STAT-5361
Statistical Computing
HW #2

Instructor
Prof. Jun Yan

Students
Hee Seung Kim (2539662)
Department of ECE, University of Connecticut
e-mail: hee.s.kim@uconn.edu

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***All figures and table are the results of R code**

Problem 1.

(a)

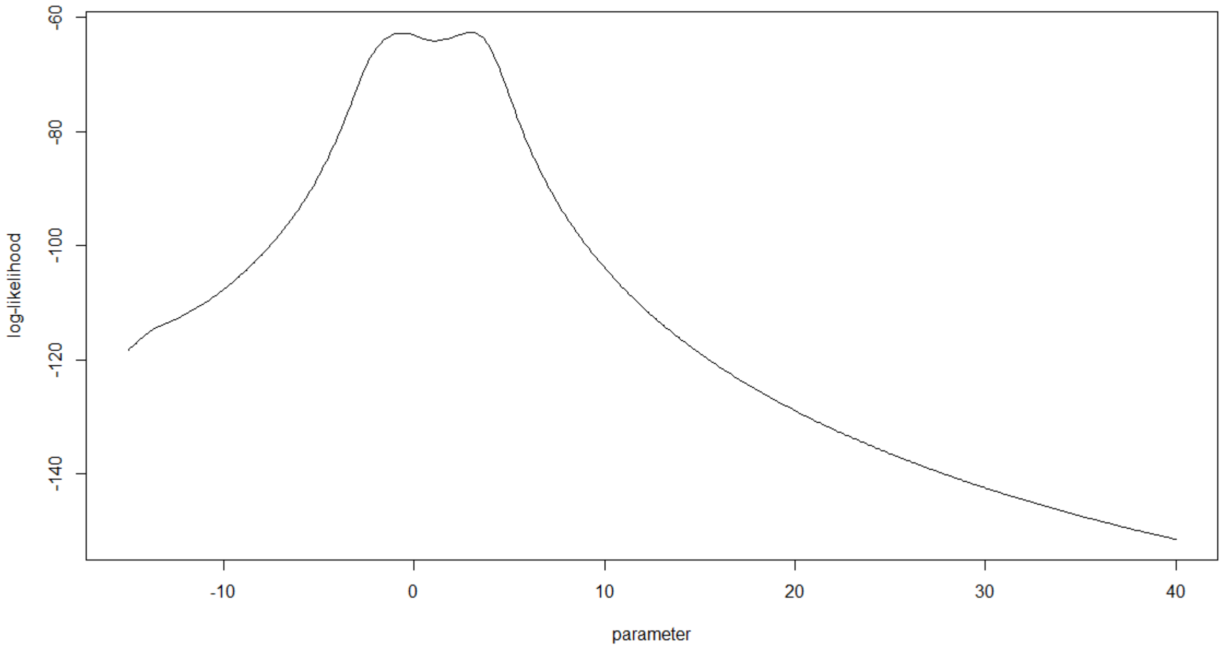
$$l(\theta) = \ln\left(\prod_{i=1}^{n} \frac{1}{\pi[1+(x_i-\theta)^2]}\right) = \sum_{i=1}^{n} \ln\left(\frac{1}{\pi[1+(x_i-\theta)^2]}\right)$$
$$= \sum_{i=1}^{n} \ln\left(\frac{1}{\pi}\right) + \sum_{i=1}^{n} \ln\left(\frac{1}{1+(x_i-\theta)^2}\right) = -n\ln\pi - \sum_{i=1}^{n} \ln[1+(\theta-x_i)^2]$$

$$l'(\theta) = \frac{\partial}{\partial\theta}\left(-n\ln\pi - \sum_{i=1}^{n} \ln[1+(\theta-x_i)^2]\right) = -\frac{\partial}{\partial\theta}\left(\sum_{i=1}^{n} \ln[1+(\theta-x_i)^2]\right)$$
$$= -\left(\sum_{i=1}^{n} \frac{2(\theta-x_i)}{1+(\theta-x_i)^2}\right) = -2\sum_{i=1}^{n} \frac{\theta-x_i}{1+(\theta-x_i)^2}$$

$$l''(\theta) = \frac{\partial}{\partial\theta}\left(-2\sum_{i=1}^{n} \frac{\theta-x_i}{1+(\theta-x_i)^2}\right) = -2\sum_{i=1}^{n}\left(\frac{1}{1+(\theta-x_i)^2} - \frac{2(\theta-x_i)^2}{(1+(\theta-x_i)^2)^2}\right)$$
$$= -2\sum_{i=1}^{n}\left(\frac{1+(\theta-x_i)^2}{(1+(\theta-x_i)^2)^2} - \frac{2(\theta-x_i)^2}{(1+(\theta-x_i)^2)^2}\right) = -2\sum_{i=1}^{n} \frac{1-(\theta-x_i)^2}{(1+(\theta-x_i)^2)^2}$$

$$I(\theta) = n\int \frac{\{p'(x)\}^2}{p(x)}\,dx = n\int_{-\infty}^{\infty} \frac{\left\{-\frac{2\pi x}{\left(\pi(1+x^2)\right)^2}\right\}^2}{\frac{1}{\pi(1+x^2)}}\,dx = n\int_{-\infty}^{\infty} \frac{4\pi^2 x^2}{\left(\pi(1+x^2)\right)^3}\,dx$$
$$= \frac{4n}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3}\,dx = \frac{4n}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\cdot\frac{1}{(1+x^2)^2}\,dx$$
$$= \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\arctan^2\theta}{1+\arctan^2\theta}\cdot\frac{1+\arctan^2\theta}{(1+\arctan^2\theta)^2}\,d\theta = \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\cdot\cos^2\theta\,d\theta$$
$$= \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 2\theta}{4}\,d\theta = \frac{n}{\pi}\left(\frac{\theta}{2} - \frac{\sin 4\theta}{8}\bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) = \frac{n}{\pi}\left(\frac{\pi}{2}\right) = \frac{n}{2}$$

(b)



The sample mean of given x (x=1.77, -0.23, 2.76, 3.80, 3.47, 56.75, -1.34, 4.24, -2.44, 3.29, 3.71, -2.40, 4.53, -0.07, -1.05, -13.87, -2.53, -1.75) is 3.2577. We have performed the Newton-Raphson method starting from 10 different points: -11, -1, 0, 1.5, 4, 4.7, 7, 8, 38 and the sample mean 3.2577. Comparing the results, we say that starting from a sample mean value is good.

int.val	**-11**	**-1**	**0**	**1.5**	**4**
NR_par	3.021345	-0.5914735	-0.5914735	3.021345	3.021345
NR_obj	-62.590323	-62.731861	-62.731861	-62.590323	-62.590323
NR_iter	6	4	4	7	4

int.val	**4.7**	**7**	**8**	**38**	**3.2577**
NR_par	3.021345	3.021345	3.021345	3.021345	3.021345
NR_obj	-62.590323	-62.590323	-62.590323	-62.590323	-62.590323
NR_iter	7	5	9	8	4

(c)

int.val	Alpha		
	1	0.64	0.25
-11	-0.59073	-0.59165	-0.5922
-1	0.103508	-0.59163	-0.59234
0	-1.10631	-0.59134	-0.59074
1.5	0.103508	3.239838	3.021302
4	-1.10631	-0.59136	3.021324
4.7	-1.17139	-0.59141	3.021312
7	-1.17139	2.591507	3.021341
8	0.241727	-0.59141	3.021317
38	0.241727	2.591627	3.021302
3.2577	-1.10631	2.591523	3.021338

(d), (e)

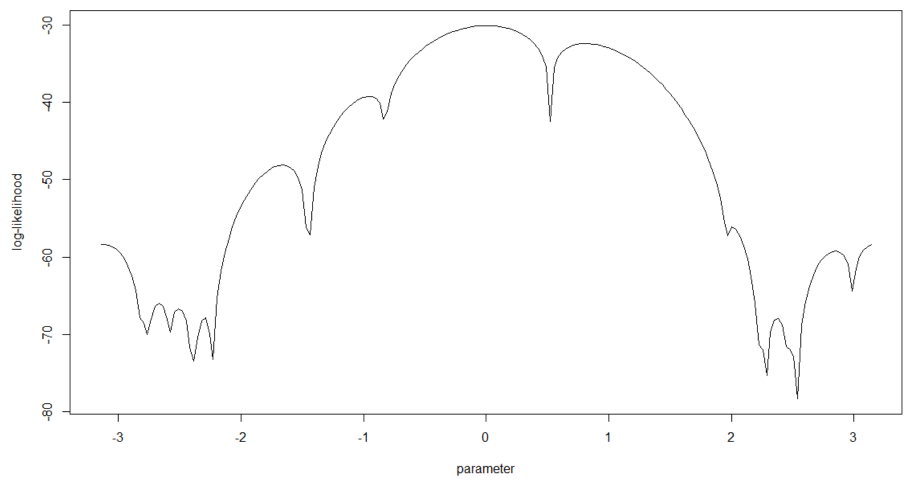
We have performed 3 iterations of the Fisher scoring method, then the Newton-Raphson method for refinement. Starting points are the same as previous method. We see that under Fisher scoring together with Newton method, the convergence points of MLE are less affected by choice of initial point. The number of iterations is not significantly different from the results of Newton-Raphson method alone. Thus, under those testing conditions, we can say that the Fisher Scoring with Newton method is more stable algorithm for global maximization overall.

int.val	-11	-1	0	1.5	4
FS_par	3.021345	-0.5914735	-0.5914735	3.021345	3.021345
FS_obj	-62.590323	-62.731861	-62.731861	-62.590323	-62.590323
FS_iter	6	4	6	7	6

int.val	4.7	7	8	38	3.2577
FS_par	3.021345	3.021345	-0.5914735	-0.5914736	3.021345
FS_obj	-62.590323	-62.590323	-62.731861	-62.731861	-62.590323
FS_iter	7	6	5	9	5

Problem 2.

(a)



(b)

$$p(x) = \frac{1-\cos(x-\theta)}{2\pi}, \quad 0 \le x \le 2\pi$$

$$E[X|\theta] = \int_0^{2\pi} x \cdot \frac{1-\cos(x-\theta)}{2\pi} dx = \frac{x^2}{4\pi}\bigg|_0^{2\pi} - \frac{x\sin(x-\theta)}{2\pi}\bigg|_0^{2\pi} + \int_0^{2\pi} \frac{\sin(x-\theta)}{2\pi} dx$$

$$= \pi - \sin(2\pi-\theta) - \frac{\cos(x-\theta)}{2\pi}\bigg|_0^{2\pi} = \pi - \sin(2\pi-\theta) - \frac{\cos(2\pi-\theta)}{2\pi} + \frac{\cos\theta}{2\pi}$$

$$= \pi - \sin(2\pi-\theta) = \pi + \sin(\theta) = \bar{x}$$

$$\hat{\theta}_{moment} = \sin^{-1}(\bar{x}-\pi) = 0.09539$$

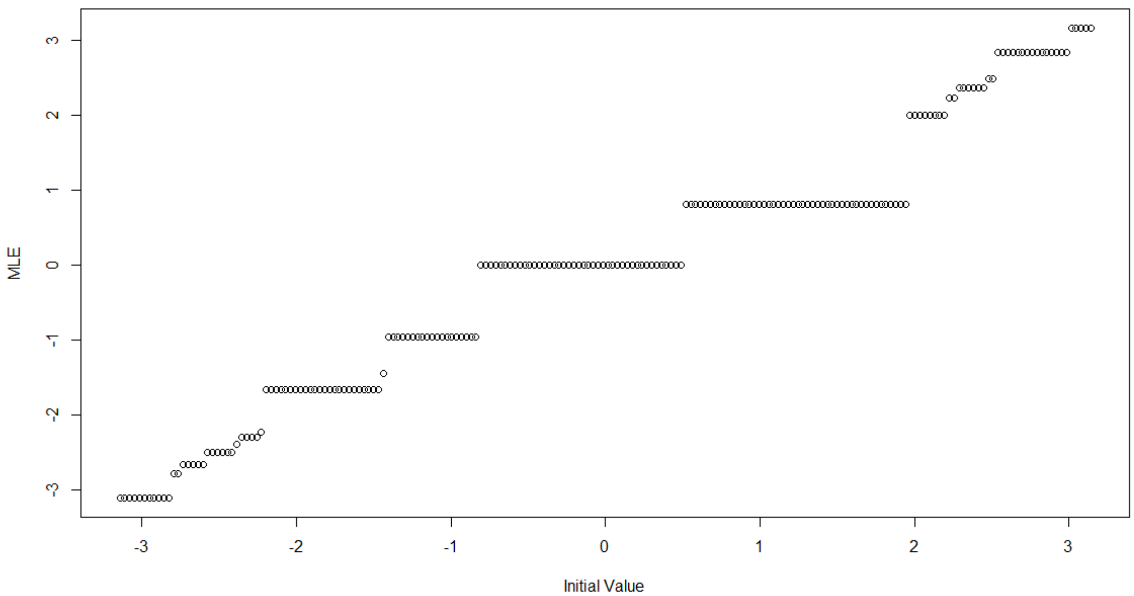
(c) – (d)

With $\theta_0 = \hat{\theta}_{moment} = 0.09539$, the global maximum is reached, and the MLE for θ is found to be 0.003118. With -2.7 and 2.7 as starting points, the MLE for θ is found to be -2.66886 and 2.848415 respectively.

int.val	MME	-2.7	2.7
trig_par	0.003118	-2.66886	2.848415
trig_obj	-30.0949	-65.9818	-59.2278
trig_iter	4	4	5

(e)

For 200 equally spaced starting points between $-\pi$ and π, we have grouped the points with each group corresponding to each unique outcome of the optimization. There are total 18 groups with 18 different outcomes of the optimization. Refer to the results below.



Unique Outcome		From	To	Unique Outcome		From	To
1	-3.112471	-3.14159	-2.82585	**10**	-0.954406	-1.40503	-0.83671
2	-2.786557	-2.79428	-2.76271	**11**	0.003118	-0.80513	0.489394
3	-2.668858	-2.73113	-2.60484	**12**	0.812637	0.520968	1.941788
4	-2.509357	-2.57326	-2.4154	**13**	2.007223	1.973362	2.194379
5	-2.388267	-2.38382	-2.38382	**14**	2.237012	2.225953	2.257526
6	-2.297926	-2.35225	-2.25753	**15**	2.374711	2.2891	2.446969
7	-2.232192	-2.22595	-2.22595	**16**	2.488449	2.478543	2.510117
8	-1.662713	-2.19438	-1.46818	**17**	2.848415	2.541691	2.983724
9	-1.447503	-1.43661	-1.43661	**18**	3.170714	3.015298	3.141593

Problem 3.

(a)

We used “nls” function from lecture slide, and here is an example of this function. According to below result, we can find about a population growth model as below.

- r = 0.118
- K=1033.5150
- Residual sum of squares: 83240

```
## ------------example------------------------------------
x <- 1:10
y <- 2*x + 3                          # perfect fit
yeps <- y + rnorm(length(y), sd = 0.01) # added noise
nls(yeps ~ a + b*x, start = list(a = 0.12345, b = 0.54321))
## -----------------------------------------------------------
```

```
3792408 :  200.0   0.2
2469890 :  376.8296589   0.1022548
656489.7 :  705.8732408   0.1155781
83443.08 :  1030.7198251   0.1192449
83254.53 :  1032.5696826   0.1182903
83241.46 :  1033.276710   0.118046
83240.55 :  1033.4530815   0.1179816
83240.49 :  1033.4989398   0.1179647
83240.49 :  1033.5110016   0.1179602
83240.49 :  1033.514183   0.117959
83240.49 :  1033.5150235   0.1179587
Nonlinear regression model
  model: x ~ ((x[1] * K)/(x[1] + (K - x[1]) * exp(-r * t)))
   data: x
       K        r
1033.515    0.118
 residual sum-of-squares: 83240

Number of iterations to convergence: 10
Achieved convergence tolerance: 3.781e-06
```