HW2-Q1

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Q1 a

Since probability density of the Cauchy (x,theta) is:

$$P(x;\theta) = \frac{1}{\pi[1+(x-\theta)^2]}$$

Let $x_1, x_2 ... x_n$be an i.i.d sample, then $l(\theta)$ the log-likelihood function is:

$$l(\theta) = \ln(\prod_{i=1}^{n} p(x_i;\theta)) \quad = \ln(\prod_{i=1}^{n} \frac{1}{\pi[1+(x_i-\theta)^2]}) \ = \sum_{i=1}^{n} \ln(\frac{1}{\pi[1+(x_i-\theta)^2]}) \quad = -n\ln\pi - \sum_{i=1}^{n} \ln[1+(\theta-x_i)^2]$$

$$l^{'}(\theta) = 0 - (\sum_{i=1}^{n} \ln(1+(x_i-\theta)^2))^{'} = -\sum_{i=1}^{n} \frac{1}{1+(x_i-\theta)^2} * (1+x_i^2-2x_i\theta+\theta^2)^{'} = -2\sum_{i=1}^{n} \frac{\theta-x_i}{1+(\theta-x_i)^2}$$

$$l^{''}(\theta) = -2\sum_{i=1}^{n} \frac{1+(\theta-x_i)^2-2(\theta-x_i)^2}{[1+(\theta-x_i)^2]^2} = -2\sum_{i=1}^{n} \frac{1-(\theta-x_i)^2}{[1+(\theta-x_i)^2]^2}$$

$$P(x) = \frac{1}{\pi(1+x^2)}$$

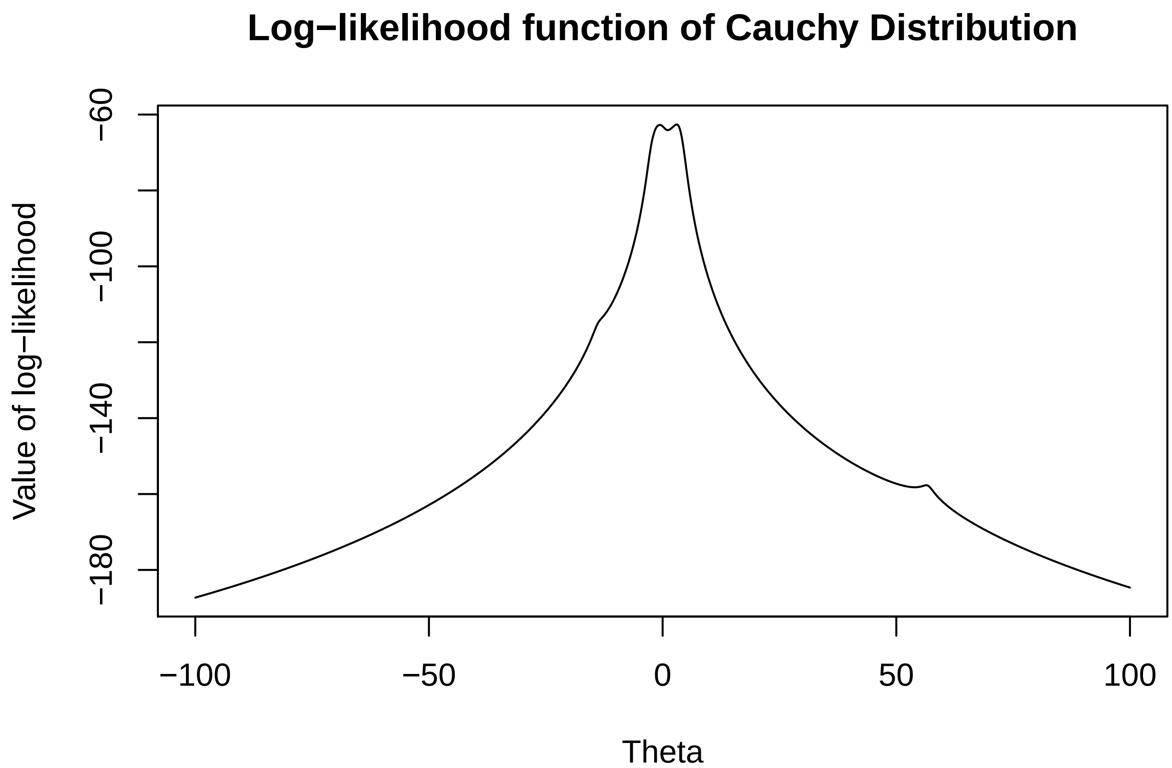
$$P^{'}(x) = -\frac{2x}{\pi(1+x^2)^2}$$

$$I(\theta) = n\int_{-\infty}^{\infty} \frac{{p^{'}(x)}^2}{p(x)} dx = \int_{-\infty}^{\infty} (\frac{4x^2}{\pi^2(1+x^2)^4}) * \frac{\pi(1+x^2)}{1} dx = \frac{4n}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3} dx$$

Set $x = \tan(\alpha); \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ Therefore,

$$I(\theta) = \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^{-2}(\alpha)-1}{(cos^{-2}(\alpha))^3} = \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\tan^2(\alpha)}{(1+tan^2(\alpha))^3}\frac{1}{\cos^2(\alpha)} d\alpha = \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^2(\alpha)*cos^2(\alpha) d\alpha = \frac{4n}{\pi}*\frac{\pi}{8} = \frac{n}{2}$$

Q1 b




```
## [1] -0.5914735 -0.5914735 -0.5914735  3.0213456  3.0213454  3.0213469
## [7] -0.5914735 -0.5914732 -0.5914735

## [1] 3.021345
```

The sample mean is a good starting point, which is relatively close to the actual value compared to most of the sample values.

Q1 C

```
## [1] 0.1035079
## [1] -0.5914734
## [1] -0.5914738

## [1] -0.5914730  0.1035079 -1.1063091  0.1035079 -1.1063091 -1.1713919
## [7] -1.1713919  0.2417269  0.2417269

## [1] -0.5914734 -0.5914734 -0.5914731  3.2398379 -0.5914731 -0.5914733
## [7]  2.5915177 -0.5914733  2.5915177

## [1] -0.5914740 -0.5914738 -0.5914730  3.0213454  3.0213454  3.0213454
## [7]  3.0213452  3.0213454  3.0213454
```

Q1 d

```
## [1] -0.5916003 -0.5915967 -0.5913588  3.0213156  3.0213928  3.0213874
## [7]  3.0213819  3.0213759 19.1751085
```

```
## [1] 3.021386
```

```
## [1] -0.5914735 -0.5914735 -0.5914735  3.0213753  3.0213473  3.0213874
## [7]  3.0213819  3.0213759  3.0213454
```

Q1 e

```
##       initial value     Newton    alpha=1 alpha=0.64 alpha=0.25     Fisher
##  [1,]         -11.0 -0.5914735 -0.5914730 -0.5914734 -0.5914740 -0.5916003
##  [2,]          -1.0 -0.5914735  0.1035079 -0.5914734 -0.5914738 -0.5915967
##  [3,]           0.0 -0.5914735 -1.1063091 -0.5914731 -0.5914730 -0.5913588
##  [4,]           1.5  3.0213456  0.1035079  3.2398379  3.0213454  3.0213156
##  [5,]           4.0  3.0213454 -1.1063091 -0.5914731  3.0213454  3.0213928
##  [6,]           4.7  3.0213469 -1.1713919 -0.5914733  3.0213454  3.0213874
##  [7,]           7.0 -0.5914735 -1.1713919  2.5915177  3.0213452  3.0213819
##  [8,]           8.0 -0.5914732  0.2417269 -0.5914733  3.0213454  3.0213759
##  [9,]          38.0 -0.5914735  0.2417269  2.5915177  3.0213454 19.1751085
##           Refine
##  [1,] -0.5914735
##  [2,] -0.5914735
##  [3,] -0.5914735
##  [4,]  3.0213753
##  [5,]  3.0213473
##  [6,]  3.0213874
##  [7,]  3.0213819
##  [8,]  3.0213759
##  [9,]  3.0213454
```

From the experiments I did above, similarly as the theory, the speed of convergence of Newton method is the fastest among all the mothods. The converge speed of Fisher method is slow. However, the stability of newton method is not as good as Fisher method. The stability of fixed point method is depends on the alpha it applied. Smaller alpha provides better result.