Theoretical Statistics: Topics for a Core Course
Problem Solutions

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Contents

Preface

This note contains the solution of the problems in the textbook, *Theoretical Statistics: Topics for a Core Course*, and it was created by Hyunsung Kim, who is a Ph.D. student. I wrote it when I study a theoretical statistics based on this textbook on my own by solving some problems and also referred to the solution manual in the textbook.
It contains a few selected problems in the textbook what I studied, and also note that it may not be the exact solutions. If you want to refer to this note, you should study with doubt about the answer.

Textbook

- Keener, *Theoretical Statistics: Topics for a Core Course.*

Reference

- Durrett, *Probability: Theory and Examples, 5th edition.*
- Royden, *Real Analysis, 4th edition.*

Chapter 1

Probability and Measure

Problem 1.1

Prove (1.1). If measurable sets $B_n,\ n \geq 1$, are increasing, with $B = \bigcup_{n=1}^{\infty} B_n$, called the limit of the sequence, then

$$\mu(B) = \lim_{n\to\infty} \mu(B_n).$$

Solution.
First, we showed that A_n's are disjoint. If $j < k$, then $B_j \subseteq B_{k-1}$.
Since $A_j \subset B_j \subseteq B_{k-1}$ and $A_k \subset B_{k-1}^c$, A_j and A_k are disjoint.
Also $B_n = \bigcup_{j=1}^{n} A_j$ and $\bigcup_{n=1}^{\infty} A_n = B$,

$$\mu(B) = \sum_{i=1}^{\infty} \mu(A_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} A_i\right) = \lim_{n\to\infty} \mu(B_n).$$

□

Problem 1.8

Prove *Boole's inequality*: For any events $B_1, B_2, \ldots$,

$$P\left(\bigcup_{i\geq 1} B_i\right) \leq \sum_{i\geq 1} P(B_i).$$

Solution.
Let $B = \bigcup_{i=1}^{\infty} B_i$, then $1_B \leq \sum 1_{B_i}$. Then by Fubini's theorem,

$$P(B) = \int 1_B dP \leq \int \sum 1_{B_i} dP = \sum \int 1_{B_i} dP = \sum P(B_i).$$

□

Problem 1.10

Let μ and ν be measures on $(\mathcal{E}, \mathcal{B})$.

a) Show that the sum η defined by $\eta(B) = \mu(B) + \nu(B)$ is also a measure.

Solution.
We should check following 2 conditions.

(i) For arbitrary set $A \in \mathcal{B}$, $\mu(A) \geq 0$ and $\nu(A) \geq 0$. $\Rightarrow \eta(A) = \mu(A) + \nu(A) \geq 0$.
$\therefore \eta : \mathcal{B} \to [0, \infty]$.

(ii) For disjoint set $B_1, B_2, \cdots \subset \mathcal{B}$, then

$$\begin{aligned}\eta\left(\bigcup_{i=1}^{\infty} B_i\right) &= \mu\left(\bigcup B_i\right) + \nu\left(\bigcup B_i\right) \\ &= \sum \mu(B_i) + \sum \nu(B_i) \\ &= \sum \{\mu(B_i) + \nu(B_i)\} \\ &= \sum \eta(B_i)\end{aligned}$$

$\therefore \eta$ is a measure. □

b) If f is a non-negative measurable function, show that

$$\int f d\eta = \int f d\mu + \int f d\nu.$$

Solution.
We show it by 2 stage.

(i) Let $f = \sum_{i=1}^{n} a_i 1_{A_i}$, the non-negative simple function. Then,

$$\begin{aligned}\int f d\eta &= \int \sum_{i=1}^{n} a_i 1_{A_i} d\eta = \sum a_i \eta(A_i) \\ &= \sum a_i \{\mu(A_i) + \nu(A_i)\} = \int f d\mu + \int f d\nu.\end{aligned}$$

(ii) For general case, let f_n is the sequence of non-negative simple functions increasing to f. (i.e. $f_1 \leq f_2 \leq \cdots \leq f$)

$$\begin{aligned}\int f d\eta &= \lim_{n\to\infty} \int f_n d\eta = \lim_{n\to\infty} \left(\int f_n d\mu + \int f_n d\nu\right) \\ &= \lim_{n\to\infty} \int f_n d\mu + \lim_{n\to\infty} \int f_n d\nu = \int f d\mu + \int f d\nu.\end{aligned}$$

□

Problem 1.11

Suppose f is the simple function $1_{(1/2,\pi]} + 21_{(1,2]}$, and let μ be a measure on $\mathbb{R}$ with $\mu\{(0, a^2]\} = a,\ a > 0$. Evaluate $\int f d\mu$.

Solution.
By the integral of simple function and the finite additivity, it can be simply computed as

$$\begin{aligned}\int f d\mu &= \mu\{(1/2,\pi]\} + 2\mu\{(1,2]\} \\ &= [\mu\{(0,\pi]\} - \mu\{(1/2,\pi]\}] + 2\left[\mu\{(0,2]\} - \mu\{(1,2]\}\right] \quad \text{(finite additivity)} \\ &= \left(\sqrt{\pi} - 1/\sqrt{2}\right) + 2\left(\sqrt{2} - 1\right)\end{aligned}$$

□

Problem 1.12

Suppose that $\mu\{(0,a)\} = a^2$ for $a > 0$ and that f is defined by

$$f(x) = \begin{cases} 0, & x \leq 0, \\ 1, & 0 < x < 2, \\ \pi, & 2 \geq x < 5, \\ 0, & x \geq 5. \end{cases}$$

Compute $\int f d\mu$.

Solution.
Since $f = 1_{(0,2)} + 1_{[2,5)}$ is simple, the integral can be computed as

$$\begin{aligned}\int f d\mu &= \mu\{(0,2)\} + \pi\mu\{[2,5)\} \\ &= \mu\{(0,2)\} + \pi\left[\mu\{(0,5)\} - \mu\{(0,2)\}\right] \\ &= 4 - \pi(25 - 4)\end{aligned}$$

□

Problem 1.13

Define the function f by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find simple functions $f_1 \leq f_2 \leq \cdots$ increasing to f (i.e. $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in \mathbb{R}$). Let μ be Lebesgue measure on $\mathbb{R}$. Using our formal definition of an integral and the faProblemct that $\mu\big((a,b]\big) = b - a$ whenever $b > a$ (this might be used to formally define Lebesgue measure), show that $\int f d\mu = 1/2$.

Solution.

Let $f_n = \lfloor 2^n x \rfloor / 2^n$ for $0 < x \le 1$ and 0 otherwise. ($\lfloor y \rfloor$ is a floor function.) Then,

$$f_1(x) = \lfloor 2x \rfloor / 2 = \begin{cases} 0, & x < 1/2, \\ 1/2, & 1/2 \le x < 1, \\ 1, & x = 1 \end{cases}$$

$$f_2(x) = \lfloor 2^2 x \rfloor / 2^2 = \begin{cases} 0, & x < 1/2^2, \\ 1/2^2, & 1/2^2 \le x < 2/2^2, \\ 2/2^2, & 2/2^2 \le x < 3/2^2, \\ 3/2^2, & 3/2^2 \le x < 1, \\ 1, & x = 1 \end{cases}$$

$$\vdots$$

$$f_n(x) = \lfloor 2^n x \rfloor / 2^n = \begin{cases} 0, & x < 1/2^n, \\ 1/2^n, & 1/2^n \le x < 2/2^n, \\ \vdots & \\ (2^n - 1)/2^n, & (2^n - 1)/2^n \le x < 1, \\ 1, & x = 1 \end{cases}$$

$$\begin{aligned} \therefore \int f_n d\mu &= \frac{1}{2^n} \left(\frac{1}{2^n} + \frac{2}{2^n} + \cdots + \frac{2^n - 1}{2^n} \right) \\ &= \frac{1 + 2 + \cdots + (2^n - 1)}{4^n} \\ &= \frac{2^n (2^n - 1)}{2 \cdot 4^n} \\ &\longrightarrow \frac{1}{2}. \end{aligned}$$

My solution

Let the simple function $f_n = \sum_{i=1}^n \frac{i}{n} 1_{(\frac{i-1}{n}, \frac{i}{n}]}$. Then,

$$\int f_n d\mu = \sum_{i=1}^n \frac{i}{n} \frac{1}{n} = \frac{1}{n^2} \frac{n(n+1)}{2} \longrightarrow \frac{1}{2}.$$

□

Problem 1.16

Define $F(a-) = \lim_{x \uparrow a} F(x)$. Then, if F is non-decreasing, $F(a-) = \lim_{n \to \infty} F(a - 1/n)$. Use (1.1)[Continuity of measure] to show that if a random variable X has cumulative distribution function F_X,

$$P(X < a) = F_X(a-).$$

Also, show that

$$P(X = a) = F_X(a) - F_X(a-).$$

Solution.

(i) Let $B_n = \{X \le a - 1/n\}$ and $\bigcup_{n=1}^{\infty} B_n = \{X < a\}$. By continuity of measure,

$$P(B) = P(X < a) = \lim_{n\to\infty} P(X \le a - 1/n) = \lim_{n\to\infty} F_X(a - 1/n) = F_X(a-).$$

(ii) Since $\{X < a\}$ and $\{X = a\}$ are disjoint with union $\{X \le a\}$,

$$P(X < a) + P(X = a) = P(X \le a).$$

$$\therefore P(X = a) = F_X(a) - F_X(a-).$$

□

Problem 1.17

Suppose X is a geometric random variable with mass function

$$p(x) = P(X = x) = \theta(1-\theta)^x, \quad x = 0, 1, \ldots,$$

where $\theta \in (0, 1)$ is a constant. Find the probability that X is even.

Solution.

$$\begin{aligned} P(X \text{ is even}) &= P(X = 0) + P(X = 2) + P(X = 4) \cdots \\ &= \theta + \theta(1-\theta)^2 + \theta(1-\theta)^4 + \cdots \\ &= \frac{\theta}{1-(1-\theta)^2} \\ &= \frac{1}{2-\theta} \end{aligned}$$

□

Problem 1.18

Let X be a function mapping $\mathcal{E}$ into $\mathbb{R}$. Recall that if B is a subset of $\mathbb{R}$, then $X^{-1}(B) = \{e \in \mathcal{E} : X(e) \in B\}$. Use this definition to prove that

$$X^{-1}(A \cap B) = X^{-1}(A) \cap X^{-1}(B),$$

$$X^{-1}(A \cup B) = X^{-1}(A) \cup X^{-1}(B),$$

and

$$X^{-1}\left(\bigcup_{i=0}^{\infty} A_i\right) = \bigcup_{i=0}^{\infty} X^{-1}(A_i).$$

Solution.

(i)

$$\begin{aligned} e \in X^{-1}(A \cap B) &\Leftrightarrow X(e) \in A \cap B \\ &\Leftrightarrow X(e) \in A \text{ and } X(e) \in B \\ &\Leftrightarrow e \in X^{-1}(A) \text{ and } e \in X^{-1}(B) \\ &\Leftrightarrow e \in X^{-1}(A) \cap X^{-1}(B). \end{aligned}$$

(ii) Similarly,

$$\begin{aligned} e \in X^{-1}(A \cup B) &\Leftrightarrow X(e) \in A \cup B \\ &\Leftrightarrow X(e) \in A \text{ or } X(e) \in B \\ &\Leftrightarrow e \in X^{-1}(A) \text{ or } e \in X^{-1}(B) \\ &\Leftrightarrow e \in X^{-1}(A) \cup X^{-1}(B). \end{aligned}$$

(iii)

$$\begin{aligned} e \in X^{-1}(\bigcup_{i=0}^{\infty} A_i) &\Leftrightarrow X(e) \in \bigcup_{i=0}^{\infty} A_i \\ &\Leftrightarrow X(e) \in A_i \text{ for some } i \\ &\Leftrightarrow e \in X^{-1}(A_i) \text{ for some } i \\ &\Leftrightarrow e \in \bigcup_{i=0}^{\infty} X^{-1}(A_i). \end{aligned}$$

□

Problem 1.19

Let P be a probability measure on $(\mathcal{E}, \mathcal{B})$, and let X be a random variable. Show that the distribution P_X of X defined by $P_X(B) = P(X \in B) = P(X^{-1}(B))$ is a measure (on the Borel sets of $\mathbb{R}$).

<u>Solution</u>.
To show that P_X is a measure, we should check 2 conditions.

(i) $P_X(B) = P\big(X^{-1}(B)\big) \geq 0$.

(ii) From Problem 1.18, $X^{-1}\left(\bigcup_i A_i\right) = \bigcup_i X^{-1}(A_i)$ is hold.
Also, if B_i and B_j are disjoint, then $X^{-1}(B_i)$ and $X^{-1}(B_j)$ are disjoint. ($\because$ If $e \in \mathcal{E}$ lie in both $X^{-1}(B_1)$ and $X^{-1}(B_2)$, $X(e)$ lies in B_1 and B_2.)
Therefore, for arbitrary disjoint set $B_1, B_2, \dots$ with union $B = \bigcup_i B_i$, $X^{-1}(B_1), X^{-1}(B_2), \dots$ are also disjoint with union $X^{-1}(B)$.

$$\therefore \sum_i P_X(B_i) = \sum_i P\big(X^{-1}(B_i)\big) = P\big(X^{-1}(B)\big) = P_X(B).$$

The above 2 conditions are hold, P_X is a measure. □

Problem 1.21

Let X have a uniform distribution on $(0,1)$; that is, X is absolutely continuous with density p defined by

$$p(x) = \begin{cases} 1, & x \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Let Y_1 and Y_2 denote the first two digits of X when X is written as a binary decimal (so $Y_1 = 0$ if $X \in (0, 1/2)$ for instance). Find $P(Y_1 = i, Y_2 = j),\ i = 0$ or $1, j = 0$ or 1.

Solution.
For all cases, the probability are same. In other words,

$$P(Y_1 = 0, Y_2 = 0) = P(Y_1 = 1, Y_2 = 0) = P(Y_1 = 0, Y_2 = 1) = P(Y_1 = 1, Y_2 = 1) = 1/4.$$

□

Problem 1.22

Let $\mathcal{E} = (0,1)$, let $\mathcal{B}$ be the Borel subsets of $\mathcal{E}$, and let $P(A)$ be the length of A for $A \in \mathcal{B}$. (P would be called the *uniform probability measure* on $(0,1)$.) Define the random variable X by

$$X(e) = \min\{e, 1/2\}.$$

Let μ be the sum of Lebesgue measure on $\mathbb{R}$ and counting measure on $\mathcal{X}_0 = \{1/2\}$. Show that the distribution P_X of X is absolutely continuous with respect to μ and find the density of P_X.

Solution.

(i) Claim: For any $B \in \mathbb{R}$, $\mu(B) = 0 \Longrightarrow P_X(B) = 0$.

$$\begin{aligned} \{X \in B\} &= \{y \in (0,1) : X(y) \in B\} \\ &= \begin{cases} B \cap (0, 1/2), & 1/2 \notin B, \\ \big(B \cap (0,1/2)\big) \cup [1/2, 1), & 1/2 \in B. \end{cases} \end{aligned}$$

Let λ be a Lebesgue measure and ν be a counting measure on $\{1/2\}$. Then,

$$\mu(B) = 0 \Longleftrightarrow \lambda(B) = 0 \text{ and } 1/2 \notin B.\ (\mu(B) = \nu(B) = 0)$$

$$\Longrightarrow P_X(B) = P(X \in B) = P\big(B \cap (0,1/2)\big) = \lambda\big(B \cap (0, 1/2)\big) = 0.$$

$$\therefore P_X \text{ is absolutely continuous w.r.t. } \lambda + \nu \overset{\text{def}}{=} \mu.$$

(ii) Find the density of P_X.
From the above equation,

$$P(X \in B) = \lambda\big(B \cap (0,1/2)\big) + \frac{1}{2} 1_B(1/2). \tag{1.1}$$

If f is the density, then (1.1) should be equal to $\int_B f d(\lambda+\nu)$.

$$\begin{aligned}
\int_B f d(\lambda+\nu) &= \int f 1_B d(\lambda+\nu) \\
&= \int \left(1_{B\cap(0,1/2)} + \frac{1}{2}1_{\{1/2\}\cap B}\right) d(\lambda+\nu) \\
&= \lambda\big(B\cap(0,1/2)\big) + \nu\big(B\cap(0,1/2)\big) + \frac{1}{2}\lambda\big(\{1/2\}\cap B\big) + \frac{1}{2}\nu\big(\{1/2\}\cap B\big) \\
&= \lambda\big(B\cap(0,1/2)\big) + \frac{1}{2}1_B(1/2).
\end{aligned}$$

$$\therefore f = 1_{(0,1/2} + \frac{1}{2}1_{\{1/2\}}.$$

□

Problem 1.23

The standard normal distribution $N(0,1)$ has density ϕ given by

$$\phi(x) = \frac{e^{-x^2/2}}{\sqrt{w\pi}}, \qquad x\in\mathbb{R},$$

with respect to Lebesgue measure λ on $\mathbb{R}$. The corresponding cumulative distribution function is Φ, so

$$\Phi(x) = \int_{-\infty}^{x} \phi(z)dz$$

for $x\in\mathbb{R}$. Suppose that $X\sim N(0,1)$ and that the random variable Y equals X when $|X|<1$ and is 0 otherwise. Let P_Y denote the distribution of Y and let μ be counting measure on $\{0\}$. Find the density of P_Y with respect to $\lambda+\mu$.

Solution.
Let $g(x) = x1_{|x|<1}$, then $Y=g(X)$.
For any integrable function f, the expectation of $f(Y)$ against the density of X is

$$Ef(Y) = Ef\big(g(X)\big) = \int f\big(g(x)\big)\phi(x)dx = \int f\big(g(x)\big)\phi(x)1_{(-1,1)}(x)dx + cf(0), \tag{1.2}$$

where $c = \int \phi(x)1_{|x|>1}(x)dx = 2\Phi(-1)$.
Similarly, the expectation of $f(Y)$ against the density p of Y is

$$Ef(Y) = \int fpd(\lambda+\mu) = \int fpd\lambda + \int fpd\mu = \int f(x)p(x)dx + f(0)p(0). \tag{1.3}$$

Then, (1.2) and (1.3) should be same. Therefore,

$$\begin{aligned}
f = 1_{\{0\}} &\Longrightarrow p(0) = 2\Phi(-1), \\
f(0) = 0 &\Longrightarrow \int f(x)\phi(x)1_{(-1,1)}(x)dx = \int f(x)p(x)dx \\
&\Longrightarrow p(x) = \phi(x) \text{ for } 0<|x|<1.
\end{aligned}$$

$$\therefore p(x) = 2\Phi(-1)1_{\{0\}}(x) + \phi(x)1_{(0,1)}(|x|) \text{ is the density of } P_Y.$$

□

Problem 1.24

Let μ be a σ-finite measure on a measurable space $(X, \mathcal{B})$. Show that μ is absolutely continuous with respect to some probability measure P.
Hint: You can use the fact that if $\mu_1, \mu_2, \dots$ are probability measures and $c_1, c_2, \dots$ are non-negative constants, then $\sum c_i\mu_i$ is a measure. (The proof for Problem 1.10 extents easily to this case.) The measure μ_i, you will want to consider, are truncations of μ to sets A_i covering X with $\mu(A_i) < \infty$, given by $\mu_i(B) = \mu(B \cap A_i)$. With the constants c_i chosen properly, $\sum c_i\mu_i$ will be a probability measure.

Solution.

(i) μ is finite $\Longrightarrow$ Trivial!
$\because$ Since $\mu(X) < \infty$, $P = \mu/\mu(X)$ be a probablity measure. Therefore, $P(N) = 0$ implies $\mu(N) = 0$.

(ii) μ is infinite but σ-finite.
$\Longrightarrow\ \exists A_1, A_2, \dots \in \mathcal{B}$ s.t. $\bigcup A_i = X$ and $0 < \mu(A_i) < \infty\ \forall i$.
From Hint, $\mu_i(B) = \mu(B \cap A_i)$ and also $\mu_i(X) = \mu(A_i) \Longrightarrow \mu_i$ is a finite measure.
Let $b_i = 1/2^i$ (or any other seq of positive constants s.t. $\sum b_i = 1$) and define $c_i = b_i/\mu(A_i)$.
Then $P = \sum_i c_i\mu_i$ is a probability measure, since $P(X) = \sum_i c_i\mu_i(X) = \sum_i \left(\frac{b_i}{\mu(A_i)}\right)\mu(A_i) = 1$.
Suppose $P(N) = \sum_i c_i\mu_i(N) = 0$. Then $\mu_i(N) = \mu(N \cap A_i) = 0\ \forall i$.
By Boole's inequality,

$$\mu(N) = \mu\big(\bigcup_i (N \cap A_i)\big) \le \sum_i \mu(N \cap A_i) = 0.$$

For any null set for P is a null set for μ.
$\therefore \mu \ll P$.

$\square$

Problem 1.25

The monotone convergence theorem states that if $0 \le f_1 \le f_2 \le \cdots$ are measurable functions and $f = \lim_{n\to\infty} f_n$, then $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$. Use this result to prove the following assertions.

a) Show that if $X \sim P_X$ is a random variable on $(\mathcal{E}, \mathcal{B}, P)$ and f is a non-negative measurable function, then

$$\int f\big(X(e)\big)dP(e) = \int f(x)dP_X(x).$$

Hint: Try it first with f an indicator function. For the general case, let f_n be a sequence of simple functions increasing to f.

Solution.
We will show the equation by 3 steps.

(i) Suppose $f = 1_A$ (indicator function).
Then, $f\big(X(e)\big) = 1$ for $X(e) \in A$.
$\Rightarrow f \circ X = 1_B$ where $B = \{e : X(e) \in A\}$.

By definition, $P_X(A) = P(B)$.

$$\int f\big(X(e)\big)dP(e) = \int 1_B(e)dP(e) = P(B),$$
$$\int f(x)dP_X(x) = \int 1_A(x)dP_X(x) = P_X(A).$$

$\therefore$ The statement is holds.

(ii) Suppose $f = \sum_{i=1}^n c_i 1_{A_i}$ (simple function).

$$\begin{aligned}
\therefore \int f\big(X(e)\big)dP(e) &= \int \sum_i c_i 1_{A_i}\big(X(e)\big)dP(e) \\
&= \sum_i c_i \int 1_{A_i}\big(X(e)\big)dP(e) \\
&= \sum_i c_i \int 1_{A_i}(x)dP_X(x) \quad \big(\text{by (i)}\big) \\
&= \int \sum_i c_i 1_{A_i}(x)dP_X(x) \\
&= \int f(x)dP_X(x).
\end{aligned}$$

(iii) Suppose f is a non-negative measurable function (general case), and f_n is non-negative simple functions increasing to f.
$\Rightarrow f_n \circ X \nearrow f \circ X$.

$$\begin{aligned}
\therefore \int f\big(X(e)\big)dP(e) &= \lim_{n\to\infty} \int f_n\big(X(e)\big)dP(e) \quad \big(\text{by M.C.T.}\big) \\
&= \lim_{n\to\infty} \int f_n(x)dP_X(x) \quad \big(\text{by (ii)}\big) \\
&= \int f(x)dP_X(x) \quad \big(\text{by M.C.T.}\big)
\end{aligned}$$

$\square$

b) Suppose that P_X has density p with respect to μ, and let f be a non-negative measurable function. Show that
$$\int f dP_X = \int f p d\mu.$$

Solution.
It can be also showed by 3 steps.

(i) Let $f = 1_A$ (indicator function).

$$\int f dP_X = \int 1_A dP_X = P_X(A) = \int_A p d\mu = \int 1_A p d\mu = \int f p d\mu.$$

(ii) Suppose $f = \sum_{i=1}^n c_i 1_{A_i}$ (simple function).

$$\int f dP_X = \int \sum_i c_i 1_{A_i} dP_X = \sum_i c_i \int 1_{A_i} dP_X \overset{\text{(i)}}{\overset{\downarrow}{=}} \sum_i c_i \int 1_{A_i} p d\mu = \int \sum_i c_i 1_{A_i} p d\mu = \int f p d\mu.$$

(iii) For general case, let a non-negative measurable function f and non-negative simple functions f_n s.t. $f_n \nearrow f$. Then by M.C.T.,

$$\int f dP_X \overset{\text{M.C.T.}}{\overset{\downarrow}{=}} \lim_{n\to\infty} \int f_n dP_X \overset{\text{(ii)}}{\overset{\downarrow}{=}} \lim_{n\to\infty} \int f_n p d\mu \overset{\text{M.C.T.}}{\overset{\downarrow}{=}} \int f p d\mu.$$

□

Problem 1.26

The gamma distribution.

a) The gamma function is defined for $\alpha > 0$ by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$. Show that $\Gamma(X+1) = x!$ for $x = 0, 1, \ldots$.

Solution.
By integration by parts,

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = -t^x e^{-t}\Big|_0^\infty + (\alpha - 1)\int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x).$$

Since $\Gamma(1) = \int_0^\infty e^{-x} dx = 1$,

$$\Gamma(x+1) = x\Gamma(x) = x(x-1)\Gamma(x-1) = \cdots = x(x-1)\cdots 2 \cdot 1 \cdot \Gamma(1) = x!.$$

□

b) Show that the function

$$p(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & \text{otherwise}, \end{cases}$$

is a (Lebesgue) probability density when $\alpha > 0$ and $\beta > 0$. This density is called the gamma density with parameters α and β. The corresponding probability distribution is denoted $\Gamma(\alpha, \beta)$.

Solution.
Using change of variable, $y = x/\beta \Rightarrow dx = \beta dy$. Therefore,

$$\begin{aligned} \int p(x)dx &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx \\ &= \int \frac{1}{\Gamma(\alpha)\beta^\alpha} (\beta y)^{\alpha-1} e^{-y} \beta dy \\ &= \frac{1}{\Gamma(\alpha)} \int y^{\alpha-1} e^{-y} dy \\ &= 1. \end{aligned}$$

□

c) Show that if $X \sim \Gamma(\alpha, \beta)$, then $EX^r = \beta^r \Gamma(\alpha + r)/\Gamma(\alpha)$. Use this formula to find the mean and variance of X.

Solution.

$$\begin{aligned}
EX^r &= \int_0^\infty x^r p(x) dx \\
&= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha+r-1} e^{-x/\beta} dx \\
&= \int_0^\infty \frac{1}{\Gamma(\alpha+r)\beta^{\alpha+r}} x^{\alpha+r-1} e^{-x/\beta} dx \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \beta^r \\
&= \beta^r \Gamma(\alpha+r)/\Gamma(\alpha).
\end{aligned}$$

By using this,

$$EX = \beta\Gamma(\alpha+1)/\Gamma(\alpha) = \alpha\beta,$$

$$EX^2 = \beta^2\Gamma(\alpha+2)/\Gamma(\alpha) = (\alpha+1)\alpha\beta^2,$$

$$Var(X) = EX^2 - \{EX\}^2 = \alpha\beta^2.$$

□

Problem 1.27

Suppose X has a uniform distribution on $(0, 1)$. Find the mean and covariance matrix of the random vector $\binom{X}{X^2}$.

Solution.
Since $X \sim$ Uniform$(0, 1)$, the density of X is $p(x) = 1$ for $x \in (0, 1)$. Thus,

$$EX^r = \int x^r dx = \frac{1}{r+1}.$$

Therefore,

$$E\left(\begin{pmatrix} X \\ X^2 \end{pmatrix}\right) = \begin{pmatrix} EX \\ EX^2 \end{pmatrix},$$

$$Cov\left(\begin{pmatrix} X \\ X^2 \end{pmatrix}\right) = \begin{pmatrix} Var(X) & Cov(X, X^2) \\ Cov(X^2, X) & Var(X^2) \end{pmatrix} = \begin{pmatrix} EX^2 - \{EX\}^2 & EX^3 - EXEX^2 \\ EX^3 - EXEX^2 & EX^4 - \{EX^2\}^2 \end{pmatrix}.$$

□

Problem 1.28

If $X \sim N(0, 1)$, find the mean and covariance matrix of the random vector $\binom{X}{I\{X>c\}}$.

Solution.
Let $\Phi(x)$ is a cumulative distribution function of X, and ϕ is a density of X. Then,

$$E\left(\begin{pmatrix} X \\ I\{X > c\} \end{pmatrix}\right) = \begin{pmatrix} EX \\ EI\{X > c\} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 - \Phi(c) \end{pmatrix},$$

$$Cov\left(\begin{pmatrix} X \\ I\{X > c\} \end{pmatrix}\right) = \begin{pmatrix} Var(X) & EX1_{X>c} - EXE1_{X>c} \\ EX1_{X>c} - EXE1_{X>c} & E1^2_{X>c} - \{E1_{X>c}\}^2 \end{pmatrix} = \begin{pmatrix} 1 & \phi(c) \\ \phi(c) & \Phi(c)(1-\Phi(c)) \end{pmatrix}.$$

$$\because EX1_{X>c} = \int_c^\infty \frac{1}{\sqrt{2\pi}} xe^{-x^2/2}dx = \phi(c) \text{ and } 1^2_{X>c} = 1_{X>c}.$$

□

Problem 1.32

Suppose $E|X| < \infty$ and let

$$h(t) = \frac{1 - E\cos(tX)}{t^2}.$$

Use Fubini's theorem to find $\int_0^\infty h(t)dt$. Hint:

$$\int_0^\infty \big(1 - \cos(u)\big)u^{-2}du = \frac{\pi}{2}.$$

Solution.
By Funini's theorem,

$$\begin{aligned}
\int_0^\infty h(t)dt &= \int_0^\infty \left(\frac{1 - E\cos(tX)}{t^2}\right) dt \\
&= \int_0^\infty E\left(\frac{1 - \cos(tX)}{t^2}\right) dt \\
&= \int_0^\infty \int \frac{1 - \cos(tx)}{t^2} dP_X(x)dt \\
&= \int \int_0^\infty \frac{1 - \cos(tx)}{t^2} dtdP_X(x) \quad \text{(by Funini's theorem)} \\
&\quad u = tx \Rightarrow du = xdt, t = x/u \\
&= \int \int_0^\infty |x| \frac{1 - \cos(u)}{u^2} dudP_X(x) \\
&= \int \frac{\pi}{2}|x|dP_X(x) \\
&= \frac{\pi}{2}E|X|.
\end{aligned}$$

□

Problem 1.33

Suppose X is absolutely continuous with density $p_X(x) = xe^{-x},\ x > 0$ and $p_X(x) = 0,\ x \le 0$. Define $c_n = E(1+X)^{-n}$. Use Funini's theorem to evaluate $\sum_{n=1}^\infty c_n$.

Solution.

By Funini's theorem,

$$\begin{aligned}
\sum_{n=1}^{\infty} c_n &= \sum_n E(1+X)^{-n} \\
&= \sum_n \int_0^{\infty} (1+x)^{-n} x e^{-x} dx \\
&= \int_0^{\infty} \sum_n (1+x)^{-n} x e^{-x} dx \\
\text{Since } & \left|\frac{1}{1+x}\right| < 1, \ \sum_{n=1}^{\infty} \left(\frac{1}{1+x}\right)^n = \frac{1/(1+x)}{1-1/(1+x)} = \frac{1}{x}. \\
&= \int_0^{\infty} \frac{1}{x} x e^{-x} dx \\
&= \int_0^{\infty} e^{-x} dx \\
&= 1.
\end{aligned}$$

□

Problem 1.36

Suppose X and Y are independent random variables, and let F_X and F_Y denote their cumulative distribution functions.

a) Use smoothing to show that the cumulative distribution function of $S = X + Y$ is

$$F_S(s) = P(X + Y \leq s) = EF_X(s - Y). \tag{1.4}$$

Solution.
Since X and Y are independent, $P(X + Y \leq s|Y = y) = P(X \leq s - y) = F_X(s - y)$.
Thus, $P(X + Y \leq s|Y) = F_X(s - Y)$.

$$\therefore F_S(s) = P(X + Y \leq s) = EP(X + Y \leq s|Y) = EF_X(s - Y).$$

□

b) If X and Y are independent and Y is almost surely positive, use smoothing to show that the cumulative distribution function of $W = XY$ is $F_W(w) = EF_X(w/Y)$ for $w > 0$.

Solution.
Since X and Y are independent, $P(XY \leq w|Y = y) = P(X \leq w/y|Y = y) = P(X \leq w/y) = F_X(w/y)$, for $y > 0$.
Thus, $P(XY \leq w|Y) = F_X(w/Y)$ a.s.

$$\therefore F_W(w) = P(XY \leq w) = EP(XY \leq w|Y) = EF_X(w/Y).$$

□

Problem 1.37

Differentiating (1.4) with respect to s one can show that if X is absolutely continuous with density p_X, then $S = X + Y$ is absolutely continuous with density

$$p_S(s) = Ep_X(s - Y)$$

for $s \in \mathbb{R}$. Use this formula to show that if X and Y are independent with $X \sim \Gamma(\alpha, 1)$ and $Y \sim \Gamma(\beta, 1)$, then $X + Y \sim \Gamma(\alpha + \beta, 1)$.

Solution.

□