Pricing

(Stochastic Processes and Financial Applications)

Hélyette GEMAN

Lecture 3 (22 Jan. 2013)

1 Recapitulation

For a given stock S, the first derivatives that we can build are the forward contracts with expiry T', whose price we denote $f^{T'}(t)$, and the future contracts with expiry T', whose price we denote $F^{T'}(t)$. Due to the absence of arbitrage, we can assert the relation of their price at maturity $f^{T'}(T') = S(T') = F^{T'}(T')$. The primary differences are:

Forwards	Futures
over-the-counter	exchange-traded
sparse quotes	continuously observed prices
customised	standardized
no deposit	margin deposits
no interim cash flows	daily cash flows (margin calls) of $F^{T'}(t+1) - F^{T'}(t)$

We would like to understand the relation of $f^{T'}(t)$ and $F^{T'}(t)$ before maturity, i.e. at $t < T'$. We recall:

Theorem 1 (Cox Ingersoll Ross (1981))**.** *Assuming no credit risk in the economy, and*

1. *constant interest rates, then* $F^{T'}(t) = f^{T'}(t)$
2. *stochastic interest rates, then* $F^{T'}(t) = f^{T'}(t)$ *as long as there is no correlation between* S *and the interest rate*

2 Effect of Interest Rates

Let us consider the assets and situations we have seen to date:

1. In the Black-Scholes (1976) setting, interest rates are constant and $F^{T'}(t) = f^{T'}(t)$
2. If interest rates are stochastic, and $T' >> t$, i.e. we are sufficiently far from maturity for interest rates to matter:
   (a) If the underlying S is a commodity such as wheat or corn, then we can safely say that $\mathrm{Corr}(S, r) = 0$, and then $F^{T'}(t) = f^{T'}(t)$.
   (b) If S is a stock, then changes in the stock price are inversely correlated to the interest rate (when interest rates decrease, stock prices increase), so $F^{T'}(t) \neq f^{T'}(t)$.

Remark 1. In banks, there is frequently a desk called "Delta One" (i.e. all derivatives must have $\Delta = 1$) that deals with this: they trade forwards and futures knowing their price differs and trying to identify mispricings. The \$6 billion loss incurred by Jérôme Kerviel at Société Générale occured while he was working on the Delta One desk. Theoretically he was hedged, long $F^{T'}$ and short $f^{T'}$, but in reality he did not have a position in $f^{T'}$.

(c) If S is a currency, then as exchange rate differentials are highly correlated to interest rate differentials, $\mathrm{Corr}(S, r) \neq 0$, so $F^{T'}(t) \neq f^{T'}(t)$.

3. If we incorporate credit risk into the analysis, then if the long counterparty has a better credit profile we will have $F^{T'}(t) > f^{T'}(t)$. The difference between the two will increse with the credit spread.

4. In the situation of stochastic interest rates and any general asset S, then $F^{T'}(t) \neq f^{T'}(t)$.

We mentioned last week (with no proof) the "spot-forward" relationship in the case of the constant interest rate. We will now extend it to stochastic interest rates.

Theorem 2. *Let S be a generic risky asset (with no cash flows and making no assumptions on the dynamics of $S(t)$) and let $B^{T'}$ be a zero-coupon bond paying \$1 at date $B^{T'}$ (with price $B^{T'}(t)$ at time t). Then*

$$f^{T'}(t) = \frac{S(t)}{B^{T'}(t)}$$

and, since $B^{T'}(t) < 1$ when $t < T'$

$$f^{T'}(t) < S(t)$$

Proof. We will invoke the no-arbitrage condition and, as always, build a replicating portfolio. At time t, we sell a forward contract written on S expiring at time T'. We buy S and have negative cash flow of $-S(t)$. We finance this by borrowing $S(t)$. At time T', we deliver S, receive our forward price $f^{T'}(t)$, and repay our loan with interest, $\frac{S(t)}{B^{T'}(t)}$. Our cash flow (and portfolio value) looks like this:

at time t		at time T'	
action	cash flow	action	cash flow
borrow cash	$+S(t)$	repay loan	$-\frac{S(t)}{B^{T'}(t)}$
buy stock	$-S(t)$	deliver stock	0
write a forward	0	recive forward price	$f^{T'}(t)$
total $V_p(t)$	0	total $V_p(T')$	$f^{T'}(t) - \frac{S(t)}{B^{T'}(t)}$

Since the value of our portfolio at time t is zero, and we have no interim cash flows, then we known by no arbitrage that the value of our portfolio at time $T' > t$ also must be zero. Therefore

$$f^{T'}(t) - \frac{S(t)}{B^{T'}(t)} = 0 \qquad \Longrightarrow \qquad f^{T'}(t) = \frac{S(t)}{B^{T'}(t)}$$

$\square$

We can now consider variations of this argument, with the same starting assumptions, for different assets:

1. Let S be a stock paying a continuous dividend q under stochastic interest rates. Recall that when you own S over the period $(t, t + \mathrm{d}t)$ you receive the dividend amount $g\,\mathrm{d}t \cdot S(t)$. We immediately reinvest this dividend, i.e. purchase more shares of S at the prevaling price $S(t + \mathrm{d}t)$. The number of shares owned thus grows at the rate g; i.e., if we start at date t with one share, than at T' we will have $e^{q(T'-t)}$ shares.

   Our replicating portfolio works as follows: at time t, we buy $e^{-q(T'-t)}$ shares for each share of S we need to deliver according to the contract f, and have negative cash flow of $-S(t)e^{-q(T'-t)}$. We finance this by borrowing $S(t)e^{-q(T'-t)}$, and sell a forward contract written on S expiring at time T'. At time T', we deliver the resulting shares of S, receive our forward price $f^{T'}(t)$, and repay our loan. Our cash flow looks like this:

at time t		at time T'	
action	cash flow	action	cash flow
borrow cash	$+S(t)e^{-q(T'-t)}$	repay loan	$-\frac{S(t)e^{-q(T'-t)}}{B^{T'}(t)}$
buy stock	$-S(t)e^{-q(T'-t)}$	deliver stock	0
write a forward	0	recive forward price	$f^{T'}(t)$
total $V_p(t)$	0	total $V_p(T')$	$f^{T'}(t) - \frac{S(t)e^{-q(T'-t)}}{B^{T'}(t)}$

   As before, the initial portfolio value is zero, so we can invoke the no-arbitrage condition to get:

$$f^{T'}(t) = \frac{S(t)e^{-q(T'-t)}}{B^{T'}(t)}$$

2. Let S be a commodity under stochastic interet rates. Ownership of a commodity gives you the benefit of a convenience yield y (Kaldoz 1949). Under stochastic interest rates we can consider y to be equivalent to the dividend yield of a stock, and so by the above argument:

$$f^{T'}(t) = \frac{S(t)e^{-y(T'-t)}}{B^{T'}(t)}$$

3. Let S be a currency under stochastic interest rates. We need to consider which economy of the currency pair we are in. As an example, let the domestic currency be U.S. dollars ($) and the foreign currency be Japanese yen (¥). Let $X(t)$ be the number of dollars I need at date t to buy one unit of yen. If I need to make a payment in yen in 6 months (at T'), I can either buy the yen today in the spot market or I can buy a forward contract with maturity T'. Under no-arbitrage, these two transactions should have the same price:

$$f^{T'}(t) = \frac{X(t)B^{T'}_{\text{foreign}}(t)}{B^{T'}_{\text{domestic}}(t)}$$

   That is, the currency behaves like a dividend-paying *domestic* stock with the dividend rate equal to the short-term rate in the *foreign* economy.

3 Valuing Risky Cash Flows Under Stochastic Interest Rates

Recall we showed that in the Black-Scholes setting, under the assumption of no arbitrage, we were able to construct a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that the discounted prices $S_j(t)e^{-rt}$ of the primitive securities S_j (including the money market account S_0) were $\mathbb{Q}$-martingales.

Recall also that the assumption of "constant rates" in the Black-Scholes setting was sloppy. In continuous time, we should specify that the interest rate of significance is the short-term (or overnight) rate $r(t)$ prevailing on $(t, t + \mathrm{d}t)$. This rate is also the base rate against which other variable rates (such as mortgages) are defined, and which is affected by monetary policy.

Consider the money market account:

$$M(t + \mathrm{d}t) = M(t) + r(t)M(t)\,\mathrm{d}t \qquad \Longrightarrow \qquad dM(t) = r(t)M(t)\,\mathrm{d}t$$

In the particular case when the short-term rate is constant, $r(t) = r$, we have the ODE

$$dM(t) = rM(t)\,\mathrm{d}t$$

which has solution

$$M(t) = e^{rt}M(0) \qquad \text{or} \qquad M(t) = e^{rt}$$

if, as usual, we define $M(0) = 1$.

For the general case $r(t)$ is not constant: it evolves randomly over time. The solution then becomes

$$M(t) = M(0)\exp\left\{\int_0^t r(s)\,\mathrm{d}s\right\} \qquad \text{or} \qquad M(t) = \exp\left\{\int_0^t r(s)\,\mathrm{d}s\right\}$$

if $M(0) = 1$. At time 0, this quantity is unknown, and so the money market account is no longer risk-free. Hence, results need to be adjusted.

Let $\widetilde{\Phi}_T$ denote a contingent claim paying a unique cashflow at date T that is *attainable*, that is, it has at least one replicating portfolio build from primitive securities. We will show that

$$V_t(\widetilde{\Phi}_T) = \mathbb{E}_{\mathbb{Q}}\left[\widetilde{\Phi}_T e^{-\int_t^T r(s)ds} \middle/ \mathcal{F}_t\right]$$

Consider two particular cases:

1. Return for a second to constant interest rates. Then clearly the discount factor can be moved outside the expectation (see the second proof of the Black-Scholes formula).
2. Consider the case when $\widetilde{\Phi}_T$ is constant. Assume that $\widetilde{\Phi}_T = 1$. Then by no-arbitrage we have:

$$V_t(\widetilde{\Phi}_T) = \mathbb{E}_{\mathbb{Q}}\left[1 \cdot e^{-\int_t^T r(s)ds} \middle/ \mathcal{F}_t\right] = B^T(t)$$

where B^T are short-term bonds, and by definition

$$B^T(t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \middle/ \mathcal{F}_t\right]$$

B^T has beautiful properties:

(a) It appears in the equation as a discount factor with no assumptions about interest rates.

(b) It is actively and continuously traded in the market, and remarkably liquid.

(c) At maturity, $B^T(T) = 1$.

We see that in contrast,

$$M(T) = M(t)\,\mathbb{E}_{\mathbb{Q}}\left[e^{\int_t^T r(s)ds} \Big/ \mathcal{F}_t\right]$$

so there is nothing remarkable about $M(T)$, and we prefer to use B_T in this case.

Recall our FFTAP,[1] which stated that by the no-arbitrage condition, $S_j e^{-rt}$ are $\mathbb{Q}$-martingales. In fact, assuming $M(0) = 1$, then

$$S_j(t)e^{-rt} = \frac{S_j(t)}{M(t)}$$

under the constant interest rate.

4 Change of Measure to Handle Stochastic Interest Rates

Recall that by the Cameron-Martin-Girsanov theorem, we could transform the measure $\mathbb{P}$ into the measure $\mathbb{Q}$, that absorbed the market price of equity risk λ.

Now under stochastic interest rates we have

$$V_t(\widetilde{\Phi}_T) = \mathbb{E}_{\mathbb{Q}}\left[\widetilde{\Phi}_T e^{-\int_t^T r(s)ds} \Big/ \mathcal{F}_t\right]$$

Consider the case when the two components are independent. Then the expectation of the product is the product of the expectations,

$$V_t(\widetilde{\Phi}_T) = \mathbb{E}_{\mathbb{Q}}\left[\widetilde{\Phi}_T \Big/ \mathcal{F}_t\right] \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \Big/ \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\widetilde{\Phi}_T \Big/ \mathcal{F}_t\right] B^T(t)$$

If they are not, then we need to separate the terms in some other way. We can do this by changing the measure on $\big(\Omega, \sigma(\Omega)\big)$. Consider the Radon-Nikodym derivative defining a new measure $\mathbb{Q}_T$ with respect to $\mathbb{Q}$ that removes the interest rate term:

$$\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{Q}} = e^{-\int_t^T r(s)ds} \qquad \Longrightarrow \qquad \mathbb{E}_{\mathbb{Q}}\left[\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{Q}}\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\right] = B^T(t)$$

It will be more convenient for us to have $\mathbb{E}_{\mathbb{Q}}\left[\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{Q}}\right] = 1$ and so we define

$$\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{Q}} = \frac{e^{-\int_t^T r(s)ds}}{B^T(t)}$$

[1] First Fundamental Theorem of Asset Pricing

We can use this to rewrite

$$V_t(\widetilde{\Phi}_T) = \mathbb{E}_{\mathbb{Q}}\left[\widetilde{\Phi}_T e^{-\int_t^T r(s)ds} \Big/ \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\widetilde{\Phi}_T B^T(t) \frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{Q}} \Big/ \mathcal{F}_t\right]$$

But $B^T(t)$ is observed and known at time t:

$$V_t(\widetilde{\Phi}_T) = B^T(t)\, \mathbb{E}_{\mathbb{Q}}\left[\widetilde{\Phi}_T \frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{Q}} \Big/ \mathcal{F}_t\right]$$

Knowing that

$$\mathbb{E}_{\mathbb{Q}}[X] = \int_\Omega X \,\mathrm{d}\mathbb{Q}$$

We can now change the measure of our expectation:

$$\mathbb{E}_{\mathbb{Q}}\left[\widetilde{\Phi}_T \frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{Q}} \Big/ \mathcal{F}_t\right] = \int_\Omega \widetilde{\Phi}_T \frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{Q}} \,\mathrm{d}\mathbb{Q} = \int_\Omega \widetilde{\Phi}_T \,\mathrm{d}\mathbb{Q}_T = \mathbb{E}_{\mathbb{Q}_T}\left[\widetilde{\Phi}_T \big/ \mathcal{F}_t\right]$$

And we end up with:

$$V_t(\widetilde{\Phi}_T) = B^T(t)\, \mathbb{E}_{\mathbb{Q}_T}\left[\widetilde{\Phi}_T \big/ \mathcal{F}_t\right]$$

Let us apply the result to a non-dividend-paying stock S. Specifically, in the above formula, we substitute $\widetilde{\Phi}_T = S(T)$. By no-arbitrage and the above result,

$$S(t) = V_t\big(S(T)\big) = B^T(t)\, \mathbb{E}_{\mathbb{Q}_T}\left[S(T)/\mathcal{F}_t\right]$$

so

$$\frac{S(t)}{B^T(t)} = \mathbb{E}_{\mathbb{Q}_T}\left[S(T)/\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}_T}\left[\frac{S(T)}{1} \Big/ \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}_T}\left[\frac{S(T)}{B_T(T)} \Big/ \mathcal{F}_t\right]$$

Remark 2 (Geman 1989). We recall that $\frac{S(t)}{B^T(t)}$ is the T-forward price of S at time t, $f^T(t)$. Thus

$$f^T(t) = \mathbb{E}_{\mathbb{Q}_T}\left[f_T(T)/\mathcal{F}_t\right]$$

That is, the T-forward price of S is a $\mathbb{Q}_T$-martingale, and we call $\mathbb{Q}_T$ the T-*forward measure* .