# Theoretical Statistics: Topics for a Core Course
# Problem Solutions

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# Contents

# Preface

This note contains the solution of the problems in the textbook, *Theoretical Statistics: Topics for a Core Course*, and it was created by Hyunsung Kim, who is a Ph.D. student. I wrote it when I study a theoretical statistics based on this textbook on my own by solving some problems and also referred to the solution manual in the textbook.
It contains a few selected problems in the textbook what I studied, and also note that it may not be the exact solutions. If you want to refer to this note, you should study with doubt about the answer.

## Textbook

- Keener, *Theoretical Statistics: Topics for a Core Course.*

## Reference


- Durrett, *Probability: Theory and Examples, 5th edition.*
- Royden, *Real Analysis, 4th edition.*

# Chapter 1

# Probability and Measure

## Problem 1.1

Prove (1.1). If measurable sets $B_n,\ n \geq 1$, are increasing, with $B = \bigcup_{n=1}^{\infty} B_n$, called the limit of the sequence, then

$$\mu(B) = \lim_{n\to\infty} \mu(B_n).$$

***Solution***.
First, we showed that $A_n$'s are disjoint. If $j < k$, then $B_j \subseteq B_{k-1}$.
Since $A_j \subset B_j \subseteq B_{k-1}$ and $A_k \subset B_{k-1}^c$, $A_j$ and $A_k$ are disjoint.
Also $B_n = \bigcup_{j=1}^{n} A_j$ and $\bigcup_{n=1}^{\infty} A_n = B$,

$$\mu(B) = \sum_{i=1}^{\infty} \mu(A_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} A_i\right) = \lim_{n\to\infty} \mu(B_n).$$

□

## Problem 1.8

Prove *Boole's inequality*: For any events $B_1, B_2, \ldots,$

$$P\left(\bigcup_{i\geq 1} B_i\right) \leq \sum_{i\geq 1} P(B_i).$$

***Solution***.
Let $B = \bigcup_{i=1}^{\infty} B_i$, then $1_B \leq \sum 1_{B_i}$. Then by Fubini's theorem,

$$P(B) = \int 1_B dP \leq \int \sum 1_{B_i} dP = \sum \int 1_{B_i} dP = \sum P(B_i).$$

□

## Problem 1.10

Let $\mu$ and $\nu$ be measures on $(\mathcal{E}, \mathcal{B})$.

a) Show that the sum $\eta$ defined by $\eta(B) = \mu(B) + \nu(B)$ is also a measure.

***Solution***.
(i) For arbitrary set $A \in \mathcal{B}$, $\mu(A) \geq 0$ and $\nu(A) \geq 0$. $\Rightarrow \eta(A) = \mu(A) + \nu(A) \geq 0$.
$\therefore \eta : \mathcal{B} \to [0, \infty]$.

(ii) For disjoint set $B_1, B_2, \cdots \subset \mathcal{B}$, then

$$\begin{aligned}\eta\left(\bigcup_{i=1}^{\infty} B_i\right) &= \mu\left(\bigcup B_i\right) + \nu\left(\bigcup B_i\right) \\ &= \sum \mu(B_i) + \sum \nu(B_i) \\ &= \sum \left\{\mu(B_i) + \nu(B_i)\right\} \\ &= \sum \eta(B_i)\end{aligned}$$

$\therefore \eta$ is a measure. □

b) If $f$ is a non-negative measurable function, show that

$$\int f d\eta = \int f d\mu + \int f d\nu.$$

***Solution***.
(i) Let $f = \sum_{i=1}^{n} a_i 1_{A_i}$, the non-negative simple function. Then,

$$\begin{aligned}\int f d\eta &= \int \sum_{i=1}^{n} a_i 1_{A_i} d\eta = \sum a_i \eta(A_i) \\ &= \sum a_i \left\{\mu(A_i) + \nu(A_i)\right\} = \int f d\mu + \int f d\nu.\end{aligned}$$

(ii) For general case, let $f_n$ is the sequence of non-negative simple functions increasing to $f$. (i.e. $f_1 \leq f_2 \leq \cdots \leq f$)

$$\begin{aligned}\int f d\eta &= \lim_{n\to\infty} \int f_n d\eta = \lim_{n\to\infty} \left(\int f_n d\mu + \int f_n d\nu\right) \\ &= \lim_{n\to\infty} \int f_n d\mu + \lim_{n\to\infty} \int f_n d\nu = \int f d\mu + \int f d\nu.\end{aligned}$$

□

## Problem 1.11

Suppose $f$ is the simple function $1_{(1/2,\pi]} + 21_{(1,2]}$, and let $\mu$ be a measure on $\mathbb{R}$ with $\mu\{(0,a^2]\} = a,\ a > 0$. Evaluate $\int f d\mu$.

***Solution***.
By the integral of simple function and the finite additivity, it can be simply computed as

$$\begin{aligned}\int f d\mu &= \mu\{(1/2,\pi]\} + 2\mu\{(1,2]\} \\ &= [\mu\{(0,\pi]\} - \mu\{(1/2,\pi]\}] + 2\,[\mu\{(0,2]\} - \mu\{(1,2]\}] \quad \text{(finite additivity)} \\ &= \left(\sqrt{\pi} - 1/\sqrt{2}\right) + 2\left(\sqrt{2} - 1\right)\end{aligned}$$

□

## Problem 1.12

Suppose that $\mu\{(0,a)\} = a^2$ for $a > 0$ and that $f$ is defined by

$$f(x) = \begin{cases} 0, & x \le 0, \\ 1, & 0 < x < 2, \\ \pi, & 2 \ge x < 5, \\ 0, & x \ge 5. \end{cases}$$

Compute $\int f d\mu$.

***Solution***.
Since $f = 1_{(0,2)} + 1_{[2,5)}$ is simple, the integral can be computed as

$$\begin{aligned}\int f d\mu &= \mu\{(0,2)\} + \pi\mu\{[2,5)\} \\ &= \mu\{(0,2)\} + \pi\,[\mu\{(0,5)\} - \mu\{(0,2)\}] \\ &= 4 - \pi(25 - 4)\end{aligned}$$

□

## Problem 1.13

Define the function $f$ by

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find simple functions $f_1 \le f_2 \le \cdots$ increasing to $f$ (i.e. $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in \mathbb{R}$). Let $\mu$ be Lebesgue measure on $\mathbb{R}$. Using our formal definition of an integral and the faProblemct that

$\mu\big((a,b]\big) = b - a$ whenever $b > a$ (this might be used to formally define Lebesgue measure), show that $\int f d\mu = 1/2$.

***Solution***.
Let $f_n = \lfloor 2^n x \rfloor / 2^n$ for $0 < x \leq 1$ and 0 otherwise. ($\lfloor y \rfloor$ is a floor function.) Then,

$$f_1(x) = \lfloor 2x \rfloor / 2 = \begin{cases} 0, & x < 1/2, \\ 1/2, & 1/2 \leq x < 1, \\ 1, & x = 1 \end{cases}$$

$$f_2(x) = \lfloor 2^2 x \rfloor / 2^2 = \begin{cases} 0, & x < 1/2^2, \\ 1/2^2, & 1/2^2 \leq x < 2/2^2, \\ 2/2^2, & 2/2^2 \leq x < 3/2^2, \\ 3/2^2, & 3/2^2 \leq x < 1, \\ 1, & x = 1 \end{cases}$$

$$\vdots$$

$$f_n(x) = \lfloor 2^n x \rfloor / 2^n = \begin{cases} 0, & x < 1/2^n, \\ 1/2^n, & 1/2^n \leq x < 2/2^n, \\ \vdots \\ (2^n - 1)/2^n, & (2^n - 1)/2^n \leq x < 1, \\ 1, & x = 1 \end{cases}$$

$$\begin{aligned} \therefore \int f_n d\mu &= \frac{1}{2^n}\left(\frac{1}{2^n} + \frac{2}{2^n} + \cdots + \frac{2^n - 1}{2^n}\right) \\ &= \frac{1 + 2 + \cdots + (2^n - 1)}{4^n} \\ &= \frac{2^n(2^n - 1)}{2 \cdot 4^n} \\ &\longrightarrow \frac{1}{2}. \end{aligned}$$

My solution
Let the simple function $f_n = \sum_{i=1}^n \frac{i}{n} 1_{(\frac{i-1}{n}, \frac{i}{n}]}$. Then,

$$\int f_n d\mu = \sum_{i=1}^n \frac{i}{n}\frac{1}{n} = \frac{1}{n^2}\frac{n(n+1)}{2} \longrightarrow \frac{1}{2}.$$

□

## Problem 1.16

Define $F(a-) = \lim_{x\uparrow a} F(x)$. Then, if $F$ is non-decreasing, $F(a-) = \lim_{n\to\infty} F(a - 1/n)$. Use (1.1)[Continuity of measure] to show that if a random variable $X$ has cumulative distribution function $F_X$,

$$P(X < a) = F_X(a-).$$

Also, show that

$$P(X = a) = F_X(a) - F_X(a-).$$

***Solution***.
(i) Let $B_n = \{X \le a - 1/n\}$ and $\bigcup_{n=1}^{\infty} B_n = \{X < a\}$. By continuity of measure,

$$P(B) = P(X < a) = \lim_{n\to\infty} P(X \le a - 1/n) = \lim_{n\to\infty} F_X(a - 1/n) = F_X(a-).$$

(ii) Since $\{X < a\}$ and $\{X = a\}$ are disjoint with union $\{X \le a\}$,

$$P(X < a) + P(X = a) = P(X \le a).$$

$$\therefore P(X = a) = F_X(a) - F_X(a-).$$

□

## Problem 1.17

Suppose $X$ is a geometric random variable with mass function

$$p(x) = P(X = x) = \theta(1-\theta)^x, \quad x = 0, 1, \ldots,$$

where $\theta \in (0, 1)$ is a constant. Find the probability that $X$ is even.

***Solution***.

$$\begin{aligned} P(X \text{ is even}) &= P(X = 0) + P(X = 2) + P(X = 4)\cdots \\ &= \theta + \theta(1-\theta)^2 + \theta(1-\theta)^4 + \cdots \\ &= \frac{\theta}{1-(1-\theta)^2} \\ &= \frac{1}{2-\theta} \end{aligned}$$

□

## Problem 1.18

Let $X$ be a function mapping $\mathcal{E}$ into $\mathbb{R}$. Recall that if $B$ is a subset of $\mathbb{R}$, then $X^{-1}(B) = \{e \in \mathcal{E} : X(e) \in B\}$. Use this definition to prove that

$$X^{-1}(A \cap B) = X^{-1}(A) \cap X^{-1}(B),$$

$$X^{-1}(A \cup B) = X^{-1}(A) \cup X^{-1}(B),$$

and

$$X^{-1}\left(\bigcup_{i=0}^{\infty} A_i\right) = \bigcup_{i=0}^{\infty} X^{-1}(A_i).$$

***Solution***.
(i)

$$\begin{aligned} e \in X^{-1}(A \cap B) &\Leftrightarrow X(e) \in A \cap B \\ &\Leftrightarrow X(e) \in A \text{ and } X(e) \in B \\ &\Leftrightarrow e \in X^{-1}(A) \text{ and } e \in X^{-1}(B) \\ &\Leftrightarrow e \in X^{-1}(A) \cap X^{-1}(B). \end{aligned}$$

(ii) Similarly,

$$\begin{aligned} e \in X^{-1}(A \cup B) &\Leftrightarrow X(e) \in A \cup B \\ &\Leftrightarrow X(e) \in A \text{ or } X(e) \in B \\ &\Leftrightarrow e \in X^{-1}(A) \text{ or } e \in X^{-1}(B) \\ &\Leftrightarrow e \in X^{-1}(A) \cup X^{-1}(B). \end{aligned}$$

(iii)

$$\begin{aligned} e \in X^{-1}(\bigcup_{i=0}^{\infty} A_i) &\Leftrightarrow X(e) \in \bigcup_{i=0}^{\infty} A_i \\ &\Leftrightarrow X(e) \in A_i \text{ for some } i \\ &\Leftrightarrow e \in X^{-1}(A_i) \text{ for some } i \\ &\Leftrightarrow e \in \bigcup_{i=0}^{\infty} X^{-1}(A_i). \end{aligned}$$

□

## Problem 1.19

Let $P$ be a probability measure on $(\mathcal{E}, \mathcal{B})$, and let $X$ be a random variable. Show that the distribution $P_X$ of $X$ defined by $P_X(B) = P(X \in B) = P(X^{-1}(B))$ is a measure (on the Borel sets of $\mathbb{R}$).

***Solution***.
(i) $P_X(B) = P\big(X^{-1}(B)\big) \geq 0$.
(ii) From Problem 1.18, $X^{-1}\left(\bigcup_i A_i\right) = \bigcup_i X^{-1}(A_i)$ is hold.
Also, if $B_i$ and $B_j$ are disjoint, then $X^{-1}(B_i)$ and $X^{-1}(B_j)$ are disjoint. ($\because$ If $e \in \mathcal{E}$ lie in both $X^{-1}(B_1)$ and $X^{-1}(B_2)$, $X(e)$ lies in $B_1$ and $B_2$.)
Therefore, for arbitrary disjoint set $B_1, B_2, \ldots$ with union $B = \bigcup_i B_i$, $X^{-1}(B_1), X^{-1}(B_2), \ldots$ are also disjoint with union $X^{-1}(B)$.

$$\therefore \sum_i P_X(B_i) = \sum_i P\big(X^{-1}(B_i)\big) = P\big(X^{-1}(B)\big) = P_X(B).$$

The above 2 conditions are hold, $P_X$ is a measure. □