\right).$$

$$\therefore p_\eta(x) = \exp\big\{-\eta x - 2\sqrt{x} - A(\eta)\big\}\frac{1}{\sqrt{x}}, \quad x>0.$$

The mean of X is

$$E_\eta(X) = -E_\eta T = -A'(\eta) = \frac{1}{2\eta} + \frac{1}{\eta^2} - \frac{\frac{\sqrt{2}}{2}\eta^{-3/2}\phi\left(-\sqrt{\frac{2}{\eta}}\right)}{\Phi\left(-\sqrt{\frac{2}{\eta}}\right)}.$$

□

Problem 2.6

Find the natural parameter space Ξ and densities p_η for a canonical two-parameter exponential family with μ counting measure on $\{0,1,2\}$, $T_1(x) = x,\ T_2(x) = x^2$, and $h(x) = 1$ for $x \in \{0,1,2\}$.

***Solution*.**

$$\begin{aligned} e^{A(\eta)} &= \sum_{x=0}^{2} e^{\eta_1 x + \eta_2 x^2} \\ &= 1 + e^{\eta_1+\eta_2} + e^{2\eta_1+4\eta_2}. \end{aligned}$$

To the sum is finite, $\eta_1 + \eta_2 < \infty$ & $2\eta_1 + 4\eta_2 < \infty \Rightarrow \eta \in \Xi = \mathbb{R}^2$.

$$\therefore p_\eta(x) = \exp\left[\eta_1 x + \eta_2 x^2 - \left(1 + e^{\eta_1+\eta_2} + e^{2\eta_1+4\eta_2}\right)\right].$$

□

Problem 2.7

Suppose $X_1, \dots, X_n$ are independent geometric variables with p_i the success probability for X_i. Suppose these success probabilities are related to a sequence of "independent" variables $t_1, \dots, t_n$, viewed as known constants, through

$$p_i = 1 - \exp(\alpha + \beta t_i), \quad i = 1, \dots, n.$$

Show that the joint densities for $X_1, \dots, X_n$ form a two-parameter exponential family, and identify the statistics T_1 and T_2.

***Solution*.**
Since $X_i \overset{\text{ind}}{\sim} Geo(p_i)$, then the joint density is represented by

$$\begin{aligned} \prod_{i=1}^{n} p(x_i) = \prod_{i=1}^{n} (1-p_i)^{x_i} p_i &= \exp[\sum_{i=1}^{n} x_i \log(1-p_i) + \sum_{i=1}^{n} \log p_i] \\ &= \exp[\sum_{i=1}^{n} x_i(\alpha+\beta t_i) + \sum_{i=1}^{n} \log(1-\exp(\alpha+\beta t_i))]. \end{aligned}$$

$\therefore T_1 = \sum_{i=1}^n X_i,\ T_2 = \sum_{i=1}^n t_i X_i.$ □

Problem 2.8

Assume that $X_1, \dots, X_n$ are independent random variables with $X_i \sim N(\alpha + \beta t_i, 1)$, where $t_1, \dots, t_n$ are observed constants and α and β are unknown parameters. Show that the joint densities for $X_1, \dots, X_n$ form a two-parameter exponential family, and identify the statistics T_1 and T_2.

***Solution*.**
The joint density of $X_1, \dots, X_n$ is

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(x_i - \alpha - \beta t_i)^2\right] = (2\pi)^{-\frac{n}{2}} \exp\left[\sum_i x_i(\alpha+\beta t_i) - \frac{1}{2}\sum_i (\alpha+\beta t_i)\right] \exp\left(-\sum_i \frac{x_i^2}{2}\right).$$

$\therefore T_1 = \sum_{i=1}^n X_i,\ T_2 = \sum_{i=1}^n t_i X_i.$ □

Problem 2.9

Suppose that $X_1, \ldots, X_n$ are independent Bernoulli variables (a random variable is Bernoulli if it only takes on values 0 and 1) with

$$P(X_i = 1) = \frac{\exp(\alpha + \beta t_i)}{1 + \exp(\alpha + \beta t_i)}.$$

Show that the joint distribution for $X_1, \ldots, X_n$ form a two-parameter exponential family, and identify the statistics T_1 and T_2.

Solution.
The joint density of $X_1, \ldots, X_n$ is

$$\begin{aligned}
\prod_{i=1}^n p_i^{x_i}(1-p_i)^{1-x_i} &= \exp\left[\sum_i x_i \log p_i + \sum_i (1-x_i)\log(1-p_i)\right] \\
&= \exp\left[\sum_i x_i \left(\alpha + \beta t_i - \log\left(1 + e^{\alpha+\beta t_i}\right)\right) + \sum_i (1-x_i)\left(1 - \log\left(1 + e^{\alpha+\beta t_i}\right)\right)\right] \\
&= \exp\left[\sum_i x_i(\alpha + \beta t_i) - \sum_i \log\left(1 + e^{\alpha+\beta t_i}\right)\right].
\end{aligned}$$

$\therefore T_1 = \sum_{i=1}^n X_i,\ T_2 = \sum_{i=1}^n t_i X_i.$ □

Problem 2.15

For an exponential family in canonical form, $ET_j = \partial A(\eta)/\partial \eta_j$. This can be written in vector form as $ET = \nabla A(\eta)$. Derive an analogous differential formula for $E_\theta T$ for an s-parameter exponential family that is not in canonical form. Assume that Ω has dimension s.
Hint: Differentiation under the integral sign should give a system of linear equations. Write these equations in matrix form.

Solution.
Note that the exponential family form is

$$\begin{aligned}
p_\eta(x) &= \exp\left[\sum_{i=1}^s \eta_i T_i(x) - A(\eta)\right] h(x) \\
&= \exp\left[\sum_{i=1}^s \eta_i(\theta) T_i(x) - B(\theta)\right] h(x) \\
&= p_\theta(x).
\end{aligned}$$

Thus,

$$e^{B(\theta)} = \int \exp\left[\sum_{i=1}^s \eta_i(\theta) T_i(x)\right] h(x) d\mu(x).$$

By differentiating with respect to θ_i,

$$e^{B(\theta)} \frac{\partial B(\theta)}{\partial \theta_i} = \int \left(\sum_{j=1}^s \frac{\partial \eta_j(\theta)}{\partial \theta_i} T_j(x)\right) \exp\left[\sum_{j=1}^s \eta_j(\theta) T_j(x)\right] h(x) d\mu(x).$$

$$\begin{aligned}\frac{\partial B(\theta)}{\partial\theta_i} &= \int\left(\sum_{j=1}^{s}\frac{\partial\eta_j(\theta)}{\partial\theta_i}T_j(x)\right)p_\theta(x)d\mu(x)\\ &= \sum_{j=1}^{s}\frac{\partial\eta_j(\theta)}{\partial\theta_i}E_\theta T_j, \qquad i=1,\dots,s.\end{aligned}$$

□

Problem 2.17

Let μ denote counting measure on $\{1,2,\dots\}$. One common definition for $\sum_{k=1}^{\infty} f(k)$ is $\lim_{n\to\infty}\sum_{k=1}^{n} f(k)$, and another definition is $\int f d\mu$.

a) Use the dominated convergence theorem to show that the two definitions give the same answer when $\int |f| d\mu < \infty$.
Hint: Find functions $f_n,\ n=1,2,\dots$, so that $\sum_{k=1}^{n} f(k) = \int f_n d\mu$.

Solution.
Let $f_n(k)=f(k),\ \forall k\le n,\ f_n(k)=0,\ \forall k>n$. Then, $f_n\to f$ pointwise and $|f_n|\le|f|$. Also f_n is simple and $\int f_n d\mu = \sum_{k=1}^{n} f(k)$.
By D.C.T., $\int f_n d\mu \to \int f d\mu$. □

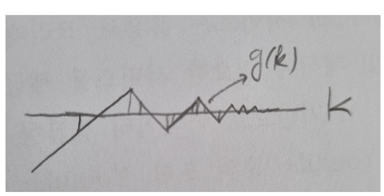
b) Use the monotone convergence theorem, give in Problem 1.25, to show the definitions agree if $f(k)\ge 0$ for all $1,2,\dots$.

Solution.
Define f_n and f like part (a). Then, $0\le f_1\le f_2\le\cdots$.
By M.C.T., $\sum_{k=1}^{n} f(k) = \int f_n d\mu \to \int f d\mu$. □

c) Suppose $\lim_{n\to\infty} f(n)=0$ and that $\int f^+ d\mu = \int f^- d\mu = \infty$ (so that $\int f d\mu$ is undefined.) Let K be an arbitrary constant. Show that the list $f(1), f(2),\dots$ can be rearranged to form a new list $g(1), g(2),\dots$ so that

$$\lim_{n\to\infty}\sum_{k=1}^{n} g(k) = K.$$

Solution.
Take positive parts of the sequence f until the sum is exceed K. And then, take negative parts of the sequence f until the sum is below K. Repeat this procedure like the figure below. (In the figure, $g(k)$ is typo. $\sum g(k)$ is correct.) Then, the sum of the sequence goes to 0 for $k>k'$ where k' is



the number when the sum of the rearranged sequence exceed K, first. Let this sequence as g. Then, $\lim_{n\to\infty}\sum_{k=1}^{n} g(k) \to K$. □

Problem 2.19

Let $p_n, n = 1, 2, \ldots$, and p be probability densities with respect to a measure μ, and let $P_n, n = 1, 2, \ldots$, and P be the corresponding probability measures.

a) Show that if $p_n(x) \to p(x)$ as $n \to \infty$, then $\int |p_n - p| d\mu \to 0$.
Hint: First use the fact that $\int (p_n - p) d\mu = 0$ to argue that $\int |p_n - p| d\mu = 2 \int (p - p_n)^+ d\mu$. Then use dominated convergence.

Solution.
Since p_n and p be probability densities, $\int (p_n - p) d\mu = \int p_n d\mu - \int p d\mu = 1 - 1 = 0$. Since $p_n - p = (p - p_n)^+ - (p - p_n)^-$,

$$\int (p_n - p) d\mu = \int (p - p_n)^+ d\mu - \int (p - p_n)^- d\mu = 0,$$

$$\therefore \int (p - p_n)^+ d\mu = \int (p - p_n)^- d\mu.$$

Also $|p_n - p| = (p - p_n)^+ + (p - p_n)^-$,

$$\int |p_n - p| d\mu = \int (p - p_n)^+ d\mu + \int (p - p_n)^- d\mu = 2 \int (p - p_n)^+ d\mu.$$

Also note that $|(p - p_n)^+| \le p \Rightarrow |(p - p_n)^+|$ is integrable. ($\because$ p is a probability density $\Rightarrow$ p is integrable.) From the condition, $(p_n \to p) \Rightarrow \big(p(x) - p_n(x)\big)^+ \to 0$.
By D.C.T.,

$$\int |p_n - p| d\mu = 2 \int (p - p_n)^+ d\mu \to 0.$$

$\square$

b) Show that $|P_n(A) - P(A)| \le \int |p_n - p| d\mu$.
Hint: Use indicators and the bound $|\int f d\mu| \le \int |f| d\mu$.

Solution.

$$\begin{aligned} LHS &= \left| \int 1_A dP_n - \int 1_A dP \right| \\ &= \left| \int 1_A (p_n - p) d\mu \right| \\ &\le \int |1_A (p_n - p)| \, d\mu \\ &\le \int |p_n - p| \, d\mu. \end{aligned}$$

$\square$

Remark: Distributions $P_n,\ n \ge 1$, are said to *converge strongly* to P if $\sup_A |P_n(A) - P(A)| \to 0$. The two parts above show that pointwise convergence of p_n to p implies strong convergence. This was discovered by Scheffé.

Problem 2.22

Suppose X is absolutely continuous with density

$$p_\theta(x) = \begin{cases} \frac{e^{-(x-\theta)^2/2}}{\sqrt{2\pi}\Phi(\theta)}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the moment generating function of X. Compute the mean and variance of X.

Solution.

$$p_\theta(x) = \exp\left[-\frac{1}{2}(x^2 - 2x\theta + \theta^2) - \log\Phi(\theta)\right]\frac{1}{\sqrt{2\pi}}, \quad x > 0$$

is the exponential family with $T = X,\ A(\theta) = \frac{\theta^2}{2} + \log\Phi(\theta)$. Therefore, the moment generating function of $T = X$ is

$$\begin{aligned} M_X(u) &= \exp\left[A(\theta+u) - A(\theta)\right] \\ &= \exp\left[\frac{1}{2}(\theta+u)^2 - \frac{1}{2}\theta^2\right]\Phi(\theta+u)/\Phi(\theta) \\ &= \exp\left(\theta u + \frac{1}{2}u^2\right)\Phi(\theta+u)/\Phi(\theta). \end{aligned}$$

Meanwhile, the *c.g.f.* of $T = X$ for exponential family is $K_X(u) = A(\theta+u) - A(\theta)$. Thus,

$$EX = K_X'(u) = A'(\theta) = \theta + \frac{\phi(\theta)}{\Phi(\theta)},$$

$$\mathrm{Var}(X) = K_X''(u) = A''(\theta) = 1 + \frac{\phi'(\theta)\Phi(\theta) - \phi(\theta)^2}{\Phi(\theta)^2}.$$

$$\left(\because \phi(x) = e^{-x^2/2}/\sqrt{2\pi} \Rightarrow \phi'(x) = -x\phi(x).\right)$$

□

Problem 2.23

Suppose $Z \sim N(0,1)$. Find the first four cumulants of Z^2.
Hint: Consider the exponential family $N(0,\sigma^2)$.

Solution.
Let $X \sim N(0,\sigma^2)$. Then, its density is

$$p_\sigma(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{s\sigma^2}} = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma^2}x^2 - \log\sigma\right],$$

and also be the exponential family.
Reparameterize $\eta = -\frac{1}{2\sigma^2}$, then $T = X^2$ and $A(\eta) = \frac{1}{2}\log -\frac{1}{2\eta} = -\frac{1}{2}\log(-2\eta)$. Thus, the cumulative generating function of $T = X^2$ is

$$\begin{aligned} K_{X^2}(u) &= A(\eta+u) - A(\eta) \\ &= -\frac{1}{2}\left[\log\big(-2(\eta+u)\big) + \log(-2\eta)\right]. \end{aligned}$$

If $\sigma = 1$, then $\eta = -\frac{1}{2}$ and $X^2 = Z^2$. Therefore,

$$K_{Z^2}(u) = -\frac{1}{2}\left[\log(-2u+1)\right].$$

$$K'_{Z^2}(u) = \frac{1}{1-2u}, \quad K''_{Z^2}(u) = \frac{2}{(1-2u)^2},$$

$$K^{(3)}_{Z^2}(u) = 8(1-2u)^{-3}, \quad K^{(4)}_{Z^2}(u) = 48(1-2u)^{-4}.$$

$\therefore$ the cumulants are 1, 2, 8, and 48, respectively. □

Problem 2.24

Find the first four cumulants of $T = XY$ when X and Y are independent standard normal variates.

Solution.
Since X and Y are independent, the function of these r.v.s are also independent. (i.e. $g(X)$ and $g(Y)$ are independent for any function g.) Then, first 2 cumulants are

$$\kappa_1 = ET = EXY = EXEY = 0,$$

$$\kappa_2 = E(T-ET)^2 = EX^2Y^2 = EX^2EY^2 = 1.$$

For third cumulants, we need to obtain EX^3.

$$\begin{aligned} EX^3 &= \int x^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &\text{Let } t = x^2/2 \Rightarrow xdx = dt \\ &= \frac{2}{\sqrt{2\pi}} \int_0^\infty te^{-t} dt \\ &= \frac{2}{\sqrt{2\pi}} \left[-te^{-t}\Big|_0^\infty + \int_0^\infty e^{-t} dt \right] \\ &= 0. \end{aligned}$$

Thus, the third cumulants is

$$\kappa_3 = E(T-ET)^3 = EX^3Y^3 = EX^3EY^3 = 0.$$

Similarly, we need to obtain EX^4. 자꾸 적분이 틀림... 다시 해보기 Gamma dist 형태 사용

$$\begin{aligned} EX^4 &= \int x^4 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &\text{Let } t = x^2/2 \Rightarrow xdx = dt \end{aligned}$$

$$\kappa_4 = E(T-ET)^4 - 3\mathrm{Var}(T)^2 = EX^4EY^4 - 3 \cdot 1^2 = 4.$$

□

Problem 2.25

Find the third and fourth cumulants of the geometric distribution.

Solution.

$$f_p(x) = (1-p)^x p = \exp\left[x\log(1-p) + \log p\right]$$

is the exponential family. Reparameterize $\eta = -\log(1-p)$ ($\Rightarrow p = 1 - e^{\eta}$), then the above form be the canonical exponential family with $T = X$ and $A(\eta) = -\log(1-e^{\eta})$.
Thus, the cumulative generating function of $T = X$ is

$$K_X(u) = A(\eta+u) - A(\eta) = -\log(1-e^{\eta+u}) + \log(1-e^{\eta}).$$

Then, the first cumulants is

$$K_X'(0) = A'(\eta) = \frac{e^{\eta}}{1-e^{\eta}} = \frac{1-p}{p}.$$

The higher order cumulants are easily obtained by using the chain rule. By using the $p' = \frac{dp}{d\eta} = -e^{\eta} = p-1$,

$$K_X''(0) = A''(\eta) = \frac{d}{dp}A'(\eta)\frac{dp}{d\eta} = \frac{-p-(1-p)}{p^2}(p-1) = \frac{1}{p^2} - \frac{1}{p},$$
$$K_X^{(3)}(0) = A^{(3)}(\eta) = \frac{d}{dp}A''(\eta)p' = -\frac{2p'}{p^3} + \frac{p'}{p^2} = -\frac{2-2p}{p^3} + \frac{p-1}{p^2} = \frac{2}{p^3} - \frac{3}{p^2} + \frac{1}{p},$$
$$K_X^{(4)}(0) = A^{(4)}(\eta) = \frac{d}{dp}A^{(3)}(\eta)p' = -\frac{6p'}{p^4} + \frac{6p'}{p^3} - \frac{p'}{p^2} = \frac{6}{p^4} - \frac{12}{p^3} + \frac{7}{p^2} - \frac{1}{p}.$$

□

Problem 2.26

Find the third cumulant and third moment of the binomial distribution with n trials and success probability p.

Solution.

$$p(x) = \binom{n}{x}p^x(1-p)^{n-x} = \binom{n}{x}\left(\frac{p}{1-p}\right)^x(1-p)^n = \exp\left[x\log\frac{p}{1-p} + n\log(1-p)\right]\binom{n}{x}$$

is the exponential family for $\eta = \log\frac{p}{1-p}$ with $T = X$ and $A(\eta) = -n\log(1-p)$. To use the chain rule,

$$p = \frac{e^{\eta}}{1+e^{\eta}} \Rightarrow \frac{dp}{d\eta} = \frac{e^{\eta}(1+e^{\eta}) - (e^{\eta})^2}{(1+e^{\eta})^2} = p(1-p).$$

By using the above fact,

$$\kappa_1 = A'(\eta) = \frac{d}{dp}A(\eta)\frac{dp}{d\eta} = \frac{n}{1-p}p(1-p) = np,$$
$$\kappa_2 = A''(\eta) = \frac{d}{dp}A'(\eta)\frac{dp}{d\eta} = np(1-p),$$
$$\kappa_3 = A'''(\eta) = \frac{d}{dp}A''(\eta)\frac{dp}{d\eta} = (n-2np)p(1-p) = np(1-p)(1-2p).$$

The third moment for $T = X$ can be obtained by

$$EX^3 = \kappa_3 + 3\kappa_1\kappa_2 + \kappa_1^3.$$

□

Problem 2.27

Let T be a random vector in $\mathbb{R}^2$.

a) Express $\kappa_{2,1}$ as a function of the moments of T.

Solution.
To make simple derivation, we denote $f_{ij}(u) = \frac{\partial^{i+j} f(u)}{\partial u_1^i \partial u_2^j}$. Note that $K(u) = \log M(u)$, by taking derivative,

$$K_{10} = \frac{M_{10}}{M}, \quad K_{20} = \frac{M_{20}M - M_{10}^2}{M^2},$$

$$\begin{aligned} K_{21} &= \frac{1}{M^4}\left[(M_{21}M + M_{20}M_{01} - 2M_{10}M_{11})M^2 - 2MM_{01}(M_{20}M - M_{10}^2)\right] \\ &= \frac{1}{M^3}[M_{21}M^2 + M_{20}M_{01}M - 2M_{10}M_{11}M - 2M_{01}M_{20}M - 2M_{01}M_{10}^2] \\ &= \frac{1}{M^3}[M_{21}M^2 - M_{20}M_{01}M - 2M_{10}M_{11}M - 2M_{01}M_{10}^2]. \end{aligned}$$

Taking $u = 0$, then

$$\kappa_{2,1} = K_{2,1}|_{u=0} = ET_1^2T_2 - (ET_2)(ET_1^2) - 2(ET_1)(ET_1T_2) - 2(ET_2)(ET_1)^2.$$

□

b) Assume $ET_1 = ET_2 = 0$ and give an expression for $\kappa_{2,2}$ in terms of moments of T.

Solution.
$\kappa_{2,2}$ can be obtain by one more derivative for K_{21}. (Too complicated. See Keener page 462.) □

Problem 2.28

Suppose $X \sim \Gamma(\alpha, 1/\lambda)$, with density

$$\frac{\lambda^\alpha x^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0.$$

Find the cumulants of $T = (X, \log X)$ of order 3 or less. The answer will involve $\psi(\alpha) = d\log\Gamma(\alpha)/d\alpha = \Gamma'(\alpha)/\Gamma(\alpha)$.

Solution.
Since Gamma distribution is the exponential family,

$$\begin{aligned} p(x) &= \exp\left[\alpha\log\lambda + (\alpha-1)\log x - \lambda x - \log\Gamma(\alpha)\right] \\ &= \exp\left[-\lambda x + \alpha\log x - \big(\log\Gamma(\alpha) - \alpha\log\lambda\big) - \log x\right]. \end{aligned}$$

Let $\eta = (-\lambda, \alpha)$, and $A(\eta) = \log\Gamma(\eta_2) - \eta_2\log(-\eta_1)$.
Then, the cumulants are obtained by taking derivatives:

$$\kappa_{1,0} = \frac{\partial A(\eta)}{\partial\eta_1} = -\frac{\eta_2}{\eta_1} = \frac{\alpha}{\lambda},$$

$$\kappa_{0,1} = \frac{\partial A(\eta)}{\partial\eta_2} = \psi(\eta_2) - \log(-\eta_1) = \psi(\alpha) - \log\lambda,$$

$$\kappa_{2,0} = \frac{\partial^2 A(\eta)}{\partial \eta_1^2} = \frac{\eta_2}{\eta_1^2} = \frac{\alpha}{\lambda^2},$$

$$\kappa_{1,1} = \frac{\partial^2 A(\eta)}{\partial \eta_1 \partial \eta_2} = -\frac{1}{\eta_1} = \frac{1}{\lambda},$$

$$\kappa_{0,2} = \frac{\partial^2 A(\eta)}{\partial \eta_2^2} = \psi'(\eta_2) = \psi'(\alpha),$$

$$\kappa_{3,0} = \frac{\partial^3 A(\eta)}{\partial \eta_1^3} = -\frac{2\eta_2}{\eta_1^3} = \frac{2\alpha}{\lambda^3},$$

$$\kappa_{2,1} = \frac{\partial^3 A(\eta)}{\partial \eta_1^2 \partial \eta_2} = \frac{1}{\eta_1^2} = frac1\lambda^2,$$

$$\kappa_{1,2} = \frac{\partial^3 A(\eta)}{\partial \eta_1 \partial \eta_2^2} = 0,$$

$$\kappa_{0,3} = \frac{\partial^3 A(\eta)}{\partial \eta_2^3} = \psi''(\eta_2) = \psi''(\alpha).$$

□

Chapter 3

Risk, Sufficiency, Completeness, and Ancillarity

Problem 3.2

Suppose data $X_1, \ldots, X_n$ are independent with

$$P_\theta(X_i \le x) = x^{t_i\theta}, \qquad x \in (0,1).$$

where $\theta > 0$ is the unknown parameter, and $t_1, \ldots, t_n$ are known positive constants. Find a one-dimensional sufficient statistic T.

Solution.
Take a first derivative for the above cdf, then the pdf of X be $p_\theta(x) = t_i\theta x_i^{t_i\theta-1}$. Then, the joint density is

$$p_\theta(x_1, \ldots, x_n) = \prod_{i=1}^n t_i\theta x_i^{t_i\theta-1} = \theta^n \left(\prod_{i=1}^n x_i^{t_i}\right)^\theta \frac{\prod_{i=1}^n t_i}{\prod_{i=1}^n x_i}.$$

By factorization theorem, the sufficient statistic for θ is

$$T = \prod_{i=1}^n x_i^{t_i}.$$

□

Problem 3.3

An object with weight θ is weighted on scales with different precision. The data $X_1, \ldots, X_n$ are independent, with $X_i \sim N(\theta, \sigma_i^2),\ i = 1, \ldots, n$, with the standard deviations $\sigma_1, \ldots, \sigma_n$ known constants. Use sufficiency to suggest a weighted average of $X_1, \ldots, X_n$ to estimate θ. (A weighted average would have form $\sum_{i=1}^n w_i X_i$, where the w_i are positive and sum to one.)

Solution.
The pdf of X_i is

$$p_\mu(x_i) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2\sigma_i^2}(x_i-\theta)^2} = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left[\frac{\theta x_i}{\sigma_i^2} - \frac{x_i^2}{2\sigma_i^2} - \frac{\theta^2}{2\sigma_i^2}\right].$$

Then, the joint density is

$$p_\mu(x_1,\ldots,x_n) = \frac{1}{(2\pi)^{n/2}\prod_i \sigma_i} \exp\left[\sum_i \frac{\theta x_i}{\sigma_i^2} - \sum_i \frac{x_i^2}{2\sigma_i^2} - \sum_i \frac{\theta^2}{2\sigma_i^2}\right].$$

By factorization theorem,

$$T = \sum_{i=1}^n \frac{x_i}{\sigma_i^2}.$$

To estimate θ using a weighted average form, let

$$w_i = \frac{T}{\sum_{i=1}^n \sigma_i^{-2}},$$

then the weighted average of $X_1,\ldots,X_n$ be an estimator for θ satisfied the sufficiency. □

Problem 3.4

Let $X_1,\ldots,X_n$ be a random sample from an arbitrary discrete distribution P on $\{1,2,3\}$. Find a two-dimensional sufficient statistic.

Solution.
Let $p_i = P(\{i\})$ and $n_i(x) = \sum_{j=1}^n 1_{\{x_j=i\}}(x)$ for $i=1,2,3$. Then, $n_1(x)+n_2(x)+n_3(x)=n$. Since X_is are i.i.d., the joint pdf is

$$\begin{aligned}P(X_1=x_1,\ldots,X_n=x_n) &= p_1^{n_1(x)}p_2^{n_2(x)}p_3^{n_3(x)} \\ &= p_1^{n_1(x)}p_2^{n_2(x)}p_3^{n-n_1(x)-n_2(x)}.\end{aligned}$$

By factorization theorem, the sufficient statistic $T=(n_1,n_2)$. □

Problem 3.6

The beta distribution with parameters $\alpha>0$ and $\beta>0$ has density

$$f_{\alpha,\beta}(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & x\in(0,1), \\ 0, & \text{otherwise.}\end{cases}$$

Suppose $X_1,\ldots,X_n$ are i.i.d. from a beta distribution.

a) Determine a minimal sufficient statistic (for the family of joint distributions) if α and β vary freely.

Solution.
The joint density

$$\begin{aligned}f_{\alpha,\beta}(x_1,\ldots,x_n) &= \exp\left[\sum_i(\alpha-1)\log x_i + \sum_i(\beta-1)\log(1-x_i) + n\log\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \\ &= \exp\left[\sum_i \alpha\log x_i + \sum_i \beta\log(1-x_i) + n\log\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} - \sum_i \log x_i(1-x_i)\right],\end{aligned}$$

is the exponential family and full-rank. Then,

$$T = \left(\sum_{i=1}^{n} \log X_i, \sum_{i=1}^{n} \log(1 - X_i) \right)$$

is complete.
By factorization theorem, T is also sufficient.
Therefore, T is the minimal sufficient statistic for (α, β). (See Theorem 3.19) □

b) Determine a minimal sufficient statistic if $\alpha = 2\beta$.

Solution.
Plug into the joint density,

$$\begin{aligned} f_\beta(x_1, \dots, x_n) &= \exp\left[\sum_i 2\beta \log x_i + \sum_i \beta \log(1 - x_i) + n \log \frac{\Gamma(2\beta + \beta)}{\Gamma(2\beta)\Gamma(\beta)} - \sum_i \log x_i(1 - x_i) \right] \\ &= \exp\left[\beta \sum_i \left(\log x_i^2 + \log(1 - x_i) \right) + n \log \frac{\Gamma(3\beta)}{\Gamma(2\beta)\Gamma(\beta)} - \sum_i \log x_i(1 - x_i) \right] \end{aligned}$$

is the one-parameter exponential family and full-rank. Thus,

$$T = \sum_i \left(\log x_i^2 + \log(1 - x_i) \right) = 2T_1 + T_2$$

is complete, and also sufficient by factorization theorem.
Therefore, T is the minimal sufficient statistic for β. □

c) Determine a minimal sufficient statistic if $\alpha = \beta^2$.

Solution.
Now, the joint density is

$$\begin{aligned} p_\beta(x_1, \dots, x_n) &= \exp\left[\sum_i (\beta^2 - 1) \log x_i + \sum_i (\beta - 1) \log(1 - x_i) + n \log \frac{\Gamma(2\beta + \beta)}{\Gamma(2\beta)\Gamma(\beta)} \right] \\ &= \exp\left[(\beta^2 - 1) T_1(x) + (\beta - 1) T_2(x) + n \log \frac{\Gamma(\beta^2 + \beta)}{\Gamma(\beta^2)\Gamma(\beta)} \right]. \end{aligned}$$

Suppose $p_\theta(x) \propto_\theta p_\theta(y)$. Then W.L.O.G. we consider 2 cases as follows:

$$\begin{aligned} \frac{p_1(y)}{p_1(x)} = \frac{p_2(y)}{p_2(x)} &\Longrightarrow \frac{p_2(x)}{p_1(x)} = \frac{p_2(y)}{p_1(y)}, \\ \frac{p_1(y)}{p_1(x)} = \frac{p_3(y)}{p_3(x)} &\Longrightarrow \frac{p_3(x)}{p_1(x)} = \frac{p_3(y)}{p_1(y)}. \end{aligned}$$

Since $p_1(x) = p_1(y) = 1$, from the above equation,

$$\begin{aligned} 3T_1(x) + T_2(x) + n \log \frac{\Gamma(2^2 + 2)}{\Gamma(2^2)\Gamma(2)} &= 3T_1(y) + T_2(y) + n \log \frac{\Gamma(2^2 + 2)}{\Gamma(2^2)\Gamma(2)}, \\ 8T_1(x) + 2T_2(x) + n \log \frac{\Gamma(3^2 + 3)}{\Gamma(3^2)\Gamma(3)} &= 8T_1(y) + 2T_2(y) + n \log \frac{\Gamma(3^2 + 3)}{\Gamma(3^2)\Gamma(3)}. \end{aligned}$$

These 2 equations imply $T(x) = T(y)$.
Therefore, T is the minimal sufficient statistic by Theorem 3.11. □

Problem 3.7

Logistic regression. Let $X_1, \ldots, X_n$ be independent Bernoulli variables, with $p_i = P(X_i = 1),\ i = 1, \ldots, n$. Let $t_1, \ldots, t_n$ be a sequence of known constants that are related to the p_i via

$$\log \frac{p_i}{1-p_i} = \alpha + \beta t_i,$$

where α and β are unknown parameters. Determine a minimal sufficient statistic for the family of joint distributions.

Solution.
From the above equation,

$$p_i = \frac{e^{\alpha+\beta t_i}}{1+e^{\alpha+\beta t_i}}.$$

Then, the density of X_i is

$$\begin{aligned}
p_{\alpha,\beta}(x_i) &= p_i^{x_i}(1-p_i)^{1-x_i} \\
&= \left(\frac{e^{\alpha+\beta t_i}}{1+e^{\alpha+\beta t_i}}\right)^{x_i} \left(\frac{1}{1+e^{\alpha+\beta t_i}}\right)^{1-x_i} \\
&= \exp\left[x_i(\alpha+\beta t_i) - x_i \log(1+e^{\alpha+\beta t_i}) - (1-x_i)\log(1+e^{\alpha+\beta t_i})\right] \\
&= \exp\left[\alpha x_i + \beta x_i t_i - \log(1+e^{\alpha+\beta t_i})\right].
\end{aligned}$$

Therefore, we can easily obtain the following sufficient statistic by factorization theorem for the joint density,

$$T = \left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i t_i\right).$$

And also, the joint density is exponential family and full-rank. Thus T is complete.
$\therefore$ T is minimal sufficient. $\square$

Problem 3.8

The multinomial distribution, derived later in Section 5.3, is a discrete distribution with mass function

$$\frac{n!}{x_1! \times \cdots \times x_s!} p_1^{x_1} \times \cdots \times p_s^{x_s},$$

where $x_1, \ldots, x_s$ are non-negative integers summing to n, where $p_1, \ldots, p_s$ are non-negative probabilities summing to one, and n is the sample size. Let N_{11}, N_{21}, N_{22} have a multinomial distribution with n trials and success probabilities $p_{11}, p_{12}, p_{21}, p_{22}$. (A common model for a two-by-two contingency table.)

a) Give a minimal sufficient statistic if the success probabilities vary freely over the unit simplex in $\mathbb{R}^4$. (The unit simplex in $\mathbb{R}^p$ is the set of all vectors with non-negative entries summing to one.)

Solution.
The density be the exponential family as follows:

$$\begin{aligned}
p(\mathbf{N}) &= \frac{n!}{N_{11}!N_{12}!N_{21}!N_{22}!} p_{11}^{N_{11}} p_{12}^{N_{12}} p_{21}^{N_{21}} p_{22}^{N_{22}} \\
&= \frac{n!}{N_{11}!N_{12}!N_{21}!N_{22}!} \exp\left[N_{11}\log p_{11} + N_{12}\log p_{12} + N_{21}\log p_{21} + N_{22}\log p_{22}\right] \qquad (*)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n!}{N_{11}!N_{12}!N_{21}!N_{22}!} \exp\left[N_{11}\log p_{11} + N_{12}\log p_{12} + N_{21}\log p_{21}\right.\\
&\qquad\qquad\qquad\left. +(n - N_{11} - N_{12} - N_{21})\log(1 - p_{11} - p_{12} - p_{21})\right]\\
&= \frac{n!}{N_{11}!N_{12}!N_{21}!N_{22}!} \exp\left[N_{11}\log\left(\frac{p_{11}}{1-p_{11}-p_{12}-p_{21}}\right) + N_{12}\log\left(\frac{p_{12}}{1-p_{11}-p_{12}-p_{21}}\right)\right.\\
&\qquad\qquad\qquad\left. +N_{21}\log\left(\frac{p_{21}}{1-p_{11}-p_{12}-p_{21}}\right) + n\log(1-p_{11}-p_{12}-p_{21})\right].
\end{aligned}
$$

Since it is the exponential family of full-rank, $T = (N_{11}, N_{12}, N_{21})$ is complete, and also sufficient by factorization theorem. Thus $T = (N_{11}, N_{12}, N_{21})$ is minimal sufficient.
If we use the equation $(*)$, the minimal sufficient statistic is $T = (N_{11}, N_{12}, N_{21}, N_{22})$. □

b) Give a minimal sufficient statistic if the success probabilities are constrained so that $p_{11}p_{22} = p_{12}p_{21}$.

Solution.
확실하지 않음...
From the constraint, plug in $p_{11} = (p_{12}p_{21})/p_{22}$ and denote C is the term which is not depend on parameters,

$$
\begin{aligned}
p(\mathbf{N}) &= Cp_{11}^{N_{11}}p_{12}^{N_{12}}p_{21}^{N_{21}}p_{22}^{N_{22}}\\
&= C\left(\frac{p_{12}p_{21}}{p_{22}}\right)^{N_{11}} p_{12}^{N_{12}}p_{21}^{N_{21}}p_{22}^{N_{22}}\\
&= Cp_{12}^{N_{11}+N_{12}}p_{21}^{N_{11}+N_{21}}p_{22}^{N_{22}-N_{11}}\\
&= Cp_{12}^{N_{11}+N_{12}}p_{21}^{N_{11}+N_{21}}(1-p_{11}-p_{12}-p_{21})^{n-N_{12}-N_{21}-2N_{11}}\\
&= C\exp\left[(N_{11}+N_{12})\log\left(\frac{p_{12}}{1-p_{11}-p_{12}-p_{21}}\right)\right.\\
&\qquad\left. +(N_{11}+N_{21})\log\left(\frac{p_{21}}{1-p_{11}-p_{12}-p_{21}}\right) + n\log(1-p_{11}-p_{12}-p_{21})\right].
\end{aligned}
$$

Since it is the exponential family of full-rank, $T = (N_{11}+N_{12}, N_{11}+N_{21})$ is complete, and also sufficient by factorization theorem. Thus $T = (N_{11}+N_{12}, N_{11}+N_{21})$ is minimal sufficient. □

Problem 3.9

Let f be a positive integrable function on $(0, \infty)$. Define

$$c(\theta) = 1/\int_\theta^\infty f(x)dx,$$

and take $p_\theta(x) = c(\theta)f(x)$ for $x > \theta$, and $p_\theta(x) = 0$ for $x \le \theta$. Let $X_1, \dots, X_n$ be i.i.d. with common density p_θ.

a) Show that $M = \min\{X_1, \dots, X_n\}$ is sufficient.

Solution.
The joint density is

$$p_\theta(x_1, \dots, x_n) = c(\theta)^n \prod_i f(x_i)$$

$$= c(\theta)^n \prod_i f(x_i) \cdot 1_{\min\{x_1,\ldots,x_n\}>\theta}.$$

By factorization theorem, $M = \min\{X_1, \ldots, X_n\}$ is sufficient. □

b) Show that M is minimal sufficient.

Solution.
Suppose $p_\theta(x) \propto_\theta p_\theta(y)$. Then, the region of 0 value should be equal. Thus $T(x) = T(y)$ should be satisfied, therefore $T = M$ is minimal sufficient. □

Problem 3.10

Suppose $X_1, \ldots, X_n$ are i.i.d. with common density $f_\theta(x) = (1 + \theta x)/2,\ |x| < 1;\ f_\theta(x) = 0$, otherwise, where $\theta \in [-1, 1]$ is an unknown parameter. Show that the order statistics are minimal sufficient. (Hint: A polynomial of degree n is uniquely determined by its value on a grid of $n + 1$ points.)

Solution.
The joint density is

$$\begin{aligned} f_\theta(x_1, \ldots, x_n) &= 2^{-n}(1 + \theta x_1) \cdots (1 + \theta x_n) \\ &= 2^{-n}(1 + \theta x_{(1)}) \cdots (1 + \theta x_{(n)}) 1_{\{-1<x_{(1)}<\cdots<x_{(n)}<1\}}. \end{aligned}$$

By factorization theorem, the order statistic $T = (X_{(1)}, \ldots, X_{(n)})$ is sufficient.
Now, suppose $p_\theta(x) \propto_\theta p_\theta(y)$. Since the joint density is the n-th order polynomials(다항식) with root(근) $-1/x_{(i)}$, the roots of $p_\theta(x)$ and $p_\theta(x)$ should be equal which implies $T(x) = T(y)$. Therefore, $T = (X_{(1)}, \ldots, X_{(n)})$ is minimal sufficient. □

Problem 3.16

Let $X_1, \ldots, X_n$ be a random sample from an absolutely continuous distribution with density

$$f_\theta(x) = \begin{cases} 2x/\theta^2, & x \in (0, \theta), \\ 0, & \text{otherwise}. \end{cases}$$

a) Find a one-dimensional sufficient statistic T.

Solution.
The joint density is factorized as follows:

$$f_\theta(x_1, \ldots, x_n) = 2^n \frac{x_1 \cdots x_n}{\theta^{2n}} = \frac{2^n}{\theta^{2n}} x_{(1)} \cdots x_{(n)} 1_{x_{(1)}>0} 1_{x_{(n)}<\theta}.$$

By factorization theorem, $T = \max\{X_1, \ldots, X_n\}$ is the sufficient statistic for θ. □

b) Determine the density of T.

***Solution*.**
Since $T = \max\{X_1, \dots, X_n\}$,

$$P(T \le t) = \prod_{i=1}^{n} P(X_i \le t)$$

$$\text{Note that } P(X_i \le t) = \int_0^t 2x/\theta^2 dx = t^2/\theta^2.$$

$$= \frac{t^{2n}}{\theta^{2n}}$$

Taking a derivative with respect to t, the density of T is

$$p_\theta(t) = \frac{2n}{\theta^{2n}} t^{2n-1}, \quad t \in (0, \theta).$$

□

c) Show directly that T is complete.

***Solution*.**
Suppose $E_\theta(T) = c, \ \forall\theta > 0$. Then,

$$\begin{aligned} E_\theta\left[f(T) - c\right] &= \frac{2n}{\theta^{2n}} \int_0^\theta [f(t) - c] t^{2n-1} dt = 0 \\ \Rightarrow \quad & [f(t) - c] t^{2n-1} = 0 \quad \text{a.e. } 0 < t < \theta. \\ \Rightarrow \quad & f(T) = c. \end{aligned}$$

Therefore, T is complete.

Other solution

$$\begin{aligned} E_\theta f(T) &= \frac{2n}{\theta^{2n}} \int_0^\theta f(t) t^{2n-1} dt = c \\ \Rightarrow \quad & \int_0^\theta f(t) t^{2n-1} dt = c\theta^{2n}/2n, \quad \forall\theta > 0 \\ \Rightarrow \quad & f(\theta)\theta^{2n-1} = c\theta^{2n-1} \quad \text{a.e. } \theta \\ \Rightarrow \quad & f(t) = c \quad \text{a.e. } t. \end{aligned}$$

□

Problem 3.17

Let $X, X_1, X_2, \dots$ be i.i.d. from an exponential distribution with failure rate λ (introduced in Problem 1.30).

a) Find the density of $Y = \lambda X$.

***Solution*.**
Note that the density of X is $p_\lambda(x) = \lambda e^{-\lambda x}, \ \lambda > 0$, and its cdf is easily obtained by integration, $P(X \le x) = 1 - e^{-\lambda x}$. Thus the cdf of Y is

$$P(Y \le y) = P(X \le y/\lambda) = 1 - e^{-y}, \ y > 0,$$

and the density is

$$p(y) = e^{-y}, \ y > 0.$$

□

b) Let $\overline{X} = (X_1 + \cdots + X_n)/n$. Show that $\overline{X}$ and $(X_1^2 + \cdots + X_n^2)/\overline{X}^2$ are independent.

Solution.
The joint density of $X_1, \ldots, X_n$ is

$$p_\lambda(x_1, \ldots, x_n) = \lambda^n e^{-\lambda \sum_i x_i} = \lambda^n e^{-n\lambda\bar{x}},$$

is the full rank exponential family, and by factorization theorem, $T = \overline{X}$ is the complete sufficient statistic.
Now, let $Y = \lambda X$, then

$$V = \frac{(Y_1^2 + \cdots + Y_n^2)}{\overline{Y}^2} = \frac{(X_1^2 + \cdots + X_n^2)}{\overline{X}^2}.$$

Since Y does not depend on parameter λ, V is ancillary for P_λ.
$\therefore$ By Basu's theorem, T and V are independent. □

Problem 3.29

Find a function on $(0, \infty)$ that is bounded and strictly convex.

Solution.
$f(x) = e^{-x}$ is bounded on range $(0, 1)$ and strictly convex.
Or $f(x) = 1/(x + 1)$ is bounded on range $(0, 1)$ and strictly convex. □

Problem 3.30

Use convexity to show that the canonical parameter space Ξ of a one-parameter exponential family must be an interval. Specifically, show that if $\eta_0 < \eta < \eta_1$, and if η_0 and η_1 both lie in Ξ, then η must lie in Ξ.

Solution.
Note that if a random variable X is a canonical exponential family, then it has the form,

$$p_\eta(x) = \exp[\eta T(x) - A(\eta)]h(x), \quad \eta \in \Xi.$$

Since $\eta_0 < \eta < \eta_1$, we define $\eta = \gamma\eta_0 + (1-\gamma)\eta_1$ for some $\gamma \in (0, 1)$. Since the exponential function is convex,

$$\exp\left[(\gamma\eta_0 + (1-\gamma)\eta_1)T(x)\right] = e^{\eta T(x)} < \gamma e^{\eta_0 T(x)} + (1-\gamma)e^{\eta_1 T(x)}.$$

From the definition of exponential family, $h(x) \geq 0$, and for the positive function, the inequality sign does not change by integration against μ. Thus, following inequality is satisfied:

$$\int e^{\eta T(x)} h(x) d\mu(x) < \gamma \int e^{\eta_0 T(x)} h(x) d\mu(x) + (1-\gamma) \int e^{\eta_1 T(x)} h(x) d\mu(x).$$

Because it should be satisfied the exponential family, RHS is finite. ($\because$ RHS $= \gamma e^{A(\eta_0)} + (1-\gamma)e^{A(\eta_1)} < \infty$.)
Therefore, η must lie in Ξ. □

Problem 3.31

Let f and g be positive probability densities on $\mathbb{R}$. Use Jensen's inequality to show that

$$\int \log\left(\frac{f(x)}{g(x)}\right) f(x)dx > 0,$$

unless $f = g$ a.e. (If $f = g$, the integral equals zero.) This integral is called the *Kullback-Liebler information*.

Solution.
Suppose X be a absolutely continuous random variable with density f. Now, define $Y = \frac{g(X)}{f(X)}$, then

$$EY = \int \frac{g(x)}{f(x)} f(x)dx = \int g(x)dx = 1.$$

Next, let $h(y) = \log\frac{1}{y} = -\log y$, then h is strictly convex on $(0, \infty)$. Then, by Jensen's inequality,

$$Eh(Y) = \int \log\left(\frac{f(x)}{g(x)}\right) f(x)dx \geq h(EY) = \log 1 = 0.$$

The equality holds when Y is a constant. (then $Y = EY = 1 \Rightarrow f(x) = g(x)$ a.e.) $\square$

Chapter 4

Unbiased Estimation

Problem 4.1

Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent variables with the X_i a random sample from an exponential distribution with failure rate λ_x, and the Y_j a random sample from an exponential distribution with failure rate λ_y.

a) Determine the UMVU estimator of λ_x/λ_y.

Solution.
Note that if $X \sim \text{Exp}(\lambda)$ with rate parameter λ, the denstiy $p_\lambda(x) = \lambda e^{-\lambda x}$.
Since $X_i, \forall i$ and $Y_j, \forall j$ are independent, the joint density is

$$p_{\lambda_x,\lambda_y}(x, y) = \lambda_x^n \lambda_y^n e^{-\lambda_x \sum_i x_i - \lambda_y \sum_i y_i},$$

which is the 2-parameter full-rank exponential family. Thus, by factorization theorem, $T = (\sum_{i=1}^m X_i, \sum_{i=1}^n Y_i)$ is the complete sufficient.
Since $X_1, \ldots, X_m$ are i.i.d. $\text{Exp}(\lambda_x) = \Gamma(1, \lambda_x)$, $T_1 = \sum_{i=1}^m X_i \sim \Gamma(m, \lambda_x)$. Then,

$$E\frac{1}{T_1} = \int \frac{1}{t} \frac{\lambda_x^m}{\Gamma(m)} t^{m-1} e^{-\lambda_x t} dt = \frac{\Gamma(m-1)\lambda_x^m}{\Gamma(m)\lambda_x^{m-1}} = \frac{\lambda_x}{m-1}.$$

Similarly, we can obtain $ET_2 = n/\lambda_y$. ($\because EY_i = 1/\lambda_y$)
From the fact of the expectation of independent random variables,

$$E\frac{T_2}{T_1} = E\frac{1}{T_1} \cdot ET_2 = \frac{n\lambda_x}{(m-1)\lambda_y}.$$

Therefore,

$$\delta(x, y) = \frac{(m-1)T_2}{nT_1}$$

is the unbiased estimator of λ_x/λ_y, and also UMVU since it is the function of complete sufficient statistic T. (By Theorem 4.4 with Rao-Blackwellization) $\square$

b) Under squared error loss, find the best estimator of λ_x/λ_y of form $\delta = c\overline{Y}/\overline{X}$.

***Solution*.**
Let $\delta = c\overline{Y}/\overline{X} = dT_2/T_1$. Then, the risk of $\delta = c\overline{Y}/\overline{X}$ and λ_x/λ_y is

$$R\left(\frac{\lambda_y}{\lambda_x}, \delta\right) = E\left[d\frac{T_2}{T_1} - \frac{\lambda_x}{\lambda_y}\right]^2 = d^2 E\frac{T_2^2}{T_1^2} - 2\frac{\lambda_x}{\lambda_y} dE\frac{T_2}{T_1} + \frac{\lambda_x^2}{\lambda_y^2}. \tag{4.1}$$

Since $T_1 \sim \Gamma(m, \lambda_x)$ and $T_2 \sim \Gamma(n, \lambda_y)$,

$$ET_2^2 = \mathrm{Var}(T_2) + \{ET_2\}^2 = n/\lambda_y^2 + n^2/\lambda_y^2,$$

$$E\frac{1}{T_1^2} = \int \frac{1}{t^2}\frac{\lambda_x^m}{\Gamma(m)} t^{m-1} e^{-\lambda_x t} dt = \frac{\Gamma(m-2)\lambda_x^2}{\Gamma(m)} = \frac{\lambda_x^2}{(m-1)(m-2)}.$$

Thus, using the fact that T_1 and T_2 are independent,

$$E\frac{T_2^2}{T_1^2} = E\frac{1}{T_1^2} \cdot ET_2^2 = \frac{n(n+1)\lambda_x^2}{(m-1)(m-2)\lambda_y^2}.$$

By using the result from a), we know $ET_2/T_1 = \big(n\lambda_x\big)/\big((m-1)\lambda_y\big)$, then the risk (4.1) be

$$\begin{aligned}
R\left(\frac{\lambda_y}{\lambda_x}, \delta\right) &= d^2 E\frac{T_2^2}{T_1^2} - 2\frac{\lambda_x}{\lambda_y} dE\frac{T_2}{T_1} + \frac{\lambda_x^2}{\lambda_y^2} \\
&= d^2 \frac{n(n+1)\lambda_x^2}{(m-1)(m-2)\lambda_y^2} - 2\frac{\lambda_x}{\lambda_y} d\frac{n\lambda_x}{(m-1)\lambda_y} + \frac{\lambda_x^2}{\lambda_y^2} \\
&= \frac{\lambda_x^2}{\lambda_y^2}\left(\frac{n(n+1)}{(m-1)(m-2)}d^2 - 2\frac{n}{(m-1)}d + 1\right) \\
&= \frac{\lambda_x^2}{\lambda_y^2}\frac{n(n+1)}{(m-1)(m-2)}\left[\left(d - \frac{m-2}{n+1}\right)^2 - \left(\frac{m-2}{n+1}\right)^2 + 1\right].
\end{aligned}$$

Thus, it has the minimum when $d = \frac{m-1}{n+1}$.
Therefore, the best estimator of λ_x/λ_y is $c\overline{Y}/\overline{X} = dT_2/T_1 = \big[(m-2)n\overline{Y}\big] \,/\, \big[(n+1)m\overline{X}\big]$.

$$\therefore c = \frac{(m-2)n}{(n+1)m}.$$

□

c) Find the UMVU estimator of $e^{-\lambda_x} = P(X_1 > 1)$.

***Solution*.**
Let δ is the unbiased estimator of $P(X_1 > 1)$. Then,

$$E\delta = P(X_1 > 1) \Rightarrow \delta = 1_{\{X_1>1\}}.$$

From the previous steps, we know T_1 is C.S.S., thus $\eta(T_1) = E(\delta|T_1) = P(X_1 > 1|T_1)$ is UMVU by Theorem 4.4. We follow by 2 steps.

(i) Find the joint density of X and T_1, where $X = 1_{\{X_1>1\}}$ and $T_1 = \sum_{i=1}^m X_i$.
First, we should find the joint density of X and S where $S = \sum_{i=2}^m X_i$. Since X and S are independent, the joint density is easily obtained as

$$f(x,s) = f(x)f(s) = \lambda_x e^{-\lambda_x x}\frac{\lambda_x^{m-1}}{\Gamma(m-1)} s^{m-2} e^{-\lambda_x s} = \frac{\lambda_x^m}{\Gamma(m-1)} s^{m-2} e^{-\lambda_x(x+2)}, \quad x > 0, s > 0.$$

By transformation of variables $X = X, T_1 = S + X$, the jacobian $|J| = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$. Then, the joint density of X and T_1 is

$$f(x,t) = \frac{\lambda_x^m}{\Gamma(m-2)}(t-x)^{m-2}e^{-\lambda_x t}, \quad 0 < x < t.$$

(ii) Derive $E(\delta|T_1)$.

To obtain the expectation, we derive the conditional density of X given T_1. Since $T_1 \sim \Gamma(m, \lambda_x)$, the conditional density be

$$f(x|t) = \frac{f(x,t)}{f(t)} = \frac{\Gamma(m)}{\Gamma(m-2)}(t-x)^{m-2}t^{-(m-1)} = (m-1)\frac{1}{t}\left(1-\frac{x}{t}\right)^{m-2}, \quad 0 < x < t.$$

Then,

$$\begin{aligned} E(\delta|T_1) &= P(X_1 > 1|T_1) = P(X|T_1) \\ &= \int_1^t f(x|t)dx \\ &= \int_1^t (m-1)\frac{1}{t}\left(1-\frac{x}{t}\right)^{m-2} dx \\ &= -\left(1-\frac{x}{t}\right)^{m-1}\Bigg|_1^t \\ &= \left(1-\frac{1}{t}\right)^{m-1} 1_{\{T_1>1\}}. \end{aligned}$$

□

Problem 4.2

Let $X_1, \ldots, X_n$ be a random sample from $N(\mu_x, \sigma^2)$, and let $Y_1, \ldots, Y_m$ be an independent random sample from $N(\mu_y, 2\sigma^2)$, with $\mu_x,\ \mu_y$, and σ^2 all unknown parameters..

a) Find a complete sufficient statisic.

Solution.

Since $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ are mutually independent, the joint density,

$$\begin{aligned} f(x,y) &= f(x)f(y) \\ &= (2\pi\sigma^2)^{-n/2}(4\pi\sigma^2)^{-m/2}\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^n(x_i-\mu_x)^2 - \frac{1}{4\sigma^2}\sum_{j=1}^m(y_j-\mu_y)^2\right] \\ &= \exp\left[-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2} - \frac{\sum_{j=1}^m y_j^2}{4\sigma^2} + \frac{\mu_x\sum_i x_i}{\sigma^2} + \frac{\mu_y\sum_j y_j}{2\sigma^2} - \frac{n\mu_x^2}{2\sigma^2} - \frac{m\mu_y^2}{4\sigma^2}\right](2\pi\sigma^2)^{-n/2}(4\pi\sigma^2)^{-m/2} \\ &= \exp\left[\frac{\mu_x\sum_i x_i}{\sigma^2} + \frac{\mu_y\sum_j y_j}{2\sigma^2} - \frac{2\sum_{i=1}^n x_i^2 + \sum_{j=1}^m y_j^2}{4\sigma^2} - \frac{n\mu_x^2}{2\sigma^2} - \frac{m\mu_y^2}{4\sigma^2}\right](2\pi\sigma^2)^{-n/2}(4\pi\sigma^2)^{-m/2}, \end{aligned}$$

is the full-rank exponential family. Therefore,

$$T = \left(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j, 2\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2\right)$$

is the complete sufficient statistic for (μ_x, μ_y, σ^2). □

b) Determine the UMVU estimator of σ^2.
Hint: Find a linear combination L of $S_x^2 = \sum_{i=1}^n (X_i - \overline{X})^2/(n-1)$ and $S_y^2 = \sum_{j=1}^m (Y_j - \overline{Y})^2/(m-1)$ so that $(\overline{X}, \overline{Y}, L)$ is complete sufficient.

Solution.
To use the C.S.S. obtained from a), let

$$2(n-1)S_x^2 + (m-1)S_y^2 = 2\left(\sum_i X_i^2 - n\overline{X}^2\right) + \left(\sum_j Y_j^2 - m\overline{Y}^2\right)$$

is the linear combination of T. Since the sample variance is unbiased, then its expectation is

$$E\left[2(n-1)S_x^2 + (m-1)S_y^2\right] = (2n+2m-4)\sigma^2.$$

Now, we let

$$S_p^2 = \frac{1}{2n+2m-4}\left[2(n-1)S_x^2 + (m-1)S_y^2\right],$$

then it is unbiased for σ^2 based on complete sufficient T.
By theorem 4.4, it is UMVU of σ^2. □

c) Find a UMVU estimator of $(\mu_x - \mu_y)^2$.

Solution.

$$\begin{aligned}
E(\overline{X} - \overline{Y})^2 &= \mathrm{Var}(\overline{X} - \overline{Y}) + \left[E(\overline{X} - \overline{Y})\right]^2 \\
&= \mathrm{Var}(\overline{X}) + \mathrm{Var}(\overline{Y}) + \left[E\overline{X} - E\overline{Y}\right]^2 \\
&= \frac{\sigma^2}{n} + \frac{2\sigma^2}{m} + (\mu_x - \mu_y)^2.
\end{aligned}$$

We obtained the unbiased estimator of σ^2 in b). Thus, we let δ as

$$\delta = (\overline{X} - \overline{Y})^2 - S_p^2\left(\frac{1}{n} - \frac{2}{m}\right),$$

then it is the unbiased estimator of $(\mu_x - \mu_y)^2$ based on complete sufficient T.
By Theorem 4.4, δ is UMVU of $(\mu_x - \mu_y)^2$. □

d) Suppose we know the $\mu_y = 3\mu_x$. What is the UMVU estimator of μ_x?

Solution.
With restriction $\mu_y = 3\mu_x$, the joint density of exponential family is changed as

$$f(x, y) = \cdots = \exp\left[\frac{\mu_x(2\sum_i x_i + 3\sum_j y_j)}{2\sigma^2} - \cdots\right]\cdots.$$

Thus, $T = \left(2\sum_i X_i + 3\sum_j Y_j,\ 2\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2\right)$ is complete sufficient.
Take expectation for T_1, then

$$ET_1 = 2n\mu_x + 3m\mu_y = (2n+9m)\mu_x,$$

Now we let δ as

$$\delta = \frac{2\sum_i x_i + 3\sum_j y_j}{2n + 9m},$$

then, it is unbiased estimator based on T.
By Theorem 4.4, δ is UMVU of μ_x. □

Problem 4.3

Let $X_1, \dots, X_n$ be a random sample from the Poisson distribution with mean λ. Find the UMVU for $\cos\lambda$. (Hint: For Taylor expansion, the identity $\cos\lambda = (e^{i\lambda} + e^{-i\lambda})/2$ may be useful.)

Solution.
The joint density is

$$p_\lambda(x) = \frac{\lambda^{\sum_i x_i} e^{-n\lambda}}{x_1! \cdots x_n!},$$

with $T = \sum_i X_i$, that is complete sufficient. ($\because$ full-rank exponential family, and by factorization theorem) Note that $T \sim Poisson(n\lambda)$. Let $\delta(t)$ is an unbiased estimator of $\cos\lambda$ based on T. Then, its expectation should satisfy follows:

$$E\delta(T) = \sum_{t=0}^{\infty} \delta(t) \frac{(n\lambda)^t e^{-n\lambda}}{t!} = \cos\lambda$$

$$\Leftrightarrow \sum_{t=0}^{\infty} \delta(t) \frac{(n\lambda)^t}{t!} = e^{n\lambda}\cos\lambda$$

Apply Talyor expansion for RHS,

$$\begin{aligned} e^{n\lambda}\cos\lambda &= e^{n\lambda}\frac{1}{2}\left(e^{i\lambda} + e^{-i\lambda}\right) \\ &= \frac{1}{2}\left(e^{(n+i)\lambda} + e^{(n-i)\lambda}\right) \\ &= \frac{1}{2}\sum_{t=0}^{\infty} \frac{\left((n+i)\lambda\right)^t + \left((n-i)\lambda\right)^t}{t!} \\ &= \sum_{t=0}^{\infty} \frac{(n\lambda)^t}{t!}\left[\frac{\left(1+\frac{i}{n}\right)^t + \left(1-\frac{i}{n}\right)^t}{2}\right]. \end{aligned}$$

Therefore,

$$\delta(t) = \frac{1}{2}\left\{\left(1+\frac{i}{n}\right)^t + \left(1-\frac{i}{n}\right)^t\right\}$$

is unbiased estimator of $\cos\lambda$ based on T.
By Theorem 4.4, $\delta(t)$ is UMVU of $\cos\lambda$. □

Problem 4.4

Let $X_1, \dots, X_n$ be independent normal variables, each with unit variance, and with $EX_i = \alpha t_i + \beta t_i^2,\ i = 1, \dots, n$, where α and β are unknown parameters and $t_1, \dots, t_n$ are known constants. Find UMVU estimators of α and β.

***Solution*.**
Since $X_i \sim N(\alpha t_i + \beta t_i^2, 1)$, the joint density is

$$\begin{aligned} f(x) &= (2\pi)^{-\frac{n}{2}} \exp\left[-\frac{1}{2}\sum_i (x_i - \alpha t_i - \beta t_i^2)^2\right] \\ &= (2\pi)^{-\frac{n}{2}} \exp\left[-\frac{1}{2}\sum_i \left(x_i^2 + \alpha^2 t_i^2 + \beta^2 t_i^4 - 2\alpha x_i t_i - 2\beta x_i t_i^2 + \alpha\beta t_i^3\right)\right]. \end{aligned}$$

Since these density is full-rank exponential family,

$$T = \left(\sum_i X_i t_i,\ \sum_i X_i t_i^2\right)$$

is complete sufficient and its expected value is

$$ET_1 = \alpha \sum t_i^2 + \beta \sum t_i^3, \quad ET_2 = \alpha \sum t_i^3 + \beta \sum t_i^4.$$

By using this,

$$\widehat{\alpha} = \frac{T_1 \sum t_i^4 - T_2 \sum t_i^3}{\sum t_i^2 \sum t_i^4 - \left(\sum t_i^3\right)^2}, \quad \widehat{\beta} = \frac{T_1 \sum t_i^3 - T_2 \sum t_i^2}{\left(\sum t_i^3\right)^2 - \sum t_i^4 \sum t_i^2}$$

are unbiased estimator of α and β. Since these are the function of T, also UMVU. □

Problem 4.5

Let $X_1, \ldots, X_n$ be i.i.d. from some distribution Q_θ, and let $\overline{X} = (X_1 + \cdots + X_n)/n$ be the sample average.

a) Show that $S^2 = \sum (X_i - \overline{X})^2/(n-1)$ is unbiased for $\sigma^2 = \sigma^2(\theta) = \mathrm{Var}_\theta(X_i)$.

***Solution*.**
Denote $\mu = EX_i$. Also note that $(n-1)S^2 = \sum_i X_i^2 - n\overline{X}^2$. Then,

$$\begin{aligned} E\left((n-1)S^2\right) &= \sum_i EX_i^2 - nE\overline{X}^2 \\ &= \sum_i \left[\mathrm{Var}X_i + (EX_i)^2\right] - n\left[\mathrm{Var}\overline{X} + (E\overline{X})^2\right] \\ &= n(\sigma^2 + \mu^2) - n(\sigma^2/n + \mu^2) \\ &= (n-1)\sigma^2. \end{aligned}$$

Therefore S^2 is unbiased for σ^2. □

b) If Q_θ is the Bernoulli distribution with success probability θ, show that S^2 from (a) is UMVU.

***Solution*.**
Note that the density of X_i is $\theta^{x_i}(1-\theta)^{1-x_i} = \left(\frac{\theta}{1-\theta}\right)^x (1-\theta)$.
The joint density,

$$p(x) = \exp\left[\sum_i x_i \log\frac{\theta}{1-\theta} + n\log(1-\theta)\right],$$

is the full-rank exponential family. Thus, $T = \overline{X}$ (or $\sum_i X_i$) is complete sufficient.
Meanwhile,

$$\begin{aligned} S^2 &= \frac{1}{n-1}\left(\sum X_i^2 - n\overline{X}^2\right) \\ &= \frac{1}{n-1}\left(\sum X_i - n\overline{X}^2\right) \quad (\because X_i^2 = X_i) \\ &= \frac{n\overline{X}(1-\overline{X})}{n-1} \end{aligned}$$

is the function of $T = \overline{X}$. Therefore, S^2 is UMVU of σ^2 by Theorem 4.4. □

c) If Q_θ is the exponential distribution with failure rate θ, find the UMVU estimator of $\sigma^2 = 1/\theta^2$. Give a formula for $E_\theta[X_i^2|\overline{X} = c]$ in this case.

Solution.
Note that the density of X_i is $\theta e^{-\theta x}$ with mean $1/\theta$ and variance $1/\theta^2$.
The joint density,

$$p(x) = \exp\left[-\theta \sum x_i + n\log\theta\right],$$

is the full-rank exponential family. Thus, $T = \overline{X}$ (or $\sum_i X_i$) is complete sufficient with $ET = 1/\theta$ and $\mathrm{Var}T = 1/(n\theta^2)$.
Meanwhile,

$$E\overline{X}^2 = \mathrm{Var}\overline{X} + (E\overline{X})^2 = \frac{1}{n\theta^2} + \frac{1}{\theta^2} = \frac{n+1}{n}\frac{1}{\theta^2} = \frac{n+1}{n}\sigma^2.$$

Thus, $\delta = \frac{n}{n+1}\overline{X}^2$ is the unbiased estimator of σ^2, and also function of $T = \overline{X}$.
By Theorem 4.4, δ is UMVU for σ^2.

Next, since X_i's are i.i.d., $E_\theta[X_1^2|\overline{X} = c] = \cdots = E_\theta[X_n^2|\overline{X} = c]$. Then,

$$\begin{aligned} E_\theta[S^2|\overline{X} = c] &= \frac{nc^2}{n+1} \\ &= E_\theta\left[\frac{\sum_i X_i^2 - n\overline{X}^2}{n-1}\middle|\overline{X} = c\right] \\ &= E_\theta\left[\frac{\sum_i X_i^2}{n-1} - \frac{nc^2}{n-1}\middle|\overline{X} = c\right] \\ &= \frac{n}{n-1}E_\theta[X_1^2|\overline{X} = c] - \frac{nc^2}{n-1}. \end{aligned}$$

$$\therefore E_\theta[X_i^2|\overline{X} = c] = nc^2\left(\frac{1}{n+1} + \frac{1}{n-1}\right)\frac{n-1}{n} = c^2\left(\frac{n-1}{n+1} + 1\right) = \frac{2nc^2}{n+1}.$$

□

Problem 4.6

Suppose δ is a UMVU estimator of $g(\theta)$; U is an unbiased estimator of zero, $E_\theta U = 0,\ \theta \in \Omega$; and that δ and U both have finite variances for all $\theta \in \Omega$. Show that U and δ are uncorrelated, $E_\theta U\delta = 0,\ \theta \in \Omega$.

Solution.
For the convenience, we drop θ in the notation of E, Var and Cov.
Let $\delta^* = \delta + cU$ where c is a constant. Then, δ^* is an unbiased estimator for $g(\theta)$ ($\because\ E\delta^* = E(\delta + cU) = E\delta + cEU = g(\theta)$).
Since δ is UMVU,

$$\mathrm{Var}(\delta + cU) = \mathrm{Var}(\delta) + c^2\mathrm{Var}(U) + 2c\mathrm{Cov}(\delta, U) \geq \mathrm{Var}(\delta),$$

$$\Rightarrow c^2\mathrm{Var}(U) + 2c\mathrm{Cov}(\delta, U) \geq 0.$$

Denote LHS as $h(c)$. Since h is convex and has minimum $h(0) = 0$, h should have a local minimum at $c = 0$. Since h is differentiable, we can obtain the following result by taking a first derivative at $c = 0$:

$$h'(0) = 2\mathrm{Cov}(\delta, U) = 0.$$

Since $EU = 0$, we finally obtain $E_\theta U\delta = 0$ for all $\theta \in \Omega$. $\square$

Problem 4.7

Suppose δ_1 is a UMVU estimator of $g_1(\theta)$, δ_2 is UMVU estimator of $g_2(\theta)$, and that δ_1 and δ_2 both have finite variance for all θ. Show that $\delta_1 + \delta_2$ is UMVU for $g_1(\theta) + g_2(\theta)$.
Hint: Use the result in the previous problem.

Solution.
We can easily show that $\delta_1 + \delta_2$ is unbiased for $g_1(\theta) + g_2(\theta)$.
Now, suppose δ is any unbiased estimator of $g_1(\theta) + g_2(\theta)$ with $\mathrm{Var}(\delta) < \infty$. Let $U = \delta - \delta_1 - \delta_2$, then U is an unbiased estimator of 0 ($\because\ EU = E\delta - E\delta_1 - E\delta_2 = 0$).
By Problem 4.6,

$$\mathrm{Cov}(U, \delta_1 + \delta_2) = \mathrm{Cov}(U, \delta_1) + \mathrm{Cov}(U, \delta_2) = 0.$$

Since $\delta = U + \delta_1 + \delta_2$, we obtain the following result:

$$\begin{aligned}\mathrm{Var}(\delta) &= \mathrm{Var}(U) + \mathrm{Var}(\delta_1 + \delta_2)\\ &\geq \mathrm{Var}(\delta_1 + \delta_2).\end{aligned}$$

Because δ is arbitrary, $\delta_1 + \delta_2$ is UMVU. $\square$

Problem 4.8

Let $X_1, \ldots, X_n$ be i.i.d. absolutely continuous variables with common density f_θ, $\theta > 0$, given by

$$f_\theta(x) = \begin{cases} \theta/x^2, & x > \theta, \\ 0, & x \leq \theta. \end{cases}$$

Find the UMVU estimator for $g(\theta)$ if $g(\theta)/\theta^n \to 0$ as $\theta \to \infty$ and g is differentiable.

Solution.
The joint density is

$$\theta^n/(x_1 \cdots x_n)^2 1_{\{\min x_i\}}.$$

By factorization theorem, $T = \min X_i$ is sufficient.
To obtain the density of T, the cumulative distribution function is

$$P(T \le t) = 1 - P(\min X_i > t) = 1 - \prod_i P(X_i > t) = 1 - \left(\frac{\theta}{t}\right)^n, \quad t > \theta.$$

Take a derivative, we can obtain the density,

$$f(t) = \frac{n\theta^n}{t^{n+1}}, \quad t > \theta.$$

Now, suppose $\delta(T)$ is the unbiased estimator of $g(\theta)$, then

$$\begin{aligned}
\int_\theta^\infty \delta(t)\frac{n\theta^n}{t^{n+1}}dt = g(\theta) &\iff n\int_\theta^\infty \delta(t)t^{-n-1}dt = \frac{g(\theta)}{\theta^n} \\
&\quad \text{Taking a derivative w.r.t. } \theta, \\
&\iff -n\delta(t)t^{-n-1} = g'(t)t^{-n} - ng(t)t^{-n-1}. \\
&\therefore\ \delta(t) = g(t) - \frac{g'(t)}{nt}.
\end{aligned}$$

From the above result, $g(t) = c$ for any constant c implies $\delta(t) = c$. Therefore T is complete.
In summary, T is complete sufficient, and $\delta(T)$ is the function of T. Therefore, by Theorem 4.4, $\delta(T)$ is UMVU. □

Problem 4.10

Suppose X is an exponential variable with density $p_\theta(x) = \theta e^{-\theta x},\ x > 0;\ p_\theta(x) = 0$, otherwise. Find the UMVU estimator for $1/(1+\theta)$.

Solution.
Since X is a full-rank exponential family, $T = X$ is complete sufficient.
Let $\delta(X)$ be an unbiased estimator of $1/(1+\theta)$, and represent as $\delta(x) = \sum_{n=0}^\infty c_n x^n$ by power series. Then,

$$\begin{aligned}
E\delta(X) &= \int_0^\infty \left(\sum_{n=0}^\infty c_n x^n\right)\theta e^{-\theta x}dx \\
&\quad \text{By Fubini theorem,} \\
&= \sum_{n=0}^\infty \int_0^\infty c_n\theta x^n e^{-\theta x}dx \\
&\quad \text{The inner integration follows } \Gamma(n+1, 1/\theta) \text{ with rate parameter.} \\
&= \sum_{n=0}^\infty \frac{c_n\Gamma(n+1)}{\theta^n} \\
&= \sum_{n=0}^\infty \frac{c_n n!}{\theta^n}. \qquad (*)
\end{aligned}$$

Meanwhile, $1/(1+\theta)$ can be written as the geometric series,

$$\frac{1}{1+\theta} = \frac{1/\theta}{1+1/\theta} = -\sum_{n=1}^\infty \left(-\frac{1}{\theta}\right)^n = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{\theta^n} \quad \text{for } \theta > 1. \qquad (**)$$

Since δ is unbiased, $(*)$ and $(**)$ should be identical. Thus,

$$c_0 = 0,\ c_n = \frac{(-1)^{n+1}}{n!},\ n = 1, 2, \ldots.$$

Therefore,

$$\delta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} x^n = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n!} = 1 - e^{-x} \quad \text{for } \theta > 1.$$

Since δ is the function of the complete sufficient statistic X, $\delta(X)$ is UMVU by Theorem 4.4.
If $\theta \leq 1$, direct하게 unbiased임을 보일 수 있다?? 어떻게?? □

Problem 4.11

Let $X_1, \ldots, X_3$ be i.i.d. geometric variables with common mass function $f_\theta(x) = P_\theta(X_i = x) = \theta(1-\theta)^x$, $x = 0, 1, \ldots$. Find the UMVU estimator of θ^2.

Solution.
Since the joint mass, $\theta^3(1-\theta)^{x_1+x_2+x_3}$, is the full-rank exponential family, $T = X_1 + X_2 + X_3$ is complete sufficient.
Denote $g(\theta) = \theta^2$, and let $\delta(X) = 1_{\{X_1=0, X_2=0\}}$, then $\delta(X)$ be an unbiased estimator of $g(\theta)$ ($\because\ E\delta(X) = P(X_1 = 0, X_2 = 0) = \theta^2$).
Define $\eta(T) = E[\delta(X)|T] = P(X_1 = 0, X_2 = 0|T)$. Then it is UMVU by Theorem 4.4. To obtain $\eta(t)$, we first find the density of T by using the transformation $T = X_1 + X_2 + X_3$ from the joint density,

$$P(T = t) = \binom{t+2}{2}\theta^3(1-\theta)^t = \binom{t+2}{2}\theta^2\Big[\theta(1-\theta)^t\Big].$$

Because we already have 2 successes in first $t+2$ trials of total $t+3$ trials, we should add the combination term. (See RHS!!)
Next, the joint mass of X_1, X_2, T is

$$P(X_1 = 0, X_2 = 0, T = t) = P(X_1 = 0, X_2 = 0, X_3 = t) = \theta^3(1-\theta)^t,$$

from the joint mass of X_1, X_2, X_3.
Therefore,

$$\eta(t) = \frac{P(X_1 = 0, X_2 = 0, T = t)}{P(T = t)} = 1\Big/\binom{t+2}{2} = \frac{2}{(t+2)(t+1)},$$

$$\therefore\ \eta(T) = \frac{2}{(T+2)(T+1)}$$

is the UMVU estimator of θ^2. □

Problem 4.12

Let X be a single observation, absolutely continuous with density

$$p_\theta(x) = \begin{cases} \frac{1}{2}(1+\theta x), & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

Here $\theta \in [-1, 1]$ is an unknown parameter.

a) Find a constant a so that aX is unbiased for θ. The expected value is

Solution.

$$EX = \int_{-1}^{1} \frac{x}{2}(1+\theta x)dx = \frac{1}{4}x^2 + \frac{\theta}{6}x^3 \Big|_{-1}^{1} = \frac{\theta}{3}.$$

Therefore, when $a = 3$, aX be unbiased. □

b) Show that $b = E_\theta|X|$ is independent of θ.

Solution.
By using the procedure in a),

$$E|X| = \int_{-1}^{1} \frac{|x|}{2}(1+\theta x)dx = \frac{1}{2}.$$

Thus, $b = E_\theta|X|$ does not depend on θ. □

c) Let θ_0 be a fixed parameter value in $[-1, 1]$. Determine the constant $c = c_{\theta_0}$ that minimizes the variance of the unbiased estimator $aX + c\big(|X| - b\big)$ when $\theta = \theta_0$. Is aX uniformly minimum variance unbiased?

Solution.
To obtain the variance of $aX + c\big(|X| - b\big)$, we first derive the second moments.

$$EX^2 = E|X|^2 = \int_{-1}^{1} \frac{x^2}{2}(1+\theta x)dx = \frac{x^3}{6} + \frac{\theta x^4}{8}\Big|_{-1}^{1} = \frac{1}{3},$$

$$EX|X| = \int_{-1}^{1} \frac{x|x|}{2}(1+\theta x)dx = \frac{\theta}{4}.$$

From the previous steps, we obtain $a = 3$ and $b = E|X|$. Under $\theta = \theta_0$, the variance be

$$\begin{aligned}
\mathrm{Var}_{\theta_0}\Big(3X + c\big(|X| - E|X|\big)\Big) &= 9\mathrm{Var}_{\theta_0}(X) + 6c\mathrm{Cov}_{\theta_0}\big(X, |X| - E|X|\big) + c^2\mathrm{Var}_{\theta_0}\big(|X| - E|X|\big) \\
&= 9\mathrm{Var}_{\theta_0}(X) + 6c\mathrm{Cov}_{\theta_0}\big(X, |X| - E|X|\big) + c^2\mathrm{Var}_{\theta_0}\big(|X|\big) \\
&= 9\left(\frac{1}{3} - \frac{\theta_0^2}{9}\right) + 6c\left(\frac{\theta_0}{4} - \frac{1}{2}\cdot\frac{\theta_0}{3}\right) + c^2\left(\frac{1}{3} - \frac{1}{4}\right) \\
&= 3 - \theta_0^2 + \frac{\theta_0}{2}c + \frac{1}{12}c^2. \qquad (*)
\end{aligned}$$

Since $(*)$ is convex with respect to c, take a derivative,

$$\frac{\theta_0}{2} + \frac{c}{6} = 0$$

Therefore, $(*)$ has minimum when $c = -3\theta_0$.
Meanwhile, $\mathrm{Var}_{\theta_0}(3X) = 3 - \theta_0^2$ which is greater than $(*)$. Therefore, $3X$ is not UMVU. □

Problem 4.24

In the normal one-sample problem, the statistic $t = \sqrt{n}\overline{X}/S$ has the non-central t-distribution on $n-1$ degrees of freedom and non-centrality parameter $\delta = \sqrt{n}\mu/\sigma$. Use our results on distribution theory for the one-sample problem to find the mean and variance of t.

Solution.
Note that $E\overline{X} = \mu = \sigma\delta/\sqrt{n}$, $\text{Var}(\overline{X}) = \sigma^2/n$, and the formula ES^r in (4.10) page 70.
Since $\overline{X}$ and S^2 are independent,

$$Et = E\left[\sqrt{n}\frac{\overline{X}}{S}\right] = \sqrt{n}(E\overline{X})(ES^{-1}) = \sqrt{n}\frac{\sigma\delta}{\sqrt{n}}\frac{\sigma^{-1}2^{-\frac{1}{2}}\Gamma\big[(-1+n-1)/2\big]}{(n-1)^{-\frac{1}{2}}\Gamma\big[(n-1)/2\big]} = \delta\frac{\sqrt{n-1}\;\Gamma\big[(n-2)/2\big]}{\sqrt{2}\;\Gamma\big[(n-1)/2\big]},$$

$$\begin{aligned}
Et^2 &= n(E\overline{X}^2)(ES^2) = n\left(\frac{\sigma^2}{n}+\frac{\sigma^2\delta^2}{n}\right)\frac{\sigma^{-2}2^{-1}\;\Gamma\big[(-2+n-1)/2\big]}{(n-1)^{-1}\;\Gamma\big[(n-1)/2\big]}\\
&= (1+\delta^2)\frac{n-1}{2}\frac{\Gamma\big[(n-3)/2\big]}{\Gamma\big[(n-3)/2+1\big]} = (1+\delta^2)\frac{n-1}{n-3}.
\end{aligned}$$

$$\text{Var}(t) = Et^2 - (Et)^2 = \frac{n-1}{n-3} + \delta^2\left[\frac{n-1}{n-3} - \frac{(n-1)\left(\Gamma\big[(n-2)/2\big]\right)^2}{2\left(\Gamma\big[(n-1)/2\big]\right)^2}\right].$$

□

Problem 4.28

Let $X_1, \dots, X_n$ be i.i.d. from the uniform distribution on $(0,\theta)$.

a) Use the Hammersley-Chapman-Robbins inequality to find a lower bound for the variance of an unbiased estimator of θ. This bound will depend on Δ. Note that Δ cannot vary freely but must lie in a suitable set.

Solution.
Since δ is the unbiased estimator of θ, $g(\theta) = \theta$ and $g(\theta+\Delta) - g(\theta) = \Delta$. By Hammersley-Chapman-Robbins inequality, the lower bound for the variance of δ be

$$LB = \frac{\Delta^2}{E_\theta\left[\frac{p_{\theta+\Delta}(X)}{p_\theta(X)} - 1\right]^2}.$$

Since $X_i \sim Uniform(0,\theta)$, $p_\theta(X) = 0 \Rightarrow p_{\theta+\Delta}(X) = 0$. Thus Δ should be negative and $\theta + \Delta > 0$, therefore $\Delta \in (-\theta, 0)$.

$$\frac{p_{\theta+\Delta}(X)}{p_\theta(X)} = \begin{cases} \frac{\theta^n}{(\theta+\Delta)^n}, & \max\{X_1,\dots,X_n\} < \theta+\Delta \text{ with probability } \frac{(\theta+\Delta)^n}{\theta^n} \text{ under } P_\theta,\\ 0, & \text{otherwise.}\end{cases}$$

For simplicity, denote $Y = p_{\theta+\Delta}(X)/p_\theta(X)$, then $EY = 1$ and $EY^2 = \frac{\theta^n}{(\theta+\Delta)^n}$. Thus, the expectation in the denominator is

$$E(Y-1)^2 = Var(Y) = EY^2 - [EY]^2 = \frac{\theta^n}{(\theta+\Delta)^n} - 1.$$

Therefore, the lower bound be

$$LB = \frac{\Delta^2}{\frac{\theta^n}{(\theta+\Delta)^n} - 1}. \tag{$*$}$$

□

b) In principle, the best lower bound can be found taking the supremum over Δ. This calculation cannot be done explicitly, but an approximation is possible. Suppose $\Delta = -c\theta/n$. Show that the lower bound for the variance can be written as $\theta^2 g_n(c)/n^2$. Determine $g(c) = \lim_{n\to\infty} g_n(c)$.

***Solution*.**
Plug in $\Delta = -c\theta/n$ to $(*)$, then

$$LB = \frac{c^2\theta^2/n^2}{\left(1-\frac{c}{n}\right)^{-n} - 1}.$$

Therefore,

$$g_n(c) = \frac{c^2}{\left(1-\frac{c}{n}\right)^{-n} - 1},$$

$$g(c) = \lim_{n\to\infty} g_n(c) = \frac{c^2}{e^c - 1}.$$

□

c) Find the value c_0 that maximizes $g(c)$ over $c \in (0, 1)$ and give an approximate lower bound for the variance of δ. (The value c_0 cannot be found explicitly, but you should be able to come up with a numerical value.)

***Solution*.**
Taking a derivative,

$$\frac{\partial g(c)}{\partial c} = \frac{\partial}{\partial c}\left(\frac{c^2}{e^c - 1}\right) = \frac{2c(e^c-1) - c^2e^c}{(e^c-1)^2} = 0$$

$$\begin{aligned}
&\iff 2c(e^c - 1) = c^2e^c \\
&\iff 2e^c - 2 = ce^c \\
&\iff 1 - e^c = \frac{c}{2} \\
&\iff e^{-c} = 1 - \frac{c}{2}.
\end{aligned}$$

By numerical method, $c_0 = 1.59362$. Therefore, an approximated lower bound of $g(c_0)\theta^2/n^2 = 0.64761\theta^2/n^2$.

□

Problem 4.30

Suppose $X_1, \dots, X_n$ are independent with $X_i \sim N(\alpha + \beta t_i, 1)$, $i = 1, \dots, n$, where $t_1, \dots, t_n$ are known constants and α, β are unknown parameters.

a) Find the Fisher information matrix $I(\alpha, \beta)$.

Solution.
The log likelihood of $\theta = (\alpha, \beta)$ is

$$l(\beta) = \frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i - \alpha - \beta t_i)^2.$$

Take derivatives,

$$\frac{\partial l}{\partial \alpha} = \sum_i (x_i - \alpha - \beta t_i), \qquad \frac{\partial l}{\partial \beta} = \sum_i (x_i - \alpha - \beta t_i)t_i,$$

$$\frac{\partial^2 l}{\partial \alpha^2} = -n, \qquad \frac{\partial^2 l}{\partial \beta^2} = -\sum_i t_i^2,$$

$$\frac{\partial^2 l}{\partial \alpha\beta} = -\sum_i t_i$$

Therefore, the Fisher information matrix be

$$I(\alpha,\beta) = -E\left[\nabla l(\alpha,\beta)\right]^2 = \begin{pmatrix} n & \sum_i t_i \\ \sum_i t_i & \sum_i t_i^2 \end{pmatrix}.$$

□

b) Give a lower bound for the variance of an unbiased estimator of α.

Solution.
Since $g(\theta) = \alpha$, by Cramer-Rao lower bound,

$$\begin{aligned} LB &= \nabla g(\theta)^T I^{-1}(\alpha,\beta)\nabla g(\theta) \\ &= (1,0)\frac{\begin{pmatrix} \sum_i t_i^2 & -\sum_i t_i \\ -\sum_i t_i & n \end{pmatrix}}{n\sum_i t_i^2 - (\sum_i t_i)^2}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{n\sum_i t_i^2 - (\sum_i t_i)^2}\left(\sum_i t_i^2, \ -\sum_i t_i\right)\begin{pmatrix}1\\0\end{pmatrix} \\ &= \frac{\sum_i t_i^2}{n\sum_i t_i^2 - (\sum_i t_i)^2} \\ &= \frac{1/n}{1 - (\sum_i t_i)^2/\sum_i t_i^2}. \end{aligned}$$

□

c) Suppose we know the value of β. Give a lower bound for the variance of an unbiased estimator of α in this case.

Solution.
Since β is known, $\theta = \alpha$. Then the lower bound be

$$LB = \frac{g'(\theta)}{I(\theta)} = \frac{1}{n}.$$

□

d) Compare the estimators in parts (b) and (c). When are the bounds the same? If the bounds are different, which is larger?

Solution.
If $\sum_i t_i = 0$, two lower bounds be same. And by comparing the lower bounds,

$$LB_{b)} = \frac{1/n}{1 - (\sum_i t_i)^2 / \sum_i t_i^2} > \frac{1}{n} = LB_{c)}.$$

□

e) Give a lower bound for the variance of an unbiased estimator of the product $\alpha\beta$.

Solution.
Since $g(\theta) = \alpha\beta$, $\nabla g(\theta) = \binom{\beta}{\alpha}$. Then the lower bound be

$$\begin{aligned}
LB &= \nabla g(\theta)^T I^{-1}(\alpha, \beta) \nabla g(\theta) \\
&= (\beta,\ \alpha) \frac{\begin{pmatrix} \sum_i t_i^2 & -\sum_i t_i \\ -\sum_i t_i & n \end{pmatrix}}{n \sum_i t_i^2 - (\sum_i t_i)^2} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \\
&= \frac{1}{n \sum_i t_i^2 - (\sum_i t_i)^2} \left(\beta \sum_i t_i^2 - \alpha \sum_i t_i,\ \ -\beta \sum_i t_i + n\alpha \right) \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \\
&= \frac{\beta^2 \sum_i t_i^2 - 2\alpha\beta \sum_i t_i + n\alpha^2}{n \sum_i t_i^2 - (\sum_i t_i)^2}.
\end{aligned}$$

□

Problem 4.31

Find the Fisher information for the Cauchy location family with densities p_θ given by

$$p_\theta(x) = \frac{1}{\pi\big[(x-\theta)^2 + 1\big]}.$$

Also, what is the Fisher information for θ^3?

Solution.
Since p_θ is the location family, by using Example 4.11 in page 75,

$$\begin{aligned}
I(\theta) = \int \frac{[f'(x)]^2}{f(x)} dx &= \int \frac{\left[-\frac{2\pi x}{\pi(x^2+1)}\right]^2}{1/[\pi(x^2+1)]} = \int \frac{4x^2}{\pi(x^2+1)^3} dx \\
&\text{Change of variable } x = \tan t \Rightarrow dx = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} dt \\
&= \frac{4}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\tan^2 t}{(\tan^2 + 1)^3} \frac{1}{\cos^2 t} dt \\
&= \frac{4}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 t}{\cos^2 t} \cos^6 t \frac{1}{\cos^2 t} dt \quad (\because \tan^2 t + 1 = 1/\cos^2 t) \\
&= \frac{4}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t \cos^2 t dt \\
&= \frac{4}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - \cos 2x}{2} \frac{1 + \cos 2x}{2} dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos^2 2x}{4} dt \\
&= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos 4x}{2} dt \\
&= \frac{1}{2\pi}\left[t - \frac{1}{4}\sin 4t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{1}{2}.
\end{aligned}
$$

Let $\xi = \theta^3$, then $\theta = \xi^{1/3} := h(\xi)$.

$$\therefore I(\xi) = [h'(\xi)]^2 I\big(h(\xi)\big) = \left(\frac{1}{3}\xi^{-2/3}\right)^2 \frac{1}{2} = \frac{1}{18\xi^{4/3}}.$$

□

Problem 4.32

Suppose X has a Poisson distribution with mean θ^2, so the parameter θ is the square root of the usual parameter $\lambda = EX$. Show that the Fisher information $I(\theta)$ is constant.

Solution.
$p_\theta(X) = (\theta^2)^x e^{-\theta^2}/x!$ and $l(\theta) = \log 1/x! + 2x\log\theta - \theta^2$.

$$\frac{\partial l(\theta)}{\partial\theta} = \frac{2x}{\theta} - 2\theta, \quad \frac{\partial^2 l(\theta)}{\partial\theta^2} = -\frac{2x}{\theta^2} - 2.$$

$$\therefore I(\theta) = E\left(-\frac{\partial^2 l(\theta)}{\partial\theta^2}\right) = \frac{2}{\theta^2}EX + 2 = 4.$$

□

Problem 4.33

Consider the exponential distribution with failure rate λ. Find a function h defining a new parameter $\theta = h(\lambda)$ so that Fisher information $I(\theta)$ is constant.

Solution.
Let the original parameter is θ, and reparameterized parameter is λ.
$p_\lambda(x) = \lambda e^{-\lambda x}$, $l(\lambda) = \log\lambda - \lambda x$.

$$\frac{\partial l(\lambda)}{\partial\lambda} = \frac{1}{\lambda} - x, \quad \frac{\partial^2 l(\lambda)}{\partial\lambda^2} = -\frac{1}{\lambda^2}.$$

$$\tilde{I}(\lambda) = \frac{1}{\lambda^2}.$$

From (4.18) in page 74,

$$I(\theta) = \frac{\tilde{I}(\lambda)}{[h'(\lambda)]^2} = \frac{1}{[\lambda h'(\lambda)]^2}.$$

Thus, if $h(\lambda) = \log\lambda$, $I(\theta)$ be constant. □

Problem 4.34

Consider an autoregressive model in which $X_1 \sim N\big(\theta, \sigma^2/(1-\rho^2)\big)$ and the conditional distribution of X_{j+1} given $X_1 = x_1, \dots X_j = x_j$, is $N\big(\theta + \rho(x_j - \theta), \sigma^2\big),\ j = 1, \dots, n-1$.

a) Find the Fisher information matrix, $I(\theta, \sigma)$.

Solution.
The joint density is

$$p(x_1, \dots, x_n) = p(x_1)p(x_2|x_1)\cdots(x_n|x_1, \dots, x_n),$$

thus the likelihood function be

$$\begin{aligned}
l(\theta,\sigma) &= l(x_1) + \sum_{j=1}^{n-1} l(x_{j+1}|x_1, \dots, x_j) \\
&= \log \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} - \log\sigma - \frac{1-\rho^2}{2\sigma^2}(x_1-\theta)^2 \\
&\quad - \frac{1}{\sigma^2}\sum_{j=1}^{n-1}[x_{j+1} - \theta - \rho(x_j - \theta)]^2 + \log\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{n-1} \\
&= -\frac{1-\rho^2}{2\sigma^2}(x_1-\theta)^2 - \frac{1}{2\sigma^2}\sum_{j=1}^{n-1}[x_{j+1} - \rho x_j - \theta(1-\rho)]^2 \\
&\quad - \frac{n}{2}\log(2\pi) - n\log\sigma + \log\sqrt{1-\rho^2}.
\end{aligned}$$

Take derivatives,

$$\frac{\partial l(\theta,\sigma)}{\partial\theta} = \frac{1-\rho^2}{\sigma^2}(x_1-\theta) + \frac{1-\rho}{\sigma^2}\sum_{j=1}^{n-1}[x_{j+1} - \rho x_j - \theta(1-\rho)],$$

$$\frac{\partial^2 l(\theta,\sigma)}{\partial\theta^2} = -\frac{1-\rho^2}{\sigma^2} - \frac{(n-1)(1-\rho)^2}{\sigma^2},$$

$$\frac{\partial^2 l(\theta,\sigma)}{\partial\theta\partial\sigma} = -\frac{2(1-\rho^2)\epsilon_1}{\sigma^3} + \frac{2(1-\rho)}{\sigma^3}\sum_{j=1}^{n-1}\eta_{j+1},$$

$$\frac{\partial l(\theta,\sigma)}{\partial\sigma} = \frac{(1-\rho^2)(x_1-\theta)^2}{\sigma^3} + \frac{1}{\sigma^3}\sum_{j=1}^{n-1}\eta_{j+1}^2 - \frac{n}{\sigma},$$

$$\frac{\partial^2 l(\theta,\sigma)}{\partial\sigma^2} = -\frac{3(1-\rho^2)(x_1-\theta)^2}{\sigma^4} - \frac{3}{\sigma^4}\sum_{j=1}^{n-1}\eta_{j+1}^2 + \frac{n}{\sigma^2},$$

where $\epsilon_j = x_j - \theta$, and $\eta_{j+1} = \epsilon_{j+1} - \rho\epsilon_j$. (See Hint in part c))
We can easily infer that $\epsilon_1 \sim N\big(0, \sigma^2/(1-\rho^2)\big)$ and $\eta_{j+1}|_{X_1=x_1,\dots,X_j=x_j} = \eta_{j+1}|_{\epsilon_1,\eta_2,\dots,\eta_j} \sim N(0,\sigma^2)$.
From this, $\eta_2, \dots, \eta_n \sim$ i.i.d. $N(0,\sigma^2)$ and these are independent of ϵ_1. (왜???)
Therefore, the information matrix is given by taking expectation,

$$I(\theta,\sigma) = \begin{pmatrix} \frac{1-\rho^2+(n-1)(1-\rho)^2}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{pmatrix}.$$

□

b) Give a lower bound for the variance of an unbiased estimator of θ.

Solution.
Since $g(\theta,\sigma)=\theta$,

$$LB = \nabla g(\theta,\sigma)^T I^{-1}(\theta,\sigma)\nabla g(\theta,\sigma) = (1,0)I^{-1}(\theta,\sigma)\begin{pmatrix}1\\0\end{pmatrix} = \frac{\sigma^2}{1-\rho^2+(n-1)(1-\rho)^2}.$$

□

c) Show that the sample average $\overline{X}=(X_1+\cdots+X_n)/n$ is an unbiased estimator of θ, compute its variance, and compare its variance with the lower bound.
<u>Hint</u>: Define $\epsilon_j = X_j-\theta$ and $\eta_{j+1}=\epsilon_{j+1}-\epsilon_j$. Use smoothing to argue that $\eta_2,\dots,\eta_n$ are i.i.d. $N(0,\sigma^2)$ and are independent of ϵ_1. Similarly, X_i is independent of $\eta_{i+1},\eta_{i+2},\dots$. Use these facts to find first $\mathrm{Var}(X_2)=\mathrm{Var}(\epsilon_2)$, then $\mathrm{Var}(X_3),\mathrm{Var}(X_4),\dots$. Finally find $\mathrm{Cov}(X_{i+1},X_i), n\mathrm{Cov}(X_{i+2},X_i)$, and so on.

Solution.
Let $\bar\epsilon=\frac{1}{n}\sum_j\epsilon_j=\overline{X}-\theta$. Since $E\epsilon_j=0$, $E(\overline{X}-\theta)=E(\bar\epsilon)=0$. Thus $\overline{X}$ is unbiased estimator of θ.
Since $\epsilon_2=\rho\epsilon_1+\eta_2$, $\mathrm{Var}(\epsilon_2)=\rho^2\mathrm{Var}(\epsilon_1)+\mathrm{Var}(\eta_2)=\sigma^2+\frac{\rho^2\sigma^2}{1-\rho^2}=\frac{\sigma^2}{1-\rho^2}$. In general, $\epsilon_i=\rho\epsilon_{i-1}+\eta_i$ and we can obtain $\mathrm{Var}(X_i)=\mathrm{Var}(\epsilon_i)=\frac{\sigma^2}{1-\rho^2}$.
If $i>j$,

$$\begin{aligned}\epsilon_i &= \rho\epsilon_{i-1}+\eta_i\\ &= \rho^2(\epsilon_{i-2}+\eta_{i-1})+\eta_i\\ &= \cdots\\ &= \rho^{i-j}\epsilon_j+\rho^{i-j+1}\eta_{j+1}+\cdots+\rho\eta_{i-1}+\eta_i.\end{aligned}$$

From this,

$$\mathrm{Cov}(X_i,X_j)=\mathrm{Cov}(\epsilon_i,\epsilon_j)=\rho^{|i-j|}\frac{\sigma^2}{1-\rho^2}.$$

Then, the variance of $\overline{X}$ is

$$\begin{aligned}\mathrm{Var}(\overline{X}) &= \mathrm{Var}(\bar\epsilon)\\ &= \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^n\epsilon_i\right) = \frac{1}{n^2}\mathrm{Cov}\left(\sum_{i=1}^n\epsilon_i,\sum_{i=1}^n\epsilon_i\right)\\ &= \frac{1}{n^2}\left[\sum_{i=1}^n\mathrm{Cov}(\epsilon_i,\epsilon_i)+2\sum_{i>j}\mathrm{Cov}(\epsilon_i,\epsilon_j)\right]\\ &= \frac{\sigma^2}{n^2(1-\rho^2)}\left[n+2\left\{(n-1)\rho+(n-2)\rho^2+\cdots+\rho^{n-1}\right\}\right]\\ &= \frac{\sigma^2}{n^2(1-\rho^2)}\left[n+2\sum_{j=1}^{n-1}(n-j)\rho^j\right]\\ &= \frac{\sigma^2}{n^2(1-\rho^2)}\left[n+2\left(n\frac{\rho-\rho^n}{1-\rho}-\frac{\rho-\rho^{n+1}}{(1-\rho)^2}+\frac{n\rho^n}{1-\rho}\right)\right] \qquad (*)\\ &= \frac{\sigma^2}{n^2(1-\rho^2)}\left[n+2\frac{n\rho-n\rho^2-\rho+\rho^{n+1}}{(1-\rho)^2}\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sigma^2}{n^2(1-\rho^2)} \left[\frac{n - 2n\rho + n\rho^2 + 2n\rho - 2n\rho^2 - 2\rho + 2\rho^{n+1}}{(1-\rho)^2} \right] \\
&= \frac{\sigma^2}{n^2(1-\rho^2)} \left[\frac{-n\rho^2 - 2\rho + n + 2\rho^{n+1}}{(1-\rho)^2} \right] \\
&= \frac{\sigma^2}{n^2(1-\rho^2)} \left[\frac{n(1-\rho^2) - 2(\rho - \rho^{n+1})}{(1-\rho)^2} \right] \\
&= \frac{\sigma^2}{n(1-\rho)^2} - \frac{2\sigma^2\rho(1-\rho^n)}{n^2(1-\rho^2)(1-\rho)^2}.
\end{aligned}
$$

<u>Note.</u> $(*)$ is obtained as follows:

$$
\begin{aligned}
\sum_{j=1}^{n-1} \rho^j &= \frac{\rho(1-\rho^{n-1})}{1-\rho}, \\
\sum_{j=1}^{n-1} j\rho^j &= \sum_j \rho \frac{d}{d\rho}\rho^j = \rho \frac{d}{d\rho} \sum_{j=1}^{n-1} \rho^j \\
&= \rho \frac{d}{d\rho} \left(\frac{\rho - \rho^n}{1-\rho} \right) \\
&= \rho \frac{(1 - n\rho^{n-1})(1-\rho) + (\rho - \rho^n)}{(1-\rho)^2} \\
&= \rho \frac{1 - \rho - n\rho^{n-1} + n\rho^n + \rho - \rho^n}{(1-\rho)^2} \\
&= \rho \frac{1 - \rho^n - n\rho^{n-1}(1-\rho)}{(1-\rho)^2} \\
&= \frac{\rho - \rho^{n+1}}{(1-\rho)^2} - \frac{n\rho^n}{1-\rho}.
\end{aligned}
$$

□

Chapter 5

Curved Exponential Families

Problem 5.1

a) ***Solution***.

$\square$

b) ***Solution***.

$\square$