Theoretical Statistics: Topics for a Core Course

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Preface

This note contains the solution of the exercises in the textbook, *Theoretical Statistics: Topics for a Core Course*, and it was created by Hyunsung Kim, who is a Ph.D. student. I wrote it when I study a theoretical statistics based on this textbook on my own by solving some exercises and also referred to the solution manual in the textbook.
It contains a few selected exercises in the textbook what I studied, and also note that it may not be the exact solutions. If you want to refer to this note, you should study with doubt about the answer.

Textbook

- Keener, *Theoretical Statistics: Topics for a Core Course.*

Reference

- Durrett, *Probability: Theory and Examples, 5th edition.*
- Royden, *Real Analysis, 4th edition.*

Chapter 1

Probability and Measure

1.1 Measures

Definition 1.1.1 (σ-algebra)**.** *A collection of $\mathcal{A}$ of subsets of a set $\mathcal{X}$ is a σ-algebra (or σ-field) if*

1. *$\mathcal{X} \in \mathcal{A}$ and $\emptyset \in \mathcal{A}$,*
2. *If $A \in \mathcal{A}$, then $A^c = \mathcal{X} - A \in \mathcal{A}$,*
3. *If $A_1, A_2, \dots \in \mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.*

Definition 1.1.2 (Measure)**.** *A function μ on a σ-algebra $\mathcal{A}$ of $\mathcal{X}$ is a measure if*

1. *For every $A \in \mathcal{A}$, $0 \le \mu(A) \le \infty$; that is, $\mu : \mathcal{A} \to [0, \infty]$.*
2. *(Countable Additivity) If $A_1, A_2, \dots$ are disjoint elements of $\mathcal{A}$, then*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Example 1.1.3. For a measure μ on a set $\mathcal{X}$ and subsets $A \in \mathcal{X}$,

1. μ is *counting measure* on $\mathcal{X}$ if $\mathcal{X}$ is countable, and define

   $$\mu(A) = \#A = \text{number of points in } A,$$

   and also its σ-algebra $\mathcal{A}$ be the *power set* of $\mathcal{X}$, denoted $\mathcal{A} = 2^{\mathcal{X}}$.

2. μ is *Lebesgue measure* on $\mathcal{X}$ if $\mathcal{X} = \mathbb{R}^n$, and define

   $$\mu(A) = \int \underset{A}{\cdots} \int \mathrm{d}x_1 \cdots dx_n,$$

   and for any set A in a σ-algebra $\mathcal{A}$ called the *Borel sets* of $\mathcal{X} = \mathbb{R}^n$, and formally $\mathcal{A}$ is the smallest σ-algebra that contains all open set in $\mathbb{R}^n$ ("rectangles").

3. μ is *probability measure* on $\mathcal{X}$ if $\mu(\mathcal{X}) = 1$.

Remark 1.1.4. If $\mathcal{A}$ is a σ-algebra of subsets of $\mathcal{X}$, the pair $(\mathcal{X}, \mathcal{A})$ is called a *measurable space*, and if μ is a measure on $\mathcal{A}$, the triple $(\mathcal{X}, \mathcal{A}, \mu)$ is called a *measure space*.

Proposition 1.1.5 (Continuity property of measures)**.** *If measurable sets B_n, $n \geq 1$, are increasing ($B_1 \subset B_2 \subset \cdots$), with $B = \bigcup_{i=1}^{\infty} B_n$, called the limit of the sequence, then*

$$\mu(B) = \lim_{n \to \infty} \mu(B_n).$$

If a measure μ is a probability measure, it also holds for decreasing sets ($B_1 \supset B_2 \supset \cdots$) and its limit $B = \bigcap_{n=1}^{\infty} B_n$.

Proof. 생략 $\square$

Definition 1.1.6 (σ-finite)**.** *There is some notations and definitions about the measure.*

1. *A measure μ is finite if $\mu(\mathcal{X}) < \infty$.*
2. *A measure μ is σ-finite if $\exists A_1, A_2, \cdots \in \mathcal{A}$ with $\mu(A_i) < \infty$ $\forall i = 1, 2, \ldots$ and $\bigcup_{i=1}^{\infty} A_i = \mathcal{X}$.*

Remark 1.1.7. A probability measure is σ-finite.

1.2 Integration

Example 1.2.1. The integral of "nice" function f against a measure μ is written as $\int f d\mu$ or $\int f(x) d\mu(x)$.

1. If μ is *counting measure* on $\mathcal{X}$, then the integral of f against μ is
$$\int f d\mu = \sum_{x \in \mathcal{X}} f(x).$$
2. If μ is *Lebesgue measure* on $\mathbb{R}^n$, then the integral of f against μ is
$$\int f d\mu = \int \cdots \int f(x_1, \ldots, x_n) dx_1 \ldots dx_n.$$

Definition 1.2.2 (Measurable)**.** *If $(\mathcal{X}, \mathcal{A})$ is a measurable space and f is a real-valued function on $\mathcal{X}$, then f is measureble if*

$$f^{-1}(B) \stackrel{\text{def}}{=} \{x \in \mathcal{X} : f(x) \in B\} \in \mathcal{A},$$

for every Borel set B.

Definition 1.2.3 (Indicator function)**.** *The indicator function 1_A of a set A is defined as*

$$1_A(x) = I\{x \in A\} = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Proposition 1.2.4. *The integral of the indicator function 1_A has following properties:*

1. *For any set* $A \in \mathcal{A}$, $\int 1_A d\mu = \mu(A)$.
2. *If* f *and* g *are non-negative measurable functions, and if* a *and* b *are positive constants,*
$$\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu.$$
3. *If* $f_1 \leq f_2 \leq \cdots$ *are non-negative measurable functions, and if* $f(x) = \lim_{n\to\infty} f_n(x)$, *then*
$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

Proof. 생략 □

Definition 1.2.5 (Simple function). *A function* f *is simple if*
$$f = \sum_{i=1}^{m} a_i 1_{A_i},$$
where $a_a, \ldots, a_m$ *are positive constants, and* $A_1, \ldots, A_m$ *are sets in* $\mathcal{A}$

Remark 1.2.6. By using 1 and 2 in Proposition **??**, the integral of simple function is obtained as
$$\int \left(\sum_{i=1}^{m} a_i 1_{A_i} \right) d\mu = \sum_{i=1}^{m} a_i \mu(A_i).$$

Theorem 1.2.7. *If* f *is non-negative and measurable, then there exist non-negative simple functions* $f_1 \leq f_2 \leq \cdots$ *with* $f = \lim_{n\to\infty} f_n$.

Proof. 생략 □

Definition 1.2.8 (Integrable). *A measurable function* f *is integrable if* $\int |f| d\mu < \infty$.

1.3 Events, Probabilities, and Random Variables

Definition 1.3.1 (Probability space). *If* $\mathcal{E}$ *is a sample space,* $\mathcal{B}$ *is a* σ*-algebra of subsets of* $\mathcal{X}$, *and* P *is a probability measure, then the triple* $(\mathcal{X}, \mathcal{A}, P)$ *is called a probability space. Also* $B \in \mathcal{B}$ *are called events, points* $e \in \mathcal{E}$ *are called outcomes, and* $P(B)$ *is called the probability of* B.

Definition 1.3.2 (Random variable). *A measurable function* $X : \mathcal{E} \to \mathbb{R}$ *is called a random variable.*

Definition 1.3.3 (Distribution). *The probability measure* P_X *is defined by*
$$P_X(A) = P\left(\{e \in \mathcal{E} : X(e) \in A\}\right) \overset{\text{def}}{=} P(X \in A),$$
for Borel sets A *is called distribution of* X, *and if we let* $P_X := Q$, *then it is denoted as* $X \sim Q$, *means that a random variable* X *has distribution* Q.

Definition 1.3.4 (Cumulative distribution function). *The cumulative distribution function of* X *is defined by*
$$F_X(x) = P(X \leq x) = P\left(\{e \in \mathcal{E} : X(e) \leq x\}\right) = P_X\left((-\infty, x]\right),$$
for $x \in \mathbb{R}$.

1.4 Null Sets

Definition 1.4.1 (Null set)**.** *For a measure μ on the measuruable space $(\mathcal{X}, \mathcal{A})$, a set N is called null with respect to μ if*

$$\mu(N) = 0.$$

Definition 1.4.2 (Almost everywhere)**.** *If a statement holds for $x \in \mathcal{X} - N$ with N null, the statement is said to hold almost everywhere (a.e.) or a.e. μ.*

Example 1.4.3. $f = 0$ a.e. $\mu \iff \mu(\{x \in \mathcal{X} : f(x) \neq 0\}) = 0$.

Remark 1.4.4 (Almost sure)**.** For the probability space, the statement holds a.e. if and only if $\mu(B^c) = 0$ or $\mu(B) = 1$. Especially, it can be denoted that the statement holds *almost surely (a.s.)* or *with probability one.*

Proposition 1.4.5. *There are a few useful facts about integration:*

1. *If $f = 0$ (a.e. μ), then $\int f d\mu = 0$.*
2. *If $f \geq 0$ and $\int f d\mu = 0$, then $f = 0$ (a.e. μ).*
3. *If $f = g$ (a.e. μ), then $\int f d\mu = \int g d\mu$ for at least of the integral exists.*
4. *If $\int 1_{(c,x)} f d\mu = 0$ for all $x > c$, then $f(x) = 0$ for a.e. $x > c$. The constant c here can be $-\infty$.*
5. *If f and g are integrable and $f > g$, then $\int f d\mu > \int g d\mu$.* (From 2)

Proof. 생략 □

1.5 Densities

Definition 1.5.1 (Absolutely continuity)**.** *Let P and μ be measures on a σ-field $\mathcal{A}$ of $\mathcal{X}$. Then P is called absolutely continuous with respect to μ, written $P \ll \mu$, if $P(A) = 0$ whenever $\mu(A) = 0$.*

Theorem 1.5.2 (Radon-Nikodym)**.** *If a finite measure P is absolutely continuous with respect to a σ-finite measure μ, then there exists a non-negative measurable function f such that*

$$P(A) = \int_A f d\mu \stackrel{\text{def}}{=} \int f 1_A d\mu.$$

Remark 1.5.3. The function f in Theorem **??** is called Radon-Nikodym derivitive of P with respect to μ, or the *density* of P with respect to μ, denoted

$$f = \frac{dP}{d\mu}.$$

Remark 1.5.4. If $X \sim P_X$ and P_X is absolutely continuous with respect to μ with density $p = dP_X/d\mu$, it is called that X *has density p with repect to μ.*

Example 1.5.5 (Absolutely continuous random variables)**.** If a random variable X has density p with respect to Lebesgue measure on $\mathbb{R}$, then X or its distribution P_X is called absolutely continuous with density p. Then, from the Radon-Nikodym theorem,

$$F_X(x) = P(X \leq x) = P_X\left((-\infty, x]\right) = \int_{-\infty}^{x} p(u)du$$

And also by fundamental theorem of calculus, $p(x) = F'_X(x)$.

Example 1.5.6 (Discrete random variables)**.** Let $\mathcal{X}_0$ be a countable subset of $\mathbb{R}$. The measure μ defined by