5361 HW3

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Question 1

We first consider the expression which contains β

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)}(y_i - x_i^T\beta_j)^2/(2\sigma^2)$$

The intuition is that β should be the weighted average of the samples for every j. The weights equals $p_{ij}^{(k+1)}x_i x^T$, and the samples are of the form $\frac{y}{x_i^T}$. We use $\frac{y}{x_i^T}$ to represent a vector $x^* \in \mathbb{R}^p s.t. x^T \cdot x^* = y$.

$$\beta_j^{(k+1)} = argmax_\beta\{-\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k+1)} x_i x_i^T(\frac{y}{x_i^T} - \beta_j)^2/(2\sigma^2)\}$$

$$= (\sum_{i=1}^{n} x_i p_{ij}^{(k+1)} y_i)/(\sum_{i=1}^{n} p_{ij}^{(k+1)} x_i x_i^T)$$

Next, we consider the expression which contains σ. Since σ^2 appears only in the density function of normal distribution and is the variance of the normal distribution models, to maximize the log likelihood we need to set σ^2 to equal the sample variance.

$$\sigma^{2(k+1)} = \frac{\sum_{j=1}^{m}\sum_{i=1}^{n} p_{ij}^{(k+1)}(y_i - x_i^T\beta_j^{(k+1)} - 0)^2}{\sum_{j=1}^{m}\sum_{i=1}^{n} p_{ij}^{(k+1)}}$$

$$= \frac{\sum_{j=1}^{m}\sum_{i=1}^{n} p_{ij}^{(k+1)}(y_i - x_i^T\beta_j^{(k+1)})^2}{\sum_{i=1}^{n} 1}$$

$$= \frac{\sum_{j=1}^{m}\sum_{i=1}^{n} p_{ij}^{(k+1)}(y_i - x_i^T\beta_j^{(k+1)})^2}{n}$$

Finally, we obtain π using Lagrangian multipliers.

$$L(\pi, \lambda) = Q(\psi|\psi(k)) - \lambda(\vec{\pi}\cdot\vec{1}) = 0$$

$$L_{\pi j} = \sum_{i=1}^{n} p_{ij}^{(k+1)}\frac{1}{\pi_j} - \lambda = 0$$

$$\pi_j = \frac{\sum_{i=1}^{n} p_{ij}^{(k+1)}}{\lambda}$$

$$\sum_{j=1}^{m}\pi_j = \frac{\sum_{j=1}^{m}\sum_{i=1}^{n} p_{ij}^{(k+1)}}{\lambda} = \frac{n}{\lambda} = 1$$

$$\lambda = n$$

$$\pi_j = \frac{\sum_{i=1}^n p_{ij}^{(k+1)}}{\lambda} = \frac{\sum_{i=1}^n p_{ij}^{(k+1)}}{n}$$

Question 2

(a)

$$C\int_0^\infty (2x^{\theta-1} + x^{\theta-\frac{1}{2}})e^{-x}dx$$

$$= 2C\int_0^\infty x^{\theta-1}e^{-x}dx + C\int_0^\infty x^{\theta-\frac{1}{2}}e^{-x}dx$$

$$= 2C\Gamma(\theta) + C\Gamma(\theta + \frac{1}{2})$$

$$= C(2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})) = 1$$

$$C = \frac{1}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})}$$

Let

$$g_1 = \frac{1}{\Gamma(\theta)}x^{\theta-1}e^{-x}$$

$$g_2 = \frac{1}{\Gamma(\theta + \frac{1}{2})}x^{\theta-\frac{1}{2}}e^{-x}$$

Then

$$g = C_1 g_1 + C_2 g_2$$

where

$$C_1 = \frac{2\Gamma(\theta)}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})}$$

$$C_2 = \frac{\Gamma(\theta + \frac{1}{2})}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})}$$

(b)

1.Obtain

$$C\int_0^x (2s^{\theta-1}+s^{\theta-\frac{1}{2}})e^{-s}ds$$

;

2.Find x such that

$$C\int_0^x (2s^{\theta-1}+s^{\theta-\frac{1}{2}})e^{-s}ds-u=0$$

where u~unif(0,1);

3.Repeat step 2 until

$$|\vec{x}|=10000$$

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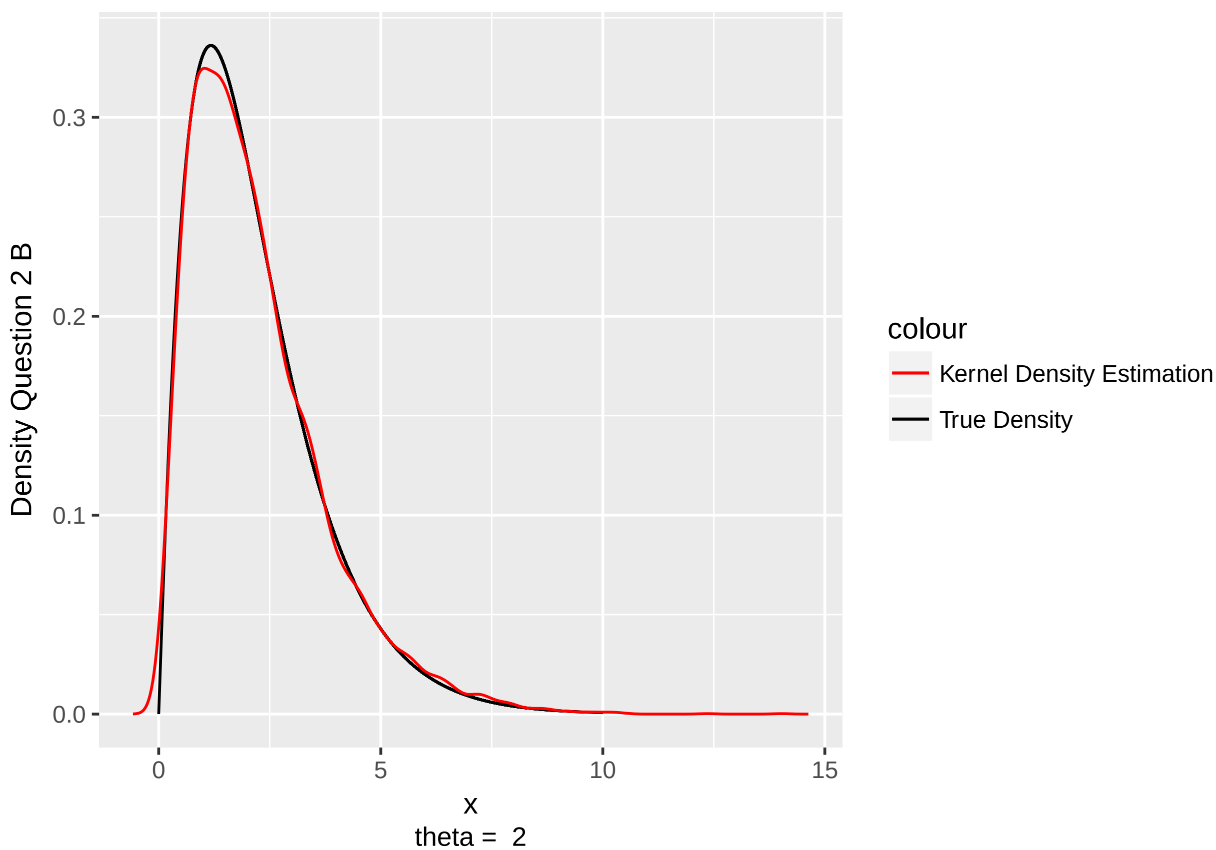

(c)

Let

$$\sqrt{4+x}\cdot x^{\theta-1}e^{-x}\le \alpha C(2x^{\theta-1}+x^{\theta-\frac{1}{2}})e^{-x}$$

$$\alpha C(2+x^{\frac{1}{2}})\ge\sqrt{4+x}$$

$$\alpha C\ge\frac{\sqrt{4+x}}{2+\sqrt{x}}$$

Notice that

$$\forall x \in \mathbb{R}^+, \frac{\sqrt{4+x}}{2+\sqrt{x}} \leq 1,$$

so we choose

$$\alpha C = 1, \alpha = \frac{1}{C}$$

1.Sample x from

$$g = C(2x^{\theta} + x^{\theta-\frac{1}{2}})e^{-x}$$

by inverse transform sampling;

2.If

$$U \leq \frac{f(x)}{\alpha g(x)}$$

where U~unif(0,1),then keep x;

3.Repeat step 1 and step 2 until

$$|\vec{x}| = 10000$$

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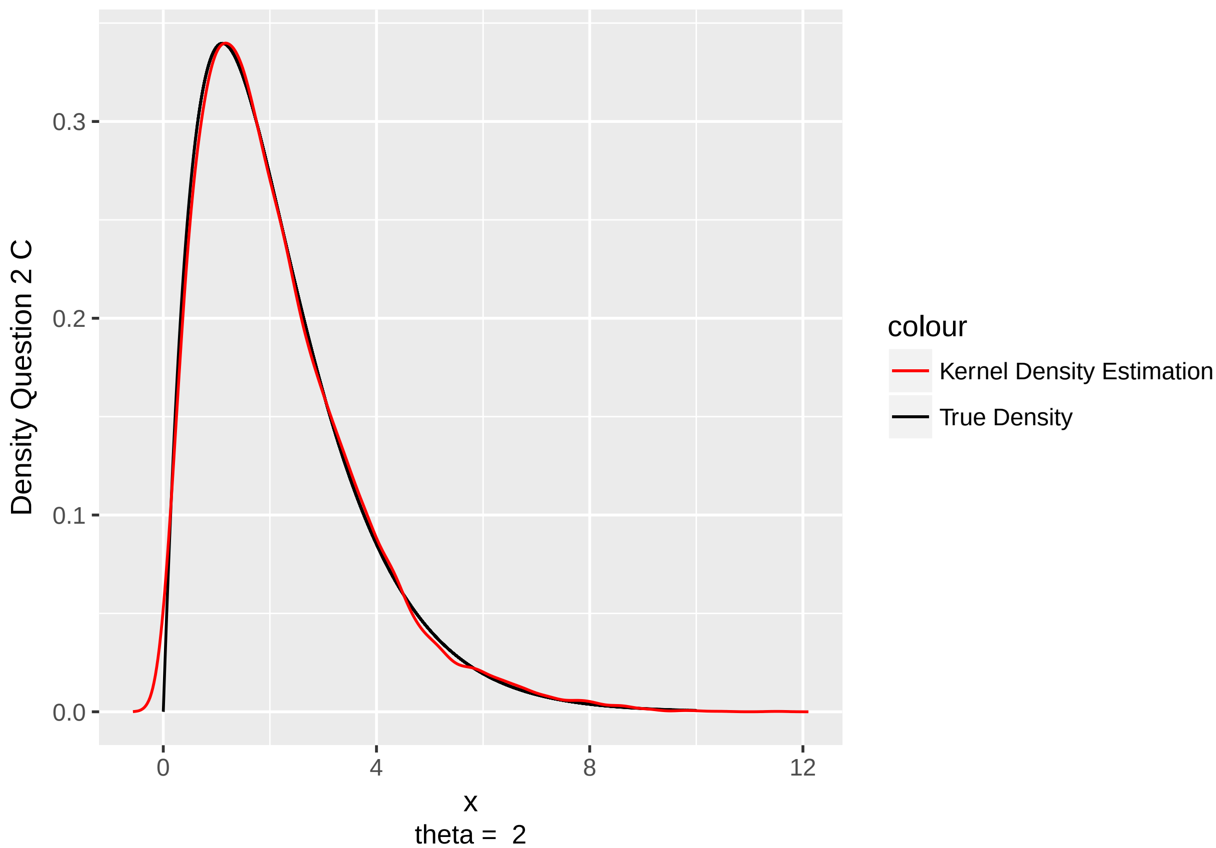

Question 3

(a)

Let

$$g = P_1(\frac{x^{\theta-1}}{B(\theta,1)}) + P_2(\frac{(1-x)^{\beta-1}}{B(2,\beta)})$$

$$P_1(\frac{x^{\theta-1}}{B(\theta,1)}) \geq \frac{x^{\theta-1}}{1+x^2} \Rightarrow P_1(\frac{x^{\theta-1}}{B(\theta,1)}) \geq x^{\theta-1} \Rightarrow P_1 \geq B(\theta,1)$$

$$P_2(\frac{(1-x)^{\beta-1}}{B(1,\beta)}) \geq \sqrt{2+x^2}(1-x)^{\beta-1} \Rightarrow P_2(\frac{(1-x)^{\beta-1}}{B(1,\beta)}) \geq \sqrt{3}(1-x)^{\beta-1} \Rightarrow P_2 \geq \sqrt{3}B(1,\beta)$$

Now, choose

$$P_1 = B(\theta,1), P_2 = \sqrt{3}B(1,\beta)$$

1.If

$$U_1 \leq \frac{P_1}{P_1+P_2}$$

where

$$U_1 \ unif(0,1)$$

set

$$g = Beta(\theta,1), f = \frac{x^{\theta-1}}{1+x^2}, \alpha = P_1$$

else set

$$g = Beta(1,\beta), f = \sqrt{2+x^2}(1-x)^{\beta-1}, \alpha = P_2$$

;

2.Sample

$$x \ gU_2 \ unif(0,1)$$

if

$$U_2 \leq \frac{f(x)}{\alpha g(x)},$$

keep x;

3.Repeat step 1 and step 2 until

$$|\vec{x}| = 10000$$

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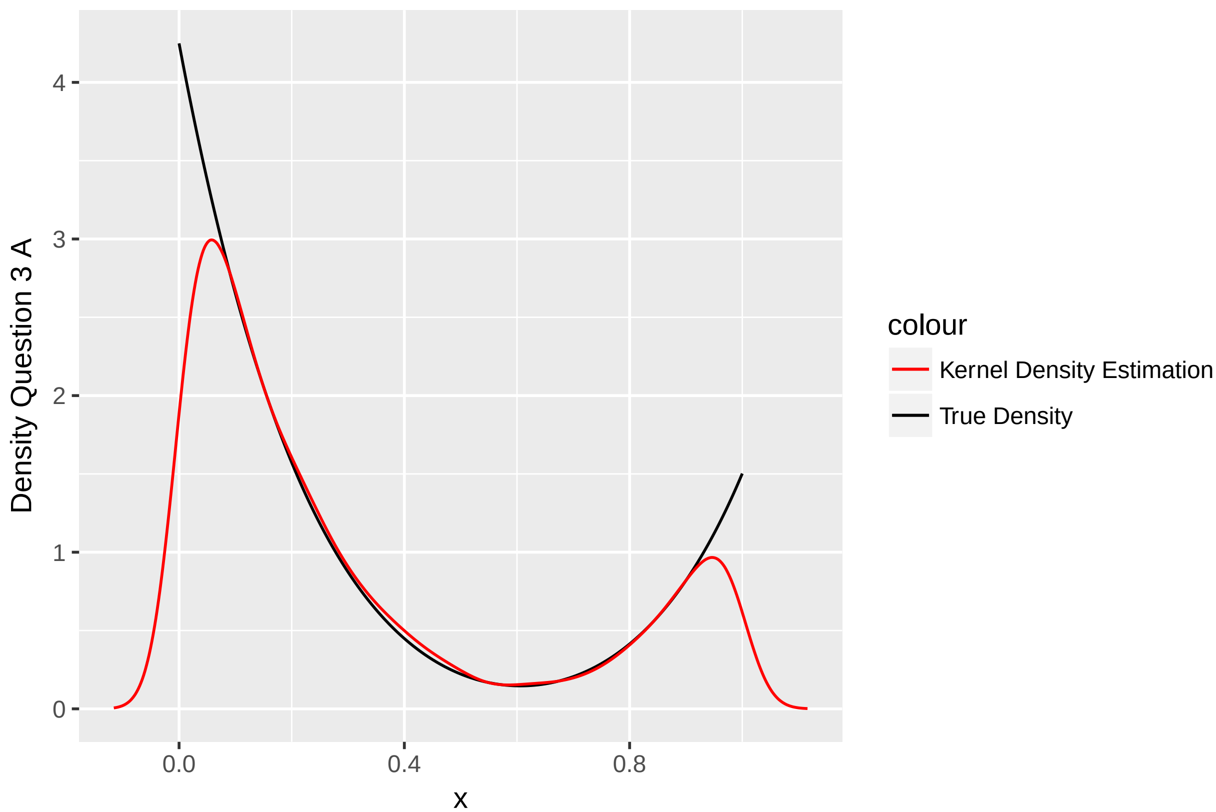

theta = 7.7 , beta = 5.5

(b)

Let

$$g = \frac{1}{2}Beta(\theta, 1) + \frac{1}{2}Beta(1, \beta)$$

$$\alpha_1 Beta(\theta, 1) \geq \frac{x^{\theta-1}}{1+x^2} \Rightarrow \alpha_1 \geq B(\theta, 1)$$

$$\alpha_2 Beta(1, \beta) \geq \sqrt{2+x^2}(1-x)^{\beta-1} \Rightarrow \alpha_2 \geq \sqrt{3}B(1, \beta)$$

Choose

$$\alpha = max(2\alpha_1, 2\alpha_2)$$

then

$$\alpha g \geq \alpha_1 Beta(\theta, 1) + \alpha_2 Beta(1, \beta) \geq f$$

1.Generate

$$x\ g$$

by inverse transform sampling;

2.If

$$U \leq \frac{f(x)}{\alpha g(x)}$$

where

$$U\ unif(0, 1)$$

, keep x;

3.Repeat step 1 and step 2 until

$$|\vec{x}| = 10000$$

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