HW3

Weixing Gu&Qi You

February 26, 2018

Question 1

a

$$Q(\psi|\psi^{(k)}) = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}^{(k+1)}\{log\pi + log\phi(y_i - x_i^T\beta_j; 0, \sigma^2\} \tag{1}$$

$$\tag{2}$$

$$\sum_{j=1}^{m} \pi_j = 1 \tag{3}$$

$$\tag{4}$$

$$L(\pi_1, \cdots, \pi_m, \lambda) = Q(\psi|\psi(k)) - \lambda(\sum_{j=1}^{m} \pi_j - 1) = 0 \tag{5}$$

$$\tag{6}$$

$$L'_{\pi_j} = 0L'_{\lambda} = 0, j = 1, 2, \cdots, m \tag{7}$$

$$\tag{8}$$

$$\sum_{i=1}^{n} p_{ij}^{(k+1)} \frac{1}{\pi_j} - \lambda = 0, j = 1, 2, \cdots, m \tag{9}$$

$$\tag{10}$$

$$\pi_j = \frac{\sum_{i=1}^{n} p_{ij}^{(k+1)}}{\lambda} \tag{11}$$

$$\tag{12}$$

$$\sum_{j=1}^{m} \pi_j = \frac{\sum_{j=1}^{m}\sum_{i=1}^{n} p_{ij}^{(k+1)}}{\lambda} \tag{13}$$

$$= \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}}{\lambda} \tag{14}$$

$$= \frac{n}{\lambda} = 1 \tag{15}$$

$$\tag{16}$$

$$\therefore \lambda = n \tag{17}$$

$$\tag{18}$$

$$\therefore \pi_j = \frac{\sum_{i=1}^{n} p_{ij}^{(k+1)}}{n} \tag{19}$$

b

$$Q(\psi|\psi^{(k)}) = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}\{log\pi_j + log(\frac{1}{\sqrt{2\pi}\sigma}exp[-\frac{(y_i - x_i^T\beta_j)^2}{2\sigma^2}])\} \tag{20}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}\{log\pi_i + log(\frac{1}{\sqrt{2\pi}\sigma}) + [-\frac{(y_i - x_i^T\beta_j)^2}{2\sigma^2}]\} \tag{21}$$

$$\tag{22}$$

It's the sum of m quadratic forms, where each form includes a single beta_j for every j.

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}(-\frac{(y_i - x_i^T\beta_j)^2}{2\sigma^2}) \tag{23}$$

$$= -\sum_{i=1}^{n} p_{ij}^{(k+1)}[\frac{(y_i - x_i^T\beta_j)^2}{2\sigma^2}] \tag{24}$$

$$= \sum_{i=1}^{n} p_{ij}^{(k+1)}[x_i^T(\frac{y_i}{x_i^T} - \beta_j)]^2 \tag{25}$$

$$= \sum_{i=1}^{n} p_{ij}^{(k+1)} x_i x_i^T(\frac{y_i}{x_i^T} - \beta_j)^2 \tag{26}$$

$$\tag{27}$$

$$\beta_j = \sum_{i=1}^{n} p_{ij} x_i x_i^T \frac{y_i}{x_i^T} = \sum_{i=1}^{n} p_{ij} x_i y_i^T \tag{28}$$

$$\tag{29}$$

c

$$\sigma^{2(k+1)} \tag{30}$$

$$= \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}(y_i - x_i^T\beta_j^{(k+1)})^2}{\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}} \tag{31}$$

$$= \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}(y_i - x_i^T\beta_j^{(k+1)})^2}{n} \tag{32}$$

$$\tag{33}$$

Question 2

a

$$g(x) \propto (2x^{\theta-1} + x^{\theta-\frac{1}{2}}e^{-x}) \tag{34}$$

$$\tag{35}$$

$$C\int_0^\infty (2x^{\theta-1} + x^{\theta-\frac{1}{2}})e^{-x}dx = 1 \tag{36}$$

$$\tag{37}$$

$$2C\Gamma(\theta) + C\Gamma(\theta + \frac{1}{2}) = 1 \tag{38}$$

$$\tag{39}$$

$$C = \frac{1}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})} \tag{40}$$

$$\tag{41}$$

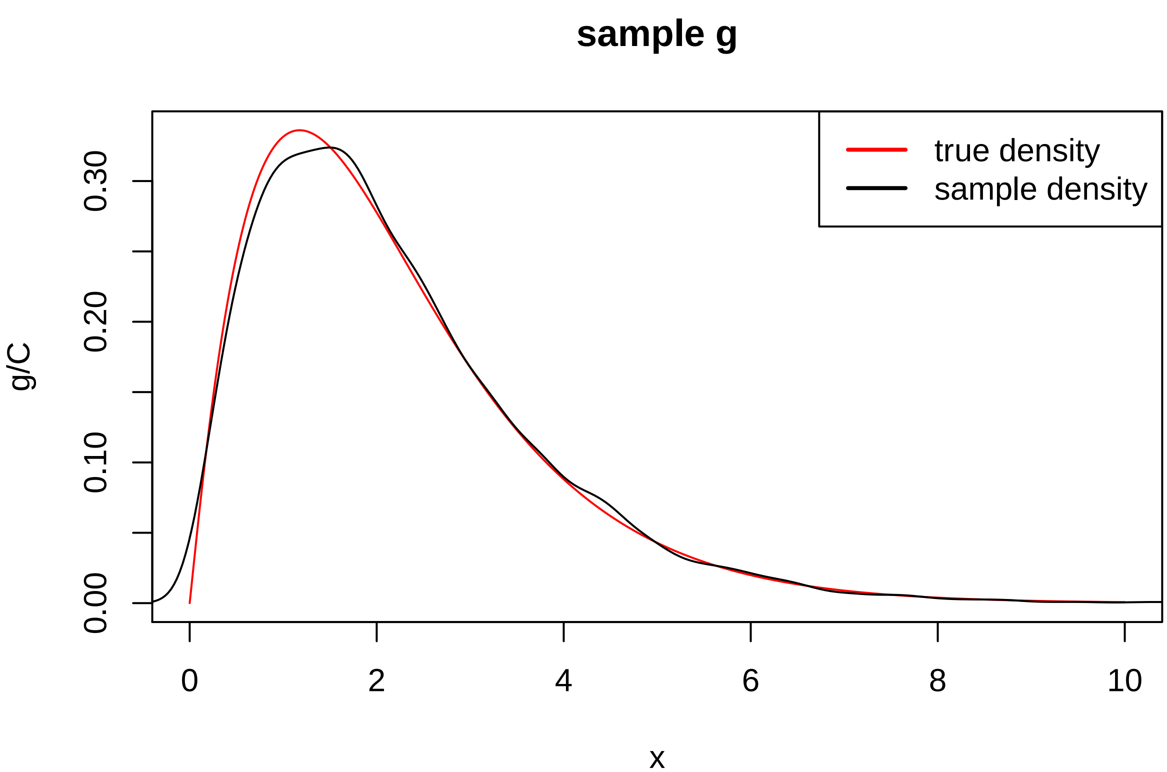
$$\therefore g(x) = \frac{2\Gamma(\theta)}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})}\frac{1}{\Gamma(\theta)}x^{\theta-1}e^{-x} + \frac{\Gamma(\theta + \frac{1}{2})}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})} * \frac{1}{\Gamma(\theta + \frac{1}{2})}x^{\theta-\frac{1}{2}}e^{-x} \tag{42}$$

$$\tag{43}$$

b

Notice that integration of g/C is 1,and this is an undecreasing function, so we use inverse function to sample g. By this way we first sample u~uniform(0,1), and compute the root of intergrate(g,0,x). In this way we can sample 'x'. Then by density function, we come out the result. We set theta to be 2

```
plot(x,tru,type='l',col='red',main='sample g',xlab = 'x',ylab='g/C')
lines(sample2)
legend('topright', c('true density','sample density'), lwd = 2, col = c('red','black'))
```



c

$$f'(x) = \frac{\frac{2+\sqrt{x}}{2\sqrt{4+x}} - \frac{\sqrt{4+x}}{2\sqrt{x}}}{(2+\sqrt{x})^2} \tag{44}$$

$$= \frac{\frac{2\sqrt{x}-4}{2\sqrt{4+x}\sqrt{x}}}{(2+\sqrt{x})^2} \tag{45}$$

$$\tag{46}$$

$$x \geq 4, f(x) \uparrow \tag{47}$$

$$\tag{48}$$

$$x \leq 4, f(x) \downarrow \tag{49}$$

$$\tag{50}$$

$$\therefore max f(x) = max(f(0), f(\infty)) \tag{51}$$

$$\tag{52}$$

$$f(\infty) = 1, f(0) = 1 \tag{53}$$

$$\tag{54}$$

$$\therefore \sqrt{4+x} \leq 2 + \sqrt{x} \tag{55}$$

$$\tag{56}$$

$$\therefore \sqrt{4+x} x^{\theta-1} e^{-x} \leq 2 + \sqrt{x} x^{\theta-1} e^{-x} \tag{57}$$

$$\tag{58}$$

$$f(x) = \sqrt{4+x} x^{\theta-1} e^{-x} \leq (2x^{\theta-1} + x^{\theta-\frac{1}{2}}) e^{-x} = g(x) \tag{59}$$

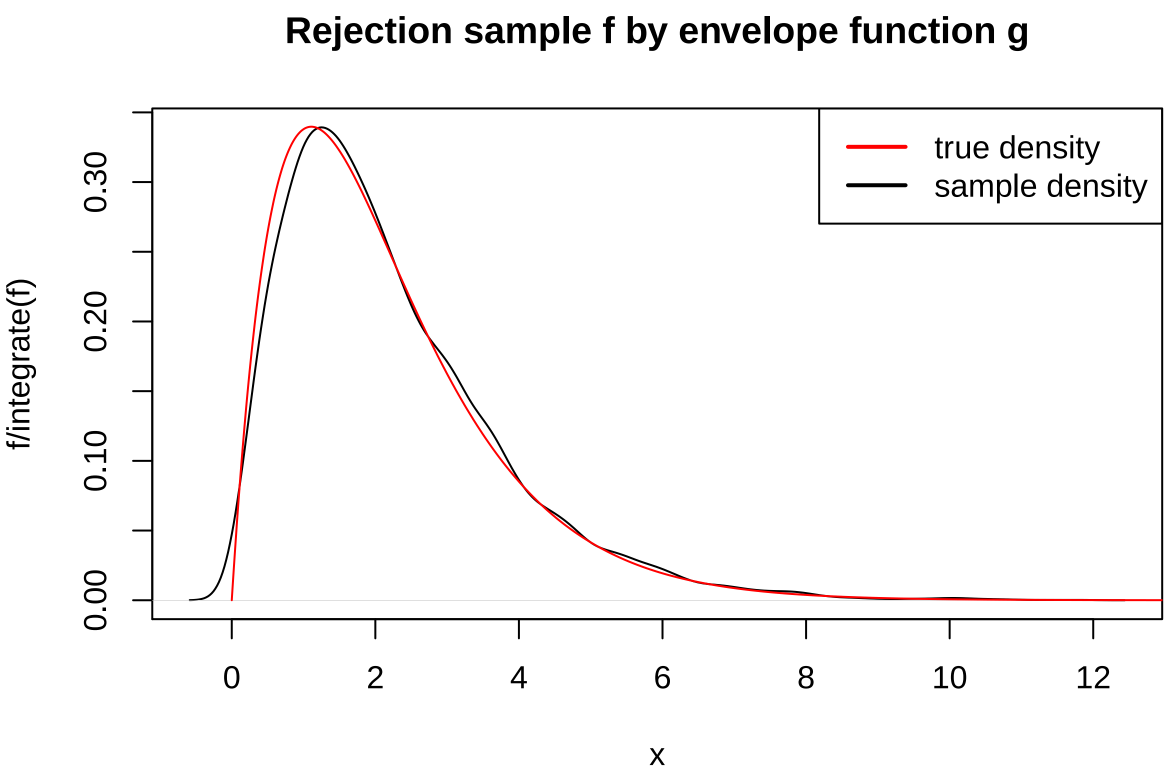
$$\tag{60}$$

Therefore, g(x) can be f(x)'s envelope.

Actually, we can see that f is smaller than g a.s. So here we set alpha to be 1 is fine.

We also set theta to be 2

```
plot(xx,main='Rejection sample f by envelope function g',xlab='x',ylab = 'f/integrate(f)')
lines(sa,saa/k,col='red')
legend('topright', c('true density','sample density'), lwd = 2, col = c('red','black'))
```



Question3

a

$$f_1(x) = \frac{x^{\theta-1}}{1+x^2} \leq x^{\theta-1} = g_1(x), 0 < x < 1 \tag{61}$$
$$\tag{62}$$

$$f_2(x) = \sqrt{2+x^2}(1-x)^{\beta-1} < 2(1-x)^{\beta-1} = g_2(x), 0 < x < 1 \tag{63}$$
$$\tag{64}$$

$$C_1 \int_0^1 g_1(x) = 1 \tag{65}$$
$$\tag{66}$$

$$C_1 \int_0^1 x^{\theta-1} dx = C_1 \frac{x^\theta}{\theta}|_0^1 = \frac{C_1}{\theta} = 1 \tag{67}$$
$$\tag{68}$$

$$\therefore C_1 = \theta \tag{69}$$
$$\tag{70}$$

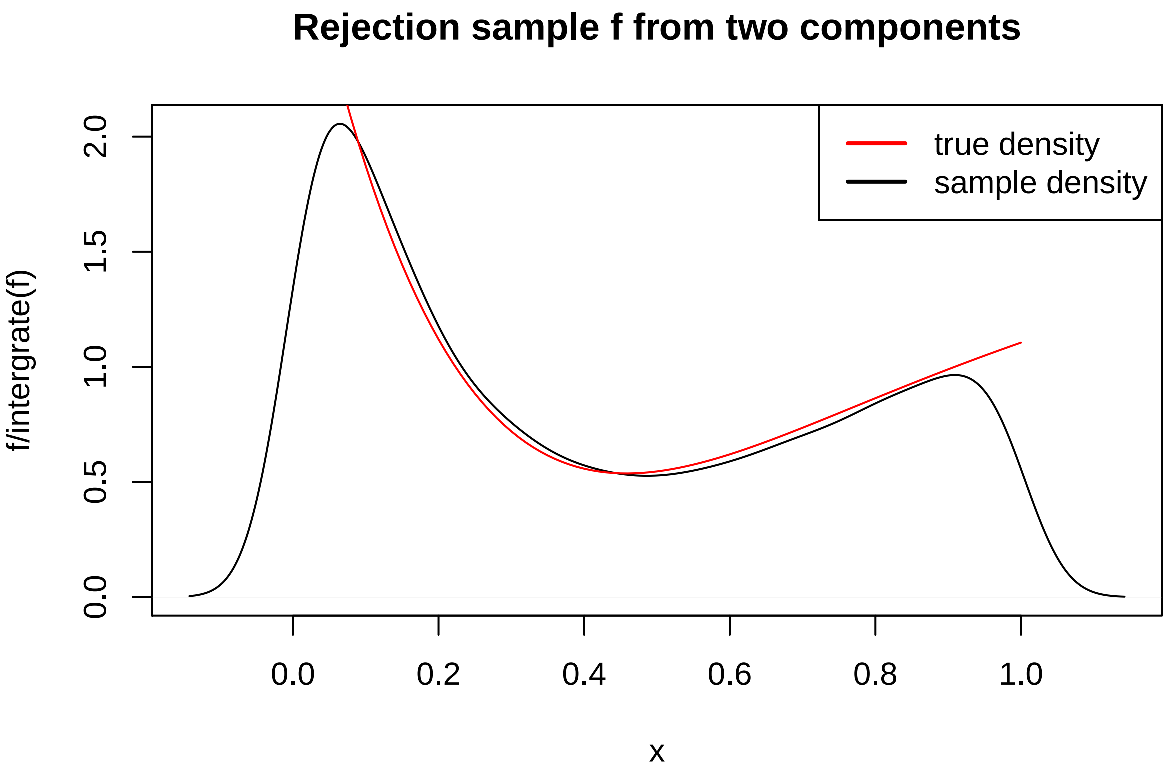
Similarily

$$C_2 = \frac{\beta}{2} \tag{71}$$

Here we sample g1 and g2 saperately and put them together to sample f. Similarly, we use inverse function to sample them.

We set coordinates (theta,beta) to be (3,9)

```
plot(sample2,main='Rejection sample f from two components',xlab='x',ylab = 'f/intergrate(f)')
lines(sa,saa/k,col='red')
legend('topright', c('true density','sample density'), lwd = 2, col = c('red','black'))
```

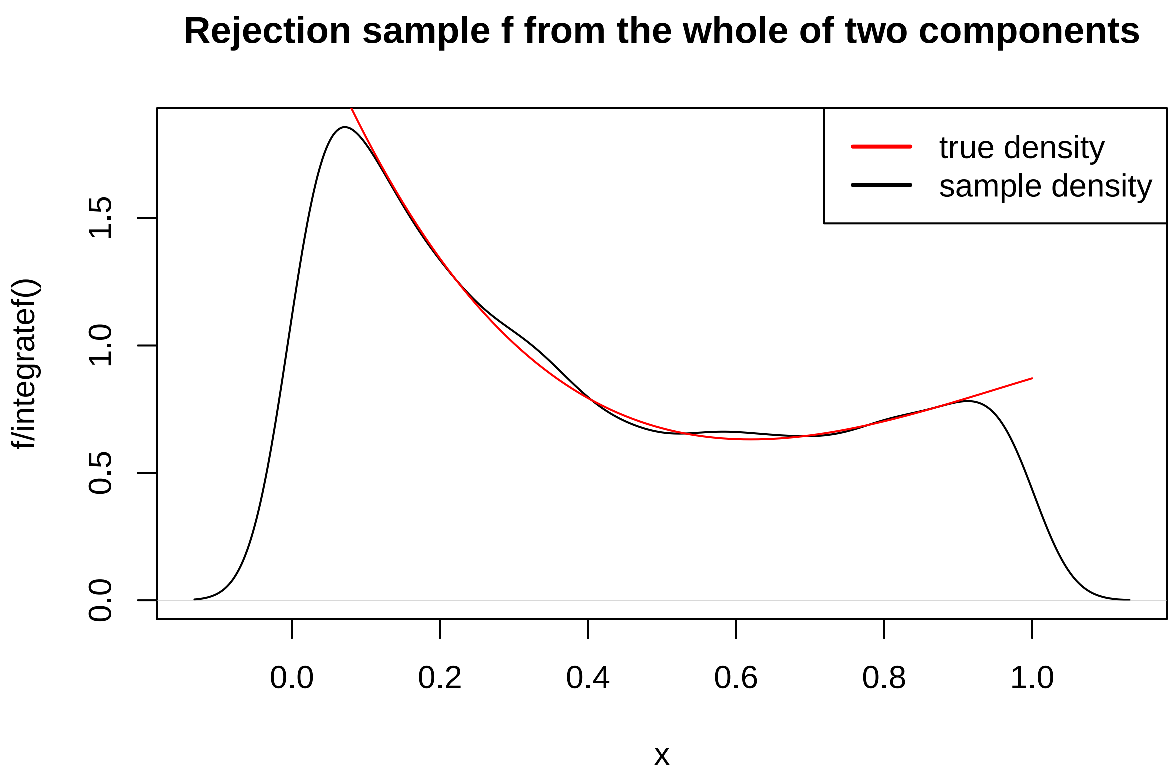


b

This section we se g to be g1+g2, and set alpha to be 1. By sampling g we can sample f.

We set coordinates (theta,beta) to be (3,4)

```
plot(sample2,main='Rejection sample f from the whole of two components',xlab = 'x',ylab = 'f/integratef
lines(sa,saa/k,col='red')
legend('topright', c('true density','sample density'), lwd = 2, col = c('red','black'))
```


Rejection sample f from the whole of two components
true density
sample density
f/integratef()
0.0
0.5
1.0
1.5
0.0
0.2
0.4
0.6
0.8
1.0
x