Homework 8

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5.3.3 Orstein–Uhlenbeck Process

Consider the Ornstein-Uhlenbeck process

$$dr(t) = \alpha(b - r(t))dt + \sigma dW(t)$$

where $\alpha > 0$, $\sigma > 0$, and b are constants.

1. Show that for t > 0, $\Delta > 0$,

$$r(t+\Delta) = e^{-\alpha\Delta}r(t) + b(1 - e^{-\alpha\Delta}) + \frac{\sigma}{\sqrt{2\alpha}}\sqrt{1 - e^{-2\alpha\Delta}}Z$$

where Z~N(0,1)

Let

$$V(t) = e^{\alpha t}(r(t) - b)$$

we can get

$$\frac{\partial V}{\partial t} = \alpha e^{\alpha t}(r(t) - b)$$
$$\frac{\partial V}{\partial r} = e^{\alpha t}$$
$$\frac{\partial^2 V}{\partial r^2} = 0$$

then dV can be expressed as

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial r}dr + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}dr^2$$

simplify it as

$$dV = \sigma e^{\alpha t}dW(t)$$

so

$$V(T) - V(t) = e^{\alpha T}(r(T) - b) - e^{\alpha t}(r(t) - b) = \int_t^T \sigma e^{\alpha u}dW(u)$$

divided by $e^{\alpha T}$

$$r(T) - b - e^{\alpha(-T+t)}(r(t) - b) = \int_t^T \sigma e^{\alpha u - \alpha T}dW(u)$$

and we've known that

$$\int_t^T \sigma e^{\alpha u - \alpha T}dW(u) = \frac{\sigma}{\sqrt{2\alpha}}\sqrt{1 - e^{-2\alpha(T-t)}}Z$$

so the solution is

$$r(T) = b(1 - e^{e^{\alpha(-T+t)}}) + e^{\alpha(-T+t)}r(t) + \frac{\sigma}{\sqrt{2\alpha}}\sqrt{1 - e^{-2\alpha(T-t)}}Z$$

2. Use the transition distribution from the last part to implement a random walk construction for the process on time interval $[0, T]$. Your code should take α, σ, b the initial value $r(0)$, T, and the time step Δ of the random walk as input arguments. For $r(0) = 1$, $T = 500$ and $\Delta = 1/500$, plot a sample path for each combination of the following values,

$$\alpha \in \{0.1, 1, 5\},\ \sigma \in \{0.1, 0.2, 0.5\},\ b \in \{-5, 5\}.$$

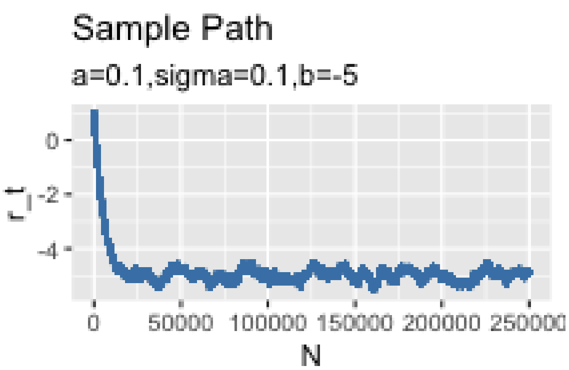
Comment on how the behavior of $r(t)$ depends on α and σ.

First, we need to get the data, then to plot graphs to find out the behavior of r dependents on α and σ. The result shows that with the increasing of the α, the path will be more quickly to become stable, and with the increase of the $sigma$, the wave will become lager.

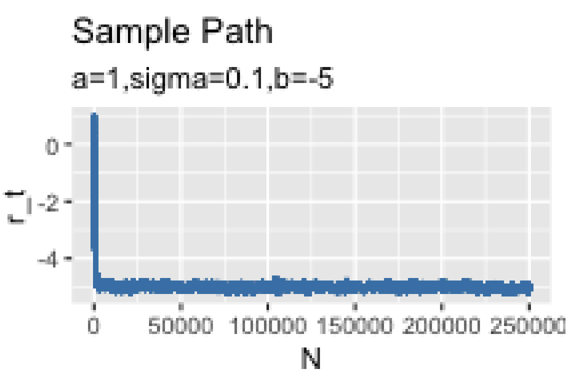
```
r_t <- function(r,a,b,sigma,delta){
  exp(-a*delta)*r+b*(1-exp(-a*delta))+sigma/sqrt(2*a)*sqrt(1-exp(-2*a*delta))*rnorm(1,0,1)
}

r_T <-function(a,b,sigma,delta,N){
  r <- 1
  data <- matrix(nrow = N+1)
  data[1] <- 1
  for(i in 1:N){
    r <- data[i+1] <- r_t(r,a,b,sigma,delta)
  }
  data
}

a_data <- c(0.1,1,5)
sigma_data <- c(0.1,0.2,0.5)
b_data <- c(-5,5)
library(ggplot2)

result <- array(0,c(3,3,2,250001))
for(i in 1:3){
  a <- a_data[i]
  for(j in 1:3){
    sigma <- sigma_data[j]
    for(k in 1:2){
      b <- b_data[k]
      result[i,j,k,] <- r_T(a,b,sigma,1/500,250000)
    }
  }
}
```

Thus, the graph will looks like

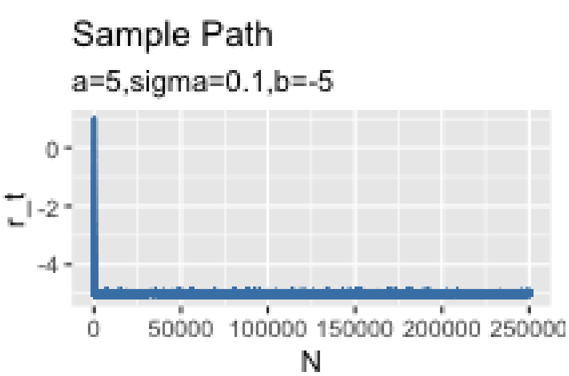
```
plot1 <- ggplot(data.frame(N = seq(1:250001),r_t = result[1,1,1,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=0.1,sigma=0.1,b=-5")
plot1
```



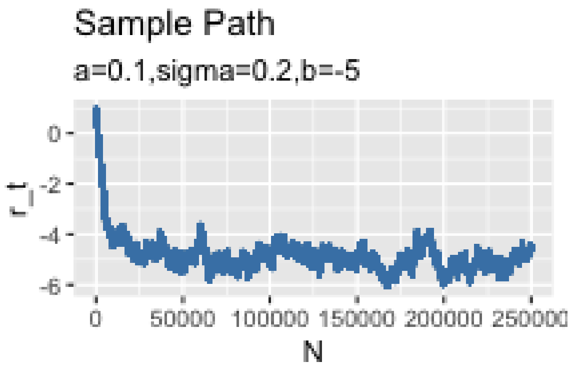

```
plot2 <- ggplot(data.frame(N = seq(1:250001),r_t = result[2,1,1,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=1,sigma=0.1,b=-5")
plot2
```



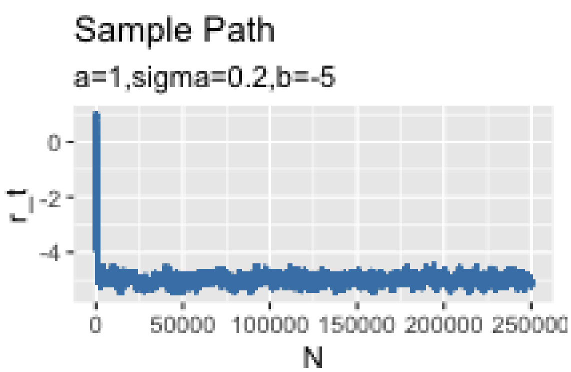

```
plot3 <- ggplot(data.frame(N = seq(1:250001),r_t = result[3,1,1,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=5,sigma=0.1,b=-5")
plot3
```



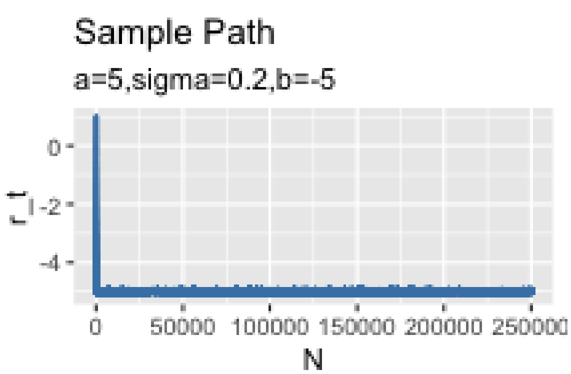

```
plot4 <- ggplot(data.frame(N = seq(1:250001),r_t = result[1,2,1,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=0.1,sigma=0.2,b=-5")
plot4
```



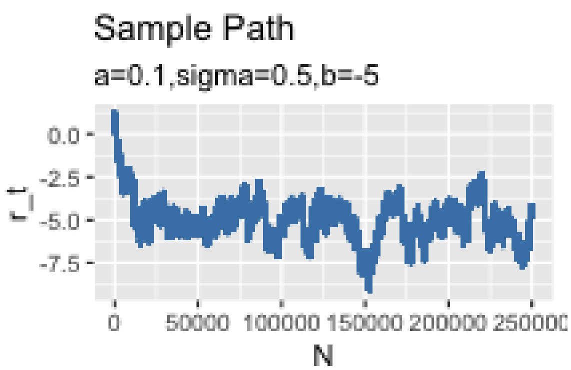

```
plot5 <- ggplot(data.frame(N = seq(1:250001),r_t = result[2,2,1,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=1,sigma=0.2,b=-5")
plot5
```



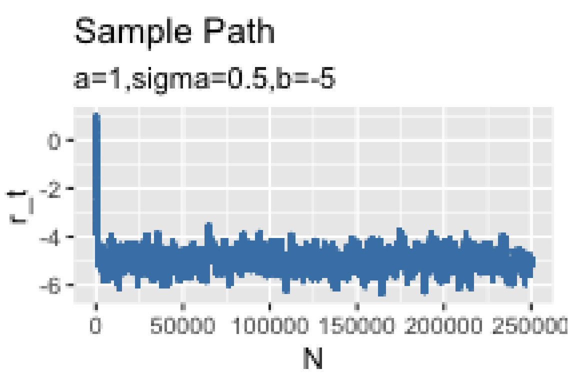

```
plot6 <- ggplot(data.frame(N = seq(1:250001),r_t = result[3,1,1,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=5,sigma=0.2,b=-5")
plot6
```



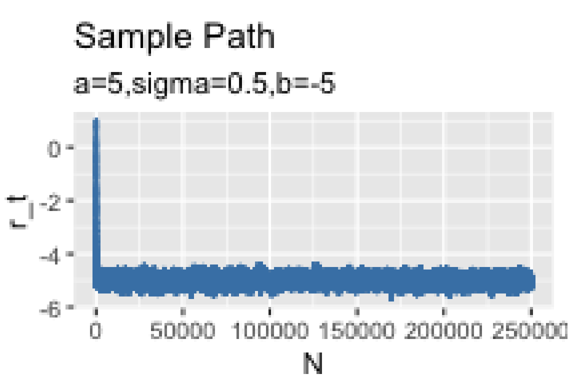

```
plot7 <- ggplot(data.frame(N = seq(1:250001),r_t = result[1,3,1,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=0.1,sigma=0.5,b=-5")
plot7
```



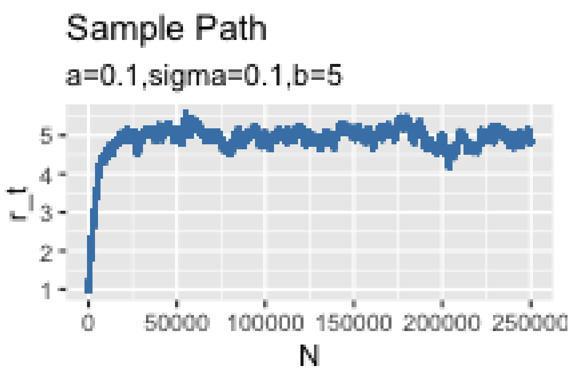

```
plot8 <- ggplot(data.frame(N = seq(1:250001),r_t = result[2,3,1,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=1,sigma=0.5,b=-5")
plot8
```



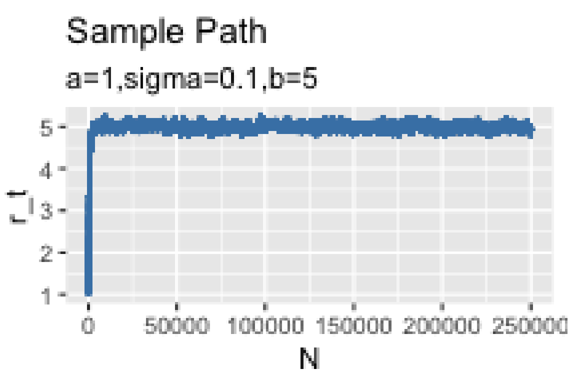

```
plot9 <- ggplot(data.frame(N = seq(1:250001),r_t = result[3,3,1,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=5,sigma=0.5,b=-5")
plot9
```



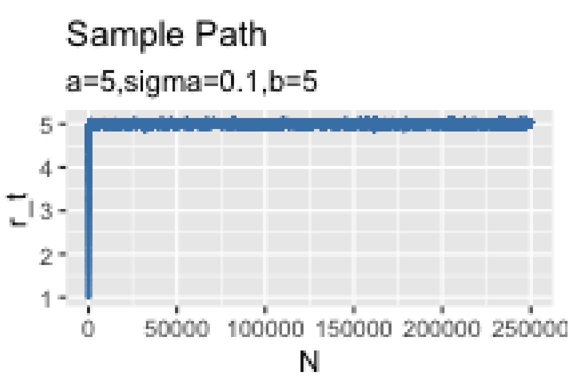

```
plot11 <- ggplot(data.frame(N = seq(1:250001),r_t = result[1,1,2,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=0.1,sigma=0.1,b=5")
plot11
```



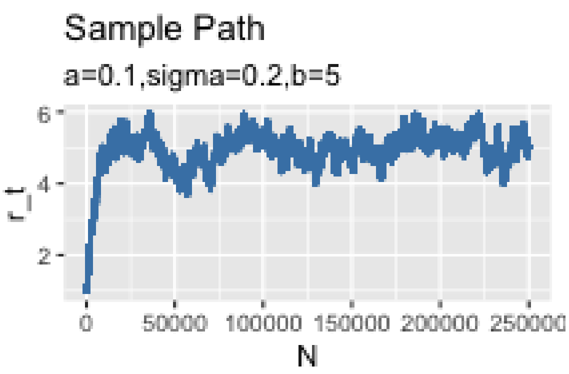

```
plot12 <- ggplot(data.frame(N = seq(1:250001),r_t = result[2,1,2,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=1,sigma=0.1,b=5")
plot12
```



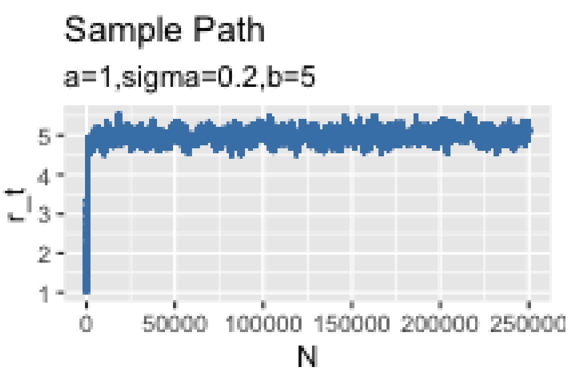

```
plot13 <- ggplot(data.frame(N = seq(1:250001),r_t = result[3,1,2,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=5,sigma=0.1,b=5")
plot13
```



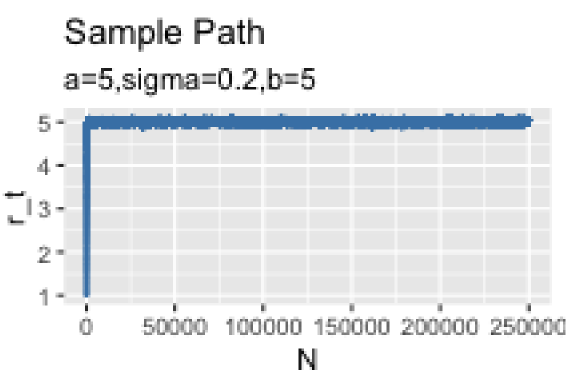

```
plot14 <- ggplot(data.frame(N = seq(1:250001),r_t = result[1,2,2,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=0.1,sigma=0.2,b=5")
plot14
```



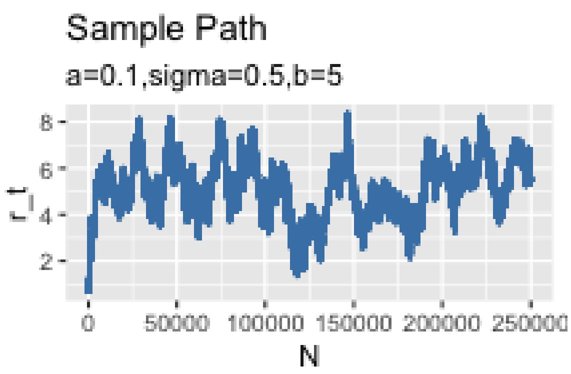

```
plot15 <- ggplot(data.frame(N = seq(1:250001),r_t = result[2,2,2,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=1,sigma=0.2,b=5")
plot15
```



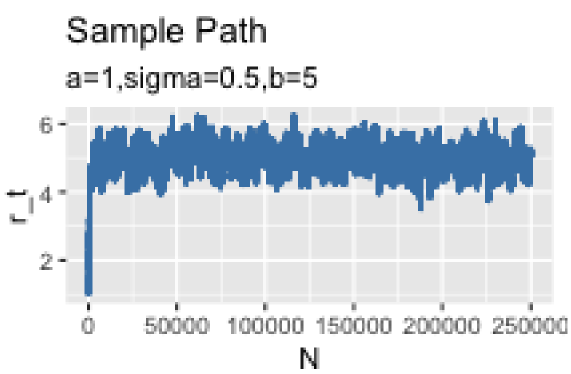

```
plot16 <- ggplot(data.frame(N = seq(1:250001),r_t = result[3,1,2,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=5,sigma=0.2,b=5")
plot16
```



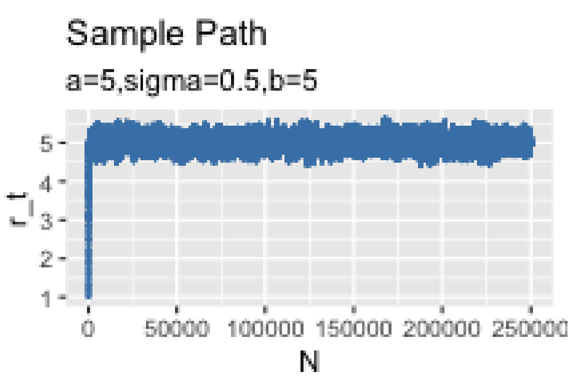

```
plot17 <- ggplot(data.frame(N = seq(1:250001),r_t = result[1,3,2,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=0.1,sigma=0.5,b=5")
plot17
```




```
plot18 <- ggplot(data.frame(N = seq(1:250001),r_t = result[2,3,2,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=1,sigma=0.5,b=5")
plot18
```



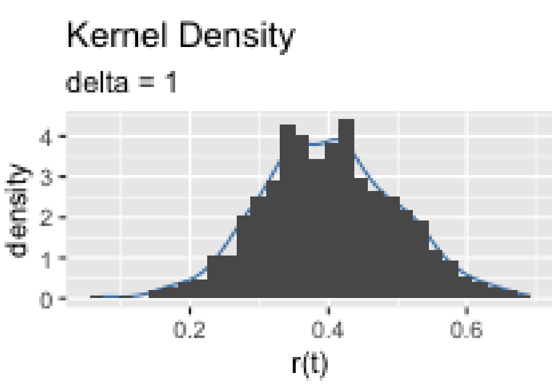

```
plot19 <- ggplot(data.frame(N = seq(1:250001),r_t = result[3,3,2,]),aes(x = N, y = r_t))+
        geom_point(col="steelblue", size=0.1)+
        labs(title="Sample Path", subtitle="a=5,sigma=0.5,b=5")
plot19
```

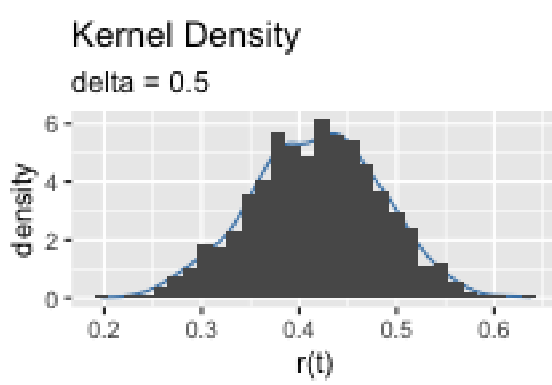


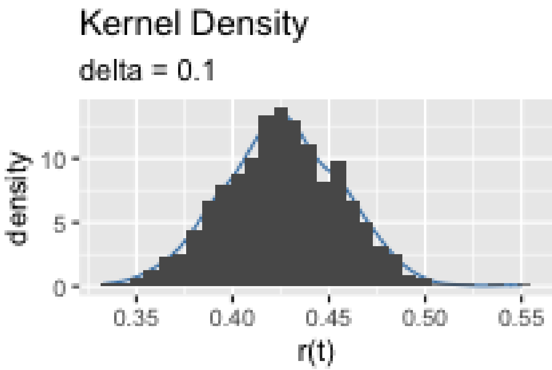
3. Use the Euler–Maruyama method (or the Euler method; see Wiki) to approximate a simulation from the process. Specifically, partition the time interval into a grid with subintervals of equal length $\delta > 0$ for a small δ; approximate $r(t+\delta)$ by a normal random variable with mean $r(t) + \alpha(b - r(t))\delta$ and standard deviation $\sigma\delta$. Write a function to implement this approximation with δ as one of the arguments. For $\delta \in \{1, 0.5, 0.1, 0.01\}$, generate a sample of size 1000 for $r(1)$. Plot the kernel densities against the true density.

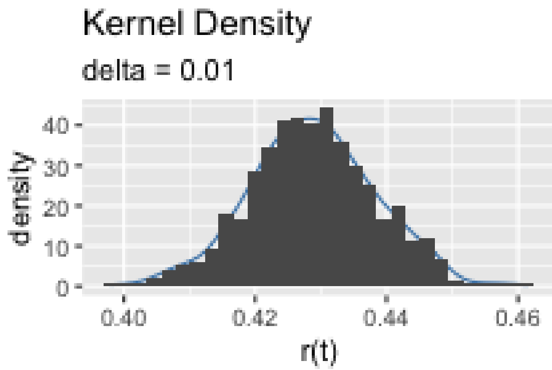
Set $r(0) = 1$, $\alpha = 0.1$, $\sigma = 0.1$, $b = -5$

```
r_t1 <- function(r,a,b,sigma,delta){
  rnorm(1,r + a*(b-r)*delta,sigma*delta)
}

r_T1 <-function(a,b,sigma,delta){
  r <- 1
  length <- 1/delta
  for(i in 1:length){
    r <-  r_t1(r,a,b,sigma,delta)
  }
  r
}

a <- 0.1
sigma <- 0.1
b <- -5
delta_data <- c(1,0.5,0.1,0.01)
result <- matrix(nrow = 4,ncol = 1000)
for(i in 1:4){
  delta <- delta_data[i]
  for(j in 1:1000){
    result[i,j] <- r_T1(a,b,sigma,delta)
  }
}

library(ggplot2)
```









and

The true value of r(1) can be calculated by

```
r_t2 <- function(r,a,b,sigma,delta){
  exp(-a*delta)*r+b*(1-exp(-a*delta))+sigma/sqrt(2*a)*sqrt(1-exp(-2*a*delta))*rnorm(1,0,1)
}

a <- 0.1
sigma <- 0.1
b <- -5
delta <- 1
result <- matrix(nrow =1,ncol = 1000)
for(i in 1:1000){
  result[1,i] <- r_t2(1,a,b,sigma,delta)
}
plot5 <- ggplot(data.frame(x = result[1,]), aes(x = x))+
  geom_density(colour ='steelblue',alpha=0.2)+
  geom_histogram(aes(y=..density..))+labs(x = 'r(t)',
  y = 'density', title='True Density')
plot5
```