

# HW3

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## Question 1

a

$$Q(\psi|\psi^{(k)}) = \sum_{i=1}^n \sum_{j=1}^n p_{ij}^{(k+1)} \{\log \pi + \log \phi(y_i - x_i^T \beta_j; 0, \sigma^2)\} \quad (1)$$

(2)

$$\sum_{j=1}^m \pi_j = 1 \quad (3)$$

(4)

$$L(\pi_1, \dots, \pi_m, \lambda) = Q(\psi|\psi^{(k)}) - \lambda \left( \sum_{j=1}^m \pi_j - 1 \right) = 0 \quad (5)$$

(6)

$$L'_{\pi_j} = 0, L'_\lambda = 0, j = 1, 2, \dots, m \quad (7)$$

(8)

$$\sum_{i=1}^n p_{ij}^{(k+1)} \frac{1}{\pi_j} - \lambda = 0, j = 1, 2, \dots, m \quad (9)$$

(10)

$$\pi_j = \frac{\sum_{i=1}^n p_{ij}^{(k+1)}}{\lambda} \quad (11)$$

(12)

$$\sum_{j=1}^m \pi_j = \frac{\sum_{j=1}^m \sum_{i=1}^n p_{ij}^{(k+1)}}{\lambda} \quad (13)$$

$$= \frac{\sum_{i=1}^n \sum_{j=1}^m p_{ij}^{(k+1)}}{\lambda} \quad (14)$$

$$= \frac{n}{\lambda} = 1 \quad (15)$$

(16)

$$\therefore \lambda = n \quad (17)$$

(18)

$$\therefore \pi_j = \frac{\sum_{i=1}^n p_{ij}^{(k+1)}}{n} \quad (19)$$

**b**

$$Q(\psi|\psi^{(k)}) = \sum_{i=1}^n \sum_{j=1}^m p_{ij}^{(k+1)} \{ \log \pi_j + \log \left( \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{(y_i - x_i^T \beta_j)^2}{2\sigma^2} \right] \right) \} \quad (20)$$

$$= \sum_{i=1}^n \sum_{j=1}^m p_{ij}^{(k+1)} \{ \log \pi_i + \log \left( \frac{1}{\sqrt{2\pi}\sigma} \right) + \left[ -\frac{(y_i - x_i^T \beta_j)^2}{2\sigma^2} \right] \} \quad (21)$$

$$(22)$$

It's the sum of m quadratic forms, where each form includes a single  $\beta_j$  for every j.

$$\sum_{i=1}^n \sum_{j=1}^m p_{ij}^{(k+1)} \left( -\frac{(y_i - x_i^T \beta_j)^2}{2\sigma^2} \right) \quad (23)$$

$$= - \sum_{i=1}^n p_{ij}^{(k+1)} \left[ \frac{(y_i - x_i^T \beta_j)^2}{2\sigma^2} \right] \quad (24)$$

$$= \sum_{i=1}^n p_{ij}^{(k+1)} \left[ x_i^T \left( \frac{y_i}{x_i^T} - \beta_j \right) \right]^2 \quad (25)$$

$$= \sum_{i=1}^n p_{ij}^{(k+1)} x_i x_i^T \left( \frac{y_i}{x_i^T} - \beta_j \right)^2 \quad (26)$$

$$(27)$$

$$\beta_j = \sum_{i=1}^n p_{ij} x_i x_i^T \frac{y_i}{x_i^T} = \sum_{i=1}^n p_{ij} x_i y_i^T \quad (28)$$

$$(29)$$

**c**

$$\sigma^{2(k+1)} \quad (30)$$

$$= \frac{\sum_{i=1}^n \sum_{j=1}^m p_{ij}^{(k+1)} (y_i - x_i^T \beta_j^{(k+1)})^2}{\sum_{i=1}^n \sum_{j=1}^m p_{ij}^{(k+1)}} \quad (31)$$

$$= \frac{\sum_{i=1}^n \sum_{j=1}^m p_{ij}^{(k+1)} (y_i - x_i^T \beta_j^{(k+1)})^2}{n} \quad (32)$$

$$(33)$$

## Question 2

**a**

$$g(x) \propto (2x^{\theta-1} + x^{\theta-\frac{1}{2}} e^{-x}) \quad (34)$$

$$(35)$$

$$C \int_0^{\infty} (2x^{\theta-1} + x^{\theta-\frac{1}{2}})e^{-x} dx = 1 \quad (36)$$

$$(37)$$

$$2C\Gamma(\theta) + C\Gamma(\theta + \frac{1}{2}) = 1 \quad (38)$$

$$(39)$$

$$C = \frac{1}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})} \quad (40)$$

$$(41)$$

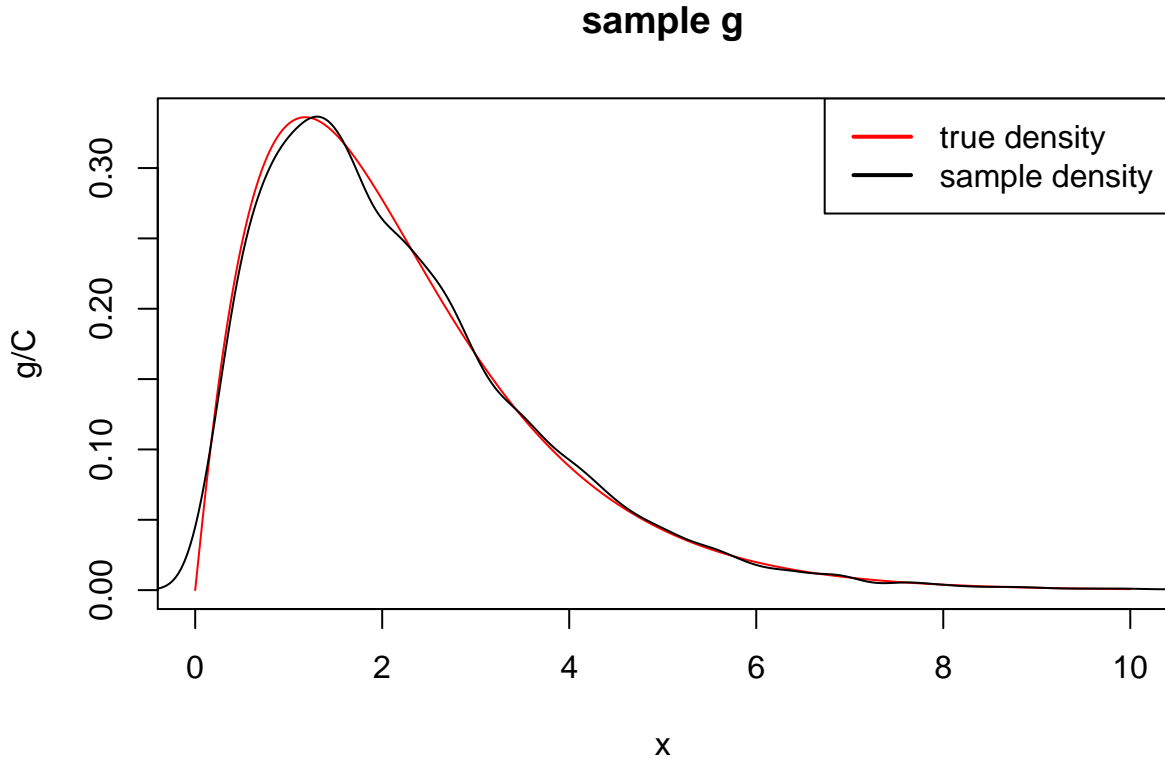
$$\therefore g(x) = \frac{2\Gamma(\theta)}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})} \frac{1}{\Gamma(\theta)} x^{\theta-1} e^{-x} + \frac{\Gamma(\theta + \frac{1}{2})}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})} * \frac{1}{\Gamma(\theta + \frac{1}{2})} x^{\theta-\frac{1}{2}} e^{-x} \quad (42)$$

$$(43)$$

**b**

Notice that integration of  $g/C$  is 1, and this is an undecreasing function, so we use inverse function to sample  $g$ . By this way we first sample  $u \sim \text{uniform}(0,1)$ , and compute the root of  $\text{intergrate}(g,0,x)$ . In this way we can sample 'x'. Then by density function, we come out the result. We set theta to be 2

```
plot(x,tru,type='l',col='red',main='sample g',xlab = 'x',ylab='g/C')
lines(sample2)
legend('topright', c('true density','sample density'), lwd = 2, col = c('red','black'))
```



**C**

$$f'(x) = \frac{\frac{2+\sqrt{x}}{2\sqrt{4+x}} - \frac{\sqrt{4+x}}{2\sqrt{x}}}{(2+\sqrt{x})^2} \quad (44)$$

$$= \frac{\frac{2\sqrt{x}-4}{2\sqrt{4+x}\sqrt{x}}}{(2+\sqrt{x})^2} \quad (45)$$

$$(46)$$

$$x \geq 4, f(x) \uparrow \quad (47)$$

$$(48)$$

$$x \leq 4, f(x) \downarrow \quad (49)$$

$$(50)$$

$$\therefore \max f(x) = \max(f(0), f(\infty)) \quad (51)$$

$$(52)$$

$$f(\infty) = 1, f(0) = 1 \quad (53)$$

$$(54)$$

$$\therefore \sqrt{4+x} \leq 2 + \sqrt{x} \quad (55)$$

$$(56)$$

$$\therefore \sqrt{4+xx}^{\theta-1}e^{-x} \leq 2 + \sqrt{xx}^{\theta-1}e^{-x} \quad (57)$$

$$(58)$$

$$f(x) = \sqrt{4+xx}^{\theta-1}e^{-x} \leq (2x^{\theta-1} + x^{\theta-\frac{1}{2}})e^{-x} = g(x) \quad (59)$$

$$(60)$$

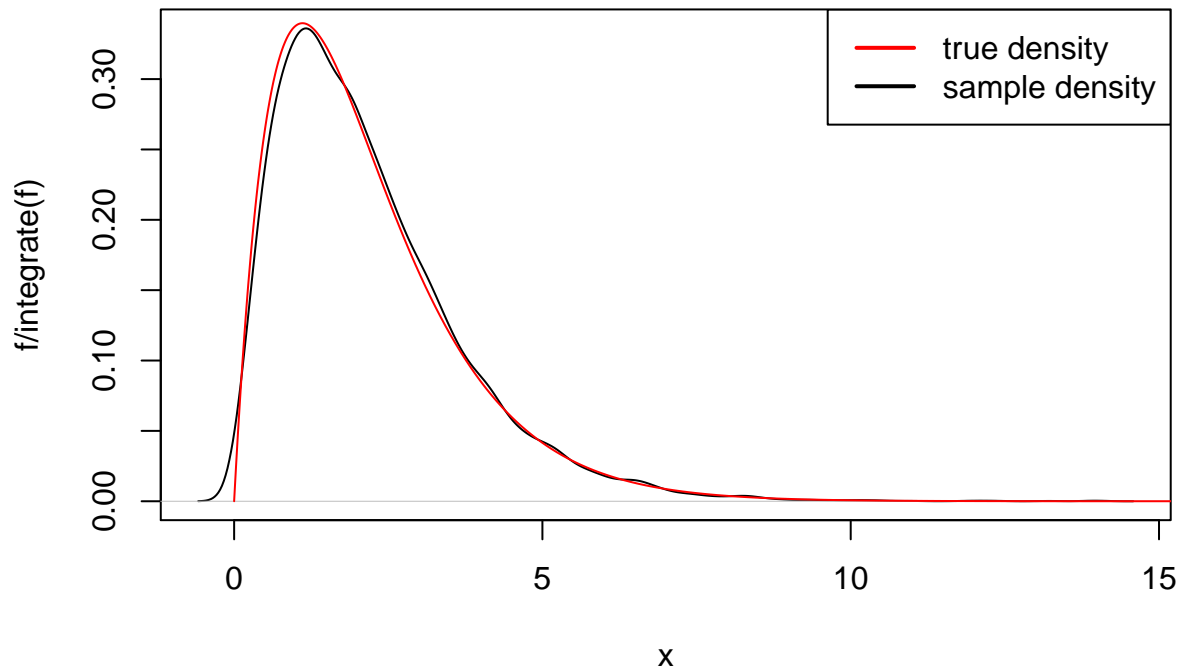
Therefore,  $g(x)$  can be  $f(x)$ 's envelope.

Actually, we can see that  $f$  is smaller than  $g$  a.s. So here we set  $\alpha$  to be 1 is fine.

We also set  $\theta$  to be 2

```
plot(xx,main='Rejection sample f by envelope function g',xlab='x',ylab = 'f/integrate(f)')
lines(sa,saa/k,col='red')
legend('topright', c('true density','sample density'), lwd = 2, col = c('red','black'))
```

## Rejection sample f by envelope function g



### Question3

a

$$f_1(x) = \frac{x^{\theta-1}}{1+x^2} \leq x^{\theta-1} = g_1(x), 0 < x < 1 \quad (61)$$

(62)

$$f_2(x) = \sqrt{2+x^2}(1-x)^{\beta-1} < 2(1-x)^{\beta-1} = g_2(x), 0 < x < 1 \quad (63)$$

(64)

$$C_1 \int_0^1 g_1(x) dx = 1 \quad (65)$$

(66)

$$C_1 \int_0^1 x^{\theta-1} dx = C_1 \frac{x^\theta}{\theta} \Big|_0^1 = \frac{C_1}{\theta} = 1 \quad (67)$$

(68)

$$\therefore C_1 = \theta \quad (69)$$

(70)

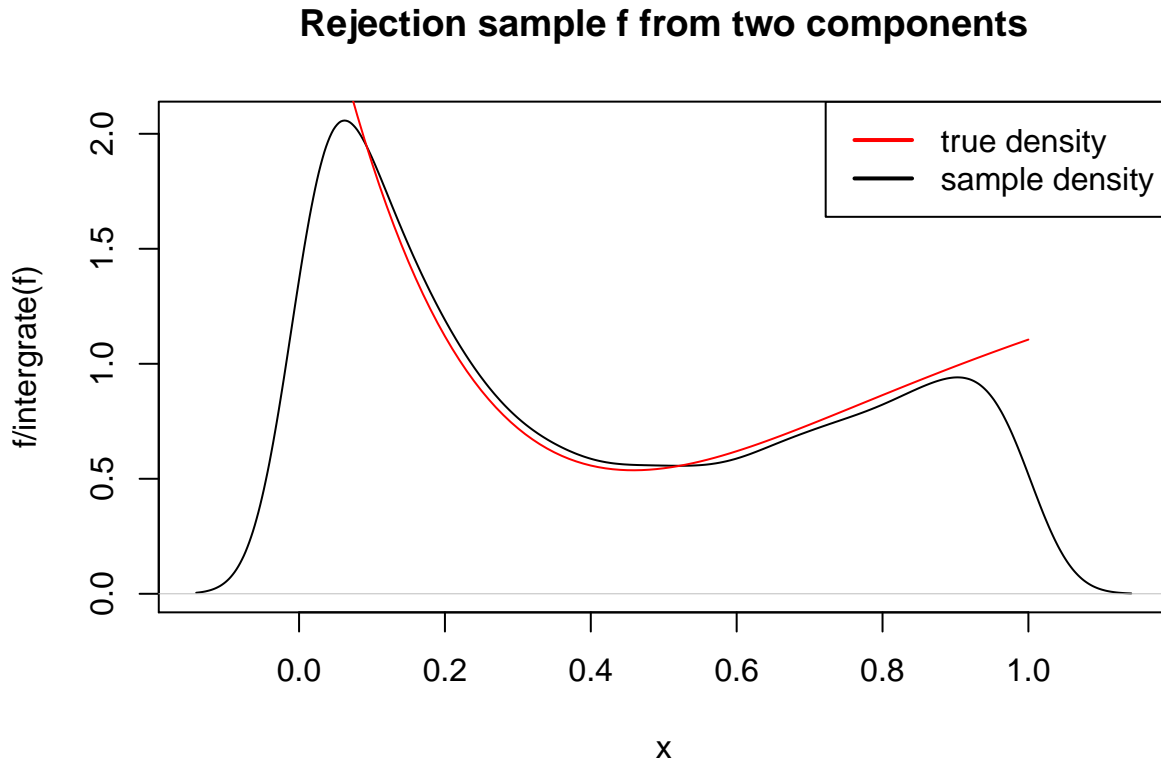
Similarly

$$C_2 = \frac{\beta}{2} \quad (71)$$

Here we sample  $g_1$  and  $g_2$  separately and put them together to sample  $f$ . Similarly, we use inverse function to sample them.

We set coordinates  $(\theta, \beta)$  to be  $(3, 9)$

```
plot(sample2, main='Rejection sample f from two components', xlab='x', ylab = 'f/integrate(f)')
lines(sa, saa/k, col='red')
legend('topright', c('true density', 'sample density'), lwd = 2, col = c('red', 'black'))
```



**b**

This section we set  $g$  to be  $g_1 + g_2$ , and set  $\alpha$  to be 1. By sampling  $g$  we can sample  $f$ .

We set coordinates  $(\theta, \beta)$  to be  $(3, 4)$

```
plot(sample2, main='Rejection sample f from the whole of two components', xlab = 'x', ylab = 'f/integrate(f)')
lines(sa, saa/k, col='red')
legend('topright', c('true density', 'sample density'), lwd = 2, col = c('red', 'black'))
```

## Rejection sample f from the whole of two components

