Theoretical Statistics: Topics for a Core Course
Problem Solutions

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Contents

Preface

This note contains the solution of the problems in the textbook, *Theoretical Statistics: Topics for a Core Course*, and it was created by Hyunsung Kim, who is a Ph.D. student. I wrote it when I study a theoretical statistics based on this textbook on my own by solving some problems and also referred to the solution manual in the textbook.

It contains a few selected problems in the textbook what I studied, and also note that it may not be the exact solutions. If you want to refer to this note, you should study with doubt about the answer.

Textbook

- Keener, *Theoretical Statistics: Topics for a Core Course.*

Reference

- Durrett, *Probability: Theory and Examples, 5th edition.*

- Royden, *Real Analysis, 4th edition.*

Chapter 1

Probability and Measure

Problem 1.1

Prove (1.1). If measurable sets B_n, $n \geq 1$, are increasing, with $B = \bigcup_{n=1}^{\infty} B_n$, called the limit of the sequence, then

$$\mu(B) = \lim_{n \to \infty} \mu(B_n).$$

Solution.
First, we showed that A_n's are disjoint. If $j < k$, then $B_j \subseteq B_{k-1}$.
Since $A_j \subset B_j \subseteq B_{k-1}$ and $A_k \subset B_{k-1}^c$, A_j and A_k are disjoint.
Also $B_n = \bigcup_{j=1}^{n} A_j$ and $\bigcup_{n=1}^{\infty} A_n = B$,

$$\mu(B) = \sum_{i=1}^{\infty} \mu(A_i) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(A_i) = \lim_{n \to \infty} \mu\left(\bigcup_{i=1}^{n} A_i\right) = \lim_{n \to \infty} \mu(B_n).$$

$\square$

Problem 1.8

Prove *Boole's inequality*: For any events $B_1, B_2, \ldots,$

$$P\left(\bigcup_{i \geq 1} B_i\right) \leq \sum_{i \geq 1} P(B_i).$$

Solution.
Let $B = \bigcup_{i=1}^{\infty} B_i$, then $1_B \leq \sum 1_{B_i}$. Then by Fubini's theorem,

$$P(B) = \int 1_B dP \leq \int \sum 1_{B_i} dP = \sum \int 1_{B_i} dP = \sum P(B_i).$$

$\square$

Problem 1.10

Let μ and ν be measures on $(\mathcal{E}, \mathcal{B})$.

a) Show that the sum η defined by $\eta(B) = \mu(B) + \nu(B)$ is also a measure.

Solution.
We should check following 2 conditions.

(i) For arbitrary set $A \in \mathcal{B}$, $\mu(A) \geq 0$ and $\nu(A) \geq 0$. $\Rightarrow \eta(A) = \mu(A) + \nu(A) \geq 0$.
$\therefore \eta : \mathcal{B} \to [0, \infty]$.

(ii) For disjoint set $B_1, B_2, \cdots \subset \mathcal{B}$, then

$$\eta\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(\bigcup B_i\right) + \nu\left(\bigcup B_i\right)$$
$$= \sum \mu(B_i) + \sum \nu(B_i)$$
$$= \sum \{\mu(B_i) + \nu(B_i)\}$$
$$= \sum \eta(B_i)$$

$\therefore \eta$ is a measure. □

b) If f is a non-negative measurable function, show that

$$\int f d\eta = \int f d\mu + \int f d\nu.$$

Solution.
We show it by 2 stage.

(i) Let $f = \sum_{i=1}^{n} a_i 1_{A_i}$, the non-negative simple function. Then,

$$\int f d\eta = \int \sum_{i=1}^{n} a_i 1_{A_i} d\eta = \sum a_i \eta(A_i)$$
$$= \sum a_i \{\mu(A_i) + \nu(A_i)\} = \int f d\mu + \int f d\nu.$$

(ii) For general case, let f_n is the sequence of non-negative simple functions increasing to f. (i.e. $f_1 \leq f_2 \leq \cdots \leq f$)

$$\int f d\eta = \lim_{n \to \infty} \int f_n d\eta = \lim_{n \to \infty} \left(\int f_n d\mu + \int f_n d\nu\right)$$
$$= \lim_{n \to \infty} \int f_n d\mu + \lim_{n \to \infty} \int f_n d\nu = \int f d\mu + \int f d\nu.$$

□

Problem 1.11

Suppose f is the simple function $1_{(1/2, \pi]} + 21_{(1,2]}$, and let μ be a measure on $\mathbb{R}$ with $\mu\{(0, a^2]\} = a$, $a > 0$. Evaluate $\int f d\mu$.

Solution.
By the integral of simple function and the finite additivity, it can be simply computed as

$$\int f d\mu = \mu\{(1/2, \pi]\} + 2\mu\{(1, 2]\}$$
$$= [\mu\{(0, \pi]\} - \mu\{(1/2, \pi]\}] + 2[\mu\{(0, 2]\} - \mu\{(1, 2]\}] \quad \text{(finite additivity)}$$
$$= \left(\sqrt{\pi} - 1/\sqrt{2}\right) + 2\left(\sqrt{2} - 1\right)$$

$\square$

Problem 1.12

Suppose that $\mu\{(0, a)\} = a^2$ for $a > 0$ and that f is defined by

$$f(x) = \begin{cases} 0, & x \leq 0, \\ 1, & 0 < x < 2, \\ \pi, & 2 \geq x < 5, \\ 0, & x \geq 5. \end{cases}$$

Compute $\int f d\mu$.

Solution.
Since $f = 1_{(0,2)} + 1_{[2,5)}$ is simple, the integral can be computed as

$$\int f d\mu = \mu\{(0, 2)\} + \pi\mu\{[2, 5)\}$$
$$= \mu\{(0, 2)\} + \pi[\mu\{(0, 5)\} - \mu\{(0, 2)\}]$$
$$= 4 - \pi(25 - 4)$$

$\square$

Problem 1.13

Define the function f by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find simple functions $f_1 \leq f_2 \leq \cdots$ increasing to f (i.e. $f(x) = \lim_{n \to \infty} f_n(x)$ for all $x \in \mathbb{R}$). Let μ be Lebesgue measure on $\mathbb{R}$. Using our formal definition of an integral and the fact that $\mu((a, b]) = b - a$ whenever $b > a$ (this might be used to formally define Lebesgue measure), show that $\int f d\mu = 1/2$.

Solution.

Let $f_n = \lfloor 2^n x \rfloor / 2^n$ for $0 < x \leq 1$ and 0 otherwise. ($\lfloor y \rfloor$ is a floor function.) Then,

$$f_1(x) = \lfloor 2x \rfloor / 2 = \begin{cases} 0, & x < 1/2, \\ 1/2, & 1/2 \leq x < 1, \\ 1, & x = 1 \end{cases}$$

$$f_2(x) = \lfloor 2^2 x \rfloor / 2^2 = \begin{cases} 0, & x < 1/2^2, \\ 1/2^2, & 1/2^2 \leq x < 2/2^2, \\ 2/2^2, & 2/2^2 \leq x < 3/2^2, \\ 3/2^2, & 3/2^2 \leq x < 1, \\ 1, & x = 1 \end{cases}$$

$$\vdots$$

$$f_n(x) = \lfloor 2^n x \rfloor / 2^n = \begin{cases} 0, & x < 1/2^n, \\ 1/2^n, & 1/2^n \leq x < 2/2^n, \\ \vdots \\ (2^n - 1)/2^n, & (2^n - 1)/2^n \leq x < 1, \\ 1, & x = 1 \end{cases}$$

$$\begin{aligned} \therefore \int f_n d\mu &= \frac{1}{2^n} \left(\frac{1}{2^n} + \frac{2}{2^n} + \cdots + \frac{2^n - 1}{2^n} \right) \\ &= \frac{1 + 2 + \cdots + (2^n - 1)}{4^n} \\ &= \frac{2^n (2^n - 1)}{2 \cdot 4^n} \\ &\longrightarrow \frac{1}{2}. \end{aligned}$$

My solution
Let the simple function $f_n = \sum_{i=1}^{n} \frac{i}{n} 1_{(\frac{i-1}{n}, \frac{i}{n}]}$. Then,

$$\int f_n d\mu = \sum_{i=1}^{n} \frac{i}{n} \frac{1}{n} = \frac{1}{n^2} \frac{n(n+1)}{2} \longrightarrow \frac{1}{2}.$$

$\square$

Problem 1.16

Define $F(a-) = \lim_{x \uparrow a} F(x)$. Then, if F is non-decreasing, $F(a-) = \lim_{n \to \infty} F(a - 1/n)$. Use (1.1)[Continuity of measure] to show that if a random variable X has cumulative distribution function F_X,

$$P(X < a) = F_X(a-).$$

Also, show that

$$P(X = a) = F_X(a) - F_X(a-).$$

Solution.

(i) Let $B_n = \{X \leq a - 1/n\}$ and $\bigcup_{n=1}^{\infty} B_n = \{X < a\}$. By continuity of measure,

$$P(B) = P(X < a) = \lim_{n \to \infty} P(X \leq a - 1/n) = \lim_{n \to \infty} F_X(a - 1/n) = F_X(a-).$$

(ii) Since $\{X < a\}$ and $\{X = a\}$ are disjoint with union $\{X \leq a\}$,

$$P(X < a) + P(X = a) = P(X \leq a).$$

$$\therefore P(X = a) = F_X(a) - F_X(a-).$$

$\square$

Problem 1.17

Suppose X is a geometric random variable with mass function

$$p(x) = P(X = x) = \theta(1 - \theta)^x, \quad x = 0, 1, \ldots,$$

where $\theta \in (0, 1)$ is a constant. Find the probability that X is even.

Solution.

$$
\begin{aligned}
P(X \text{ is even}) &= P(X = 0) + P(X = 2) + P(X = 4) \cdots \\
&= \theta + \theta(1 - \theta)^2 + \theta(1 - \theta)^4 + \cdots \\
&= \frac{\theta}{1 - (1 - \theta)^2} \\
&= \frac{1}{2 - \theta}
\end{aligned}
$$

$\square$

Problem 1.18

Let X be a function mapping $\mathcal{E}$ into $\mathbb{R}$. Recall that if B is a subset of $\mathbb{R}$, then $X^{-1}(B) = \{e \in \mathcal{E} : X(e) \in B\}$. Use this definition to prove that

$$X^{-1}(A \cap B) = X^{-1}(A) \cap X^{-1}(B),$$

$$X^{-1}(A \cup B) = X^{-1}(A) \cup X^{-1}(B),$$

and

$$X^{-1}\left(\bigcup_{i=0}^{\infty} A_i\right) = \bigcup_{i=0}^{\infty} X^{-1}(A_i).$$

Solution.

(i)

$$e \in X^{-1}(A \cap B) \Leftrightarrow X(e) \in A \cap B$$
$$\Leftrightarrow X(e) \in A \text{ and } X(e) \in B$$
$$\Leftrightarrow e \in X^{-1}(A) \text{ and } e \in X^{-1}(B)$$
$$\Leftrightarrow e \in X^{-1}(A) \cap X^{-1}(B).$$

(ii) Similarly,

$$e \in X^{-1}(A \cup B) \Leftrightarrow X(e) \in A \cup B$$
$$\Leftrightarrow X(e) \in A \text{ or } X(e) \in B$$
$$\Leftrightarrow e \in X^{-1}(A) \text{ or } e \in X^{-1}(B)$$
$$\Leftrightarrow e \in X^{-1}(A) \cup X^{-1}(B).$$

(iii)

$$e \in X^{-1}\left(\bigcup_{i=0}^{\infty} A_i\right) \Leftrightarrow X(e) \in \bigcup_{i=0}^{\infty} A_i$$
$$\Leftrightarrow X(e) \in A_i \text{ for some } i$$
$$\Leftrightarrow e \in X^{-1}(A_i) \text{ for some } i$$
$$\Leftrightarrow e \in \bigcup_{i=0}^{\infty} X^{-1}(A_i).$$

$\square$

Problem 1.19

Let P be a probability measure on $(\mathcal{E}, \mathcal{B})$, and let X be a random variable. Show that the distribution P_X of X defined by $P_X(B) = P(X \in B) = P(X^{-1}(B))$ is a measure (on the Borel sets of $\mathbb{R}$).

Solution.
To show that P_X is a measure, we should check 2 conditions.

(i) $P_X(B) = P(X^{-1}(B)) \geq 0$.

(ii) From Problem 1.18, $X^{-1}(\bigcup_i A_i) = \bigcup_i X^{-1}(A_i)$ is hold.
Also, if B_i and B_j are disjoint, then $X^{-1}(B_i)$ and $X^{-1}(B_j)$ are disjoint. ($\because$ If $e \in \mathcal{E}$ lie in both $X^{-1}(B_1)$ and $X^{-1}(B_2)$, $X(e)$ lies in B_1 and B_2.)
Therefore, for arbitrary disjoint set $B_1, B_2, \ldots$ with union $B = \bigcup_i B_i$, $X^{-1}(B_1), X^{-1}(B_2), \ldots$ are also disjoint with union $X^{-1}(B)$.

$$\therefore \sum_i P_X(B_i) = \sum_i P(X^{-1}(B_i)) = P(X^{-1}(B)) = P_X(B).$$

The above 2 conditions are hold, P_X is a measure. $\square$

Problem 1.21

Let X have a uniform distribution on $(0, 1)$; that is, X is absolutely continuous with density p defined by

$$p(x) = \begin{cases} 1, & x \in (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Let Y_1 and Y_2 denote the first two digits of X when X is written as a binary decimal (so $Y_1 = 0$ if $X \in (0, 1/2)$ for instance). Find $P(Y_1 = i, Y_2 = j)$, $i = 0$ or $1, j = 0$ or 1.

Solution.
For all cases, the probability are same. In other words,

$$P(Y_1 = 0, Y_2 = 0) = P(Y_1 = 1, Y_2 = 0) = P(Y_1 = 0, Y_2 = 1) = P(Y_1 = 1, Y_2 = 1) = 1/4.$$

$\square$

Problem 1.22

Let $\mathcal{E} = (0, 1)$, let $\mathcal{B}$ be the Borel subsets of $\mathcal{E}$, and let $P(A)$ be the length of A for $A \in \mathcal{B}$. (P would be called the *uniform probability measure* on $(0, 1)$.) Define the random variable X by

$$X(e) = \min\{e, 1/2\}.$$

Let μ be the sum of Lebesgue measure on $\mathbb{R}$ and counting measure on $\mathcal{X}_0 = \{1/2\}$. Show that the distribution P_X of X is absolutely continuous with respect to μ and find the density of P_X.

Solution.

(i) Claim: For any $B \in \mathbb{R}$, $\mu(B) = 0 \implies P_X(B) = 0$.

$$\{X \in B\} = \{y \in (0, 1) : X(y) \in B\}$$
$$= \begin{cases} B \cap (0, 1/2), & 1/2 \notin B, \\ \big(B \cap (0, 1/2)\big) \cup [1/2, 1), & 1/2 \in B. \end{cases}$$

Let λ be a Lebesgue measure and ν be a counting measure on $\{1/2\}$. Then,

$$\mu(B) = 0 \iff \lambda(B) = 0 \text{ and } 1/2 \notin B. \ (\mu(B) = \nu(B) = 0)$$

$$\implies P_X(B) = P(X \in B) = P\big(B \cap (0, 1/2)\big) = \lambda\big(B \cap (0, 1/2)\big) = 0.$$

$$\therefore P_X \text{ is absolutely continuous w.r.t. } \lambda + \nu \overset{\text{def}}{=} \mu.$$

(ii) Find the density of P_X.
From the above equation,

$$P(X \in B) = \lambda\big(B \cap (0, 1/2)\big) + \frac{1}{2}1_B(1/2). \tag{1.1}$$

If f is the density, then (1.1) should be equal to $\int_B f d(\lambda + \nu)$.

$$
\begin{aligned}
\int_B f d(\lambda + \nu) &= \int f 1_B d(\lambda + \nu) \\
&= \int \left(1_{B \cap (0,1/2)} + \frac{1}{2} 1_{\{1/2\} \cap B} \right) d(\lambda + \nu) \\
&= \lambda \big(B \cap (0, 1/2) \big) + \nu \big(B \cap (0, 1/2) \big) + \frac{1}{2} \lambda \big(\{1/2\} \cap B \big) + \frac{1}{2} \nu \big(\{1/2\} \cap B \big) \\
&= \lambda \big(B \cap (0, 1/2) \big) + \frac{1}{2} 1_B(1/2).
\end{aligned}
$$

$$
\therefore f = 1_{(0,1/2} + \frac{1}{2} 1_{\{1/2\}}.
$$

$\square$

Problem 1.23

The standard normal distribution $N(0,1)$ has density ϕ given by

$$
\phi(x) = \frac{e^{-x^2/2}}{\sqrt{w\pi}}, \qquad x \in \mathbb{R},
$$

with respect to Lebesgue measure λ on $\mathbb{R}$. The corresponding cumulative distribution function is Φ, so

$$
\Phi(x) = \int_{-\infty}^x \phi(z) dz
$$

for $x \in \mathbb{R}$. Suppose that $X \sim N(0,1)$ and that the random variable Y equals X when $|X| < 1$ and is 0 otherwise. Let P_Y denote the distribution of Y and let μ be counting measure on $\{0\}$. Find the density of P_Y with respect to $\lambda + \mu$.

Solution.
Let $g(x) = x 1_{|x|<1}$, then $Y = g(X)$.
For any integrable function f, the expectation of $f(Y)$ against the density of X is

$$
Ef(Y) = Ef\big(g(X)\big) = \int f\big(g(x)\big) \phi(x) dx = \int f\big(g(x)\big) \phi(x) 1_{(-1,1)}(x) dx + cf(0), \tag{1.2}
$$

where $c = \int \phi(x) 1_{|x|>1}(x) dx = 2\Phi(-1)$.
Similarly, the expectation of $f(Y)$ against the density p of Y is

$$
Ef(Y) = \int f p \, d(\lambda + \mu) = \int f p \, d\lambda + \int f p \, d\mu = \int f(x) p(x) dx + f(0) p(0). \tag{1.3}
$$

Then, (1.2) and (1.3) should be same. Therefore,

$$
\begin{aligned}
f &= 1_{\{0\}} \Longrightarrow p(0) = 2\Phi(-1), \\
f(0) &= 0 \Longrightarrow \int f(x) \phi(x) 1_{(-1,1)}(x) dx = \int f(x) p(x) dx \\
&\qquad\quad \Longrightarrow p(x) = \phi(x) \text{ for } 0 < |x| < 1.
\end{aligned}
$$

$$
\therefore p(x) = 2\Phi(-1) 1_{\{0\}}(x) + \phi(x) 1_{(0,1)}(|x|) \text{ is the density of } P_Y.
$$

$\square$

Problem 1.24

Let μ be a σ-finite measure on a measurable space $(X, \mathcal{B})$. Show that μ is absolutely continuous with respect to some probability measure P.

<u>Hint</u>: You can use the fact that if $\mu_1, \mu_2, \dots$ are probability measures and $c_1, c_2, \dots$ are non-negative constants, then $\sum c_i \mu_i$ is a measure. (The proof for Problem 1.10 extents easily to this case.) The measure μ_i, you will want to consider, are truncations of μ to sets A_i covering X with $\mu(A_i) < \infty$, given by $\mu_i(B) = \mu(B \cap A_i)$. With the constants c_i chosen properly, $\sum c_i \mu_i$ will be a probability measure.

Solution.

(i) μ is finite $\Longrightarrow$ Trivial.

$\because$ Since $\mu(X) < \infty$, $P = \mu/\mu(X)$ be a probablity measure. Therefore, $P(N) = 0$ implies $\mu(N) = 0$.

(ii) μ is infinite but σ-finite.

$\Longrightarrow \exists A_1, A_2, \dots \in \mathcal{B}$ s.t. $\bigcup A_i = X$ and $0 < \mu(A_i) < \infty$ $\forall i$.

From <u>Hint</u>, $\mu_i(B) = \mu(B \cap A_i)$ and also $\mu_i(X) = \mu(A_i) \Longrightarrow \mu_i$ is a finite measure.

Let $b_i = 1/2^i$ (or any other seq of positive constants s.t. $\sum b_i = 1$) and define $c_i = b_i/\mu(A_i)$.

Then $P = \sum_i c_i \mu_i$ is a probability measure, since $P(X) = \sum_i c_i \mu_i(X) = \sum_i \left(\frac{b_i}{\mu(A_i)} \right) \mu(A_i) = 1$.

Suppose $P(N) = \sum_i c_i \mu_i(N) = 0$. Then $\mu_i(N) = \mu(N \cap A_i) = 0$ $\forall i$.

By Boole's inequality,

$$\mu(N) = \mu\left(\bigcup_i (N \cap A_i) \right) \leq \sum_i \mu(N \cap A_i) = 0.$$

For any null set for P is a null set for μ.

$\therefore \mu \ll P$.

$\square$

Problem 1.25

The monotone convergence theorem states that if $0 \leq f_1 \leq f_2 \leq \cdots$ are measurable functions and $f = \lim_{n \to \infty} f_n$, then $\int f d\mu = \lim_{n \to \infty} \int f_n d\mu$. Use this result to prove the following assertions.

a) Show that if $X \sim P_X$ is a random variable on $(\mathcal{E}, \mathcal{B}, P)$ and f is a non-negative measurable function, then

$$\int f(X(e)) dP(e) = \int f(x) dP_X(x).$$

<u>Hint</u>: Try it first with f an indicator function. For the general case, let f_n be a sequence of simple functions increasing to f.

Solution.
We will show the equation by 3 steps.

(i) Suppose $f = 1_A$ (indicator function).

Then, $f(X(e)) = 1$ for $X(e) \in A$.

$\Rightarrow f \circ X = 1_B$ where $B = \{e : X(e) \in A\}$.

By definition, $P_X(A) = P(B)$.

$$\int f(X(e))dP(e) = \int 1_B(e)dP(e) = P(B),$$
$$\int f(x)dP_X(x) = \int 1_A(x)dP_X(x) = P_X(A).$$

$\therefore$ The statement is holds.

(ii) Suppose $f = \sum_{i=1}^{n} c_i 1_{A_i}$ (simple function).

$$\therefore \int f(X(e))dP(e) = \int \sum_i c_i 1_{A_i}(X(e))dP(e)$$
$$= \sum_i c_i \int 1_{A_i}(X(e))dP(e)$$
$$= \sum_i c_i \int 1_{A_i}(x)dP_X(x) \quad \text{(by (i))}$$
$$= \int \sum_i c_i 1_{A_i}(x)dP_X(x)$$
$$= \int f(x)dP_X(x).$$

(iii) Suppose f is a non-negative measurable function (general case), and f_n is non-negative simple functions increasing to f.
$\Rightarrow f_n \circ X \nearrow f \circ X$.

$$\therefore \int f(X(e))dP(e) = \lim_{n\to\infty} \int f_n(X(e))dP(e) \quad \text{(by M.C.T.)}$$
$$= \lim_{n\to\infty} \int f_n(x)dP_X(x) \quad \text{(by (ii))}$$
$$= \int f(x)dP_X(x) \quad \text{(by M.C.T.)}$$

$\square$

b) Suppose that P_X has density p with respect to μ, and let f be a non-negative measurable function. Show that
$$\int f dP_X = \int f p d\mu.$$

Solution.
It can be also showed by 3 steps.

(i) Let $f = 1_A$ (indicator function).
$$\int f dP_X = \int 1_A dP_X = P_X(A) = \int_A p d\mu = \int 1_A p d\mu = \int f p d\mu.$$

(ii) Suppose $f = \sum_{i=1}^{n} c_i 1_{A_i}$ (simple function).
$$\int f dP_X = \int \sum_i c_i 1_{A_i} dP_X = \sum_i c_i \int 1_{A_i} dP_X \overset{(i)}{=} \sum_i c_i \int 1_{A_i} p d\mu = \int \sum_i c_i 1_{A_i} p d\mu = \int f p d\mu.$$

(iii) For general case, let a non-negative measurable function f and non-negative simple functions f_n s.t. $f_n \nearrow f$. Then by M.C.T.,

$$\int f dP_X \overset{\text{M.C.T.}}{=} \lim_{n \to \infty} \int f_n dP_X \overset{\text{(ii)}}{=} \lim_{n \to \infty} \int f_n p d\mu \overset{\text{M.C.T.}}{=} \int f p d\mu.$$

$\square$

Problem 1.26

The gamma distribution.

a) The gamma function is defined for $\alpha > 0$ by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$. Show that $\Gamma(X+1) = x!$ for $x = 0, 1, \ldots$.

<u>Solution</u>.
By integration by parts,

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = -t^x e^{-t} \Big|_0^\infty + (\alpha-1) \int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x).$$

Since $\Gamma(1) = \int_0^\infty e^{-x} dx = 1$,

$$\Gamma(x+1) = x\Gamma(x) = x(x-1)\Gamma(x-1) = \cdots = x(x-1)\cdots 2 \cdot 1 \cdot \Gamma(1) = x!.$$

$\square$

b) Show that the function

$$p(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & \text{otherwise}, \end{cases}$$

is a (Lebesgue) probability density when $\alpha > 0$ and $\beta > 0$. This density is called the gamma density with parameters α and β. The corresponding probability distribution is denoted $\Gamma(\alpha, \beta)$.

<u>Solution</u>.
Using change of variable, $y = x/\beta \Rightarrow dx = \beta dy$. Therefore,

$$\int p(x) dx = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \int \frac{1}{\Gamma(\alpha)\beta^\alpha} (\beta y)^{\alpha-1} e^{-y} \beta dy$$

$$= \frac{1}{\Gamma(\alpha)} \int y^{\alpha-1} e^{-y} dy$$

$$= 1.$$

$\square$

c) Show that if $X \sim \Gamma(\alpha, \beta)$, then $EX^r = \beta^r \Gamma(\alpha + r)/\Gamma(\alpha)$. Use this formula to find the mean and variance of X.

Solution.

$$EX^r = \int_0^\infty x^r p(x) dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha+r-1} e^{-x/\beta} dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha+r)\beta^{\alpha+r}} x^{\alpha+r-1} e^{-x/\beta} dx \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \beta^r$$

$$= \beta^r \Gamma(\alpha+r)/\Gamma(\alpha).$$

By using this,

$$EX = \beta\Gamma(\alpha+1)/\Gamma(\alpha) = \alpha\beta,$$

$$EX^2 = \beta^2 \Gamma(\alpha+2)/\Gamma(\alpha) = (\alpha+1)\alpha\beta^2,$$

$$Var(X) = EX^2 - \{EX\}^2 = \alpha\beta^2.$$

$\square$

Problem 1.27

Suppose X has a uniform distribution on $(0,1)$. Find the mean and covariance matrix of the random vector $\binom{X}{X^2}$.

Solution.
Since $X \sim \text{Uniform}(0,1)$, the density of X is $p(x) = 1$ for $x \in (0,1)$. Thus,

$$EX^r = \int x^r dx = \frac{1}{r+1}.$$

Therefore,

$$E\left(\binom{X}{X^2}\right) = \binom{EX}{EX^2},$$

$$Cov\left(\binom{X}{X^2}\right) = \begin{pmatrix} Var(X) & Cov(X, X^2) \\ Cov(X^2, X) & Var(X^2) \end{pmatrix} = \begin{pmatrix} EX^2 - \{EX\}^2 & EX^3 - EXEX^2 \\ EX^3 - EXEX^2 & EX^4 - \{EX^2\}^2 \end{pmatrix}.$$

$\square$

Problem 1.28

If $X \sim N(0,1)$, find the mean and covariance matrix of the random vector $\binom{X}{I\{X>c\}}$.

Solution.
Let $\Phi(x)$ is a cumulative distribution function of X, and ϕ is a density of X. Then,

$$E\left(\binom{X}{I\{X > c\}}\right) = \binom{EX}{EI\{X > c\}} = \binom{0}{1 - \Phi(c)},$$

$$Cov\left(\begin{pmatrix} X \\ I\{X > c\} \end{pmatrix}\right) = \begin{pmatrix} Var(X) & EX1_{X>c} - EXE1_{X>c} \\ EX1_{X>c} - EXE1_{X>c} & E1^2_{X>c} - \{E1_{X>c}\}^2 \end{pmatrix} = \begin{pmatrix} 1 & \phi(c) \\ \phi(c) & \Phi(c)(1 - \Phi(c)) \end{pmatrix}.$$

$$\because EX1_{X>c} = \int_c^\infty \frac{1}{\sqrt{2\pi}} xe^{-x^2/2}dx = \phi(c) \text{ and } 1^2_{X>c} = 1_{X>c}.$$

$\square$

Problem 1.32

Suppose $E|X| < \infty$ and let

$$h(t) = \frac{1 - E\cos(tX)}{t^2}.$$

Use Fubini's theorem to find $\int_0^\infty h(t)dt$. <u>Hint</u>:

$$\int_0^\infty \left(1 - \cos(u)\right)u^{-2}du = \frac{\pi}{2}.$$

<u>*Solution*</u>.
By Funini's theorem,

$$\begin{aligned}
\int_0^\infty h(t)dt &= \int_0^\infty \left(\frac{1 - E\cos(tX)}{t^2}\right)dt \\
&= \int_0^\infty E\left(\frac{1 - \cos(tX)}{t^2}\right)dt \\
&= \int_0^\infty \int \frac{1 - \cos(tx)}{t^2}dP_X(x)dt \\
&= \int \int_0^\infty \frac{1 - \cos(tx)}{t^2}dtdP_X(x) \quad \text{(by Funini's theorem)} \\
&\quad u = tx \Rightarrow du = xdt, t = x/u \\
&= \int \int_0^\infty |x|\frac{1 - \cos(u)}{u^2}dudP_X(x) \\
&= \int \frac{\pi}{2}|x|dP_X(x) \\
&= \frac{\pi}{2}E|X|.
\end{aligned}$$

$\square$

Problem 1.33

Suppose X is absolutely continuous with density $p_X(x) = xe^{-x}$, $x > 0$ and $p_X(x) = 0$, $x \leq 0$. Define $c_n = E(1 + X)^{-n}$. Use Funini's theorem to evaluate $\sum_{n=1}^\infty c_n$.

<u>*Solution*</u>.

By Funini's theorem,

$$\sum_{n=1}^{\infty} c_n = \sum_n E(1+X)^{-n}$$

$$= \sum_n \int_0^{\infty} (1+x)^{-n} x e^{-x} dx$$

$$= \int_0^{\infty} \sum_n (1+x)^{-n} x e^{-x} dx$$

$$\text{Since } \left| \frac{1}{1+x} \right| < 1, \ \sum_{n=1}^{\infty} \left(\frac{1}{1+x} \right)^n = \frac{1/(1+x)}{1 - 1/(1+x)} = \frac{1}{x}.$$

$$= \int_0^{\infty} \frac{1}{x} x e^{-x} dx$$

$$= \int_0^{\infty} e^{-x} dx$$

$$= 1.$$

$\square$

Problem 1.36

Suppose X and Y are independent random variables, and let F_X and F_Y denote their cumulative distribution functions.

a) Use smoothing to show that the cumulative distribution function of $S = X + Y$ is

$$F_S(s) = P(X + Y \leq s) = EF_X(s - Y). \tag{1.4}$$

__Solution__.
Since X and Y are independent, $P(X + Y \leq s | Y = y) = P(X \leq s - y) = F_X(s - y)$.
Thus, $P(X + Y \leq s | Y) = F_X(s - Y)$.

$$\therefore F_S(s) = P(X + Y \leq s) = EP(X + Y \leq s | Y) = EF_X(s - Y).$$

$\square$

b) If X and Y are independent and Y is almost surely positive, use smoothing to show that the cumulative distribution function of $W = XY$ is $F_W(w) = EF_X(w/Y)$ for $w > 0$.

__Solution__.
Since X and Y are independent, $P(XY \leq w | Y = y) = P(X \leq w/y | Y = y) = P(X \leq w/y) = F_X(w/y)$, for $y > 0$.
Thus, $P(XY \leq w | Y) = F_X(w/Y)$ a.s.

$$\therefore F_W(w) = P(XY \leq w) = EP(XY \leq w | Y) = EF_X(w/Y).$$

$\square$

Problem 1.37

Differentiating (1.4) with respect to s one can show that if X is absolutely continuous with density p_X, then $S = X + Y$ is absolutely continuous with density

$$p_S(s) = E p_X(s - Y)$$

for $s \in \mathbb{R}$. Use this formula to show that if X and Y are independent with $X \sim \Gamma(\alpha, 1)$ and $Y \sim \Gamma(\beta, 1)$, then $X + Y \sim \Gamma(\alpha + \beta, 1)$.

Solution.
Since $X \sim \Gamma(\alpha, 1)$, $P_X(x - Y) = 0$ for $Y \geq x$ ($\because p_X(x) = 0, \ \forall x < 0$). Then,

$$P_S(x) = E P_X(x - Y)$$

$$= E\left[\frac{1}{\Gamma(\alpha)} (x - Y)^{\alpha - 1} e^{-(x-Y)} 1_{(0,x)}(Y) \right]$$

$$= \int_0^x \frac{1}{\Gamma(\alpha)} (x - y)^{\alpha - 1} e^{-(x-y)} \frac{1}{\Gamma(\beta)} y^{\beta - 1} e^{-y} dy$$

$$= \int_0^x \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (x - y)^{\alpha - 1} y^{\beta - 1} e^{-x} dy$$

Change the variable by $u = y/x \Rightarrow du = 1/x\,dy$.

$$= \int_0^1 \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (x - xu)^{\alpha - 1} (xu)^{\beta - 1} e^{-x} x\,du$$

$$= \int_0^1 \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha + \beta - 1} (1 - u)^{\alpha - 1} u^{\beta - 1} e^{-x} du$$

$$= \frac{1}{\Gamma(\alpha + \beta)} x^{\alpha + \beta - 1} e^{-x}, \quad \text{for } x > 0.$$

$\therefore S = X + Y \sim \Gamma(\alpha + \beta, 1)$. $\qquad \square$

Problem 1.38

Let Q_λ denote the exponential distribution with failure rate λ, given in Problem 1.30. Let X be a discrete random variable taking values in $\{1, \ldots, n\}$ with mass function

$$P(X = k) = \frac{2k}{n(n+1)}, \quad k = 1, \ldots, n,$$

and assume that the conditional distribution of Y given $X = x$ is exponential with failure rate x,

$$Y | X = x \sim Q_x.$$

a) Find $E[Y | X]$.

  Solution.
  From Problem 1.30, $q_\lambda(x) = \lambda 3^{-\lambda x}, \ \lambda > 0$. Then, the mean of Q_λ is

  $$\int x\,dQ_\lambda(x) = \int x q_\lambda(x)\,dx = \int_0^\infty x\lambda e^{-\lambda x}\,dx = \frac{1}{\lambda}.$$

  $$\therefore E(Y | X = x) = \frac{1}{x} \Rightarrow EY | X = \frac{1}{X}.$$

  $\qquad \square$

b) Use smoothing to compute EY.

> *__Solution.__*
>
> $$EY = E[E(Y|X)] = E\frac{1}{X} = \sum_{x=1}^{n} \frac{1}{x} \frac{2x}{n(n+1)} = \sum_{x=1}^{n} \frac{2}{n(n+1)} = \frac{2}{n+1}.$$

<div align="right">□</div>

Problem 1.39

Let X be a discrete random variable uniformly distributed on $\{1, \ldots, n\}$, so $P(X = k) = 1/n, \ k = 1, \ldots, n$, and assume that the conditional distribution of Y given $X = x$ is exponential with failure rate x.

a) For $y > 0$ find $P[Y > y|X]$.

> *__Solution.__*
>
> $$P(Y > y|X = x) = \int_{y}^{\infty} xe^{-xt}dt = -e^{-xt}\Big|_{y}^{\infty} = e^{-xy}.$$
>
> $$\therefore P(X > y|X) = e^{-Xy}.$$

<div align="right">□</div>

b) Use smoothing to compute $P(Y > y)$.

> *__Solution.__*
>
> Using the sum of the geometric sequence (등비급수의 합),
>
> $$P(Y > y) = EP(Y > y|X) = Ee^{-Xy} = \sum_{x=1}^{n} e^{-xy}\frac{1}{n} = \frac{1}{n}\frac{e^{-y}(1 - e^{-ny})}{1 - e^{-y}} = \frac{1 - e^{-ny}}{n(e^{y} - 1)}.$$

<div align="right">□</div>

c) Determine the density of Y.

> *__Solution.__*
>
> Let the density of Y is p_Y. Then,
>
> $$\begin{aligned}
> p_Y(y) &= \frac{d}{dy}\left(1 - P(Y > y)\right) \\
> &= \frac{d}{dy}\left(\frac{1 - e^{-ny}}{n(e^{y} - 1)}\right) \\
> &= \frac{-ne^{-ny}(ne^{y} - n) + ne^{y}(1 - e^{-ny})}{n^2(e^{y} - 1)^2} \\
> &= \frac{-ne^{y}e^{-ny} + ne^{-ny} + e^{y} - e^{y}e^{-ny}}{n(e^{y} - 1)^2} \\
> &= \frac{e^{y}(1 - e^{-ny})}{n(e^{y} - 1)^2} - \frac{e^{-ny}}{e^{y} - 1}.
> \end{aligned}$$

<div align="right">□</div>

Chapter 2

Exponential Families

Problem 2.1

Consider independent Bernoulli trials with success probability p and let X be the number of failures before the first success. Then $P(X = x) = p(1 - p)^x$, for $x = 0, 1, \ldots$, and X has the geometric distribution with parameter p, introduced in Problem 1.17.

    a) Show that the geometric distributions form an exponential family.

        Solution.
$$P(X = x) = p(1 - p)^x = \exp\left\{ x \log(1 - p) - (-\log p) \right\}.$$

        $\therefore$ the geometric distribution is an exponential family. $\square$

    b) Write the densities for the family in canonical form, identifying the canonical parameter η, and the function $A(\eta)$.

        Solution.
        Let $\eta(p) = \log(1 - p)$. Then, $P(X = x) = \exp\left\{ \eta x - \left(-\log(1 - e^\eta) \right) \right\}$.
        $\therefore A(\eta) = -\log(1 - e^\eta)$ with $T(x) = x$. $\square$

    c) Find the mean of the geometric distribution using a differential identity.

        Solution.
$$E_\eta(T) = A'(\eta) = \frac{e^\eta}{1 - e^\eta} = \frac{1 - p}{p}.$$

        $\square$

    d) Suppose $X_1, \ldots, X_n$ are i.i.d. from a geometric distribution. Show that the joint distributions form an exponential family, and find the mean and variance of T.

        Solution.

$$P(X_1 = x_1, \ldots, X_n = x_n) = \prod_{i=1}^n P(X_i = x_i) = p^n (1 - p)^{\sum_{i=1}^n x_i} = \exp\left\{ \sum_i x_i \log(1 - p) - (-n \log p) \right\}.$$

Therefore, the joint distribution is also exponential family.

Now, let $\eta = \log(1-p)$. Then, the canonical exponential family is obtained with $A(\eta) = -n\log(1-e^\eta)$ and $T = \sum_i x_i$.

$$ET = \kappa_1 = A'(\eta) = \frac{ne^\eta}{1-e^\eta} = \frac{n(1-p)}{p},$$

$$Var(T) = \kappa_2 = A''(\eta) = \frac{ne^\eta(1-e^\eta) + ne^{2\eta}}{(1-e^\eta)^2} = \frac{np(1-p) + n(1-p)^2}{p^2} = \frac{n(1-p)}{p^2}.$$

$\square$

Problem 2.2

Determine the canonical parameter space Ξ, and find densities for the one-parameter exponential family with μ Lebesgue measure on $\mathbb{R}^2$, $h(x,y) = \exp\left[-(x^2+y^2)/2\right]/(2\pi)$, and $T(x,y) = xy$.

Solution.

By definition of the canonical exponential family, the pdf be represented as

$$p(x,y) = \exp\left(\eta T(x,y) - A(\eta)\right)h(x,y) = \exp\left\{\eta xy - A(\eta) - (x^2+y^2)/2\right\}/(2\pi).$$

Thus,

$$\int p(x,y)d\mu(x,y) = 1 \iff \int\int \exp\left\{\eta xy - A(\eta) - (x^2+y^2)/2\right\}/(2\pi)dxdy = 1$$

$$\iff e^{A(\eta)} = \int\int \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(x-\eta y)^2\right\}dx \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y^2 - \eta^2 y^2)}dy$$

$$\iff e^{A(\eta)} = \frac{1}{\sqrt{2\pi}}\int e^{-\frac{1}{2}y^2(1-\eta^2)}dy \underset{\substack{\uparrow \\ \text{the integral is finite} \Leftrightarrow |\eta| < 1}}{=} (1-\eta^2)^{-1/2}.$$

$$\therefore A(\eta) = -\frac{1}{2}\log(1-\eta^2) \text{ for } \eta \in \Xi = (-1,1).$$

$\therefore$ The canonical exponential density is

$$\exp\left\{\eta xy + \frac{1}{2}\log(1-\eta^2) - (x^2+y^2)/2\right\}/(2\pi).$$

$\square$

Problem 2.4

Find the natural parameter space Ξ and densities p_η for a canonical one-parameter exponential family with μ Lebesgue measure on $\mathbb{R}$, $T_1(x) = \log x$, and $h(x) = (1-x)^2$, $x \in (0,1)$, and $h(x) = 0$, $x \notin (0,1)$.

<u>*Solution*</u>.

$$e^{A(\eta)} = \int_0^1 e^{\eta \log x}(1-x)^2 dx$$

$$\text{Let } t = \log x \Rightarrow dt = 1/x dx = e^{-t}dx, \quad -\infty < t < 0$$

$$= \int_{-\infty}^0 e^{\eta t}(1-e^t)^2 e^t dt$$

$$= \int_{-\infty}^0 \left(e^{t(\eta+1)} - 2e^{t(\eta+2)} + e^{t(\eta+3)} \right) dt$$

$$= \left(\frac{1}{\eta+1}e^{t(\eta+1)} - \frac{2}{\eta+2}e^{t(\eta+2)} + \frac{1}{\eta+3}e^{t(\eta+3)} \right) \Big|_{-\infty}^0$$

$$= \frac{1}{\eta+1} - \frac{2}{\eta+2} + \frac{1}{\eta+3}$$

$$= \frac{2}{(\eta+1)(\eta+2)(\eta+3)}, \quad \eta \in \Xi = (-1, \infty).$$

($\because$ For a finite integral, $\eta + 1 > 0$, $\eta + 2 > 0$, and $\eta + 3 > 0 \Rightarrow \eta > -1$.)

$$\therefore p_\eta(x) = \exp\left(\eta T(x) - A(\eta) \right) h(x) = \frac{1}{2}(\eta+1)(\eta+2)(\eta+3)(1-x)^2 x^\eta, \quad x \in (0,1).$$

$\square$

Problem 2.5

Find the natural parameter space Ξ and densities p_η for a canonical one-parameter exponential family with μ Lebesgue measure on $\mathbb{R}$, $T_1(x) = -x$, and $h(x) = e^{-2\sqrt{x}}/\sqrt{x}$, $x > 0$, and $h(x) = 0$, $x \le 0$. (<u>Hint</u>: After a change of variables, relevant integrals will look like integrals against a normal density. You should be able to express the answer using Φ, the standard normal cumulative distribution function.) Also, determine the mean and variance for a variable X with this density.

<u>*Solution*</u>.

$$e^{A(\eta)} = \int_0^\infty e^{-\eta x - 2\sqrt{x}} \frac{1}{\sqrt{x}} dx$$

To the integral is finite, $\eta \in \Xi = [0, \infty)$,

Let $t = \sqrt{x} \Rightarrow t^2 = x$, $2t dt = dx$

$$= \int_0^\infty \frac{1}{t} e^{-\eta t^2 - 2t} 2t dt$$

$$= 2 \int_0^\infty e^{-\eta\left(t^2 + \frac{2}{\eta} t\right)} dt$$

$$= 2 \int_0^\infty e^{-\frac{2\eta}{2}\left(t + \frac{1}{\eta}\right)^2} dt \cdot e^{-\eta\left(-\frac{1}{\eta^2}\right)}$$

Since inner term of integration $\sim N(-1/\eta, 1/2\eta)$,

$$= 2\sqrt{2\pi}/\sqrt{2\eta} \cdot e^{\frac{1}{\eta}} \Phi\left(\frac{-1/\eta}{\sqrt{1/(2\eta)}}\right)$$

$$= \frac{2\sqrt{\pi}}{\sqrt{\eta}} e^{1/\eta} \Phi(-\sqrt{2/\eta})$$

$$\Rightarrow A(\eta) = \log 2\sqrt{\pi} - \frac{1}{2}\log \eta + \frac{1}{\eta} + \log \Phi\left(-\sqrt{\frac{2}{\eta}}\right).$$

$$\therefore p_\eta(x) = \exp\left\{-\eta x - 2\sqrt{x} - A(\eta)\right\} \frac{1}{\sqrt{x}}, \quad x > 0.$$

The mean of X is

$$E_\eta(X) = -E_\eta T = -A'(\eta) = \frac{1}{2\eta} + \frac{1}{\eta^2} - \frac{\frac{\sqrt{2}}{2}\eta^{-3/2}\phi\left(-\sqrt{\frac{2}{\eta}}\right)}{\Phi\left(-\sqrt{\frac{2}{\eta}}\right)}.$$

$\square$

Problem 2.6

Find the natural parameter space Ξ and densities p_η for a canonical two-parameter exponential family with μ counting measure on $\{0, 1, 2\}$, $T_1(x) = x$, $T_2(x) = x^2$, and $h(x) = 1$ for $x \in \{0, 1, 2\}$.

Solution.

$\square$

Problem 2.7

Suppose $X_1, \ldots, X_n$ are independent geometric variables with p_i the success probability for X_i. Suppose these success probabilities are related to a sequence of "independent" variables $t_1, \ldots, t_n$, viewed as known constants, through

$$p_i = 1 - \exp(\alpha + \beta t_i), \quad i = 1, \ldots, n.$$

Show that the joint densities for $X_1, \ldots, X_n$ form a two-parameter exponential family, and identify the statistics T_1 and T_2.

Solution.

$\square$

Problem 2.8

Assume that $X_1, \ldots, X_n$ are independent random variables with $X_i \sim N(\alpha + \beta t_i, 1)$, where $t_1, \ldots, t_n$ are observed constants and α and β are unknown parameters. Show that the joint densities for $X_1, \ldots, X_n$ form a two-parameter exponential family, and identify the statistics T_1 and T_2.

Solution.

$\square$

Problem 2.9

Suppose that $X_1, \ldots, X_n$ are independent Bernoulli variables (a random variable is Bernoulli if it only takes on values 0 and 1) with

$$P(X_i = 1) = \frac{\exp(\alpha + \beta t_i)}{1 + \exp(\alpha + \beta t_i)}.$$

Show that the joint distribution for $X_1, \ldots, X_n$ form a two-parameter exponential family, and identify the statistics T_1 and T_2.

Solution.

$\square$

Problem 2.15

For an exponential family in canonical form, $ET_j = \partial A(\eta)/\partial \eta_j$. This can be written in vector form as $ET = \nabla A(\eta)$. Derive an analogous differential formula for $E_\theta T$ for an s-parameter exponential family that is not in canonical form. Assume that Ω has dimension s.
<u>Hint</u>: Differentiation under the integral sign should give a system of linear equations. Write these equations in matrix form.

Solution.

$\square$

Problem 2.17

Let μ denote counting measure on $\{1, 2, \ldots\}$. One common definition for $\sum_{k=1}^{\infty} f(k)$ is $\lim_{n \to \infty} \sum_{k=1}^{n} f(k)$, and another definition is $\int f d\mu$.

  a) Use the dominated convergence theorem to show that the two definitions give the same answer when $\int |f| d\mu < \infty$.
   <u>Hint</u>: Find functions $f_n, \ n = 1, 2, \ldots$, so that $\sum_{k=1}^{n} f(k) = \int f_n d\mu$.

Solution.

☐

b) Use the monotone convergence theorem, give in Problem 1.25, to show the definitions agree if $f(k) \geq 0$ for all $1, 2, \ldots$.

Solution.

☐

c) Suppose $\lim_{n \to \infty} f(n) = 0$ and that $\int f^+ d\mu = \int f^- d\mu = \infty$ (so that $\int f d\mu$ is undefined.) Let K be an arbitrary constant. Show that the list $f(1), f(2), \ldots$ can be rearranged to form a new list $g(1), g(2), \ldots$ so that

$$\lim_{n \to \infty} \sum_{k=1}^{n} g(k) = K.$$

Solution.

☐

Problem 2.19

Let $p_n, n = 1, 2, \ldots$, and p be probability densities with respect to a measure μ, and let $P_n, n = 1, 2, \ldots$, and P be the corresponding probability measures.

a) Show that if $p_n(x) \to p(x)$ as $n \to \infty$, then $\int |p_n - p| d\mu \to 0$.
<u>Hint</u>: First use the fact that $\int (p_n - p) d\mu = 0$ to argue that $\int |p_n - p| d\mu = 2 \int (p - p_n)^+ d\mu$. Then use dominated convergence.

Solution.

☐

b) Show that $|P_n(A) - P(A)| \leq \int |p_n - p| d\mu$.
<u>Hint</u>: Use indicators and the bound $|\int f d\mu| \leq \int |f| d\mu$.

Solution.

☐

Remark: Distributions P_n, $n \geq 1$, are said to *converge strongly* to P if $\sup_A |P_n(A) - P(A)| \to 0$. The two parts above show that pointwise convergence of p_n to p implies strong convergence. This was discovered by Scheffé.

Problem 2.22

Suppose X is absolutely continuous with density

$$p_\theta(x) = \begin{cases} \frac{e^{-(x-\theta)^2/2}}{\sqrt{2\pi}\Phi(\theta)}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the moment generating function of X. Compute the mean and variance of X.

Solution.

$\square$

Problem 2.23

Suppose $Z \sim N(0,1)$. Find the first four cumulants of Z^2.
<u>Hint</u>: Consider the exponential family $N(0, \sigma^2)$.

Solution.

$\square$

Problem 2.24

Find the first four cumulants of $T = XY$ when X and Y are independent standard normal variates.

Solution.

$\square$

Problem 2.25

Find the third and fourth cumulants of the geometric distribution.

Solution.

$\square$

Problem 2.26

Find the third cumulant and third moment of the binomial distribution with n trials and success probability p.

Solution.

$\square$

Problem 2.27

Let T be a random vector in $\mathbb{R}^2$.

    a) Express $\kappa_{2,1}$ as a function of the moments of T.

       Solution.

                    □

    b) Assume $ET_1 = ET_2 = 0$ and give an expression for $\kappa_{2,2}$ in terms of moments of T.

       Solution.

                    □

Problem 2.28

Suppose $X \sim \Gamma(\alpha, 1/\lambda)$, with density

$$\frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \qquad x > 0.$$

Find the cumulants of $T = (X, \log X)$ of order 3 or less. The answer will involve $\psi(\alpha) = d\log\Gamma(\alpha)/d\alpha = \Gamma'(\alpha)/\Gamma(\alpha)$.

Solution.

                    □

Chapter 3

Risk, Sufficiency, Completeness, and Ancillarity