

Theoretical Statistics: Topics for a Core Course

Problem Solutions

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Chapter 1

Probability and Measure

Problem 1.1

Prove (1.1). If measurable sets B_n , $n \geq 1$, are increasing, with $B = \bigcup_{n=1}^{\infty} B_n$, called the limit of the sequence, then

$$\mu(B) = \lim_{n \rightarrow \infty} \mu(B_n).$$

Solution.

First, we showed that A_n 's are disjoint. If $j < k$, then $B_j \subseteq B_{k-1}$.

Since $A_j \subset B_j \subseteq B_{k-1}$ and $A_k \subset B_{k-1}^c$, A_j and A_k are disjoint.

Also $B_n = \bigcup_{j=1}^n A_j$ and $\bigcup_{n=1}^{\infty} A_n = B$,

$$\mu(B) = \sum_{i=1}^{\infty} \mu(A_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(A_i) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{i=1}^n A_i\right) = \lim_{n \rightarrow \infty} \mu(B_n).$$

□

Problem 1.8

Prove *Boole's inequality*: For any events B_1, B_2, \dots ,

$$P\left(\bigcup_{i \geq 1} B_i\right) \leq \sum_{i \geq 1} P(B_i).$$

Solution.

Let $B = \bigcup_{i=1}^{\infty} B_i$, then $1_B \leq \sum 1_{B_i}$. Then by Fubini's theorem,

$$P(B) = \int 1_B dP \leq \int \sum 1_{B_i} dP = \sum \int 1_{B_i} dP = \sum P(B_i).$$

□

Problem 1.10

Let μ and ν be measures on $(\mathcal{E}, \mathcal{B})$.

a) Show that the sum η defined by $\eta(B) = \mu(B) + \nu(B)$ is also a measure.

Solution.

We should check following 2 conditions.

- (i) For arbitrary set $A \in \mathcal{B}$, $\mu(A) \geq 0$ and $\nu(A) \geq 0$. $\Rightarrow \eta(A) = \mu(A) + \nu(A) \geq 0$.
 $\therefore \eta : \mathcal{B} \rightarrow [0, \infty]$.
- (ii) For disjoint set $B_1, B_2, \dots \subset \mathcal{B}$, then

$$\begin{aligned} \eta\left(\bigcup_{i=1}^{\infty} B_i\right) &= \mu\left(\bigcup_{i=1}^{\infty} B_i\right) + \nu\left(\bigcup_{i=1}^{\infty} B_i\right) \\ &= \sum \mu(B_i) + \sum \nu(B_i) \\ &= \sum \{\mu(B_i) + \nu(B_i)\} \\ &= \sum \eta(B_i) \end{aligned}$$

$\therefore \eta$ is a measure. □

b) If f is a non-negative measurable function, show that

$$\int f d\eta = \int f d\mu + \int f d\nu.$$

Solution.

We show it by 2 stage.

- (i) Let $f = \sum_{i=1}^n a_i 1_{A_i}$, the non-negative simple function. Then,

$$\begin{aligned} \int f d\eta &= \int \sum_{i=1}^n a_i 1_{A_i} d\eta = \sum a_i \eta(A_i) \\ &= \sum a_i \{\mu(A_i) + \nu(A_i)\} = \int f d\mu + \int f d\nu. \end{aligned}$$

- (ii) For general case, let f_n is the sequence of non-negative simple functions increasing to f . (i.e. $f_1 \leq f_2 \leq \dots \leq f$)

$$\begin{aligned} \int f d\eta &= \lim_{n \rightarrow \infty} \int f_n d\eta = \lim_{n \rightarrow \infty} \left(\int f_n d\mu + \int f_n d\nu \right) \\ &= \lim_{n \rightarrow \infty} \int f_n d\mu + \lim_{n \rightarrow \infty} \int f_n d\nu = \int f d\mu + \int f d\nu. \end{aligned}$$

□

Problem 1.11

Suppose f is the simple function $1_{(1/2, \pi]} + 21_{(1, 2]}$, and let μ be a measure on \mathbb{R} with $\mu\{(0, a^2]\} = a$, $a > 0$. Evaluate $\int f d\mu$.

Solution.

By the integral of simple function and the finite additivity, it can be simply computed as

$$\begin{aligned}
 \int f d\mu &= \mu\{(1/2, \pi]\} + 2\mu\{(1, 2]\} \\
 &= [\mu\{(0, \pi]\} - \mu\{(1/2, \pi]\}] + 2[\mu\{(0, 2]\} - \mu\{(1, 2]\}] \quad (\text{finite additivity}) \\
 &= (\sqrt{\pi} - 1/\sqrt{2}) + 2(\sqrt{2} - 1)
 \end{aligned}$$

□

Problem 1.12

Suppose that $\mu\{(0, a)\} = a^2$ for $a > 0$ and that f is defined by

$$f(x) = \begin{cases} 0, & x \leq 0, \\ 1, & 0 < x < 2, \\ \pi, & 2 \leq x < 5, \\ 0, & x \geq 5. \end{cases}$$

Compute $\int f d\mu$.

Solution.

Since $f = 1_{(0,2)} + \pi 1_{[2,5)}$ is simple, the integral can be computed as

$$\begin{aligned}
 \int f d\mu &= \mu\{(0, 2)\} + \pi\mu\{[2, 5)\} \\
 &= \mu\{(0, 2)\} + \pi[\mu\{(0, 5)\} - \mu\{(0, 2)\}] \\
 &= 4 - \pi(25 - 4)
 \end{aligned}$$

□

Problem 1.13

Define the function f by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find simple functions $f_1 \leq f_2 \leq \dots$ increasing to f (i.e. $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for all $x \in \mathbb{R}$). Let μ be Lebesgue measure on \mathbb{R} . Using our formal definition of an integral and the fact that $\mu((a, b]) = b - a$ whenever $b > a$ (this might be used to formally define Lebesgue measure), show that $\int f d\mu = 1/2$.

Solution.

Let $f_n = \lfloor 2^n x \rfloor / 2^n$ for $0 < x \leq 1$ and 0 otherwise. ($\lfloor y \rfloor$ is a floor function.) Then,

$$f_1(x) = \lfloor 2x \rfloor / 2 = \begin{cases} 0, & x < 1/2, \\ 1/2, & 1/2 \leq x < 1, \\ 1, & x = 1 \end{cases}$$

$$\begin{aligned}
f_2(x) &= \lfloor 2^2 x \rfloor / 2^2 = \begin{cases} 0, & x < 1/2^2, \\ 1/2^2, & 1/2^2 \leq x < 2/2^2, \\ 2/2^2, & 2/2^2 \leq x < 3/2^2, \\ 3/2^2, & 3/2^2 \leq x < 1, \\ 1, & x = 1 \end{cases} \\
&\vdots \\
f_n(x) &= \lfloor 2^n x \rfloor / 2^n = \begin{cases} 0, & x < 1/2^n, \\ 1/2^n, & 1/2^n \leq x < 2/2^n, \\ \vdots \\ (2^n - 1)/2^n, & (2^n - 1)/2^n \leq x < 1, \\ 1, & x = 1 \end{cases}
\end{aligned}$$

$$\begin{aligned}
\therefore \int f_n d\mu &= \frac{1}{2^n} \left(\frac{1}{2^n} + \frac{2}{2^n} + \cdots + \frac{2^n - 1}{2^n} \right) \\
&= \frac{1 + 2 + \cdots + (2^n - 1)}{4^n} \\
&= \frac{2^n(2^n - 1)}{2 \cdot 4^n} \\
&\longrightarrow \frac{1}{2}.
\end{aligned}$$

My solution

Let the simple function $f_n = \sum_{i=1}^n \frac{i}{n} 1_{(\frac{i-1}{n}, \frac{i}{n}]}$. Then,

$$\int f_n d\mu = \sum_{i=1}^n \frac{i}{n} \frac{1}{n} = \frac{1}{n^2} \frac{n(n+1)}{2} \longrightarrow \frac{1}{2}.$$

□

Problem 1.16

Define $F(a-) = \lim_{x \uparrow a} F(x)$. Then, if F is non-decreasing, $F(a-) = \lim_{n \rightarrow \infty} F(a - 1/n)$. Use (1.1)[Continuity of measure] to show that if a random variable X has cumulative distribution function F_X ,

$$P(X < a) = F_X(a-).$$

Also, show that

$$P(X = a) = F_X(a) - F_X(a-).$$

Solution.

(i) Let $B_n = \{X \leq a - 1/n\}$ and $\bigcup_{n=1}^{\infty} B_n = \{X < a\}$. By continuity of measure,

$$P(B) = P(X < a) = \lim_{n \rightarrow \infty} P(X \leq a - 1/n) = \lim_{n \rightarrow \infty} F_X(a - 1/n) = F_X(a-).$$

(ii) Since $\{X < a\}$ and $\{X = a\}$ are disjoint with union $\{X \leq a\}$,

$$P(X < a) + P(X = a) = P(X \leq a).$$

$$\therefore P(X = a) = F_X(a) - F_X(a-).$$

□

Problem 1.17

Suppose X is a geometric random variable with mass function

$$p(x) = P(X = x) = \theta(1 - \theta)^x, \quad x = 0, 1, \dots,$$

where $\theta \in (0, 1)$ is a constant. Find the probability that X is even.

Solution.

$$\begin{aligned} P(X \text{ is even}) &= P(X = 0) + P(X = 2) + P(X = 4) \cdots \\ &= \theta + \theta(1 - \theta)^2 + \theta(1 - \theta)^4 + \cdots \\ &= \frac{\theta}{1 - (1 - \theta)^2} \\ &= \frac{1}{2 - \theta} \end{aligned}$$

□

Problem 1.18

Let X be a function mapping \mathcal{E} into \mathbb{R} . Recall that if B is a subset of \mathbb{R} , then $X^{-1}(B) = \{e \in \mathcal{E} : X(e) \in B\}$. Use this definition to prove that

$$X^{-1}(A \cap B) = X^{-1}(A) \cap X^{-1}(B),$$

$$X^{-1}(A \cup B) = X^{-1}(A) \cup X^{-1}(B),$$

and

$$X^{-1}\left(\bigcup_{i=0}^{\infty} A_i\right) = \bigcup_{i=0}^{\infty} X^{-1}(A_i).$$

Solution.

(i)

$$\begin{aligned} e \in X^{-1}(A \cap B) &\Leftrightarrow X(e) \in A \cap B \\ &\Leftrightarrow X(e) \in A \text{ and } X(e) \in B \\ &\Leftrightarrow e \in X^{-1}(A) \text{ and } e \in X^{-1}(B) \\ &\Leftrightarrow e \in X^{-1}(A) \cap X^{-1}(B). \end{aligned}$$

(ii) Similarly,

$$\begin{aligned}
 e \in X^{-1}(A \cup B) &\Leftrightarrow X(e) \in A \cup B \\
 &\Leftrightarrow X(e) \in A \text{ or } X(e) \in B \\
 &\Leftrightarrow e \in X^{-1}(A) \text{ or } e \in X^{-1}(B) \\
 &\Leftrightarrow e \in X^{-1}(A) \cup X^{-1}(B).
 \end{aligned}$$

(iii)

$$\begin{aligned}
 e \in X^{-1}\left(\bigcup_{i=0}^{\infty} A_i\right) &\Leftrightarrow X(e) \in \bigcup_{i=0}^{\infty} A_i \\
 &\Leftrightarrow X(e) \in A_i \text{ for some } i \\
 &\Leftrightarrow e \in X^{-1}(A_i) \text{ for some } i \\
 &\Leftrightarrow e \in \bigcup_{i=0}^{\infty} X^{-1}(A_i).
 \end{aligned}$$

□

Problem 1.19

Let P be a probability measure on $(\mathcal{E}, \mathcal{B})$, and let X be a random variable. Show that the distribution P_X of X defined by $P_X(B) = P(X \in B) = P(X^{-1}(B))$ is a measure (on the Borel sets of \mathbb{R}).

Solution.

To show that P_X is a measure, we should check 2 conditions.

(i) $P_X(B) = P(X^{-1}(B)) \geq 0$.

(ii) From Problem 1.18, $X^{-1}(\bigcup_i A_i) = \bigcup_i X^{-1}(A_i)$ is hold.

Also, if B_i and B_j are disjoint, then $X^{-1}(B_i)$ and $X^{-1}(B_j)$ are disjoint. (\because If $e \in \mathcal{E}$ lie in both $X^{-1}(B_1)$ and $X^{-1}(B_2)$, $X(e)$ lies in B_1 and B_2 .)

Therefore, for arbitrary disjoint set B_1, B_2, \dots with union $B = \bigcup_i B_i$, $X^{-1}(B_1), X^{-1}(B_2), \dots$ are also disjoint with union $X^{-1}(B)$.

$$\therefore \sum_i P_X(B_i) = \sum_i P(X^{-1}(B_i)) = P(X^{-1}(B)) = P_X(B).$$

The above 2 conditions are hold, P_X is a measure.

□

Problem 1.21

Let X have a uniform distribution on $(0, 1)$; that is, X is absolutely continuous with density p defined by

$$p(x) = \begin{cases} 1, & x \in (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Let Y_1 and Y_2 denote the first two digits of X when X is written as a binary decimal (so $Y_1 = 0$ if $X \in (0, 1/2)$ for instance). Find $P(Y_1 = i, Y_2 = j)$, $i = 0$ or $1, j = 0$ or 1 .

Solution.

For all cases, the probability are same. In other words,

$$P(Y_1 = 0, Y_2 = 0) = P(Y_1 = 1, Y_2 = 0) = P(Y_1 = 0, Y_2 = 1) = P(Y_1 = 1, Y_2 = 1) = 1/4.$$

□

Problem 1.22

Let $\mathcal{E} = (0, 1)$, let \mathcal{B} be the Borel subsets of \mathcal{E} , and let $P(A)$ be the length of A for $A \in \mathcal{B}$. (P would be called the *uniform probability measure* on $(0, 1)$.) Define the random variable X by

$$X(e) = \min\{e, 1/2\}.$$

Let μ be the sum of Lebesgue measure on \mathbb{R} and counting measure on $\mathcal{X}_0 = \{1/2\}$. Show that the distribution P_X of X is absolutely continuous with respect to μ and find the density of P_X .

Solution.

(i) Claim: For any $B \in \mathbb{R}$, $\mu(B) = 0 \implies P_X(B) = 0$.

$$\begin{aligned} \{X \in B\} &= \{y \in (0, 1) : X(y) \in B\} \\ &= \begin{cases} B \cap (0, 1/2), & 1/2 \notin B, \\ (B \cap (0, 1/2)) \cup [1/2, 1), & 1/2 \in B. \end{cases} \end{aligned}$$

Let λ be a Lebesgue measure and ν be a counting measure on $\{1/2\}$. Then,

$$\begin{aligned} \mu(B) = 0 &\iff \lambda(B) = 0 \text{ and } 1/2 \notin B. \quad (\mu(B) = \nu(B) = 0) \\ \implies P_X(B) &= P(X \in B) = P(B \cap (0, 1/2)) = \lambda(B \cap (0, 1/2)) = 0. \\ \therefore P_X &\text{ is absolutely continuous w.r.t. } \lambda + \nu \stackrel{\text{def}}{=} \mu. \end{aligned}$$

(ii) Find the density of P_X .

From the above equation,

$$P(X \in B) = \lambda(B \cap (0, 1/2)) + \frac{1}{2}1_B(1/2). \quad (1.1)$$

If f is the density, then (1.1) should be equal to $\int_B f d(\lambda + \nu)$.

$$\begin{aligned} \int_B f d(\lambda + \nu) &= \int f 1_B d(\lambda + \nu) \\ &= \int \left(1_{B \cap (0, 1/2)} + \frac{1}{2}1_{\{1/2\} \cap B} \right) d(\lambda + \nu) \\ &= \lambda(B \cap (0, 1/2)) + \nu(B \cap (0, 1/2)) + \frac{1}{2}\lambda(\{1/2\} \cap B) + \frac{1}{2}\nu(\{1/2\} \cap B) \\ &= \lambda(B \cap (0, 1/2)) + \frac{1}{2}1_B(1/2). \end{aligned}$$

$$\therefore f = 1_{(0, 1/2)} + \frac{1}{2}1_{\{1/2\}}.$$

□

Problem 1.23

The standard normal distribution $N(0, 1)$ has density ϕ given by

$$\phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \quad x \in \mathbb{R},$$

with respect to Lebesgue measure λ on \mathbb{R} . The corresponding cumulative distribution function is Φ , so

$$\Phi(x) = \int_{-\infty}^x \phi(z) dz$$

for $x \in \mathbb{R}$. Suppose that $X \sim N(0, 1)$ and that the random variable Y equals X when $|X| < 1$ and is 0 otherwise. Let P_Y denote the distribution of Y and let μ be counting measure on $\{0\}$. Find the density of P_Y with respect to $\lambda + \mu$.

Solution.

Let $g(x) = x1_{|x|<1}$, then $Y = g(X)$.

For any integrable function f , the expectation of $f(Y)$ against the density of X is

$$Ef(Y) = Ef(g(X)) = \int f(g(x))\phi(x)dx = \int f(g(x))\phi(x)1_{(-1,1)}(x)dx + cf(0), \quad (1.2)$$

where $c = \int \phi(x)1_{|x|>1}(x)dx = 2\Phi(-1)$.

Similarly, the expectation of $f(Y)$ against the density p of Y is

$$Ef(Y) = \int f p d(\lambda + \mu) = \int f p d\lambda + \int f p d\mu = \int f(x)p(x)dx + f(0)p(0). \quad (1.3)$$

Then, (1.2) and (1.3) should be same. Therefore,

$$\begin{aligned} f = 1_{\{0\}} &\implies p(0) = 2\Phi(-1), \\ f(0) = 0 &\implies \int f(x)\phi(x)1_{(-1,1)}(x)dx = \int f(x)p(x)dx \\ &\implies p(x) = \phi(x) \text{ for } 0 < |x| < 1. \end{aligned}$$

$\therefore p(x) = 2\Phi(-1)1_{\{0\}}(x) + \phi(x)1_{(0,1)}(|x|)$ is the density of P_Y .

□

Problem 1.24

Let μ be a σ -finite measure on a measurable space (X, \mathcal{B}) . Show that μ is absolutely continuous with respect to some probability measure P .

Hint: You can use the fact that if μ_1, μ_2, \dots are probability measures and c_1, c_2, \dots are non-negative constants, then $\sum c_i \mu_i$ is a measure. (The proof for Problem 1.10 extends easily to this case.) The measure μ_i , you will want to consider, are truncations of μ to sets A_i covering X with $\mu(A_i) < \infty$, given by $\mu_i(B) = \mu(B \cap A_i)$. With the constants c_i chosen properly, $\sum c_i \mu_i$ will be a probability measure.

Solution.

(i) μ is finite \implies Trivial.

\therefore Since $\mu(X) < \infty$, $P = \mu/\mu(X)$ be a probability measure. Therefore, $P(N) = 0$ implies $\mu(N) = 0$.

(ii) μ is infinite but σ -finite.

$\implies \exists A_1, A_2, \dots \in \mathcal{B}$ s.t. $\bigcup A_i = X$ and $0 < \mu(A_i) < \infty \forall i$.

From Hint, $\mu_i(B) = \mu(B \cap A_i)$ and also $\mu_i(X) = \mu(A_i) \implies \mu_i$ is a finite measure.

Let $b_i = 1/2^i$ (or any other seq of positive constants s.t. $\sum b_i = 1$) and define $c_i = b_i/\mu(A_i)$.

Then $P = \sum_i c_i \mu_i$ is a probability measure, since $P(X) = \sum_i c_i \mu_i(X) = \sum_i \left(\frac{b_i}{\mu(A_i)} \right) \mu(A_i) = 1$.

Suppose $P(N) = \sum_i c_i \mu_i(N) = 0$. Then $\mu_i(N) = \mu(N \cap A_i) = 0 \forall i$.

By Boole's inequality,

$$\mu(N) = \mu\left(\bigcup_i (N \cap A_i)\right) \leq \sum_i \mu(N \cap A_i) = 0.$$

For any null set for P is a null set for μ .

$\therefore \mu \ll P$.

□

Problem 1.25

The monotone convergence theorem states that if $0 \leq f_1 \leq f_2 \leq \dots$ are measurable functions and $f = \lim_{n \rightarrow \infty} f_n$, then $\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$. Use this result to prove the following assertions.

- a) Show that if $X \sim P_X$ is a random variable on $(\mathcal{E}, \mathcal{B}, P)$ and f is a non-negative measurable function, then

$$\int f(X(e)) dP(e) = \int f(x) dP_X(x).$$

Hint: Try it first with f an indicator function. For the general case, let f_n be a sequence of simple functions increasing to f .

Solution.

We will show the equation by 3 steps.

- (i) Suppose $f = 1_A$ (indicator function).

Then, $f(X(e)) = 1$ for $X(e) \in A$.

$\implies f \circ X = 1_B$ where $B = \{e : X(e) \in A\}$.

By definition, $P_X(A) = P(B)$.

$$\begin{aligned} \int f(X(e)) dP(e) &= \int 1_B(e) dP(e) = P(B), \\ \int f(x) dP_X(x) &= \int 1_A(x) dP_X(x) = P_X(A). \end{aligned}$$

\therefore The statement is holds.

- (ii) Suppose $f = \sum_{i=1}^n c_i 1_{A_i}$ (simple function).

$$\begin{aligned} \therefore \int f(X(e)) dP(e) &= \int \sum_i c_i 1_{A_i}(X(e)) dP(e) \\ &= \sum_i c_i \int 1_{A_i}(X(e)) dP(e) \\ &= \sum_i c_i \int 1_{A_i}(x) dP_X(x) \quad (\text{by (i)}) \end{aligned}$$

$$\begin{aligned}
 &= \int \sum_i c_i 1_{A_i}(x) dP_X(x) \\
 &= \int f(x) dP_X(x).
 \end{aligned}$$

- (iii) Suppose f is a non-negative measurable function (general case), and f_n is non-negative simple functions increasing to f .

$$\Rightarrow f_n \circ X \nearrow f \circ X.$$

$$\begin{aligned}
 \therefore \int f(X(e)) dP(e) &= \lim_{n \rightarrow \infty} \int f_n(X(e)) dP(e) \quad (\text{by M.C.T.}) \\
 &= \lim_{n \rightarrow \infty} \int f_n(x) dP_X(x) \quad (\text{by (ii)}) \\
 &= \int f(x) dP_X(x) \quad (\text{by M.C.T.})
 \end{aligned}$$

□

- b) Suppose that P_X has density p with respect to μ , and let f be a non-negative measurable function. Show that

$$\int f dP_X = \int f p d\mu.$$

Solution.

It can be also showed by 3 steps.

- (i) Let $f = 1_A$ (indicator function).

$$\int f dP_X = \int 1_A dP_X = P_X(A) = \int_A p d\mu = \int 1_A p d\mu = \int f p d\mu.$$

- (ii) Suppose $f = \sum_{i=1}^n c_i 1_{A_i}$ (simple function).

$$\int f dP_X = \int \sum_i c_i 1_{A_i} dP_X = \sum_i c_i \int 1_{A_i} dP_X \stackrel{(i)}{=} \sum_i c_i \int 1_{A_i} p d\mu = \int \sum_i c_i 1_{A_i} p d\mu = \int f p d\mu.$$

- (iii) For general case, let a non-negative measurable function f and non-negative simple functions f_n s.t. $f_n \nearrow f$. Then by M.C.T.,

$$\int f dP_X \stackrel{\text{M.C.T.}}{=} \lim_{n \rightarrow \infty} \int f_n dP_X \stackrel{(ii)}{=} \lim_{n \rightarrow \infty} \int f_n p d\mu \stackrel{\text{M.C.T.}}{=} \int f p d\mu.$$

□

Problem 1.26

The gamma distribution.

- a) The gamma function is defined for $\alpha > 0$ by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$. Show that $\Gamma(x+1) = x!$ for $x = 0, 1, \dots$

Solution.

By integration by parts,

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = -t^x e^{-t} \Big|_0^\infty + (\alpha-1) \int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x).$$

Since $\Gamma(1) = \int_0^\infty e^{-x} dx = 1$,

$$\Gamma(x+1) = x\Gamma(x) = x(x-1)\Gamma(x-1) = \cdots = x(x-1)\cdots 2 \cdot 1 \cdot \Gamma(1) = x!.$$

□

b) Show that the function

$$p(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

is a (Lebesgue) probability density when $\alpha > 0$ and $\beta > 0$. This density is called the gamma density with parameters α and β . The corresponding probability distribution is denoted $\Gamma(\alpha, \beta)$.

Solution.

Using change of variable, $y = x/\beta \Rightarrow dx = \beta dy$. Therefore,

$$\begin{aligned} \int p(x) dx &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx \\ &= \int \frac{1}{\Gamma(\alpha)\beta^\alpha} (\beta y)^{\alpha-1} e^{-y} \beta dy \\ &= \frac{1}{\Gamma(\alpha)} \int y^{\alpha-1} e^{-y} dy \\ &= 1. \end{aligned}$$

□

c) Show that if $X \sim \Gamma(\alpha, \beta)$, then $EX^r = \beta^r \Gamma(\alpha + r)/\Gamma(\alpha)$. Use this formula to find the mean and variance of X .

Solution.

$$\begin{aligned} EX^r &= \int_0^\infty x^r p(x) dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha+r-1} e^{-x/\beta} dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha+r)\beta^{\alpha+r}} x^{\alpha+r-1} e^{-x/\beta} dx \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \beta^r \\ &= \beta^r \Gamma(\alpha+r)/\Gamma(\alpha). \end{aligned}$$

By using this,

$$\begin{aligned} EX &= \beta \Gamma(\alpha+1)/\Gamma(\alpha) = \alpha\beta, \\ EX^2 &= \beta^2 \Gamma(\alpha+2)/\Gamma(\alpha) = (\alpha+1)\alpha\beta^2, \\ Var(X) &= EX^2 - \{EX\}^2 = \alpha\beta^2. \end{aligned}$$

□

Problem 1.27

Suppose X has a uniform distribution on $(0, 1)$. Find the mean and covariance matrix of the random vector $\begin{pmatrix} X \\ X^2 \end{pmatrix}$.

Solution.

Since $X \sim \text{Uniform}(0, 1)$, the density of X is $p(x) = 1$ for $x \in (0, 1)$. Thus,

$$EX^r = \int x^r dx = \frac{1}{r+1}.$$

Therefore,

$$E\left(\begin{pmatrix} X \\ X^2 \end{pmatrix}\right) = \begin{pmatrix} EX \\ EX^2 \end{pmatrix},$$

$$\text{Cov}\left(\begin{pmatrix} X \\ X^2 \end{pmatrix}\right) = \begin{pmatrix} \text{Var}(X) & \text{Cov}(X, X^2) \\ \text{Cov}(X^2, X) & \text{Var}(X^2) \end{pmatrix} = \begin{pmatrix} EX^2 - \{EX\}^2 & EX^3 - EXEX^2 \\ EX^3 - EXEX^2 & EX^4 - \{EX^2\}^2 \end{pmatrix}.$$

□

Problem 1.28

If $X \sim N(0, 1)$, find the mean and covariance matrix of the random vector $\begin{pmatrix} X \\ I_{\{X > c\}} \end{pmatrix}$.

Solution.

Let $\Phi(x)$ is a cumulative distribution function of X , and ϕ is a density of X . Then,

$$E\left(\begin{pmatrix} X \\ I_{\{X > c\}} \end{pmatrix}\right) = \begin{pmatrix} EX \\ EI_{\{X > c\}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 - \Phi(c) \end{pmatrix},$$

$$\text{Cov}\left(\begin{pmatrix} X \\ I_{\{X > c\}} \end{pmatrix}\right) = \begin{pmatrix} \text{Var}(X) & EX1_{X>c} - EXEI_{X>c} \\ EX1_{X>c} - EXEI_{X>c} & E1_{X>c}^2 - \{EI_{X>c}\}^2 \end{pmatrix} = \begin{pmatrix} 1 & \phi(c) \\ \phi(c) & \Phi(c)(1 - \Phi(c)) \end{pmatrix}.$$

$$\because EX1_{X>c} = \int_c^\infty \frac{1}{\sqrt{2\pi}} xe^{-x^2/2} dx = \phi(c) \text{ and } 1_{X>c}^2 = 1_{X>c}.$$

□

Problem 1.32

Suppose $E|X| < \infty$ and let

$$h(t) = \frac{1 - E \cos(tX)}{t^2}.$$

Use Fubini's theorem to find $\int_0^\infty h(t) dt$. Hint:

$$\int_0^\infty (1 - \cos(u)) u^{-2} du = \frac{\pi}{2}.$$

Solution.

By Fubini's theorem,

$$\int_0^\infty h(t) dt = \int_0^\infty \left(\frac{1 - E \cos(tX)}{t^2} \right) dt$$

$$\begin{aligned}
 &= \int_0^\infty E\left(\frac{1 - \cos(tX)}{t^2}\right) dt \\
 &= \int_0^\infty \int \frac{1 - \cos(tx)}{t^2} dP_X(x) dt \\
 &= \int \int_0^\infty \frac{1 - \cos(tx)}{t^2} dt dP_X(x) \quad (\text{by Fubini's theorem}) \\
 &\quad u = tx \Rightarrow du = x dt, t = x/u \\
 &= \int \int_0^\infty |x| \frac{1 - \cos(u)}{u^2} du dP_X(x) \\
 &= \int \frac{\pi}{2} |x| dP_X(x) \\
 &= \frac{\pi}{2} E|X|.
 \end{aligned}$$

□

Problem 1.33

Suppose X is absolutely continuous with density $p_X(x) = xe^{-x}$, $x > 0$ and $p_X(x) = 0$, $x \leq 0$. Define $c_n = E(1 + X)^{-n}$. Use Fubini's theorem to evaluate $\sum_{n=1}^\infty c_n$.

Solution.

By Fubini's theorem,

$$\begin{aligned}
 \sum_{n=1}^\infty c_n &= \sum_n E(1 + X)^{-n} \\
 &= \sum_n \int_0^\infty (1 + x)^{-n} x e^{-x} dx \\
 &= \int_0^\infty \sum_n (1 + x)^{-n} x e^{-x} dx \\
 \text{Since } \left| \frac{1}{1+x} \right| &< 1, \quad \sum_{n=1}^\infty \left(\frac{1}{1+x} \right)^n = \frac{1/(1+x)}{1 - 1/(1+x)} = \frac{1}{x}. \\
 &= \int_0^\infty \frac{1}{x} x e^{-x} dx \\
 &= \int_0^\infty e^{-x} dx \\
 &= 1.
 \end{aligned}$$

□

Problem 1.36

Suppose X and Y are independent random variables, and let F_X and F_Y denote their cumulative distribution functions.

- a) Use smoothing to show that the cumulative distribution function of $S = X + Y$ is

$$F_S(s) = P(X + Y \leq s) = EF_X(s - Y). \quad (1.4)$$

Solution.

Since X and Y are independent, $P(X + Y \leq s | Y = y) = P(X \leq s - y) = F_X(s - y)$.

Thus, $P(X + Y \leq s | Y) = F_X(s - Y)$.

$$\therefore F_S(s) = P(X + Y \leq s) = EP(X + Y \leq s | Y) = EF_X(s - Y).$$

□

- b) If X and Y are independent and Y is almost surely positive, use smoothing to show that the cumulative distribution function of $W = XY$ is $F_W(w) = EF_X(w/Y)$ for $w > 0$.

Solution.

Since X and Y are independent, $P(XY \leq w | Y = y) = P(X \leq w/y | Y = y) = P(X \leq w/y) = F_X(w/y)$, for $y > 0$.

Thus, $P(XY \leq w | Y) = F_X(w/Y)$ a.s.

$$\therefore F_W(w) = P(XY \leq w) = EP(XY \leq w | Y) = EF_X(w/Y).$$

□

Problem 1.37

Differentiating (1.4) with respect to s one can show that if X is absolutely continuous with density p_X , then $S = X + Y$ is absolutely continuous with density

$$p_S(s) = Ep_X(s - Y)$$

for $s \in \mathbb{R}$. Use this formula to show that if X and Y are independent with $X \sim \Gamma(\alpha, 1)$ and $Y \sim \Gamma(\beta, 1)$, then $X + Y \sim \Gamma(\alpha + \beta, 1)$.

Solution.

Since $X \sim \Gamma(\alpha, 1)$, $P_X(x - Y) = 0$ for $Y \geq x$ ($\because p_X(x) = 0, \forall x < 0$). Then,

$$\begin{aligned} P_S(x) &= EP_X(x - Y) \\ &= E \left[\frac{1}{\Gamma(\alpha)} (x - Y)^{\alpha-1} e^{-(x-Y)} 1_{(0,x)}(Y) \right] \\ &= \int_0^x \frac{1}{\Gamma(\alpha)} (x - y)^{\alpha-1} e^{-(x-y)} \frac{1}{\Gamma(\beta)} y^{\beta-1} e^{-y} dy \\ &= \int_0^x \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (x - y)^{\alpha-1} y^{\beta-1} e^{-x} dy \end{aligned}$$

Change the variable by $u = y/x \Rightarrow du = 1/x dy$.

$$\begin{aligned} &= \int_0^1 \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (x - xu)^{\alpha-1} (xu)^{\beta-1} e^{-x} x du \\ &= \int_0^1 \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha+\beta-1} (1 - u)^{\alpha-1} u^{\beta-1} e^{-x} du \\ &= \frac{1}{\Gamma(\alpha + \beta)} x^{\alpha+\beta-1} e^{-x}, \quad \text{for } x > 0. \end{aligned}$$

$\therefore S = X + Y \sim \Gamma(\alpha + \beta, 1)$.

□

Problem 1.38

Let Q_λ denote the exponential distribution with failure rate λ , given in Problem 1.30. Let X be a discrete random variable taking values in $\{1, \dots, n\}$ with mass function

$$P(X = k) = \frac{2k}{n(n+1)}, \quad k = 1, \dots, n,$$

and assume that the conditional distribution of Y given $X = x$ is exponential with failure rate x ,

$$Y|X = x \sim Q_x.$$

- a) Find $E[Y|X]$.

Solution.

From Problem 1.30, $q_\lambda(x) = \lambda e^{-\lambda x}$, $\lambda > 0$. Then, the mean of Q_λ is

$$\begin{aligned} \int x dQ_\lambda(x) &= \int x q_\lambda(x) dx = \int_0^\infty x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}. \\ \therefore E(Y|X = x) &= \frac{1}{x} \Rightarrow EY|X = \frac{1}{X}. \end{aligned}$$

□

- b) Use smoothing to compute EY .

Solution.

$$EY = E[E(Y|X)] = E\frac{1}{X} = \sum_{x=1}^n \frac{1}{x} \frac{2x}{n(n+1)} = \sum_{x=1}^n \frac{2}{n(n+1)} = \frac{2}{n+1}.$$

□

Problem 1.39

Let X be a discrete random variable uniformly distributed on $\{1, \dots, n\}$, so $P(X = k) = 1/n$, $k = 1, \dots, n$, and assume that the conditional distribution of Y given $X = x$ is exponential with failure rate x .

- a) For $y > 0$ find $P[Y > y|X]$.

Solution.

$$\begin{aligned} P(Y > y|X = x) &= \int_y^\infty x e^{-xt} dt = -e^{-xt} \Big|_y^\infty = e^{-xy}. \\ \therefore P(Y > y|X) &= e^{-XY}. \end{aligned}$$

□

- b) Use smoothing to compute $P(Y > y)$.

Solution.

Using the sum of the geometric sequence (등비급수의 합),

$$P(Y > y) = EP(Y > y|X) = Ee^{-XY} = \sum_{x=1}^n e^{-xy} \frac{1}{n} = \frac{1}{n} \frac{e^{-y}(1 - e^{-ny})}{1 - e^{-y}} = \frac{1 - e^{-ny}}{n(e^y - 1)}.$$

□

c) Determine the density of Y .

Solution.

Let the density of Y is p_Y . Then,

$$\begin{aligned}
 p_Y(y) &= \frac{d}{dy} (1 - P(Y > y)) \\
 &= \frac{d}{dy} \left(\frac{1 - e^{-ny}}{n(e^y - 1)} \right) \\
 &= \frac{-ne^{-ny}(ne^y - n) + ne^y(1 - e^{-ny})}{n^2(e^y - 1)^2} \\
 &= \frac{-ne^ye^{-ny} + ne^{-ny} + e^y - e^ye^{-ny}}{n(e^y - 1)^2} \\
 &= \frac{e^y(1 - e^{-ny})}{n(e^y - 1)^2} - \frac{e^{-ny}}{e^y - 1}.
 \end{aligned}$$

□

Chapter 2

Exponential Families

Problem 2.1

Consider independent Bernoulli trials with success probability p and let X be the number of failures before the first success. Then $P(X = x) = p(1 - p)^x$, for $x = 0, 1, \dots$, and X has the geometric distribution with parameter p , introduced in Problem 1.17.

- a) Show that the geometric distributions form an exponential family.

Solution.

$$P(X = x) = p(1 - p)^x = \exp \{x \log(1 - p) - (-\log p)\}.$$

\therefore the geometric distribution is an exponential family. \square

- b) Write the densities for the family in canonical form, identifying the canonical parameter η , and the function $A(\eta)$.

Solution.

Let $\eta(p) = \log(1 - p)$. Then, $P(X = x) = \exp \{ \eta x - (-\log(1 - e^\eta)) \}$.

$\therefore A(\eta) = -\log(1 - e^\eta)$ with $T(x) = x$. \square

- c) Find the mean of the geometric distribution using a differential identity.

Solution.

$$E_\eta(T) = A'(\eta) = \frac{e^\eta}{1 - e^\eta} = \frac{1 - p}{p}.$$

\square

- d) Suppose X_1, \dots, X_n are i.i.d. from a geometric distribution. Show that the joint distributions form an exponential family, and find the mean and variance of T .

Solution.

$$P(X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^n P(X_i = x_i) = p^n (1 - p)^{\sum_{i=1}^n x_i} = \exp \left\{ \sum_i x_i \log(1 - p) - (-n \log p) \right\}.$$

Therefore, the joint distribution is also exponential family.

Now, let $\eta = \log(1 - p)$. Then, the canonical exponential family is obtained with $A(\eta) = -n \log(1 - e^\eta)$ and $T = \sum_i x_i$.

$$ET = \kappa_1 = A'(\eta) = \frac{ne^\eta}{1 - e^\eta} = \frac{n(1 - p)}{p},$$

$$Var(T) = \kappa_2 = A''(\eta) = \frac{ne^\eta(1 - e^\eta) + ne^{2\eta}}{(1 - e^\eta)^2} = \frac{np(1 - p) + n(1 - p)^2}{p^2} = \frac{n(1 - p)}{p^2}.$$

□

Problem 2.2

Determine the canonical parameter space Ξ , and find densities for the one-parameter exponential family with μ Lebesgue measure on \mathbb{R}^2 , $h(x, y) = \exp[-(x^2 + y^2)/2]/(2\pi)$, and $T(x, y) = xy$.

Solution.

By definition of the canonical exponential family, the pdf be represented as

$$p(x, y) = \exp(\eta T(x, y) - A(\eta))h(x, y) = \exp\{\eta xy - A(\eta) - (x^2 + y^2)/2\}/(2\pi).$$

Thus,

$$\begin{aligned} \int p(x, y) d\mu(x, y) = 1 &\iff \int \int \exp\{\eta xy - A(\eta) - (x^2 + y^2)/2\}/(2\pi) dx dy = 1 \\ &\iff e^{A(\eta)} = \int \int \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x - \eta y)^2\right\} dx \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y^2 - \eta^2 y^2)} dy \\ &\iff e^{A(\eta)} = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}y^2(1 - \eta^2)} dy \quad \quad \quad \begin{matrix} \uparrow \\ \text{the integral is finite} \iff |\eta| < 1 \end{matrix} \quad (1 - \eta^2)^{-1/2}. \end{aligned}$$

$$\therefore A(\eta) = -\frac{1}{2} \log(1 - \eta^2) \text{ for } \eta \in \Xi = (-1, 1).$$

\therefore The canonical exponential density is

$$\exp\left\{\eta xy + \frac{1}{2} \log(1 - \eta^2) - (x^2 + y^2)/2\right\}/(2\pi).$$

□

Problem 2.4

Find the natural parameter space Ξ and densities p_η for a canonical one-parameter exponential family with μ Lebesgue measure on \mathbb{R} , $T_1(x) = \log x$, and $h(x) = (1 - x)^2$, $x \in (0, 1)$, and $h(x) = 0$, $x \notin (0, 1)$.

Solution.

$$\begin{aligned} e^{A(\eta)} &= \int_0^1 e^{\eta \log x} (1 - x)^2 dx \\ \text{Let } t &= \log x \Rightarrow dt = 1/x dx = e^{-t} dx, \quad -\infty < t < 0 \\ &= \int_{-\infty}^0 e^{\eta t} (1 - e^t)^2 e^t dt \end{aligned}$$

$$\begin{aligned}
 &= \int_{-\infty}^0 \left(e^{t(\eta+1)} - 2e^{t(\eta+2)} + e^{t(\eta+3)} \right) dt \\
 &= \left(\frac{1}{\eta+1} e^{t(\eta+1)} - \frac{2}{\eta+2} e^{t(\eta+2)} + \frac{1}{\eta+3} e^{t(\eta+3)} \right) \Big|_{-\infty}^0 \\
 &= \frac{1}{\eta+1} - \frac{2}{\eta+2} + \frac{1}{\eta+3} \\
 &= \frac{2}{(\eta+1)(\eta+2)(\eta+3)}, \quad \eta \in \Xi = (-1, \infty).
 \end{aligned}$$

(\because For a finite integral, $\eta+1 > 0$, $\eta+2 > 0$, and $\eta+3 > 0 \Rightarrow \eta > -1$.)

$$\therefore p_\eta(x) = \exp(\eta T(x) - A(\eta)) h(x) = \frac{1}{2}(\eta+1)(\eta+2)(\eta+3)(1-x)^2 x^\eta, \quad x \in (0, 1).$$

□

Problem 2.5

Find the natural parameter space Ξ and densities p_η for a canonical one-parameter exponential family with μ Lebesgue measure on \mathbb{R} , $T_1(x) = -x$, and $h(x) = e^{-2\sqrt{x}}/\sqrt{x}$, $x > 0$, and $h(x) = 0$, $x \leq 0$. (Hint: After a change of variables, relevant integrals will look like integrals against a normal density. You should be able to express the answer using Φ , the standard normal cumulative distribution function.) Also, determine the mean and variance for a variable X with this density.

Solution.

$$e^{A(\eta)} = \int_0^\infty e^{-\eta x - 2\sqrt{x}} \frac{1}{\sqrt{x}} dx$$

To the integral is finite, $\eta \in \Xi = [0, \infty)$,

Let $t = \sqrt{x} \Rightarrow t^2 = x$, $2tdt = dx$

$$\begin{aligned}
 &= \int_0^\infty \frac{1}{t} e^{-\eta t^2 - 2t} 2tdt \\
 &= 2 \int_0^\infty e^{-\eta(t^2 + \frac{2}{\eta}t)} dt \\
 &= 2 \int_0^\infty e^{-\frac{2\eta}{2}(t + \frac{1}{\eta})^2} dt \cdot e^{-\eta(-\frac{1}{\eta^2})}
 \end{aligned}$$

Since inner term of integration $\sim N(-1/\eta, 1/2\eta)$,

$$\begin{aligned}
 &= 2\sqrt{2\pi}/\sqrt{2\eta} \cdot e^{\frac{1}{\eta}} \Phi\left(\frac{-1/\eta}{\sqrt{1/(2\eta)}}\right) \\
 &= \frac{2\sqrt{\pi}}{\sqrt{\eta}} e^{1/\eta} \Phi(-\sqrt{2/\eta})
 \end{aligned}$$

$$\Rightarrow A(\eta) = \log 2\sqrt{\pi} - \frac{1}{2} \log \eta + \frac{1}{\eta} + \log \Phi\left(-\sqrt{\frac{2}{\eta}}\right).$$

$$\therefore p_\eta(x) = \exp\left\{-\eta x - 2\sqrt{x} - A(\eta)\right\} \frac{1}{\sqrt{x}}, \quad x > 0.$$

The mean of X is

$$E_\eta(X) = -E_\eta T = -A'(\eta) = \frac{1}{2\eta} + \frac{1}{\eta^2} - \frac{\frac{\sqrt{2}}{2}\eta^{-3/2}\phi\left(-\sqrt{\frac{2}{\eta}}\right)}{\Phi\left(-\sqrt{\frac{2}{\eta}}\right)}.$$

□

Problem 2.6

Find the natural parameter space Ξ and densities p_η for a canonical two-parameter exponential family with μ counting measure on $\{0, 1, 2\}$, $T_1(x) = x$, $T_2(x) = x^2$, and $h(x) = 1$ for $x \in \{0, 1, 2\}$.

Solution.

$$\begin{aligned} e^{A(\eta)} &= \sum_{x=0}^2 e^{\eta_1 x + \eta_2 x^2} \\ &= 1 + e^{\eta_1 + \eta_2} + e^{2\eta_1 + 4\eta_2}. \end{aligned}$$

To the sum is finite, $\eta_1 + \eta_2 < \infty$ & $2\eta_1 + 4\eta_2 < \infty \Rightarrow \eta \in \Xi = \mathbb{R}^2$.

$$\therefore p_\eta(x) = \exp \left[\eta_1 x + \eta_2 x^2 - \left(1 + e^{\eta_1 + \eta_2} + e^{2\eta_1 + 4\eta_2} \right) \right].$$

□

Problem 2.7

Suppose X_1, \dots, X_n are independent geometric variables with p_i the success probability for X_i . Suppose these success probabilities are related to a sequence of "independent" variables t_1, \dots, t_n , viewed as known constants, through

$$p_i = 1 - \exp(\alpha + \beta t_i), \quad i = 1, \dots, n.$$

Show that the joint densities for X_1, \dots, X_n form a two-parameter exponential family, and identify the statistics T_1 and T_2 .

Solution.

Since $X_i \stackrel{\text{ind}}{\sim} \text{Geo}(p_i)$, then the joint density is represented by

$$\begin{aligned} \prod_{i=1}^n p(x_i) &= \prod_{i=1}^n (1 - p_i)^{x_i} p_i = \exp \left[\sum_{i=1}^n x_i \log(1 - p_i) + \sum_{i=1}^n \log p_i \right] \\ &= \exp \left[\sum_{i=1}^n x_i (\alpha + \beta t_i) + \sum_{i=1}^n \log(1 - \exp(\alpha + \beta t_i)) \right]. \end{aligned}$$

$$\therefore T_1 = \sum_{i=1}^n X_i, \quad T_2 = \sum_{i=1}^n t_i X_i.$$

□

Problem 2.8

Assume that X_1, \dots, X_n are independent random variables with $X_i \sim N(\alpha + \beta t_i, 1)$, where t_1, \dots, t_n are observed constants and α and β are unknown parameters. Show that the joint densities for X_1, \dots, X_n form a two-parameter exponential family, and identify the statistics T_1 and T_2 .

Solution.

The joint density of X_1, \dots, X_n is

$$\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{1}{2} (x_i - \alpha - \beta t_i)^2 \right] = (2\pi)^{-\frac{n}{2}} \exp \left[\sum_i x_i (\alpha + \beta t_i) - \frac{1}{2} \sum_i (\alpha + \beta t_i) \right] \exp \left(-\sum_i \frac{x_i^2}{2} \right).$$

$$\therefore T_1 = \sum_{i=1}^n X_i, \quad T_2 = \sum_{i=1}^n t_i X_i.$$

□

Problem 2.9

Suppose that X_1, \dots, X_n are independent Bernoulli variables (a random variable is Bernoulli if it only takes on values 0 and 1) with

$$P(X_i = 1) = \frac{\exp(\alpha + \beta t_i)}{1 + \exp(\alpha + \beta t_i)}.$$

Show that the joint distribution for X_1, \dots, X_n form a two-parameter exponential family, and identify the statistics T_1 and T_2 .

Solution.

The joint density of X_1, \dots, X_n is

$$\begin{aligned} \prod_{i=1}^n p_i^{x_i} (1 - p_i)^{1-x_i} &= \exp \left[\sum_i x_i \log p_i + \sum_i (1 - x_i) \log(1 - p_i) \right] \\ &= \exp \left[\sum_i x_i (\alpha + \beta t_i - \log(1 + e^{\alpha + \beta t_i})) + \sum_i (1 - x_i) (1 - \log(1 + e^{\alpha + \beta t_i})) \right] \\ &= \exp \left[\sum_i x_i (\alpha + \beta t_i) - \sum_i \log(1 + e^{\alpha + \beta t_i}) \right]. \end{aligned}$$

$$\therefore T_1 = \sum_{i=1}^n X_i, \quad T_2 = \sum_{i=1}^n t_i X_i. \quad \square$$

Problem 2.15

For an exponential family in canonical form, $ET_j = \partial A(\eta)/\partial \eta_j$. This can be written in vector form as $ET = \nabla A(\eta)$. Derive an analogous differential formula for $E_\theta T$ for an s -parameter exponential family that is not in canonical form. Assume that Ω has dimension s .

Hint: Differentiation under the integral sign should give a system of linear equations. Write these equations in matrix form.

Solution.

Note that the exponential family form is

$$\begin{aligned} p_\eta(x) &= \exp \left[\sum_{i=1}^s \eta_i T_i(x) - A(\eta) \right] h(x) \\ &= \exp \left[\sum_{i=1}^s \eta_i(\theta) T_i(x) - B(\theta) \right] h(x) \\ &= p_\theta(x). \end{aligned}$$

Thus,

$$e^{B(\theta)} = \int \exp \left[\sum_{i=1}^s \eta_i(\theta) T_i(x) \right] h(x) d\mu(x).$$

By differentiating with respect to θ_i ,

$$e^{B(\theta)} \frac{\partial B(\theta)}{\partial \theta_i} = \int \left(\sum_{j=1}^s \frac{\partial \eta_j(\theta)}{\partial \theta_i} T_j(x) \right) \exp \left[\sum_{j=1}^s \eta_j(\theta) T_j(x) \right] h(x) d\mu(x).$$

$$\begin{aligned}\frac{\partial B(\theta)}{\partial \theta_i} &= \int \left(\sum_{j=1}^s \frac{\partial \eta_j(\theta)}{\partial \theta_i} T_j(x) \right) p_\theta(x) d\mu(x) \\ &= \sum_{j=1}^s \frac{\partial \eta_j(\theta)}{\partial \theta_i} E_\theta T_j, \quad i = 1, \dots, s.\end{aligned}$$

□

Problem 2.17

Let μ denote counting measure on $\{1, 2, \dots\}$. One common definition for $\sum_{k=1}^{\infty} f(k)$ is $\lim_{n \rightarrow \infty} \sum_{k=1}^n f(k)$, and another definition is $\int f d\mu$.

- a) Use the dominated convergence theorem to show that the two definitions give the same answer when $\int |f| d\mu < \infty$.

Hint: Find functions f_n , $n = 1, 2, \dots$, so that $\sum_{k=1}^n f(k) = \int f_n d\mu$.

Solution.

Let $f_n(k) = f(k)$, $\forall k \leq n$, $f_n(k) = 0$, $\forall k > n$. Then, $f_n \rightarrow f$ pointwise and $|f_n| \leq |f|$. Also f_n is simple and $\int f_n d\mu = \sum_{k=1}^n f(k)$.

By D.C.T., $\int f_n d\mu \rightarrow \int f d\mu$.

□

- b) Use the monotone convergence theorem, give in Problem 1.25, to show the definitions agree if $f(k) \geq 0$ for all $1, 2, \dots$.

Solution.

Define f_n and f like part (a). Then, $0 \leq f_1 \leq f_2 \leq \dots$.

By M.C.T., $\sum_{k=1}^n f(k) = \int f_n d\mu \rightarrow \int f d\mu$.

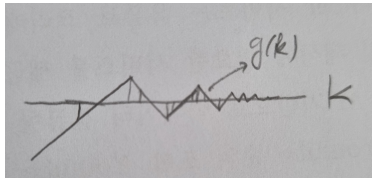
□

- c) Suppose $\lim_{n \rightarrow \infty} f(n) = 0$ and that $\int f^+ d\mu = \int f^- d\mu = \infty$ (so that $\int f d\mu$ is undefined.) Let K be an arbitrary constant. Show that the list $f(1), f(2), \dots$ can be rearranged to form a new list $g(1), g(2), \dots$ so that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n g(k) = K.$$

Solution.

Take positive parts of the sequence f until the sum is exceed K . And then, take negative parts of the sequence f until the sum is below K . Repeat this procedure like the figure below. (In the figure, $g(k)$ is typo. $\sum g(k)$ is correct.) Then, the sum of the sequence goes to 0 for $k > k'$ where k' is



the number when the sum of the rearranged sequence exceed K , first. Let this sequence as g . Then, $\lim_{n \rightarrow \infty} \sum_{k=1}^n g(k) \rightarrow K$.

□

Problem 2.19

Let $p_n, n = 1, 2, \dots$, and p be probability densities with respect to a measure μ , and let $P_n, n = 1, 2, \dots$, and P be the corresponding probability measures.

- a) Show that if $p_n(x) \rightarrow p(x)$ as $n \rightarrow \infty$, then $\int |p_n - p| d\mu \rightarrow 0$.

Hint: First use the fact that $\int (p_n - p) d\mu = 0$ to argue that $\int |p_n - p| d\mu = 2 \int (p - p_n)^+ d\mu$. Then use dominated convergence.

Solution.

Since p_n and p be probability densities, $\int (p_n - p) d\mu = \int p_n d\mu - \int p d\mu = 1 - 1 = 0$. Since $p_n - p = (p - p_n)^+ - (p - p_n)^-$,

$$\begin{aligned} \int (p_n - p) d\mu &= \int (p - p_n)^+ d\mu - \int (p - p_n)^- d\mu = 0, \\ \therefore \int (p - p_n)^+ d\mu &= \int (p - p_n)^- d\mu. \end{aligned}$$

Also $|p_n - p| = (p - p_n)^+ + (p - p_n)^-$,

$$\int |p_n - p| d\mu = \int (p - p_n)^+ d\mu + \int (p - p_n)^- d\mu = 2 \int (p - p_n)^+ d\mu.$$

Also note that $|(p - p_n)^+| \leq p \Rightarrow |(p - p_n)^+|$ is integrable. ($\because p$ is a probability density $\Rightarrow p$ is integrable.) From the condition, $(p_n \rightarrow p) \Rightarrow (p(x) - p_n(x))^+ \rightarrow 0$.

By D.C.T.,

$$\int |p_n - p| d\mu = 2 \int (p - p_n)^+ d\mu \rightarrow 0.$$

□

- b) Show that $|P_n(A) - P(A)| \leq \int |p_n - p| d\mu$.

Hint: Use indicators and the bound $|\int f d\mu| \leq \int |f| d\mu$.

Solution.

$$\begin{aligned} LHS &= \left| \int 1_A dP_n - \int 1_A dP \right| \\ &= \left| \int 1_A (p_n - p) d\mu \right| \\ &\leq \int |1_A (p_n - p)| d\mu \\ &\leq \int |p_n - p| d\mu. \end{aligned}$$

□

Remark: Distributions $P_n, n \geq 1$, are said to *converge strongly* to P if $\sup_A |P_n(A) - P(A)| \rightarrow 0$. The two parts above show that pointwise convergence of p_n to p implies strong convergence. This was discovered by Scheffé.

Problem 2.22

Suppose X is absolutely continuous with density

$$p_\theta(x) = \begin{cases} \frac{e^{-(x-\theta)^2/2}}{\sqrt{2\pi}\Phi(\theta)}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the moment generating function of X . Compute the mean and variance of X .

Solution.

$$p_\theta(x) = \exp \left[-\frac{1}{2}(x^2 - 2x\theta + \theta^2) - \log \Phi(\theta) \right] \frac{1}{\sqrt{2\pi}}, \quad x > 0$$

is the exponential family with $T = X$, $A(\theta) = \frac{\theta^2}{2} + \log \Phi(\theta)$. Therefore, the moment generating function of $T = X$ is

$$\begin{aligned} M_X(u) &= \exp [A(\theta + u) - A(\theta)] \\ &= \exp \left[\frac{1}{2}(\theta + u)^2 - \frac{1}{2}\theta^2 \right] \Phi(\theta + u)/\Phi(\theta) \\ &= \exp \left(\theta u + \frac{1}{2}u^2 \right) \Phi(\theta + u)/\Phi(\theta). \end{aligned}$$

Meanwhile, the *c.g.f.* of $T = X$ for exponential family is $K_X(u) = A(\theta + u) - A(\theta)$. Thus,

$$\begin{aligned} EX &= K'_X(u) = A'(\theta) = \theta + \frac{\phi(\theta)}{\Phi(\theta)}, \\ \text{Var}(X) &= K''_X(u) = A''(\theta) = 1 + \frac{\phi'(\theta)\Phi(\theta) - \phi(\theta)^2}{\Phi(\theta)^2}. \\ &\left(\because \phi(x) = e^{-x^2/2}/\sqrt{2\pi} \Rightarrow \phi'(x) = -x\phi(x). \right) \end{aligned}$$

□

Problem 2.23

Suppose $Z \sim N(0, 1)$. Find the first four cumulants of Z^2 .

Hint: Consider the exponential family $N(0, \sigma^2)$.

Solution.

Let $X \sim N(0, \sigma^2)$. Then, its density is

$$p_\sigma(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{1}{2\sigma^2} x^2 - \log \sigma \right],$$

and also be the exponential family.

Reparameterize $\eta = -\frac{1}{2\sigma^2}$, then $T = X^2$ and $A(\eta) = \frac{1}{2} \log -\frac{1}{2\eta} = -\frac{1}{2} \log(-2\eta)$. Thus, the cumulative generating function of $T = X^2$ is

$$\begin{aligned} K_{X^2}(u) &= A(\eta + u) - A(\eta) \\ &= -\frac{1}{2} [\log(-2(\eta + u)) + \log(-2\eta)]. \end{aligned}$$

If $\sigma = 1$, then $\eta = -\frac{1}{2}$ and $X^2 = Z^2$. Therefore,

$$K_{Z^2}(u) = -\frac{1}{2} [\log(-2u + 1)].$$

$$K'_{Z^2}(u) = \frac{1}{1-2u}, \quad K''_{Z^2}(u) = \frac{2}{(1-2u)^2},$$

$$K^{(3)}_{Z^2}(u) = 8(1-2u)^{-3}, \quad K^{(4)}_{Z^2}(u) = 48(1-2u)^{-4}.$$

\therefore the cumulants are 1, 2, 8, and 48, respectively. □

Problem 2.24

Find the first four cumulants of $T = XY$ when X and Y are independent standard normal variates.

Solution.

Since X and Y are independent, the function of these r.v.s are also independent. (i.e. $g(X)$ and $g(Y)$ are independent for any function g .) Then, first 2 cumulants are

$$\kappa_1 = ET = EXY = EXEY = 0,$$

$$\kappa_2 = E(T - ET)^2 = EX^2Y^2 = EX^2EY^2 = 1.$$

For third cumulants, we need to obtain EX^3 .

$$\begin{aligned} EX^3 &= \int x^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ \text{Let } t &= x^2/2 \Rightarrow x dx = dt \\ &= \frac{2}{\sqrt{2\pi}} \int_0^\infty t e^{-t} dt \\ &= \frac{2}{\sqrt{2\pi}} \left[-te^{-t} \Big|_0^\infty + \int_0^\infty e^{-t} dt \right] \\ &= 0. \end{aligned}$$

Thus, the third cumulants is

$$\kappa_3 = E(T - ET)^3 = EX^3Y^3 = EX^3EY^3 = 0.$$

Similarly, we need to obtain EX^4 . [자꾸 적분이 틀림... 다시 해보기](#) Gamma dist 형태 사용

$$\begin{aligned} EX^4 &= \int x^4 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ \text{Let } t &= x^2/2 \Rightarrow x dx = dt \end{aligned}$$

$$\kappa_4 = E(T - ET)^4 - 3\text{Var}(T)^2 = EX^4EY^4 - 3 \cdot 1^2 = 4.$$

□

Problem 2.25

Find the third and fourth cumulants of the geometric distribution.

Solution.

$$f_p(x) = (1-p)^x p = \exp[x \log(1-p) + \log p]$$

is the exponential family. Reparameterize $\eta = -\log(1-p)$ ($\Rightarrow p = 1 - e^{-\eta}$), then the above form be the canonical exponential family with $T = X$ and $A(\eta) = -\log(1 - e^{-\eta})$.

Thus, the cumulative generating function of $T = X$ is

$$K_X(u) = A(\eta + u) - A(\eta) = -\log(1 - e^{\eta+u}) + \log(1 - e^{-\eta}).$$

Then, the first cumulants is

$$K'_X(0) = A'(\eta) = \frac{e^\eta}{1 - e^{-\eta}} = \frac{1-p}{p}.$$

The higher order cumulants are easily obtained by using the chain rule. By using the $p' = \frac{dp}{d\eta} = -e^{-\eta} = p - 1$,

$$\begin{aligned} K''_X(0) &= A''(\eta) = \frac{d}{dp} A'(\eta) \frac{dp}{d\eta} = \frac{-p - (1-p)}{p^2} (p-1) = \frac{1}{p^2} - \frac{1}{p}, \\ K^{(3)}_X(0) &= A^{(3)}(\eta) = \frac{d}{dp} A''(\eta) p' = -\frac{2p'}{p^3} + \frac{p'}{p^2} = -\frac{2-2p}{p^3} + \frac{p-1}{p^2} = \frac{2}{p^3} - \frac{3}{p^2} + \frac{1}{p}, \\ K^{(4)}_X(0) &= A^{(4)}(\eta) = \frac{d}{dp} A^{(3)}(\eta) p' = -\frac{6p'}{p^4} + \frac{6p'}{p^3} - \frac{p'}{p^2} = \frac{6}{p^4} - \frac{12}{p^3} + \frac{7}{p^2} - \frac{1}{p}. \end{aligned}$$

□

Problem 2.26

Find the third cumulant and third moment of the binomial distribution with n trials and success probability p .

Solution.

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} \left(\frac{p}{1-p} \right)^x (1-p)^n = \exp \left[x \log \frac{p}{1-p} + n \log(1-p) \right] \binom{n}{x}$$

is the exponential family for $\eta = \log \frac{p}{1-p}$ with $T = X$ and $A(\eta) = -n \log(1-p)$. To use the chain rule,

$$p = \frac{e^\eta}{1 + e^\eta} \Rightarrow \frac{dp}{d\eta} = \frac{e^\eta(1 + e^\eta) - (e^\eta)^2}{(1 + e^\eta)^2} = p(1-p).$$

By using the above fact,

$$\begin{aligned} \kappa_1 &= A'(\eta) = \frac{d}{dp} A(\eta) \frac{dp}{d\eta} = \frac{n}{1-p} p(1-p) = np, \\ \kappa_2 &= A''(\eta) = \frac{d}{dp} A'(\eta) \frac{dp}{d\eta} = np(1-p), \\ \kappa_3 &= A'''(\eta) = \frac{d}{dp} A''(\eta) \frac{dp}{d\eta} = (n - 2np)p(1-p) = np(1-p)(1-2p). \end{aligned}$$

The third moment for $T = X$ can be obtained by

$$EX^3 = \kappa_3 + 3\kappa_1\kappa_2 + \kappa_1^3.$$

□

Problem 2.27

Let T be a random vector in \mathbb{R}^2 .

- a) Express $\kappa_{2,1}$ as a function of the moments of T .

Solution.

To make simple derivation, we denote $f_{ij}(u) = \frac{\partial^{i+j} f(u)}{\partial u_1^i \partial u_2^j}$. Note that $K(u) = \log M(u)$, by taking derivative,

$$K_{10} = \frac{M_{10}}{M}, \quad K_{20} = \frac{M_{20}M - M_{10}^2}{M^2},$$

$$\begin{aligned} K_{21} &= \frac{1}{M^4} [(M_{21}M + M_{20}M_{01} - 2M_{10}M_{11})M^2 - 2MM_{01}(M_{20}M - M_{10}^2)] \\ &= \frac{1}{M^3} [M_{21}M^2 + M_{20}M_{01}M - 2M_{10}M_{11}M - 2M_{01}M_{20}M - 2M_{01}M_{10}^2] \\ &= \frac{1}{M^3} [M_{21}M^2 - M_{20}M_{01}M - 2M_{10}M_{11}M - 2M_{01}M_{10}^2]. \end{aligned}$$

Taking $u = 0$, then

$$\kappa_{2,1} = K_{2,1}|_{u=0} = ET_1^2T_2 - (ET_2)(ET_1^2) - 2(ET_1)(ET_1T_2) - 2(ET_2)(ET_1)^2.$$

□

- b) Assume $ET_1 = ET_2 = 0$ and give an expression for $\kappa_{2,2}$ in terms of moments of T .

Solution.

$\kappa_{2,2}$ can be obtain by one more derivative for K_{21} . ([Too complicated. See Keener page 462.](#))

□

Problem 2.28

Suppose $X \sim \Gamma(\alpha, 1/\lambda)$, with density

$$\frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0.$$

Find the cumulants of $T = (X, \log X)$ of order 3 or less. The answer will involve $\psi(\alpha) = d \log \Gamma(\alpha) / d\alpha = \Gamma'(\alpha) / \Gamma(\alpha)$.

Solution.

Since Gamma distribution is the exponential family,

$$\begin{aligned} p(x) &= \exp [\alpha \log \lambda + (\alpha - 1) \log x - \lambda x - \log \Gamma(\alpha)] \\ &= \exp [-\lambda x + \alpha \log x - (\log \Gamma(\alpha) - \alpha \log \lambda) - \log x]. \end{aligned}$$

Let $\eta = (-\lambda, \alpha)$, and $A(\eta) = \log \Gamma(\eta_2) - \eta_2 \log(-\eta_1)$.

Then, the cumulants are obtained by taking derivatives:

$$\begin{aligned} \kappa_{1,0} &= \frac{\partial A(\eta)}{\partial \eta_1} = -\frac{\eta_2}{\eta_1} = \frac{\alpha}{\lambda}, \\ \kappa_{0,1} &= \frac{\partial A(\eta)}{\partial \eta_2} = \psi(\eta_2) - \log(-\eta_1) = \psi(\alpha) - \log \lambda, \end{aligned}$$

$$\kappa_{2,0} = \frac{\partial^2 A(\eta)}{\partial \eta_1^2} = \frac{\eta_2}{\eta_1^2} = \frac{\alpha}{\lambda^2},$$

$$\kappa_{1,1} = \frac{\partial^2 A(\eta)}{\partial \eta_1 \partial \eta_2} = -\frac{1}{\eta_1} = \frac{1}{\lambda},$$

$$\kappa_{0,2} = \frac{\partial^2 A(\eta)}{\partial \eta_2^2} = \psi'(\eta_2) = \psi'(\alpha),$$

$$\kappa_{3,0} = \frac{\partial^3 A(\eta)}{\partial \eta_1^3} = -\frac{2\eta_2}{\eta_1^3} = \frac{2\alpha}{\lambda^3},$$

$$\kappa_{2,1} = \frac{\partial^3 A(\eta)}{\partial \eta_1^2 \partial \eta_2} = \frac{1}{\eta_1^2} = \frac{1}{\lambda^2},$$

$$\kappa_{1,2} = \frac{\partial^3 A(\eta)}{\partial \eta_1 \partial \eta_2^2} = 0,$$

$$\kappa_{0,3} = \frac{\partial^3 A(\eta)}{\partial \eta_2^3} = \psi''(\eta_2) = \psi''(\alpha).$$

□

Chapter 3

Risk, Sufficiency, Completeness, and Ancillarity

Problem 3.2

Suppose data X_1, \dots, X_n are independent with

$$P_\theta(X_i \leq x) = x^{t_i \theta}, \quad x \in (0, 1).$$

where $\theta > 0$ is the unknown parameter, and t_1, \dots, t_n are known positive constants. Find a one-dimensional sufficient statistic T .

Solution.

Take a first derivative for the above cdf, then the pdf of X be $p_\theta(x) = t_i \theta x_i^{t_i \theta - 1}$. Then, the joint density is

$$p_\theta(x_1, \dots, x_n) = \prod_{i=1}^n t_i \theta x_i^{t_i \theta - 1} = \theta^n \left(\prod_{i=1}^n x_i^{t_i} \right)^\theta \frac{\prod_{i=1}^n t_i}{\prod_{i=1}^n x_i}.$$

By factorization theorem, the sufficient statistic for θ is

$$T = \prod_{i=1}^n x_i^{t_i}.$$

□

Problem 3.3

An object with weight θ is weighted on scales with different precision. The data X_1, \dots, X_n are independent, with $X_i \sim N(\theta, \sigma_i^2)$, $i = 1, \dots, n$, with the standard deviations $\sigma_1, \dots, \sigma_n$ known constants. Use sufficiency to suggest a weighted average of X_1, \dots, X_n to estimate θ . (A weighted average would have form $\sum_{i=1}^n w_i X_i$, where the w_i are positive and sum to one.)

Solution.

The pdf of X_i is

$$p_\mu(x_i) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2\sigma_i^2}(x_i - \theta)^2} = \frac{1}{\sqrt{2\pi}\sigma_i} \exp \left[\frac{\theta x_i}{\sigma_i^2} - \frac{x_i^2}{2\sigma_i^2} - \frac{\theta^2}{2\sigma_i^2} \right].$$

Then, the joint density is

$$p_{\mu}(x_1, \dots, x_n) = \frac{1}{(2\pi)^{n/2} \prod_i \sigma_i} \exp \left[\sum_i \frac{\theta x_i}{\sigma_i^2} - \sum_i \frac{x_i^2}{2\sigma_i^2} - \sum_i \frac{\theta^2}{2\sigma_i^2} \right].$$

By factorization theorem,

$$T = \sum_{i=1}^n \frac{x_i}{\sigma_i^2}.$$

To estimate θ using a weighted average form, let

$$w_i = \frac{T}{\sum_{i=1}^n \sigma_i^{-2}},$$

then the weighted average of X_1, \dots, X_n be an estimator for θ satisfied the sufficiency. \square

Problem 3.4

Let X_1, \dots, X_n be a random sample from an arbitrary discrete distribution P on $\{1, 2, 3\}$. Find a two-dimensional sufficient statistic.

Solution.

Let $p_i = P(\{i\})$ and $n_i(x) = \sum_{j=1}^n 1_{\{x_j=i\}}(x)$ for $i = 1, 2, 3$. Then, $n_1(x) + n_2(x) + n_3(x) = n$. Since X_i s are i.i.d., the joint pdf is

$$\begin{aligned} P(X_1 = x_1, \dots, X_n = x_n) &= p_1^{n_1(x)} p_2^{n_2(x)} p_3^{n_3(x)} \\ &= p_1^{n_1(x)} p_2^{n_2(x)} p_3^{n - n_1(x) - n_2(x)}. \end{aligned}$$

By factorization theorem, the sufficient statistic $T = (n_1, n_2)$. \square

Problem 3.6

The beta distribution with parameters $\alpha > 0$ and $\beta > 0$ has density

$$f_{\alpha, \beta}(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}, & x \in (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Suppose X_1, \dots, X_n are i.i.d. from a beta distribution.

- a) Determine a minimal sufficient statistic (for the family of joint distributions) if α and β vary freely.

Solution.

The joint density

$$\begin{aligned} f_{\alpha, \beta}(x_1, \dots, x_n) &= \exp \left[\sum_i (\alpha - 1) \log x_i + \sum_i (\beta - 1) \log(1 - x_i) + n \log \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \right] \\ &= \exp \left[\sum_i \alpha \log x_i + \sum_i \beta \log(1 - x_i) + n \log \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} - \sum_i \log x_i (1 - x_i) \right], \end{aligned}$$

is the exponential family and full-rank. Then,

$$T = \left(\sum_{i=1}^n \log X_i, \sum_{i=1}^n \log(1 - X_i) \right)$$

is complete.

By factorization theorem, T is also sufficient.

Therefore, T is the minimal sufficient statistic for (α, β) . (See Theorem 3.19) \square

- b) Determine a minimal sufficient statistic if $\alpha = 2\beta$.

Solution.

Plug into the joint density,

$$\begin{aligned} f_{\beta}(x_1, \dots, x_n) &= \exp \left[\sum_i 2\beta \log x_i + \sum_i \beta \log(1 - x_i) + n \log \frac{\Gamma(2\beta + \beta)}{\Gamma(2\beta)\Gamma(\beta)} - \sum_i \log x_i(1 - x_i) \right] \\ &= \exp \left[\beta \sum_i (\log x_i^2 + \log(1 - x_i)) + n \log \frac{\Gamma(3\beta)}{\Gamma(2\beta)\Gamma(\beta)} - \sum_i \log x_i(1 - x_i) \right] \end{aligned}$$

is the one-parameter exponential family and full-rank. Thus,

$$T = \sum_i (\log x_i^2 + \log(1 - x_i)) = 2T_1 + T_2$$

is complete, and also sufficient by factorization theorem.

Therefore, T is the minimal sufficient statistic for β . \square

- c) Determine a minimal sufficient statistic if $\alpha = \beta^2$.

Solution.

Now, the joint density is

$$\begin{aligned} p_{\beta}(x_1, \dots, x_n) &= \exp \left[\sum_i (\beta^2 - 1) \log x_i + \sum_i (\beta - 1) \log(1 - x_i) + n \log \frac{\Gamma(2\beta + \beta)}{\Gamma(2\beta)\Gamma(\beta)} \right] \\ &= \exp \left[(\beta^2 - 1)T_1(x) + (\beta - 1)T_2(x) + n \log \frac{\Gamma(\beta^2 + \beta)}{\Gamma(\beta^2)\Gamma(\beta)} \right]. \end{aligned}$$

Suppose $p_{\theta}(x) \propto_{\theta} p_{\theta}(y)$. Then W.L.O.G. we consider 2 cases as follows:

$$\begin{aligned} \frac{p_1(y)}{p_1(x)} = \frac{p_2(y)}{p_2(x)} &\implies \frac{p_2(x)}{p_1(x)} = \frac{p_2(y)}{p_1(y)}, \\ \frac{p_1(y)}{p_1(x)} = \frac{p_3(y)}{p_3(x)} &\implies \frac{p_3(x)}{p_1(x)} = \frac{p_3(y)}{p_1(y)}. \end{aligned}$$

Since $p_1(x) = p_1(y) = 1$, from the above equation,

$$\begin{aligned} 3T_1(x) + T_2(x) + n \log \frac{\Gamma(2^2 + 2)}{\Gamma(2^2)\Gamma(2)} &= 3T_1(y) + T_2(y) + n \log \frac{\Gamma(2^2 + 2)}{\Gamma(2^2)\Gamma(2)}, \\ 8T_1(x) + 2T_2(x) + n \log \frac{\Gamma(3^2 + 3)}{\Gamma(3^2)\Gamma(3)} &= 8T_1(y) + 2T_2(y) + n \log \frac{\Gamma(3^2 + 3)}{\Gamma(3^2)\Gamma(3)}. \end{aligned}$$

These 2 equations imply $T(x) = T(y)$.

Therefore, T is the minimal sufficient statistic by Theorem 3.11. \square

Problem 3.7

Logistic regression. Let X_1, \dots, X_n be independent Bernoulli variables, with $p_i = P(X_i = 1)$, $i = 1, \dots, n$. Let t_1, \dots, t_n be a sequence of known constants that are related to the p_i via

$$\log \frac{p_i}{1 - p_i} = \alpha + \beta t_i,$$

where α and β are unknown parameters. Determine a minimal sufficient statistic for the family of joint distributions.

Solution.

From the above equation,

$$p_i = \frac{e^{\alpha + \beta t_i}}{1 + e^{\alpha + \beta t_i}}.$$

Then, the density of X_i is

$$\begin{aligned} p_{\alpha, \beta}(x_i) &= p_i^{x_i} (1 - p_i)^{1 - x_i} \\ &= \left(\frac{e^{\alpha + \beta t_i}}{1 + e^{\alpha + \beta t_i}} \right)^{x_i} \left(\frac{1}{1 + e^{\alpha + \beta t_i}} \right)^{1 - x_i} \\ &= \exp [x_i(\alpha + \beta t_i) - x_i \log(1 + e^{\alpha + \beta t_i}) - (1 - x_i) \log(1 + e^{\alpha + \beta t_i})] \\ &= \exp [\alpha x_i + \beta x_i t_i - \log(1 + e^{\alpha + \beta t_i})]. \end{aligned}$$

Therefore, we can easily obtain the following sufficient statistic by factorization theorem for the joint density,

$$T = \left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i t_i \right).$$

And also, the joint density is exponential family and full-rank. Thus T is complete.

$\therefore T$ is minimal sufficient. □

Problem 3.8

The multinomial distribution, derived later in Section 5.3, is a discrete distribution with mass function

$$\frac{n!}{x_1! \times \dots \times x_s!} p_1^{x_1} \times \dots \times p_s^{x_s},$$

where x_1, \dots, x_s are non-negative integers summing to n , where p_1, \dots, p_s are non-negative probabilities summing to one, and n is the sample size. Let N_{11}, N_{21}, N_{22} have a multinomial distribution with n trials and success probabilities $p_{11}, p_{12}, p_{21}, p_{22}$. (A common model for a two-by-two contingency table.)

- a) Give a minimal sufficient statistic if the success probabilities vary freely over the unit simplex in \mathbb{R}^4 . (The unit simplex in \mathbb{R}^p is the set of all vectors with non-negative entries summing to one.)

Solution.

The density be the exponential family as follows:

$$\begin{aligned} p(\mathbf{N}) &= \frac{n!}{N_{11}! N_{12}! N_{21}! N_{22}!} p_{11}^{N_{11}} p_{12}^{N_{12}} p_{21}^{N_{21}} p_{22}^{N_{22}} \\ &= \frac{n!}{N_{11}! N_{12}! N_{21}! N_{22}!} \exp [N_{11} \log p_{11} + N_{12} \log p_{12} + N_{21} \log p_{21} + N_{22} \log p_{22}] \end{aligned} \quad (*)$$

$$\begin{aligned}
 &= \frac{n!}{N_{11}!N_{12}!N_{21}!N_{22}!} \exp [N_{11} \log p_{11} + N_{12} \log p_{12} + N_{21} \log p_{21} \\
 &\quad + (n - N_{11} - N_{12} - N_{21}) \log(1 - p_{11} - p_{12} - p_{21})] \\
 &= \frac{n!}{N_{11}!N_{12}!N_{21}!N_{22}!} \exp \left[N_{11} \log \left(\frac{p_{11}}{1 - p_{11} - p_{12} - p_{21}} \right) + N_{12} \log \left(\frac{p_{12}}{1 - p_{11} - p_{12} - p_{21}} \right) \right. \\
 &\quad \left. + N_{21} \log \left(\frac{p_{21}}{1 - p_{11} - p_{12} - p_{21}} \right) + n \log(1 - p_{11} - p_{12} - p_{21}) \right].
 \end{aligned}$$

Since it is the exponential family of full-rank, $T = (N_{11}, N_{12}, N_{21})$ is complete, and also sufficient by factorization theorem. Thus $T = (N_{11}, N_{12}, N_{21})$ is minimal sufficient.

If we use the equation (*), the minimal sufficient statistic is $T = (N_{11}, N_{12}, N_{21}, N_{22})$. \square

- b) Give a minimal sufficient statistic if the success probabilities are constrained so that $p_{11}p_{22} = p_{12}p_{21}$.

Solution.

확실하지 않음...

From the constraint, plug in $p_{11} = (p_{12}p_{21})/p_{22}$ and denote C is the term which is not depend on parameters,

$$\begin{aligned}
 p(\mathbf{N}) &= Cp_{11}^{N_{11}} p_{12}^{N_{12}} p_{21}^{N_{21}} p_{22}^{N_{22}} \\
 &= C \left(\frac{p_{12}p_{21}}{p_{22}} \right)^{N_{11}} p_{12}^{N_{12}} p_{21}^{N_{21}} p_{22}^{N_{22}} \\
 &= Cp_{12}^{N_{11}+N_{12}} p_{21}^{N_{11}+N_{21}} p_{22}^{N_{22}-N_{11}} \\
 &= Cp_{12}^{N_{11}+N_{12}} p_{21}^{N_{11}+N_{21}} (1 - p_{11} - p_{12} - p_{21})^{n - N_{12} - N_{21} - 2N_{11}} \\
 &= C \exp \left[(N_{11} + N_{12}) \log \left(\frac{p_{12}}{1 - p_{11} - p_{12} - p_{21}} \right) \right. \\
 &\quad \left. + (N_{11} + N_{21}) \log \left(\frac{p_{21}}{1 - p_{11} - p_{12} - p_{21}} \right) + n \log(1 - p_{11} - p_{12} - p_{21}) \right].
 \end{aligned}$$

Since it is the exponential family of full-rank, $T = (N_{11} + N_{12}, N_{11} + N_{21})$ is complete, and also sufficient by factorization theorem. Thus $T = (N_{11} + N_{12}, N_{11} + N_{21})$ is minimal sufficient. \square

Problem 3.9

Let f be a positive integrable function on $(0, \infty)$. Define

$$c(\theta) = 1 / \int_{\theta}^{\infty} f(x) dx,$$

and take $p_{\theta}(x) = c(\theta)f(x)$ for $x > \theta$, and $p_{\theta}(x) = 0$ for $x \leq \theta$. Let X_1, \dots, X_n be i.i.d. with common density p_{θ} .

- a) Show that $M = \min\{X_1, \dots, X_n\}$ is sufficient.

Solution.

The joint density is

$$p_{\theta}(x_1, \dots, x_n) = c(\theta)^n \prod_i f(x_i)$$

$$= c(\theta)^n \prod_i f(x_i) \cdot 1_{\min\{x_1, \dots, x_n\} > \theta}.$$

By factorization theorem, $M = \min\{X_1, \dots, X_n\}$ is sufficient. \square

b) Show that M is minimal sufficient.

Solution.

Suppose $p_\theta(x) \propto_\theta p_\theta(y)$. Then, the region of 0 value should be equal. Thus $T(x) = T(y)$ should be satisfied, therefore $T = M$ is minimal sufficient. \square

Problem 3.10

Suppose X_1, \dots, X_n are i.i.d. with common density $f_\theta(x) = (1 + \theta x)/2$, $|x| < 1$; $f_\theta(x) = 0$, otherwise, where $\theta \in [-1, 1]$ is an unknown parameter. Show that the order statistics are minimal sufficient. (Hint: A polynomial of degree n is uniquely determined by its value on a grid of $n + 1$ points.)

Solution.

The joint density is

$$\begin{aligned} f_\theta(x_1, \dots, x_n) &= 2^{-n} (1 + \theta x_1) \cdots (1 + \theta x_n) \\ &= 2^{-n} (1 + \theta x_{(1)}) \cdots (1 + \theta x_{(n)}) 1_{\{-1 < x_{(1)} < \cdots < x_{(n)} < 1\}}. \end{aligned}$$

By factorization theorem, the order statistic $T = (X_{(1)}, \dots, X_{(n)})$ is sufficient.

Now, suppose $p_\theta(x) \propto_\theta p_\theta(y)$. Since the joint density is the n -th order polynomials (다항식) with root($\frac{1}{\theta}$) $-1/x_{(i)}$, the roots of $p_\theta(x)$ and $p_\theta(y)$ should be equal which implies $T(x) = T(y)$. Therefore, $T = (X_{(1)}, \dots, X_{(n)})$ is minimal sufficient. \square

Problem 3.16

Let X_1, \dots, X_n be a random sample from an absolutely continuous distribution with density

$$f_\theta(x) = \begin{cases} 2x/\theta^2, & x \in (0, \theta), \\ 0, & \text{otherwise.} \end{cases}$$

a) Find a one-dimensional sufficient statistic T .

Solution.

The joint density is factorized as follows:

$$f_\theta(x_1, \dots, x_n) = 2^n \frac{x_1 \cdots x_n}{\theta^{2n}} = \frac{2^n}{\theta^{2n}} x_{(1)} \cdots x_{(n)} 1_{x_{(1)} > 0} 1_{x_{(n)} < \theta}.$$

By factorization theorem, $T = \max\{X_1, \dots, X_n\}$ is the sufficient statistic for θ . \square

b) Determine the density of T .

Solution.

Since $T = \max\{X_1, \dots, X_n\}$,

$$P(T \leq t) = \prod_{i=1}^n P(X_i \leq t)$$

$$\begin{aligned} \text{Note that } P(X_i \leq t) &= \int_0^t 2x/\theta^2 dx = t^2/\theta^2. \\ &= \frac{t^{2n}}{\theta^{2n}} \end{aligned}$$

Taking a derivative with respect to t , the density of T is

$$p_\theta(t) = \frac{2n}{\theta^{2n}} t^{2n-1}, \quad t \in (0, \theta).$$

□

c) Show directly that T is complete.

Solution.

Suppose $E_\theta(T) = c$, $\forall \theta > 0$. Then,

$$\begin{aligned} E_\theta [f(T) - c] &= \frac{2n}{\theta^{2n}} \int_0^\theta [f(t) - c] t^{2n-1} dt = 0 \\ \Rightarrow [f(t) - c] t^{2n-1} &= 0 \quad \text{a.e. } 0 < t < \theta. \\ \Rightarrow f(T) &= c. \end{aligned}$$

Therefore, T is complete.

Other solution

$$\begin{aligned} E_\theta f(T) &= \frac{2n}{\theta^{2n}} \int_0^\theta f(t) t^{2n-1} dt = c \\ \Rightarrow \int_0^\theta f(t) t^{2n-1} dt &= c \theta^{2n} / 2n, \quad \forall \theta > 0 \\ \Rightarrow f(\theta) \theta^{2n-1} &= c \theta^{2n-1} \quad \text{a.e. } \theta \\ \Rightarrow f(t) &= c \quad \text{a.e. } t. \end{aligned}$$

□

Problem 3.17

Let X, X_1, X_2, \dots be i.i.d. from an exponential distribution with failure rate λ (introduced in Problem 1.30).

a) Find the density of $Y = \lambda X$.

Solution.

Note that the density of X is $p_\lambda(x) = \lambda e^{-\lambda x}$, $\lambda > 0$, and its cdf is easily obtained by integration, $P(X \leq x) = 1 - e^{-\lambda x}$. Thus the cdf of Y is

$$P(Y \leq y) = P(X \leq y/\lambda) = 1 - e^{-y}, \quad y > 0,$$

and the density is

$$p(y) = e^{-y}, \quad y > 0.$$

□

b) Let $\bar{X} = (X_1 + \cdots + X_n)/n$. Show that \bar{X} and $(X_1^2 + \cdots + X_n^2)/\bar{X}^2$ are independent.

Solution.

The joint density of X_1, \dots, X_n is

$$p_\lambda(x_1, \dots, x_n) = \lambda^n e^{-\lambda \sum_i x_i} = \lambda^n e^{-n\lambda \bar{x}},$$

is the full rank exponential family, and by factorization theorem, $T = \bar{X}$ is the complete sufficient statistic.

Now, let $Y = \lambda X$, then

$$V = \frac{(Y_1^2 + \cdots + Y_n^2)}{\bar{Y}^2} = \frac{(X_1^2 + \cdots + X_n^2)}{\bar{X}^2}.$$

Since Y does not depend on parameter λ , V is ancillary for P_λ .

\therefore By Basu's theorem, T and V are independent.

□

Problem 3.29

Find a function on $(0, \infty)$ that is bounded and strictly convex.

Solution.

$f(x) = e^{-x}$ is bounded on range $(0, 1)$ and strictly convex.

Or $f(x) = 1/(x+1)$ is bounded on range $(0, 1)$ and strictly convex.

□

Problem 3.30

Use convexity to show that the canonical parameter space Ξ of a one-parameter exponential family must be an interval. Specifically, show that if $\eta_0 < \eta < \eta_1$, and if η_0 and η_1 both lie in Ξ , then η must lie in Ξ .

Solution.

Note that if a random variable X is a canonical exponential family, then it has the form,

$$p_\eta(x) = \exp[\eta T(x) - A(\eta)]h(x), \quad \eta \in \Xi.$$

Since $\eta_0 < \eta < \eta_1$, we define $\eta = \gamma\eta_0 + (1-\gamma)\eta_1$ for some $\gamma \in (0, 1)$. Since the exponential function is convex,

$$\exp[(\gamma\eta_0 + (1-\gamma)\eta_1)T(x)] = e^{\eta T(x)} < \gamma e^{\eta_0 T(x)} + (1-\gamma)e^{\eta_1 T(x)}.$$

From the definition of exponential family, $h(x) \geq 0$, and for the positive function, the inequality sign does not change by integration against μ . Thus, following inequality is satisfied:

$$\int e^{\eta T(x)} h(x) d\mu(x) < \gamma \int e^{\eta_0 T(x)} h(x) d\mu(x) + (1-\gamma) \int e^{\eta_1 T(x)} h(x) d\mu(x).$$

Because it should be satisfied the exponential family, RHS is finite. (\because RHS = $\gamma e^{A(\eta_0)} + (1-\gamma)e^{A(\eta_1)} < \infty$.)

Therefore, η must lie in Ξ .

□

Problem 3.31

Let f and g be positive probability densities on \mathbb{R} . Use Jensen's inequality to show that

$$\int \log \left(\frac{f(x)}{g(x)} \right) f(x) dx > 0,$$

unless $f = g$ a.e. (If $f = g$, the integral equals zero.) This integral is called the *Kullback-Liebler information*.

Solution.

Suppose X be a absolutely continuous random variable with density f . Now, define $Y = \frac{g(X)}{f(X)}$, then

$$EY = \int \frac{g(x)}{f(x)} f(x) dx = \int g(x) dx = 1.$$

Next, let $h(y) = \log \frac{1}{y} = -\log y$, then h is strictly convex on $(0, \infty)$. Then, by Jensen's inequality,

$$Eh(Y) = \int \log \left(\frac{f(x)}{g(x)} \right) f(x) dx \geq h(EY) = \log 1 = 0.$$

The equality holds when Y is a constant. (then $Y = EY = 1 \Rightarrow f(x) = g(x)$ a.e.)

□

Chapter 4

Unbiased Estimation

Problem 4.1

Let X_1, \dots, X_m and Y_1, \dots, Y_n be independent variables with the X_i a random sample from an exponential distribution with failure rate λ_x , and the Y_j a random sample from an exponential distribution with failure rate λ_y .

- a) Determine the UMVU estimator of λ_x/λ_y .

Solution.

Note that if $X \sim \text{Exp}(\lambda)$ with rate parameter λ , the density $p_\lambda(x) = \lambda e^{-\lambda x}$.

Since $X_i, \forall i$ and $Y_j, \forall j$ are independent, the joint density is

$$p_{\lambda_x, \lambda_y}(x, y) = \lambda_x^m \lambda_y^n e^{-\lambda_x \sum_{i=1}^m x_i - \lambda_y \sum_{j=1}^n y_j},$$

which is the 2-parameter full-rank exponential family. Thus, by factorization theorem, $T = (\sum_{i=1}^m X_i, \sum_{j=1}^n Y_j)$ is the complete sufficient.

Since X_1, \dots, X_m are i.i.d. $\text{Exp}(\lambda_x) = \Gamma(1, \lambda_x)$, $T_1 = \sum_{i=1}^m X_i \sim \Gamma(m, \lambda_x)$. Then,

$$E \frac{1}{T_1} = \int \frac{1}{t} \frac{\lambda_x^m}{\Gamma(m)} t^{m-1} e^{-\lambda_x t} dt = \frac{\Gamma(m-1) \lambda_x^m}{\Gamma(m) \lambda_x^{m-1}} = \frac{\lambda_x}{m-1}.$$

Similarly, we can obtain $ET_2 = n/\lambda_y$. ($\because EY_i = 1/\lambda_y$)

From the fact of the expectation of independent random variables,

$$E \frac{T_2}{T_1} = E \frac{1}{T_1} \cdot ET_2 = \frac{n \lambda_x}{(m-1) \lambda_y}.$$

Therefore,

$$\delta(x, y) = \frac{(m-1)T_2}{nT_1}$$

is the unbiased estimator of λ_x/λ_y , and also UMVU since it is the function of complete sufficient statistic T . (By Theorem 4.4 with Rao-Blackwellization) \square

- b) Under squared error loss, find the best estimator of λ_x/λ_y of form $\delta = c\bar{Y}/\bar{X}$.

Solution.

Let $\delta = c\bar{Y}/\bar{X} = dT_2/T_1$. Then, the risk of $\delta = c\bar{Y}/\bar{X}$ and λ_x/λ_y is

$$R\left(\frac{\lambda_y}{\lambda_x}, \delta\right) = E\left[d\frac{T_2}{T_1} - \frac{\lambda_x}{\lambda_y}\right]^2 = d^2 E\frac{T_2^2}{T_1^2} - 2\frac{\lambda_x}{\lambda_y} dE\frac{T_2}{T_1} + \frac{\lambda_x^2}{\lambda_y^2}. \quad (4.1)$$

Since $T_1 \sim \Gamma(m, \lambda_x)$ and $T_2 \sim \Gamma(n, \lambda_y)$,

$$ET_2^2 = \text{Var}(T_2) + \{ET_2\}^2 = n/\lambda_y^2 + n^2/\lambda_y^2,$$

$$E\frac{1}{T_1^2} = \int \frac{1}{t^2} \frac{\lambda_x^m}{\Gamma(m)} t^{m-1} e^{-\lambda_x t} dt = \frac{\Gamma(m-2)\lambda_x^2}{\Gamma(m)} = \frac{\lambda_x^2}{(m-1)(m-2)}.$$

Thus, using the fact that T_1 and T_2 are independent,

$$E\frac{T_2^2}{T_1^2} = E\frac{1}{T_1^2} \cdot ET_2^2 = \frac{n(n+1)\lambda_x^2}{(m-1)(m-2)\lambda_y^2}.$$

By using the result from a), we know $ET_2/T_1 = (n\lambda_x)/((m-1)\lambda_y)$, then the risk (4.1) be

$$\begin{aligned} R\left(\frac{\lambda_y}{\lambda_x}, \delta\right) &= d^2 E\frac{T_2^2}{T_1^2} - 2\frac{\lambda_x}{\lambda_y} dE\frac{T_2}{T_1} + \frac{\lambda_x^2}{\lambda_y^2} \\ &= d^2 \frac{n(n+1)\lambda_x^2}{(m-1)(m-2)\lambda_y^2} - 2\frac{\lambda_x}{\lambda_y} d \frac{n\lambda_x}{(m-1)\lambda_y} + \frac{\lambda_x^2}{\lambda_y^2} \\ &= \frac{\lambda_x^2}{\lambda_y^2} \left(\frac{n(n+1)}{(m-1)(m-2)} d^2 - 2\frac{n}{(m-1)} d + 1 \right) \\ &= \frac{\lambda_x^2}{\lambda_y^2} \frac{n(n+1)}{(m-1)(m-2)} \left[\left(d - \frac{m-2}{n+1} \right)^2 - \left(\frac{m-2}{n+1} \right)^2 + 1 \right]. \end{aligned}$$

Thus, it has the minimum when $d = \frac{m-1}{n+1}$.

Therefore, the best estimator of λ_x/λ_y is $c\bar{Y}/\bar{X} = dT_2/T_1 = [(m-2)n\bar{Y}] / [(n+1)m\bar{X}]$.

$$\therefore c = \frac{(m-2)n}{(n+1)m}.$$

□

c) Find the UMVU estimator of $e^{-\lambda_x} = P(X_1 > 1)$.

Solution.

Let δ is the unbiased estimator of $P(X_1 > 1)$. Then,

$$E\delta = P(X_1 > 1) \Rightarrow \delta = 1_{\{X_1 > 1\}}.$$

From the previous steps, we know T_1 is C.S.S., thus $\eta(T_1) = E(\delta|T_1) = P(X_1 > 1|T_1)$ is UMVU by Theorem 4.4. We follow by 2 steps.

(i) Find the joint density of X and T_1 , where $X = 1_{\{X_1 > 1\}}$ and $T_1 = \sum_{i=1}^m X_i$.

First, we should find the joint density of X and S where $S = \sum_{i=2}^m X_i$. Since X and S are independent, the joint density is easily obtained as

$$f(x, s) = f(x)f(s) = \lambda_x e^{-\lambda_x x} \frac{\lambda_x^{m-1}}{\Gamma(m-1)} s^{m-2} e^{-\lambda_x s} = \frac{\lambda_x^m}{\Gamma(m-1)} s^{m-2} e^{-\lambda_x(x+s)}, \quad x > 0, s > 0.$$

By transformation of variables $X = X, T_1 = S + X$, the jacobian $|J| = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$. Then, the joint density of X and T_1 is

$$f(x, t) = \frac{\lambda_x^m}{\Gamma(m-2)} (t-x)^{m-2} e^{-\lambda_x t}, \quad 0 < x < t.$$

(ii) Derive $E(\delta|T_1)$.

To obtain the expectation, we derive the conditional density of X given T_1 . Since $T_1 \sim \Gamma(m, \lambda_x)$, the conditional density be

$$f(x|t) = \frac{f(x, t)}{f(t)} = \frac{\Gamma(m)}{\Gamma(m-2)} (t-x)^{m-2} t^{-(m-1)} = (m-1) \frac{1}{t} \left(1 - \frac{x}{t}\right)^{m-2}, \quad 0 < x < t.$$

Then,

$$\begin{aligned} E(\delta|T_1) &= P(X_1 > 1|T_1) = P(X|T_1) \\ &= \int_1^t f(x|t) dx \\ &= \int_1^t (m-1) \frac{1}{t} \left(1 - \frac{x}{t}\right)^{m-2} dx \\ &= -\left(1 - \frac{x}{t}\right)^{m-1} \Big|_1^t \\ &= \left(1 - \frac{1}{t}\right)^{m-1} 1_{\{T_1 > 1\}}. \end{aligned}$$

□

Problem 4.2

Let X_1, \dots, X_n be a random sample from $N(\mu_x, \sigma^2)$, and let Y_1, \dots, Y_m be an independent random sample from $N(\mu_y, 2\sigma^2)$, with μ_x , μ_y , and σ^2 all unknown parameters..

a) Find a complete sufficient statistic.

Solution.

Since $X_1, \dots, X_n, Y_1, \dots, Y_m$ are mutually independent, the joint density,

$$\begin{aligned} f(x, y) &= f(x)f(y) \\ &= (2\pi\sigma^2)^{-n/2} (4\pi\sigma^2)^{-m/2} \exp \left[-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu_x)^2 - \frac{1}{4\sigma^2} \sum_{j=1}^m (y_j - \mu_y)^2 \right] \\ &= \exp \left[-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2} - \frac{\sum_{j=1}^m y_j^2}{4\sigma^2} + \frac{\mu_x \sum_{i=1}^n x_i}{\sigma^2} + \frac{\mu_y \sum_{j=1}^m y_j}{2\sigma^2} - \frac{n\mu_x^2}{2\sigma^2} - \frac{m\mu_y^2}{4\sigma^2} \right] (2\pi\sigma^2)^{-n/2} (4\pi\sigma^2)^{-m/2} \\ &= \exp \left[\frac{\mu_x \sum_{i=1}^n x_i}{\sigma^2} + \frac{\mu_y \sum_{j=1}^m y_j}{2\sigma^2} - \frac{2 \sum_{i=1}^n x_i^2 + \sum_{j=1}^m y_j^2}{4\sigma^2} - \frac{n\mu_x^2}{2\sigma^2} - \frac{m\mu_y^2}{4\sigma^2} \right] (2\pi\sigma^2)^{-n/2} (4\pi\sigma^2)^{-m/2}, \end{aligned}$$

is the full-rank exponential family. Therefore,

$$T = \left(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j, 2 \sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2 \right)$$

is the complete sufficient statistic for (μ_x, μ_y, σ^2) . \square

b) Determine the UMVU estimator of σ^2 .

Hint: Find a linear combination L of $S_x^2 = \sum_{i=1}^n (X_i - \bar{X})^2 / (n-1)$ and $S_y^2 = \sum_{j=1}^m (Y_j - \bar{Y})^2 / (m-1)$ so that (\bar{X}, \bar{Y}, L) is complete sufficient.

Solution.

To use the C.S.S. obtained from a), let

$$2(n-1)S_x^2 + (m-1)S_y^2 = 2 \left(\sum_i X_i^2 - n\bar{X}^2 \right) + \left(\sum_j Y_j^2 - m\bar{Y}^2 \right)$$

is the linear combination of T . Since the sample variance is unbiased, then its expectation is

$$E[2(n-1)S_x^2 + (m-1)S_y^2] = (2n + 2m - 4)\sigma^2.$$

Now, we let

$$S_p^2 = \frac{1}{2n + 2m - 4} [2(n-1)S_x^2 + (m-1)S_y^2],$$

then it is unbiased for σ^2 based on complete sufficient T .

By theorem 4.4, it is UMVU of σ^2 . \square

c) Find a UMVU estimator of $(\mu_x - \mu_y)^2$.

Solution.

$$\begin{aligned} E(\bar{X} - \bar{Y})^2 &= \text{Var}(\bar{X} - \bar{Y}) + [E(\bar{X} - \bar{Y})]^2 \\ &= \text{Var}(\bar{X}) + \text{Var}(\bar{Y}) + [E\bar{X} - E\bar{Y}]^2 \\ &= \frac{\sigma^2}{n} + \frac{2\sigma^2}{m} + (\mu_x - \mu_y)^2. \end{aligned}$$

We obtained the unbiased estimator of σ^2 in b). Thus, we let δ as

$$\delta = (\bar{X} - \bar{Y})^2 - S_p^2 \left(\frac{1}{n} - \frac{2}{m} \right),$$

then it is the unbiased estimator of $(\mu_x - \mu_y)^2$ based on complete sufficient T .

By Theorem 4.4, δ is UMVU of $(\mu_x - \mu_y)^2$. \square

d) Suppose we know the $\mu_y = 3\mu_x$. What is the UMVU estimator of μ_x ?

Solution.

With restriction $\mu_y = 3\mu_x$, the joint density of exponential family is changed as

$$f(x, y) = \cdots = \exp \left[\frac{\mu_x (2 \sum_i x_i + 3 \sum_j y_j)}{2\sigma^2} - \cdots \right] \cdots$$

Thus, $T = \left(2 \sum_i X_i + 3 \sum_j Y_j, 2 \sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2 \right)$ is complete sufficient.

Take expectation for T_1 , then

$$ET_1 = 2n\mu_x + 3m\mu_y = (2n + 9m)\mu_x,$$

Now we let δ as

$$\delta = \frac{2 \sum_i x_i + 3 \sum_j y_j}{2n + 9m},$$

then, it is unbiased estimator based on T .

By Theorem 4.4, δ is UMVU of μ_x . □

Problem 4.3

Let X_1, \dots, X_n be a random sample from the Poisson distribution with mean λ . Find the UMVU for $\cos \lambda$. (Hint: For Taylor expansion, the identity $\cos \lambda = (e^{i\lambda} + e^{-i\lambda})/2$ may be useful.)

Solution.

The joint density is

$$p_\lambda(x) = \frac{\lambda^{\sum_i x_i} e^{-n\lambda}}{x_1! \cdots x_n!},$$

with $T = \sum_i X_i$, that is complete sufficient. (\because full-rank exponential family, and by factorization theorem) Note that $T \sim \text{Poisson}(n\lambda)$. Let $\delta(t)$ is an unbiased estimator of $\cos \lambda$ based on T . Then, its expectation should satisfy follows:

$$\begin{aligned} E\delta(T) &= \sum_{t=0}^{\infty} \delta(t) \frac{(n\lambda)^t e^{-n\lambda}}{t!} = \cos \lambda \\ \Leftrightarrow \sum_{t=0}^{\infty} \delta(t) \frac{(n\lambda)^t}{t!} &= e^{n\lambda} \cos \lambda \end{aligned}$$

Apply Talyor expansion for RHS,

$$\begin{aligned} e^{n\lambda} \cos \lambda &= e^{n\lambda} \frac{1}{2} (e^{i\lambda} + e^{-i\lambda}) \\ &= \frac{1}{2} (e^{(n+i)\lambda} + e^{(n-i)\lambda}) \\ &= \frac{1}{2} \sum_{t=0}^{\infty} \frac{((n+i)\lambda)^t + ((n-i)\lambda)^t}{t!} \\ &= \sum_{t=0}^{\infty} \frac{(n\lambda)^t}{t!} \left[\frac{(1 + \frac{i}{n})^t + (1 - \frac{i}{n})^t}{2} \right]. \end{aligned}$$

Therefore,

$$\delta(t) = \frac{1}{2} \left\{ \left(1 + \frac{i}{n}\right)^t + \left(1 - \frac{i}{n}\right)^t \right\}$$

is unbiased estimator of $\cos \lambda$ based on T .

By Theorem 4.4, $\delta(t)$ is UMVU of $\cos \lambda$. □

Problem 4.4

Let X_1, \dots, X_n be independent normal variables, each with unit variance, and with $EX_i = \alpha t_i + \beta t_i^2$, $i = 1, \dots, n$, where α and β are unknown parameters and t_1, \dots, t_n are known constants. Find UMVU estimators of α and β .

Solution.

Since $X_i \sim N(\alpha t_i + \beta t_i^2, 1)$, the joint density is

$$\begin{aligned} f(x) &= (2\pi)^{-\frac{n}{2}} \exp \left[-\frac{1}{2} \sum_i (x_i - \alpha t_i - \beta t_i^2)^2 \right] \\ &= (2\pi)^{-\frac{n}{2}} \exp \left[-\frac{1}{2} \sum_i (x_i^2 + \alpha^2 t_i^2 + \beta^2 t_i^4 - 2\alpha x_i t_i - 2\beta x_i t_i^2 + \alpha\beta t_i^3) \right]. \end{aligned}$$

Since these density is full-rank exponential family,

$$T = \left(\sum_i X_i t_i, \sum_i X_i t_i^2 \right)$$

is complete sufficient and its expected value is

$$ET_1 = \alpha \sum t_i^2 + \beta \sum t_i^3, \quad ET_2 = \alpha \sum t_i^3 + \beta \sum t_i^4.$$

By using this,

$$\hat{\alpha} = \frac{T_1 \sum t_i^4 - T_2 \sum t_i^3}{\sum t_i^2 \sum t_i^4 - (\sum t_i^3)^2}, \quad \hat{\beta} = \frac{T_1 \sum t_i^3 - T_2 \sum t_i^2}{(\sum t_i^3)^2 - \sum t_i^4 \sum t_i^2}$$

are unbiased estimator of α and β . Since these are the function of T , also UMVU. □

Problem 4.5

Let X_1, \dots, X_n be i.i.d. from some distribution Q_θ , and let $\bar{X} = (X_1 + \dots + X_n)/n$ be the sample average.

- a) Show that $S^2 = \sum (X_i - \bar{X})^2 / (n-1)$ is unbiased for $\sigma^2 = \sigma^2(\theta) = \text{Var}_\theta(X_i)$.

Solution.

Denote $\mu = EX_i$. Also note that $(n-1)S^2 = \sum_i X_i^2 - n\bar{X}^2$. Then,

$$\begin{aligned} E((n-1)S^2) &= \sum_i EX_i^2 - nE\bar{X}^2 \\ &= \sum_i [\text{Var}X_i + (EX_i)^2] - n[\text{Var}\bar{X} + (E\bar{X})^2] \\ &= n(\sigma^2 + \mu^2) - n(\sigma^2/n + \mu^2) \\ &= (n-1)\sigma^2. \end{aligned}$$

Therefore S^2 is unbiased for σ^2 . □

- b) If Q_θ is the Bernoulli distribution with success probability θ , show that S^2 from (a) is UMVU.

Solution.

Note that the density of X_i is $\theta^{x_i}(1-\theta)^{1-x_i} = \left(\frac{\theta}{1-\theta}\right)^{x_i} (1-\theta)$.

The joint density,

$$p(x) = \exp \left[\sum_i x_i \log \frac{\theta}{1-\theta} + n \log(1-\theta) \right],$$

is the full-rank exponential family. Thus, $T = \bar{X}$ (or $\sum_i X_i$) is complete sufficient. Meanwhile,

$$\begin{aligned} S^2 &= \frac{1}{n-1} \left(\sum X_i^2 - n\bar{X}^2 \right) \\ &= \frac{1}{n-1} \left(\sum X_i - n\bar{X}^2 \right) \quad (\because X_i^2 = X_i) \\ &= \frac{n\bar{X}(1-\bar{X})}{n-1} \end{aligned}$$

is the function of $T = \bar{X}$. Therefore, S^2 is UMVU of σ^2 by Theorem 4.4. \square

- c) If Q_θ is the exponential distribution with failure rate θ , find the UMVU estimator of $\sigma^2 = 1/\theta^2$. Give a formula for $E_\theta[X_i^2|\bar{X} = c]$ in this case.

Solution.

Note that the density of X_i is $\theta e^{-\theta x}$ with mean $1/\theta$ and variance $1/\theta^2$.

The joint density,

$$p(x) = \exp \left[-\theta \sum x_i + n \log \theta \right],$$

is the full-rank exponential family. Thus, $T = \bar{X}$ (or $\sum_i X_i$) is complete sufficient with $ET = 1/\theta$ and $\text{Var}T = 1/(n\theta^2)$.

Meanwhile,

$$E\bar{X}^2 = \text{Var}\bar{X} + (E\bar{X})^2 = \frac{1}{n\theta^2} + \frac{1}{\theta^2} = \frac{n+1}{n} \frac{1}{\theta^2} = \frac{n+1}{n} \sigma^2.$$

Thus, $\delta = \frac{n}{n+1} \bar{X}^2$ is the unbiased estimator of σ^2 , and also function of $T = \bar{X}$.

By Theorem 4.4, δ is UMVU for σ^2 .

Next, since X_i 's are i.i.d., $E_\theta[X_1^2|\bar{X} = c] = \cdots = E_\theta[X_n^2|\bar{X} = c]$. Then,

$$\begin{aligned} E_\theta[S^2|\bar{X} = c] &= \frac{nc^2}{n+1} \\ &= E_\theta \left[\frac{\sum_i X_i^2 - n\bar{X}^2}{n-1} \middle| \bar{X} = c \right] \\ &= E_\theta \left[\frac{\sum_i X_i^2}{n-1} - \frac{nc^2}{n-1} \middle| \bar{X} = c \right] \\ &= \frac{n}{n-1} E_\theta[X_1^2|\bar{X} = c] - \frac{nc^2}{n-1}. \end{aligned}$$

$$\therefore E_\theta[X_i^2|\bar{X} = c] = nc^2 \left(\frac{1}{n+1} + \frac{1}{n-1} \right) \frac{n-1}{n} = c^2 \left(\frac{n-1}{n+1} + 1 \right) = \frac{2nc^2}{n+1}.$$

\square

Problem 4.6

Suppose δ is a UMVU estimator of $g(\theta)$; U is an unbiased estimator of zero, $E_\theta U = 0$, $\theta \in \Omega$; and that δ and U both have finite variances for all $\theta \in \Omega$. Show that U and δ are uncorrelated, $E_\theta U \delta = 0$, $\theta \in \Omega$.

Solution.

For the convenience, we drop θ in the notation of E , Var and Cov .

Let $\delta^* = \delta + cU$ where c is a constant. Then, δ^* is an unbiased estimator for $g(\theta)$ ($\because E\delta^* = E(\delta + cU) = E\delta + cEU = g(\theta)$).

Since δ is UMVU,

$$\begin{aligned}\text{Var}(\delta + cU) &= \text{Var}(\delta) + c^2\text{Var}(U) + 2c\text{Cov}(\delta, U) \geq \text{Var}(\delta), \\ &\Rightarrow c^2\text{Var}(U) + 2c\text{Cov}(\delta, U) \geq 0.\end{aligned}$$

Denote LHS as $h(c)$. Since h is convex and has minimum $h(0) = 0$, h should have a local minimum at $c = 0$. Since h is differentiable, we can obtain the following result by taking a first derivative at $c = 0$:

$$h'(0) = 2\text{Cov}(\delta, U) = 0.$$

Since $EU = 0$, we finally obtain $E_\theta U \delta = 0$ for all $\theta \in \Omega$. □

Problem 4.7

Suppose δ_1 is a UMVU estimator of $g_1(\theta)$, δ_2 is UMVU estimator of $g_2(\theta)$, and that δ_1 and δ_2 both have finite variance for all θ . Show that $\delta_1 + \delta_2$ is UMVU for $g_1(\theta) + g_2(\theta)$.

Hint: Use the result in the previous problem.

Solution.

We can easily show that $\delta_1 + \delta_2$ is unbiased for $g_1(\theta) + g_2(\theta)$.

Now, suppose δ is any unbiased estimator of $g_1(\theta) + g_2(\theta)$ with $\text{Var}(\delta) < \infty$. Let $U = \delta - \delta_1 - \delta_2$, then U is an unbiased estimator of 0 ($\because EU = E\delta - E\delta_1 - E\delta_2 = 0$).

By Problem 4.6,

$$\text{Cov}(U, \delta_1 + \delta_2) = \text{Cov}(U, \delta_1) + \text{Cov}(U, \delta_2) = 0.$$

Since $\delta = U + \delta_1 + \delta_2$, we obtain the following result:

$$\begin{aligned}\text{Var}(\delta) &= \text{Var}(U) + \text{Var}(\delta_1 + \delta_2) \\ &\geq \text{Var}(\delta_1 + \delta_2).\end{aligned}$$

Because δ is arbitrary, $\delta_1 + \delta_2$ is UMVU. □

Problem 4.8

Let X_1, \dots, X_n be i.i.d. absolutely continuous variables with common density f_θ , $\theta > 0$, given by

$$f_\theta(x) = \begin{cases} \theta/x^2, & x > \theta, \\ 0, & x \leq \theta. \end{cases}$$

Find the UMVU estimator for $g(\theta)$ if $g(\theta)/\theta^n \rightarrow 0$ as $\theta \rightarrow \infty$ and g is differentiable.

Solution.

The joint density is

$$\theta^n / (x_1 \cdots x_n)^2 1_{\{\min x_i\}}.$$

By factorization theorem, $T = \min X_i$ is sufficient.

To obtain the density of T , the cumulative distribution function is

$$P(T \leq t) = 1 - P(\min X_i > t) = 1 - \prod_i P(X_i > t) = 1 - \left(\frac{\theta}{t}\right)^n, \quad t > \theta.$$

Take a derivative, we can obtain the density,

$$f(t) = \frac{n\theta^n}{t^{n+1}}, \quad t > \theta.$$

Now, suppose $\delta(T)$ is the unbiased estimator of $g(\theta)$, then

$$\int_{\theta}^{\infty} \delta(t) \frac{n\theta^n}{t^{n+1}} dt = g(\theta) \iff n \int_{\theta}^{\infty} \delta(t) t^{-n-1} dt = \frac{g(\theta)}{\theta^n}$$

Taking a derivative w.r.t. θ ,

$$\iff -n\delta(t)t^{-n-1} = g'(t)t^{-n} - ng(t)t^{-n-1}.$$

$$\therefore \delta(t) = g(t) - \frac{g'(t)}{nt}.$$

From the above result, $g(t) = c$ for any constant c implies $\delta(t) = c$. Therefore T is complete.

In summary, T is complete sufficient, and $\delta(T)$ is the function of T . Therefore, by Theorem 4.4, $\delta(T)$ is UMVU. \square

Problem 4.10

Suppose X is an exponential variable with density $p_{\theta}(x) = \theta e^{-\theta x}$, $x > 0$; $p_{\theta}(x) = 0$, otherwise. Find the UMVU estimator for $1/(1 + \theta)$.

Solution.

Since X is a full-rank exponential family, $T = X$ is complete sufficient.

Let $\delta(X)$ be an unbiased estimator of $1/(1 + \theta)$, and represent as $\delta(x) = \sum_{n=0}^{\infty} c_n x^n$ by power series. Then,

$$E\delta(X) = \int_0^{\infty} \left(\sum_{n=0}^{\infty} c_n x^n \right) \theta e^{-\theta x} dx$$

By Fubini theorem,

$$= \sum_{n=0}^{\infty} \int_0^{\infty} c_n \theta x^n e^{-\theta x} dx$$

The inner integration follows $\Gamma(n + 1, 1/\theta)$ with rate parameter.

$$= \sum_{n=0}^{\infty} \frac{c_n \Gamma(n + 1)}{\theta^n}$$

$$= \sum_{n=0}^{\infty} \frac{c_n n!}{\theta^n}. \tag{*}$$

Meanwhile, $1/(1 + \theta)$ can be written as the geometric series,

$$\frac{1}{1 + \theta} = \frac{1/\theta}{1 + 1/\theta} = - \sum_{n=1}^{\infty} \left(-\frac{1}{\theta}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\theta^n} \quad \text{for } \theta > 1. \tag{**}$$

Since δ is unbiased, $(*)$ and $(**)$ should be identical. Thus,

$$c_0 = 0, \quad c_n = \frac{(-1)^{n+1}}{n!}, \quad n = 1, 2, \dots$$

Therefore,

$$\delta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} x^n = - \sum_{n=1}^{\infty} \frac{(-x)^n}{n!} = 1 - e^{-x} \quad \text{for } \theta > 1.$$

Since δ is the function of the complete sufficient statistic X , $\delta(X)$ is UMVU by Theorem 4.4.

If $\theta \leq 1$, direct하게 unbiased임을 보일 수 있다?? 어떻게?? □

Problem 4.11

Let X_1, \dots, X_3 be i.i.d. geometric variables with common mass function $f_\theta(x) = P_\theta(X_i = x) = \theta(1-\theta)^x$, $x = 0, 1, \dots$. Find the UMVU estimator of θ^2 .

Solution.

Since the joint mass, $\theta^3(1-\theta)^{x_1+x_2+x_3}$, is the full-rank exponential family, $T = X_1 + X_2 + X_3$ is complete sufficient.

Denote $g(\theta) = \theta^2$, and let $\delta(X) = 1_{\{X_1=0, X_2=0\}}$, then $\delta(X)$ be an unbiased estimator of $g(\theta)$ ($\because E\delta(X) = P(X_1=0, X_2=0) = \theta^2$).

Define $\eta(T) = E[\delta(X)|T] = P(X_1=0, X_2=0|T)$. Then it is UMVU by Theorem 4.4. To obtain $\eta(t)$, we first find the density of T by using the transformation $T = X_1 + X_2 + X_3$ from the joint density,

$$P(T=t) = \binom{t+2}{2} \theta^3(1-\theta)^t = \binom{t+2}{2} \theta^2 [\theta(1-\theta)^t].$$

Because we already have 2 successes in first $t+2$ trials of total $t+3$ trials, we should add the combination term. (See RHS!!)

Next, the joint mass of X_1, X_2, T is

$$P(X_1=0, X_2=0, T=t) = P(X_1=0, X_2=0, X_3=t) = \theta^3(1-\theta)^t,$$

from the joint mass of X_1, X_2, X_3 .

Therefore,

$$\begin{aligned} \eta(t) &= \frac{P(X_1=0, X_2=0, T=t)}{P(T=t)} = 1 / \binom{t+2}{2} = \frac{2}{(t+2)(t+1)}, \\ \therefore \eta(T) &= \frac{2}{(T+2)(T+1)} \end{aligned}$$

is the UMVU estimator of θ^2 . □

Problem 4.12

Let X be a single observation, absolutely continuous with density

$$p_\theta(x) = \begin{cases} \frac{1}{2}(1+\theta x), & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

Here $\theta \in [-1, 1]$ is an unknown parameter.

- a) Find a constant a so that aX is unbiased for θ . The expected value is

Solution.

$$EX = \int_{-1}^1 \frac{x}{2}(1 + \theta x)dx = \frac{1}{4}x^2 + \frac{\theta}{6}x^3 \Big|_{-1}^1 = \frac{\theta}{3}.$$

Therefore, when $a = 3$, aX be unbiased. □

- b) Show that $b = E_\theta|X|$ is independent of θ .

Solution.

By using the procedure in a),

$$E|X| = \int_{-1}^1 \frac{|x|}{2}(1 + \theta x)dx = \frac{1}{2}.$$

Thus, $b = E_\theta|X|$ does not depend on θ . □

- c) Let θ_0 be a fixed parameter value in $[-1, 1]$. Determine the constant $c = c_{\theta_0}$ that minimizes the variance of the unbiased estimator $aX + c(|X| - b)$ when $\theta = \theta_0$. Is aX uniformly minimum variance unbiased?

Solution.

To obtain the variance of $aX + c(|X| - b)$, we first derive the second moments.

$$EX^2 = E|X|^2 = \int_{-1}^1 \frac{x^2}{2}(1 + \theta x)dx = \frac{x^3}{6} + \frac{\theta x^4}{8} \Big|_{-1}^1 = \frac{1}{3},$$

$$EX|X| = \int_{-1}^1 \frac{x|x|}{2}(1 + \theta x)dx = \frac{\theta}{4}.$$

From the previous steps, we obtain $a = 3$ and $b = E|X|$. Under $\theta = \theta_0$, the variance be

$$\begin{aligned} \text{Var}_{\theta_0}(3X + c(|X| - E|X|)) &= 9\text{Var}_{\theta_0}(X) + 6c\text{Cov}_{\theta_0}(X, |X| - E|X|) + c^2\text{Var}_{\theta_0}(|X| - E|X|) \\ &= 9\text{Var}_{\theta_0}(X) + 6c\text{Cov}_{\theta_0}(X, |X| - E|X|) + c^2\text{Var}_{\theta_0}(|X|) \\ &= 9\left(\frac{1}{3} - \frac{\theta_0^2}{9}\right) + 6c\left(\frac{\theta_0}{4} - \frac{1}{2} \cdot \frac{\theta_0}{3}\right) + c^2\left(\frac{1}{3} - \frac{1}{4}\right) \\ &= 3 - \theta_0^2 + \frac{\theta_0}{2}c + \frac{1}{12}c^2. \end{aligned} \tag{*}$$

Since (*) is convex with respect to c , take a derivative,

$$\frac{\theta_0}{2} + \frac{c}{6} = 0$$

Therefore, (*) has minimum when $c = -3\theta_0$.

Meanwhile, $\text{Var}_{\theta_0}(3X) = 3 - \theta_0^2$ which is greater than (*). Therefore, $3X$ is not UMVU. □

Problem 4.24

In the normal one-sample problem, the statistic $t = \sqrt{n}\bar{X}/S$ has the non-central t -distribution on $n - 1$ degrees of freedom and non-centrality parameter $\delta = \sqrt{n}\mu/\sigma$. Use our results on distribution theory for the one-sample problem to find the mean and variance of t .

Solution.

Note that $E\bar{X} = \mu = \sigma\delta/\sqrt{n}$, $\text{Var}(\bar{X}) = \sigma^2/n$, and the formula ES^r in (4.10) page 70.

Since \bar{X} and S^2 are independent,

$$Et = E\left[\sqrt{n}\frac{\bar{X}}{S}\right] = \sqrt{n}(E\bar{X})(ES^{-1}) = \sqrt{n}\frac{\sigma\delta}{\sqrt{n}}\frac{\sigma^{-1}2^{-\frac{1}{2}}\Gamma[(-1+n-1)/2]}{(n-1)^{-\frac{1}{2}}\Gamma[(n-1)/2]} = \delta\frac{\sqrt{n-1}\Gamma[(n-2)/2]}{\sqrt{2}\Gamma[(n-1)/2]},$$

$$\begin{aligned} Et^2 &= n(E\bar{X}^2)(ES^2) = n\left(\frac{\sigma^2}{n} + \frac{\sigma^2\delta^2}{n}\right)\frac{\sigma^{-2}2^{-1}\Gamma[(-2+n-1)/2]}{(n-1)^{-1}\Gamma[(n-1)/2]} \\ &= (1+\delta^2)\frac{n-1}{2}\frac{\Gamma[(n-3)/2]}{\Gamma[(n-3)/2+1]} = (1+\delta^2)\frac{n-1}{n-3}. \end{aligned}$$

$$\text{Var}(t) = Et^2 - (Et)^2 = \frac{n-1}{n-3} + \delta^2\left[\frac{n-1}{n-3} - \frac{(n-1)(\Gamma[(n-2)/2])^2}{2(\Gamma[(n-1)/2])^2}\right].$$

□

Problem 4.28

Let X_1, \dots, X_n be i.i.d. from the uniform distribution on $(0, \theta)$.

- a) Use the Hammersley-Chapman-Robbins inequality to find a lower bound for the variance of an unbiased estimator of θ . This bound will depend on Δ . Note that Δ cannot vary freely but must lie in a suitable set.

Solution.

Since δ is the unbiased estimator of θ , $g(\theta) = \theta$ and $g(\theta + \Delta) - g(\theta) = \Delta$. By Hammersley-Chapman-Robbins inequality, the lower bound for the variance of δ be

$$LB = \frac{\Delta^2}{E_\theta \left[\frac{p_{\theta+\Delta}(X)}{p_\theta(X)} - 1 \right]^2}.$$

Since $X_i \sim \text{Uniform}(0, \theta)$, $p_\theta(X) = 0 \Rightarrow p_{\theta+\Delta}(X) = 0$. Thus Δ should be negative and $\theta + \Delta > 0$, therefore $\Delta \in (-\theta, 0)$.

$$\frac{p_{\theta+\Delta}(X)}{p_\theta(X)} = \begin{cases} \frac{\theta^n}{(\theta+\Delta)^n}, & \max\{X_1, \dots, X_n\} < \theta + \Delta \text{ with probability } \frac{(\theta+\Delta)^n}{\theta^n} \text{ under } P_\theta, \\ 0, & \text{otherwise.} \end{cases}$$

For simplicity, denote $Y = p_{\theta+\Delta}(X)/p_\theta(X)$, then $EY = 1$ and $EY^2 = \frac{\theta^n}{(\theta+\Delta)^n}$. Thus, the expectation in the denominator is

$$E(Y-1)^2 = \text{Var}(Y) = EY^2 - [EY]^2 = \frac{\theta^n}{(\theta+\Delta)^n} - 1.$$

Therefore, the lower bound be

$$LB = \frac{\Delta^2}{\frac{\theta^n}{(\theta+\Delta)^n} - 1}. \quad (*)$$

□

- b) In principle, the best lower bound can be found taking the supremum over Δ . This calculation cannot be done explicitly, but an approximation is possible. Suppose $\Delta = -c\theta/n$. Show that the lower bound for the variance can be written as $\theta^2 g_n(c)/n^2$. Determine $g(c) = \lim_{n \rightarrow \infty} g_n(c)$.

Solution.

Plug in $\Delta = -c\theta/n$ to $(*)$, then

$$LB = \frac{c^2 \theta^2 / n^2}{\left(1 - \frac{c}{n}\right)^{-n} - 1}.$$

Therefore,

$$g_n(c) = \frac{c^2}{\left(1 - \frac{c}{n}\right)^{-n} - 1},$$

$$g(c) = \lim_{n \rightarrow \infty} g_n(c) = \frac{c^2}{e^c - 1}.$$

□

- c) Find the value c_0 that maximizes $g(c)$ over $c \in (0, 1)$ and give an approximate lower bound for the variance of δ . (The value c_0 cannot be found explicitly, but you should be able to come up with a numerical value.)

Solution.

Taking a derivative,

$$\frac{\partial g(c)}{\partial c} = \frac{\partial}{\partial c} \left(\frac{c^2}{e^c - 1} \right) = \frac{2c(e^c - 1) - c^2 e^c}{(e^c - 1)^2} = 0$$

$$\iff 2c(e^c - 1) = c^2 e^c$$

$$\iff 2e^c - 2 = ce^c$$

$$\iff 1 - e^c = \frac{c}{2}$$

$$\iff e^{-c} = 1 - \frac{c}{2}.$$

By numerical method, $c_0 = 1.59362$. Therefore, an approximated lower bound of $g(c_0)\theta^2/n^2 = 0.64761\theta^2/n^2$.

□

Problem 4.30

Suppose X_1, \dots, X_n are independent with $X_i \sim N(\alpha + \beta t_i, 1)$, $i = 1, \dots, n$, where t_1, \dots, t_n are known constants and α, β are unknown parameters.

- a) Find the Fisher information matrix $I(\alpha, \beta)$.

Solution.

The log likelihood of $\theta = (\alpha, \beta)$ is

$$l(\beta) = \frac{n}{2} \log(2\pi) - \frac{1}{2} \sum_{i=1}^n (x_i - \alpha - \beta t_i)^2.$$

Take derivatives,

$$\begin{aligned} \frac{\partial l}{\partial \alpha} &= \sum_i (x_i - \alpha - \beta t_i), & \frac{\partial l}{\partial \beta} &= \sum_i (x_i - \alpha - \beta t_i) t_i, \\ \frac{\partial^2 l}{\partial \alpha^2} &= -n, & \frac{\partial^2 l}{\partial \beta^2} &= -\sum_i t_i^2, \\ \frac{\partial^2 l}{\partial \alpha \beta} &= -\sum_i t_i \end{aligned}$$

Therefore, the Fisher information matrix be

$$I(\alpha, \beta) = -E [\nabla l(\alpha, \beta)]^2 = \begin{pmatrix} n & \sum_i t_i \\ \sum_i t_i & \sum_i t_i^2 \end{pmatrix}.$$

□

- b) Give a lower bound for the variance of an unbiased estimator of α .

Solution.

Since $g(\theta) = \alpha$, by Cramer-Rao lower bound,

$$\begin{aligned} LB &= \nabla g(\theta)^T I^{-1}(\alpha, \beta) \nabla g(\theta) \\ &= (1, 0) \frac{\begin{pmatrix} \sum_i t_i^2 & -\sum_i t_i \\ -\sum_i t_i & n \end{pmatrix}}{n \sum_i t_i^2 - (\sum_i t_i)^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{n \sum_i t_i^2 - (\sum_i t_i)^2} \begin{pmatrix} \sum_i t_i^2 & -\sum_i t_i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{\sum_i t_i^2}{n \sum_i t_i^2 - (\sum_i t_i)^2} \\ &= \frac{1/n}{1 - (\sum_i t_i)^2 / \sum_i t_i^2}. \end{aligned}$$

□

- c) Suppose we know the value of β . Give a lower bound for the variance of an unbiased estimator of α in this case.

Solution.

Since β is known, $\theta = \alpha$. Then the lower bound be

$$LB = \frac{g'(\theta)}{I(\theta)} = \frac{1}{n}.$$

□

- d) Compare the estimators in parts (b) and (c). When are the bounds the same? If the bounds are different, which is larger?

Solution.

If $\sum_i t_i = 0$, two lower bounds be same. And by comparing the lower bounds,

$$LB_b) = \frac{1/n}{1 - (\sum_i t_i)^2 / \sum_i t_i^2} > \frac{1}{n} = LB_c).$$

□

e) Give a lower bound for the variance of an unbiased estimator of the product $\alpha\beta$.

Solution.

Since $g(\theta) = \alpha\beta$, $\nabla g(\theta) = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$. Then the lower bound be

$$\begin{aligned} LB &= \nabla g(\theta)^T I^{-1}(\alpha, \beta) \nabla g(\theta) \\ &= (\beta, \alpha) \frac{\begin{pmatrix} \sum_i t_i^2 & -\sum_i t_i \\ -\sum_i t_i & n \end{pmatrix}}{n \sum_i t_i^2 - (\sum_i t_i)^2} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \\ &= \frac{1}{n \sum_i t_i^2 - (\sum_i t_i)^2} \begin{pmatrix} \beta \sum_i t_i^2 - \alpha \sum_i t_i, & -\beta \sum_i t_i + n\alpha \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \\ &= \frac{\beta^2 \sum_i t_i^2 - 2\alpha\beta \sum_i t_i + n\alpha^2}{n \sum_i t_i^2 - (\sum_i t_i)^2}. \end{aligned}$$

□

Problem 4.31

Find the Fisher information for the Cauchy location family with densities p_θ given by

$$p_\theta(x) = \frac{1}{\pi[(x - \theta)^2 + 1]}.$$

Also, what is the Fisher information for θ^3 ?

Solution.

Since p_θ is the location family, by using Example 4.11 in page 75,

$$\begin{aligned} I(\theta) &= \int \frac{[f'(x)]^2}{f(x)} dx = \int \frac{\left[-\frac{2\pi x}{\pi(x^2+1)}\right]^2}{1/[\pi(x^2+1)]} = \int \frac{4x^2}{\pi(x^2+1)^3} dx \\ \text{Change of variable } x &= \tan t \Rightarrow dx = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} dt \\ &= \frac{4}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\tan^2 t}{(\tan^2 t + 1)^3} \frac{1}{\cos^2 t} dt \\ &= \frac{4}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\sin^2 t}{\cos^2 t} \cos^6 t \frac{1}{\cos^2 t} dt \quad (\because \tan^2 t + 1 = 1/\cos^2 t) \\ &= \frac{4}{\pi} \int_{-\pi/2}^{\pi/2} \sin^2 t \cos^2 t dt \\ &= \frac{4}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1 - \cos 2x}{2} \frac{1 + \cos 2x}{2} dt \end{aligned}$$

$$\begin{aligned}
&= \frac{4}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - \cos^2 2x}{4} dt \\
&= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - \cos 4x}{2} dt \\
&= \frac{1}{2\pi} \left[t - \frac{1}{4} \sin 4t \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{1}{2}.
\end{aligned}$$

Let $\xi = \theta^3$, then $\theta = \xi^{1/3} := h(\xi)$.

$$\therefore I(\xi) = [h'(\xi)]^2 I(h(\xi)) = \left(\frac{1}{3} \xi^{-2/3} \right)^2 \frac{1}{2} = \frac{1}{18 \xi^{4/3}}.$$

□

Problem 4.32

Suppose X has a Poisson distribution with mean θ^2 , so the parameter θ is the square root of the usual parameter $\lambda = EX$. Show that the Fisher information $I(\theta)$ is constant.

Solution.

$p_\theta(X) = (\theta^2)^x e^{-\theta^2} / x!$ and $l(\theta) = \log 1/x! + 2x \log \theta - \theta^2$.

$$\frac{\partial l(\theta)}{\partial \theta} = \frac{2x}{\theta} - 2\theta, \quad \frac{\partial^2 l(\theta)}{\partial \theta^2} = -\frac{2x}{\theta^2} - 2.$$

$$\therefore I(\theta) = E \left(-\frac{\partial^2 l(\theta)}{\partial \theta^2} \right) = \frac{2}{\theta^2} EX + 2 = 4.$$

□

Problem 4.33

Consider the exponential distribution with failure rate λ . Find a function h defining a new parameter $\theta = h(\lambda)$ so that Fisher information $I(\theta)$ is constant.

Solution.

Let the original parameter is θ , and reparameterized parameter is λ .

$p_\lambda(x) = \lambda e^{-\lambda x}$, $l(\lambda) = \log \lambda - \lambda x$.

$$\frac{\partial l(\lambda)}{\partial \lambda} = \frac{1}{\lambda} - x, \quad \frac{\partial^2 l(\lambda)}{\partial \lambda^2} = -\frac{1}{\lambda^2}.$$

$$\tilde{I}(\lambda) = \frac{1}{\lambda^2}.$$

From (4.18) in page 74,

$$I(\theta) = \frac{\tilde{I}(\lambda)}{[h'(\lambda)]^2} = \frac{1}{[\lambda h'(\lambda)]^2}.$$

Thus, if $h(\lambda) = \log \lambda$, $I(\theta)$ be constant.

□

Problem 4.34

Consider an autoregressive model in which $X_1 \sim N(\theta, \sigma^2/(1-\rho^2))$ and the conditional distribution of X_{j+1} given $X_1 = x_1, \dots, X_j = x_j$, is $N(\theta + \rho(x_j - \theta), \sigma^2)$, $j = 1, \dots, n-1$.

a) Find the Fisher information matrix, $I(\theta, \sigma)$.

Solution.

The joint density is

$$p(x_1, \dots, x_n) = p(x_1)p(x_2|x_1) \cdots p(x_n|x_1, \dots, x_{n-1}),$$

thus the likelihood function be

$$\begin{aligned} l(\theta, \sigma) &= l(x_1) + \sum_{j=1}^{n-1} l(x_{j+1}|x_1, \dots, x_j) \\ &= \log \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} - \log \sigma - \frac{1-\rho^2}{2\sigma^2}(x_1 - \theta)^2 \\ &\quad - \frac{1}{\sigma^2} \sum_{j=1}^{n-1} [x_{j+1} - \theta - \rho(x_j - \theta)]^2 + \log \left(\frac{1}{\sqrt{2\pi}\sigma} \right)^{n-1} \\ &= -\frac{1-\rho^2}{2\sigma^2}(x_1 - \theta)^2 - \frac{1}{2\sigma^2} \sum_{j=1}^{n-1} [x_{j+1} - \rho x_j - \theta(1-\rho)]^2 \\ &\quad - \frac{n}{2} \log(2\pi) - n \log \sigma + \log \sqrt{1-\rho^2}. \end{aligned}$$

Take derivatives,

$$\begin{aligned} \frac{\partial l(\theta, \sigma)}{\partial \theta} &= \frac{1-\rho^2}{\sigma^2}(x_1 - \theta) + \frac{1-\rho}{\sigma^2} \sum_{j=1}^{n-1} [x_{j+1} - \rho x_j - \theta(1-\rho)], \\ \frac{\partial^2 l(\theta, \sigma)}{\partial \theta^2} &= -\frac{1-\rho^2}{\sigma^2} - \frac{(n-1)(1-\rho)^2}{\sigma^2}, \\ \frac{\partial^2 l(\theta, \sigma)}{\partial \theta \partial \sigma} &= -\frac{2(1-\rho^2)\epsilon_1}{\sigma^3} + \frac{2(1-\rho)}{\sigma^3} \sum_{j=1}^{n-1} \eta_{j+1}, \\ \frac{\partial l(\theta, \sigma)}{\partial \sigma} &= \frac{(1-\rho^2)(x_1 - \theta)^2}{\sigma^3} + \frac{1}{\sigma^3} \sum_{j=1}^{n-1} \eta_{j+1}^2 - \frac{n}{\sigma}, \\ \frac{\partial^2 l(\theta, \sigma)}{\partial \sigma^2} &= -\frac{3(1-\rho^2)(x_1 - \theta)^2}{\sigma^4} - \frac{3}{\sigma^4} \sum_{j=1}^{n-1} \eta_{j+1}^2 + \frac{n}{\sigma^2}, \end{aligned}$$

where $\epsilon_j = x_j - \theta$, and $\eta_{j+1} = \epsilon_{j+1} - \rho\epsilon_j$. (See [Hint](#) in part c))

We can easily infer that $\epsilon_1 \sim N(0, \sigma^2/(1-\rho^2))$ and $\eta_{j+1}|X_1=x_1, \dots, X_j=x_j = \eta_{j+1}|\epsilon_1, \eta_2, \dots, \eta_j \sim N(0, \sigma^2)$. From this, $\eta_2, \dots, \eta_n \sim \text{i.i.d. } N(0, \sigma^2)$ and these are independent of ϵ_1 . (???)

Therefore, the information matrix is given by taking expectation,

$$I(\theta, \sigma) = \begin{pmatrix} \frac{1-\rho^2+(n-1)(1-\rho)^2}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{pmatrix}.$$

□

- b) Give a lower bound for the variance of an unbiased estimator of θ .

Solution.

Since $g(\theta, \sigma) = \theta$,

$$LB = \nabla g(\theta, \sigma)^T I^{-1}(\theta, \sigma) \nabla g(\theta, \sigma) = (1, 0) I^{-1}(\theta, \sigma) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\sigma^2}{1 - \rho^2 + (n-1)(1-\rho)^2}.$$

□

- c) Show that the sample average $\bar{X} = (X_1 + \cdots + X_n)/n$ is an unbiased estimator of θ , compute its variance, and compare its variance with the lower bound.

Hint: Define $\epsilon_j = X_j - \theta$ and $\eta_{j+1} = \epsilon_{j+1} - \epsilon_j$. Use smoothing to argue that η_2, \dots, η_n are i.i.d. $N(0, \sigma^2)$ and are independent of ϵ_1 . Similarly, X_i is independent of $\eta_{i+1}, \eta_{i+2}, \dots$. Use these facts to find first $\text{Var}(X_2) = \text{Var}(\epsilon_2)$, then $\text{Var}(X_3), \text{Var}(X_4), \dots$. Finally find $\text{Cov}(X_{i+1}, X_i), n\text{Cov}(X_{i+2}, X_i)$, and so on.

Solution.

Let $\bar{\epsilon} = \frac{1}{n} \sum_j \epsilon_j = \bar{X} - \theta$. Since $E\epsilon_j = 0$, $E(\bar{X} - \theta) = E(\bar{\epsilon}) = 0$. Thus \bar{X} is unbiased estimator of θ .

Since $\epsilon_2 = \rho\epsilon_1 + \eta_2$, $\text{Var}(\epsilon_2) = \rho^2 \text{Var}(\epsilon_1) + \text{Var}(\eta_2) = \sigma^2 + \frac{\rho^2 \sigma^2}{1 - \rho^2} = \frac{\sigma^2}{1 - \rho^2}$. In general, $\epsilon_i = \rho\epsilon_{i-1} + \eta_i$ and we can obtain $\text{Var}(X_i) = \text{Var}(\epsilon_i) = \frac{\sigma^2}{1 - \rho^2}$.

If $i > j$,

$$\begin{aligned} \epsilon_i &= \rho\epsilon_{i-1} + \eta_i \\ &= \rho^2(\epsilon_{i-2} + \eta_{i-1}) + \eta_i \\ &= \cdots \\ &= \rho^{i-j}\epsilon_j + \rho^{i-j+1}\eta_{j+1} + \cdots + \rho\eta_{i-1} + \eta_i. \end{aligned}$$

From this,

$$\text{Cov}(X_i, X_j) = \text{Cov}(\epsilon_i, \epsilon_j) = \rho^{|i-j|} \frac{\sigma^2}{1 - \rho^2}.$$

Then, the variance of \bar{X} is

$$\begin{aligned} \text{Var}(\bar{X}) &= \text{Var}(\bar{\epsilon}) \\ &= \frac{1}{n^2} \text{Var} \left(\sum_{i=1}^n \epsilon_i \right) = \frac{1}{n^2} \text{Cov} \left(\sum_{i=1}^n \epsilon_i, \sum_{i=1}^n \epsilon_i \right) \\ &= \frac{1}{n^2} \left[\sum_{i=1}^n \text{Cov}(\epsilon_i, \epsilon_i) + 2 \sum_{i>j} \text{Cov}(\epsilon_i, \epsilon_j) \right] \\ &= \frac{\sigma^2}{n^2(1 - \rho^2)} [n + 2 \{ (n-1)\rho + (n-2)\rho^2 + \cdots + \rho^{n-1} \}] \\ &= \frac{\sigma^2}{n^2(1 - \rho^2)} \left[n + 2 \sum_{j=1}^{n-1} (n-j)\rho^j \right] \\ &= \frac{\sigma^2}{n^2(1 - \rho^2)} \left[n + 2 \left(n \frac{\rho - \rho^n}{1 - \rho} - \frac{\rho - \rho^{n+1}}{(1 - \rho)^2} + \frac{n\rho^n}{1 - \rho} \right) \right] \quad (*) \\ &= \frac{\sigma^2}{n^2(1 - \rho^2)} \left[n + 2 \frac{n\rho - n\rho^2 - \rho + \rho^{n+1}}{(1 - \rho)^2} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{\sigma^2}{n^2(1-\rho^2)} \left[\frac{n - 2n\rho + n\rho^2 + 2n\rho - 2n\rho^2 - 2\rho + 2\rho^{n+1}}{(1-\rho)^2} \right] \\
&= \frac{\sigma^2}{n^2(1-\rho^2)} \left[\frac{-n\rho^2 - 2\rho + n + 2\rho^{n+1}}{(1-\rho)^2} \right] \\
&= \frac{\sigma^2}{n^2(1-\rho^2)} \left[\frac{n(1-\rho^2) - 2(\rho - \rho^{n+1})}{(1-\rho)^2} \right] \\
&= \frac{\sigma^2}{n(1-\rho)^2} - \frac{2\sigma^2\rho(1-\rho^n)}{n^2(1-\rho^2)(1-\rho)^2}.
\end{aligned}$$

Note. (*) is obtained as follows:

$$\begin{aligned}
\sum_{j=1}^{n-1} \rho^j &= \frac{\rho(1-\rho^{n-1})}{1-\rho}, \\
\sum_{j=1}^{n-1} j\rho^j &= \sum_j \rho \frac{d}{d\rho} \rho^j = \rho \frac{d}{d\rho} \sum_{j=1}^{n-1} \rho^j \\
&= \rho \frac{d}{d\rho} \left(\frac{\rho - \rho^n}{1-\rho} \right) \\
&= \rho \frac{(1-n\rho^{n-1})(1-\rho) + (\rho - \rho^n)}{(1-\rho)^2} \\
&= \rho \frac{1-\rho - n\rho^{n-1} + n\rho^n + \rho - \rho^n}{(1-\rho)^2} \\
&= \rho \frac{1-\rho^n - n\rho^{n-1}(1-\rho)}{(1-\rho)^2} \\
&= \frac{\rho - \rho^{n+1}}{(1-\rho)^2} - \frac{n\rho^n}{1-\rho}.
\end{aligned}$$

□

Chapter 5

Curved Exponential Families

Problem 5.1

Suppose X has a binomial distribution with m trials and success probability θ , Y has a binomial distribution with n trials and success probability θ^2 , and X and Y are independent.

- a) Find a minimal sufficient statistic T .

Solution.

Note that $f(x) = \binom{m}{x}\theta^x(1-\theta)^{m-x}$ and $f(y) = \binom{n}{y}(\theta^2)^y(1-\theta^2)^{n-y}$. Then the joint density is

$$f(x, y) = \binom{m}{x} \binom{n}{y} \exp \left[x \log \frac{\theta}{1-\theta} + y \log \frac{\theta^2}{1-\theta^2} + m \log(1-\theta) + n \log(1-\theta^2) \right].$$

Since the canonical parameter $\eta = \left(\log \frac{\theta}{1-\theta}, \log \frac{\theta^2}{1-\theta^2} \right)$ does not satisfy a linear constraint, the joint density is the curved exponential family. Therefore, $T = (X, Y)$ is minimal sufficient. \square

- b) Show that T is not complete, providing a non-trivial function f with $E_\theta f(T) = 0$.

Solution.

Note that $ET_1 = EX = m\theta$, $ET_1^2 = EX^2 = m\theta(1-\theta) + m^2\theta^2$ and $ET_2 = EY = n\theta^2$.

Now we let

$$f(T) = T_1^2 - T_1 - \frac{m(m-1)}{n}T_2,$$

then $E_\theta f(T) = 0$ for all $\theta \in (0, 1)$. But $f(T)$ is not always 0. Thus T is not complete. \square

Problem 5.2

Let X and Y be independent Bernoulli variables with $P(X = 1) = p$ and $P(Y = 1) = h(p)$ for some known function h .

- a) Show that the family of joint distributions is a curved exponential family unless

$$h(p) = \frac{1}{1 + \exp \left\{ a + b \log \frac{p}{1-p} \right\}}$$

for some constants a and b .

Solution.

The joint density is

$$f(x, y) = p^x(1-p)^{1-x}h(p)^y(1-h(p))^{1-y} = \exp \left[x \log \left(\frac{p}{1-p} \right) + y \log \left(\frac{h(p)}{1-h(p)} \right) \right] (1-p)(1-h(p)).$$

If $h(p)$ defined as the above for some constants a and b , then

$$\log \left(\frac{h(p)}{1-h(p)} \right) = - \left(a + b \log \left(\frac{p}{1-p} \right) \right)$$

which is the linear constraint. Therefore, unless the above case, the family of joint densities is a curved exponential family. \square

- b) Give two functions h , one where (X, Y) is minimal but not complete, and one where (X, Y) is minimal and complete.

Solution.

- (i) Minimal but not complete

From a), the joint density is a curved exponential family except on the above h form.

If we let a function $h(p) = \frac{p}{2}$, $E_p(X - 2Y) = 0$ for all $p \in (0, 1)$. But $X - 2Y$ is not always 0. Therefore T is not complete.

- (ii) Minimal and complete

For some function g ,

$$E_p g(X, Y) = g(1, 1)ph(p) + g(1, 0)p(1-h(p)) + g(0, 1)(1-p)h(p) + g(0, 0)(1-p)(1-h(p)).$$

To show T is complete, we need to find a function h that $ph(p), p(1-h(p)), (1-p)h(p)$ are linearly independent.

If we let $h(p) = p^2$, then

$$\begin{aligned} E_p g(X, Y) &= p^3 [g(1, 1) - g(1, 0) - g(0, 1) + g(0, 0)] \\ &\quad + p^2 [g(0, 1) - g(0, 0)] + p [g(1, 0) - g(0, 0)] + g(0, 0). \end{aligned}$$

If this is 0 for all $p \in (0, 1)$, it must be $g(0, 0) = g(1, 0) = g(0, 1) = g(1, 1) = 0$. Thus $g(X, Y) = 0$ and T is complete. \square

Problem 5.6

Two teams A and B play a series of games, stopping as soon as one of the team has 4 wins. Assume that game outcomes are independent and that on any given game team A has a fixed chance θ of winning. Let X and Y denote the number of games won by the first and second team, respectively.

- a) Find the joint mass function for X and Y . Show that as θ varies these mass functions form a curved exponential family.

Solution.

First, suppose team A has 4 wins which is the event $X = 4$ and $Y = y$. Then, the first $(3 + y)$ trials are random and last trial should be the winning of team A, thus the probability be

$$P(X = 4, Y = y) = \binom{3+y}{3} \theta^4 (1-\theta)^y, \quad y = 0, 1, 2, 3.$$

Similarly,

$$P(X = x, Y = 4) = \binom{3+x}{3} \theta^x (1-\theta)^4, \quad x = 0, 1, 2, 3.$$

Therefore, the joint mass have the form $h(x, y) \exp [x \log \theta + y \log(1-\theta)]$. Since the canonical parameters $\log \theta$ and $\log(1-\theta)$ have non-linear relationship, the family of the joint mass is a curved exponential family. \square

b) Show that $T = (X, Y)$ is complete.

Solution.

Let g be an arbitrary function of X and Y , and suppose

$$E_{\theta} g(X, Y) = \sum_{x=0}^3 g(x, 4) \binom{3+x}{3} \theta^x (1-\theta)^4 + \sum_{y=0}^3 g(4, y) \binom{3+y}{3} \theta^4 (1-\theta)^y = 0, \quad \theta \in (0, 1).$$

Since it is the polynomial in θ , if $\theta \rightarrow 0$, then the only constant term $g(0, 4)$ is remain (other terms are tend to 0). Thus $g(0, 4)$ should be 0.

If $g(0, 4) = 0$, then the linear term $g(1, 4)$ also should be 0 (dividing by θ and letting $\theta \rightarrow 0$). Similarly $g(2, 4) = g(3, 4) = 0$.

Then, now remain only

$$\frac{E_{\theta} g(X, Y)}{\theta^4} = \sum_{y=0}^3 g(4, y) \binom{3+y}{3} (1-\theta)^y = 0,$$

and this is the polynomial in $1-\theta$. Similarly, we can obtain $g(4, 0) = \dots = g(4, 3) = 0$. Therefore, $g(X, Y) = 0$ almost surely.

By definition of completeness, T is complete. \square

c) Find a UMVU estimator of θ .

Solution.

Let $\delta = 1_{\{\text{team A wins the first game}\}}$. Then δ is unbiased because $E\delta = P(\text{team A wins the first game}) = \theta$. Thus the UMVU estimator of θ be $E[\delta|T] = P[\delta = 1|X, Y]$. By using the joint mass function,

$$\begin{aligned} E[\delta = 1|X = 4, Y = y] &= P(\delta = 1|X = 4, Y = y) \\ &= \frac{P(\delta = 1, X = 4, Y = y)}{P(X = 4, Y = y)} \\ &= \frac{\theta^2 \binom{2+y}{2} \theta^2 (1-\theta)^y}{\binom{3+y}{3} \theta^4 (1-\theta)^y} \\ &= \frac{3}{3+y}, \end{aligned}$$

and

$$P(\delta = 1|X = x, Y = 4) = \frac{\theta(1-\theta)\binom{2+x}{x-1}\theta^{x-1}(1-\theta)^3}{\binom{3+x}{3}\theta^x(1-\theta)^4} = \frac{x}{3+x}.$$

By combining 2 results, the UMVU estimator of θ is

$$\frac{X - 1_{\{X=4\}}}{X + Y - 1}.$$

□

Problem 5.7

Consider a sequential experiment in which observations are i.i.d. from a Poisson distribution with mean λ . If the first observation X is zero, the experiment stops, and if $X > 0$, a second observation Y is observed. Let $T = 0$ if $X = 0$, and let $T = 1 + X + Y$ if $X > 0$.

- a) Find the mass function for T .

Solution.

$$P(T = 0) = P(X = 0) = e^{-\lambda}.$$

For $k = 0, 1, \dots$,

$$\begin{aligned} P(T = k + 1) &= P(X + Y = k, X > 0) \\ &= P(X + Y = k) - P(X + Y = k, X = 0) \\ &= \frac{(2\lambda)^k e^{-2\lambda}}{k!} - \frac{\lambda^k e^{-2\lambda}}{k!}. \end{aligned}$$

이게 왜 아니지??

$$\begin{aligned} P(T = k + 1) &= P(X + Y = k, X > 0) \\ &= P(X + Y = k) - P(X + Y = k, X = 0) \\ &= P(X + Y = k) - P(Y = k) \\ &= \frac{(2\lambda)^k e^{-2\lambda}}{k!} - \frac{\lambda^k e^{-\lambda}}{k!}. \end{aligned}$$

□

- b) Show that T is minimal sufficient.

Solution.

Part c) first!!

From c), (W, N) is minimal sufficient. Therefore T that is one-to-one function of (W, N) is also minimal sufficient. □

- c) Does this experiment give a curved two-parameter exponential family or full rank one-parameter exponential family?

Solution.

Let N is the sample size(or # of experiments), and W as follows:

$$N = \begin{cases} 1, & \text{if } X = 0, \\ 2, & \text{if } X > 0, \end{cases} \quad W = \begin{cases} 0, & \text{if } X = 0, \\ X + Y, & \text{if } X > 0. \end{cases}$$

Note that X and Y are from i.i.d. $\text{Poisson}(\lambda)$ and it satisfies the condition in Theorem 5.4.

By Theorem 5.4, the joint density is an exponential family with canonical parameter $\eta = (\log \lambda, -\lambda)$ and sufficient statistic $(X + Y, N) = (W, N)$.

Since η does not satisfy a linear constraint, the density of W and N is a curved exponential family. Thus, (W, N) is minimal sufficient. \square

d) Is T a complete sufficient statistic?

Hint: Write $e^\lambda E_\lambda g(T)$ as a power series in λ and derive equations for g setting coefficients for λ^x to zero.

Solution.

Suppose $E_\lambda g(T) = 0$ for all $\lambda > 0$ and any function g . Then

$$E_\lambda g(T) = g(0)e^{-\lambda} + \sum_{k=0}^{\infty} g(k+1) \frac{(2\lambda)^k e^{-2\lambda} - \lambda^k e^{-\lambda}}{k!} = 0.$$

Multiplying $e^{2\lambda}$,

$$e^{2\lambda} E_\lambda g(T) = g(0)e^\lambda + \sum_{k=0}^{\infty} \frac{g(k+1)(2^k - 1)}{k!} \lambda^k = 0.$$

This is the power series in λ , so the coefficients of the power should be 0. ($g(0) = 0$ and $g(k+1) = 0$ for $k = 1, 2, \dots$)

Note that $T \neq 1$ (· If $X > 0$, then $Y \geq 0$ and $T > 1$). Therefore, $g(T) = 0$ thus T is complete. \square

Problem 5.13

Consider a single two-way contingency table and define $R = N_{11} + N_{12}$ (the first row sum), $C = N_{11} + N_{21}$ (the first column sum), and $D = N_{11} + N_{22}$ (the sum of the diagonal entries).

a) Show that the joint mass function can be written as a full rank three-parameter exponential family with $T = (R, C, D)$ as the canonical sufficient statistic.

Solution.

The contingency table be as follows:

N_{11}	N_{12}	R
N_{21}	N_{22}	
C		n

By representing the elements with R, C, D , and n ,

$$N_{11} = (R + C + D - n)/2, \quad N_{12} = (n - D - C + R)/2,$$

$$N_{21} = (n - D - R + C)/2, \quad N_{22} = (n + D - R - C)/2.$$

Thus the joint mass function is

$$\begin{aligned} \binom{n}{n_{11}, \dots, n_{22}} p_{11}^{n_{11}} \cdots p_{22}^{n_{22}} &= \binom{n}{n_{11}, \dots, n_{22}} (\sqrt{p_{11}})^{r+c+d-n} (\sqrt{p_{12}})^{n-d-c+r} (\sqrt{p_{21}})^{n-d-r+c} (\sqrt{p_{22}})^{n+d-r-c} \\ &= \binom{n}{n_{11}, \dots, n_{22}} \sqrt{\frac{p_{11}p_{12}}{p_{21}p_{22}}}^r \sqrt{\frac{p_{11}p_{21}}{p_{12}p_{22}}}^c \sqrt{\frac{p_{11}p_{22}}{p_{12}p_{21}}}^d \sqrt{\frac{p_{12}p_{21}p_{22}}{p_{11}}}^n \\ &= \binom{n}{n_{11}, \dots, n_{22}} \exp \left[r \log \sqrt{\frac{p_{11}p_{12}}{p_{21}p_{22}}} + c \log \sqrt{\frac{p_{11}p_{21}}{p_{12}p_{22}}} \right. \\ &\quad \left. + d \log \sqrt{\frac{p_{11}p_{22}}{p_{12}p_{21}}} + n \log \sqrt{\frac{p_{12}p_{21}p_{22}}{p_{11}}} \right], \end{aligned}$$

where r, c , and d are defined as before.

The density is a full-rank(3-parameter) exponential family, and the sufficient statistic (R, C, D) do not satisfy a linear constraint because there is a one-to-one linear association between it and (N_{11}, N_{12}, N_{21}) , and the 3 canonical parameters η_r, η_c , and η_d can vary freely over \mathbb{R}^3 . **마지막 부분 잘 이해 안됨,,,** \square

- b) Relate the canonical parameter associated with D to the "cross-product ratio" α defined as $\alpha = p_{11}p_{22}/(p_{12}p_{21})$.

Solution.

$$\eta_d = \log \sqrt{\alpha} \quad \square$$

- c) Suppose we observe m independent two-by-two contingency tables. Let n_i , $i = 1, \dots, m$, denote the trials for table i . Assume that cell probabilities for the tables may differ, but that the cross-product ratios for all m tables are all the same. Show that the joint mass functions form a full rank exponential family. Express the sufficient statistic as a function of the variables $R_1, \dots, R_m, C_1, \dots, C_m$, and D_1, \dots, D_m .

Solution.

The joint mass can be obtained as

$$\exp \left[\sum_{i=1}^m r_i \eta_{r,i} + \sum_{i=1}^m c_i \eta_{c,i} + \eta_d \sum_{i=1}^m d_i - \sum_{i=1}^m A_i(\eta_i) \right] h(x),$$

which is a full-rank($2m + 1$ parameter) exponential family. Therefore, the complete sufficient statistic is

$$T = \left(R_1, \dots, R_m, C_1, \dots, C_m, \sum_{i=1}^m D_i \right),$$

which is a function of $R_1, \dots, R_m, C_1, \dots, C_m$. \square

Problem 5.16

For an $I \times J$ contingency table with independence, the UMVU estimator of p_{ij} is $\hat{p}_{i+}\hat{p}_{+j} = N_{i+}N_{+j}/n^2$.

- a) Determine the variance of this estimator, $\text{Var}(\hat{p}_{i+}\hat{p}_{+j})$.

Solution.

Since N_{i+} and N_{+j} are independent with

$$N_{i+} \sim \text{Binomial}(n, p_{i+}), \quad N_{+j} \sim \text{Binomial}(n, p_{+j}).$$

$$E\hat{p}_{i+} = \frac{1}{n}EN_{i+} = p_{i+} \text{ and similarly, } E\hat{p}_{+j} = p_{+j}.$$

$$\begin{aligned} E\hat{p}_{i+}^2\hat{p}_{+j}^2 &= \frac{1}{n^4}EN_{i+}^2N_{+j}^2 \\ &= \frac{1}{n^4} [np_{i+}(1-p_{i+}) + n^2p_{i+}^2] [np_{+j}(1-p_{+j}) + n^2p_{+j}^2] \\ &= \frac{1}{n^2} [p_{i+}(1-p_{i+}) + np_{i+}^2] [p_{+j}(1-p_{+j}) + np_{+j}^2]. \end{aligned}$$

$$\begin{aligned} \therefore \text{Var}(\hat{p}_{i+}\hat{p}_{+j}) &= E\hat{p}_{i+}^2\hat{p}_{+j}^2 - [E\hat{p}_{i+}\hat{p}_{+j}]^2 \\ &= \frac{1}{n^2} [p_{i+}(1-p_{i+}) + np_{i+}^2] [p_{+j}(1-p_{+j}) + np_{+j}^2] - p_{i+}^2p_{+j}^2 \\ &= \frac{1}{n^2} [p_{i+}(1-p_{i+})p_{+j}(1-p_{+j}) + np_{i+}(1-p_{i+})p_{+j}^2 + np_{i+}^2p_{+j}(1-p_{+j})] \\ &= \frac{p_{i+}(1-p_{i+})p_{+j}^2 + p_{+j}(1-p_{+j})p_{i+}^2}{n} + \frac{p_{i+}(1-p_{i+})p_{+j}(1-p_{+j})}{n^2}. \end{aligned}$$

□

b) Find the UMVU estimator of the variance in (a).

Solution.

First, we should find the unbiased estimators in the numerator term.

$$\begin{aligned} EN_{i+}^2 &= np_{i+}(1-p_{i+}) + n^2p_{i+}^2 \\ &= np_{i+} - np_{i+}^2 + n^2p_{i+}^2 \\ &= EN_{i+} + (n^2 - n)p_{i+}^2 \\ \Rightarrow E\left(\frac{N_{i+}(N_{i+} - 1)}{n^2 - n}\right) &= p_{i+}^2, \end{aligned}$$

$$\begin{aligned} p_{i+}(1-p_{i+}) &= p_{i+} - p_{i+}^2 \\ &= E\frac{N_{i+}}{n} - E\left(\frac{N_{i+}(N_{i+} - 1)}{n^2 - n}\right) \\ &= E\left(\frac{N_{i+}(n - N_{i+})}{n^2 - n}\right), \end{aligned}$$

From the above, we obtain the unbiased estimators as follows:

$$\begin{aligned} E\left(\frac{N_{i+}(N_{i+} - 1)}{n^2 - n}\right) &= p_{i+}^2, \\ E\left(\frac{N_{i+}(n - N_{i+})}{n^2 - n}\right) &= p_{i+}(1-p_{i+}), \\ E\left(\frac{N_{+j}(N_{+j} - 1)}{n^2 - n}\right) &= p_{+j}^2, \end{aligned}$$

$$E \left(\frac{N_{+j}(n - N_{+j})}{n^2 - n} \right) = p_{+j}(1 - p_{+j}).$$

By using these, the expectation of $\text{Var}(\hat{p}_{i+}\hat{p}_{+j})$ is

$$\begin{aligned} E[\text{Var}(\hat{p}_{i+}\hat{p}_{+j})] &= E \left[\frac{p_{i+}(1 - p_{i+})p_{+j}^2 + p_{+j}(1 - p_{+j})p_{i+}^2}{n} + \frac{p_{i+}(1 - p_{i+})p_{+j}(1 - p_{+j})}{n^2} \right] \\ &= \frac{N_{i+}(n - N_{i+})N_{+j}(N_{+j} - 1) + N_{+j}(n - N_{+j})N_{i+}(N_{i+} - 1)}{n(n^2 - n)^2} \\ &\quad + \frac{N_{i+}(n - N_{i+})N_{+j}(n - N_{+j})}{n^2(n^2 - n)^2}. \end{aligned} \quad (*)$$

Therefore, $(*)$ is UMVU estimator of $\text{Var}(\hat{p}_{i+}\hat{p}_{+j})$. □

Chapter 6

Conditional Distributions

Problem 6.1

a) Solution.

□

b) Solution.

□

Problem 6.2

Solution.

□