Homework 2

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Question 1

(a)

Proof

$$
\begin{aligned}
l(\theta) &= \ln \prod_{i=1}^{n} \frac{1}{\pi[1+(x_i-\theta)^2]} && (1)\\
&= \sum_{i=1}^{n} \ln[\frac{1}{\pi[1+(x_i-\theta)^2]}] && (2)\\
&= n\ln\pi^{-1} + \sum_{i=1}^{n} \ln[1+(\theta-x_i)^2]^{-1} && (3)\\
&= -n\ln\pi - \sum_{i=1}^{n} \ln[1+(\theta-x_i)^2]. && (4)
\end{aligned}
$$

$$l'(\theta) = -\sum_{i=1}^{n} \frac{1}{1+(\theta-x_i)^2} \times 2(\theta-x_i) = -2\sum_{i=1}^{n} \frac{\theta-x_i}{1+(\theta-x_i)^2}$$

. We know

$$p(x) = \frac{1}{\pi(1+x^2)},$$

and

$$p'(x) = -\frac{2x}{\pi(1+x^2)^2},$$

then

$$
\begin{aligned}
I(\theta) &= n\int \frac{[p'(x)]^2}{p(x)}dx && (5)\\
&= \frac{4n}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3}dx, && (6)
\end{aligned}
$$

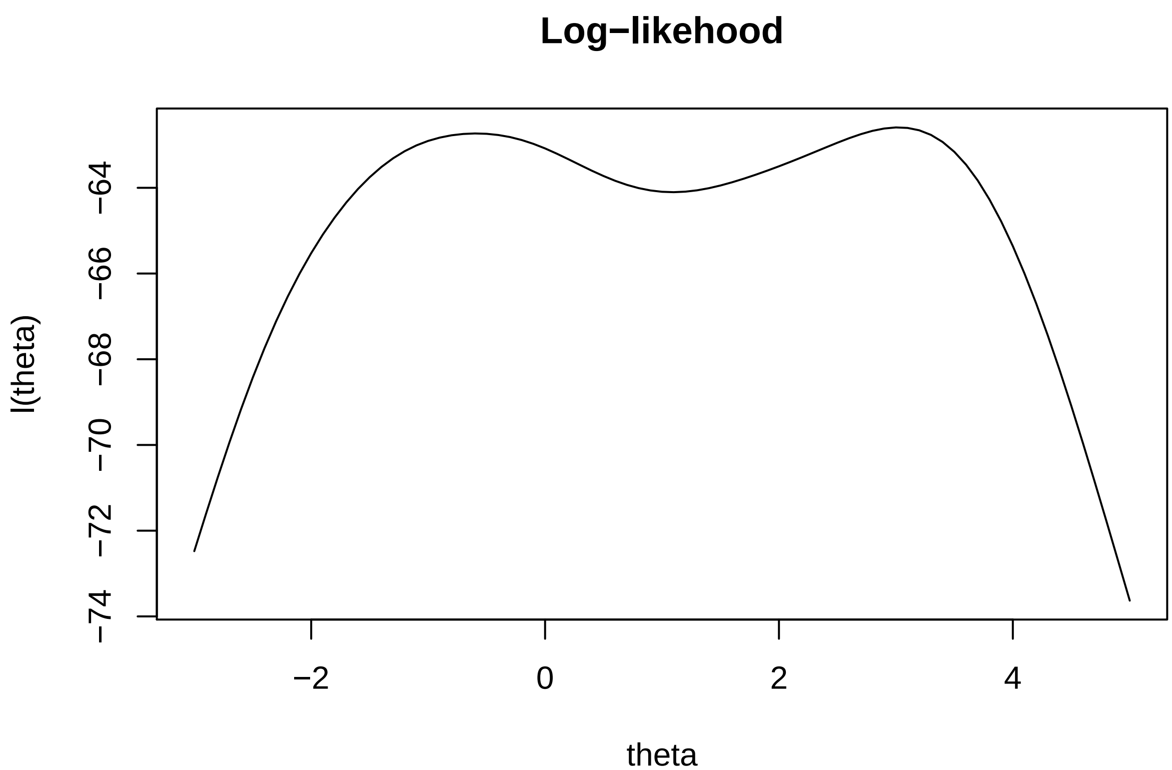
Let $x = \tan y$, so $dx = \frac{1}{\cos^2 x}$ then,

$$
\begin{aligned}
I(\theta) &= \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\tan^2 x}{(1+\tan^2 x)^3} \times \frac{1}{\cos^2 x} dx && (7)\\
&= \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^2 x \cos^4 x dx && (8)\\
&= \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^2 x dx && (9)\\
&= \frac{4n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x(1-\sin^2 x) dx && (10)\\
&= \frac{4n}{\pi}(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x dx) && (11)\\
&= \frac{8n}{\pi}(\int_{0}^{\frac{\pi}{2}} \sin^2 x dx - \int_{0}^{\frac{\pi}{2}} \sin^4 x dx) && (12)\\
&= \frac{8n}{\pi} \times (\frac{\pi}{4} - \frac{3\pi}{16}) && (13)\\
&= \frac{n}{2} && (14)
\end{aligned}
$$

(b)

The graph of the log-likehood function

```
x <- c(1.77, -0.23, 2.76, 3.80, 3.47, 56.75, -1.34, 4.24, -2.44,
       3.29, 3.71, -2.40, 4.53, -0.07, -1.05, -13.87, -2.53, -1.75)
n <- length(x)
theta <- seq(-3, 5, 0.1)
loglikehood <- function(theta){
  return(-n*log(pi) - sum(log(1 + (theta - x)^2)))
}
b <- lapply(theta, loglikehood) #b is l(theta)
plot(theta, b, type = 'l', main = 'Log-likehood', xlab = 'theta', ylab = 'l(theta)')
```



The MLE of θ

```
theta_0 <- c(-11, -1, 0, 1.5, 4, 4.7, 7, 8, 38)
Dloglikehood <- function(theta){
  return(-2*sum((theta - x)/(1 + (theta - x)^2)))
}
DDloglikehood <- function(theta){
  return(-2*sum((1 - (theta - x)^2)/(1 + (theta - x)^2)^2))
}
Newton_Raphson <- function(theta_t){
  while(abs(Dloglikehood(theta_t))>10^(-10)){
    theta_t <- theta_t - Dloglikehood(theta_t)/DDloglikehood(theta_t)
  }
  return(theta_t)
}
c <- lapply(theta_0, Newton_Raphson)
c
```

```
## [[1]]
## [1] -396859755588
##
## [[2]]
## [1] -0.5914735
##
## [[3]]
## [1] -0.5914735
##
## [[4]]
## [1] 1.09273
##
## [[5]]
```

```
## [1] 3.021345
##
## [[6]]
## [1] -0.5914735
##
## [[7]]
## [1] 662415349525
##
## [[8]]
## [1] 701220414344
##
## [[9]]
## [1] 615420525724
```

When the start point is 4, the results is the θ of MLE.

```
x_mean <- mean(x)
Newton_Raphson(3.25)
```

```
## [1] 3.021345
```

It has the same result with start point 4, so the sample mean is a good starting point.

(c)

```
alpha <- c(1, 0.64, 0.25)
fixedpoint_1 <- function(theta_t){
  while(abs(Dloglikehood(theta_t))>10^(-10)){
    theta_t <- theta_t - Dloglikehood(theta_t)
  }
  return(theta_t)
}
fixedpoint_2 <- function(theta_t){
  while(abs(Dloglikehood(theta_t))>10^(-10)){
    theta_t <- theta_t - Dloglikehood(theta_t)*0.64
  }
  return(theta_t)
}
fixedpoint_3 <- function(theta_t){
  while(abs(Dloglikehood(theta_t))>10^(-10)){
    theta_t <- theta_t - Dloglikehood(theta_t)*0.25
  }
  return(theta_t)
}
```

The resluts went wrong.

(d)

```
Fisher_Scoring <- function(theta_t){
  while(abs(Dloglikehood(theta_t))>10^(-10)){
    theta_t <- theta_t + 2*Dloglikehood(theta_t)/n
  }
```

```
  return(theta_t)
}
d <- lapply(theta_0, Fisher_Scoring)
d
```

```
## [[1]]
## [1] -0.5914735
##
## [[2]]
## [1] -0.5914735
##
## [[3]]
## [1] -0.5914735
##
## [[4]]
## [1] 3.021345
##
## [[5]]
## [1] 3.021345
##
## [[6]]
## [1] 3.021345
##
## [[7]]
## [1] 3.021345
##
## [[8]]
## [1] 3.021345
##
## [[9]]
## [1] 3.021345
```

(e)

The Newton method is the fastest, and the Fisher scoring is the most stable.

Question 2

(a)

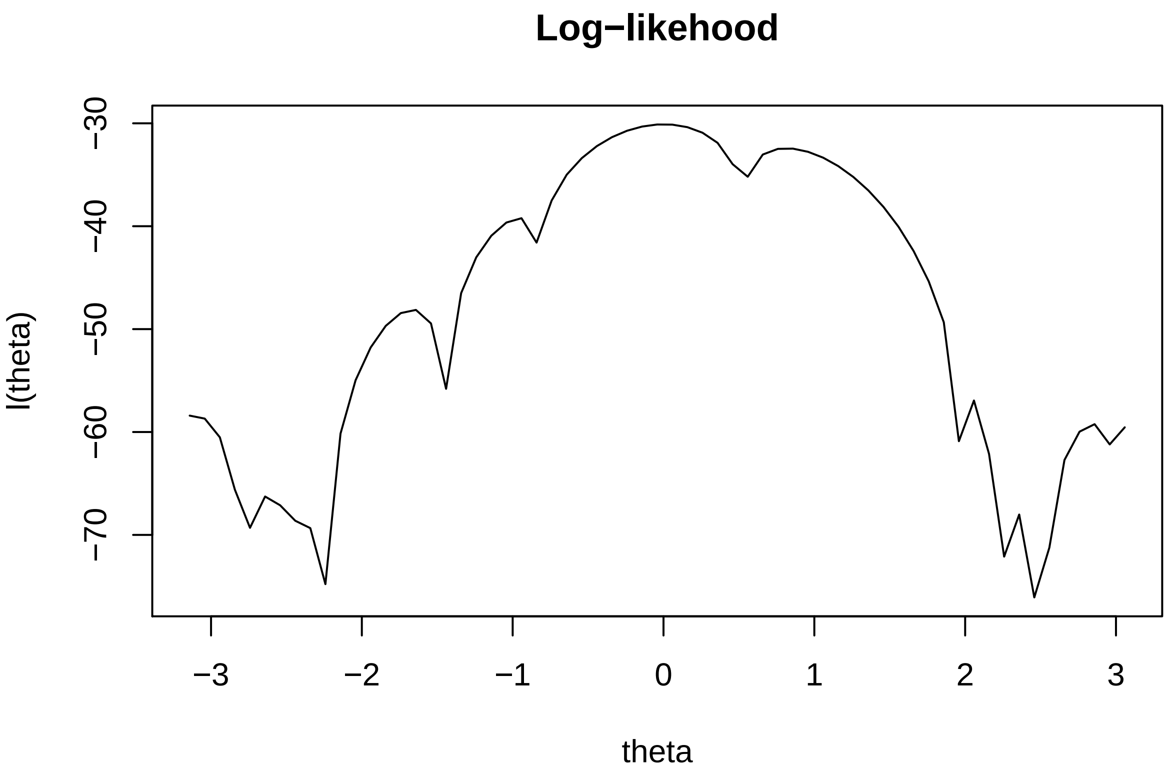
The log-likehood function of θ is

$$l(\theta) = \sum_{i=1}^{n} \ln[1 - \cos(x_i - \theta)] - n \ln 2\pi$$

Graph

```
x <- c(3.91, 4.85, 2.28, 4.06, 3.70, 4.04, 5.46, 3.53, 2.28, 1.96,
       2.53, 3.88, 2.22, 3.47, 4.82, 2.46, 2.99, 2.54, 0.52)
n <- length(x)
theta <- seq(-pi, pi, 0.1)
loglikehood <- function(theta){
```

```
  return(-n*log(2*pi) + sum(log(1 - cos(x - theta))))
}
b <- lapply(theta, loglikehood) #b is l(theta)
plot(theta, b, type = 'l', main = 'Log-likehood', xlab = 'theta', ylab = 'l(theta)')
```



(b)

$$E[X|\theta] = \frac{1}{2\pi}\int_0^{2\pi} x[1-\cos(x-\theta)]dx \tag{15}$$

$$= \frac{1}{2\pi}[\frac{1}{2}x^2|_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} x\cos(x-\theta)dx] \tag{16}$$

$$= \pi - \frac{1}{2\pi}\int_0^{2\pi} xd\sin(x-\theta) \tag{17}$$

$$= \pi - \frac{1}{2\pi}x\sin(x-\theta)|_0^{2\pi} + \frac{1}{2\pi}\int_0^{2\pi}\sin(x-\theta)dx \tag{18}$$

$$= \pi + \sin\theta - \frac{1}{2\pi}\cos(x-\theta)|_0^{2\pi} \tag{19}$$

$$= \pi + \sin\theta = \overline{x} \tag{20}$$

So, $\hat{\theta}_{moment} = \arcsin(\overline{x} - \pi)$

```
theta_m <- asin(mean(x) - pi)
theta_m
```

```
## [1] 0.09539407
```

(c)

```r
Dloglikehood <- function(theta){
  return(sum(sin(theta - x)/(1 - cos(theta - x))))
}
DDloglikehood <- function(theta){
  return(sum(1/(1 - cos(theta - x))))
}
Newton_Raphson <- function(theta){
  while(abs(Dloglikehood(theta_m))>10^-10){
    theta <- theta - (Dloglikehood(theta))/(DDloglikehood(theta))
  }
  return(theta)
}
```

The result went wrong.

(d)

No results

(e)

```r
thetaSeq <- seq(-pi, pi, by = 2*pi/199)
```

Question 3

(a)

```r
beetles <- data.frame(
  days    = c(0,  8,  28,  41,  63,  69,   97, 117,  135,  154),
  beetles = c(2, 47, 192, 256, 768, 896, 1120, 896, 1184, 1024))
t <- beetles$days
N <- beetles$beetles
K <- seq(0, 10, 0.01)
r <- seq(-5, 5, 0.01)
```