Homework2

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question1

(a)

$$L(\theta) = \prod_{i=1}^{n} p(x_i;\theta) = \prod_{i=1}^{n} \frac{1}{\pi[1+(x_i-\theta)^2]} = \frac{1}{\pi^n}\prod_{i=1}^{n}\frac{1}{1+(x_i-\theta)^2}$$

$$l(\theta) = \ln L(\theta) = \ln(\frac{1}{\pi^n}\prod_{i=1}^{n}\frac{1}{1+(x_i-\theta)^2}) = -n\ln\pi - \sum_{i=1}^{n}\ln(1+(\theta-x_i)^2)$$

$$l'(\theta) = (-\sum_{i=1}^{n}\ln(1+(\theta-x_i)^2))' = -\sum_{i=1}^{n}\frac{2(\theta-x_i)}{1+(\theta-x_i)^2} = -2\sum_{i=1}^{n}\frac{\theta-x_i}{1+(\theta-x_i)^2}$$

$$l''(\theta) = -2\sum_{i=1}^{n}\frac{1+(\theta-x_i)^2-2(\theta-x_i)^2}{[1+(\theta-x_i)^2]^2} = -2\sum_{i=1}^{n}\frac{1-(\theta-x_i)^2}{[1+(\theta-x_i)^2]^2}$$

$$P(x) = \frac{1}{\pi(1+x^2)}$$

$$P'(x) = -\frac{1}{\pi}\ \frac{2x}{(1+x^2)^2}$$

$$P'(x)^2 = \frac{4}{\pi^2}\ \frac{x^2}{(1+x^2)^4}$$

$$I(\theta) = n\int\frac{p'(x)^2}{p(x)}dx = \frac{4n}{\pi}\int_{-\infty}^{\infty}\frac{x^2dx}{(1+x^2)^3}$$

Let $x = \tan\beta$*, then we have* $1+\tan^2\beta = \operatorname{arcsin}^2\beta = \frac{1}{\cos^2\beta}$*, s.t.*

$$I(\theta) = \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\tan^2\beta}{(1+tan^2\beta)^3}\frac{1}{cos^2\beta}d\beta = \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\beta\cos^2\beta\ d\beta = \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\beta(1-\sin^2\beta)\ d\beta = \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\beta-\sin^4\beta\ d\beta$$

As $1-\cos 2\beta = 2\sin^2\beta$*, then we have*

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\beta\ d\beta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos 2\beta)d\beta = \frac{1}{2}(\beta-\frac{1}{2}\sin 2\beta)|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2}$$

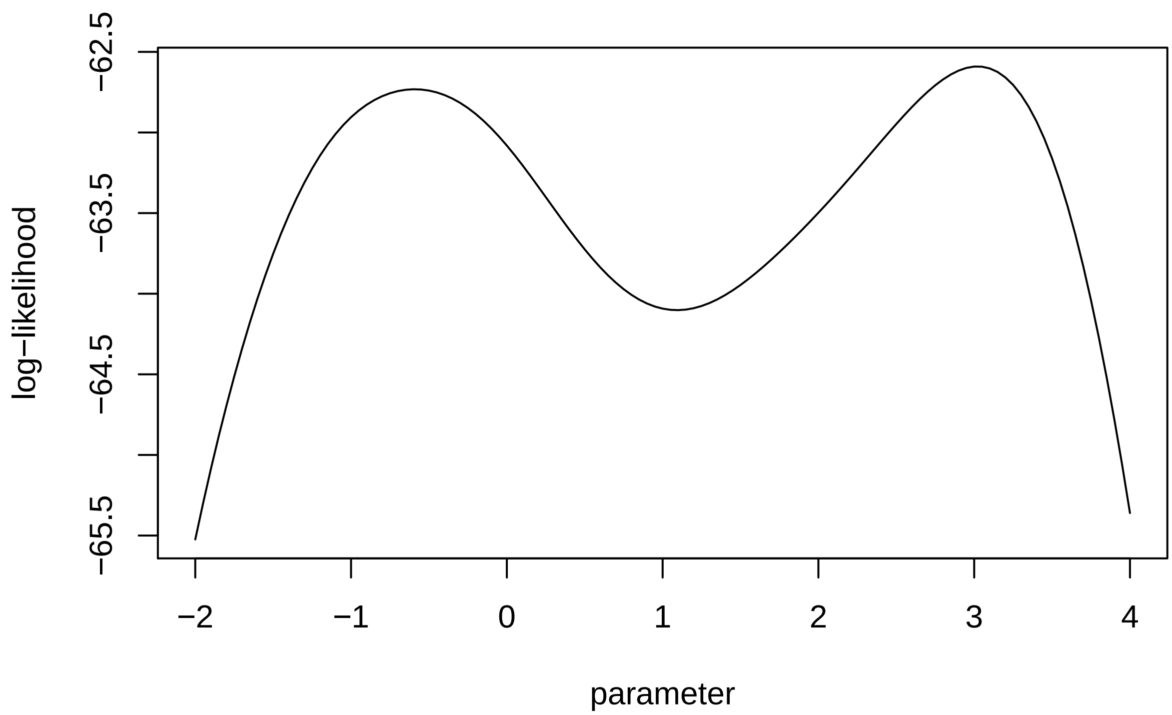
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^4\beta\ d\beta = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos 2\beta)^2d\beta = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 2\beta - 2\cos 2\beta + 1\ d\beta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos 4\beta}{2} - 2\cos 2\beta + \frac{3}{2}\ d\beta = \frac{1}{4}(\frac{1}{8}\sin 4\beta - \sin 2\beta + \frac{3}{2}\beta)|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{3\pi}{8}$$

Finally we have:

$$I(\theta) = \frac{4n}{\pi}(\frac{\pi}{2} - \frac{3\pi}{8}) = \frac{n}{2}$$

(b)




```
## [1] 3.021345
## [1] -0.5914735
## [1] -0.5914735
## [1] 3.021345
## [1] 3.021345
## [1] 3.021345
## [1] 3.021345
## [1] 3.021345
## [1] 3.021345
```

(c)

```
## [1] -0.5914827  3.0213332 -0.5915249
## [1] -0.5914791 -0.5914824 -0.5915051
```

```
## [1] -0.5914717 -0.5914659 -0.5914218
## [1] 3.021345 3.021345 3.021344
## [1] 3.021345 3.021338 3.021345
## [1] 3.021345 3.021328 3.021344
## [1] 3.021335 3.021345 3.021343
## [1] -0.5914408 -0.5914885  3.0213434
## [1] -0.5914796  3.0213451  3.0213435
```

(d)

```
## [1] -0.5915031
## [1] -0.5915022
## [1] -0.5914394
## [1] 3.021336
## [1] 3.021354
## [1] 3.021359
## [1] 3.021358
## [1] 3.021356
## [1] 3.021357
```

(e)

According to the computation above we find some special properties of these methods:

Newton Method: the speed of convergence of this method is the fastest while the stability is the worst.
Fisher Method: The stability of fisher method is good while the speed of convergence is as satisfied as its stability.
Fixed-point Method: the speed of convergence and stability of this method depend on α. We find that the larger the $\alpha$is, the faster the speed of convergence will be and the worse the stability is.

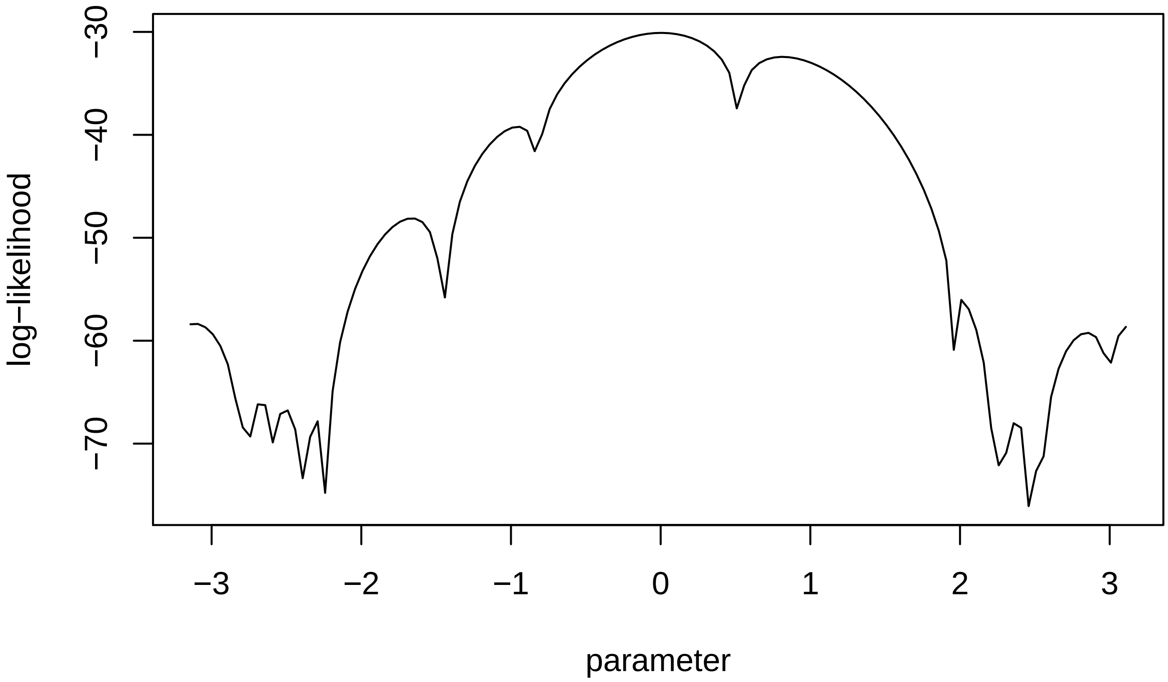
question 2

(a)

As $p(x;\theta) = \frac{1-\cos(x-\theta)}{2\pi},\ 0 \leqslant x \leqslant 2\pi,\ \theta \in (-\pi, \pi),$ *then we have*

$$L(\theta) = \prod_{i=1}^{n} \frac{1-\cos(x_i-\theta)}{2\pi} = \frac{1}{(2\pi)^n} \prod_{i=1}^{n} [1-\cos(x_i-\theta)]$$

$$l(\theta) = -n\log(2\pi) + \sum_{i=1}^{n} \log[1-\cos(x_i-\theta)]$$



(b)

$$E(x|\theta) = \int_0^{2\pi} xP(x|\theta)\,dx = \int_0^{2\pi} x\,\frac{1-\cos(x-\theta)}{2\pi}\,dx = \frac{1}{2\pi}\int_0^{2\pi} x - x\cos(x-\theta)\,dx = \pi - \frac{1}{2\pi}\int_0^{2\pi} x\cos(x-\theta)\,dx$$

$$\int_0^{2\pi} x\cos(x-\theta)\ dx = x\sin(x-\theta)|_0^{2\pi} - \int_0^{2\pi} \sin(x-\theta)\ dx = -2\pi\sin\theta + \cos(x-\theta)|_0^{2\pi} = -2\pi\sin\theta$$

$$E(x|\theta) = \pi + \sin\theta = \bar{x}$$

$$\hat{\theta}_{moment} = \arcsin(\bar{x} - \pi)$$

```
## [1] 0.09539407
```

(c)

```
## [1] 0.003118157
```

(d)

```
## [1] -2.668857
```

```
## [1] 2.848415
```

(e)

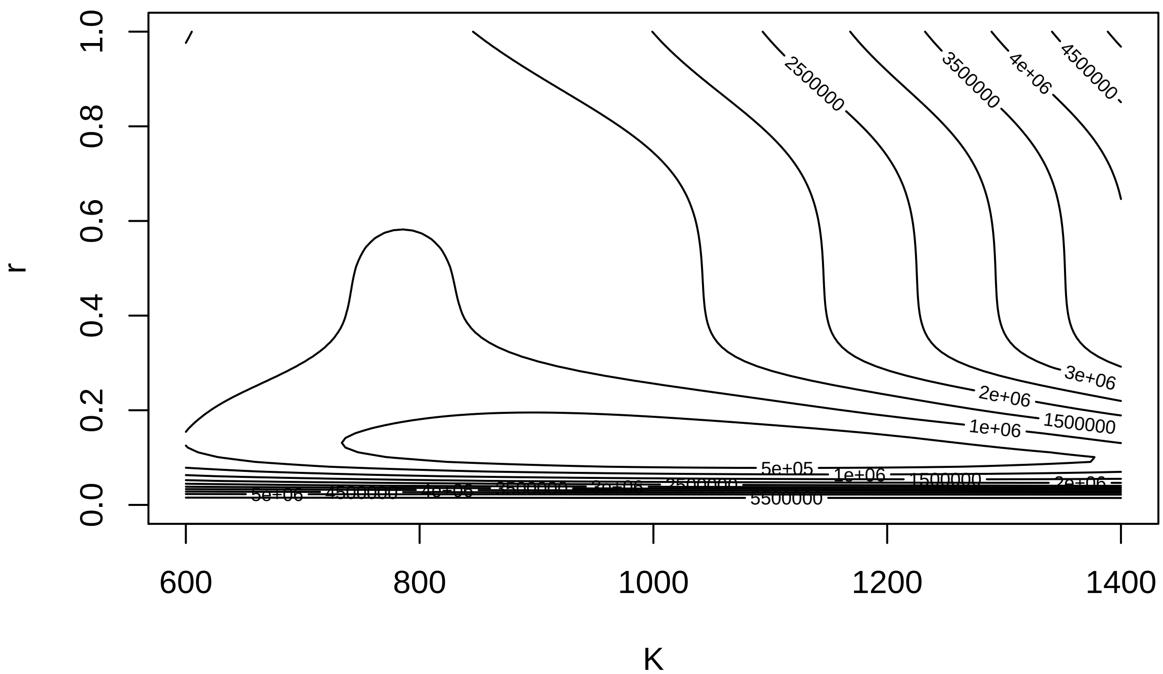
1	2	3	4	5	6	7	8	9	10	11	12
-2.38382											
-2.22595											
2.47854											
2.51012											
2.22595											
2.25753											
-2.79428											
-2.76271											
2.32067											
2.44697											
2.2891	2.35225	2.38382	2.4154								
-2.35225	-2.2891	-2.25753									
-2.32067											
-2.4154											
-2.57326	-2.54169	-2.47854									
-2.51012	-2.44697										
-2.73113	-2.69956	-2.66799									
-2.63641	-2.60484										
2.54169	2.57326	2.63641	2.66799	2.73113	2.76271	2.85743	2.889	2.92058	2.98372		
2.60484	2.69956	2.79428	2.82585	2.95215							
-3.14159	-3.11002	-3.04687	-3.0153	-2.95215	-2.92058	-2.889	-2.85743	-2.82585			
-3.07845	-2.98372										
3.0153	3.04687	3.07845	3.11002	3.14159							
2.1628											
1.97336											
2.13123											
2.00494	2.06808	2.19438									
2.09966											
2.03651											
-1.43661											
-2.13123	-2.09966	-1.97336	-1.94179	-1.87864	-1.84707	-1.75235	-1.65762	-1.5629	-1.53133	-1.49976	
-2.19438	-2.1628	-2.06808	-2.03651	-2.00494	-1.91021	-1.81549	-1.78392	-1.72077	-1.6892	-1.62605	-1.5
-1.18402											
-1.40503	-1.37346	-1.34189	-1.31031	-1.27874	-1.24716	-1.21559	-1.15244	-1.12087	-1.0893	-1.05772	-1.0
0.52097	0.55254	0.58412	0.61569	0.64726	0.67884	0.71041	0.74198	0.77356	0.80513	0.83671	0.8
-0.80513	-0.77356	-0.74198	-0.71041	-0.67884	-0.64726	-0.61569	-0.58412	-0.55254	-0.52097	-0.48939	-0.4

question3

(a)

```
##              [,1]
## [1,] 1049.4038970
## [2,]    0.1182693
```

(b)



(c)

```
##              [,1]
## [1,] 820.5349872
## [2,]   0.1926176
## [3,]   0.6441323
```

```
##               [,1]          [,2]          [,3]
## [1,]  6.248530e+04 -9.054089e+00  1.051866e-07
## [2,] -9.054089e+00  3.974068e-03 -5.353628e-11
## [3,]  1.051866e-07 -5.353628e-11  2.074532e-02
```

So $K = 820.5349872,\ r = 0.1926176,\ \sigma = 0.6441323.$
The variance of $K,\ r,\ \sigma^2$ are $6.248530 * 10^4,\ 3.974068 * 10^{-3},\ 2.074532 * 10^{-2}$